Complementary Filters Shaping Using $\mathcal{H}_\infty$ Synthesis

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Abstract For many applications, large bandwidth and dynamic ranges are requiring to use several sensors, whose signals are combined using complementary filters. This paper presents a method for designing these complementary filters using $\mathcal{H}_\infty$ synthesis that allows to shape the filter norms. This method is shown to be easily applicable for the synthesis of complex complementary filters.

Introduction

Complementary filters are used when two or more sensors are measuring the same quantity with different noise characteristics. Unreliable frequencies of each sensor are filtered out and then **combined to form a super sensor giving a better estimate over a wider bandwidth**. This technique is called **sensor fusion** and is used in **many applications** ranging from the attitude estimation of UAVs [1] to the isolation systems for the LIGO [2]. As the super sensor characteristics largely depend on the **complementary filter norms**, their proper design is of primary importance for sensor fusion. Although many design methods of complementary filters have been proposed in the literature [3], [4], no simple method that allows to shape the norm of the complementary filters is available. Such method is proposed here and is based on the $\mathcal{H}_\infty$ synthesis.

Sensor Fusion Architecture

Let's consider **two sensors** measuring the same physical quantity x with dynamics $G_1(s)$ and $G_2(s)$, and with uncorrelated noise characteristics n_1 and n_2.

The signals from both sensors are fed into two **complementary filters** $H_1(s)$ and $H_2(s)$ and then combined to form a **super sensor** that yield an estimate $\hat{x}$ of x (Fig. 1).

$$\hat{x} = (G_1 H_1 + G_2 H_2)\, x + H_1 n_1 + H_2 n_2$$

The **complementary property** of $H_1(s)$ and $H_2(s)$ implies that their transfer function sum is equal to one at all frequencies:

$$\boxed{H_1(s) + H_2(s) = 1}$$

Complementary Filters Requirements

Noise Property

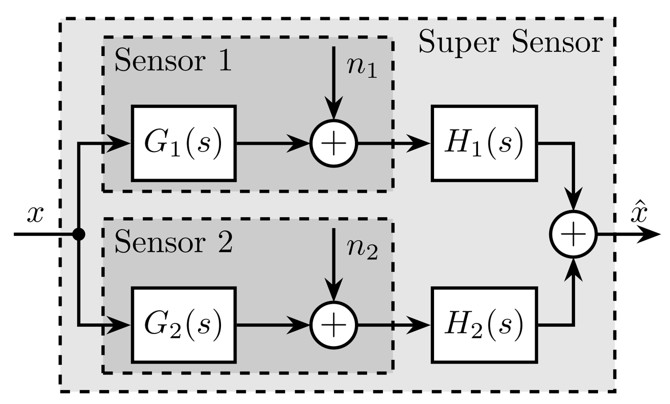

Fig. 1: Sensor fusion architecture

First suppose **perfectly known sensor dynamics** such that is can be inverted:

$$G_1(s) = G_2(s) = 1$$

The estimate $\hat{x}$ is then:

$$\boxed{\hat{x} = x + H_1 n_1 + H_2 n_2}$$

The signal x is kept **undistorted** while the noises n_1 and n_2 are **filtered out by the complementary filters**.

The estimate error δx is:

$$\delta x \triangleq \hat{x} - x = H_1 n_1 + H_2 n_2$$

The PSD of the super sensor noise depends on the **norm** of the complementary filters:

