

GooseSolid/NonLinearElastic

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Abstract

This constitutive model encompasses a non-linear, but history independent, relation between the Cauchy stress, σ , and the linear strain tensor, ε , i.e.:

$$\sigma = f(\varepsilon)$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

Keywords: non-linear elasticity

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1 Constitutive model

The following strain-energy is defined:

$$W(\boldsymbol{\varepsilon}) = \frac{9}{2} K \varepsilon_m^2 + \frac{\sigma_0 \varepsilon_0}{n+1} \left(\frac{\varepsilon_{eq}}{\varepsilon_0} \right)^{n+1} \quad (1)$$

where K is the bulk modulus, ε_0 and σ_0 are a reference strain and stress respectively, and n is an exponent that sets the degree of non-linearity. Finally ε_m and ε_{eq} are the hydrostatic and equivalent strains (see Appendix B).

This leads to the following stress-strain relation:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = 3K \varepsilon_m \mathbf{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \varepsilon_{eq}^{n-1} \boldsymbol{\varepsilon}_d \quad (2)$$

see Appendix A for nomenclature.

2 Consistent tangent

The consistent tangent maps a variation in strain, $\delta \boldsymbol{\varepsilon}$, to a variation in stress, $\delta \boldsymbol{\sigma}$, as follows

$$\delta \boldsymbol{\sigma} = \mathbb{C} : \delta \boldsymbol{\varepsilon} \quad (3)$$

The tangent, \mathbb{C} , thus corresponds to the derivative of (2) w.r.t. strain. For this, the chain rule is employed:

$$\mathbb{C} = \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[3K \varepsilon_m \mathbf{I} \right] + \frac{\partial}{\partial \boldsymbol{\varepsilon}_d} \left[\frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \varepsilon_{eq}^{n-1} \boldsymbol{\varepsilon}_d \right] : \frac{\partial \boldsymbol{\varepsilon}_d}{\partial \boldsymbol{\varepsilon}} \quad (4)$$

Where:

- the derivative of the volumetric part reads

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[3K \varepsilon_m \mathbf{I} \right] = K \mathbf{I} \otimes \mathbf{I} \quad (5)$$

- the chain rule for the deviatoric part reads

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_d} \left[\varepsilon_{eq}^{n-1} \boldsymbol{\varepsilon}_d \right] = \frac{\partial [\varepsilon_{eq}^{n-1}]}{\partial \boldsymbol{\varepsilon}_d} \otimes \boldsymbol{\varepsilon}_d + \varepsilon_{eq}^{n-1} \frac{\partial \boldsymbol{\varepsilon}_d}{\partial \boldsymbol{\varepsilon}_d} \quad (6)$$

$$= \frac{2}{3} (n-1) \varepsilon_{eq}^{n-3} \boldsymbol{\varepsilon}_d \otimes \boldsymbol{\varepsilon}_d + \varepsilon_{eq}^{n-1} \mathbb{I} \quad (7)$$

- and it has been used that

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_d} \left[\varepsilon_{eq}^{n-1} \right] = (n-1) \varepsilon_{eq}^{n-2} \frac{2}{3} \frac{\boldsymbol{\varepsilon}_d}{\varepsilon_{eq}} \quad (8)$$

$$= \frac{2}{3} (n-1) \varepsilon_{eq}^{n-3} \boldsymbol{\varepsilon}_d \quad (9)$$

Combining the above yields:

$$\mathbb{C} = K \mathbf{I} \otimes \mathbf{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \left(\frac{2}{3} (n-1) \varepsilon_{eq}^{n-3} \boldsymbol{\varepsilon}_d \otimes \boldsymbol{\varepsilon}_d + \varepsilon_{eq}^{n-1} \mathbb{I} \right) : \mathbb{I}_d \quad (10)$$

$$= K \mathbf{I} \otimes \mathbf{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \left(\frac{2}{3} (n-1) \varepsilon_{eq}^{n-3} \boldsymbol{\varepsilon}_d \otimes \boldsymbol{\varepsilon}_d + \varepsilon_{eq}^{n-1} \mathbb{I}_d \right) \quad (11)$$

A Nomenclature

- Dyadic tensor product

$$\mathbb{C} = \mathbf{A} \otimes \mathbf{B} \quad (12)$$

$$C_{ijkl} = A_{ij} B_{kl} \quad (13)$$

- Double tensor contraction

$$C = \mathbf{A} : \mathbf{B} \quad (14)$$

$$= A_{ij} B_{ji} \quad (15)$$

- Deviatoric projection tensor

$$\mathbb{I}_d = \mathbb{I}_s - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \quad (16)$$

B Strain measures

- Mean strain

$$\varepsilon_m = \frac{1}{3} \text{tr}(\boldsymbol{\varepsilon}) = \frac{1}{3} \boldsymbol{\varepsilon} : \mathbf{I} \quad (17)$$

- Strain deviator

$$\boldsymbol{\varepsilon}_d = \boldsymbol{\varepsilon} - \varepsilon_m \mathbf{I} = \mathbb{I}_d : \boldsymbol{\varepsilon} \quad (18)$$

- Equivalent strain

$$\varepsilon_{eq} = \sqrt{\frac{2}{3} \boldsymbol{\varepsilon}_d : \boldsymbol{\varepsilon}_d} \quad (19)$$

C Variations

- Strain deviator

$$\delta \boldsymbol{\varepsilon}_d = \left(\mathbb{I}_s - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) : \delta \boldsymbol{\varepsilon} = \mathbb{I}_d : \delta \boldsymbol{\varepsilon} \quad (20)$$

- Mean equivalent strain

$$\delta \varepsilon_m = \frac{1}{3} \mathbf{I} : \delta \boldsymbol{\varepsilon} \quad (21)$$

- Von Mises equivalent strain

$$\delta \varepsilon_{eq} = \frac{1}{3} \frac{1}{\varepsilon_{eq}} (\boldsymbol{\varepsilon}_d : \delta \boldsymbol{\varepsilon}_d + \delta \boldsymbol{\varepsilon}_d : \boldsymbol{\varepsilon}_d) \quad (22)$$

$$= \frac{2}{3} \frac{1}{\varepsilon_{eq}} (\boldsymbol{\varepsilon}_d : \delta \boldsymbol{\varepsilon}_d) \quad (23)$$

$$= \frac{2}{3} \frac{\boldsymbol{\varepsilon}_d}{\varepsilon_{eq}} : \delta \boldsymbol{\varepsilon}_d \quad (24)$$