Answers: PMFs, PDFs, and CDFs

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Summary

Answers to questions relating to the guide on PMFs, PDFs, and CDFs.

These are the answers to Questions: PMFs, PDFs, and CDFs.

Please attempt the questions before reading these answers!

Q1

1.1.

The given PMF is valid because:

Non-negativity: All $P(X = x) \geq 0$

Honesty: The sum of all probabilities equals 1:

$$\sum_{x=1}^{4} p(x) = \sum_{x=1}^{4} P(X = x) = \frac{1}{10} + \frac{1}{5} + \frac{1}{2} + \frac{1}{5} = 1$$

$P(X = 4) = \frac{1}{5}$.

1.2.

The given PMF is valid because:

Non-negativity: All $P(X = x) \geq 0$

Honesty: The sum of all probabilities equals 1:

$$\sum_{x=1}^{4} p(x) = \sum_{x=1}^{4} P(X = x) = 0.25 + 0.35 + 0.05 + 0.2 + 0.1 = 1$$

$P(X = 3 \text{ or } X = 4) = 0.05 + 0.2 = 0.25$

1.3.

The completed PMF table for the biased coin toss is:

x	Heads	Tails
$P(X = x)$	0.3	0.7

This is a valid PMF because:

Non-negativity: Both $P(X = x) \geq 0$

Honesty: The sum of both probabilities equal 1:

$$\sum_x p(x) = \sum_x P(X = x) = 0.3 + 0.7 = 1$$

1.4. {-}

This is not a valid PMF since it fails the honesty condition:

Honesty: The sum of the given probabilities does not equal 1:

$$\sum_{x=1}^{7} p(x) = \sum_{x=1}^{7} P(X = x) = 0.1 + 0.05 + 0.05 + 0.3 + 0.25 + 0.75 + 0.35 = 1.85 \neq 1$$

1.5.

(a) $P(\text{Blue}) = \dfrac{3}{10} = 0.3$

(b) The PMF for the given scenario is:

x	Red	Blue	Green
$P(X = x)$	0.5	0.3	0.2

This is a valid PMF because:

Non-negativity: All $P(X = x) \geq 0$

Honesty: The sum of all three probabilities equals to 1:

$$\sum_x p(x) = \sum_x P(X = x) = 0.5 + 0.3 + 0.2 = 1$$

1.6.

(a) For the given PMF to be valid, you must have $p = \dfrac{1}{10}$.

(b) For $p = \dfrac{1}{10}$, then $P(X = 3) = \dfrac{3}{10}$.

Q2

2.1.

This is a valid PDF because:

Non-negativity: $f(x) \geq 0$ for all values of x.

Honesty: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{0}^{2} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_{0}^{2} = 1$

$P(1 \leq x \leq 2) = \int_{1}^{2} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_{1}^{2} = \frac{1}{2}$

2.2.

This is a valid PDF because:

Non-negativity: $f(x) \geq 0$ for all values of x

Honesty: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{0}^{1} \frac{x}{2}\,\mathrm{d}x = \left[x^2\right]_{0}^{1} = 1$

(a) $P(0.5 \leq X \leq 1) = \int_{0.5}^{1} 2x\,\mathrm{d}x = [x^2]_{0.5}^{1} = 1^2 - (0.5)^2 = 1 - 0.25 = 0.75$

(b) $P(0.25 \leq X \leq 0.75) = \int_{0.25}^{0.75} 2x\,\mathrm{d}x = [x^2]_{0.25}^{0.75} = (0.75)^2 - (0.25)^2 = 0.5625 - 0.0625 = 0.5$

2.3.

This is a valid PDF because:

Non-negativity: $f(x) \geq 0$ for all values of x

Honesty: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{3}^{7} \frac{1}{4}\,\mathrm{d}x = \left[\frac{x}{4}\right]_{3}^{7} = 1$

$P(3 \leq X \leq 6) = \int_{3}^{6} \frac{1}{4}\,\mathrm{d}x = \left[\frac{x}{4}\right]_{3}^{6} = \frac{6}{4} - \frac{3}{4} = \frac{3}{4}$

2.4.

