Law of total probability and Bayes' theorem

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Summary

The law of total probability and Bayes' theorem build on the fundamentals of conditional probability. They are used to calculate probabilities across different scenarios and to update predictions when information is incomplete. This guide outlines both methods and explains how and when to apply them.

Introduction

The law of total probability and Bayes' theorem extend the basic ideas of conditional probability. They provide a way to calculate probabilities when outcomes depend on multiple possible scenarios or when information is incomplete or indirect. These results are widely used across statistics, particularly in problems involving uncertainty, prediction, and statistical inference. This guide introduces both the law of total probability and Bayes' theorem, explains how they are derived, and shows how to apply them in practical contexts.

Law of total probability

In everyday life, an event can often happen in more ways than one. For example, the chance that a package arrives late might depend on whether it is delivered by courier A or courier B. If each courier is used for a different proportion of deliveries, and each one has a different chance of causing a delay, then the overall chance of delay depends on both which courier is used and how often each one is chosen.

The **law of total probability** allows you to calculate the total probability of an event by combining its conditional probabilities across several possible scenarios. This approach is especially helpful when outcomes depend on mutually exclusive events (only one possible scenario can occur at a time), each with a known probability.

Definition of the law of total probability

Suppose an event B depends on several possible scenarios. These scenarios can be described by events A_1, A_2, ..., A_n, that are:

- **Mutually exclusive**: they cannot occur at the same time, and
- **Exhaustive**: one of them must always occur.

Then, the **law of total probability** states that the probability of event B is:

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

Each term represents the probability of B in one particular scenario.

Tip

The symbol $\sum$ is called **sigma notation**. In this case, it is adding the probabilities for each possible scenario where B could happen. For more, see [Guide: Introduction to sigma notation].

Tip

To apply the law of total probability:

1. Calculate each scenario's conditional probability, $P(B \mid A_i)$.
2. Multiply these conditional probabilities by their scenario likelihoods, $P(A_i)$.
3. Add these results to find the total probability of event B.

This accounts for each possible scenario exactly once.

Note

The law of total probability comes directly from conditional probability:

$$P(B \mid A_i) = \frac{P(B \cap A_i)}{P(A_i)}$$

Multiplying by $P(A_i)$ gives the multiplication rule:

$$P(B \cap A_i) = P(B \mid A_i)P(A_i)$$

As scenarios $A_1, A_2, \dots, A_n$ are mutually exclusive and exhaustive:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$$

Substituting the above expressions gives:

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_n)P(A_n)$$

Which results in the law of total probability:

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

Example 1

At a hospital:

- 70% of patients are treated in the general ward,
- 30% of patients are treated in the emergency unit,
- The probability a patient in the general ward recovers in 2 days is 0.8.
- The probability a patient in the emergency unit recovers in 2 days is 0.6.

What is the probability that a randomly selected patient recovers in 2 days?
Let B be the event the patient recovers in 2 days. Let A_G be the event they are treated in the general ward, and A_E in the emergency unit.
You know:

- $P(A_G) = 0.7$,
- $P(A_E) = 0.3$,
- $P(B \mid A_G) = 0.8$,
- $P(B \mid A_E) = 0.6$.

Using the law of total probability:

$$P(B) = (0.7)(0.8) + (0.3)(0.6) = 0.56 + 0.18 = 0.74$$

So the probability that a patient recovers in 2 days is 0.74, or 74%.

Example 2

As a student, you can choose to study in three different locations. Most students spend:

- 50% of the time in the library,
- 30% in a café,
- 20% at home.

It is said they concentrate well:

- 90% of the time in the library,
- 60% of the time in the café,
- 40% of the time at home.

What is the probability that the student concentrates well on any given day?
Let B be the event the student concentrates well. Let A_L, A_C, and A_H be the events that the student studies in the library, café, and home, respectively.
You know:

- $P(A_L) = 0.5$, $P(B \mid A_L) = 0.9$
- $P(A_C) = 0.3$, $P(B \mid A_C) = 0.6$
- $P(A_H) = 0.2$, $P(B \mid A_H) = 0.4$

Using the law of total probability:

$$P(B) = (0.5)(0.9) + (0.3)(0.6) + (0.2)(0.4) = 0.45 + 0.18 + 0.08 = 0.71$$

So, the probability that the student concentrates well on any given day is 0.71, or 71%.

Example 3

Imagine at your school, students receive emails from three types of senders:

- 40% of the time from teachers,
- 35% from classmates,
- 25% from student societies.

The chance of an email containing a typo is:

- 5% for teacher emails,
- 2% for classmate emails,
- 8% for society emails.

What is the probability that a randomly selected email contains a typo?
Let B be the event that the email contains a typo. Let A_T, A_C, and A_S represent the sender: teacher, classmate, or society.
You know:

- $P(A_T) = 0.4$, $P(B \mid A_T) = 0.05$
- $P(A_C) = 0.35$, $P(B \mid A_C) = 0.02$
- $P(A_S) = 0.25$, $P(B \mid A_S) = 0.08$

Using the law of total probability:

$$P(B) = (0.4)(0.05) + (0.35)(0.02) + (0.25)(0.08) = 0.02 + 0.007 + 0.02 = 0.047$$

So, the probability that a school email contains a typo is 0.047, or 4.7%.

