On the Design of Complementary Filters for Control

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***Abstract*— Abstract text to be done**

complementary filters, h-infinity, feedback control

I. Introduction

The basic idea of a complementary filter involves taking two or more sensors, filtering out unreliable frequencies for each sensor and combining the filtered outputs to get a better estimate throughout the entire bandwidth of the system. To achieve this, the sensors included in the filter should complement one another by performing better over specific parts of the system bandwidth. A set of filters is said to be complementary if the sum of their transfer functions is equal to one at all frequencies, (i.e.) its magnitude is one and its phase is zero.

The proper design of this particular kind of filter is of primary importance in a wide range of applications. Often, multiple sensors with different noise or dynamical properties are used to measure the same physical quantity. In such case, complementary filters can be used to merge the sensors and forms a "super sensor" that has gives a better estimate of the physical quantity over a wider bandwidth. This is called sensor blending or sensor fusion.

This is widely used for the attitude estimation of unmanned aerial vehicles using various kind of sensors (accelerometers, gyroscopes, vision sensors, inclinometer) [1], [2], [3].

[4] Fast position measurement of flexible structure

[5] (relative displacement measurement at low frequencies with inertial at high frequencies)

[6]

[7] The design methods for such filters goes from simple analytical formulas

[2]

[3] [8]

[4] [1] [5] [7]

[9] [6] [5]

[10]

[11]

[8] (feedback system, P, PI, classical control theory for filter design) [12]

[11]

[3] Although In this paper, we propose The body of the paper consists of five parts followed by a conclusion.

II. Requirements on the Design of Complementary Filters

A. Sensor Fusion

Let's consider two sensors measuring the physical quantity x with dynamics $G_1(s)$ and $G_2(s)$ and with noise n_1 and n_2 respectively.

$H_1(s)$ and $H_2(s)$ are complementary filters:

$$H_1(s) + H_2(s) = 1 \tag{1}$$

$$\hat{x} = (G_1 H_1 + G_2 H_2)\, x + H_1 n_1 + H_2 n_2 \tag{2}$$

If we now consider sensors with perfect dynamics ($G_1(s) = G_2(s) = 1$), we have that the estimate of x using the two sensors are shown on figure 1 is:

$$\hat{x} = x + H_1 n_1 + H_2 n_2 \tag{3}$$

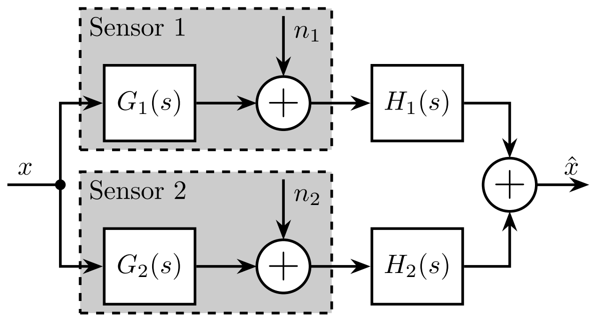

Fig. 1. Sensor Fusion Architecture

We see that the complementary filters $H_1(s)$ and $H_2(s)$ operates only on the noise of the sensors.

Thus, this architecture permits to filter each of the sensors without introducing any distortion in the physical quantity to measure.

$$\delta x = \hat{x} - x = H_1 n_1 + H_2 n_2 \tag{4}$$

Usually, the two sensors have higher noise levels over distinct yet complementary frequency regions. The two complementary filters are used to combine the filtered noise and yield to a better estimate $\hat{x}$ over a larger bandwidth.

The noise of the super sensor is determine by the norm of the complementary filters.

B. Noise Sensor Filtering

C. Robustness of the Fusion

D. Upper bounds as a mathematical translation of the requirements

The conclusion of the section should be that it is the norm of the complementary filter that is important and that is why we propose a method of synthesis based on H-infinity

III. Shaping of Complementary Filters using the $\mathcal{H}_\infty$ Synthesis

As shown in Sec. …, most of the performance requirements can be expressed as upper bounds on the magnitude of the complementary filters. As presented in Sec. **??**, almost all the requirements can be specified with upper bounds on the complementary filters.

