# Rationalizing the denominator

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**Summary**

Rationalizing the denominator is a technique for simplifying fractions involving square roots in the denominator. This study guide also explains how to rationalize denominators through the use of the quadratic conjugate.

*Before reading this guide, it is recommended that you read Guide: Laws of indices and [Guide: Expanding brackets]. The only irrational numbers you will see in this guide are square roots. If you want to learn more about irrational numbers, have a look at [Guide: Overview of numbers].*

## What is rationalizing the denominator?

Rationalizing the denominator is helpful for doing operations like addition or subtraction on fractions and is also useful when trying to approximate fractions.

When you rationalize the denominator, you rewrite a fraction so that the denominator contains no square roots or other $n$th roots. This guide only focuses on rationalizing the denominator of fractions that have square roots in the denominator.

For example, in the fraction $\frac{3}{\sqrt{2}}$, the denominator contains a square root of $2$. To rationalize the denominator, you want to rewrite the fraction so that the denominator is free of square roots. After applying the techniques you will learn in this guide, the fraction will look like this: $\frac{3\sqrt{2}}{2}$, where the numerator contains a square root, but crucially, the denominator does not.

> **Note**
>
> When you rewrite fractions you can multiply the numerator and denominator by the same value $k$, where $k$ is any non-zero number. When you are doing this, you are multiplying the fraction by $\frac{k}{k}$ which is the same as multiplying the fraction by $1$.
> You are therefore not changing the value of the fraction, which is the key concept behind rationalizing the denominator.

## Expressions of the form $\frac{a}{b\sqrt{c}}$

For fractions like $\dfrac{a}{b\sqrt{c}}$, where $a$ is any number, $b$ is an integer and $c$ is an integer that is not a square number, you can rationalize the denominator by multiplying both the numerator and denominator by the square root in the denominator. This gives you:

$$\frac{a}{b\sqrt{c}} \cdot \frac{\sqrt{c}}{\sqrt{c}} = \frac{a\sqrt{c}}{b(\sqrt{c})^2}$$

Since $(\sqrt{c})^2 = c$ by laws of indices, it follows that:

$$\frac{a\sqrt{c}}{b(\sqrt{c})^2} = \frac{a\sqrt{c}}{bc}$$

The denominator is now rational and free of square roots. As $b$ and $c$ are integers, so is $bc$. Example 1 shows you how to rationalize the denominator of the fraction you saw in the introduction.

**Example 1**

Simplify $\dfrac{3}{\sqrt{2}}$ by rationalizing the denominator.

To rationalize this, you can multiply both the numerator and denominator by $\sqrt{2}$ , giving you:

$$\frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{(\sqrt{2})^2}$$

Simplifying the denominator then gives you:

$$\frac{3\sqrt{2}}{(\sqrt{2})^2} = \frac{3\sqrt{2}}{2}$$

Therefore $\dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$, which has a rational denominator.

**Example 2**

Simplify $\frac{3}{2\sqrt{5}}$ by rationalizing the denominator.
You can multiply both the numerator and denominator by $\sqrt{5}$, giving you:

$$\frac{3}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{2(\sqrt{5})^2}$$

Since $(\sqrt{5})^2 = 5$, it follows that:

$$\frac{3\sqrt{5}}{2(\sqrt{5})^2} = \frac{3\sqrt{5}}{10}$$

.
Therefore $\frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{10}$, which has a rational denominator.

**Example 3**

Simplify $\frac{5+\sqrt{2}}{7\sqrt{3}}$ by rationalizing the denominator.
You can multiply both the numerator and denominator by $\sqrt{3}$ , giving you:

$$\frac{5+\sqrt{2}}{7\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{(5+\sqrt{2})\sqrt{3}}{7(\sqrt{3})^2}$$

Expanding the brackets in the numerator and simplifying the denominator then gives you:

$$\frac{(5+\sqrt{2})\sqrt{3}}{7(\sqrt{3})^2} = \frac{5\sqrt{3}+\sqrt{2}\sqrt{3}}{7(3)}$$

