

# Proof: The square root of 2 is irrational

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## Summary

Proof by contradiction of the irrationality of  $\sqrt{2}$

Before reading this proof sheet, it is recommended that you read [Overview: Number sets].

## $\sqrt{2}$ is irrational

You can remember from [Overview: Number sets] that an irrational number is a number that cannot be represented as a fraction of integers,  $p/q$  where  $q \neq 0$ . Here you can prove that  $\sqrt{2}$  is irrational. This particular proof dates back to the ancient Greeks and relies on a method of proof called **proof by contradiction**. In a proof by contradiction you begin by assuming that what you're trying to prove is false, then you show that from that assumption you can derive a contradiction, so your assumption must have been false.

Let's prove that  $\sqrt{2}$  is irrational by contradiction.

Suppose  $\sqrt{2}$  is rational. Then it can be expressed as a fraction:

$$\sqrt{2} = p/q,$$

where  $p$  and  $q$  are integers with no common factors other than 1, meaning the fraction is in its simplest form, and  $q \neq 0$ .

Then you can square both sides:

$$2 = p^2/q^2,$$

and multiply both sides by  $q^2$ :

$$2q^2 = p^2.$$

This implies that  $p^2$  is even (since it is divisible by 2). Since the square of an odd number is odd,  $p$  must be even. Let  $p = 2k$  for some integer  $k$ .

You can then substitute  $p = 2k$  into the equation:

$$2q^2 = (2k)^2 = 4k^2.$$

Dividing both sides by 2:

$$q^2 = 2k^2.$$

This shows that  $q^2$  is also even, which means  $q$  must be even.

Since both  $p$  and  $q$  are even, they share a common factor of 2, contradicting the assumption that  $p$  and  $q$  have no common factors other than 1.

This contradiction implies that your initial assumption was false,  $\sqrt{2}$  cannot be written as a fraction of integers, and so,  $\sqrt{2}$  is irrational.

## **$\sqrt{p}$ is irrational**

You can extend this proof to show that  $\sqrt{p}$  is irrational for any prime  $p$ .

Suppose  $\sqrt{p}$  is rational for some prime  $p$ . Then it can be expressed as a fraction:

$$\sqrt{p} = a/b,$$

where  $a$  and  $b$  are integers with no common factors other than 1, meaning the fraction is in its simplest form, and  $b \neq 0$ .

Squaring both sides:

$$p = a^2/b^2.$$

Multiplying both sides by  $b^2$ :

$$pb^2 = a^2.$$

This implies that  $a^2$  is divisible by  $p$ . So  $a$  must also be divisible by  $p$  (since  $p$  is prime). Let  $a = pk$  for some integer  $k$ .

Substituting into the equation:

$$pb^2 = (pk)^2 = p^2k^2.$$

Dividing both sides by  $p$ :

$$b^2 = pk^2.$$

This implies that  $b^2$  is also divisible by  $p$ , so  $b$  is divisible by  $p$ .

So, both  $a$  and  $b$  are divisible by  $p$ , contradicting the assumption that  $a/b$  is in its simplest

form.

Therefore, the assumption must be false, and  $\sqrt{p}$  is irrational for any prime number  $p$ .

## Further reading

For more on this topic, please go to [Overview: Number sets].

## Version history

v1.0: initial version created 04/25 by Jessica Taberner as part of a University of St Andrews VIP project.

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