Lab 09: Frequency Response: Bandpass Filters

1 Pre-Lab

The goal of this lab is to study the response of FIR filters to inputs such as complex exponentials and sinusoids. In the experiments of this lab, you will use `firfilt()`, or `conv()`, to implement filters and `freqz()` to obtain the filter's frequency response. As a result, you should learn how to characterize a filter by knowing how it reacts to different frequency components in the input. This lab also introduces bandpass filters, which can be used to detect and extract information from sinusoidal signals, e.g., tones in a touch-tone telephone dialer.

1.1 Frequency Response of FIR Filters

The output or response of a filter for a complex sinusoid input, $e^{j\hat{\omega}}$ depends on the frequency, $\hat{\omega}$. Often a filter is described solely by how it affects different input frequencies — this is called the frequency response. For example, the frequency response of the two-point averaging filter

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

can be found by using a general complex exponential as an input and observing the output or response.

$$\begin{aligned} x[n] &= Ae^{j(\hat{\omega}n+\phi)} && (1) \\ y[n] &= \frac{1}{2}Ae^{j(\hat{\omega}n+\phi)} + \frac{1}{2}Ae^{j(\hat{\omega}(n-1)+\phi)} && (2) \\ &= Ae^{j(\hat{\omega}n+\phi)}\frac{1}{2}\left\{1+e^{-j\hat{\omega}}\right\} = Ae^{j(\hat{\omega}n+\phi)} \cdot H(e^{j\hat{\omega}}). && (3) \end{aligned}$$

In (3) there are two terms, the original input, and a term that is a function of $\hat{\omega}$. This second term is the frequency response and it is commonly denoted by $H(e^{j\hat{\omega}})$, which in this case is

$$H(e^{j\hat{\omega}}) = \frac{1}{2}\left\{1+e^{-j\hat{\omega}}\right\}. \quad (4)$$

Once the frequency response, $H(e^{j\hat{\omega}})$, has been determined, the effect of the filter on any complex exponential may be determined by evaluating $H(e^{j\hat{\omega}})$ at the corresponding frequency. The output signal $y[n]$, will be a complex exponential whose complex amplitude has a constant magnitude and phase. The phase describes the phase change of the complex sinusoid and the magnitude describes the gain applied to the complex sinusoid. The frequency response of a general FIR linear time-invariant system is

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}. \quad (5)$$

In the example above, $M = 1$, and $b_0 = \frac{1}{2}$ and $b_1 = \frac{1}{2}$.

1.1.1 MATLAB Function for Frequency Response

MATLAB has a built-in function called `freqz()` for computing the frequency response of a discrete-time LTI system. The following MATLAB statements show how to use `freqz` to compute and plot both the magnitude (absolute value) and the phase of the frequency response of a two-point averaging system as a function of $\hat{\omega}$ in the range $-\pi \leq \hat{\omega} \leq \pi$:

```
bb = [0.5, 0.5];        %-- Filter Coefficients
ww = -pi:(pi/100):pi;   %-- omega hat
HH = freqz(bb, 1, ww);  %<--freekz.m is an alternative
subplot(2,1,1);
plot(ww, abs(HH))
subplot(2,1,2);
plot(ww, angle(HH))
xlabel('Normalized Radian Frequency')
```

For FIR filters, the second argument of `freqz(-, 1, -)` must always be equal to 1[1]. The frequency vector `ww` should cover an interval of length 2π for $\hat{\omega}$, and its spacing must be fine enough to give a smooth curve for $H(e^{j\hat{\omega}})$. Note: we will always use capital `HH` for the frequency response.

1.2 Periodicity of the Frequency Response

The frequency responses of discrete-time filters are always periodic with period equal to 2π. Explain why this is the case by stating a definition of the frequency response and then considering two input sinusoids whose frequencies are $\hat{\omega}$ and $\hat{\omega} + 2\pi$.

$$x_1[n] = e^{\hat{\omega}n} \quad \text{versus} x_2[n] = e^{j(\hat{\omega}+2\pi)n}$$

Consult Chapter 6 for a mathematical proof that the outputs from each of these signals will be identical (basically because $x_1[n]$ is equal to $x_2[n]$.)
The implication of periodicity is that a plot of $H(e^{j\hat{\omega}})$ only needs to extend over the interval $-\pi \leq \hat{\omega} \leq \pi$ or any other interval of length 2π.

