Introduction to 3D Polar Coordinate Systems: Spherical & Cylindrical Coordinates

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Summary

In three-dimensional space, points can be described using various coordinate systems. While Cartesian coordinates (x, y, z) are common, polar-based systems like spherical and cylindrical coordinates provide alternative ways to represent positions – particularly when dealing with problems involving symmetry.

Before reading this guide, it is recommended that you read Guide: Introduction to polar coordinates.

What are cylindrical coordinates?

Cylindrical coordinates can be seen as an **extension** of 2D polar coordinates by adding rotation around the z-axis. They are particularly useful for problems with rotation symmetry about an axis – for example cylinders, pipes, or beams. Each point in 3D space is described by:

- r - the radial distance from the z-axis
- θ - the angle from the x-axis (in the xy-plane)
- z - the height along the z-axis

The following diagram shows an example of how to visualise this.

Figure 1: Cylindrical coordinates showing a point (x, y, z) located by radial distance r from the z-axis, angle in the xy-plane, and height z.

The conversion between cylindrical and Cartesian coordinates follows:

Definition of cylindrical coordinates:

- $x = r\cos(\theta)$,
- $y = r\sin(\theta)$,
- $z = z$.

Such that:

- $r = \sqrt{(x^2 + y^2)}$,
- $\theta = \tan^{-1}(\frac{y}{x})$,
- $z = z$.

Geometrically, cylindrical coordinates can be imagined as stacking circles (each defined by r and θ) along the z-axis. A useful analogy is a can of soup: each horizontal slice is a circle, and stacking these circular slices vertically forms the cylinder.

Interactive Exploration (Desmos):

Use the following Desmos tool to explore how the cylindrical variables and determine a point's location.

Tip

Try keeping two variable's contant at a time to trace single-variable curves or surfaces. What can you see when doing this? (hint: to visualise cylinders, spirals, and vertical planes.)

Example 1

Consider the following point in cylindrical coordinates:

$$(r, \theta, z) = \left(4, \frac{\pi}{3}, 2\right)$$

This means:

- The point is 4 units from the origin.
- It is rotated $45°$ around the z-axis
- It is 2 units above the xy-plane

Using the cylindrical-Cartesian conversion of:

$x = rcos(\theta)$, $y = rsin(\theta)$, $z = z$

You can calculate x, y, and z to be:

$$x = 4\cos\left(\frac{\pi}{3}\right) = 4 \cdot \left(\frac{1}{2}\right) = 2$$

$$y = 4\sin\left(\frac{\pi}{3}\right) = 4 \cdot \left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$z = 2$$

So, in cartesian coodinates, the point is:

$$(x, y, z) = (2, 2\sqrt{3}, 2)$$

The geometric interpretation of this is as follows:

- Fixing $r = 4$ gives a vertical cylinder
- Fixing $\theta = \left(\frac{\pi}{3}\right)$ gives a horizontal plane
- Fixing $z = 2$ gives a horizontal plane

The point lies where these three surfaces interact.

What are spherical coordinates?

Contrastingly, spherical coordinates describe points relative to a central point – ideal for systems with radial symmetry – like stars, atoms, or fields. If you imagine a sphere, you can think of it as being composed of many 2D circles or cross-sections. Depending on how we slice the sphere - along the x, y, or z axes - we can visualize different orientations of these circular cross-sections.

In spherical coordinates, each point in 3D space is defined by three quantities:

- $r =$ the radial distance from the origin to the point
- $\theta =$ the azimuthal angle (measured in the xy-plane from the x-axis)
- $\phi =$ the polar angle (measured from the positive z-axis)

These coordinates relate to Cartesian coordinates as follows:

- $x = r \sin(\phi) \cos(\theta)$,
- $y = r \sin(\phi) \sin(\theta)$,
- $z = r \cos(\phi)$.

where, $r^2 = x^2 + y^2 + z^2$.

The range of these variables is:

- $\theta \in [0, 2\pi)$. This is the full rotation around the z-axis.
- $\phi \in [0, \pi]$ This is from the north pole to the south pole of the sphere.

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You can think of spherical coordinates as an extension of polar coordinates into 3D - adding a second angle to capture depth or height (aka vertical tilt) relative to the z-axis. For instance, visualizing Earth as a perfect sphere, a point in earth can be described by its distance from the centre (r), the latitude-like polar angle (ϕ), and the longitude-like azimuthal angle (θ).

Figure 2: Spherical coordinates showing a point (x, y, z) located by radial distance r from the origin, azimuthal angle in the xy-plane from the x-axis, and polar angle measured down from the positive z-axis.

Interactive Exploration (Desmos):

Use the linked Desmos tool to investigate how spherical coordinates position a point in space. Change to move closer to or farther from the origin

Tip

Fix one variable at a time. What can you see by doing this? (hint: cones, circular paths, or spherical shells?)

Example 2

Consider the following point in spherical coordinates:

$$(r, \theta, \phi) = \left(6, \frac{\pi}{6}, \frac{\pi}{4}\right)$$

This means:

- The point is 6 units from the origin.
- It is rotated $30°$ around the z-axis.
- It is tilted $45°$ down from the positive z-axis.

Using the spherical-Cartesian conversion of:

$$x = r\sin(\phi)\cos(\theta), \;\; y = r\sin(\phi)\sin(\theta), \;\; z = r\cos(\phi)$$

You can calculate x, y, and z to be:

$$x = 6\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) = 6 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{6}}{2}$$
$$y = 6\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = 6 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{3\sqrt{2}}{2}$$
$$z = 6\cos\left(\frac{\pi}{4}\right) = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$$

So, in cartesian coodinates, the point is:

$$(x, y, z) = \left(\frac{3\sqrt{6}}{2}, \frac{3\sqrt{2}}{2}, 3\sqrt{2}\right)$$

The geometric interpretation of this is as follows:

- Fixing $r = 6$ gives a vertical cylinder,
- Fixing $\theta = \frac{\pi}{6}$ gives a vertical half-plane,
- Fixing $\phi = \frac{\pi}{4}$ gives a cone.

The point lies where these three surfaces interact.

Quick check problems:

1. In spherical coordinates, which angle sweeps around the z-axis, and which tilts down from the z-axis?

2. For a problem with a long straight pipe, which system is likely the better option to solve: spherical or cylindrical? Explain why.

3. What symmetry suggests using spherical coordinates? What symmetry suggests using cylindrical coordinates?

4. In cylindrical coordinates, what does r measure?

5. If $\phi = 0$ in spherical coordinates, where is the point located relative to the axes?

6. Convert the cylindrical coordinates

$$(r, \theta, z) = \left(3,\ \frac{\pi}{6},\ 2\right)$$

to Cartesian coordinates.

7. Convert the Cartesian coordinates

$$(x, y, z) = \left(2,\ 2\sqrt{3},\ 1\right)$$

to cylindrical coordinates. Give r and θ in exact exact form.

8. Convert the spherical coordinates

$$(r, \theta, \phi) = \left(5,\ \frac{\pi}{3},\ \frac{\pi}{6}\right)$$

to Cartesian coordinates.

9. Given the Cartesian coordinates

$$(x, y, z) = (0,\ 0,\ 4),$$

find the corresponding spherical coordinates using the standard variable ranges.

10. For the cylindrical coordinates

$$(r, \theta, z) = \left(4,\ \frac{3\pi}{2},\ -1\right),$$

identify the point's location in the xy-plane (quadrant or axis) and give the corresponding Cartesian coordinates.