Introduction to Gaussian Elimination

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Summary

You can use Gaussian Elimination to simplify matrices and get the inverse or solve equations. Using Gaussian Elimination you can solve systems of linear equations written as matrices.

Before reading this guide, it is recommended that you read Guide: Introduction to Matrices and Guide: Introduction to fractions. For the last section of this guide (Using Gaussian elimination to solve systems of equations) it is recommended that you read Guide: Introdution to Solving Simultaneous Equations.

In Introduction to Matrices you saw what a Matrix is and how to do arithmetic operations with it such as summing, subtracting and multiplying. In this guide, you will learn about operations that are done with the rows of the matrix to change the matrix.

What is Gaussian Elimination?

Gaussian Elimination is a method used to transform matrices into 'nicer' forms so that you can get some information from the matrix. This nicer form is usually an 'upper triangle' where you only have non-zeros in the entries to right of the main diagonal, and the entries to the left are zeros (sometimes you can have more zeros than that). You can also use it to get the identity matrix. This guide will show you how to do Gaussian elimination and some of the applications of this technique. You will first study the row echelon form. Then you will learn the three types of row operations that you need to do Gaussian elimination. Finally you will discover the augmented matrix and how to apply Gaussian elimination to it to find the inverse of a matrix and solve many equations at once.

Definition of row echelon form and of reduced row echelon form

A matrix is in row echelon form (or upper triangular form) if:
(i)All the zero rows are at the bottom (this does not mean there are any zero rows!)
(ii)The first entry in a (non-zero) row - this is the leading entry - is to the right of the first entry of the row above it.

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

The leading entries of A are 2, 5 and 9.

A matrix is in reduced row echelon form if:

(i)In each non-zero row, the first entry of the row is 1, and

(ii)In each column that contains a first entry for some row, all other entries in that column are zero.

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The leading entries of B are the two 1s of the matrix and B has one zero-row.

Definition of elementary row operations (e.r.os)

In order to get to the (reduced) echelon form you have to do row operations. There are three elementary operations:

(1)Multiplying a row number by a scalar λ

When multiplying a row number by a scalar you 'freeze' all the other rows of the matrix, and multiply the values of the row you have chosen by the scalar λ. This gives you a modified matrix where only the row you choose to multiplicate has changed.

(2)Add one multiple of one row to another row

This can be very useful when you want to get zeros in certain entries of your matrix. You can multiply one of the rows by a scalar λ and then add it or subtract it to another row.

(3)Interchanging two rows

Altering the order of the rows does not change the essential properties of the matrix. You can exchange the order of the rows in your matrix to get to the upper triangular form or to move the zero-rows to the bottom.

Example 1

Let

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix}$$

You want to multiply the second row by 2:

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_2 \to 2r_2 \begin{bmatrix} 3 & 0 & 3 \\ 4 & 4 & 12 \\ 4 & 0 & 9 \end{bmatrix}$$

Hence only the second row has changed and now is a scalar multiple of the original second row.

Example 2

Let A be the same matrix as in Example 1. You can now try the second elementary row operation: add one multiple of one row to another row. Suppose you decide to add $\frac{1}{2}r_2$ to r_1,you will be adding $[1 \quad 1 \quad 3]$ to $[3 \quad 0 \quad 3]$ such that:

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_1 \to \frac{1}{2}r_2 + r_1 \begin{bmatrix} 4 & 1 & 6 \\ 4 & 4 & 12 \\ 4 & 0 & 9 \end{bmatrix}$$

The matrix remains unchanged except for the first row, that is now a sum of the original row 1 and a half of row 2.

Example 3

Finally you can interchange two rows. In this example you interchange row 3 and row 1.

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_1 \leftrightarrow r_3 \begin{bmatrix} 4 & 0 & 9 \\ 2 & 2 & 6 \\ 3 & 0 & 3 \end{bmatrix}$$

Reducing a matrix to echelon form (or reduced echelon form)

You can use elementary row operations to find the echelon form or the reduced echelon form a matrix.

You can do this in a series of steps:

Step 1: Identify the zeros of your Matrix. If there are any all-zero rows interchange rows until all the zero rows are at the bottom of the matrix.

Step 2: You can save some time if before doing other e.r.o.s you look at the leading entries of each row. Again interchange the rows of your Matrix until the first entry in each row is under or to the right of the first entry in the row above it.

Step 3: Now it is time to start using other e.r.os. Focus on the leading entries, add or subtract multiples of different rows so that, for any leading entries in the same column, the lower leading entry becomes zero. Don't worry about having negative numbers or fractions in the other entries of such rows when you do this.

Step 4: Repeat step 3 until you have the echelon form.

> **Tip**
>
> These steps are interchangeable and sometimes different steps work for different people! Maybe you prefer having all leading entries as single entries per column before you start interchanging rows, or you spot a e.r.o. that would work before another one. Many approaches can work!