1.3 Frequency Response of the Four-Point Averager

In Chapter 6 we examined filters that average input samples over a certain interval. These filters are called "running average" filters or "averagers" and they have the following form for the L-point averager:

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \tag{6}$$

(a) Use Euler's formula and complex number manipulations to show that the frequency response for the 4-point running average operator is given by:

$$H(e^{j\hat{\omega}}) = \frac{2\cos(0.5\hat{\omega}) + 2\cos(1.5\hat{\omega})}{4} e^{-1.5\hat{\omega}} \tag{7}$$

[1]If the output of the `freqz` function is not assigned, then plots are generated automatically; however, the magnitude is given in decibels which is a logarithmic scale. For linear magnitude plots a separate call to plot is necessary.

(b) Implement (7) directly in MATLAB. Use a vector that includes 400 samples between $-\pi$ and π for $\hat{\omega}$. Since the frequency response is a complex-valued quantity, use `abs()` and `angle()` to extract the magnitude and phase of the frequency response for plotting. Plotting the real and imaginary parts of $H(e^{j\hat{\omega}})$ is not very informative.

(c) In this part, use `freqz.m` in MATLAB to compute $H(e^{j\hat{\omega}})$ numerically (from the filter coefficients) and plot its magnitude and phase versus $\hat{\omega}$. Write the appropriate MATLAB code to plot both the magnitude and phase of $H(e^{j\hat{\omega}})$. Follow the example in Section 1.1.1. The filter coefficient vector for the 4-point averager is defined via:

```
bb = 1/4*ones(1,4);
```

Note: the function `freqz(bb,1,ww)` evaluates the frequency response for all frequencies in the vector `ww`. It uses the summation in (5), not the formula in (7). The filter coefficients are defined in the assignment to vector `bb`. How do your results compare with part (b)?

1.4 The MATLAB `FIND` Function

Often signal processing functions are performed in order to extract information that can be used to make a decision. The decision process inevitably requires logical tests, which might be done with `if-then` constructs in MATLAB. However, MATLAB permits vectorization of such tests, and the `find` function is one way to do lots of tests at once. In the following example, find extracts all the numbers that "round" to 3:

```
xx = 1.4:0.33:5, jkl = find(round(xx)==3), xx(jkl)
```

The argument of the `find` function can be any logical expression. Notice that find returns a list of indices where the logical condition is true. See `help` on `relop` for information.
Now, suppose that you have a frequency response:

```
ww = -pi:(pi/500):pi; HH = freqz( 1/4*ones(1,4), 1, ww );
```

Use the `find` command to determine the indices where `HH` is zero, and then use those indices to display the list of frequencies where `HH` is zero. Since there might be round-off error in calculating `HH`, the logical test should probably be a test for those indices where the magnitude (absolute value in MATLAB) of `HH` is less than some rather small number, e.g., 1×10^{-6}. Compare your answer to the frequency response that you plotted for the four-point averager in Section 1.3.

2 Lab Exercises

2.1 Simple Bandpass Filter Design

The L-point averaging filter is a lowpass filter. Its passband width is controlled by L, being inversely proportional to L. In fact, you can use the GUI `dltidemo` (see CD-ROM of the textbook, Chapter 6 demos) to view the frequency response for different averagers and measure the passband widths. It is also possible to create a filter whose passband is centered around some frequency other than zero. One simple way to do this is to define the impulse response of an L-point FIR as:

$$h[n] = \frac{2}{L}\,\cos(\hat{\omega}_c n), \quad 0 \le n < L$$

where L is the filter length, and $\hat{\omega}_c$ is the center frequency that defines the frequency location of the passband. For example, we would pick $\hat{\omega}_c = 0.44\pi$ if we want the peak of the filter's passband to be centered at 0.44π.

The bandwidth of the bandpass filter is controlled by L; the larger the value of L, the narrower the bandwidth. This particular filter is also discussed in the section on useful filters in Chapter 7 of the text.

(a) Generate a bandpass filter that will pass a frequency component at $\hat{\omega} = 0.44\pi$. Make the filter length (L) equal to 10. Since we are going to be filtering the signal defined below,

$$x[n] = 5\cos(0.3\pi n) + 22\cos(0.44\pi n - \pi/3) + 22\cos(0.7\pi n - \pi/4) \tag{8}$$

measure the gain of the filter at the three frequencies of interest: $\hat{\omega} = 0.3\pi$, $\hat{\omega} = 0.44\pi$ and $\hat{\omega} = 0.7\pi$.