Thus, the $\mathcal{H}_\infty$ framework seems adapted and we here propose a technique to synthesis complementary filters while specifying uppers bounds on their magnitudes.

A. $\mathcal{H}_\infty$ problem formulation

In this section, we formulate the $\mathcal{H}_\infty$ problem for the synthesis of complementary filters.

The synthesis objective is to shape an high pass filter H_H and a low pass filter H_L while ensuring their complementary property ($H_H + H_L = 1$).

To do so, we define two weighting functions w_L and w_H that will respectively used to shape H_L and H_H.

The synthesis problem is then

$$\text{Find } H_L, H_H \text{ such that } \begin{cases} H_L \text{ and } H_H \text{ are stable} & \text{(5a)} \\ H_L + H_H = 1 & \text{(5b)} \\ |w_L H_L| \le 1 \quad \forall \omega & \text{(5c)} \\ |w_H H_H| \le 1 \quad \forall \omega & \text{(5d)} \end{cases}$$

To express this synthesis problem into an $\mathcal{H}_\infty$ synthesis problem, we define the following generalized plant P (also shown on Fig. **??**):

$$\begin{bmatrix} w \\ u \end{bmatrix} = P \begin{bmatrix} z_H \\ z_L \\ v \end{bmatrix}; \quad P = \begin{bmatrix} w_H & -w_H \\ 0 & w_L \\ 1 & 0 \end{bmatrix} \tag{6}$$

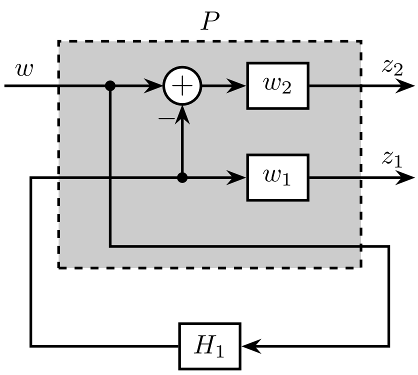

Fig. 2. Architecture used for the $\mathcal{H}_\infty$ synthesis of complementary filters

The $\mathcal{H}_\infty$ synthesis objective is then to design a stable filter H_L (Fig. **??**) such that the $\mathcal{H}_\infty$ norm of the transfer function from w to $[z_H,\ z_L]$ is less than 1:

$$\left\| \begin{matrix} (1 - H_L) w_H \\ H_L w_L \end{matrix} \right\|_\infty \leq 1 \tag{7}$$

Which is equivalent to

$$\left\| \begin{matrix} H_H w_H \\ H_L w_L \end{matrix} \right\|_\infty < 1 \text{ by choosing } H_H = 1 - H_L \tag{8}$$

Performance conditions (5c) and (5c) are satisfied by (8). Complementary condition (5b) is satisfied by design: $H_H = 1 - H_L$ and thus $H_L + H_H = 1$. The stability condition (5a) is guaranteed by the H_∞ synthesis (**reference**).

Using this synthesis method, we are then able to shape at the same time the high pass and low pass filters while ensuring their complementary.

B. Choice of the weighting functions

We here give some advice on the design of the weighting functions used for the synthesis of the complementary filters using the $\mathcal{H}_\infty$ method.

The weighting functions should be such that the performance requirements are met as explain in Sec. **??**.

However, one should be careful when designing the complementary filters, and should only use stable and minimum phase transfer functions. The order of the weights should stay reasonably small as this will increase the complexity of the optimization problem.

Moreover, the order of the complementary filters will be equal to the sum of the order of the weighting functions used.

One should not forget the fundamental limitations imposed by the synthesis: $H_L(s) + H_H(s) = 1$. This implies that H_L and H_H cannot be made small at the same time.

