Introduction to differentiation and the derivative

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Summary

The idea of differentiation

Before reading this guide, it is recommended that you read [Guide: Introduction to limits], [Guide: Introduction to continuity] and [Guide: Properties of functions].

What is differentiation?

Functions can be used to model many real-life processes; such as the speed of a car, the performance of a stock market over time, and the growth rate of animal populations. You can use mathematical tools examined in the first three chapters to solve some of these problems (as well as a host of others).

Sometimes it is necessary to examine the rate of change of a particular process or phenomenon. For instance, the rate of change of the speed of a car is acceleration. Studying different rates of change of such a real life problem is often useful; for instance, by looking at acceleration of a car over time, you can determine how far the car goes. Rates of change are common in studying other, real-life processes, such as: the change in value of a company in a stock exchange in economics, the change in population of a species of animal in biology, or the rate of decay of radioactive material in chemistry.

The idea of investigating the rate of change of functions and their associated applications is known as **differential calculus**. The process of **differentiation** allows you to examine these properties of rates of change of functions. Importantly, this process is *general*, allowing you to investigate the rate of change of a function at any point. Differentiating a function $y = f(x)$ gives its *derivative*, which is written as $\dfrac{dy}{dx}$ or $f'(x)$.

This guide will look at the idea of differentiation; where it comes from

Gradients of a graph

The rate of change of a straight line $y = mx + c$ is its gradient m, and this doesn't change wherever you are on the line. However, the rate of change of a function like $f(x) = x^2$ (see

Figure @ref(fig:l311fig)) is dependent on what point you are at — when x is small, the rate of change is correspondingly small; when x is large, the rate of change is larger. You can see this in the variable 'steepness' of the curve.

Definition of quadratic equation, root

A **quadratic equation** is an equation that can be rearranged into the form

$$ax^2 + bx + c = 0$$

where x is a variable and a, b, c are real numbers with $a \neq 0$.
Values of x that satisfy the equation $ax^2 + bx + c = 0$ are known as **roots** of the equation. Typically, roots of a quadratic are expressed in the form of the variable. So here, the roots of $ax^2 + bx + c = 0$ are 'roots in x'.

Here, $a \neq 0$ as if it was, then the equation would no longer be a quadratic equation!

The general shape of a quadratic equation is known as a **parabola**. A figure of two parabolas is given in Figure 1; the left hand graph is if $a > 0$, whereas the right hand graph is when $a < 0$.

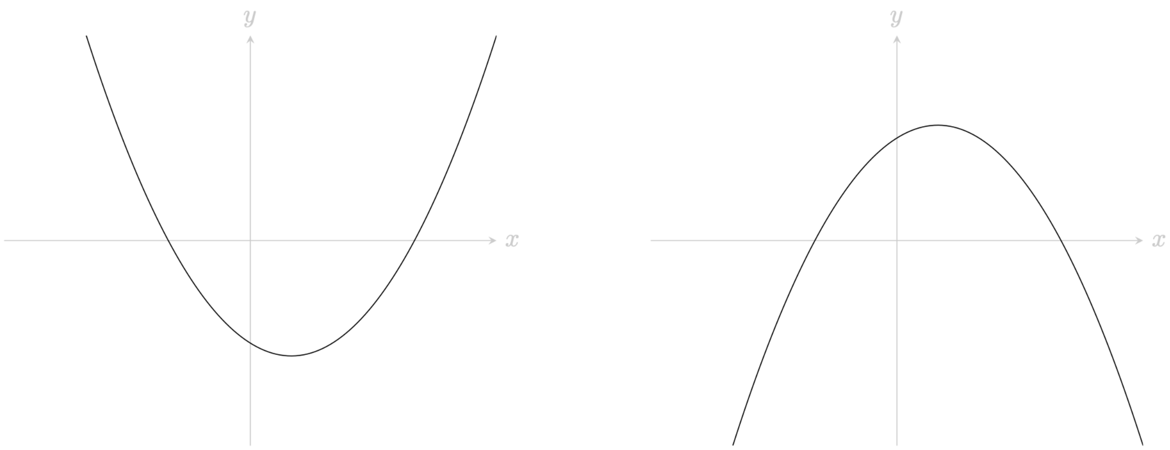

Figure 1: A pair of parabolas. (left) A graph of a quadratic ax^2+bx+c where $a > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$.