(b) The passband of the BPF filter is defined by the region of the frequency response where $|H(e^{j\hat{\omega}})|$ is close to its maximum value. If we define the maximum to be $H_{\max}$, then the passband width is defined as the length of the frequency region where the ratio $|H(e^{j\hat{\omega}})|/H_{\max}$ is greater than $1/\sqrt{2} = 0.707$. The *stopband* of the BPF filter is defined by the region of the frequency response where $|H(e^{j\hat{\omega}})|$ is close to zero. In this case, we will define the stopband as the region where $|H(e^{j\hat{\omega}})|$ is less than 25% of the maximum.

Figure 3 shows how to define the passband and stopband. Note: you can use MATLAB's `find` function to locate those frequencies where the magnitude satisfies $|H(e^{j\hat{\omega}})| \geq 0.707 H_{\max}$.

Make a plot of the frequency response for the $L = 10$ bandpass filter from part (a), and determine the passband width (at the 0.707 level). Repeat the plot for $L = 20$ and $L = 40$, so you can explain how the width of the passband is related to filter length L, i.e., what happens when L is doubled or halved.

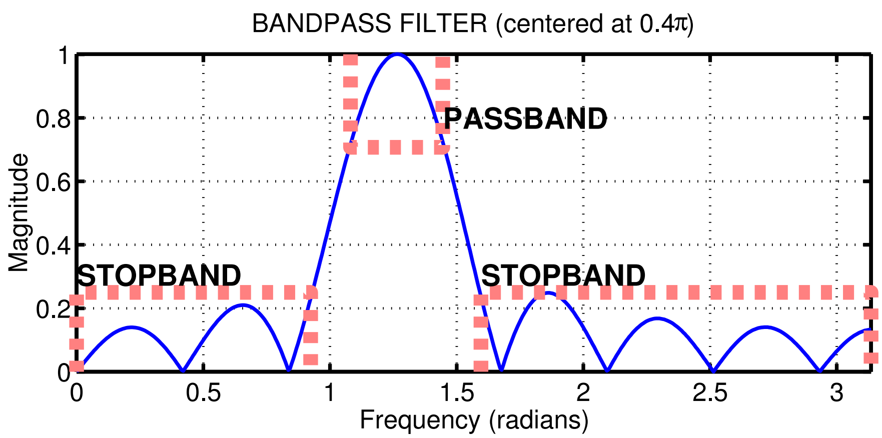

Figure 1: Passband and Stopband for a typical FIR bandpass filter. In this case, the maximum value is 1, the passband is the region where the frequency response is greater than $1/\sqrt{2} = 0.707$, and the stopband is defined as the region where the frequency response is less than 25% of the maximum.

(c) Comment on the selectivity of the $L = 10$ bandpass filter. In other words, which frequencies are "passed by the filter"? Use the frequency response to explain how the filter can pass one component at $\hat{\omega} = 0.44\pi$, while reducing or rejecting the others at $\hat{\omega} = 0.3\pi$ and $\hat{\omega} = 0.7\pi$.

(d) Generate a bandpass filter that will pass the frequency component at $\hat{\omega} = 0.44\pi$, but now make the filter length (L) long enough so that it will also greatly reduce frequency components at (or near) $\hat{\omega} = 0.3\pi$ and $\hat{\omega} == 0.7\pi$. Determine the smallest value of L so that

- Any frequency component satisfying $|\hat{\omega}| \leq 0.3\pi$ will be reduced by a factor of 10 or more[2].

[2]For example, the input amplitude of the 0.7π component is 22, so its output amplitude must be less than 2.2.

- Any frequency component satisfying $0.7\pi \leq |\hat{\omega}| \leq \pi$ will be reduced by a factor of 10 or more.

This can be done by making the passband width very small.

(e) Use the filter from the previous part to filter the "sum of 3 sinusoids" signal in (8). Make a plot of 100 points of the input and output signals, and explain how the filter has reduced or removed two of the three sinusoidal components.

(f) Make a plot of the frequency response (magnitude only) for the filter from part (d), and explain how $H(e^{j\hat{\omega}})$ can be used to determine the relative size of each sinusoidal component in the output signal. In other words, connect a mathematical description of the output signal to the values that can be obtained from the frequency response plot.