Introduction to partial differentiation

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Summary

Partial differentiation is a key technique used in calculus when dealing with functions that depend on more than one variable. Rather than considering changes to all variables at once, partial differentiation focuses on how a function changes as one variable changes while the others are kept the same.

Before reading this guide, it is recommended that you read Guide: Introduction to differentiation and the derivative.

What is partial differentiation?

In regular (single-variable) calculus, a derivative measures how a function changes as its input changes. For example, if you have a function $f(x)$, the derivative $f'(x)$ describes the rate at which f changes with respect to x.

However, in many real-world examples, functions depend on more than one variable. Consider a function $f(x, y)$ that depends on both x and y. **Partial differentiation** allows you to study the change in f with respect to one variable (say, x) while treating the other variable (y) as if it were a constant. This process is essential when you want to understand the influence of each variable separately.

Definition of the partial derivative

For a function $f(x, y)$, the **partial derivative with respect to** x is denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x(x, y)$$

It is defined by the limit

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \left(\frac{f(x+h,\, y) - f(x, y)}{h} \right)$$

This definition is very similar to that of single-variable differentiation, with the key difference that y is treated as a constant during the process.
Similarly, the **partial derivative with respect to** y is defined by the limit

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \left(\frac{f(x, y+h) - f(x, y)}{h} \right)$$

When differentiating with respect to one variable, all other variables are held fixed (they do not change).

! Important

Partial derivatives are written using a curly ∂ as opposed to a straight d in single-variable differentiation.

How to compute partial derivatives

1. Identify the variable of interest:

   Decide which variable you are differentiating with respect to. For example, if you want $\frac{\partial f}{\partial x}$, treat x as the variable and y (and any other variables) as constants.

2. Apply differentiation rules:

   Use rules of differentiation (such as the power rule, chain rule, product rule and quotient rule) while treating the other variables as constants.

3. Simplify the result:

   After applying the rules, simplify your expression if possible.

Example 1

Consider the function $f(x, y) = x^2 + y^2$.
To find the partial derivative with respect to x, first differentiate x^2 with respect to x using the power rule to get $2x$. The term y^2 is a constant with respect to x so its derivative is 0. This gives:

$$\frac{\partial f}{\partial x} = 2x$$

To find the partial derivative with respect to y, first differentiate y^2 with respect to y using the power rule to get $2y$. The term x^2 is a constant with respect to y so its derivative is 0. This gives:

$$\frac{\partial f}{\partial y} = 2y$$

Example 2

Consider the function $f(x, y) = 3x^2y + 4xy^3$.
To find the partial derivative with respect to x, first differentiate $3x^2y$ with respect to x to get $6xy$. Now differentiate $4xy^3$ with respect to x to get $4y^3$. Notice y is treated as if it is a constant.

$$\frac{\partial f}{\partial x} = 6xy + 4y^3$$

To find the partial derivative with respect to y, first differentiate $3x^2y$ with respect to y to get $3x^2$. Now differentiate $4xy^3$ with respect to y to get $12xy^2$. Notice x is treated as if it is a constant.

$$\frac{\partial f}{\partial y} = 3x^2 + 12xy^2$$

You can also use the chain, product and quotient rules when finding partial derivatives of a function. This can make problems complex very quickly so it is important to take your time, particularly when using combinations of these rules. As shown in the next example, the product rule has been used in combination with the chain rule.

