Deep learning
Feed-forward neural networks

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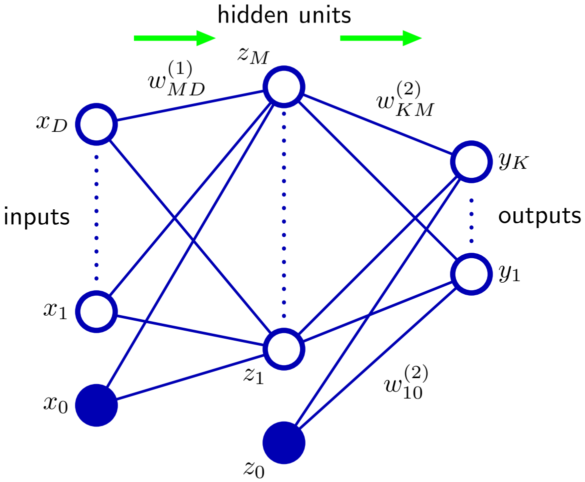
Objectives of lecture

- Feed-forward neural network (FFNN)
- Next week:
- Training with error back-propagation
- We only need to understand the principles
- Autograd - automated differentiation handles the derivation for us!

Many thanks to Tapani Raiko for making and sharing first version of these slides!

Part 1:
Feed-forward neural networks

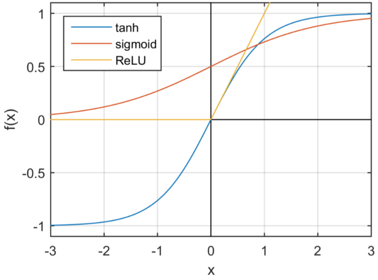
Feed forward neural networks



$$y_k(\mathbf{x}, \mathbf{w}) = \sigma\left(\sum_{j=0}^{M} w_{kj}^{(2)} \underbrace{f\left(\sum_{i=0}^{D} w_{ji}^{(1)} x_i\right)}_{z_j}\right)$$

Non-linearity and training

- Linear activation functions will give a linear network.
- Logistic function $\sigma(a) = \frac{1}{1+e^{-a}}$
- Hyperbolic tangent $\tanh(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}}$
- Rectified linear $\mathrm{relu}(a) = \max(0, a)$



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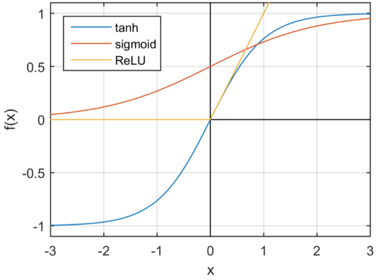

- Supervised learning
- Labeled training set

$$\mathcal{D} = \{(x_i, t_i) | i = 1, \ldots, n\} .$$

- Input x_i and target t_i.

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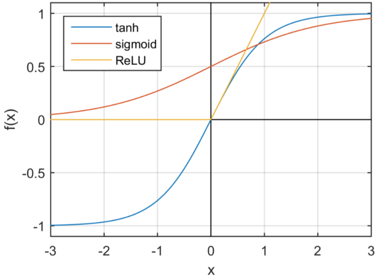

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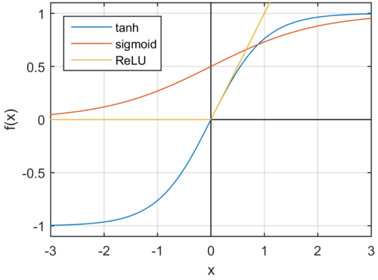

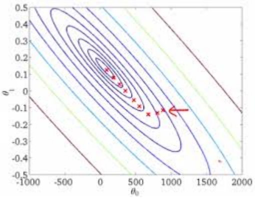

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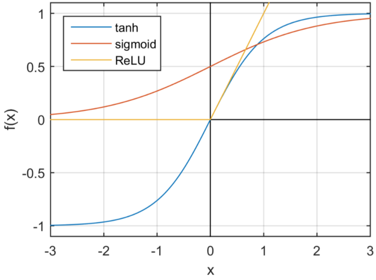

