How to connect a model to a flexible support

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1 Method 1 - Connection of multiple Mass-Spring-Damper systems

1.1 General representation

Let's consider a simple Mass-Spring-Damper system that is on top of an other mecanical system (see figure 1).

$$
\begin{cases}
x_i & \text{is the displacment of the system i} \\
x_{i-1} & \text{is the displacment of the flexible support} \\
f_{i+1} & \text{is the force applied by the system on top of this one (could be zero)} \\
g_i & \text{is the force applied to the system (actuator forces for instance)} \\
f_i & \text{is the force that the system applies on the flexible support}
\end{cases}
$$

Second Newton's laws of motion on the mass m_i:

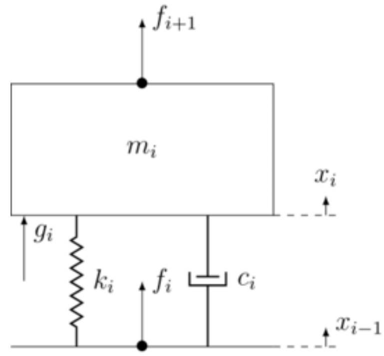

Figure 1: Basic Mass-Spring-Damper

$$m_i\ddot{x} = k_i(x_{i-1} - x_i) + c_i(\dot{x}_{i-1} - \dot{x}_i) + g_i + f_{i+1}$$

$$m_i\ddot{x} + c_i\dot{x}_i + k_i x_i = c_i\dot{x}_{i-1} + k_i x_{i-1} + g_i + f_{i+1}$$

By doing a Laplace tranformation, we have:

$$(m_i s^2 + c_i s + k_i)X_i = (c_i s + k_i)X_{i-1} + G_i + F_{i+1}$$

Finally:

$$X_i = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i}X_{i-1} + \frac{1}{m_i s^2 + c_i s + k_i}G_i + \frac{1}{m_i s^2 + c_i s + k_i}F_{i+1} \quad (1)$$

$$X_i = T_{x_{i-1}\to x_i}X_{i-1} + T_{g_i\to x_i}G_i + T_{f_{i+1}\to x_i}F_{i+1}$$

With:

$$\begin{cases} T_{x_{i-1}\to x_i} & = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} \\ T_{g_i\to x_i} & = \frac{1}{m_i s^2 + c_i s + k_i} \\ T_{f_{i+1}\to x_i} & = \frac{1}{m_i s^2 + c_i s + k_i} \end{cases}$$

Now, let's express the force f_i:

$$f_i = k_i(x_i - x_{i-1}) + c_i(\dot{x}_i - \dot{x}_{i-1})$$

By doing a Laplace tranformation, we have:

$$F_i = (X_i - X_{i-1})(c_i s + k_i)$$

And finally by reinjecting X_i into the last equation, we have the following equation:

$$F_i = \frac{-m_i s^2 (c_i s + k_i)}{m_i s^2 + c_i s + k_i} X_{i-1} + \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} G_i + \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} F_{i+1} \quad (2)$$

$$F_i = T_{x_{i-1} \to f_i} X_{i-1} + T_{g_i \to f_i} G_i + T_{f_{i+1} \to f_i} F_{i+1}$$

With:

$$\begin{cases} T_{x_{i-1} \to f_i} & = \frac{-m_i s^2 (c_i s + k_i)}{m_i s^2 + c_i s + k_i} \\ T_{g_i \to f_i} & = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} \\ T_{f_{i+1} \to f_i} & = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} \end{cases}$$

So we have a representation of the system with 3 inputs and 2 outputs (figure 2). This system is governed by the following equation:

$$\begin{pmatrix} x_i \\ f_i \end{pmatrix} = \begin{pmatrix} T_{x_{i-1} \to x_i} & T_{g_i \to x_i} & T_{f_{i+1} \to x_i} \\ T_{x_{i-1} \to f_i} & T_{g_i \to f_i} & T_{f_{i+1} \to f_i} \end{pmatrix} \begin{pmatrix} x_{i-1} \\ g_i \\ f_{i+1} \end{pmatrix} \quad (3)$$

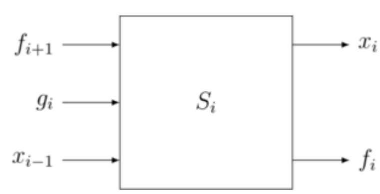

Figure 2: Basic Mass-Spring-Damper - Input/Output representation

That representation of a system with 3 inputs and 2 outputs can be generalized to any system as long as we have all the transfer functions $T_{g_i \to x_i}, \ldots$

1.2 System connection - General Case

Now that we have a system definition with the 3 inputs and 2 outputs. We can try to connect multiple systems (figure 3).