Example 3

Consider the function $f(x,y) = x^2(3x+y^2)^3$.
Firstly, find the partial derivative with respect to x. Notice that $f(x,y)$ is a product of the two functions x^2 and $(3x+y^2)^3$. You need to use the product rule to differentiate the product $x^2(3x+y^2)^3$ with respect to x. Then you need to differentiate $(3x+y^2)^3$ with respect to x using the chain rule. Remember to treat the y^2 as a constant.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2(3x+y^2)^3\right]$$
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2\right]\cdot(3x+y^2)^3 + x^2\cdot\frac{\partial}{\partial x}\left[(3x+y^2)^3\right]$$
$$\frac{\partial f}{\partial x} = 2x\cdot(3x+y^2)^3 + x^2\cdot 9(3x+y^2)^2$$
$$\frac{\partial f}{\partial x} = 2x(3x+y^2)^3 + 9x^2(3x+y^2)^2$$

Now find the partial derivative with respect to y. Notice that $f(x,y)$ is a product of the two functions x^2 and $(3x+y^2)^3$ however only the second function is a function of y. Therefore, this doesn't require the product rule to differentiate as x^2 is treated as a constant. Still, the second function requires you to use the chain rule.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2(3x+y^2)^3\right]$$
$$\frac{\partial f}{\partial y} = x^2\frac{\partial}{\partial y}\left[(3x+y^2)^3\right]$$
$$\frac{\partial f}{\partial y} = x^2\cdot 6y(3x+y^2)^2$$
$$\frac{\partial f}{\partial y} = 6x^2y(3x+y^2)^2$$

Higher-order partial derivatives

Just as with single-variable functions, you can take partial derivatives of partial derivatives. For functions of two variables, you can compute second-order partial derivatives such as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

The first of these is computed as the partial derivative of $\frac{\partial f}{\partial x}$ with respect to x. The second of these is computed as the partial derivative of $\frac{\partial f}{\partial y}$ with respect to y. These are called the pure second-order partial derivatives.

There is another type of higher-order partial derivative called the **mixed partial derivative**. For a function of two variables, there are two mixed partial derivatives.

$$
\begin{aligned}
f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)
\end{aligned}
$$

The first of these f_{yx} is computed by first differentiating f with respect to y (treating x as a constant) and then differentiating the resulting expression f_y with respect to x (treating y as a constant) to get f_{yx}.

Tip

The order in which the subscript variables are written above is not a typo!
For the two notation systems, there are different conventions for denoting the order in which differentiation occurs. They can be remembered because the variable nearest the function f is the one that undergoes differentiation first.

Under certain conditions, these mixed partial derivatives are equal. This is called *Schwarz's Theorem*.

Schwarz' Theorem

If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous and differentiable then

$$
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
$$

Example 4

Consider the function $f(x, y) = \sin(xy) + x^4y^2 + x^3y^3 + 3x$.
The first-order partial derivatives are given by

$$\frac{\partial f}{\partial x} = y\cos(xy) + 4x^3y^2 + 3x^2y^3 + 3$$
$$\frac{\partial f}{\partial y} = x\cos(xy) + 2x^4y + 3x^3y^2$$

The second-order partial derivatives are calculated from the first-order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -y^2\sin(xy) + 12x^2y^2 + 6xy^3$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x^2\sin(xy) + 2x^4 + 6x^3y$$
$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \cos(xy) - xy\sin(xy) + 8x^3y + 9x^2y^2$$
$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(xy) - xy\sin(xy) + 8x^3y + 9x^2y^2$$

Notice in Example 3 that the *Schwarz's Theorem* holds as $\dfrac{\partial^2 f}{\partial x\partial y} = \dfrac{\partial^2 f}{\partial y\partial x}$ and the first-order partial derivatives are continuous and differentiable.

Quick check problems

Consider the function $f(x, y) = x^3y^2 + y^2 + x$. Are the following statements true or false?

The partial derivative with respect to x is given by $\frac{\partial f}{\partial x} = 3x^2y^2 + y^2 + 1$.

The partial derivative with respect to y is given by $\frac{\partial f}{\partial y} = 2x^3y^2 + 2y$.

Further reading

For more questions on the subject, please go to Questions: Introduction to partial differentiation.

For a way to differentiate functions of more than one variable, please see Guide: Multivariate chain rule.

Version history

v1.0: initial version created 05/25 by Donald Campbell as part of a University of St Andrews VIP project.

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