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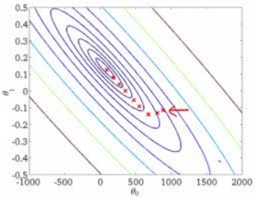
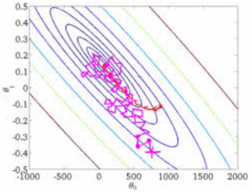
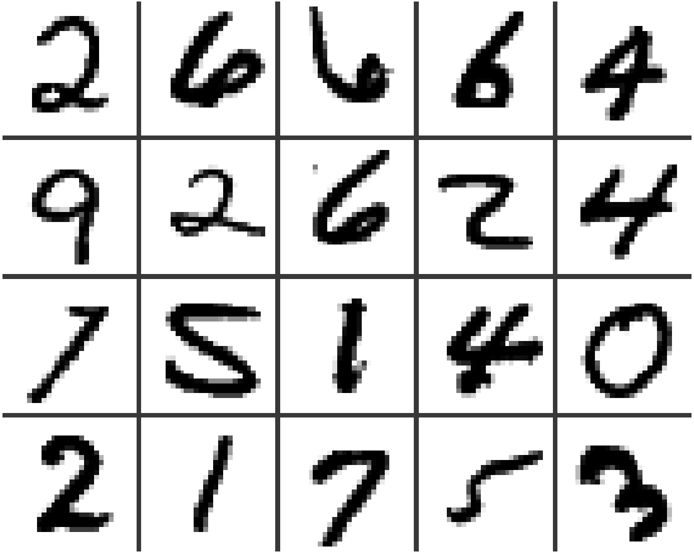


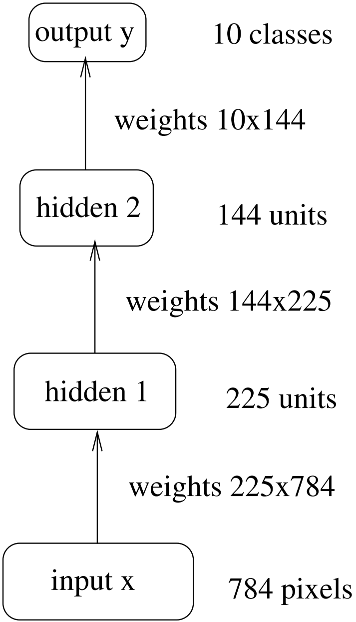


Example: MNIST handwritten digits



Train a network to classify 28×28 images.
Data: 60000 input images $\mathbf{x}(n)$ and labels $t(n)$.
Example model gives around 1.2% test error.

Example Network



$$\mathbf{y} = \mathbf{h}^{(3)} = \text{softmax}(\mathbf{W}^{(3)}\mathbf{h}^{(2)} + \mathbf{b}^{(3)})$$

$$\mathbf{h}^{(2)} = \text{relu}(\mathbf{W}^{(2)}\mathbf{h}^{(1)} + \mathbf{b}^{(2)})$$

$$\mathbf{h}^{(1)} = \text{relu}(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)})$$

$$\text{softmax}(\mathbf{z})_i = \frac{\exp(z_i)}{\sum_j \exp(z_j)}$$

$$\text{relu}(z) = max(0, z)$$

Softmax

- Softmax function

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  has two nice properties:
- $\mathrm{softmax}(\mathbf{z})_i \geq 0$
- $\sum_i \mathrm{softmax}(\mathbf{z})_i = 1$

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  has two nice properties:
- $\text{softmax}(\mathbf{z})_i \geq 0$
- $\sum_i \text{softmax}(\mathbf{z})_i = 1$
- MNIST, output labels: $0, 1, \ldots, 9$.
- Output of network

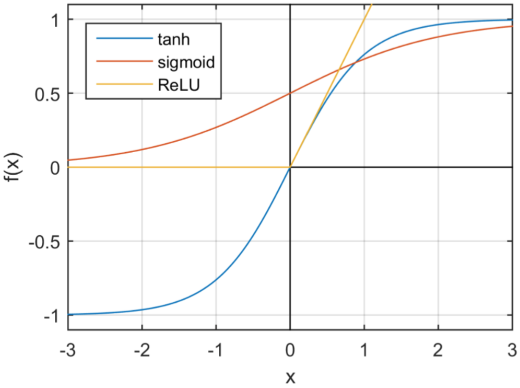
$$\mathbf{y} = \text{softmax}(\mathbf{W}^{(3)}\mathbf{h}^{(2)} + \mathbf{b}^{(3)})$$

  interpreted as class(-conditional) probabilities:
- So given input $\mathbf{x}$, according to the model, the probability of digit i is:

$$p(\text{digit} = i|\mathbf{x}) = y_{i+1}$$

- with $i = 0, \ldots, 9$.