We here propose a formula for the design of the weighting function (9).

$$W(s) = \left(\frac{\frac{1}{\omega_0} \sqrt{\frac{1 - \left(\frac{G_0}{G_c}\right)^{\frac{2}{n}}}{1 - \left(\frac{G_c}{G_\infty}\right)^{\frac{2}{n}}}} s + \left(\frac{G_0}{G_c}\right)^{\frac{1}{n}}}{\left(\frac{1}{G_\infty}\right)^{\frac{1}{n}} \frac{1}{\omega_0} \sqrt{\frac{1 - \left(\frac{G_0}{G_c}\right)^{\frac{2}{n}}}{1 - \left(\frac{G_c}{G_\infty}\right)^{\frac{2}{n}}}} s + \left(\frac{1}{G_c}\right)^{\frac{1}{n}}} \right)^n \tag{9}$$

with:

- G_0 is the absolute gain at low frequency
- G_∞ is the absolute gain at high frequency
- ω_0 and G_c define the absolute value of the filter at $\omega = \omega_0$: $|W(j\omega_0)| = G_c$
- n is the absolute slope of the filter, it is also equal to the order of the filter

The constrains are that $G_0 < 1 < G_\infty$ and $G_0 < G_c < G_\infty$ or that $G_\infty < 1 < G_0$ and $G_\infty < G_c < G_0$.

The shape of the weight generated using the formula is shown on figure 3.

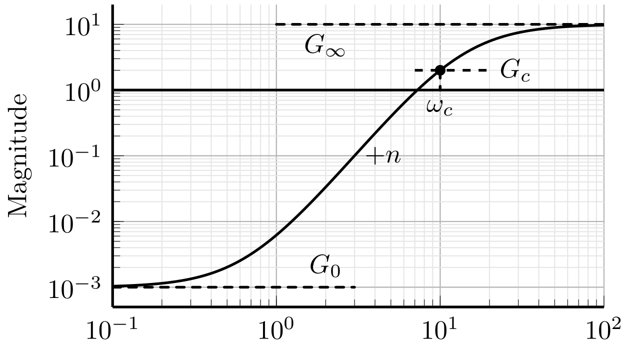

Fig. 3. Amplitude of the proposed formula for the weighting functions, $G_0 = 1e^{-3}$, $G_\infty = 10$, $\omega_c = 10\,\text{Hz}$, $G_c = 2$, $n = 3$

C. Example

We are now using the proposed $\mathcal{H}_\infty$ complementary filters synthesis method for a simple example.

The goal is to design

We use the formula (9) for both $w_L(s)$ and $w_H(s)$. The parameters used are summarized on table I. And the magnitude of the weighting functions are shown on figure 4.

TABLE I
PARAMETERS USED FOR THE WEIGHTING FUNCTIONS

Parameters	w_L	w_H
G_0	0.1	1000
G_∞	1000	0.1
ω_c [Hz]	11	10
G_c	2	2
n	2	3

After synthesis, the obtain filters are:

$$H_L(s) = \frac{10^{-8}(s + 6.6e^9)(s + 3450)^2(s^2 + 49s + 895)}{(s + 6.6e^4)(s^2 + 106s + 3000)(s^2 + 72s + 3580)} \tag{10}$$

$$H_H(s) = \frac{(s + 6.6e^4)(s + 160)(s + 4)^3}{(s + 6.6e^4)(s^2 + 106s + 3000)(s^2 + 72s + 3580)} \tag{11}$$

Their bode plot is shown on figure 5.