Example 4

Let

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Look at row 3. It only has one entry and it is on the far left, you can use this to get 0 on the other entries of the first column and, since it only has one entry, you can focus on only that column for each operation you do with row 3. Start by doing this with row 1:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \to r_1 - r_3 \quad \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

And now do the same for row 2. Note that because the leading entry of row 2 is 2, you need to multiply row 3 by a scalar 2 when you subtract it from row 2.

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \to r_2 - 2r_3 \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Great! Now there is only one entry on row 3, if we interchange row 1 and row 3, we can get the first line of the upper triangle form (left corner of the triangle)!

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \leftrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, if you look at four, you can see that it only has one entry, in the middle column. This is good since it means you don't need to use the other rows to make sure the first entry is to the right of the row above it. Finally row 3 has its leading entry on the same column as the leading entry of 4, so you need to make that entry zero. To get the row echelon form, subtract $\frac{1}{4}$ of row 2 from row 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix} r_3 \to r_3 - \tfrac{1}{4}r_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the echelon form of C is

Example 5

Let

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Following the first step identify all the zeros of the matrix. Since the first row is a zero-row, you need to move it to the bottom. So the first e.r.o. you perform is interchanging rows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} r_1 \leftrightarrow r_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now row 1 has the most-left entry so it is ok for row echelon form. However, row 2 has its leading entry directly under the leading entry of row 1. To fix this, perform the second e.r.o., add a scalar multiple (-5) of row 1 to row 2.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} r_2 \rightarrow r_2 - 5r_1 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

So the row echelon form of D is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Now you will see a couple of examples where you get the reduced echelon form.

Example 6

Let

$$E = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

There are no zero-rows in this matrix, so you can start looking at leading entries. Using the first entry of row 3 you can make the rest of the first column zero (you use row 3 because the leading is one).

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{bmatrix} r_2 \to r_2 - 2r_3 \begin{bmatrix} 4 & 2 & 1 \\ 0 & -4 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

and similarly with row 1:

$$\begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -10 & 1 \end{bmatrix}$$

Now you can try an example with a matrix that is larger. This usually means more e.r.o.s!

Using the row echelon form to find the inverse of a matrix

Definition of the Augmented Matrix

You construct the augmented matrix of a matrix A by combining it with any other matrix which has the same number of rows as A. You show it is an augmented matrix by leaving a separation bar | between the two matrices. Yprefer to the left block (matrix values before the bar) and the right block (matrix values after the bar) of the matrix.

You can construct the augmented matrix $n \times 2n$ from the original matrix A and the $n \times n$ identity matrix I_n. The left block is the original matrix A before the separation bar | and the right block is the identity matrix I_n.

Example

Let

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

be a 2×2 matrix. Then we can build the augmented matrix M' of M with the identity matrix I_n such that:

$$M' = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$$

To find the inverse of matrices using e.r.o.s you use Gaussian Elimination on the extended matrix (with the identity matrix) of the matrix whose inverse you want to find.

Warning

You can only invert matrices that are square (same number of rows and columns), so this strategy also works only with square matrices!

E.r.o.s on extended matrices

When performing elementary row operations on an extended matrix, each row operation acts equally on both sides of the extended matrix. That is, if you do e.r.o.s on the left block you have to do the same e.r.o.s on the right block.

Example 8

Let

$$G = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

be the matrix you want to invert. You start by extending it such that:

$$G' = \left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right].$$

Now you want to perform e.r.o.s until the left block is I_2. First subtract row 2 to row 1 so as to get a zero above the leading entry of row 1:

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] r_1 \to r_1 - 2r_2 \left[\begin{array}{cc|cc} 0 & -5 & 1 & -2 \\ 2 & 6 & 0 & 1 \end{array}\right].$$

You can multiply row 1 by $\frac{-1}{5}$ to get the left block in the right form:

$$\left[\begin{array}{cc|cc} 0 & -5 & 1 & -2 \\ 2 & 6 & 0 & 1 \end{array}\right] r_1 \to r_1 - 2r_2 \left[\begin{array}{cc|cc} 0 & 1 & \frac{-1}{5} & \frac{2}{5} \\ 2 & 6 & 0 & 1 \end{array}\right]$$

Now you will do two operations at the same time: first, you will multiply row 2 by $\frac{1}{2}$ to get a leading entry of one, then you will exchange row 1 and row 2 to have the diagonal of 1s you need for the identity matrix form:

$$\left[\begin{array}{cc|cc} 0 & 1 & \frac{-1}{5} & \frac{2}{5} \\ 2 & 6 & 0 & 1 \end{array}\right] \begin{array}{c} r_1 \to r_2 - \frac{1}{2}r_2 \\ r_1 \leftrightarrow r_2 \end{array} \left[\begin{array}{cc|cc} 1 & 3 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} \end{array}\right].$$