It is very important to be able to identify the variable in a quadratic equation, as well as the coefficients a, b, c. In the course of your mathematical study, it may be that the variable of a quadratic equation is not only letters like x, y, z, but squares or cubes like x^2 and y^3, or even functions like e^x, $\cos(x)$, and $\sin(y)$.

Example 1

You are given the quadratic equation $2x^2+4x-8=0$. The variable of the quadratic equation is x, and the coefficients are $a=2, b=4, c=-8$.

Example 2

Here, the equation $y^4-10y^2+25=0$ may look like a quartic equation, but it is actually a quadratic equation. Using the laws of indices, you can rewrite the equation as $(y^2)^2-10y^2+25=0$. Therefore, the variable of the quadratic equation is y^2, and the coefficients are $a=1$, $b=-10$, $c=25$.

Example 3

You are given the equation $-e^{2x}+4e^x-5=0$. Using the laws of indices, you can rewrite the equation as $-(e^x)^2+4e^x-5=0$. The variable of the quadratic equation is e^x, and the coefficients are $a=-1$, $b=4$, $c=-5$. This is not the only solution to the coefficients; since the right-hand side is equal to 0, you can multiply the equation through by -1 to get $(e^x)^2-4e^x+5=0$, which gives $a=1$, $b=-4$ and $c=5$. Both solutions are equally valid.

Example 4

You are given the equation $t+1=\frac{4}{t-3}$. This really is a quadratic equation! You can multiply both sides by $t-3$ to get

$$(t+1)(t-3)=4$$

You can then expand the brackets to get

$$t^2+t+3t+3=4$$

and so $t^2+4t+3=4$. Finally, you are able to subtract 4 from both sides to get $t^2+4t-1=0$. It follows that the variable of the quadratic equation is t, and the coefficients are $a=1$, $b=4$, $c=-1$.

Solving a quadratic equation

To solve the quadratic equation, you could use one of three methods:

- You could **factorise** the quadratic equation $ax^2 + bx + c = 0$ into linear equations $(mx+n)(px+q)$, then work out the roots when each of these linear equations is zero. See (Guide: Factorisation) for more.

- You could **complete the square** in order to reduce the quadratic equation $ax^2 + bx + c = 0$ into the form $(x + b/2a)^2 = d$, and then solve from there (not forgetting the negative root). See Guide: Completing the square for more.

- You could **use the quadratic formula**; for a quadratic equation $ax^2 + bx + c = 0$, the two roots to the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Each method is equally valid, but some may involve more work than others. It is up to you to decide which method is best for each quadratic you encounter; but it is thoroughly advised that if you are not sure which method is best, then the quadratic formula is the one to choose. See Guide: Using the quadratic formula for more.

The discriminant

What the roots of the quadratic formula look like are determined by the term $b^2 - 4ac$; this term has a special name.

> **i The discriminant**
>
> The term $D = b^2 - 4ac$ is known as the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

There are then three separate cases for solutions to quadratic equations.

- If $D = b^2 - 4ac$ is positive, then $\sqrt{D}$ is a real number and the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{D}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{D}}{2a}$$

  These two roots are both real numbers and distinct from each other. You can observe this behaviour on a graph in Figure 2; the parabola crosses the x-axis in two places.

