Design of Complementary Filters Using the $\mathcal{H}_\infty$ Synthesis

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Abstract— **Abstract text to be done**

I. INTRODUCTION

A set of filters is said to be complementary if the sum of their transfer functions is equal to one at all frequencies. These filters are used when two or more sensors are measuring the same physical quantity with different noise characteristics. Unreliable frequencies of each sensors are filtered out by the complementary filters and combined to form a "super sensor" giving a better estimate of the physical quantity over a wider bandwidth. This technique is called sensor blending or sensor fusion and used for many applications.

In [1–3], such technique is used for the attitude estimation of Unmanned Aerial Vehicles (UAV) using various sensors (accelerometers, gyroscopes, vision sensors, ...). In [4], several sensor fusion configurations using different types of sensors (inertial, relative motion and force) have been discussed in order to increase the control bandwidth of active vibration isolation systems. Furthermore, sensor fusion is used for the low frequency isolation at the Laser Interferometer Gravitational-Wave Observator (LIGO), where inertial sensors are merged with relative sensors [5, 6].

As the super sensor noise characteristics depends largely on the complementary filter norms, their proper design is of primary importance for sensor fusion performance. Different design methods have been developed. In [2, 3, 7], analytical formulas for complementary filter transfer functions of first and second order have been proposed. Higher order complementary filters have been used in [1, 4, 8]. In [7], the sensitivity and complementary sensitivity transfer functions of a feedback architecture have been proposed to be used for complementary filters. The design of such filters can then benefits from the developments of classical control theory. Linear Matrix Inequalities (LMIs) are used in [9] for the synthesis of complementary filters satisfying some frequency-like performance. A design method using linear least squares to optimize the coefficients of complementary filters is proposed in [3]. Finally, high order Finite Impulse Response (FIR) complementary filters are used to merge sensors at LIGO [5, 6, 10]. The synthesis of such filter using convex optimization is presented in [10].

Although many design methods of complementary filters have been proposed, we believe that it lacks a simple method

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of designing complementary filters [...] specifying upper bounds.

In this paper, we propose a synthesis method for the shaping of complementary filters using the $\mathcal{H}_\infty$ norm.

The section II gives a brief overview of sensor fusion using complementary filters and the associated requirements on their norms. In section III, a new method for designing complementary filters using the $\mathcal{H}_\infty$ synthesis is proposed. In section IV, two high order complementary filters are designed using the proposed method and compared with FIR filters used for sensor fusion at the LIGO. Our conclusions are drawn in the final section.

II. COMPLEMENTARY FILTERS REQUIREMENTS

A. Sensor Fusion Architecture

Let's consider two sensors measuring the same physical quantity x but with different dynamics ($G_1(s)$ and $G_2(s)$) and noise characteristics (n_1 and n_2).

The signals from both sensors are fed into two complementary filters $H_1(s)$ and $H_2(s)$ and then combined to yield an estimate $\hat{x}$ of x as shown on Fig. 1.

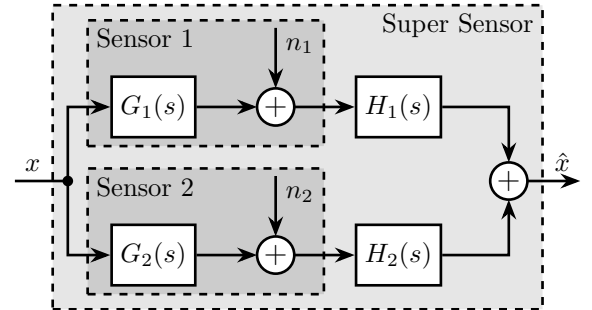$$\hat{x} = (G_1 H_1 + G_2 H_2)\, x + H_1 n_1 + H_2 n_2 \qquad (1)$$

Fig. 1.   Sensor Fusion Architecture

The complementary property of $H_1(s)$ and $H_2(s)$ implies that their transfer function sum is equal to one at all frequencies (2).

$$H_1(s) + H_2(s) = 1 \qquad (2)$$

B. Noise Sensor Filtering

Let's first consider sensors with perfect dynamics

$$G_1(s) = G_2(s) = 1 \qquad (3)$$

The estimate $\hat{x}$ is then described by

$$\hat{x} = x + H_1 n_1 + H_2 n_2 \qquad (4)$$

The complementary filters $H_1(s)$ and $H_2(s)$ only operates on the noise of the sensors. Thus, this sensor fusion architecture permits to filter the noise of both sensors without introducing any distortion in the physical quantity to measure.

