GooseSolid/PlasticLinearElastic

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Abstract

History dependent elasto-plastic material. This corresponds to a non-linear relation between the Cauchy stress $\boldsymbol{\sigma}$ and the linear strain $\boldsymbol{\varepsilon}$, depending on the history (the accumulated plastic strain $\varepsilon_\mathrm{p}^{(t)}$, and the elastic and total strain, $\boldsymbol{\varepsilon}_\mathrm{e}^{(t)}$ and $\boldsymbol{\varepsilon}^{(t)}$). I.e.

$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}, \varepsilon_\mathrm{p}^{(t)}, \boldsymbol{\varepsilon}_\mathrm{e}^{(t)}, \boldsymbol{\varepsilon}^{(t)}\right)$$

The plasticity follows a power-law hardening, which is captured by the following yield stress evolution:

$$\sigma_\mathrm{y}(\varepsilon_\mathrm{p}) = \sigma_\mathrm{y0} + \varepsilon_\mathrm{p}^m$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

Keywords: elasto-plasticity; linear elasticity

Contents

1 Constitutive model

The model consists of the following ingredients:

(i) The strain, $\boldsymbol{\varepsilon}$, is additively split in an elastic part, $\boldsymbol{\varepsilon}_\mathrm{e}$, and a plastic plastic, $\boldsymbol{\varepsilon}_\mathrm{p}$. I.e.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\mathrm{e} + \boldsymbol{\varepsilon}_\mathrm{p} \tag{1}$$

(ii) The stress, $\boldsymbol{\sigma}$, is set by to the elastic strain, $\boldsymbol{\varepsilon}_\mathrm{e}$, through the following linear relation:

$$\boldsymbol{\sigma} = \mathbb{C}_\mathrm{e} : \boldsymbol{\varepsilon}_\mathrm{e} \tag{2}$$

wherein $\mathbb{C}_\mathrm{e}$ is the elastic stiffness, which reads:

$$\mathbb{C}_\mathrm{e} = K \boldsymbol{I} \otimes \boldsymbol{I} + 2G(\mathbb{I}_\mathrm{s} - \tfrac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I}) \tag{3}$$

$$= K \boldsymbol{I} \otimes \boldsymbol{I} + 2G\, \mathbb{I}_\mathrm{d} \tag{4}$$

with K and G the bulk and shear modulus respectively. See Appendix A for nomenclature.

(iii) The elastic domain is bounded by the following yield function

$$\Phi(\boldsymbol{\sigma}, \varepsilon_\mathrm{p}) = \sigma_\mathrm{eq} - \sigma_\mathrm{y}(\varepsilon_\mathrm{p}) \leq 0 \tag{5}$$

where σ_eq the equivalent stress (see Appendix B), and σ_y the yield stress which is a non-linear function of the equivalent plastic strain ε_p. The implementation is based on power-law hardening:

$$\sigma_\mathrm{y}(\varepsilon_\mathrm{p}) = \sigma_\mathrm{y0} + H \varepsilon_\mathrm{p}^m \tag{6}$$

where σ_y0 is the initial yield stress, H is the hardening modulus, and m is the hardening exponent.

(iv) To determine the direction of plastic flow, normality is assumed. This corresponds to the following associative flow rule:

$$\dot{\boldsymbol{\varepsilon}}_\mathrm{p} = \dot{\gamma} \boldsymbol{N} \tag{7}$$

where $\dot{\gamma}$ is the plastic multiplier, and $\boldsymbol{N}$ is the Prandtl–Reuss flow vector, which is defined through normality:

$$\boldsymbol{N} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{\boldsymbol{\sigma}_\mathrm{d}}{\sigma_\mathrm{eq}} \tag{8}$$

(v) Finally, associative hardening is assumed:

$$\dot{\varepsilon}_\mathrm{p} = \sqrt{\tfrac{2}{3}\, \dot{\boldsymbol{\varepsilon}}_\mathrm{p} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p}} = \dot{\gamma} \tag{9}$$

The equivalent plastic strain then reads

$$\varepsilon_\mathrm{p} = \int\limits_0^t \dot{\varepsilon}_\mathrm{p} \; \mathrm{d}\tau \tag{10}$$

For a more detailed description the reader is referred to de Souza Neto et al. (2008, p. 216-234).

2 Numerical implementation: implicit

For the numerical implementation, first of all a numerical time integration scheme has to be selected. Here, an implicit time discretization is used, which has the favorable property of being unconditionally stable. To this end, the commonly used return-map algorithm is used. An increment in strain is first assumed fully elastic (elastic predictor). Then, if needed, a return-map is utilized to return to a physically admissible state (plastic corrector). The benefit of this scheme as that (i) the evolution of the plasticity (the state variable) can be determined by solving a single, albeit non-linear, equation; and that (ii) the linearization to obtain the consistent tangent operator is relatively straightforward.

