Introduction to Gaussian elimination

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Summary

Gaussian elimination is integral to solving systems of simultaneous linear equations. You can use Gaussian elimination to get a 'nicer' equivalent of a matrix.

Before reading this guide, it is recommended that you read Guide: Introduction to Matrices and Guide: Introduction to fractions. For the last section of this guide (Using Gaussian elimination to solve systems of equations) it is recommended that you read Guide: Introdution to solving simultaneous equations.

What is Gaussian elimination?

Gaussian Elimination is a method used to transform matrices into 'nicer' forms so that you can get some information from the matrix. This nicer form is usually an 'upper triangle' where you only have non-zeros in the entries to right of the main diagonal, and the entries to the left are zeros (sometimes you can have more zeros than that). This is very usefull to find the inverse of a matrix or to solve many linear equations at once. In this guide you will study the operations you can use to do Gaussian elimination and then you will discover the row echelon form and the reduced row echelon form, and how to get them using Gaussian elimination.

Definition of row echelon form and of reduced row echelon form

A matrix is in **row echelon form** (or upper triangular form) if:

(a) All the zero rows are at the bottom (this does not mean there are any zero rows!).

(b) The first entry in a (non-zero) row - this is the leading entry - is to the right of the first entry of the row above it.

Example 1

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

The leading entries of A are 2, 5 and 9.

Note

A matrix is in **reduced row echelon form** if:

(a)In each non-zero row, the leading entry of the row is 1, and

(b)In each column that contains a leading entry for some row, all other entries in that column are zero.

Example 2

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The leading entries of B are the two 1s of the matrix and B has one zero-row.

Elementary row operations (e.r.o.s)

In order to get to the row echelon form or reduced echelon form you have to do **row operations**. There are three elementary row operations that make sure the new matrix is 'equivalent' to the original one:

(E1) Multiplying a row number by a scalar λ

When multiplying a row number by a scalar you 'freeze' all the other rows of the matrix, and multiply the values of the row you have chosen by the scalar λ. This gives you a modified matrix where only the row you choose to multiply has changed.

(E2) Add one multiple of one row to another row

You can multiply one of the rows by a scalar λ and then add it or subtract it to another row. This can be very useful when you want to get zeros in certain entries of your matrix.

(E3) Interchanging any two rows

You can exchange the order of the rows in your matrix to get to the upper triangular form or to move the zero-rows to the bottom.Altering the order of the rows does not change the essential properties of the matrix.

Fun fact

It is also possible to do column operations! These are less used, and you usually transpose (turn columns into rows) the matrix and do row operations, but column operations exist!

Example 3

Let

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix}$$

Multiply the second row by 2 (E1):

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_2 \to 2r_2 \begin{bmatrix} 3 & 0 & 3 \\ 4 & 4 & 12 \\ 4 & 0 & 9 \end{bmatrix}$$

You can see only row 2 has changed and now is a scalar multiple of the original row 2.

Example 4

Let A be the same matrix as in Example 1. You can now try E2: adding one multiple of one row to another row. Suppose you decide to add $\frac{1}{2}r_2$ to r_1,you will be adding $[1 \quad 1 \quad 3]$ to $[3 \quad 0 \quad 3]$ such that:

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_1 \to r_1 + \tfrac{1}{2}r_2 \begin{bmatrix} 4 & 1 & 6 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix}$$

The matrix remains unchanged except for the first row, that is now a sum of the original row 1 and a half of row 2.

Example 5

Finally you can interchange two rows (E3). In this example you interchange row 3 and row 1 of matrix A.

$$\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 6 \\ 4 & 0 & 9 \end{bmatrix} r_1 \leftrightarrow r_3 \quad \begin{bmatrix} 4 & 0 & 9 \\ 2 & 2 & 6 \\ 3 & 0 & 3 \end{bmatrix}$$

Warning

The matrix you get after doing e.r.o.s on a matrix is not equal to the original one! You can say they are equivalent and write $A \sim B$, where B is the matrix in row echelon form.

Reducing a matrix to echelon form (or reduced echelon form)

All these e.r.o.s can always be undone (swap back, divide, subtract), so no information about the original matrix disappears. You can use elementary row operations to find the echelon form or the reduced echelon form of a matrix.

You can do this in a series of steps:

Step 1: Identify the zeros of your matrix. If there are any all-zero rows interchange rows using E3 until all the zero rows are at the bottom of the matrix, so that the rows with values can be part of the upper triangle in the top rows.

Step 2: You can save some time if before doing other e.r.o.s you look at the leading entries of each row. Again interchange the rows of your matrix until the first entry in each row is under or to the right of the first entry in the row above it.

Step 3: Now it is time to start using other e.r.o.s. Focus on the leading entries, add or subtract multiples of different rows using E2 so that, for any leading entries in the same column, the lower leading entry becomes zero. Try to use rows with more zero-entries to start doing these operations so that you have to make less calculations. You don't need to worry about having negative numbers or fractions in the other entries of such rows when you do this. You're writing the row different, but it is still equivalent to the original row.