$$\boxed{\Phi_{\delta x} = |H_1|^2 \Phi_{n_1} + |H_2|^2 \Phi_{n_2}}$$

Dynamical Uncertainty

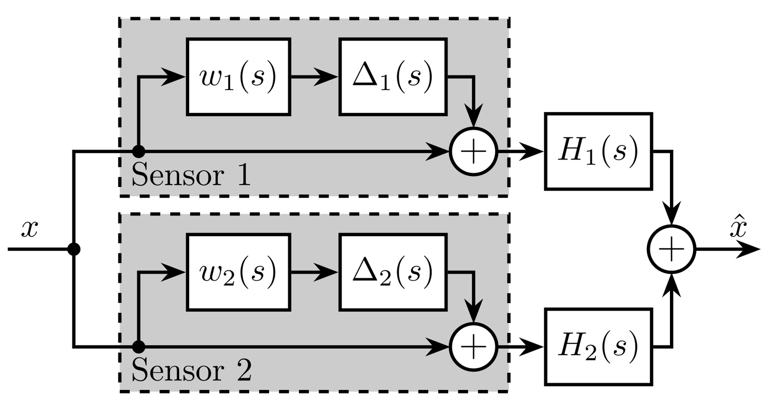

Fig. 2: Fusion of sensors with dynamics uncertainty

Let's represent sensor dynamic uncertainty by **multiplicative input uncertainty**:

$$G_i'(s) = G_i(s)[1 + w_i(s)\Delta_i(s)], \quad \forall \Delta_i, \|\Delta_i\|_\infty < 1$$

The dynamics of the super sensor is:

$$\boxed{\frac{\hat{x}}{x} = 1 + w_1 H_1 \Delta_1 + w_2 H_2 \Delta_2}$$

The obtained dynamic uncertainty depends on the **norm** of the filters (Fig. 3).

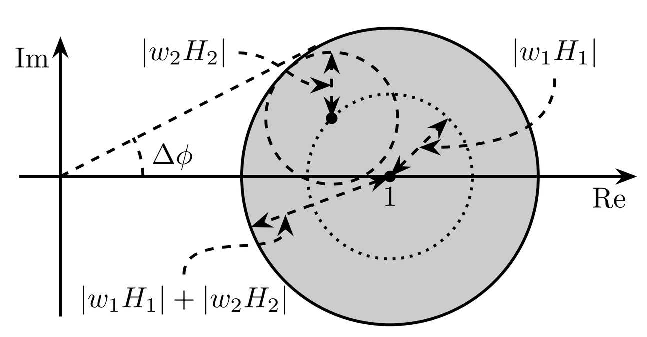

Fig. 3: Uncertainty set of the super sensor dynamics

As shown in the analysis above, **the performance and robustness of the sensor fusion architecture depends on the complementary filters norms**. Therefore, the development of a synthesis method of complementary filters that allows the shaping of their norm is necessary.

Complementary Filters Shaping using $\mathcal{H}_\infty$ Synthesis

The **synthesis objective** is to **shape the norm of two filters** while ensuring their **complementary property**. This is equivalent to the conditions on the right where $H_1(s)$ and $H_2(s)$ are stable transfer function.

$$\begin{aligned} & H_1(s) + H_2(s) = 1 \\ & |H_1(j\omega)| \le \frac{1}{|W_1(j\omega)|} \quad \forall \omega \\ & |H_2(j\omega)| \le \frac{1}{|W_2(j\omega)|} \quad \forall \omega \end{aligned}$$

$W_1(s)$ and $W_2(s)$ are **weighting functions** that are used to define wanted **upper bound of the complementary filter norms**. They should be **proper**, **stable** and **minimum phase** transfer functions.

This optimization problem is written as a **standard $\mathcal{H}_\infty$ problem** (Fig. 4).
The $\mathcal{H}_\infty$ synthesis applied to $P(s)$ generates a stable filter $H_2(s)$ such that the $\mathcal{H}_\infty$ norm from w to $[z_1,\ z_2]$ is less than one. By defining $H_1(s) \triangleq 1 - H_2(s)$, this is equivalent to the synthesis objective described above.

