Proof: The quadratic formula

Tom Coleman

Summary

An explanation as to why the quadratic formula is true.

Before reading this proof sheet, it is recommended that you read Guide: Completing the square.

Proof of the quadratic formula

Remember from Guide: Using the quadratic formula that the **quadratic formula** is used to find roots of any quadratic equation:

The quadratic formula

Let $ax^2+bx+c=0$ be a quadratic equation (where $a \neq 0$). The roots to this quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

where one of the roots is given by the term $(-b+\sqrt{b^2-4ac})/2a$ and the other given by the term $(-b-\sqrt{b^2-4ac})/2a$.

In order to prove that these really are the solutions to the quadratic, you can **complete the square** on ax^2+bx+c using the fact that $a \neq 0$. See Guide: Completing the square for why this works.

Proof of the quadratic formula

First of all, as $a \neq 0$ you can divide $ax^2+bx+c=0$ through by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Taking the c/a term over to the other side gives

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Completing the square (see Guide: Completing the square) gives

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}=-\frac{c}{a}$$

You can rearrange to get

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

Now the result is starting to come together. Taking square roots of both sides (not forgetting that it could be positive or negative) gives

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

and rearranging gives

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

as required.

Further reading

Guide: Using the quadratic formula

Questions: Using the quadratic formula

Version history and licensing

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