GooseSolid/NonLinearElastic

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Abstract

This constitutive model encompasses a non-linear, but history independent, relation between the Cauchy stress, $\boldsymbol{\sigma}$, and the linear strain tensor, $\boldsymbol{\varepsilon}$, i.e.:

$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}\right)$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

Keywords: non-linear elasticity

Contents

1 Constitutive model

The following strain-energy is defined:

$$W(\boldsymbol{\varepsilon}) = \frac{9}{2}K\varepsilon_{\mathrm{m}} + \frac{\sigma_0\,\varepsilon_0}{n+1}\left(\frac{\varepsilon_{\mathrm{eq}}}{\varepsilon_0}\right)^{n+1} \tag{1}$$

where K is the bulk modulus, ε_0 and σ_0 are a reference strain and stress respectively, and n is an exponent that sets the degree of non-linearity. Finally ε_{m} and $\varepsilon_{\mathrm{eq}}$ are the hydrostatic and equivalent strains (see Appendix B).

This leads to the following stress–strain relation:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = 3K\varepsilon_{\mathrm{m}}\,\boldsymbol{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\,\varepsilon_{\mathrm{eq}}^{n-1}\,\boldsymbol{\varepsilon}_{\mathrm{d}} \tag{2}$$

see Appendix A for nomenclature.

2 Consistent tangent

The consistent tangent maps a variation in strain, $\delta\boldsymbol{\varepsilon}$, to a variation in stress, $\delta\boldsymbol{\sigma}$, as follows

$$\delta\boldsymbol{\sigma} = \mathbb{C} : \delta\boldsymbol{\varepsilon} \tag{3}$$

The tangent, $\mathbb{C}$, thus corresponds to the derivative of (2) w.r.t. strain. For this, the chain rule is employed:

$$\mathbb{C} = \frac{\partial}{\partial\boldsymbol{\varepsilon}}\left[\,3K\varepsilon_{\mathrm{m}}\,\boldsymbol{I}\,\right] + \frac{\partial}{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}\left[\,\frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\,\varepsilon_{\mathrm{eq}}^{n-1}\,\boldsymbol{\varepsilon}_{\mathrm{d}}\,\right] : \frac{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}{\partial\boldsymbol{\varepsilon}} \tag{4}$$

Where:

- the derivative of the volumetric part reads

$$\frac{\partial}{\partial\boldsymbol{\varepsilon}}\left[\,3K\varepsilon_{\mathrm{m}}\,\boldsymbol{I}\,\right] = K\boldsymbol{I}\otimes\boldsymbol{I} \tag{5}$$

- the chain rule for the deviatoric part reads

$$\frac{\partial}{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}\left[\,\varepsilon_{\mathrm{eq}}^{n-1}\,\boldsymbol{\varepsilon}_{\mathrm{d}}\,\right] = \frac{\partial\left[\varepsilon_{\mathrm{eq}}^{n-1}\right]}{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\frac{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}} \tag{6}$$

$$= \tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\,\boldsymbol{\varepsilon}_{\mathrm{d}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\,\mathbb{I} \tag{7}$$

- and it has been used that

$$\frac{\partial}{\partial\boldsymbol{\varepsilon}_{\mathrm{d}}}\left[\,\varepsilon_{\mathrm{eq}}^{n-1}\,\right] = (n-1)\,\varepsilon_{\mathrm{eq}}^{n-2}\frac{2}{3}\frac{\boldsymbol{\varepsilon}_{\mathrm{d}}}{\varepsilon_{\mathrm{eq}}} \tag{8}$$

$$= \tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\,\boldsymbol{\varepsilon}_{\mathrm{d}} \tag{9}$$

Combining the above yields:

$$\mathbb{C} = K\boldsymbol{I}\otimes\boldsymbol{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\left(\tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\boldsymbol{\varepsilon}_{\mathrm{d}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\mathbb{I}\right) : \mathbb{I}_{\mathrm{d}} \tag{10}$$

$$= K\boldsymbol{I}\otimes\boldsymbol{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\left(\tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\boldsymbol{\varepsilon}_{\mathrm{d}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\,\mathbb{I}_{\mathrm{d}}\right) \tag{11}$$

A   Nomenclature

- Dyadic tensor product

$$\mathbb{C} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{12}$$

$$C_{ijkl} = A_{ij}\, B_{kl} \tag{13}$$

- Double tensor contraction

$$C = \boldsymbol{A} : \boldsymbol{B} \tag{14}$$

$$= A_{ij}\, B_{ji} \tag{15}$$

- Deviatoric projection tensor

$$\mathbb{I}_\mathrm{d} = \mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \tag{16}$$

B   Strain measures

- Mean strain

$$\varepsilon_\mathrm{m} = \tfrac{1}{3}\operatorname{tr}(\boldsymbol{\varepsilon}) = \tfrac{1}{3}\,\boldsymbol{\varepsilon} : \boldsymbol{I} \tag{17}$$

- Strain deviator

$$\boldsymbol{\varepsilon}_\mathrm{d} = \boldsymbol{\varepsilon} - \varepsilon_\mathrm{m}\,\boldsymbol{I} = \mathbb{I}_\mathrm{d} : \boldsymbol{\varepsilon} \tag{18}$$

- Equivalent strain

$$\varepsilon_\mathrm{eq} = \sqrt{\tfrac{2}{3}\,\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}} \tag{19}$$

C   Variations

- Strain deviator

$$\delta\boldsymbol{\varepsilon}_\mathrm{d} = \left(\mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}\right) : \delta\boldsymbol{\varepsilon} = \mathbb{I}_\mathrm{d} : \delta\boldsymbol{\varepsilon} \tag{20}$$

- Mean equivalent strain

$$\delta\varepsilon_\mathrm{m} = \tfrac{1}{3}\boldsymbol{I} : \delta\boldsymbol{\varepsilon} \tag{21}$$

- Von Mises equivalent strain

$$\delta\varepsilon_\mathrm{eq} = \frac{1}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d} + \delta\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}\right) \tag{22}$$

$$= \frac{2}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d}\right) \tag{23}$$

$$= \frac{2}{3}\frac{\boldsymbol{\varepsilon}_\mathrm{d}}{\varepsilon_\mathrm{eq}} : \delta\boldsymbol{\varepsilon}_\mathrm{d} \tag{24}$$