Introduction to integration

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Summary

As the reverse process of differentiation, integration is a key area of mathematics. It has uses in almost all places that requires calculus, such as in the sciences, social sciences, and particularly in statistics. This guide introduces the idea of integral calculus through the definite and indefinite integrals, and illustrates the connection between the two using the Fundamental Theorem of Calculus.

Before reading this guide, it is highly recommended that you read Guide: Introduction to differentiation and the derivative.

What is integration?

You have seen in Guide: Introduction to differentiation and the derivative that **differential calculus** is the study of the rate of change of quantities. In differential calculus, the **derivative** $f'(x)$ of a function $f(x)$ is a function that measures the rate of change of $f(x)$. The derivative of a function has an application in geometry: evaluating $f'(x)$ at a point a allows you to find the gradient of the tangent to the curve of $f(x)$ at a.

Differential calculus is only half of the story though! The other half is **integral calculus**; the opposite process of differentiation. Integration is used in all sorts of mathematical processes; including in statistics, physics, economics, biology, chemistry, and in mathematics for its own sake.

This guide will introduce you to the idea of **integration** of functions. There are two types of integration on a function $f(x)$.

- the **indefinite integral** of $f(x)$ with respect to x, written as

$$\int f(x)\,\mathrm{d}x$$

- the **definite integral** of $f(x)$ between limits a and b with respect to x, written as

$$\int_a^b f(x)\,\mathrm{d}x$$

The difference between the two types of integral is given by their uses. The indefinite integral is precisely the **reverse process** of differentiation. The definite integral represents (amongst other things) the signed area bounded by the curve $f(x)$ and the lines $x = a$ and $x = b$, and is related to the indefinite integral by the **Fundamental Theorem of Calculus**

This guide first explains the role of the definite integral, then introduces the indefinite integral together with the antiderivative and the constant of integration. Finally, the Fundamental Theorem of Calculus is introduced and some initial antiderivatives are given.

Definite integrals

You have learned in previous education to work out areas of shapes using geometric formulas. For instance, the area of a circle is πr^2, where r is the radius of the circle.

But what about areas of more complicated shapes? Can you define the area using any function $f(x)$, and find areas related to those? This is where the idea of a **definite integral** comes in. In many areas of the sciences, the area under a curve can often represent a quantity that you are going to find. For instance, in statistics, the area under a *probability density function* represents the probability that a particular event occurs; see Guide: PMFs, PDFs, CDFs for more.

So what does a definite integral look like? These are **always** written like

$$\int_a^b f(x)\,\mathrm{d}x,$$

which is read out loud as 'the integral of $f(x)$ with respect to x from a to b' (or words similar to that effect).

Each of the pieces of notation plays a role:

- The $\int$ symbol is called the **integral sign**, and it tells you that you are integrating. This is adapted from a letter S for sum, as the definite integral is more formally defined to be a limit of sums. (See [Guide: Definite integrals] for more.)
- The value a is called the **lower limit** and the value b is known as the **upper limit**; together, these are the **limits** or **bounds** of the integral.
- The function $f(x)$ is called the **integrand**, and this is the function you are integrating
- The $\mathrm{d}x$ denotes the variable you are integrating **with respect to**. Here, the variable x doesn't have to be called x; for instance it can be called t or u.

There are a few important things to remember about the definite integral

! Important

- The definite integral doesn't change its value if you change the name of the variable throughout. For instance

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(\theta)\,\mathrm{d}\theta$$

  and many more.

- It is important to remember that in every integral, the integrand $f(x)$ and $\mathrm{d}x$ are *multiplied together*.

- In **every** definite integral you write, you should include **every** one of these pieces of notation.

- The answer to a definite integral is always a **number**, and not an expression involving a variable.

- The numerical answer represents the *signed* area bounded by the curve $f(x)$, the x-axis, and the two lines $x = a$ and $x = b$. That means that portions of the area above the x-axis contribute positively to the answer, and portions of the area below the x-axis contribute negatively to the answer.

  – This means that the definite integral is only equal to the area 'under' the curve when the outputs of the function $f(x)$ are always positive (above the x-axis) between the limits $x = a$ and $x = b$. This is the case in Example 1.

