

# Proof: PMFs, PDFs, CDFs

Sophie Chowgule, Tom Coleman

## Summary

Explanations as to why some PMF's and PDF's are valid.

*Before reading this proof sheet, it is recommended that you read [Guide: PMFs, PDFs, CDFs](#). Other recommended reading material will be said when it is needed.*

## Proof that the binomial distribution is a PMF

*Before reading this section, you may find it useful to read [Guide: The binomial theorem].*

Remember from [Guide: PMFs, PDFs, CDFs](#) that the **binomial distribution** is given by the following.

### Binomial distribution

$$P(X = x) = \binom{n}{x} p^x q^{(n-x)} = \frac{n!}{(n-x)!x!} p^x q^{(n-x)}$$

where:

- the random variable  $X = x$  measures the number of success in a set of  $n$  trials
  - $x$  is number of successes
  - $n$  is number of trials
- $p$  is the probability of success in a single trial
- $q = 1 - p$  is the probability of failure in a single trial

Also from [Guide: PMFs, PDFs, CDFs](#), the two conditions to be a valid PMF are the following:

- **Non-negativity:** The probability assigned to each possible outcome must be greater than or equal to zero, that is:

$$p(x) = P(X = x) \geq 0 \text{ for all values of } x.$$

- **Honesty condition:** The sum of probabilities of all possible outcomes  $x$  of a discrete random variable  $X$  must be equal to one:

$$\sum_x p(x) = \sum_x P(X = x) = 1.$$

First of all, every term in the PMF for the binomial distribution above is non-negative, and the product of non-negative numbers is non-negative, so  $P(X = x) \geq 0$  for any  $x$ .

The honesty condition comes about because binomial distributions follow the **binomial theorem**. The binomial theorem states that:

$$\sum_{x=0}^n \binom{n}{x} p^x q^{(n-x)} = (p + q)^n$$

(See [Guide: The binomial theorem] for more.)

The number of successes  $x$  ranges from 0 (total failure) to  $n$  (complete success). Therefore, the sum of all possible probabilities  $P(X = x)$  is:

$$\sum_x P(X = x) = \sum_{x=0}^n \binom{n}{x} p^x q^{(n-x)}$$

which is the left-hand side of the binomial theorem. Using the binomial theorem with  $q = 1-p$ :

$$\sum_x P(X = x) = (p + q)^n = (p + (1-p))^n = (1)^n = 1$$

So, the sum of the probabilities over all possible values of  $x$  equals 1, satisfying the honesty condition.

## Proof that the uniform distribution is a PDF

*Before reading this section, you may find it useful to read [Guide: Introduction to integration] and [Guide: Properties of integration].*

Remember from [Guide: PMFs, PDFs, CDFs](#) that the **uniform distribution** over the interval  $[a, b]$  is given by the following.

## Uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where  $a, b$  are real numbers such that  $a < b$ .

Also from [Guide: PMFs, PDFs, CDFs](#), the two conditions to be a valid PDF are the following:

- **Non-negativity:** The PDF  $f(x)$  must be greater than or equal to zero over its entire range of possible values:

$$f(x) \geq 0 \text{ for all values of } x.$$

- **Honesty condition:** The area under the entire PDF  $f(x)$  must be equal to 1, so:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

To check if this is a valid PDF, you need to confirm that it satisfies these two key conditions.

**Non-negativity:**  $f(x) \geq 0$  for all values of  $x$ , as  $f(x) = \frac{1}{b-a}$  in  $[a, b]$  and 0 otherwise.

**Honesty:** To satisfy the honesty condition, the integral of the PDF over the interval  $[a, b]$  must equal 1. Using the properties of integration, you can split the integral into three parts along the lines of the PDF:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^b f(x) dx + \int_b^{\infty} f(x) dx$$

Using the definition of  $f(x)$  on these intervals gives

$$\int_{-\infty}^a f(x) dx + \int_a^b f(x) dx + \int_b^{\infty} f(x) dx = \int_{-\infty}^a 0 dx + \int_a^b \frac{1}{b-a} dx + \int_b^{\infty} 0 dx$$

Since the integral of 0 over any limits is zero, this reduces to

$$\int_{-\infty}^{\infty} f(x) dx = 0 + \int_a^b \frac{1}{b-a} dx + 0 = \int_a^b \frac{1}{b-a} dx$$

Working out this integral dives

$$\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b 1 dx = \frac{1}{b-a} [x]_a^b = \frac{1}{b-a} (b-a) = 1$$

And so you can see that all uniform distributions are valid PDFs.

