

# Further sigma notation

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## Summary

Sigma notation is used to express many additions at once. Understanding what this notation is, how it works, and how to manipulate them is a valuable skill to learn for use in almost any area of mathematics.

## Properties

In this section you will learn about a few properties of sigma notation which means you'll have a toolkit to rearrange sums!

The first property you'll learn about sigma notation is *distributivity*. This property allows you to take constants from inside the sigma notation to outside the summation.

### **i** Distributivity

Let  $a_k, a_{k+1}, \dots, a_n$  be a sequence of numbers (where  $k$  and  $n$  are integers with  $k \leq n$ ) and  $C$  be any constant. Then

$$\sum_{i=k}^n C a_i = C \sum_{i=k}^n a_i.$$

You can see this is true by writing the entire sum out, like this:

$$\begin{aligned} \sum_{i=k}^n C a_i &= C a_k + C a_{k+1} + C a_{k+2} + \dots + C a_n \\ &= C(a_k + a_{k+1} + a_{k+2} + \dots + a_n) \\ &= C \sum_{i=k}^n a_i \end{aligned}$$

### **i** Example 7

What is the value of  $\sum_{n=2}^5 6n^2$ ?

Using distributivity,  $\sum_{n=2}^5 6n^2 = 6 \sum_{n=2}^5 n^2$ . From Example 2, you know that  $\sum_{n=2}^5 n^2 = 54$ . Therefore,  $6 \sum_{n=2}^5 n^2 = 6 \times 54 = 324$ .

## Double sums

Sometimes, you'll want to multiply two sums together. This can be written succinctly using something called *double sums*.

### Double sums

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  be two sequences of numbers (where  $k, n, t$ , and  $m$  are integers with  $k \leq n$  and  $t \leq m$ ). Then the **double sum**  $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$  is defined as

$$\begin{aligned} \sum_{i=k}^n \sum_{j=t}^m a_i b_j &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} \\ &\quad + \dots + a_{k+1} b_m + \dots + a_n b_m. \end{aligned}$$

### Tip

You might find it easier to remember the above by thinking of  $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$  as  $a_1(\sum_{j=t}^m b_j) + a_2(\sum_{j=t}^m b_j) + \dots + a_n(\sum_{j=t}^m b_j)$ .

You will now see how this relates to multiplying two sums together. Suppose that  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  are like above, and consider the product  $(\sum_{i=k}^n a_i)(\sum_{j=t}^m b_j)$ . Writing it all out and performing the multiplication, you get

$$\begin{aligned} \left(\sum_{i=k}^n a_i\right) \left(\sum_{j=t}^m b_j\right) &= (a_k + a_{k+1} + \dots + a_n)(b_t + b_{t+1} + \dots + b_m) \\ &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} + \\ &\quad \dots + a_{k+1} b_m + a_{k+2} b_t + \dots + a_n b_m \\ &= \sum_{i=k}^n \sum_{j=t}^m a_i b_j \end{aligned}$$

You can write this as a result:

### Double sums and products of two sums

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  be two sequences of numbers (where  $k, n, t$ , and  $m$  are integers with  $k \leq n$  and  $t \leq m$ ). Then

$$\sum_{i=k}^n \sum_{j=t}^m a_i b_j = \left(\sum_{i=k}^n a_i\right) \left(\sum_{j=t}^m b_j\right).$$

### **i Example 10**

Write  $(1 + 2 + 3 + 4 + 5 + 6)(2 + 4 + 6 + 8 + 10 + 12)$  as a double sum and as a product of two sums.

First, notice you can write out the above expression in the form  $(1)(2) + (1)(4) + \dots(1)(12) + (2)(2) + (2)(4) \dots(3)(2) + \dots(6)(12)$

From the definition above you may now rewrite the expression to the double sum

$$\sum_{i=1}^6 \sum_{j=1}^6 i * 2j$$

using the distributivity property this can be written as

$$2 \sum_{i=1}^6 \sum_{j=1}^6 ij$$

This can then be written using the product of two sums rule above to

$$2 \sum_{i=1}^6 i \sum_{j=1}^6 j$$

It is evident that the two sums are the same with different index variables this means that they can be combined to form

$$2 \sum_{k=1}^6 k^2$$

$k$  has been used to differentiate the new sum from the ones involving  $i$  and  $j$  before but as always the choice of index variable is relatively unimportant

### **Quick check problems**

1. What is the value of  $\sum_{i=2}^6 i$ .

Answer: The value of the above is: \_\_\_\_.

2. Given  $\sum_{j=1}^{100} i$  Identify the index of the sum.

Answer: The index is \_\_

3. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) The sum  $3 + 6 + 9 + 12$  can be expressed as  $\sum_{i=0}^4 3i$  Answer: TRUE / FALSE.

(b) The sum  $-1 + 1 - 1 + 1$  can be expressed as  $\sum_{i=1}^4 -i$  Answer: TRUE / FALSE.

(c)  $\sum_{i=1}^{100} i = \sum_{i=0}^{101} i$  Answer: TRUE / FALSE.

(d)  $\sum_{i=1}^{100} 6i = 6 \sum_{i=0}^{100} i$  Answer: TRUE / FALSE.

(e)  $\sum_{i=1}^{100} 9i + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 27i^2$  Answer: TRUE / FALSE.

(f)  $\sum_{i=1}^{100} 12i - \sum_{i=1}^{100} 4i = 8 \sum_{i=1}^{100} i$  Answer: TRUE / FALSE.

4. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a)  $\sum_{i=1}^{10} \sum_{j=2}^6 ij$  can be expressed as  $\left( \sum_{i=2}^6 i \right) \left( \sum_{j=1}^{10} j \right)$  Answer: TRUE / FALSE.

(b)  $\left( \sum_{i=1}^5 2i \right) \left( \sum_{j=5}^{10} 3j \right)$  can be expressed as  $6 \left( \sum_{i=1}^5 \sum_{j=5}^{10} ij \right)$  Answer: TRUE / FALSE.

(c) The sum  $(1+2+3+4+5+6)(-1-2)(3+6+9)$  can be expressed as  $\sum_{i=1}^6 \sum_{j=1}^2 \sum_{k=1}^3 -3ijk$   
Answer: TRUE / FALSE.

## Further reading