The figure below shows a representation of a definite integral. Here, the value of the integral A corresponds to the area shaded in blue, as the function is always above the x-axis.

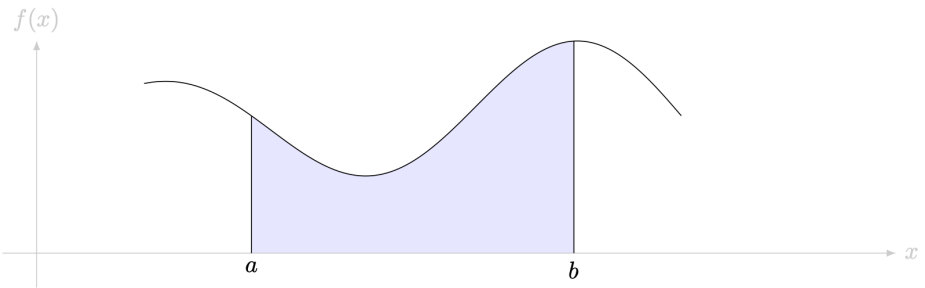

Figure 1: A representation of a definite integral. The area of the shaded region is equal to the integral $\int_a^b f(x)\,\mathrm{d}x$.

You can work out the value of definite integrals using geometry if the curve is sufficiently

well-behaved - a straight line or a circular arc.

Example 1

(a) Consider the area bounded by $f(x) = 2$, the x-axis, and the lines $x = 1$ and $x = 3$.

The distance between the two bounds is 2, and the distance between the lines $y = 0$ and $y = f(x)$ is also 2, so this area describes a square with side length 2. The area of this region is $A = 2 \cdot 2 = 4$. You can use the idea of the definite integral to say that

$$\int_1^3 2 \, \mathrm{d}x = 4.$$

(b) You are given the area bounded by $f(x) = x$, the x-axis, and the lines $x = 0$ and $x = 3$.

As the shaded region is a triangle with base and height both 3, the area is $A = (3 \cdot 3)/2 = 9/2$. You can use the idea of a definite integral to say that

$$\int_0^2 x \, \mathrm{d}x = 2.$$

Important

Both examples here involve functions which take **positive outputs**, and so the definite integral is actually equal to the area. It is worth reminding yourself at this point that the answer is the **signed** area bounded by the curve, the limits, and the y-axis. Here's another figure which demonstrates this.

The figure below shows a representation of a definite integral. Here, the value of the integral A corresponds to the area shaded in blue, as the function is always above the x-axis.

Indefinite integrals and antiderivatives

Evaluating areas gets more complicated if your curve is not a straight line. For instance, how would you work out the area bounded by $f(x) = x^2$, the limits $x = 0$ and $x = 1$ and the x-axis? In other words, what is the value of the definite integral of $f(x) = x^2$ between 0 and 1 with respect to x? You can work this out by finding the **antiderivative**.

Finding the derivative $f'(x)$ of a function $f(x)$ is well defined; see Guide: Introduction to

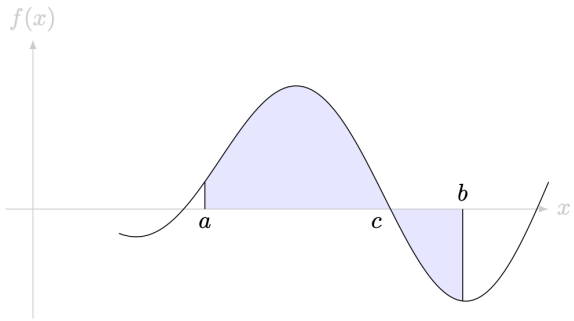

Figure 2: A representation of a definite integral. The integral $\int_a^b f(x)\,\mathrm{d}x$ is equal to the bounded area above the curve minus the bounded area below the curve.

differentiation and the derivative for more. The question of finding the antiderivative is as follows: given a function $f(x)$, is there a function $F(x)$ such that $F'(x) = f(x)$? If this happens, then $F(x)$ is the **antiderivative of** $f(x)$.

