The product rule

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Summary

The product rule is one of the three central techniques of differentiation, allowing you to differentiate any product of two differentiable functions. This guide introduces the product rule and explains examples of where it is used.

Before reading this guide, it is recommended that you read Guide: Introduction to differentiation and the derivative.

What is the product rule?

[MOTIVATION AND BLURB]

This guide will look at the product rule for differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

The summary table of key derivatives from Guide: Introduction to differentiation and the derivative is reproduced here for reference:

Function $f(x)$	Derivative $f'(x)$	Notes
$f(x) = c$	$f'(x) = 0$	
$f(x) = ax + b$	$f'(x) = a$	
$f(x) = ax^n$	$f'(x) = anx^{n-1}$	$n \neq 0$
$f(x) = ae^{bx}$	$f'(x) = abe^{bx}$	
$f(x) = a\sin(bx)$	$f'(x) = ab\cos(bx)$	
$f(x) = a\cos(bx)$	$f'(x) = -ab\sin(bx)$	
$f(x) = a\ln(bx)$	$f'(x) = \dfrac{a}{x}$	

Statement of the product rule

The product rule

Let $u(x)$ and $v(x)$ be two differentiable functions. Then the **product rule** says that

$$(uv)'(x) = \frac{d}{dx}\left(u(x)v(x)\right) = u(x)v'(x) + u'(x)v(x)$$

that is, the derivative of the product of $u(x)$ and $v(x)$ is equal to the product of $u(x)$ and the derivative of $v(x)$, plus the product of $v(x)$ and the derivative of $u(x)$.
This can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Sometimes $f(x)$ and $g(x)$ are written instead of $u(x)$ and $v(x)$ in the statement of the product rule. The reason that $u(x)$ and $v(x)$ is used here is that sometimes the product function is called $f(x)$; and you then can't use it again!

It does not matter which function you pick to be $u(x)$ and which function you pick to be $v(x)$; this is because $u(x)v(x) = v(x)u(x)$.

To see why this really is the derivative of the product of two functions, please see [Proof sheet: Rules of differentiation.]

Examples

$f(x) = e^x\sqrt{x} + x^3$

$f(x) = \ln(x)/x^5$

$f(x) = 4x^4\cos(4x) + e^{-4x}\cos(4x)$

$f(x) = e^x\sin(2x)\cos(4x)$

Example 1

What is the derivative of $y = x^3e^x$?

In this case, you have two functions multiplied together to make y. The two functions are $u(x) = x^3$ and $v(x) = e^x$. In order to use the product rule on y, you could differentiate $u(x)$ and $v(x)$ beforehand and then substitute them into the product rule. Doing this gives

- For $u(x) = x^3$, then $u'(x) = 3x^2$ by the power rule.
- For $v(x) = e^x$, then $v'(x) = e^x$.

Putting these into the statement of the product rule gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= u(x)v'(x) + u'(x)v(x) \\ &= (x^3)(e^x) + (3x^2)(e^x) \\ &= x^3e^x + 3x^2e^x\end{aligned}$$

and this is your answer. You could also factorise the answer if you wanted to to get $y'(x) = (x^3 + 3x^2)e^x$.

Here's another example of the product rule, which is included solely for a chance to remember the signs for the derivatives of $\sin(x)$ and $\cos(x)$!

Example 2

Find the derivative of the function $f(x) = 2\sin(x)\cos(x)$.
In this case, there are two options. You can either

- use the constant rule to take the 2 out and differentiate $\sin(x)\cos(x)$ using the product rule, or
- take $u(x) = 2\sin(x)$ and $v(x) = \cos(x)$ and differentiate directly.