Since $\sqrt{2}\sqrt{3} = \sqrt{6}$ by laws of indices and $7(3) = 21$, it follows that:

$$\frac{5\sqrt{3}+\sqrt{2}\sqrt{3}}{7(3)} = \frac{5\sqrt{3}+\sqrt{6}}{21}$$

Therefore $\frac{5+\sqrt{2}}{7\sqrt{3}} = \frac{5\sqrt{3}+\sqrt{6}}{21}$, which has a rational denominator

## What is a quadratic conjugate?

How would you rationalize the denominator of $\frac{1}{7+2\sqrt{3}}$?

You could try by multiplying the numerator and denominator by $\sqrt{3}$, but then you will end up with a $7\sqrt{3}$ term in the denominator, which isn't rational. It is not possible to do this with the techniques you have seen so far. In this case, more specifically, the quadratic conjugate is needed!

**Definition of a quadratic conjugate**

Given an expression of the form $b + c\sqrt{d}$, where $b$ and $c$ are integers and $d$ is an integer that is not a square number, the **quadratic conjugate** is the expression with the same terms but with the opposite sign in front of the term with a square root.
The quadratic conjugate of the expression $b+c\sqrt{d}$ is therefore $b-c\sqrt{d}$, and the quadratic conjugate of $b - c\sqrt{d}$ is $b + c\sqrt{d}$.

**Example 4**

What is the quadratic conjugate of $7 + 2\sqrt{3}$ ?
$2\sqrt{3}$ is the term involving the square root. Changing the sign of this term from positive to negative gives you the quadratic conjugate:

$$7 - 2\sqrt{3}$$

**Example 5**

What is the quadratic conjugate of $-2 - 3\sqrt{5}$ ?
$-3\sqrt{5}$ is the term involving the square root. Changing the sign of this term from negative to positive gives you the quadratic conjugate:

$$-2 + 3\sqrt{5} = 3\sqrt{5} - 2$$

These conjugates are useful for eliminating square roots when rationalizing denominators of a particular form. Multiplying a quadratic expression $b+c\sqrt{d}$ by its quadratic conjugate $b-c\sqrt{d}$ eliminates the square root and gives a rational number as an answer. Doing this gives you:

$$(b + c\sqrt{d})(b - c\sqrt{d}) = b^2 - bc\sqrt{d} + bc\sqrt{d} - (c\sqrt{d})^2$$

Simplifying this then gives you

$$b^2 - c^2 d$$

This result is rational, free of square roots, as $b, c, d$ are all integers.

## Expressions of the form $\frac{a}{b+c\sqrt{d}}$

Here $a$ is any number, $b$ and $c$ are integers and $d$ is an integer that is not a square number. In this case, rationalizing the denominator involves multiplying the numerator and the denominator by the quadratic conjugate of $b + c\sqrt{d}$, which would be $b - c\sqrt{d}$. As you saw above, the denominator will become $b^2 - c^2d$. The entire fraction therefore becomes:

$$\frac{a}{b+c\sqrt{d}} \cdot \frac{b-c\sqrt{d}}{b-c\sqrt{d}} = \frac{a(b-c\sqrt{d})}{b^2-c^2d}$$

The denominator is now $b^2 - c^2d$ and has been successfully rationalized.

**Example 6**

Simplify $\dfrac{2}{7+2\sqrt{5}}$ by rationalizing the denominator.
The quadratic conjugate of the denominator is:

$$7 - 2\sqrt{5}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{2}{7+2\sqrt{5}} \cdot \frac{7-2\sqrt{5}}{7-2\sqrt{5}} = \frac{2(7-2\sqrt{5})}{49-14\sqrt{5}+14\sqrt{5}-(2\sqrt{5})^2}$$

Since $(2\sqrt{5})^2 = 4(5)$ and $-14\sqrt{5} + 14\sqrt{5} = 0$, it follows that:

$$\frac{2(7-2\sqrt{5})}{49-14\sqrt{5}+14\sqrt{5}-(2\sqrt{5})^2} = \frac{2(7-2\sqrt{5})}{49-4(5)}$$

Expanding the brackets in the numerator and further simplifying the denominator then gives you:

$$\frac{2(7-2\sqrt{5})}{49-4(5)} = \frac{14-4\sqrt{5}}{29}$$

Therefore $\dfrac{2}{7+2\sqrt{5}} = \dfrac{14-4\sqrt{5}}{29}$, which has a rational denominator. Here, the answer is in its simplest form and with a positive denominator.