Bayes' theorem

Often, the probability of one event given another is known, but the reverse is needed. **Bayes' theorem** lets you reverse conditional probabilities, calculating the probability of event A given event B, based on the probability of B given A. Bayes' Theorem is especially useful for updating predictions as new information becomes available and when it is difficult to calculate probabilities directly.

Definition of Bayes' Theorem

If A and B are events with $P(B) > 0$, then Bayes' Theorem states:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

where:

- $P(A \mid B)$ is the probability of A given B,
- $P(B \mid A)$ is the probability of B given A,
- $P(A)$ and $P(B)$ are the individual probabilities of A and B, respectively.

Note

Bayes' Theorem is derived directly from the definition of conditional probability. Start with the conditional probabilities of two events A and B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Rearrange the second equation by multiplying both sides by $P(A)$, giving the multiplication rule:

$$P(A \cap B) = P(B \mid A)P(A)$$

Substitute this result into the first equation to get:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

This gives Bayes' Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

Example 4

A school library survey found:

- 15% of students regularly borrow mystery books.
- Of students who regularly borrow mystery books, 80% also borrow science fiction books.
- In general, 40% of students regularly borrow science fiction books.

A student is randomly selected and found to regularly borrow science fiction books. What is the probability they also regularly borrow mystery books?
First, define the events:

- M: A student regularly borrows mystery books.
- S: A student regularly borrows science fiction books.

You know:

- $P(M) = 0.15$,
- $P(S) = 0.40$,
- $P(S \mid M) = 0.80$.

So applying Bayes' theorem gives:

$$P(M \mid S) = \frac{P(S \mid M)P(M)}{P(S)} = \frac{(0.80)(0.15)}{0.40} = 0.30$$

So, given the student borrows science fiction books regularly, there is a 30% chance they also regularly borrow mystery books.

Example 5

In a city, commuters choose transport as follows:

- 60% travel by bus, and 40% travel by car.
- Bus commuters are late 25% of the time.
- Car commuters are late 10% of the time.

You randomly select a commuter who arrived late today. What's the probability they commute by bus?
First, define events:

- B: Commuter travels by bus.
- C: Commuter travels by car.
- L: Commuter arrives late.

Then calculate $P(L)$ using the law of total probability:

$$P(L) = P(L \mid B)P(B)+P(L \mid C)P(C) = (0.25)(0.60)+(0.10)(0.40) = 0.15+0.04 = 0.19$$

Now applying Bayes' theorem:

$$P(B \mid L) = \frac{P(L \mid B)P(B)}{P(L)} = \frac{(0.25)(0.60)}{0.19} \approx 0.7895$$

So given the commuter arrived late, there's about a 79% chance they commute by bus.

Example 6

A factory uses three machines to make its products:

- Machine A produces 50% of the items and has a 2% defect rate,
- Machine B produces 30% of the items and has a 5% defect rate,
- Machine C produces 20% of the items and has a 10% defect rate.

You select an item at random and find that it is defective. What is the probability that it was made by Machine C?
Let:

- A, B, and C be the events that the item came from Machine A, B, or C,
- D be the event that the item is defective.

You know:

- $P(A) = 0.5$, $P(B) = 0.3$, $P(C) = 0.2$,
- $P(D \mid A) = 0.02$, $P(D \mid B) = 0.05$, $P(D \mid C) = 0.10$.

First, use the law of total probability to find $P(D)$:

$$P(D) = (0.02)(0.5) + (0.05)(0.3) + (0.10)(0.2) = 0.01 + 0.015 + 0.02 = 0.045$$

Now apply Bayes' theorem to get:

$$P(C \mid D) = \frac{P(D \mid C)P(C)}{P(D)} = \frac{(0.10)(0.20)}{0.045} \approx 0.444$$

So, given that the item is defective, there is about a 44% chance it came from Machine C.

Quick Check Problems

1. A hospital treats patients in two wards:

- 70% in the general ward, 30% in emergency

- Recovery rates are 80% (general) and 60% (emergency)

What is the probability a randomly selected patient recovers in 2 days?

2. A student studies:

- 50% in the library, 30% in a café, 20% at home
- Concentration rates are 90%, 60%, and 40%, respectively

What is the probability that the student concentrates well on a given day?

3. In a city:

- 60% commute by bus, 40% by car
- Bus lateness: 25%, Car lateness: 10%

A commuter is late. What is the probability they took the bus?

4. A factory has 3 machines:

- A: 50% of production, 2% defect rate
- B: 30%, 5% defect
- C: 20%, 10% defect

If an item is defective, what is the probability it came from Machine C?

5. True or false:

(a) The Law of Total Probability can only be used when events are mutually exclusive and exhaustive.

(b) Bayes' Theorem allows reversing conditional probabilities.

(c) The Law of Total Probability requires $P(B \mid A_i)$ for every outcome.

Further reading

[For more questions on the subject, please go to Questions: The law of total probability and Bayes' theorem.]

Version history and licensing

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