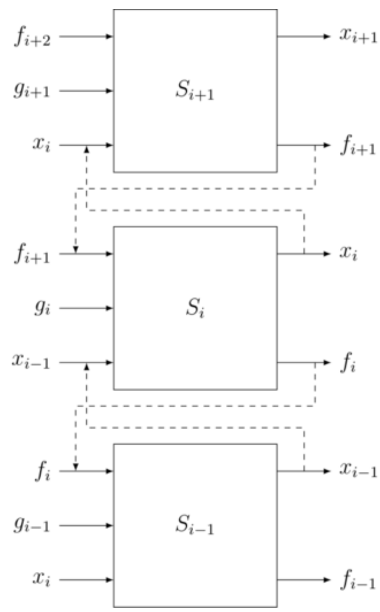

Figure 3: Basic Mass-Spring-Damper - Input/Output representation

1.3 System connection - 2 System case

For now, let's ignore:

- f_0 the force applied on the system bellow S_0 (probably the ground)
- f_2 the force applied one S_1 because we suppose that there is now system of top of S_1

We have a connected system with 3 inputs ($x_{-1} \quad g_0 \quad g_1$) and 1 outputs x_1.

Analytically:

$$x_1 = T^1_{g_1 \to x_1} g_1 + T^1_{x_0 \to x_1} x_0$$

$$\begin{aligned} x_0 &= T^0_{g_0 \to x_0} g_0 + T^0_{x_{-1} \to x_0} x_{-1} + T^0_{f_1 \to x_0} f_1 \\ &= T^0_{g_0 \to x_0} g_0 + T^0_{x_{-1} \to x_0} x_{-1} + T^0_{f_1 \to x_0} (T^1_{x_0 \to f_1} x_0 + T^1_{g_1 \to f_1} g_1) \end{aligned}$$

$$x_0 = \frac{T^0_{g_0 \to x_0}}{1 - T^0_{f_1 \to x_0} T^1_{x_0 \to f_1}} g_0 + \frac{T^0_{x_{-1} \to x_0}}{1 - T^0_{f_1 \to x_0} T^1_{x_0 \to f_1}} x_{-1} + \frac{T^0_{f_1 \to x_0} T^1_{g_1 \to f_1}}{1 - T^0_{f_1 \to x_0} T^1_{x_0 \to f_1}} g_1$$

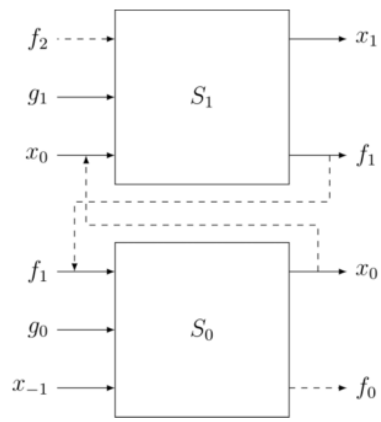

Figure 4: Connect 2 systems

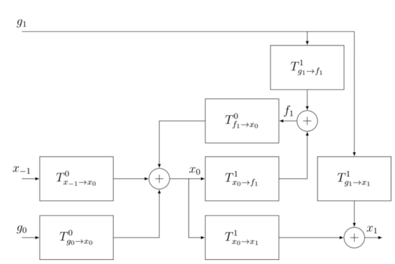

Figure 5: Connect 2 systems - Bloc representation

By reinjecting x_0 into the first equation, we have an expression of the output x_1 function of the 3 inputs $x_{-1} \quad g_0 \quad g_1$.

$$x_1 = (T^1_{g_1\to x_1} + \frac{T^1_{x_0\to x_1} T^0_{f_1\to x_0} T^1_{g_1\to f_1}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}}) g_1 + \frac{T^1_{x_0\to x_1} T^0_{g_0\to x_0}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}} g_0 + \frac{T^1_{x_0\to x_1} T^0_{x_{-1}\to x_0}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}} x_{-1}$$

$$x_1 = T_{g_1\to x_1} g_1 + T_{g_0\to x_1} g_0 + T_{x_{-1}\to x_1} x_{-1}$$

With:

$$\begin{cases} T_{g_1\to x_1} & = T^1_{g_1\to x_1} + \frac{T^1_{x_0\to x_1} T^0_{f_1\to x_0} T^1_{g_1\to f_1}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}} \\ T_{g_0\to x_1} & = \frac{T^1_{x_0\to x_1} T^0_{g_0\to x_0}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}} \\ T_{x_{-1}\to x_1} & = \frac{T^1_{x_0\to x_1} T^0_{x_{-1}\to x_0}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}} \end{cases}$$

