Arithmetic on algebraic fractions

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Summary

Algebraic fractions extend numerical fractions to expressions that include variables. This guide explains how to add, subtract, multiply and divide algebraic fractions, including how to simplify compound fractions (fractions within fractions). You will see how the familiar rules for numerical fractions still apply, provided you factorize carefully and keep track of any restricted values where the denominator would be zero.

Before reading this guide, it is recommended that you read Guide: Introduction to numerical fractions, Guide: Arithmetic on numerical fractions, Guide: Introduction to algebraic fractions and Guide: Factorization.

In Guide: Introduction to algebraic fractions, you learned how to:

- identify restrictions on the denominator for values that make it zero
- rewrite algebraic fractions in equivalent forms
- simplify algebraic fractions by factorizing and cancelling common factors

In this guide, you will see how the arithmetic rules from Guide: Arithmetic on numerical fractions still apply to algebraic fractions. You will learn how to add, subtract, multiply and divide algebraic fractions, and simplify compound fractions (fractions within fractions), while keeping track of any restricted values.

Adding and subtracting algebraic fractions

When adding or subtracting algebraic fractions, the key idea is that the denominators of both fractions must be the same. The denominator tells you the size of each part, and you can only combine parts that are of equal size.

As with numerical fractions, you cannot add directly if the denominators are different. Instead, you first rewrite the fractions as equivalent fractions with a common denominator, then add or subtract the numerators.

Adding and subtracting fractions

To add or subtract fractions, the denominators of both fractions must be the same.

- If the fractions already have the same denominator, add or subtract the numerators and keep the same denominator.
- If the fractions have different denominators, rewrite them as equivalent fractions so that the two new fractions have the same denominator. Then, add or subtract the numerators and keep the same denominator.

Example 1

Find $\frac{2}{x} + \frac{5}{x}$.
Both fractions already have denominator x, so you can add the numerators and keep the same denominator.

$$\frac{2}{x} + \frac{5}{x} = \frac{2+5}{x} = \frac{7}{x} \qquad \text{if } x \neq 0$$

There are no common factors other than 1, so the fraction is already in its simplest form.

When the denominators are different, you need to find a common denominator. For algebraic fractions, this usually means multiplying the denominators together or using their lowest common multiple, similar to what you did for numerical fractions.

Example 2

Find $\dfrac{1}{x} + \dfrac{1}{x+1}$.
Both fractions have different denominators. The denominators are x and $x+1$.
A common denominator for the fractions is found by multiplying the denominators together:

$$x(x+1)$$

You can write both fractions as equivalent fractions with this common denominator. Remember that what you do to the denominator must also be done to the numerator to keep the fractions equivalent.

$$\frac{1}{x} = \frac{x+1}{x(x+1)}$$
$$\frac{1}{x+1} = \frac{x}{x(x+1)}$$

Now add the numerators:

$$\frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} = \frac{x+1+x}{x(x+1)} = \frac{2x+1}{x(x+1)} \qquad \text{if } x \neq 0 \text{ and } x \neq -1$$

This fraction cannot be simplified further as there are no common factors other than 1. The fraction is in its simplest form.

Multiplying algebraic fractions

When you multiply two fractions, you are finding a fraction of another fraction. The method used for numerical fractions still works for algebraic fractions.

Multiplying algebraic fractions

To multiply two algebraic fractions together:

1. Factorize where possible.
2. Multiply the numerators together.
3. Multiply the denominators together.
4. Simplify, if possible, by cancelling any common factors.

Unlike with adding and subtracting fractions, you do not need to find a common denominator before multiplying.