## Proof that the normal distribution is a PDF

*Before reading this section, you may find it useful to read [Guide: Properties of integration], [Guide: Integration by substitution], [Guide: Introduction to double integration], and [Guide: Co-ordinate changes in double integration].*

Remember from [Guide: PMFs, PDFs, CDFs](#) that the **normal distribution** is given by the following.

### i Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

where  $\mu, \sigma$  are real numbers such that  $\sigma > 0$ . (Here,  $\mu$  is the mean and  $\sigma$  is the standard deviation.)

To check if this is a valid PDF, you need to confirm that it satisfies the two key conditions.

**Non-negativity:** As an exponential function,  $\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) > 0$ , and  $1/\sigma\sqrt{2\pi} > 0$  as  $\sigma > 0$ . So  $f(x) > 0$ .

**Honesty:** Here's the fun part.

The idea is to show that this integral  $I$ , given by

$$I = \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

is equal to 1. To tackle this integral, it needs to look a little nicer; you can use integration by substitution to do this (see [\[Guide: Integration by substitution\]](#)). Let  $u = \frac{x-\mu}{\sigma\sqrt{2}}$ . Then  $\frac{du}{dx} = \frac{1}{\sigma\sqrt{2}}$ , and so  $dx = \sigma\sqrt{2} du$ . As  $x \rightarrow \pm\infty$ , it follows that  $u \rightarrow \pm\infty$ . Since  $u^2 = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$ , the integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx = \int_{-\infty}^{\infty} \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

Next, you can use the fact that  $\exp(-u^2)$  is an even function to change the limits. Using the property of even function about symmetric limits (see [Guide: Properties of integration]), the integral becomes

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du = I$$

All that you have done so far has not changed the value of the integral, so this is still equal to  $I$ . Now, the choice of variables in an integral doesn't matter, so  $I = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-v^2} dv$  as well. Multiplying both together gives

$$I^2 = \frac{4}{\pi} \left( \int_0^{\infty} e^{-u^2} du \right) \left( \int_0^{\infty} e^{-v^2} dv \right)$$

Now, the variables here are independent, so you can combine this into a double integral. Doing this gives

$$I^2 = \frac{1}{\pi} \left( \int_0^{\infty} e^{-u^2} du \right) \left( \int_0^{\infty} e^{-v^2} dv \right) = \frac{1}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)} du dv$$

You can now change the co-ordinates to polar co-ordinates (see [Guide: Changing co-ordinates in double integrals] for more). By setting  $u = r \cos(\theta)$  and  $v = r \sin(\theta)$ , it follows that  $u^2 + v^2 = r^2$ . The region of integration is  $0 \leq u < \infty$  and  $0 \leq v < \infty$ , which corresponds to the first quadrant of the plane; this is represented in polar co-ordinates by  $0 \leq r < \infty$  and  $0 \leq \theta \leq \pi/2$ . Finally,  $du dv$  becomes  $r dr d\theta$  by using the Jacobian. Therefore, the integral becomes

$$I^2 = \frac{4}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)} du dv = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\infty} r e^{-r^2} dr d\theta$$

Now you can evaluate this double integral. The derivative of  $e^{-r^2}$  with respect to  $r$  is  $-2re^{-r^2}$ ; so that means that the integral of  $re^{-r^2}$  is  $-\frac{1}{2}e^{-r^2}$  (you can get this result by substitution if you wanted). Using the fact that  $e^{-r^2}$  is equal to 1 when  $r = 0$  and tends to 0 as  $r$  tends to infinity, you can get

$$I^2 = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\infty} r e^{-r^2} dr d\theta = \frac{4}{\pi} \int_0^{\pi/2} \left[ -\frac{1}{2}e^{-r^2} \right]_0^{\infty} d\theta = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2} d\theta$$

Evaluating this final integral gives

$$I^2 = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{4}{\pi} \left[ \frac{\theta}{2} \right]_0^{\pi/2} = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1$$

So  $I^2 = 1$ , implying that  $I = \pm 1$ . But  $I$  cannot be  $-1$ , as  $f(x)$  is a positive function and the integral of a positive function is always positive. So  $I = 1$  and therefore the normal distribution really is a PDF.

## Further reading

[Guide: PMFs, PDFs, CDFs](#)

[Questions: PMFs, PDFs, CDFs](#)

## Version history and licensing

v1.0: initial version created in 04/25 by tdhc and Sophie Chowgule as part of a University of St Andrews VIP project.

[This work is licensed under CC BY-NC-SA 4.0.](#)