Proof: Properties of matrix arithmetic

Jessica Taberner

Summary

Proof of commutativity of addition, associativity of addition and multiplication, and distributivity for matrices.

Before reading this proofsheet, it is recommended that you read [Guide: Introduction to matrices].

Proof

Remember from [Guide: Introduction to matrices] that a matrix is a rectangular array with entries in rows and columns.

> **A matrix**
>
> A $m \times n$ **matrix** is a rectangular array of mn entries set out in m **rows** and n **columns**. You can write it like so:
>
> $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
>
> This matrix has **dimension** $m \times n$.

In this sheet you will prove how some axioms hold when working with matrices, assuming those same axioms hold in the underlying set, that is the set the entries in our matrix are from. In [Guide: Introduction to matrices] you assumed that the entries in our matrix were from the real numbers, and that same assumption is used here.

Commutativity of addition proof

> **i** Commutativity of addition
>
> For any two matrices A and B of the same dimensions, matrix addition is commutative, that is,
> $$A + B = B + A.$$

Let A and B be two $m \times n$ matrices, where:

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}].$$

By definition of matrix addition, the sum of A and B is given by:

$$(A + B)_{ij} = a_{ij} + b_{ij},$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Since you are assuming the underlying set is the real numbers, it follows from the commutativity of the real numbers that:

$$a_{ij} + b_{ij} = b_{ij} + a_{ij} \quad \text{for all} \quad ij.$$

Using the definition of matrix addition, you can recognize:

$$b_{ij} + a_{ij} = (B + A)_{ij}.$$

So, for all entries, $(A + B)_{ij} = (B + A)_{ij}$, which implies:

$$A + B = B + A.$$

So, matrix addition is commutative.

Associativity of addition proof

> **Associativity of addition**
>
> For any three matrices A, B, and C of the same dimensions, matrix addition is associative, that is,
>
> $$(A+B)+C=A+(B+C).$$

Let A, B, and C be three $m \times n$ matrices, where:

$$A=[a_{ij}], \quad B=[b_{ij}], \quad C=[c_{ij}].$$

By definition of matrix addition:

$$(A+B)+C=[(a_{ij}+b_{ij})]+C.$$

Applying matrix addition again:

$$(A+B)+C=[(a_{ij}+b_{ij})+c_{ij}].$$

Since you are assuming that the underlying set is the real numbers, it follows by associativity in the real numbers that:

$$[(a_{ij}+b_{ij})+c_{ij}]=[a_{ij}+(b_{ij}+c_{ij})].$$

Using the definition of matrix addition again, you can notice:

$$[a_{ij}+(b_{ij}+c_{ij})]=A+[(b_{ij}+c_{ij})].$$

$$A+[(b_{ij}+c_{ij})]=A+(B+C).$$

Finally, you have that:

$$(A+B)+C=A+(B+C).$$

So matrix addition is associative.

Associativity of multiplication proof

> **Associativity of multiplication**
>
> For any three matrices A, B, and C with dimensions such that the following operations make sense, matrix multiplication is associative, that is,
>
> $$(AB)C = A(BC).$$

Let A be an $m \times n$ matrix, B be an $n \times p$ matrix, and C be a $p \times q$ matrix.

By definition of matrix multiplication, the product (AB) is an $m \times p$ matrix, whose entries are given by:

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Multiplying (AB) with C, you get:

$$((AB)C)_{il} = \sum_{k=1}^{p} (AB)_{ik} c_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl}.$$

Since i and j aren't dependent on k, and k and l aren't dependent on j, you can rearrange the summation order:

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} = \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kl} \right).$$

For a more in-depth look at how, when, and why you can do this, please read [Guide: Further summation notation].

Using the definition of matrix multiplication, you can notice that:

$$\sum_{r=1}^{p} b_{jr} c_{rk} = (BC)_{jk}.$$

So, substituting that into your previous equation:

$$\sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kl} \right) = \sum_{j=1}^{n} a_{ij} (BC)_{jl}$$

Using the definition of matrix addition, you have:

$$\sum_{j=1}^{n} a_{ij} (BC)_{jl} = (A(BC))_{il}$$

Finally, you can conclude:

$$(AB)C = A(BC).$$

So matrix multiplication is associative where it makes sense.

Distributive property for matrices proof

> **Distributive property for matrices**
>
> For any three matrices A, B, and C with compatible dimensions, matrix multiplication has the distributive property:
>
> $$A(B + C) = AB + AC \quad \text{or} \quad (B + C)A = BA + CA.$$

Let A be an $m \times n$ matrix, B be an $n \times p$ matrix, and C be a $n \times p$ matrix.

By definition of matrix addition:

$$(B + C)_{ij} = b_{ij} + c_{ij}.$$

Multiplying by A:

$$(A(B + C))_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}).$$

Using the distributive property of real numbers:

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj}).$$

Splitting the sum:

$$\sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}.$$

Recognizing the definitions of matrix products:

$$\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} = (AB)_{ij} + (AC)_{ij}.$$

Finally, you have:

$$(A(B + C))_{ij} = (AB)_{ij} + (AC)_{ij}$$

So, $A(B + C) = AB + AC$. Similarly, you can prove $(B + C)A = BA + CA$. So, matrix multiplication distributes over addition.

Further reading

For more on this topic, please go to [Guide: Introduction matrices].

Version history

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