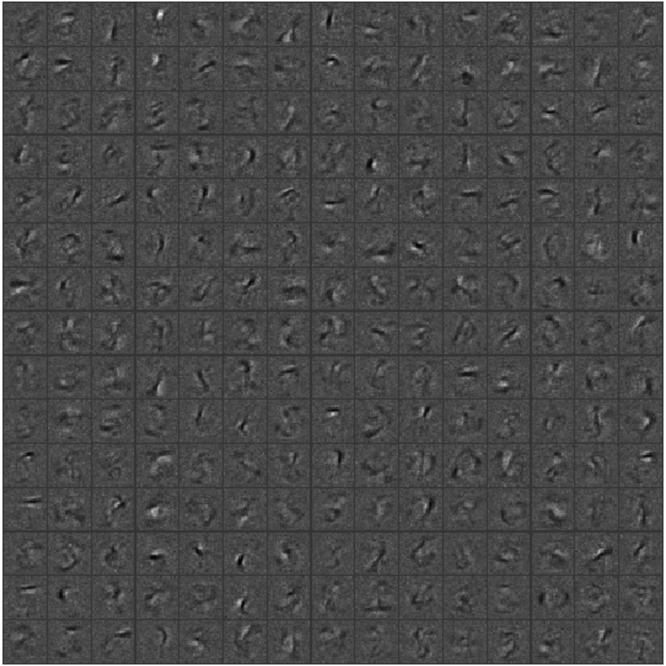
On activation functions



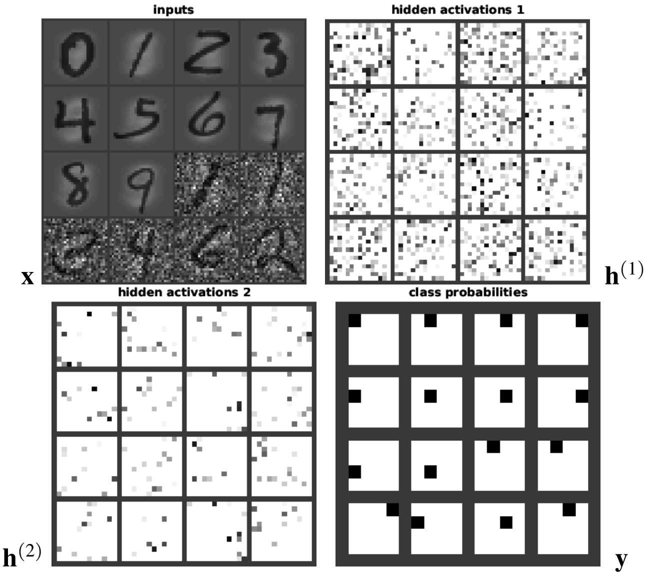
- $\mathrm{relu}(z) = \max(0, z)$ is replacing old sigmoid and tanh.
- Note that identity function would lead into:

$$\begin{aligned}
\mathbf{h}^{(2)} &= \mathbf{W}^{(2)}\mathbf{h}^{(1)} + \mathbf{b}^{(2)} \\
&= \mathbf{W}^{(2)}(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}) + \mathbf{b}^{(2)} \\
&= (\mathbf{W}^{(2)}\mathbf{W}^{(1)})\mathbf{x} + (\mathbf{W}^{(2)}\mathbf{b}^{(1)} + \mathbf{b}^{(2)}) \\
&= \mathbf{W}'\mathbf{x} + \mathbf{b}'
\end{aligned}$$

Weight matrix $\mathbf{W}^{(1)}$ size 225×784



Signals $\mathbf{x} \to \mathbf{h}^{(1)} \to \mathbf{h}^{(2)} \to \mathbf{h}^{(3)}$



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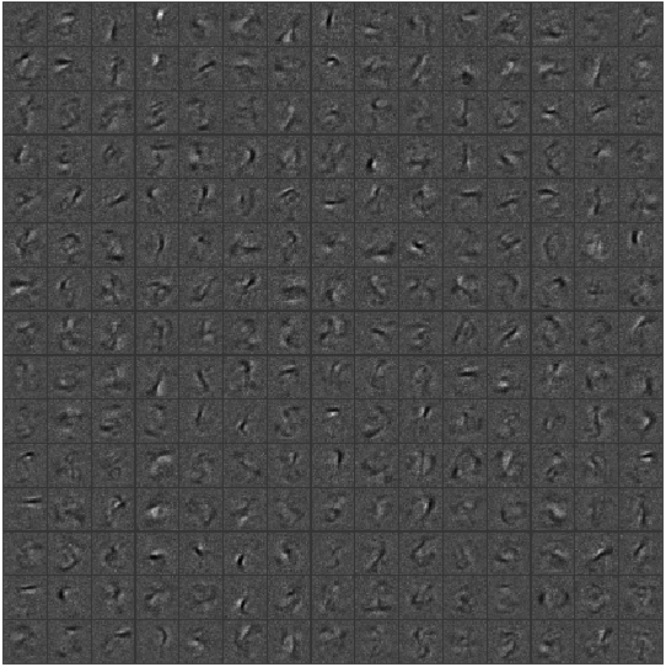