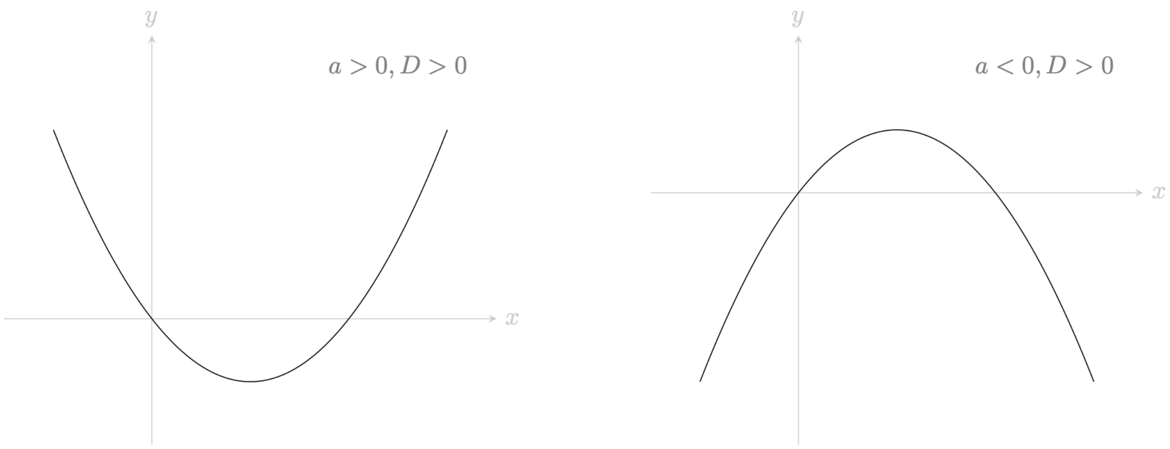

Figure 2: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D > 0$.

- If $D = b^2 - 4ac = 0$ is zero, then $\sqrt{D} = 0$. In this case, the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

  $$x = \frac{-b}{2a} \qquad \text{and} \qquad x = \frac{-b}{2a}$$

  These two roots are given by the same real number. To be sure that you express both roots, you can write '$x = -b/2a$ twice'. You can observe this behaviour on a graph in Figure 3; the parabola touches the x-axis in exactly one place.

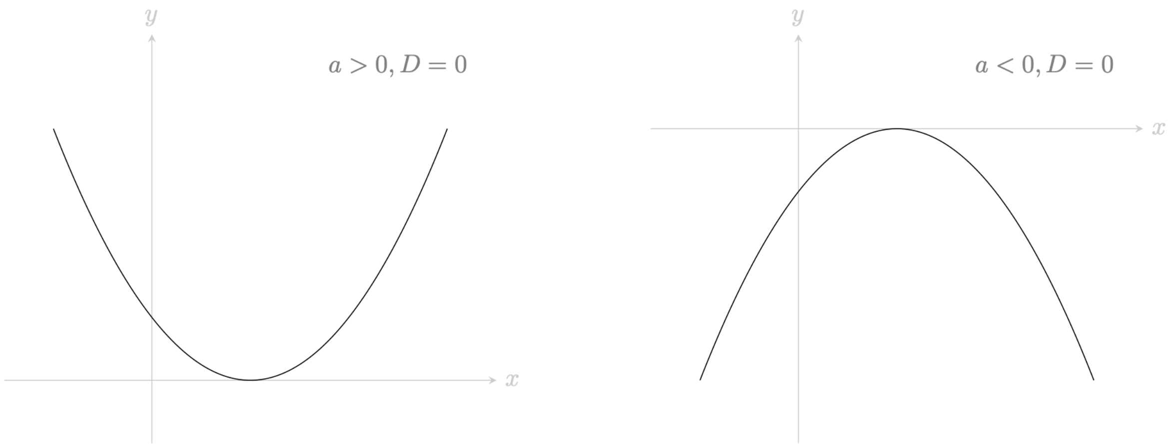

Figure 3: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D = 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D = 0$.

- If $D = b^2 - 4ac$ is negative, then $\sqrt{D}$ is not a real number. In this case, the two roots of the quadratic equation are **complex numbers**. You can express the two roots of the quadratic equation by

$$x = \frac{-b + i\sqrt{-D}}{2a} \qquad \text{and} \qquad x = \frac{-b - i\sqrt{-D}}{2a}$$

where i is the imaginary unit (so $i^2 = -1$; see Guide: Introduction to complex numbers). In a graph, the parabola does not cross the x-axis at all; this indicates that there are no real solutions to this quadratic equation. See Figure 4 for a picture.