Example 3

Find $\frac{x}{2y} \cdot \frac{4y}{x^2}$.
No expressions can be factorized, so start by multiplying the numerators and denominators together.

$$\frac{x}{2y} \cdot \frac{4y}{x^2} = \frac{x \cdot 4y}{2y \cdot x^2} = \frac{4xy}{2x^2y}$$

Now simplify by cancelling any common factors. There is a factor of 2 common to 4 and 2, and a factor of xy common to the numerator and the denominator.

$$\frac{4xy}{2x^2y} = \frac{2}{x} \qquad \text{if } x \neq 0 \text{ and } y \neq 0$$

Example 4

Find $\frac{x^2-9}{x^2+2x+1} \cdot \frac{x+1}{x-3}$.
Start by factorizing where possible:

$$x^2 - 9 = (x+3)(x-3)$$

$$x^2 + 2x + 1 = (x+1)^2$$

Now substitute these into the problem, and multiply the numerators and denominators together.

$$\frac{x^2-9}{x^2+2x+1} \cdot \frac{x+1}{x-3} = \frac{(x+3)(x-3)}{(x+1)^2} \cdot \frac{x+1}{x-3} = \frac{(x+3)(x-3)(x+1)}{(x+1)^2(x-3)}$$

Now simplify by cancelling common factors. There are factors of $x-3$ and $x+1$ common to the numerator and the denominator.

$$\frac{(x+3)(x-3)(x+1)}{(x+1)^2(x-3)} = \frac{x+3}{x+1} \qquad \text{if } x \neq -1 \text{ and } x \neq 3$$

Warning

Cancelling factors across the numerator and denominator only works for multiplication and division. You cannot cancel in this way when adding or subtracting algebraic fractions.

Dividing algebraic fractions

As seen in Guide: Arithmetic on numerical fractions, when you divide one fraction by another, you are asking "How many times does this fraction fit into the other?"

Similar to numerical fractions, a good way to handle this for algebraic fractions is to use the reciprocal of the second fraction. The reciprocal of a fraction is found by swapping its numerator and denominator.

For example, the reciprocal of $\frac{x}{3}$ is $\frac{3}{x}$, providing $x \neq 0$.

Dividing algebraic fractions

To divide algebraic fractions, follow the **Keep, Change, Flip** method:

1. **Keep** the first fraction the same.
2. **Change** division to multiplication.
3. **Flip** the second fraction (take its reciprocal).

This turns the division problem into a multiplication problem.

Example 5

Find $\dfrac{2x}{3} \div \dfrac{4}{x}$.

Keep the first fraction the same, change the division sign to a multiplication sign, and flip the second fraction so that $\frac{4}{x}$ becomes $\frac{x}{4}$. Then, factorize, multiply the fractions together, and simplify by cancelling common factors.

$$\frac{2x}{3} \div \frac{4}{x} = \frac{2x}{3} \cdot \frac{x}{4} = \frac{2x \cdot x}{3 \cdot 4} = \frac{2x^2}{12} = \frac{x^2}{6} \qquad \text{if } x \neq 0$$

Example 6

Find $\dfrac{x^2-9}{x^2+3x} \div \dfrac{x-3}{x}$.

Keep the first fraction the same, change the division sign to a multiplication sign, and flip the second fraction so that $\frac{x-3}{x}$ becomes $\frac{x}{x-3}$. Then, factorize, multiply the fractions together, and simplify by cancelling common factors.

$$\begin{aligned}
\frac{x^2-9}{x^2+3x} \div \frac{x-3}{x} &= \frac{x^2-9}{x^2+3x} \cdot \frac{x}{x-3} \\
&= \frac{(x+3)(x-3)}{x(x+3)} \cdot \frac{x}{x-3} \\
&= \frac{(x+3)(x-3)x}{x(x+3)(x-3)} \\
&= 1 \qquad \text{if } x \notin \{0, -3, 3\}
\end{aligned}$$

Here, $x \neq 0$ and $x \neq -3$ come from the denominator $x(x+3)$, and $x \neq 3$ comes from the denominator $x-3$ in the fraction you are multiplying by.

Compound algebraic fractions

Sometimes, fractions can have other fractions inside them. These are called **compound fractions**.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}} \qquad\qquad \frac{\frac{x}{y}}{\frac{a}{b}}$$

It is better to rewrite these fractions in a more standard fractional form. The goal is always the same. You need to simplify them so that there is only one single fraction left.