Finally you subtract 3 times row 2 to row 1 to get rid of the non-zero values in the column of the leading entry:

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} \end{array}\right] r_1 \to r_1 - 3r_2 \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{-7}{10} \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} \end{array}\right]$$

Since you have the identity matrix in the left block, this means the matrix you have in the right block is the inverse of G! So

$$G^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{7}{10} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix}.$$

Now you will see another example with a bigger matrix, where the identity matrix you will use is I_3 instead of I_2

Example 9

The square matrix you want to invert is
H=

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 6 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

So the extended matrix will be:

$$H' = \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 1 & 3 & 3 & 0 & 0 & 1 \end{array}\right]$$

You can see that row 1 has a leading entry of 1, this is good because we can use that to make all the other entries in the first column 0. Row 2 has already 0, this is good! Now subtract row 1 to row 3 to have zero in the first column (notice by doing this you will also get a zero in the second column, which is helpful since row 3 only needs a non-zero entry in the third column).

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 1 & 3 & 3 & 0 & 0 & 1 \end{array}\right] r_1 \rightarrow r_3 - r_1 \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The only non zero entry in row 3 is the leading entry in the third column, row 3 is ready! Now you want to have row 2 in 'identity form' too. To do this, use row 3. Again you can do two e.r.o.s at the same time. First, subtract two times row 3 to row 2 to get a zero in the third entry, then divide row 2 by six so that the leading entry of the row is 1:

Using Gaussian elimination to solve systems of equations

You can use inverse matrices to solve systems of linear equations, following these steps:

Step 1 Write the system of linear equations as a matrix, by writing the coefficients of each variable as an entry in the matrix (each column corresponds to one variable), the variables X as a column vector and the coefficients (on the other side of the equality of each equation) as another column vector.

Step 2 Find the inverse of the matrix using the row echelon form.

Step 3 Perform matrix multiplication to find X, where X is an array giving the solution to all the variables.

> **Why does this work?**
>
> If A is the matrix with all the coefficients of the system of linear equations, and X is the $(1 \times n)$ vector with all the variables the matrix has, the system can be written as:
> $AX = B$. By multiplying both sides by the inverse A^{-1} we get:
> $A^{-1}AX = A^{-1}B$. But we know by definition of the inverse matrix that $A^{-1}A = I_n$ so
> $I_nX = A^{-1}B$, and by definition of the identity matrix $I_nX = X$.
> So the solution to the array of variables X is:
> $X = A^{-1}B$. Since you know be, by finding A^{-1} you can get the solution to the system of equations.

The following examples use the inverses of previous examples to show how this can work:

Example 10

Suppose you have this system of equations:

$$4x + 7y = 1$$
$$2x + 6y = 0$$

Following the steps above, start by taking the coefficients of the variables in the system and organising them by columns. So in the first column you will the coefficients of the x variable (that is, 4 and 2) and in the second column you will have the coefficients of the y variable (that is, 2 and 6):

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Notice this is the same matrix as G from example 8! So, we know its inverse is:

$$G^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{7}{10} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

B is given by the constants of the system of equations, so B=

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly as X depends on the variables, X=

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Now that you have X, B, G and G^{-1}, we can find the values of the variables! Recall you will use the equation $GX = B$ to get $X = G^{-1}B$.

So you have:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Another option is to turn your system of linear equations into an extended matrix, where the left block represents the coefficients of the variables and the right block represents the constants, and with Gaussian elimination turn the left block into the reduced echelon form. Then you can rewrite the system of equations with the new coefficients and the new constants and solve for the variables (start from the bottom, there is at least one row with just one entry in the left block). To show this, you will work through an example with a matrix you have already seen, but this time you will be doing the e.r.o.s on the augmented form.

Example 11

We want to solve this system of equations:\ $4x + 2y = 1$
$2x + 2y = 2$
$x+3y = 0$. Let's build the augmented matrix, the left block is given by the coefficients so:

$$E_{Left} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

where the first column gives the values of the coefficients of the x-variables and the second column gives the values of the y-variables.
Now you can obtain the right block from the coefficients:

$$E_{Right} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Now we combine the two in the augmented matrix:

$$\left[\begin{array}{cc|c} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{array}\right]$$

Notice this is the matrix from **Example 6**:

$$E = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

with slightly different notation (two blocks instead of one).
But this does not change how we get the row echelon form, since with e.r.o.s we perform the same operation simultaneously on both blocks. In fact, with e.r.o.s we 'ignore' the middle bar making the matrix. It is just there to remind us which part corresponds to the coefficients, and which part corresponds to the constants at the other side of the equality.

The `echo: false` option disables the printing of code (only output is displayed).

Further reading

[For more questions on the subject, please go to Questions: Gaussian elimination.]

For an introduction on matrices, please see Guide: Introduction to Matrices.

[For an introduction systems of linear equations, please see Guide: Solving systems of linear equations.]

Version history

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