On the Design of Complementary Filters for Control

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***Abstract*— Abstract text to be done**

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I. Introduction

Complementary filters: [1].

II. H-Infinity synthesis of complementary filters

First order complementary filters are easy to synthesize. For instance, one can use the following filters

$$H_H(s) = \frac{s/\omega_0}{1 + s/\omega_0}; \quad H_L(s) = \frac{1}{1 + s/\omega_0} \tag{1}$$

with ω_0 is the tuning parameter corresponding to the crossover frequency of the filters.

However, the manual design of higher order complementary filters is far more complex and we have to use an automatic synthesis technique.

As shown in Sec. **??**, most of the performance requirements can be expressed as upper bounds on the magnitude of the complementary filters.

Thus, the $\mathcal{H}_\infty$ framework seems adapted and we here propose a technique to synthesis complementary filters while specifying uppers bounds on their magnitudes.

A. $\mathcal{H}_\infty$ problem formulation

In this section, we formulate the $\mathcal{H}_\infty$ problem for the synthesis of complementary filters.

The synthesis objective is to shape an high pass filter H_H and a low pass filter H_L while ensuring their complementary property ($H_H + H_L = 1$).

To do so, we define two weighting functions w_L and w_H that will respectively used to shape H_L and H_H.

The synthesis problem is then

$$\text{Find } H_L, H_H \text{ such that } \begin{cases} H_L \text{ and } H_H \text{ are stable} & \text{(2a)} \\ H_L + H_H = 1 & \text{(2b)} \\ |w_L H_L| \le 1 \quad \forall \omega & \text{(2c)} \\ |w_H H_H| \le 1 \quad \forall \omega & \text{(2d)} \end{cases}$$

To express this synthesis problem into an $\mathcal{H}_\infty$ synthesis problem, we define the following generalized plant P (also shown on Fig. 1):

$$\begin{bmatrix} w \\ u \end{bmatrix} = P \begin{bmatrix} z_H \\ z_L \\ v \end{bmatrix}; \quad P = \begin{bmatrix} w_H & -w_H \\ 0 & w_L \\ 1 & 0 \end{bmatrix} \tag{3}$$

Fig. 1. Generalized plant for the synthesis of the complementary filters

The $\mathcal{H}_\infty$ synthesis objective is then to design a stable filter H_L (Fig. 2) such that the $\mathcal{H}_\infty$ norm of the transfer function from w to $[z_H, \ z_L]$ is less than 1:

$$\left\| \begin{matrix} (1 - H_L) w_H \\ H_L w_L \end{matrix} \right\|_\infty \le 1 \tag{4}$$

Which is equivalent to

$$\left\| \begin{matrix} H_H w_H \\ H_L w_L \end{matrix} \right\|_\infty < 1 \text{ by choosing } H_H = 1 - H_L \tag{5}$$

Fig. 2. $\mathcal{H}_\infty$-synthesis of complementary filters

Performance conditions (2c) and (2c) are satisfied by (5). Complementary condition (2b) is satisfied by design: $H_H = 1 - H_L$ and thus $H_L + H_H = 1$. The stability condition (2a) is guaranteed by the H_∞ synthesis (**reference**).

Using this synthesis method, we are then able to shape at the same time the high pass and low pass filters while ensuring their complementary.

B. Control requirements as $\mathcal{H}_\infty$ norm of complementary filters

As presented in Sec. **??**, almost all the requirements can be specified with upper bounds on the complementary filters. However, robust performance condition (**??**) is not.

With the $\mathcal{H}_\infty$ synthesis the condition (5) only ensure

$$\begin{aligned} \left\| \begin{matrix} H_H w_H \\ H_L w_L \end{matrix} \right\|_\infty \le 1 &\Leftrightarrow \max_\omega \sqrt{|w_L H_L|^2 + |w_H H_H|^2} \le 1 \\ &\Rightarrow |w_L H_L| + |w_H H_H| \le \sqrt{2} \quad \forall \omega \end{aligned}$$

And thus we have almost robust stability.

C. Choice of the weighting functions

We here give some advice on the choice of the weighting functions used for the synthesis of the complementary filters.

The shape should be such that the performance requirements are met as explain in Sec. **??**.

However, one should be careful when designing the complementary filters, and should only use stable and minimum phase transfer functions. The order of the weights should stay reasonably small as this will increase the complexity of the optimization problem.

One should not forget the fundamental limitations of feedback control such that $S + T = 1$. Similarly, we here have that $H_L + H_H = 1$ which implies that H_L and H_H cannot be made small at the same time.

D. Trade-off between performance and robustness

E. Analytical formula of complementary filters

To simplify the synthesis, one can use already synthesized filters

$$H_L(s) = \frac{1}{1 + \frac{s}{\omega_0}} \tag{6}$$

$$H_H(s) = \frac{\frac{s}{\omega_0}}{1 + \frac{s}{\omega_0}} \tag{7}$$

$$H_L(s) = \frac{(1+\alpha)(\frac{s}{\omega_0}) + 1}{\left((\frac{s}{\omega_0}) + 1\right)\left((\frac{s}{\omega_0})^2 + \alpha(\frac{s}{\omega_0}) + 1\right)} \tag{8}$$

$$H_H(s) = \frac{(\frac{s}{\omega_0})^2 \left((\frac{s}{\omega_0}) + 1 + \alpha\right)}{\left((\frac{s}{\omega_0}) + 1\right)\left((\frac{s}{\omega_0})^2 + \alpha(\frac{s}{\omega_0}) + 1\right)} \tag{9}$$

$$H_L(s) = \frac{(1+(\alpha+1)(\beta+1))\,(\frac{s}{\omega_0})^2 + (1+\alpha+\beta)(\frac{s}{\omega_0}) + 1}{\left(\frac{s}{\omega_0}+1\right)\left((\frac{s}{\omega_0})^2+\alpha(\frac{s}{\omega_0})+1\right)\left((\frac{s}{\omega_0})^2+\beta(\frac{s}{\omega_0})+1\right)} \tag{10}$$

$$H_H(s) = \frac{(\frac{s}{\omega_0})^3\left((\frac{s}{\omega_0})^2 + (1+\alpha+\beta)(\frac{s}{\omega_0}) + (1+(\alpha+1)(\beta+1))\right)}{\left(\frac{s}{\omega_0}+1\right)\left((\frac{s}{\omega_0})^2+\alpha(\frac{s}{\omega_0})+1\right)\left((\frac{s}{\omega_0})^2+\beta(\frac{s}{\omega_0})+1\right)} \tag{11}$$

III. Discussion

IV. Conclusion

V. Acknowledgment

References

[1] W. Hua, “Low frequency vibration isolation and alignment system for advanced ligo,” Ph.D. dissertation, stanford university, 2005.