

Proof: PMFs, PDFs, CDFs

Sophie Chowgule, Tom Coleman

Summary

Explanations as to why some PMF's and PDF's are valid.

Before reading this proof sheet, it is recommended that you read [Guide: PMFs, PDFs, CDFs](#). Other recommended reading material will be said when it is needed.

Proof that the binomial distribution is a PMF

Before reading this section, you may find it useful to read [Guide: The binomial theorem].

Remember from [Guide: PMFs, PDFs, CDFs](#) that the **binomial distribution** is given by the following.

Binomial distribution

$$P(X = x) = \binom{n}{x} p^x q^{(n-x)} = \frac{n!}{(n-x)!x!} p^x q^{(n-x)}$$

where:

- the random variable $X = x$ measures the number of success in a set of n trials
 - x is number of successes
 - n is number of trials
- p is the probability of success in a single trial
- $q = 1 - p$ is the probability of failure in a single trial

Also from [Guide: PMFs, PDFs, CDFs](#), the two conditions to be a valid PMF are the following:

- **Non-negativity:** The probability assigned to each possible outcome must be greater than or equal to zero, that is:

$$p(x) = P(X = x) \geq 0 \text{ for all values of } x.$$

- **Honesty condition:** The sum of probabilities of all possible outcomes x of a discrete random variable X must be equal to one:

$$\sum_x p(x) = \sum_x P(X = x) = 1.$$

First of all, every term in the PMF for the binomial distribution above is non-negative, and the product of non-negative numbers is non-negative, so $P(X = x) \geq 0$ for any x .

The honesty condition comes about because binomial distributions follow the **binomial theorem**. The binomial theorem states that:

$$\sum_{x=0}^n \binom{n}{x} p^x q^{(n-x)} = (p + q)^n$$

(See [Guide: The binomial theorem] for more.)

The number of successes x ranges from 0 (total failure) to n (complete success). Therefore, the sum of all possible probabilities $P(X = x)$ is:

$$\sum_x P(X = x) = \sum_{x=0}^n \binom{n}{x} p^x q^{(n-x)}$$

which is the left-hand side of the binomial theorem. Using the binomial theorem with $q = 1-p$:

$$\sum_x P(X = x) = (p + q)^n = (p + (1-p))^n = (1)^n = 1$$

So, the sum of the probabilities over all possible values of x equals 1, satisfying the honesty condition.

Proof that the uniform distribution is a PDF

Before reading this section, you may find it useful to read [Guide: Introduction to integration] and [Guide: Properties of integration].

Remember from [Guide: PMFs, PDFs, CDFs](#) that the **uniform distribution** over the interval $[a, b]$ is given by the following.

i Uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where a, b are real numbers such that $a < b$.

Also from [Guide: PMFs, PDFs, CDFs](#), the two conditions to be a valid PDF are the following:

- **Non-negativity:** The PDF $f(x)$ must be greater than or equal to zero over its entire range of possible values:

$$f(x) \geq 0 \text{ for all values of } x.$$

- **Honesty condition:** The area under the entire PDF $f(x)$ must be equal to 1, so:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

To check if this is a valid PDF, you need to confirm that it satisfies these two key conditions.

Non-negativity: $f(x) \geq 0$ for all values of x , as $f(x) = \frac{1}{b-a}$ in $[a, b]$ and 0 otherwise.

Honesty: To satisfy the honesty condition, the integral of the PDF over the interval $[a, b]$ must equal 1. Using the properties of integration, you can split the integral into three parts along the lines of the PDF:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^b f(x) dx + \int_b^{\infty} f(x) dx$$

Using the definition of $f(x)$ on these intervals gives

$$\int_{-\infty}^a f(x) dx + \int_a^b f(x) dx + \int_b^{\infty} f(x) dx = \int_{-\infty}^a 0 dx + \int_a^b \frac{1}{b-a} dx + \int_b^{\infty} 0 dx$$

Since the integral of 0 over any limits is zero, this reduces to

$$\int_{-\infty}^{\infty} f(x) dx = 0 + \int_a^b \frac{1}{b-a} dx + 0 = \int_a^b \frac{1}{b-a} dx$$

Working out this integral dives

$$\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b 1 dx = \frac{1}{b-a} [x]_a^b = \frac{1}{b-a} (b-a) = 1$$

And so you can see that all uniform distributions are valid PDFs.