The estimation error δx is defined by (5).

$$\delta x \triangleq \hat{x} - x = H_1 n_1 + H_2 n_2 \quad (5)$$

As shown in (6), the Power Spectral Density (PSD) of the estimation error $\Phi_{\delta x}$ depends both on the norms of the complementary filters and on the PSD of the noise sources Φ_{n_1} and Φ_{n_2}.

$$\Phi_{\delta x} = |H_1|^2 \Phi_{n_1} + |H_2|^2 \Phi_{n_2} \quad (6)$$

Usually, the two sensors have higher noise levels over distinct yet complementary frequency regions. In order to lower the noise present in the estimation $\hat{x}$, the norm $|H_1|$ has to be made small when Φ_{n_1} is larger than Φ_{n_2} and $|H_2|$ small when Φ_{n_2} is larger than Φ_{n_1}.

C. Robustness of the Fusion

In practical systems, the sensors dynamics has always some level of uncertainty and cannot be inverted perfectly such that $G_i(s) = 1$.

This uncertainty can be represented as input multiplicative uncertainty as shown on Fig. 2 where Δ_i is any transfer function satisfying $\|\Delta_i(j\omega)\|_\infty \leq 1$, $\forall\omega$ and where $|W_i(s)|$ represents the frequency dependent uncertainty level.
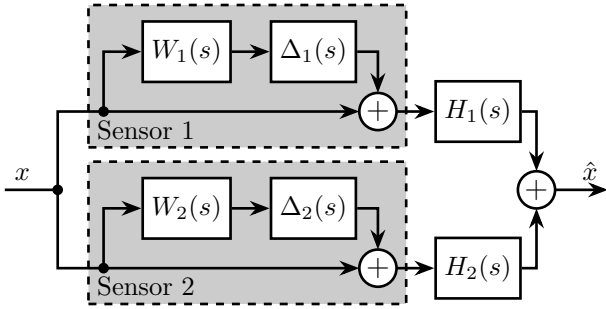
Fig. 2.   Sensor Fusion Architecture with Sensor Dynamical Uncertainty

The super sensor dynamics (7) is not longer equal to 1 and now depends on the sensor dynamic uncertainties $W_i(s)$ as well as on the complementary filters $H_i(s)$.

$$\frac{\hat{x}}{x} = 1 + W_1(s)H_1(s)\Delta_1(s) + W_2(s)H_2(s)\Delta_2(s) \quad (7)$$

In order to limit the phase and gain uncertainty of the super sensor, one may want to design the complementary filters to such that (8) is satisfied.

$$|W_1 H_1 \Delta_1| + |W_2 H_2 \Delta_2| \leq \epsilon \quad \forall\omega, \forall\Delta_i$$
$$\Leftrightarrow |W_1 H_1| + |W_2 H_2| \leq \epsilon \quad \forall\omega \quad (8)$$

Condition (8) is equivalent as to bound the uncertainty set of the super sensor dynamics in the complex plane by a circle centered on 1 with a radius equal to ϵ (Fig. 3).
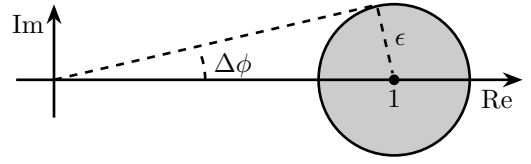
Fig. 3.   Uncertainty set of the super sensor dynamics

The maximum phase added by the super sensor uncertainty $\Delta\phi$ is then equal to (9).

$$\Delta\phi = \arcsin(\epsilon) \quad (9)$$

Limiting the phase added by the super sensor to $\Delta\phi = 30°$ requires that $H_1(s)$ and $H_2(s)$ are designed such that (8) is satisfied with $\epsilon = \sin(\Delta\phi = 30°) = 0.5$. Thus the norm of the complementary filter $|H_i|$ for sensor i should be made small at frequencies where its dynamic uncertainty $|W_i|$ is large.

As stated above, the requirements in terms of noise attenuation and robustness of the sensor fusion architecture can be termed as upper bounds on the norm of the complementary filters.