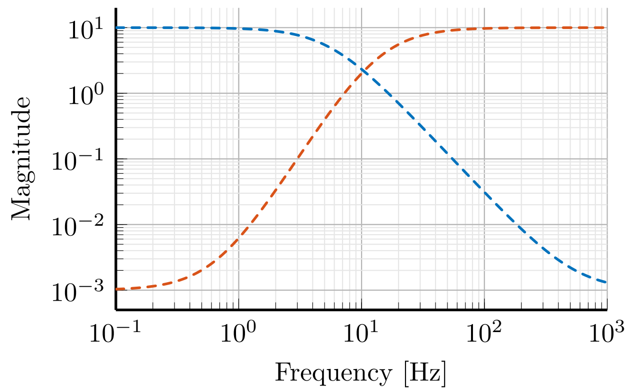

Fig. 4. Weighting Functions used for the $\mathcal{H}_\infty$ Synthesis

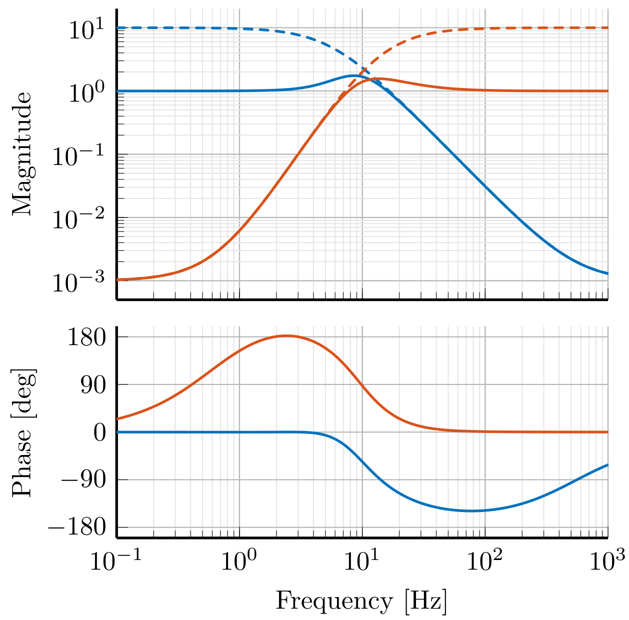

Fig. 5. Obtain Complementary Filters

D. Synthesis of Three Complementary Filters

We want:

$$
\begin{aligned}
&|H_1 w_1| < 1, \quad \forall \omega \\
&|H_2 w_2| < 1, \quad \forall \omega \\
&|H_3 w_3| < 1, \quad \forall \omega \\
&H_1 + H_2 + H_3 = 1
\end{aligned}
$$

The $\mathcal{H}_\infty$ objective is:

$$
\begin{aligned}
&|H_1 w_1| < 1, \quad \forall \omega \\
&|H_2 w_2| < 1, \quad \forall \omega \\
&|(1 - H_1 - H_2) w_3| < 1, \quad \forall \omega
\end{aligned}
$$

And thus if we choose $H_3 = 1 - H_1 - H_2$ we have solved the problem.

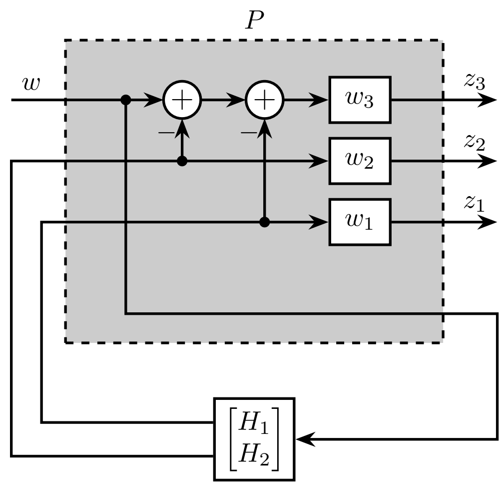

Fig. 6. Architecture for the $\mathcal{H}_\infty$ synthesis of three complementary filters