1.4 System connection - Example

Let's say that we have a flexible support S_0. The only thing we know about this support is the relation between a force applied on top of it to its displacement: $T^0_{f_1\to x_0}$.

Know, we want to add a mass-spring-damper system on top of this flexible support. We would like to know the displacement of the mass-spring-damper system from a force applied to it, that is to say we want $T_{g_1\to x_1}$.

$$T_{g_1\to x_1} = T^1_{g_1\to x_1} + \frac{T^1_{x_0\to x_1} T^0_{f_1\to x_0} T^1_{g_1\to f_1}}{1 - T^0_{f_1\to x_0} T^1_{x_0\to f_1}}$$

$$\begin{cases} T^1_{x_0\to x_1} & = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} \\ T^1_{g_1\to x_1} & = \frac{1}{m_i s^2 + c_i s + k_i} \\ T^1_{g_1\to f_1} & = \frac{c_i s + k_i}{m_i s^2 + c_i s + k_i} \\ T^1_{x_0\to f_1} & = \frac{-m_i s^2 (c_i s + k_i)}{m_i s^2 + c_i s + k_i} \end{cases}$$

We can calculate $T_{g_1\to x_1}$.

1.5 Matlab Implementation

A small matlab toolbox have been developed and is accessible on Github. This toolbox contains many functions that easily permit to create systems using the presented architecture and connect them.

- `createElement` permits to create a mass-spring-damper system (2nd, 3rd and 4th arguments, the 1st on is the number i). It creates a state space system with named inputs and outputs based on i
- `connectElements` permits to connect all the elements
- `createForceActuator` permits to create a force actuator between 2 systems (1st and 2nd arguments)
- `connectForceActuator` permits to connect the actuator to the system
- `createDisplacementSensor` permits to create a displacement sensor between 2 elements
- `connectDisplacementSensor` permits to connect the sensor to the system

All functions are well documented, you can type `help functionName` to have some help. Also, you should check the demo files inside the demo folder.

2 Method 2 - How to connect a model on top of a flexible support

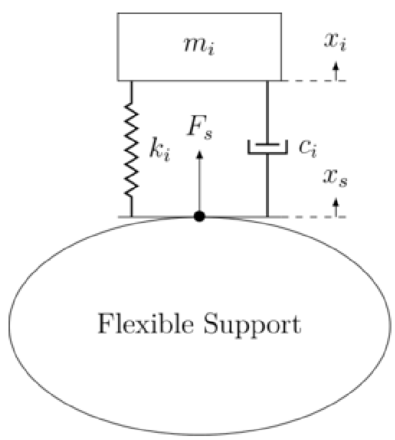

Figure 6: Flexible support

$$\begin{cases} x_i & \text{is the displacment of the system i} \\ x_s & \text{is the displacment of the flexible support} \\ f_s & \text{is the force applied by the system on the flexible support} \end{cases}$$

Second Newton's laws of motion on the mass m_i:

$$m_i \ddot{x_i} = k_i(x_s - x_i) + c_i(\dot{x_s} - \dot{x_i})$$

After Laplace transformation:

$$m_i s^2 x_s = (c_i s + k_i)(x_s - x_i)$$

Let's express F_s:

$$F_s = (c_i s + k_i)(x_i - x_s)$$

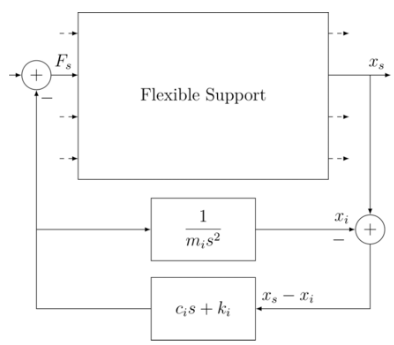

Figure 7: Flexible support Connection

In order to connect the system to the flexible support, we only need the transfer function between F_s and x_s. This is usually done by identification.