III. COMPLEMENTARY FILTERS SHAPING USING THE $\mathcal{H}_\infty$ SYNTHESIS

As shown in Sec. II, most of the performance requirements for the design of the complementary filters can be expressed as upper bounds on the magnitude of the filters.

Therefore, the $\mathcal{H}_\infty$ Loop Shaping framework seems adapted for the synthesis of complementary filters. In this section, a technique for the synthesis complementary filters while specifying uppers bounds on their magnitudes using the $\mathcal{H}_\infty$ synthesis is presented.

A. Synthesis of Complementary Filters as a $\mathcal{H}_\infty$ problem

The synthesis objective is to shape the norm of two filters $H_1(s)$ and $H_2(s)$ while ensuring their complementary property (2).

The synthesis problem is then to find stable transfer functions $H_1(s)$ and $H_2(s)$ such that conditions (10) are satisfied.

$$H_1(s) + H_2(s) = 1 \quad (10a)$$

$$|H_1(j\omega)| \leq \frac{1}{|W_1(j\omega)|} \quad \forall\omega \quad (10b)$$

$$|H_2(j\omega)| \leq \frac{1}{|W_2(j\omega)|} \quad \forall\omega \quad (10c)$$

where $W_1(s)$ and $W_2(s)$ are two weighting transfer functions chosen to shape the corresponding filters.

In order to express this synthesis problem into a standard $\mathcal{H}_\infty$ problem, we use the standard architecture shown on Fig. 4 where the generalized plant P is described by (11).

$$\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix}; \quad P(s) = \begin{bmatrix} W_1(s) & -W_1(s) \\ 0 & W_2(s) \\ 1 & 0 \end{bmatrix} \quad (11)$$
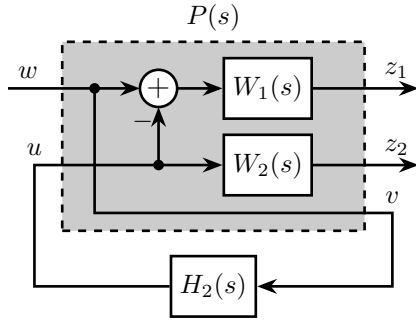
Fig. 4. Architecture used for the $\mathcal{H}_\infty$ synthesis of complementary filters

The $\mathcal{H}_\infty$ filter design problem is then to find a stable filter $H_1(s)$ which based on v, generates a signal u such that the $\mathcal{H}_\infty$ norm from w to $[z_1, \ z_2]$ is less than one (12).

$$\left\| \begin{matrix} [1 - H_2(s)] W_1(s) \\ H_2(s)W_2(s) \end{matrix} \right\|_\infty \leq 1 \tag{12}$$

Which is equivalent to (13) by defining $H_1(s) \triangleq 1 - H_2(s)$.

$$\left\| \begin{matrix} H_1(s)W_1(s) \\ H_2(s)W_2(s) \end{matrix} \right\|_\infty \leq 1 \tag{13}$$

The complementary condition (10a) is ensured by the definition of $H_1(s)$. The conditions (10b) and (10c) on the shape of the filters are satisfied by (13).

Using this $\mathcal{H}_\infty$ synthesis method, we are then able to shape complementary filters.

B. Choice of the weighting functions

The choice of the weighting functions is of primary importance for the success of the presented $\mathcal{H}_\infty$ synthesis of complementary filters.

First, only proper, stable and minimum phase transfer functions should be used. Second, the order of the weights should stay reasonably small as this will increase both the complexity of the optimization problem and the order of the synthesized complementary filters, the latter begin equal to the sum of the weighting functions orders. Third, one should not forget the fundamental limitations imposed by the complementary property: $H_1(s) + H_2(s) = 1$. This implies for instance that $|H_1(j\omega)|$ and $|H_2(j\omega)|$ cannot be made small at the same time.

