Proof: The square root of 2 is irrational

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Summary

Proof by contradiction of the irrationality of $\sqrt{2}$

Before reading this proof sheet, it is recommended that you read [Overview: Number sets].

$\sqrt{2}$ is irrational

You can remember from [Overview: Number sets] that an irrational number is a number that cannot be represented as a fraction of integers, p/q where $q \neq 0$. Here you can prove that $\sqrt{2}$ is irrational. This particular proof dates back to the ancient Greeks and relies on a method of proof called **proof by contradiction**. In a proof by contradiction you begin by assuming that what you're trying to prove is false, then you show that from that assumption you can derive a contradiction, so your assumption must have been false.

Let's prove that $\sqrt{2}$ is irrational by contradiction.

Suppose $\sqrt{2}$ is rational. Then it can be expressed as a fraction:

$$\sqrt{2} = p/q,$$

where p and q are integers with no common factors other than 1, meaning the fraction is in its simplest form, and $q \neq 0$.

Then you can square both sides:

$$2 = p^2/q^2,$$

and multiply both sides by q^2:

$$2q^2 = p^2.$$

This implies that p^2 is even (since it is divisible by 2). Since the square of an odd number is odd, p must be even. Let $p = 2k$ for some integer k.

You can then substitute $p = 2k$ into the equation:

$$2q^2 = (2k)^2 = 4k^2.$$

Dividing both sides by 2:

$$q^2 = 2k^2.$$

This shows that q^2 is also even, which means q must be even.

Since both p and q are even, they share a common factor of 2, contradicting the assumption that p and q have no common factors other than 1.

This contradiction implies that your initial assumption was false, $\sqrt{2}$ cannot be written as a fraction of integers, and so, $\sqrt{2}$ is irrational.

$\sqrt{p}$ is irrational

You can extend this proof to show that $\sqrt{p}$ is irrational for any prime p.

Suppose $\sqrt{p}$ is rational for some prime p. Then it can be expressed as a fraction:

$$\sqrt{p} = a/b,$$

where a and b are integers with no common factors other than 1, meaning the fraction is in its simplest form, and $b \neq 0$.

Squaring both sides:

$$p = a^2/b^2.$$

Multiplying both sides by b^2:

$$pb^2 = a^2.$$

This implies that a^2 is divisible by p. So a must also be divisible by p (since p is prime). Let $a = pk$ for some integer k.

Substituting into the equation:

$$pb^2 = (pk)^2 = p^2k^2.$$

Dividing both sides by p:

$$b^2 = pk^2.$$

This implies that b^2 is also divisible by p, so b is divisible by p.

So, both a and b are divisible by p, contradicting the assumption that a/b is in its simplest form.

Therefore, the assumption must be false, and $\sqrt{p}$ is irrational for any prime number p.

Further reading

For more on this topic, please go to [Overview: Number sets].

Version history

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