Part 2:

Doing yourself - understand how feed-forward neural networks approximate functions

How a feed-forward neural network learns XOR

- Consider two-layer network with two hidden units:

$$y(\mathbf{x}) = \Theta\left(W_1^{(2)} h_1^{(1)} + W_2^{(2)} h_2^{(1)} + b_1^{(1)}\right)$$

$$h_1^{(1)} = \Theta\left(W_{11}^{(1)} x_1 + W_{12}^{(1)} x_2 + b_1^{(1)}\right)$$

$$h_2^{(1)} = \Theta\left(W_{21}^{(1)} x_1 + W_{22}^{(1)} x_2 + b_2^{(1)}\right)$$

- Step function activation function: $\Theta(z) = 1$ if $z \geq 0$ and 0 otherwise

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\begin{aligned}
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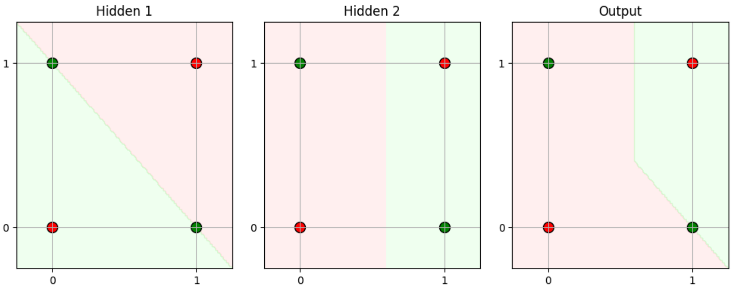
$$

- Step function activation function: $\Theta(z) = 1$ if $z \geq 0$ and 0 otherwise
- Fetch XOR_NNSandbox.ipynb from Absalon and turn the knobs to learn XOR



x1	x2	h1	h2	target	pred	OK
0	0	1	0	0	0	✓
0	1	0	0	1	0	✗
1	0	0	1	1	1	✓
1	1	0	1	0	1	✗

- Red is output 1 and green is output 0.
- We will also consider this problem in Assignment 1.

TensorFlow Playground

playground.tensorflow.org

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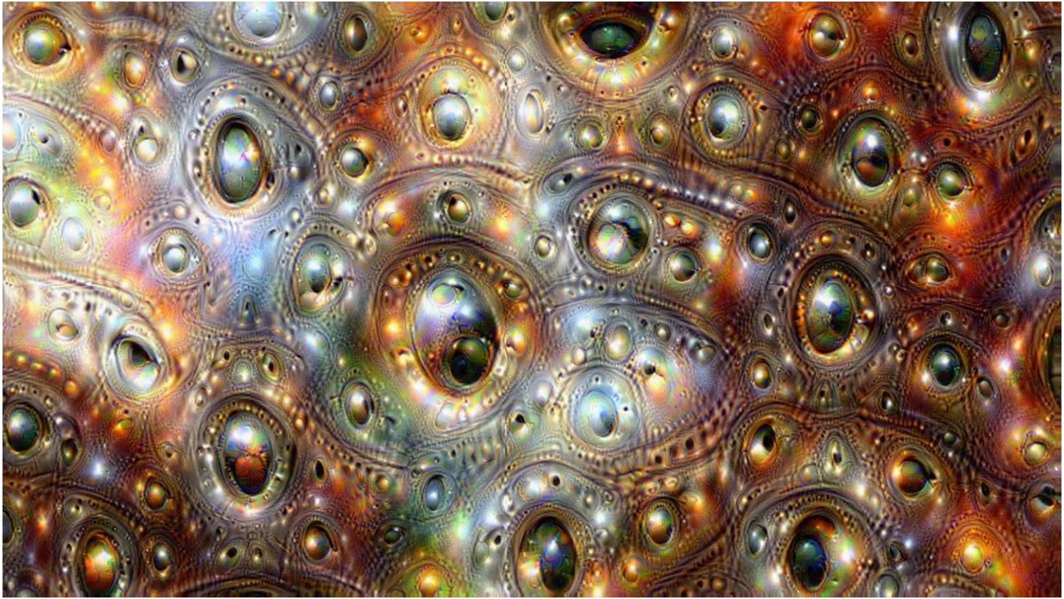

Thanks!
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