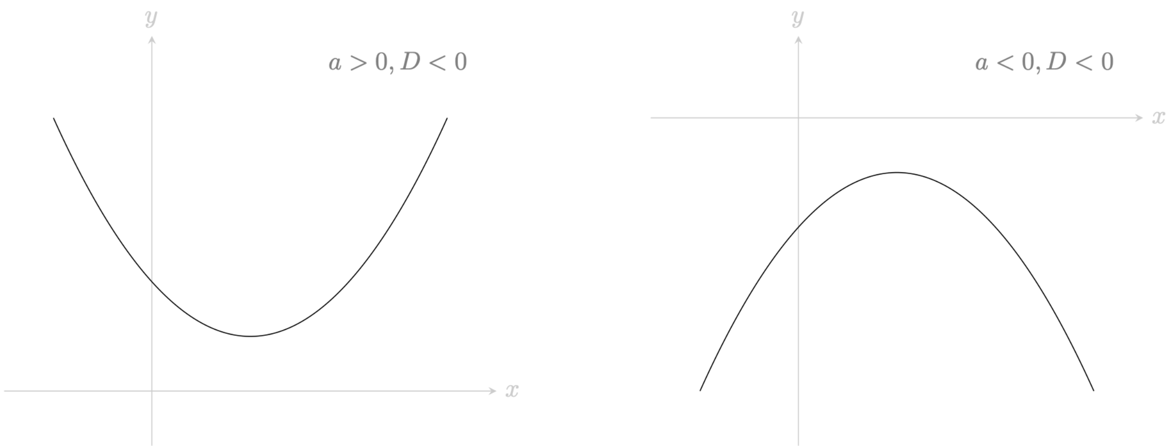

Figure 4: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D < 0$.

> **Warning**
>
> You can use the discriminant to check how many roots a quadratic equation has in the variable given to you. However, this is at most a maximum number of solutions. Conditions on that variable may also reduce the number of valid solutions, particularly if you have real valued functions. For instance, since $e^x > 0$ for all real number x, there are no solutions in x if you find $e^x = -1$.

Here's some examples of using the discriminant and known properties of functions to rule out solutions.

Example 5

In Example 1, you identified the coefficients of $2x^2+4x-8=0$ as $a=2, b=4, c=-8$. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(4)^2-4(2)(-8)=16+64=80.$$

Since $D=80$, you can say that this quadratic equation has two distinct real roots in $x=r_1$ and $x=r_2$.

Example 6

In Example 2, you identified the coefficients of $y^4-10y^2+25=0$ as $a=1, b=-10, c=25$, and the variable as y^2. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(-10)^2-4(1)(25)=100-100=0.$$

Since $D=0$, you can say that this quadratic equation has at most one real root r in terms of y^2.
Whether or not the equation itself has real solutions in y depends on whether r is positive or negative! You cannot take the square root of a negative number, so if r is negative the equation has no real solutions. If r is positive, then the equation has two real roots in y; that is, $y=\pm\sqrt{r}$.

Example 7

In Example 3, you identified the coefficients of $-e^{2x}+4e^x-5=0$ as $a=-1, b=4, c=-5$, and the variable as e^x. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(4)^2-4(-1)(-5)=16-20=-4.$$

Since $D=-4$, you can say that this quadratic equation has complex roots.
This equation therefore has no real solutions in x. This is because e^x is real for any real x; if e^x is complex, it follows that x cannot be real.

Example 8

In Example 4, you rearranged the equation $t+1=\frac{4}{t-3}$ to $t^2+4t-1=0$, and therefore identified the coefficients as $a=1, b=4, c=-1$, and the variable as t. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(4)^2-4(1)(-1)=16+4=20.$$

Since $D=20$, you can say that this quadratic equation has two distinct real roots in t.

Quick check problems

1. What is the discriminant of the quadratic equation $x^2-x-1=0$?
2. You are given the quadratic equation $4h^2-h+101=0$. Identify the variable, and the coefficients a, b, c.
3. You are given three statements below. Decide whether they are true or false.

(a) The quadratic equation $m^2+4m+4=0$ has two distinct real roots.

(b) The quadratic equation $m^2-4m-4=0$ has exactly one real root.

(c) The quadratic equation $4m^2+4m+4=0$ has no real roots.

Further reading

For more questions on the subject, please go to Questions: Introduction to quadratic equations.

For a way to solve quadratic equations, please see Guide: Using the quadratic formula.

Version history

v1.0: initial version created 11/24 by tdhc.