Proof that the normal distribution is a PDF

Before reading this section, you may find it useful to read [Guide: Properties of integration], [Guide: Integration by substitution], [Guide: Introduction to double integration], and [Guide: Co-ordinate changes in double integration].

Remember from Guide: PMFs, PDFs, CDFs that the **normal distribution** is given by the following.

i Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

where μ, σ are real numbers such that $\sigma > 0$. (Here, μ is the mean and σ is the standard deviation.)

To check if this is a valid PDF, you need to confirm that it satisfies the two key conditions.

Non-negativity: As an exponential function, $\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) > 0$, and $1/\sigma\sqrt{2\pi} > 0$ as $\sigma > 0$. So $f(x) > 0$.

Honesty: Here's the fun part.

The idea is to show that this integral I , given by

$$I = \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

is equal to 1. To tackle this integral, it needs to look a little nicer; you can use integration by substitution to do this (see [Guide: Integration by substitution]). Let $u = \frac{x-\mu}{\sigma\sqrt{2}}$. Then $\frac{du}{dx} = \frac{1}{\sigma\sqrt{2}}$, and so $dx = \sigma\sqrt{2} du$. As $x \rightarrow \pm\infty$, it follows that $u \rightarrow \pm\infty$. Since $u^2 = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$, the integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx = \int_{-\infty}^{\infty} \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

Next, you can use the fact that $\exp(-u^2)$ is an even function to change the limits. Using the property of even function about symmetric limits (see [Guide: Properties of integration]), the integral becomes

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du = I$$

All that you have done so far has not changed the value of the integral, so this is still equal to I . Now, the choice of variables in an integral doesn't matter, so $I = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-v^2} dv$ as well. Multiplying both together gives

$$I^2 = \frac{4}{\pi} \left(\int_0^{\infty} e^{-u^2} du \right) \left(\int_0^{\infty} e^{-v^2} dv \right)$$

Now, the variables here are independent, so you can combine this into a double integral. Doing this gives

$$I^2 = \frac{1}{\pi} \left(\int_0^{\infty} e^{-u^2} du \right) \left(\int_0^{\infty} e^{-v^2} dv \right) = \frac{1}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)} du dv$$

You can now change the co-ordinates to polar co-ordinates (see [Guide: Changing co-ordinates in double integrals] for more). By setting $u = r \cos(\theta)$ and $v = r \sin(\theta)$, it follows that $u^2 + v^2 = r^2$. The region of integration is $0 \leq u < \infty$ and $0 \leq v < \infty$, which corresponds to the first quadrant of the plane; this is represented in polar co-ordinates by $0 \leq r < \infty$ and $0 \leq \theta \leq \pi/2$. Finally, $du dv$ becomes $r dr d\theta$ by using the Jacobian. Therefore, the integral becomes

$$I^2 = \frac{4}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)} du dv = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\infty} r e^{-r^2} dr d\theta$$

Now you can evaluate this double integral. The derivative of e^{-r^2} with respect to r is $-2re^{-r^2}$; so that means that the integral of re^{-r^2} is $-\frac{1}{2}e^{-r^2}$ (you can get this result by substitution if you wanted). Using the fact that e^{-r^2} is equal to 1 when $r = 0$ and tends to 0 as r tends to infinity, you can get

$$I^2 = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\infty} r e^{-r^2} dr d\theta = \frac{4}{\pi} \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2} \right]_0^{\infty} d\theta = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2} d\theta$$

Evaluating this final integral gives

$$I^2 = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{4}{\pi} \left[\frac{\theta}{2} \right]_0^{\pi/2} = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1$$

So $I^2 = 1$, implying that $I = \pm 1$. But I cannot be -1 , as $f(x)$ is a positive function and the integral of a positive function is always positive. So $I = 1$ and therefore the normal distribution really is a PDF.

Further reading

[Guide: PMFs, PDFs, CDFs](#)

[Questions: PMFs, PDFs, CDFs](#)

Version history and licensing

v1.0: initial version created in 04/25 by tdhc and Sophie Chowgule as part of a University of St Andrews VIP project.

[This work is licensed under CC BY-NC-SA 4.0.](#)