This is not a valid PDF since it does not meet the honesty condition:

Honesty: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{1}^{4} \frac{1}{9}\,\mathrm{d}x + \int_{5}^{7} \frac{1}{4}\,\mathrm{d}x \neq 1$

Calculating the individual integrals:

- $\displaystyle\int_1^4 \frac{1}{9}\,\mathrm{d}x = \frac{1}{9}\left[x\right]_1^4 = \frac{1}{3}$
- $\displaystyle\int_5^7 \frac{1}{4}\,\mathrm{d}x = \frac{1}{4}\left[x\right]_5^7 = \frac{1}{2}$

And adding them together:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \neq 1$$

2.5.

(a) For the given PDF to be valid, you must have $k = 3$.

(b) $P(0.2 \leq X \leq 0.3) = \displaystyle\int_{0.2}^{0.3} 3x^2\,\mathrm{d}x = 3\left[\frac{x^3}{3}\right]_{0.2}^{0.3} = \left[x^3\right]_{0.2}^{0.3} = 0.019$

2.6.

This is a valid PDF because:

Non-negativity: $f(x) \geq 0$ for all values of x

Honesty: $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^{0.5} 4x\,\mathrm{d}x + \int_{0.5}^{0.75} (4-4x)\,\mathrm{d}x + \int_{0.75}^{1} 0.5\,\mathrm{d}x$

Calculating the individual integrals:

- $\displaystyle\int_0^{0.5} 4x\,\mathrm{d}x = \left[2x^2\right]_0^{0.5} = 0.5$
- $\displaystyle\int_{0.5}^{0.75} (4-4x)\,\mathrm{d}x = \left[4x - 2x^2\right]_{0.5}^{0.75} = 0.375$
- $\displaystyle\int_{0.75}^{1} 0.5\,\mathrm{d}x = \left[0.5x\right]_{0.75}^{1} = 0.125$

and adding them together gives $0.5 + 0.375 + 0.125 = 1$.

Q3

3.1.

(a) $F(3) = P(X \leq 3) = 0.1 + 0.3 + 0.5 = 0.9$

(b) $P(X > 2) = 1 - P(X \leq 2) = 1 - (0.1 + 0.3 + 0.5) = 1 - 0.9 = 0.1$

3.2.

(a) The CDF for values 0.5, 1, and 2:

- $F(0.5) = \int_0^{0.5} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_0^{0.5} = \frac{0.5}{2} = 0.25$
- $F(1) = \int_0^{1} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_0^{1} = \frac{1}{2} = 0.5$
- $F(2) = \int_0^{2} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_0^{2} = \frac{2}{2} = 1$

(b) $F(3) = 1$ (since the CDF for any $x \geq 2$ is 1.)

3.3.

(a) The CDF at points 4, 5, and 6:

- $F(4) = \int_3^{4} \frac{1}{4}\,\mathrm{d}x = \left[\frac{x}{4}\right]_3^{4} = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}$
- $F(5) = \int_3^{5} \frac{1}{4}\,\mathrm{d}x = \left[\frac{x}{4}\right]_3^{5} = \frac{5}{4} - \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$
- $F(6) = \int_3^{6} \frac{1}{4}\,\mathrm{d}x = \left[\frac{x}{4}\right]_3^{6} = \frac{6}{4} - \frac{3}{4} = \frac{3}{4}$

(b) $P(X > 5) = 1 - F(5) = 1 - \frac{1}{2} = \frac{1}{2}$.

3.4.

This is not a valid CDF because the CDF should be non-decreasing as x increases.

Version history and licensing

v1.0: initial version created 12/24 by Sophie Chowgule as part of a University of St Andrews VIP project.