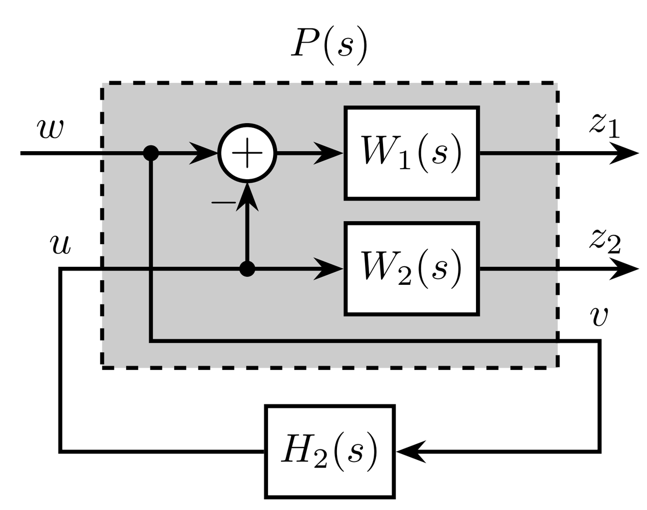

Fig. 4: $\mathcal{H}_\infty$ synthesis of complementary filters

This $\mathcal{H}_\infty$ synthesis is first applied for the design of simple complementary filters (Fig. 5).

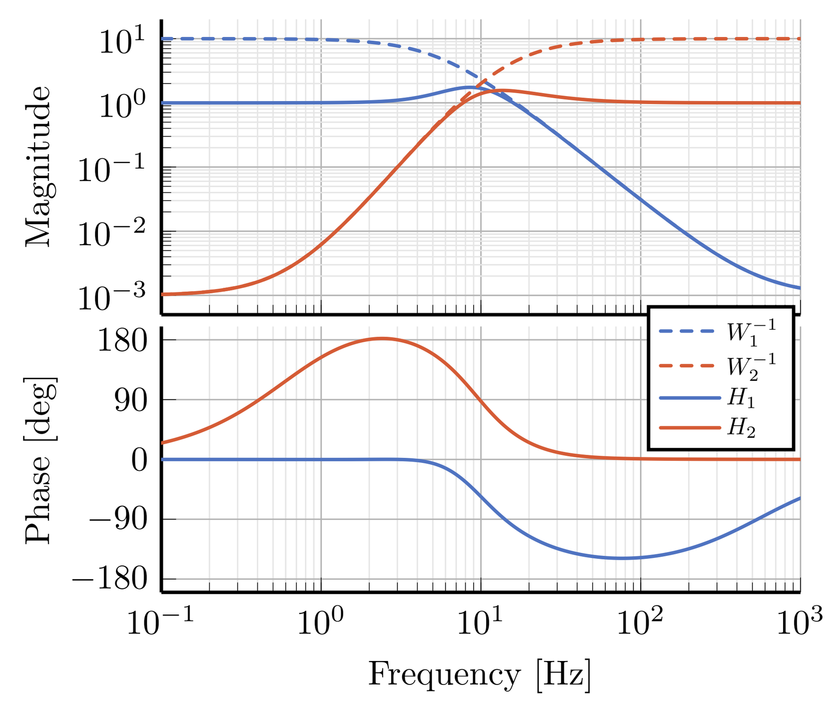

Fig. 5: Frequency response of the weighting functions and complementary filters obtained using $\mathcal{H}_\infty$ synthesis

Design of Complementary Filters used in the Active Vibration Isolation System at the LIGO

The effectiveness of the proposed method is demonstrated by designing a complementary filter pair that is used in the active isolation system at the LIGO [5]. The requirements are very tight (shown by dashed upper bounds in Fig. 6) and thus their design is complex.

The weights are designed such that their inverse magnitude is as close as possible to the specifications while being of reasonably **small order** (Fig. 6).