Step 4: Repeat Step 3 until you have the echelon form.

Tip

These steps are interchangeable and sometimes different steps work for different people! Experience helps you to shape your own approach and decide which e.r.o.s to do first. Many approaches can work! But the above guide is useful if you are very unsure about how to proceed.

Now you are going to see a few examples of how you can use e.r.o.s to get the reduced echelon form. The first two will be very clearly following the steps mentioned above, the others will have a more flexible approach.

Example 6

Let

$$C = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Step 1: Identify the zeros of your matrix and if there are any zero rows, move these to the bottom. Notice row 2 has only zeros, so you can interchange it with row 3 it using E3.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} r_2 \leftrightarrow r_3 \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2: Look at the leading entries and see if there is any that could be the upper corner corner of the upper triangular form. Row 2 has leading entry 1 in the first column, this works! Interchange row 2 and row 1 using E3.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row 1 is in the first line and has a leading entry in the first column, so that row is already in the right form. Continue following the steps to know what to do next.
Step 3: Now you need to focus on the leading entries, to make sure each leading entry is to the right of the leading entry above it. The leading entry in row 2 is 2, and is in the first column, so you need to perform e.r.o.s to fix this. If you subtract two times row 1 to row 2 (E2), the leading entry of row 2 will change:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} r_2 \to r_2 - 2r_1 \begin{bmatrix} 1 & 0 & 2 \\ 0 & -6 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 4 Repeat Step 3 to check all the leading entries. The leading entry of row 1 is ok - it is the first leading entry of the matrix and in the first column. The leading entry in row 2 is -6 and is in column three, so on the right of the leading entry above it, so row 2 is ok. Row 3 is an all zeros row so it has no leading entry. So you are done! The row echelon form of the matrix is:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Example 7

Let

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There are no zero-rows so **Step 1** is done. All the rows have leading entry in the first column, so you can wait before doing **Step 2**. Now you can start **Step 3**. Any leading entries that can be useful? Look at row 3. It only has one entry and it is on the far left, you can use this to get 0 on the other entries of the first column and, since it only has one entry, you can focus on only that column for each operation you do with row 3. Start by doing this with row 1:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \to r_1 - r_3 \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Following **Step 4** repeat **Step 3** until you have the row echelon form. Do the same for row 2. Because the leading entry of row 2 is 2, you need to multiply row 3 by a scalar 2 when you subtract it from row 2.

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \to r_2 - 2r_3 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Great! Now there is only one entry on row 3, if you interchange row 1 and row 3, you can get the first line of the upper triangle form (left corner of the triangle)!

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} r_1 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, if you look at four in row 2, you can see that it only has one entry, in the middle column. This is good since it means you don't need to use the other rows to make sure the first entry is to the right of the row above it. Finally row 3 has its leading entry on the same column as the leading entry of 4, so you need to make that entry zero. To get the row echelon form, subtract $\frac{1}{4}$ of row 2 from row 3:

You don't always need to be so rigurous with the steps when doing Gaussian elimination, here are two examples where the approach is a bit more flexible. ::: {.callout-note appearance="simple"} ## Example 8

Let

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Following the first step identify all the zeros of the matrix. Since the first row is a zero-row, you should move it to the bottom. So the first e.r.o. you perform is interchanging rows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} r_1 \leftrightarrow r_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now row 1 has the leading entry on the first column so it is ok for row echelon form. However, row 2 has its leading entry directly under the leading entry of row 1. To fix this, you can perform the second e.r.o., subtract 5 times row 1 to row 2.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} r_2 \to r_2 - 5r_1 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the row echelon form of D is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Now you will see a couple of examples where you get the reduced echelon form. To do this, once you get the row echelon form, you will have to keep doing e.r.o.s until all leading entries are 1 and the other entries in the columns where there are leading entries are 0.

Example 9

Let

$$F = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

There are no zero-rows in this matrix, so you can start looking at leading entries. Using the first entry of row 3 you can make the rest of the first column 0 (you use row 3 because the leading entry is 1).

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 0 \end{bmatrix} r_2 \to r_2 - 2r_3 \quad \begin{bmatrix} 4 & 2 & 1 \\ 0 & -4 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

and similarly with row 1:

$$\begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -10 & 1 \end{bmatrix}$$

Now you can try an example with a matrix that is larger. This usually means more e.r.o.s!

Quick check problems

The `echo: false` option disables the printing of code (only output is displayed).

Further reading

For more questions on the subject, please go to Questions: Gaussian elimination.

For an introduction on matrices, please see Guide: Introduction to Matrices. For an introduction systems of linear equations, please see Guide: Solving systems of linear equations.For examples of using Gaussian elimination, please see Guide: Applications of Gaussian elimination

Version history

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