2.1 Elastic predictor

Given an increment in strain

$$\boldsymbol{\varepsilon}_\Delta = \boldsymbol{\varepsilon}^{(t+\Delta t)} - \boldsymbol{\varepsilon}^{(t)} \tag{11}$$

and the state variables, the *elastic trial state* reads:

$$^\star\boldsymbol{\varepsilon}_\mathrm{e} = \boldsymbol{\varepsilon}_\mathrm{e}^{(t)} + \boldsymbol{\varepsilon}_\Delta \tag{12}$$

$$^\star\boldsymbol{\sigma} = \mathbb{C}_\mathrm{e} : {}^\star\boldsymbol{\varepsilon}_\mathrm{e} \tag{13}$$

$$^\star\varepsilon_\mathrm{p} = \varepsilon_\mathrm{p}^{(t)} \tag{14}$$

$$^\star\Phi = \Phi\big({}^\star\boldsymbol{\sigma}, {}^\star\varepsilon_\mathrm{p}\big) = {}^\star\sigma_\mathrm{eq} - \sigma_\mathrm{y}\big({}^\star\varepsilon_\mathrm{p}\big) \tag{15}$$

where the notation $^\star(.)$ has been used to denote a trial value for $(.)^{(t+\Delta t)}$.

2.2 Trial state: elastic

If the trial state is within or on the (current) yield surface, i.e. when

$$^\star\Phi \leq 0 \tag{16}$$

the trial state coincides with the actual state, and:

$$\boldsymbol{\varepsilon}_\mathrm{e}^{(t+\Delta t)} = {}^\star\boldsymbol{\varepsilon}_\mathrm{e} = \boldsymbol{\varepsilon}_\mathrm{e}^{(t)} + \boldsymbol{\varepsilon}_\Delta \tag{17}$$

$$\varepsilon_\mathrm{p}^{(t+\Delta t)} = {}^\star\varepsilon_\mathrm{p} = \varepsilon_\mathrm{p}^{(t)} \tag{18}$$

$$\boldsymbol{\sigma}^{(t+\Delta t)} = {}^\star\boldsymbol{\sigma} \tag{19}$$

2.3 Trial state elasto-plastic: return-map

If the trial state is outside the (current) yield surface, i.e. when

$$^\star\Phi > 0 \tag{20}$$

plastic flow occurs in the increment. A return-map is needed to return to the admissible state. It has to satisfy the following system of equations:

$$\begin{cases} \boldsymbol{\varepsilon}_\mathrm{e}^{(t+\Delta t)} &= {}^\star\boldsymbol{\varepsilon}_\mathrm{e} - \Delta\gamma\ \boldsymbol{N}^{(t+\Delta t)} \\ \varepsilon_\mathrm{p}^{(t+\Delta t)} &= \varepsilon_\mathrm{p}^{(t)} + \Delta\gamma \\ \Phi^{(t+\Delta t)} &= \sigma_\mathrm{eq}^{(t+\Delta t)} - \sigma_\mathrm{y}\big(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma\big) = 0 \end{cases} \tag{21}$$

This can be reduced by using that

$$\boldsymbol{\sigma}_\mathrm{d}^{(t+\Delta t)} = {}^\star\boldsymbol{\sigma}_\mathrm{d} - 2G\Delta\gamma\ \boldsymbol{N}^{(t+\Delta t)} \tag{22}$$

$$= {}^\star\boldsymbol{\sigma}_\mathrm{d} - 3G\Delta\gamma\ \frac{\boldsymbol{\sigma}_\mathrm{d}^{(t+\Delta t)}}{\sigma_\mathrm{eq}^{(t+\Delta t)}} \tag{23}$$

i.e. the trial and updated deviatoric stresses are *co-linear*. This implies that

$$\frac{\boldsymbol{\sigma}_\mathrm{d}^{(t+\Delta t)}}{\sigma_\mathrm{eq}^{(t+\Delta t)}} = \frac{{}^\star\boldsymbol{\sigma}_\mathrm{d}}{{}^\star\sigma_\mathrm{eq}} \qquad \text{or} \qquad \boldsymbol{N}^{(t+\Delta t)} = {}^\star\boldsymbol{N} \tag{24}$$