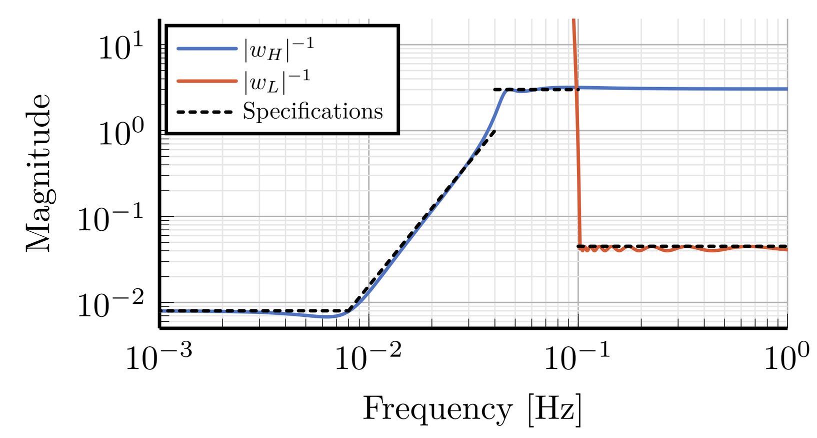

Fig. 6: Specifications and weights used for the $\mathcal{H}_\infty$ synthesis

Custom designed 7th Order Transfer Function

Type I Chebyshev Filter (Order 20)

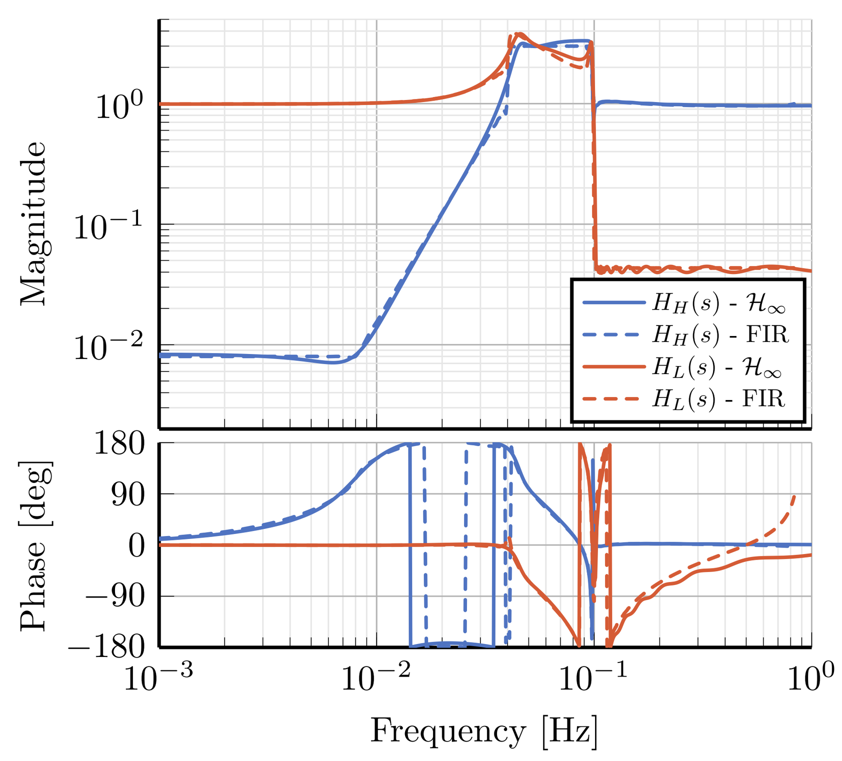

Fig. 7: Comparison of the FIR filters designed in [5] (order 512) with the filters obtained with $\mathcal{H}_\infty$ synthesis (order 27)

After synthesis, the obtained complementary filters are compared with the FIR filters [5] and are found to be very close to each other (Fig. 7).

Conclusion

Complementary filters can be used to **combine multiple sensors** in order to obtain a **super sensor**. Specification on the super sensor **noise** and on the **robustness of the sensor fusion** are linked to the **norm of the complementary filters**. A synthesis method that permits the **shaping of the complementary filters norms** has been proposed and has been successfully applied for the design of complex filters.

Reference

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