It turns out that antiderivatives are very closely related to integration. To find the antiderivative $F(x)$ of $f(x)$, you can find the **indefinite integral of** $f(x)$ **with respect to** x; in other words

$$F(x) = \int f(x)\,\mathrm{d}x.$$

This means that indefinite integration is the **reverse process** of differentiation: to undo differentiation, do an indefinite integral; to undo an indefinite integral, find the derivative.

The constant of integration

You can ask yourself; is the antiderivative $F(x)$ of a function $f(x)$ unique? The answer here is **no**, it isn't. Here's an example to illustrate this process.

Example 2

You know from Guide: Introduction to differentiation and the derivative that the derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$. Because antiderivatives reverse differentiation, it follows that the antiderivative $F(x)$ of $\cos(x)$ is equal to $F(x) = \sin(x)$. You can check your answer to see that $F'(x) = \cos(x)$.
However, $F(x) = \sin(x)$ is not the only function that differentiates to give $\cos(x)$. Because the derivative of every constant real number with respect to x is equal to 0, it follows (for instance) that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x) + 19) = \cos(x) + 0 = \cos(x).$$

So the antiderivative for $\cos(x)$ could be equal to $\sin(x) + 19$ as well. Because the choice of 19 was random, **any** real number could take its place, and so $\cos(x)$ has **infinitely many antiderivatives**.
How can you encapsulate these infinitely in a single answer? The answer is to write $F(x) = \sin(x) + C$, where the C is any real number. This C is called the **constant of integration**, and is used to represent the infinitely many possible antiderivatives. Therefore, you can write that

$$\int \cos(x)\,\mathrm{d}x = \sin(x) + C.$$

Let's look at this more generally. If $F(x)$ is a function such that $F'(x) = f(x)$, then the indefinite integral (antiderivative) of $f(x)$ with respect to x is given by

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

where C, the **constant of integration**, is **any** real number.

In summary:

Important

When you work out a **indefinite integral** $\int f(x)\,\mathrm{d}x$, your final answer should be a **function** plus a **constant of integration** C. This expression represents all the possible antiderivatives of $f(x)$. You should **always** remember to add a $+C$ at the very end when finding an indefinite integral.

Tip

You can work out the value of the constant of integration if you have some extra information about x and $f(x)$. This extra information is known as a **boundary condition**.

The Fundamental Theorem of Calculus

Connecting the idea of the definite and indefinite integrals is the **Fundamental Theorem of Calculus**.

The Fundamental Theorem of Calculus

Suppose that $F(x)$ is a function with derivative $F'(x) = f(x)$. Then the following are true:

1. The antiderivative of $f(x)$ is given by

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

   where C is the constant of integration.

2. The value of the definite integral of $f(x)$ between the limits $x = a$ and $x = b$ is equal to

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

   that is, the difference between the values of $F(a)$ and $F(b)$.

Tip

For why the Fundamental Theorem of Calculus works, please go to [Proof sheet: The Fundamental Theorem of Calculus]

Part 1 of this theorem is covered in the section on indefinite integration above, and is a concrete statement that integration is the reverse process of differentiation. Part 2 connects the idea of definite and indefinite integration. So to find the value of a definite integral, you could follow these steps:

Step 1: Find an antiderivative $F(x)$ of $f(x)$. You do not need a $+C$.

Step 2: The result $F(x)$ can be put into square brackets, with the upper limit at the top

right and the lower limit at the bottom right. So you can write:

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b$$

Step 3: Work out $F(x)$ at $x = b$ and then work out $F(x)$ at $x = a$. You can then subtract $F(a)$ from $F(b)$ to get

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

The number at the end is the answer.

In summary:

> **!** Important
>
> For an **indefinite integral** $\int f(x)\,\mathrm{d}x$, your final answer should be a **function** plus a **constant of integration** C.
> For a **definite integral** $\int_a^b f(x)\,\mathrm{d}x$, your final answer should be a **number**. There should be **no** variables in your answer, and you do **not** need a $+C$ at the end.

Here's an example of the power of the Fundamental Theorem of Calculus.