Let's do the first of these. To use the product rule take $u(x) = \sin(x)$ and $v(x) = \cos(x)$. Then $u'(x) = \cos(x)$ and $v'(x) = -\sin(x)$. Therefore, using the product rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)\cos(x)) &= u(x)v'(x) + u'(x)v(x) \\ &= (\sin(x))(-\sin(x)) + (\cos(x))(\cos(x)) \\ &= \cos^2(x) - \sin^2(x)\end{aligned}$$

Therefore, the derivative of $f(x) = 2\sin(x)\cos(x)$ is

$$f'(x) = 2\left(\cos^2(x) - \sin^2(x)\right).$$

Tip

You don't always need the product rule to differentiate the product of two functions if that product of two functions can be expressed in a different way.
For instance, the function in Example 2 is $f(x) = 2\sin(x)\cos(x)$, which by a trigonometric identity is equal to $f(x) = \sin(2x)$. (See Guide: Trigonometric identities (radians) for more.) You can then differentiate $\sin(2x)$ by the rules above to get $2\cos(2x)$. By trigonometric identities, this is also equal to $2\left(\cos^2(x) - \sin^2(x)\right)$, which is the same answer as in Example 2.
This shows you should explore all potential avenues before starting to use differentiation techniques.

As a rule of thumb, you should always write an nth root using a fractional power instead. Here's another example of where you should write a term involving a power of x:

Example 3

Find the derivative of the function $f(x) = \dfrac{1}{x^4} + 4\cos(x) - \sin(4x)$.
Again, the cosine and sine terms are manageable from the derivative above, it is only the $1/x^4$ term that needs a little thought. In fact, this is yet *another* case of x^n; this time when $n = -4$. (See Guide: Laws of indices for why this is.)
You can then rewrite the function as

$$f(x) = x^{-4} + 4\sin(x) - \cos(4x)$$

and use the familiar rules of differentiation above. Using the results above with $n = -4$, the derivative of x^{-4} is $-4x^{-4-1} = -4x^{-5}$. The derivative of $4\sin(x)$ is $4(\cos(x)) = 4\cos(x)$. The derivative of $\cos(4x)$ is $-4\sin(4x)$. Taking care of your signs you can write the derivative of $\dfrac{1}{x^4} + 4\cos(x) - \sin(4x)$ as

$$f'(x) = -4x^{-5} + 4\cos(x) - (-4\sin(4x)) = -4x^{-5} + 4\cos(x) + 4\sin(4x).$$

Example 4

Determine the behaviour of the function $f(x) = 2\ln(3x) - x$ when $x = 1$.
Here, you will first need to differentiate the function $f(x)$ to find $f'(x)$. Then, you will need to evaluate the derivative $f'(x)$ when $x = 1$ to see how the function behaves.
Using your rules of differentiation as you found above, you can say that the derivative of $2\ln(3x)$ is $2/x$, and the derivative of x is 1. Therefore, the derivative of the function $f(x) = 2\ln(3x) - x$ is

$$f'(x) = \frac{2}{x} - 1.$$

You can evaluate the derivative $f'(x)$ at $x = 1$ to get

$$f'(1) = \frac{2}{1} - 1 = 2 - 1 = 1$$

and so the derivative is positive at $x = 1$. This implies that the function $f(x) = 2\ln(3x) - x$ is increasing at the point $x = 1$.
It also means that the gradient of the tangent to the function $f(x)$ at the point $(1, 2\ln(3) - 1)$ is 1. You can see this in the figure below.

Example 5

Determine the behaviour of the function $f(x) = 2\ln(3x) - x$ when $x = 1$.
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Example 6

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Quick check problems

1. Answer the following questions true or false:

(a) The derivative of a function at $x = a$ is equal to the gradient of the tangent to $f(x)$ at $x = a$.

(b) If $f'(a) < 0$, then the function is increasing at $x = a$.

(c) If $f(x) = c$, then the derivative $f'(x) = c - 1$.

(d) The derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.

(e) The derivative of $f(x) = \dfrac{1}{x}$ is $f'(x) = \ln(x)$.

(f) The power of x in the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$ is $-3/2$.

2. Differentiate the following functions with respect to x.

(a) $f(x) = 3x^7 - 14x$

(b) $f(x) = -4\cos(3x)$

(c) $f(x) = -15\sin(x) + e^{8x}$

Further reading

For more questions on the subject, please go to Questions: The product rule

For more about techniques of differentiation, please see [Guide: The quotient rule], and [Guide: The chain rule].

For more about why the rules and techniques of differentiation are true, please see [Proof sheet: Rules of differentiation].

Version history

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