GooseMaterial/Metal/LinearStrain/NonLinearElastic

T.W.J. de Geus

*Contact: tom@geus.me – www.geus.me – https://github.com/tdegeus/GooseMaterial

Abstract

This constitutive model encompasses a non-linear, but history independent, relation between the Cauchy stress, $\boldsymbol{\sigma}$, and the linear strain tensor, $\boldsymbol{\varepsilon}$, i.e.:

$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}\right)$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

Keywords: non-linear elasticity

Contents

1 Constitutive model

The following strain-energy is defined:

$$W(\boldsymbol{\varepsilon}) = \frac{9}{2} K \varepsilon_\mathrm{m}^2 + \frac{\sigma_0\, \varepsilon_0}{n+1} \left(\frac{\varepsilon_\mathrm{eq}}{\varepsilon_0} \right)^{n+1} \tag{1}$$

where K is the bulk modulus, ε_0 and σ_0 are a reference strain and stress respectively, and n is an exponent that sets the degree of non-linearity. Finally ε_m and ε_eq are the hydrostatic and equivalent strains (see Appendix B).

This leads to the following stress–strain relation:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = 3K\varepsilon_\mathrm{m}\, \boldsymbol{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n}\, \varepsilon_\mathrm{eq}^{n-1}\, \boldsymbol{\varepsilon}_\mathrm{d} \tag{2}$$

see Appendix A for nomenclature.

2 Consistent tangent

The consistent tangent maps a variation in strain, $\delta\boldsymbol{\varepsilon}$, to a variation in stress, $\delta\boldsymbol{\sigma}$, as follows

$$\delta\boldsymbol{\sigma} = \mathbb{C} : \delta\boldsymbol{\varepsilon} \tag{3}$$

The tangent, $\mathbb{C}$, thus corresponds to the derivative of (2) w.r.t. strain. For this, the chain rule is employed:

$$\mathbb{C} = \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[3K\varepsilon_\mathrm{m}\, \boldsymbol{I} \right] + \frac{\partial}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \left[\frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n}\, \varepsilon_\mathrm{eq}^{n-1}\, \boldsymbol{\varepsilon}_\mathrm{d} \right] : \frac{\partial \boldsymbol{\varepsilon}_\mathrm{d}}{\partial \boldsymbol{\varepsilon}} \tag{4}$$

Where:

- the derivative of the volumetric part reads

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[3K\varepsilon_\mathrm{m}\, \boldsymbol{I} \right] = K \boldsymbol{I} \otimes \boldsymbol{I} \tag{5}$$

- the chain rule for the deviatoric part reads

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \left[\varepsilon_\mathrm{eq}^{n-1}\, \boldsymbol{\varepsilon}_\mathrm{d} \right] = \frac{\partial \left[\varepsilon_\mathrm{eq}^{n-1} \right]}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1} \frac{\partial \boldsymbol{\varepsilon}_\mathrm{d}}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \tag{6}$$

$$= \tfrac{2}{3}(n-1)\, \varepsilon_\mathrm{eq}^{n-3}\, \boldsymbol{\varepsilon}_\mathrm{d} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1}\, \mathbb{I} \tag{7}$$

- and it has been used that

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \left[\varepsilon_\mathrm{eq}^{n-1} \right] = (n-1)\, \varepsilon_\mathrm{eq}^{n-2}\, \frac{2}{3} \frac{\boldsymbol{\varepsilon}_\mathrm{d}}{\varepsilon_\mathrm{eq}} \tag{8}$$

$$= \tfrac{2}{3}(n-1)\, \varepsilon_\mathrm{eq}^{n-3}\, \boldsymbol{\varepsilon}_\mathrm{d} \tag{9}$$

Combining the above yields:

$$\mathbb{C} = K \boldsymbol{I} \otimes \boldsymbol{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \left(\tfrac{2}{3}(n-1)\, \varepsilon_\mathrm{eq}^{n-3} \boldsymbol{\varepsilon}_\mathrm{d} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1} \mathbb{I} \right) : \mathbb{I}_\mathrm{d} \tag{10}$$

$$= K \boldsymbol{I} \otimes \boldsymbol{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \left(\tfrac{2}{3}(n-1)\, \varepsilon_\mathrm{eq}^{n-3} \boldsymbol{\varepsilon}_\mathrm{d} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1}\, \mathbb{I}_\mathrm{d} \right) \tag{11}$$

A Nomenclature

- Dyadic tensor product

$$\mathbb{C} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{12}$$
$$C_{ijkl} = A_{ij}\, B_{kl} \tag{13}$$

- Double tensor contraction

$$C = \boldsymbol{A} : \boldsymbol{B} \tag{14}$$
$$= A_{ij}\, B_{ji} \tag{15}$$

- Deviatoric projection tensor

$$\mathbb{I}_\mathrm{d} = \mathbb{I}_\mathrm{s} - \tfrac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I} \tag{16}$$

B Strain measures

- Mean strain

$$\varepsilon_\mathrm{m} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon}) = \tfrac{1}{3}\,\boldsymbol{\varepsilon} : \boldsymbol{I} \tag{17}$$

- Strain deviator

$$\boldsymbol{\varepsilon}_\mathrm{d} = \boldsymbol{\varepsilon} - \varepsilon_\mathrm{m}\,\boldsymbol{I} = \mathbb{I}_\mathrm{d} : \boldsymbol{\varepsilon} \tag{18}$$

- Equivalent strain

$$\varepsilon_\mathrm{eq} = \sqrt{\tfrac{2}{3}\,\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}} \tag{19}$$

C Variations

- Strain deviator

$$\delta\boldsymbol{\varepsilon}_\mathrm{d} = \big(\mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}\big) : \delta\boldsymbol{\varepsilon} = \mathbb{I}_\mathrm{d} : \delta\boldsymbol{\varepsilon} \tag{20}$$

- Mean equivalent strain

$$\delta\varepsilon_\mathrm{m} = \tfrac{1}{3}\boldsymbol{I} : \delta\boldsymbol{\varepsilon} \tag{21}$$

- Von Mises equivalent strain

$$\delta\varepsilon_\mathrm{eq} = \frac{1}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d} + \delta\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}\right) \tag{22}$$
$$= \frac{2}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d}\right) \tag{23}$$
$$= \frac{2}{3}\frac{\boldsymbol{\varepsilon}_\mathrm{d}}{\varepsilon_\mathrm{eq}} : \delta\boldsymbol{\varepsilon}_\mathrm{d} \tag{24}$$