> **i** **Example 3**
>
> Find the definite integral $\int_0^{\pi/4} \cos(x)\,\mathrm{d}x$.
> You could use the step-by-step process in finding the value of this definite integral.
> **Step 1:** As seen in Example 2, a function $F(x)$ such that $F'(x) = \cos(x)$ is $F(x) = \sin(x)$.
> **Step 2:** Now, you can write :
>
> $$\int_0^{\pi/4} \cos(x)\,\mathrm{d}x = \big[\sin(x)\big]_0^{\pi/4}$$
>
> **Step 3:** Here, $\sin(0) = 0$ and $\sin(\pi/4) = \sqrt{2}/2$. Therefore
>
> $$\int_0^{\pi/4} \cos(x)\,\mathrm{d}x = \big[\sin(x)\big]_0^{\pi/4} = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}$$
>
> The number at the end is the answer!

> **!** Important
>
> When working with trigonometric functions in calculus, you should **always use radians as your angular measure**. See Guide: Radians and Guide: Trigonometry (radians) for more.

Finding antiderivatives

From the fundamental theorem of calculus, the key skill in integration is finding antiderivatives of functions.

Here's the summary table of derivatives from Guide: Introduction to differentiation and the derivative

Function $f(x)$	Derivative $f'(x)$	Notes
$f(x) = c$	$f'(x) = 0$	
$f(x) = ax + b$	$f'(x) = a$	
$f(x) = ax^n$	$f'(x) = anx^{n-1}$	$n \neq 0$
$f(x) = ae^{bx}$	$f'(x) = abe^{bx}$	
$f(x) = a\sin(bx)$	$f'(x) = ab\cos(bx)$	
$f(x) = a\cos(bx)$	$f'(x) = -ab\sin(bx)$	
$f(x) = a\ln(bx)$	$f'(x) = \dfrac{a}{x}$	

You can use this table to reverse engineer antiderivatives for common functions.

For instance, suppose you wanted to find the antiderivative of the function $f(x) = ax^n$, where a is a constant and $n \neq -1$. You can use the fact that integration is the opposite of differentiation to say that the solution will be a function $F(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = ax^n$$

Taking $F(x) = \frac{ax^{n+1}}{n+1}$, and using the power rule for differentiation gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax^{n+1}}{n+1}\right) = \frac{a}{n+1} \cdot (n+1)x^n = ax^n$$

which is $f(x)$. Therefore

The antiderivative of ax^n for $n \neq -1$ is

$$\int ax^n \, \mathrm{d}x = \frac{ax^{n+1}}{n+1} + C \quad \text{for } n \neq -1.$$

This is sometimes known as the **power rule for integration**.

A special case happens when $n = 0$; if this happens, then $f(x) = ax^0 = a$ is a constant. You can use the formula above with $n = 0$ to say s $\int a \, \mathrm{d}x = \int ax^0 \, \mathrm{d}x = \frac{ax^1}{1} + C$ and so

The antiderivative of a for a a constant is is

$$\int a \, \mathrm{d}x = ax + C.$$

You may have asked yourself already why doesn't this process work for $n = -1$? These are functions of the form $f(x) = \frac{a}{x}$. It is because if you do the process with $n = -1$, you end up dividing by 0; which is not good. However, you know from the table of derivatives above that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a \ln(x)\right) = \frac{a}{x}$$

However, you need to make sure that $\ln(x)$ is defined; you know that the logarithm of a negative number is not defined (see Guide: Logarithms for more). You can get around this by writing $\ln|x|$ (where $|x|$ is the absolute value of x) instead of $\ln(x)$; this ensures the value of the input is positive. This means you can write

The antiderivative of $ax^{-1} = \dfrac{a}{x}$ is

$$\int ax^{-1} \, \mathrm{d}x = a \ln|x| + C.$$

Now, what about $f(x) = ae^{kx}$? You are looking for a function $F(x)$ such that $\dfrac{\mathrm{d}}{\mathrm{d}x}(F(x)) = ae^{kx}$. Here, taking $F(x) = \frac{1}{k}ae^{kx} + C$, and differentiating with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{k}ae^{kx} + C\right) = ae^{kx}$$

and so you can write

The antiderivative of ae^{kx} is

$$\int ae^{kx}\,\mathrm{d}x = \frac{1}{k}ae^{kx} + C.$$

Finally, you can consider antiderivatives of $a\sin(kx)$ and $a\cos(kx)$. Using rules of differentiation, you can say that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{k}a\cos kx\right) = a\sin(kx) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{k}a\sin kx\right) = a\cos(kx)$$