This allows the reorganization of the above to

$$\boldsymbol{\sigma}_\mathrm{d}^{(t+\Delta t)} = \left(1 - \frac{3G\ \Delta\gamma}{{}^\star\sigma_\mathrm{eq}}\right){}^\star\boldsymbol{\sigma}_\mathrm{d} \tag{25}$$

from which it also follows that

$$\sigma_\mathrm{eq}^{(t+\Delta t)} = {}^\star\sigma_\mathrm{eq} - 3G\ \Delta\gamma \tag{26}$$

Instead of the system return the plastic multiplier $\Delta\gamma$ can directly be found by enforcing the yield surface

$$\Phi = {}^\star\sigma_\mathrm{eq} - 3G\Delta\gamma - \sigma_\mathrm{y}\big(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma\big) = 0 \tag{27}$$

This (non-linear) equation has to be solved for the unknown plastic multiplier $\Delta\gamma$.

2.3.1 Linear hardening

Linear hardening reads

$$\sigma_\mathrm{y} = \sigma_\mathrm{y0} + H\varepsilon_\mathrm{p} \tag{28}$$

In this case (27) can be solved analytically. The solution reads

$$\Delta\gamma = \frac{{}^\star\Phi}{3G + H} \tag{29}$$

2.3.2 Non-linear hardening

(1) Initial guess:

$$\Delta\gamma := 0 \tag{30}$$

and evaluate

$$\tilde{\Phi} := {}^\star\Phi \tag{31}$$

(2) Perform Newton-Raphson iteration:

- Hardening slope

$$H := \left.\frac{\mathrm{d}\sigma_\mathrm{y}}{\mathrm{d}\varepsilon_\mathrm{p}}\right|_{\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma} \tag{32}$$

- Residual derivative:

$$d := \frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}\Delta\gamma} = -3G - H \tag{33}$$

- Update guess for the plastic multiplier:

$$\Delta\gamma := \Delta\gamma - \frac{\tilde{\Phi}}{d} \tag{34}$$

(3) Check for convergence:

$$\tilde{\Phi} := {}^\star\sigma_\mathrm{eq} - 3G\Delta\gamma - \sigma_\mathrm{y}(\varepsilon_\mathrm{p}^{(t)} + \Delta\gamma) \tag{35}$$

Stop if:

$$\big|\tilde{\Phi}\big| \leq \epsilon_\mathrm{tol} \tag{36}$$

Otherwise continue with (2)

2.3.3 Trial state update

Finally, the trial state is updated:

- The updated stress tensor

$$\boldsymbol{\sigma}^{(t+\Delta t)} = \sigma_{\mathrm{m}}^{(t+\Delta t)} \boldsymbol{I} + \boldsymbol{\sigma}_{\mathrm{d}}^{(t+\Delta t)} \tag{37}$$

with

$$\sigma_{\mathrm{m}}^{(t+\Delta t)} = {}^{\star}\sigma_{\mathrm{m}} \tag{38}$$

$$\boldsymbol{\sigma}_{\mathrm{d}}^{(t+\Delta t)} = \left(1 - \frac{3G\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}}\right) {}^{\star}\boldsymbol{\sigma}_{\mathrm{d}} \tag{39}$$

- The updated elastic strain tensor

$$\boldsymbol{\varepsilon}_{\mathrm{e}}^{(t+\Delta t)} = \frac{1}{2G}\, \boldsymbol{\sigma}_{\mathrm{d}}^{(t+\Delta t)} + \frac{1}{3}\,\mathrm{tr}({}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}})\, \boldsymbol{I} \tag{40}$$

- The updated equivalent plastic strain:

$$\varepsilon_{\mathrm{p}}^{(t+\Delta t)} = \varepsilon_{\mathrm{p}}^{(t)} + \Delta\gamma \tag{41}$$

2.4 Consistent tangent operator

The consistent constitutive tangent operator is defined as

$$\mathbb{C}_{\mathrm{ep}} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}} \tag{42}$$

For the case that the trail state coincides with the actual state (i.e. ${}^{\star}\Phi \leq 0$), $\mathbb{C}_{\mathrm{ep}} = \mathbb{C}_{\mathrm{e}}$. Otherwise, it can be obtained from[1]

$$\mathbb{C}_{\mathrm{ep}} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}} : \frac{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}}{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}} : \mathbb{I} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}} \tag{43}$$