There are two methods you can use to simplify compound algebraic fractions.

Simplifying compound algebraic fractions (method 1)

To simplify a compound algebraic fraction, you can multiply the fraction by the lowest common multiple of the denominator of the fraction on the bottom.

1. Find the lowest common multiple of all the denominators of the fractions inside the large fraction.
2. Multiply the numerator and the denominator of the large fraction by this lowest common multiple.
3. Factorize and simplify, if possible.

This clears the denominators of the smaller fractions and leaves an ordinary fraction.

Simplifying compound algebraic fractions (method 2)

An alternative method to simplifying a compound algebraic fraction is to rewrite the large fraction as a division problem and use the Keep, Change, Flip rule.

1. Remember that "a fraction over a fraction" means division.
2. Rewrite the compound fraction as a division problem.
3. Apply the Keep, Change, Flip rule.
4. Multiply the fractions together.
5. Factorize and simplify, if possible.

This also clears the denominators of the smaller fractions and leaves an ordinary fraction.

These methods are best illustrated using some examples.

Example 7

This example demonstrates the use of method 1.

Simplify $\dfrac{\frac{1}{x}+\frac{2}{y^2}}{\frac{3}{xy}}$.

The lowest common multiple of the denominators x, y^2 and xy is xy. This means you need to multiply the top and bottom of the larger fraction by xy^2, and simplify.

$$\begin{aligned}
\frac{\frac{1}{x}+\frac{2}{y^2}}{\frac{3}{xy}} &= \frac{xy^2\left(\frac{1}{x}+\frac{2}{y^2}\right)}{xy^2\left(\frac{3}{xy}\right)} \\
&= \frac{\frac{xy^2}{x}+\frac{2xy^2}{y^2}}{\frac{3xy^2}{xy}} \\
&= \frac{y^2+2x}{3y} \qquad \text{if } x \neq 0 \text{ and } y \neq 0
\end{aligned}$$

Example 8

This example demonstrates the use of method 2.

Simplify $\frac{\frac{2x}{3}}{\frac{x+1}{6}}$.

Rewrite the compound algebraic fraction as a division problem. Then, apply the Keep, Change, Flip rule, changing the division problem into a multiplication one. Multiply the fractions together, and simplify.

$$\begin{aligned}
\frac{\frac{2x}{3}}{\frac{x+1}{6}} &= \frac{2x}{3} \div \frac{x+1}{6} \\
&= \frac{2x}{3} \cdot \frac{6}{x+1} \\
&= \frac{2x \cdot 6}{3(x+1)} \\
&= \frac{12x}{3(x+1)} \\
&= \frac{4x}{x+1} \qquad \text{if } x \neq -1
\end{aligned}$$

Quick check problems

1. Calculate $\dfrac{2}{x} + \dfrac{3}{x}$.

2. Calculate $\dfrac{1}{x} + \dfrac{1}{2x}$.

(a) $\dfrac{1}{3x}$

(b) $\dfrac{2}{3x}$

(c) $\dfrac{3}{2x}$

(d) $\dfrac{1}{2x}$

(e) $\dfrac{x+1}{2x}$

3. Calculate $\dfrac{1}{x} - \dfrac{3}{x^2}$.

4. Calculate $\frac{2}{x} \cdot \frac{x}{5}$.

(a) $\frac{x}{10}$

(b) $\frac{2x}{5}$

(c) $\frac{x}{5}$

(d) $\frac{10}{x}$

(e) $\frac{2}{5}$

5. Calculate $\frac{3x}{4} \div \frac{x}{2}$.

6. Simplify $\frac{\frac{2x}{3}}{\frac{x}{6}}$.

(a) 2

(b) 4

(c) $\frac{x}{2}$

(d) $\frac{6}{x}$

(e) $4x$

Further reading

For more questions on the subject, please go to Questions: Arithmetic on algebraic fractions.

Version history

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