This means that you can write:

The antiderivative of $a\cos(kx)$ is

$$\int a\cos(kx)\,\mathrm{d}x = \frac{1}{k}a\sin kx + C$$

and the antiderivative of $a\sin(kx)$ is

$$\int a\sin(kx)\,\mathrm{d}x = -\frac{1}{k}a\cos kx + C.$$

Here's another table to summarize the work on antiderivatives.

Function $f(x)$	Antiderivative $\int f(x)\,\mathrm{d}x$	Notes
a	$ax + C$	a real
ax^n	$\frac{ax^{(n+1)}}{n+1} + C$	a real, $n \neq -1$
ax^{-1}	$a\ln\lvert x\rvert + C$	a real
ae^{kx}	$\frac{1}{k}ae^{kx} + C$	a, k real
$a\cos(kx)$	$\frac{1}{k}a\sin(kx) + C$	a, k real
$a\sin(kx)$	$-\frac{1}{k}a\cos(kx) + C$	a, k real

You can use the work done on antiderivatives to work out the elusive definite integral given at the start of the section on indefinite integration:

Example 4

Find the definite integral $\int_0^1 x^2 \, \mathrm{d}x$.

You could use the step-by-step process in finding the value of this definite integral.

Step 1: You can use the table of antiderivatives to find a function $F(x)$ such that $F'(x) = x^2$. Here, you have $a = 1$ and $n = 2$. Therefore, you can take $F(x) = \frac{x^3}{3}$.

Step 2: Now, you can write :

$$\int_0^1 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1$$

Step 3: Here, $0^3/3 = 0$ and $1^3/3 = 1/3$. Therefore

$$\int_0^1 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

and this is your answer!

Finally, here's an example of finding an antiderivative that may look like it hasn't been covered, but you can use the **laws of indices** (see Guide: Laws of indices) to write the integrand in a form which you can integrate.

Example 5

Find the indefinite integral $\int 2\sqrt{x} \, \mathrm{d}x$.

You can use the laws of indices to write the integral in terms of a power of x. Since $\sqrt{x} = x^{1/2}$, you can write

$$\int 2\sqrt{x} \, \mathrm{d}x = \int 2x^{1/2} \, \mathrm{d}x.$$

Next, you can use the power rule with $a = 2$ and $n = 1/2$ to write

$$\int 2x^{1/2} \, \mathrm{d}x = 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = 2\frac{x^{3/2}}{3/2} + C = \frac{4}{3}x^{3/2} + C.$$

Since this is an indefinite integral, **don't forget that constant of integration**!

Quick check problems

1. Answer the following questions true or false:

(a) The indefinite integral can be used to find the area under a curve.

(b) The answer to a definite integral is always a number.

(c) If $f(x) = a$, then the antiderivative of $f(x) = ax$.

(d) Indefinite integrals have limits.

(e) Differentiation is the reverse process of integration.

(f) The power of x in the antiderivative of $f(x) = \dfrac{1}{x^4}$ is -3.

2. Find the antiderivative of the following functions with respect to x.

(a) $f(x) = 3x^7$

(b) $f(x) = 4\cos(3x)$

(c) $f(x) = e^{-8x}$

Further reading

For more questions on the subject, please go to Questions: Introduction to integration.

For more about properties of integration, please see [Guide: Properties of integration].

For more about the theory of definite integration, including a formal definition and how to find a definite integral from first principles, please see [Proof sheet: Definite integrals]. For more about why the Fundamental Theorem of Calculus works, please see [Proof sheet: Fundamental Theorem of Calculus].

For more about techniques of integration, please see [Guide: Integration by substitution], [Guide: Integration by parts], and [Guide: Trigonometry and integration].

Version history

v1.0: initial version created 08/25 by tdhc.

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