To proceed, the first step is write an explicit relation between the (actual) stress $\boldsymbol{\sigma}^{(t+\Delta t)}$ and the trial elastic strain ${}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}$:

$$\boldsymbol{\sigma}^{(t+\Delta t)} = \sigma_{\mathrm{m}}^{(t+\Delta t)} \boldsymbol{I} + \boldsymbol{\sigma}_{\mathrm{d}}^{(t+\Delta t)} \tag{44}$$

$$= {}^{\star}\sigma_{\mathrm{m}} \boldsymbol{I} + \left(1 - \frac{3G\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}}\right) {}^{\star}\boldsymbol{\sigma}_{\mathrm{d}} \tag{45}$$

$$= {}^{\star}\sigma_{\mathrm{m}} \boldsymbol{I} + \left(1 - \frac{3G\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}}\right) 2G\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}^{d} \tag{46}$$

$$= \left[\mathbb{C}_{\mathrm{e}} - \frac{6G^2\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}} \mathbb{I}_{\mathrm{d}}\right] : {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}} \tag{47}$$

The tangent then follows from

$$\mathbb{C}_{\mathrm{ep}} = \frac{\partial\, \boldsymbol{\sigma}^{(t+\Delta t)}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}} = \mathbb{C}_{\mathrm{e}} - \frac{6G^2\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}} \mathbb{I}^{d} - \frac{6G^2}{{}^{\star}\sigma_{\mathrm{eq}}} \left(\frac{\partial \Delta\gamma}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}}\right) \otimes {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}^{\mathrm{d}} + \frac{6G^2\Delta\gamma}{{}^{\star}\sigma_{\mathrm{eq}}^2} \left(\frac{\partial\, {}^{\star}\sigma_{\mathrm{eq}}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}}\right) \otimes {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}^{\mathrm{d}} \tag{48}$$

Now apply the following:

- Equivalent stress

$$\frac{\partial\, \sigma_{\mathrm{eq}}}{\partial\, \boldsymbol{\sigma}} = \frac{\partial\, \sigma_{\mathrm{eq}}}{\partial\, \boldsymbol{\sigma}_{\mathrm{d}}} = \frac{\partial}{\partial\, \boldsymbol{\sigma}_{\mathrm{d}}} \left(\sqrt{\tfrac{3}{2} \boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}}}\right) = \frac{1}{2\, \sigma_{\mathrm{eq}}} \frac{\partial}{\partial\, \boldsymbol{\sigma}_{\mathrm{d}}} \left(\tfrac{3}{2} \boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}}\right) = \frac{3}{2} \frac{\boldsymbol{\sigma}_{\mathrm{d}}}{\sigma_{\mathrm{eq}}} = \boldsymbol{N} = {}^{\star}\boldsymbol{N} \tag{49}$$

Hence:

$$\frac{\partial\, {}^{\star}\sigma_{\mathrm{eq}}}{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}} = 2G\, {}^{\star}\boldsymbol{N} \tag{50}$$

[1] To show this one has to employ (11,12) and realize that $\boldsymbol{\varepsilon}^{(t)}$ and $\boldsymbol{\varepsilon}_{\mathrm{e}}^{(t)}$ are constant during the increment. I.e. $\dfrac{\partial\, {}^{\star}\boldsymbol{\varepsilon}_{\mathrm{e}}}{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}} = \dfrac{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}}{\partial\, \boldsymbol{\varepsilon}^{(t+\Delta t)}} = \mathbb{I}$.

- Plastic multiplier

$$\frac{\partial \Delta\gamma}{\partial {}^\star\!\boldsymbol{\varepsilon}_{\mathrm{e}}} = \left(\frac{\partial\ \Delta\gamma}{\partial\ \Phi} \right) \left(\frac{\partial\ \Phi}{\partial {}^\star\sigma_{\mathrm{eq}}} \right) \left(\frac{\partial {}^\star\sigma_{\mathrm{eq}}}{\partial {}^\star\!\boldsymbol{\varepsilon}_{\mathrm{e}}} \right) = \frac{2G}{3G + \tilde{H}}\, {}^\star\!\boldsymbol{N} \tag{51}$$

where the tangent hardening modulus

$$\tilde{H} = \left. \frac{\partial\, \sigma_{\mathrm{y}}}{\partial \Delta\gamma} \right|_{\varepsilon_{\mathrm{p}}^{(t+\Delta t)}} = \left. \frac{\partial\, \sigma_{\mathrm{y}}}{\partial\, \varepsilon_{\mathrm{p}}} \right|_{\varepsilon_{\mathrm{p}}^{(t+\Delta t)}} = mH \big(\varepsilon_{\mathrm{p}}^{(t)} + \Delta\Gamma \big)^{m-1} \tag{52}$$

The final result then reads:

$$\mathbb{C}_{\mathrm{ep}} = \mathbb{C}_{\mathrm{e}} - \frac{6G^2 \Delta\gamma}{{}^\star\sigma_{\mathrm{eq}}} \mathbb{I}_{\mathrm{d}} + 4G^2 \left[\frac{\Delta\gamma}{{}^\star\sigma_{\mathrm{eq}}} - \frac{1}{3G + H} \right] {}^\star\!\boldsymbol{N} \otimes {}^\star\!\boldsymbol{N} \tag{53}$$