**Example 7**

Simplify $\dfrac{3}{2+5\sqrt{3}}$ by rationalizing the denominator.
The quadratic conjugate of the denominator is:

$$2-5\sqrt{3}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{3}{2+5\sqrt{3}} \cdot \frac{2-5\sqrt{3}}{2-5\sqrt{3}} = \frac{3(2-5\sqrt{3})}{4-10\sqrt{3}+10\sqrt{3}-(5\sqrt{3})^2}$$

Simplifying the denominator and expanding the brackets in the numerator gives you:

$$\frac{3(2-5\sqrt{3})}{4-10\sqrt{3}+10\sqrt{3}-(5\sqrt{3})^2} = \frac{6-15\sqrt{3}}{4-25(3)}$$

Further simplifying the denominator then gives you:

$$\frac{6-15\sqrt{3}}{4-25(3)} = \frac{6-15\sqrt{3}}{-71}$$

Here, the answer is in its simplest form but does not have a positive denominator. You can multiply both the numerator and the denominator by $-1$ to get a positive denominator. Doing this gives you:

$$\frac{6-15\sqrt{3}}{-71} = \frac{15\sqrt{3}-6}{71}$$

Therefore $\dfrac{3}{2+5\sqrt{3}} = \dfrac{15\sqrt{3}-6}{71}$, which has a rational denominator. The answer is also in its simplest form and has a positive denominator.

**Example 8**

Simplify $\dfrac{2+\sqrt{7}}{5-\sqrt{3}}$ by rationalizing the denominator.
The quadratic conjugate of the denominator is:

$$5+\sqrt{3}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{2+\sqrt{7}}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}} = \frac{(2+\sqrt{7})(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})}$$

Expanding the brackets in both the numerator and denominator gives you:

$$\frac{(2+\sqrt{7})(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})} = \frac{2(5)+2\sqrt{3}+5\sqrt{7}+\sqrt{7}(\sqrt{3})}{25+5\sqrt{3}-5\sqrt{3}-(\sqrt{3})^2}$$

Simplifying the denominator then gives you:

$$\frac{2(5)+2\sqrt{3}+5\sqrt{7}+\sqrt{7}(\sqrt{3})}{25+5\sqrt{3}-5\sqrt{3}-(\sqrt{3})^2} = \frac{2(5)+2\sqrt{3}+5\sqrt{7}+\sqrt{7}(\sqrt{3})}{25-9}$$

Simplifying the numerator and further simplifying the denominator then gives you:

$$\frac{2(5)+2\sqrt{3}+5\sqrt{7}+\sqrt{7}(\sqrt{3})}{25-9} = \frac{10+2\sqrt{3}+5\sqrt{7}+\sqrt{21}}{16}$$

Therefore $\dfrac{2+\sqrt{7}}{5-\sqrt{3}} = \dfrac{10+2\sqrt{3}+5\sqrt{7}+\sqrt{21}}{16}$, which has a rational denominator.
The answer is in its simplest form and with a positive denominator.

## Quick check problems

Rationalize the denominator for each of the following expressions. Provide your answers in their simplest form and with a positive denominator.

1. $\dfrac{5}{\sqrt{2}}$
2. $\dfrac{7}{5\sqrt{3}}$
3. $\dfrac{7}{2+\sqrt{5}}$
4. $\dfrac{5\sqrt{2}}{7-2\sqrt{2}}$

# Further reading

For more questions on the subject, please go to Questions: Rationalizing the denominator.

## Version history and licensing

v1.0: initial version created 01/25 by Maximilian Volmar.

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