When designing complementary filters, it is usually desired to specify the slope of the filter, its crossover frequency and its low and high frequency gains. To help with the design of the weighting functions such that the above specification are easily expressed, the formula (14) is proposed.

$$W(s) = \left(\frac{\frac{1}{\omega_0}\sqrt{\frac{1-\left(\frac{G_0}{G_c}\right)^{\frac{2}{n}}}{1-\left(\frac{G_c}{G_\infty}\right)^{\frac{2}{n}}}}s + \left(\frac{G_0}{G_c}\right)^{\frac{1}{n}}}{\left(\frac{1}{G_\infty}\right)^{\frac{1}{n}}\frac{1}{\omega_0}\sqrt{\frac{1-\left(\frac{G_0}{G_c}\right)^{\frac{2}{n}}}{1-\left(\frac{G_c}{G_\infty}\right)^{\frac{2}{n}}}}s + \left(\frac{1}{G_c}\right)^{\frac{1}{n}}} \right)^n \tag{14}$$

where:
- G_0 is the absolute gain at low frequency

- G_∞ is the absolute gain at high frequency
- ω_0 and G_c define the absolute value of the filter at $\omega = \omega_0$: $|W(j\omega_0)| = G_c$
- n is the order of the weighting function as well as its slope between high and low frequency

The parameters G_0, G_c and G_∞ should either satisfy condition (15a) or (15b).

$$G_0 < 1 < G_\infty \text{ and } G_0 < G_c < G_\infty \tag{15a}$$

$$G_\infty < 1 < G_0 \text{ and } G_\infty < G_c < G_0 \tag{15b}$$

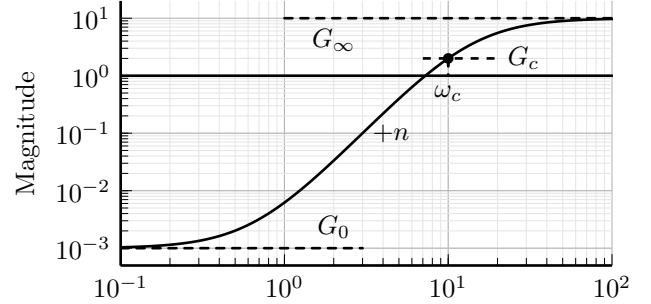The shape of the weighting function generated using (14) is shown on Fig. 5.

Fig. 5. Amplitude of the proposed formula for the weighting functions, $G_0 = 1e^{-3}$, $G_\infty = 10$, $\omega_c = 10 \text{ Hz}$, $G_c = 2$, $n = 3$

C. Example

Let's validate the proposed design method of complementary filters using the $\mathcal{H}_\infty$ synthesis with a simple example.

Both weighting functions $W_1(s)$ and $W_2(s)$ are designed using (14). The parameters used are summarized on table I and the magnitude of the weighting functions are shown on Fig. 6.

The blending frequency is chosen to be around 10 Hz. The slope of $|H_1(j\omega)|$ is chosen to be -2 above 10 Hz by choosing $n = 2$ for $W_1(s)$. The slope of $|H_2(j\omega)|$ is chosen to be $+3$ below 10 Hz by choosing $n = 3$ for $W_2(s)$. The order of the obtained complementary filters will thus be of order 5.

TABLE I
PARAMETERS USED FOR $W_1(s)$ AND $W_2(s)$

Parameter	$W_1(s)$	$W_2(s)$
G_0	0.1	1000
G_∞	1000	0.1
ω_c [Hz]	11	10
G_c	2	2
n	2	3

The bode plot of the obtained complementary filters is shown on Fig. 6 and their transfer functions in the Laplace domain are shown below.

$$H_1(s) = \frac{10^{-8}(s + 6.6e^9)(s + 3450)^2(s^2 + 49s + 895)}{(s + 6.6e^4)(s^2 + 106s + 3e^3)(s^2 + 72s + 3580)}$$

$$H_2(s) = \frac{(s + 6.6e^4)(s + 160)(s + 4)^3}{(s + 6.6e^4)(s^2 + 106s + 3e^3)(s^2 + 72s + 3580)}$$
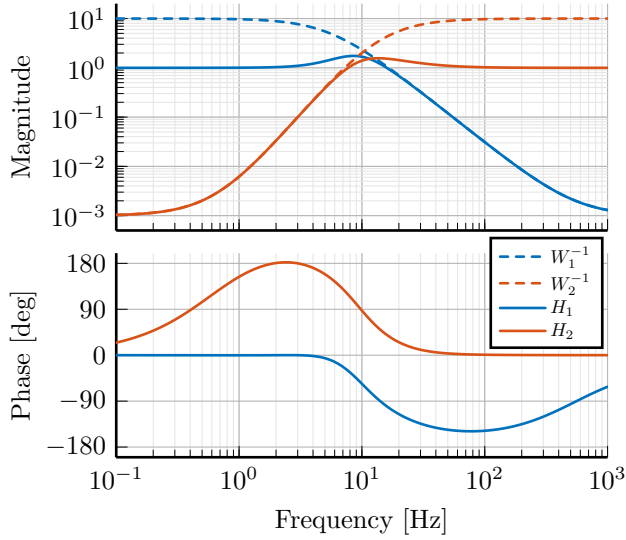
Fig. 6. Weighting functions and Obtain Complementary Filters using the $\mathcal{H}_\infty$ Synthesis