3 Examples

3.1 Linear vs. non-linear hardening

In this example we consider a material point subject to simple shear deformation

$$\boldsymbol{\varepsilon}(t) = \gamma(t)\,[\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x] \tag{54}$$

The response is a stress of the form

$$\boldsymbol{\sigma}(t) = \tau(t)\,[\vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x] \tag{55}$$

For a specific set of parameters response is shown in Figure 1. Notice that a normalization factor has been introduced such that the values along the axis are those of the equivalent stress $\sigma_{\text{eq}} = \tau\sqrt{3}$.

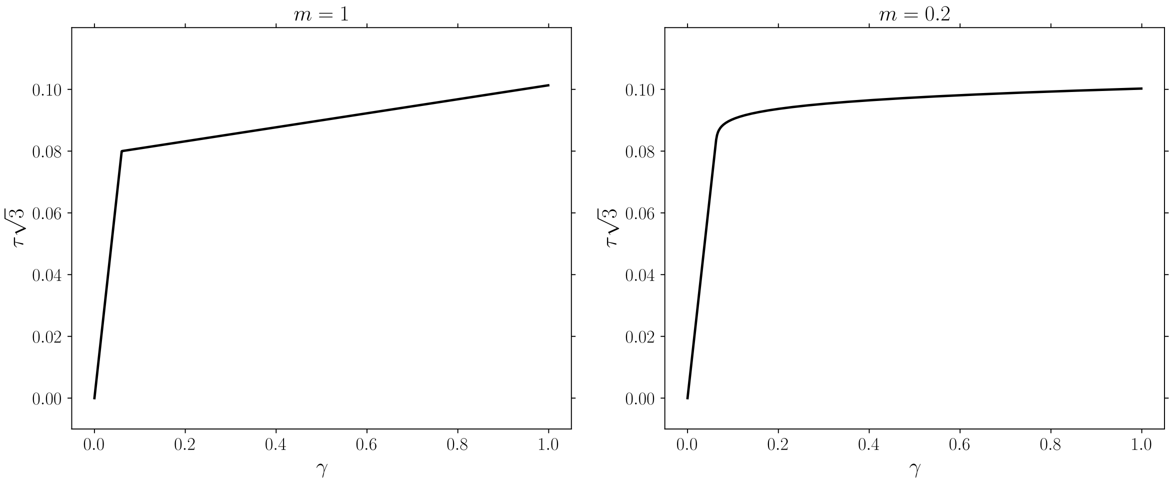

Figure 1. Linear vs. power-law hardening

A Nomenclature

- Dyadic tensor product

$$\mathbb{C} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{56}$$

$$C_{ijkl} = A_{ij}\, B_{kl} \tag{57}$$

- Double tensor contraction

$$C = \boldsymbol{A} : \boldsymbol{B} \tag{58}$$

$$= A_{ij}\, B_{ji} \tag{59}$$

- Deviatoric projection tensor

$$\mathbb{I}_{\mathrm{d}} = \mathbb{I}_{\mathrm{s}} - \tfrac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I} \tag{60}$$

B Stress measures

- Mean stress

$$\sigma_{\mathrm{m}} = \tfrac{1}{3}\, \mathrm{tr}(\boldsymbol{\sigma}) = \tfrac{1}{3}\, \boldsymbol{\sigma} : \boldsymbol{I} \tag{61}$$

- Stress deviator

$$\boldsymbol{\sigma}_{\mathrm{d}} = \boldsymbol{\sigma} - \sigma_{\mathrm{m}}\, \boldsymbol{I} = \mathbb{I}_{\mathrm{d}} : \boldsymbol{\sigma} \tag{62}$$

- Von Mises equivalent stress

$$\sigma_{\mathrm{eq}} = \sqrt{\tfrac{3}{2}\, \boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}}} = \sqrt{3 J_2(\boldsymbol{\sigma})} \tag{63}$$

where the second-stress invariant

$$J_2 = \tfrac{1}{2}\, ||\, \boldsymbol{\sigma}_{\mathrm{d}}\, ||^2 = \tfrac{1}{2}\, \boldsymbol{\sigma}_{\mathrm{d}} : \boldsymbol{\sigma}_{\mathrm{d}} \tag{64}$$

References

de Souza Neto, E., Perić, D., and Owen, D. (2008). *Computational Methods for Plasticity*. John Wiley & Sons, Ltd.