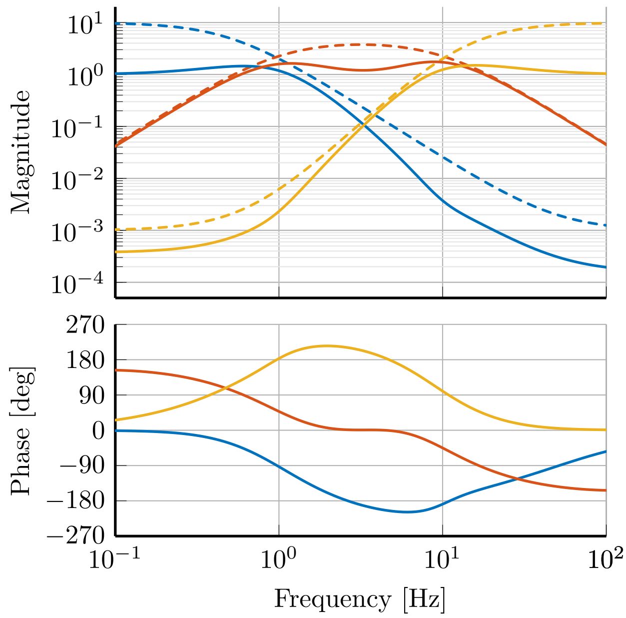

Fig. 7. Figure caption

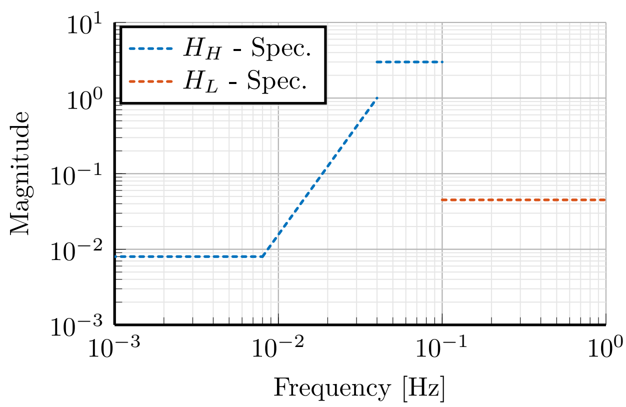

Fig. 8. Specifications on the norms of the complementary filters

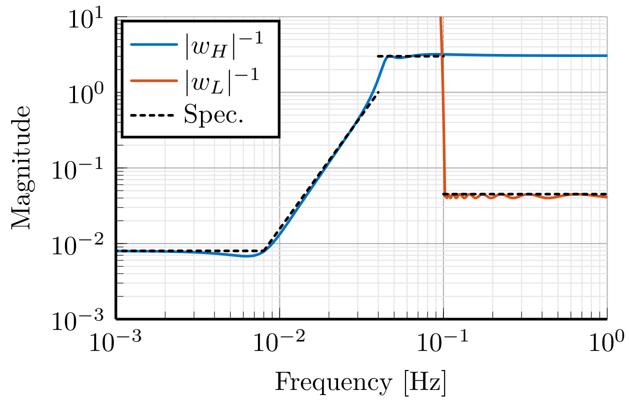

Fig. 9. Weighting Functions used for the $\mathcal{H}_\infty$ synthesis

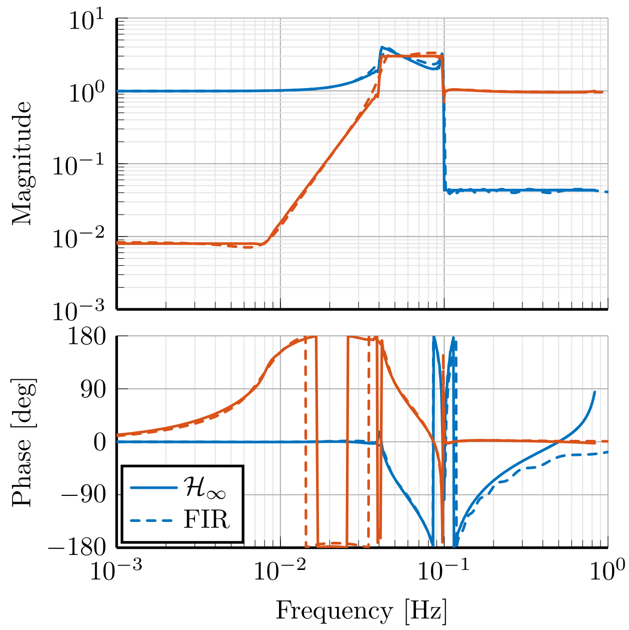

Fig. 10. Comparison of the filters obtain with the $\mathcal{H}_\infty$ synthesis and the FIR filters designed in [9]

IV. Application to the design of

A. Specifications

B. Weighting functions design

C. Comparison

V. Conclusion

VI. Acknowledgment

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