D. Generalization to the synthesis of Three Complementary Filters

In some applications, it may be needed to merge more than two sensors. In such case, it is necessary to design as many complementary filters $H_i(s)$ as the number of sensors used. The synthesis problem is then to compute n stable transfer functions $H_i(s)$ such that (16) is satisfied.

$$\sum_{i=0}^{n} H_i(s) = 1 \tag{16a}$$

$$|H_i(j\omega)| < \frac{1}{|W_i(j\omega)|}, \quad \forall\omega, \; i = 1 \ldots n \tag{16b}$$

The synthesis architecture on Fig. 4 can be generalized for the synthesis of a set of n complementary filters. For the synthesis of three complementary filters, the architecture used is shown on Fig. 7.

The $\mathcal{H}_\infty$ synthesis objective applied on $P(s)$ is to design two stable filters $H_2(s)$ and $H_3(s)$ such that the $\mathcal{H}_\infty$ norm of the transfer function from w to $[z_1, \; z_2, \; z_3]$ is less than one (17).

$$\left\| \begin{matrix} [1 - H_2(s) - H_3(s)] \, W_1(s) \\ H_2(s)W_2(s) \\ H_3(s)W_3(s) \end{matrix} \right\|_\infty \le 1 \tag{17}$$

By choosing $H_1(s) \triangleq 1 - H_2(s) - H_3(s)$, the proposed $\mathcal{H}_\infty$ synthesis solves the design problem (16).

An example is given to validate the method where three sensors are used in different frequency bands (up to $1\,\mathrm{Hz}$, from 1 to $10\,\mathrm{Hz}$ and above $10\,\mathrm{Hz}$ respectively). Three weighting functions are designed using (14) and shown by dashed curves on Fig. 8. The obtained complementary filters after synthesis are shown on Fig. 8.
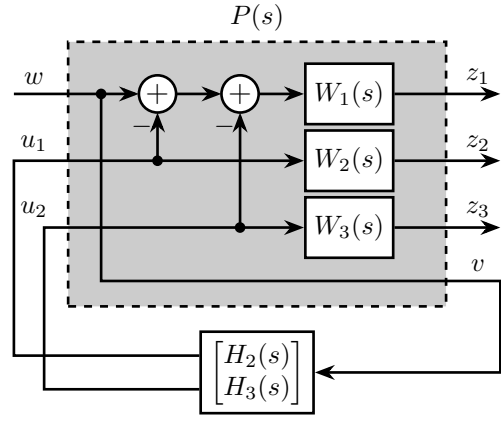

Fig. 7. Architecture for the $\mathcal{H}_\infty$ synthesis of three complementary filters
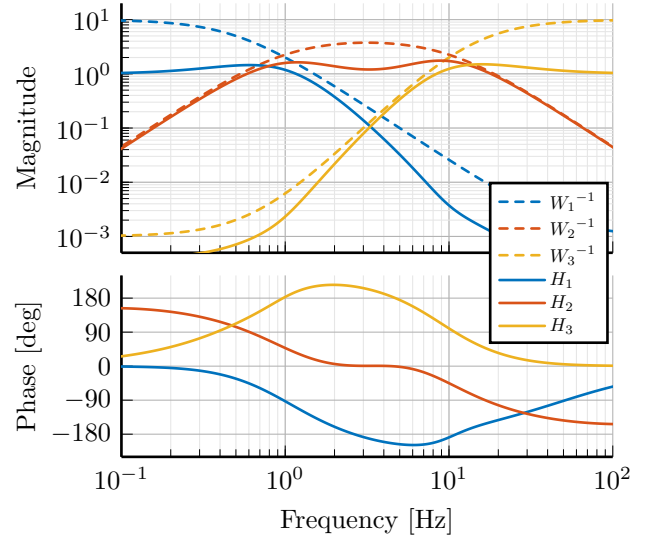

Fig. 8. Obtained three complementary filters

IV. APPLICATION: DESIGN OF COMPLEMENTARY FILTERS USED FOR THE ACTIVE VIBRATION ISOLATION SYSTEM AT THE LIGO

Several complementary filters are used for the active isolation system at the LIGO [6, 10]. The requirements on those filters are very tight and thus their design is complex. The approach taken in [10] for the design of those filters is to use high order FIR filters and write the synthesis objective as a convex optimization problem. The obtained FIR filters are compliant with the requirements, however their implementation is complex as it requires a lot of computation power.

In order to demonstrate the effectiveness of the synthesis method presented in section III, complementary filters subject to the same requirements are synthesize.

A. Specifications

As explained in section II, typical specifications for the complementary filters are on their magnitude.

The specifications are summarized below and explained in details in [6]:

- From 0 to $0.008\,\text{Hz}$, the magnitude of the filter's transfer function should be less or equal to 8×10^{-4}
- Between $0.008\,\text{Hz}$ to $0.04\,\text{Hz}$, the filter should attenuate the input signal proportional to frequency cubed
- Between $0.04\,\text{Hz}$ to $0.1\,\text{Hz}$, the magnitude of the transfer function should be less than 3
- Above $0.1\,\text{Hz}$, the magnitude of the complementary filter should be less than 0.045

The specifications are represented by the dashed black curves on the bode plot in Fig. 9.

B. Weighting functions design

The weighting functions should be designed so that their inverse amplitude is as close as possible to the specifications in order to not over-constrain the synthesis problem. However, their order should not be too high as to limit both the complexity of the optimization problem and the order of the synthesize complementary filters.

For the weight function corresponding to the low pass filter $w_L(s)$, a Type I Chebyshev filter of order 20 is used. For the one corresponding to the high pass filter $w_H(s)$, a 7th order transfer function gives satisfactory results. The magnitudes of the inverse of the weighting functions are shown on Fig. 9.

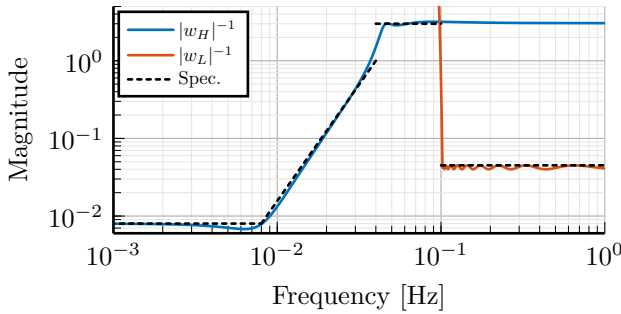Fig. 9. Specification and Weighting Functions used for the $\mathcal{H}_\infty$ synthesis

C. $\mathcal{H}_\infty$ Synthesis

The $\mathcal{H}_\infty$ synthesis is performed using the architecture of Fig. 11. The complementary filters obtained are of order 27. Their bode plot is compared on Fig. 10 with the FIR filters of order 512 obtained in [10].

V. CONCLUSION

VI. ACKNOWLEDGMENT

Fig. 10. Comparison of the FIR filters (solid) designed in [10] with the filters obtained with the $\mathcal{H}_\infty$ synthesis (dashed)

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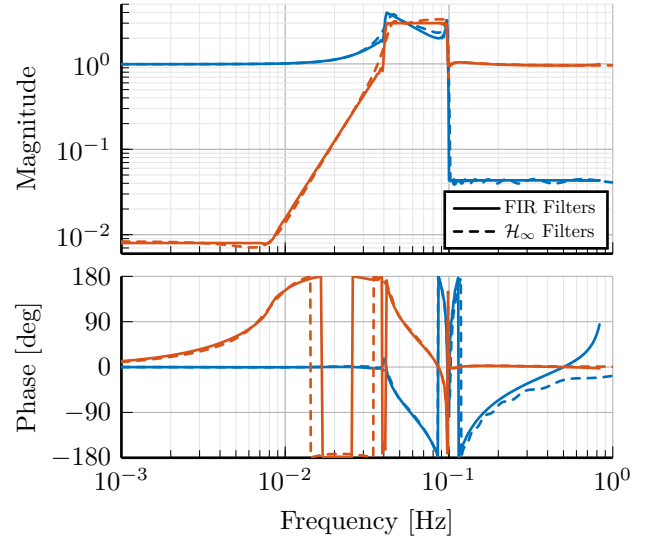
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