Solving equations involving logarithms

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Summary

This guide discusses the idea of rearranging and solving equations involving logarithms, which is a key skill in both mathematics and many areas of science.

Before reading this, it is strongly recommended that you read Guide: Introduction to logarithms. Examples 7 and 8 require solving quadratic equations: see Guide: Factorization or Guide: Using the quadratic formula for more on this.

In Guide: Introduction to logarithms, you learned about the different ways that you can manipulate expressions involving logarithms, such as $\log_2(15)$ and $\ln(x^3)$. In a more general setting however, this knowledge may not be enough; you will need to apply the ideas of logarithms to help you solve equations.

Solving equations involving logarithms is a key skill in areas whenever logarithmic concepts play a role: economics, physics (particularly acoustics, where the decibel is a logarithmic measure), chemistry (pH is a logarithmic measure), biology, mathematics and statistics, computational complexity, and even music theory!

Before you dive in to solving logarithms, it is worth restating the recommendation above:

> ❗ Important
>
> **This guide assumes a prior knowledge of logarithms and the laws of logarithms. Please make sure that you have read Guide: Introduction to logarithms before continuing.**
>
> The numbering of the laws will follow the numbering in this guide and on Factsheet: Laws of logarithms.

Discussion of techniques

In general, the first golden rule of solving equations involving logarithms is the following:

> 💡 **First golden rule**
>
> **Make sure all logarithms are in the same base before simplifying**. To do this (if applicable) you could use the **change of base rule**.
> This is because Laws 1 and 2 of the laws of logarithms can only be applied when the logarithms have the same base.

It may be that you have logarithms in different bases in your equation; so how can you make them to be in the same base? The answer is the **change of base** rule (Law 6) which you can use in this scenario.

You have seen in Guide: Introduction to logarithms that **logarithms undo exponentiation and exponentiation undoes logarithms**. More formally, you have that for all $a > 0$ with $a \neq 1$ and all real numbers y (where the expression is defined):

$$a^{\log_a(y)} = y \qquad \text{and} \qquad \log_a(a^y) = y$$

However, this only works if there is one term on each side of the equation. For instance, you can notice that

$$\log_a(x) + \log_a(y) = \log_a(z)$$

does **not** automatically imply that $x + y = z$. This is because looking at Law 1 of the laws of logarithms, it follows that

$$\log_a(x) + \log_a(y) = \log_a(xy) = \log_a(z)$$

and so it actually implies that $xy = z$.

So once the first golden rule has been applied and all logarithms are in the same base, then the second golden rule of solving equations involving logarithms is the following:

> 💡 **Second golden rule**
>
> **There should not be a coefficient in front of any logarithms, and you need exactly one term on each side of the equation. Then, and only then, can you undo the logarithm by exponentiation.**

So for instance, if you are solving an equation involving logarithms, you can isolate the variable on one side of the equation, and have a constant on the other (see Guide: Introduction to rearranging equations for more). Perhaps it looks like $\log_a(x) = b$. From there, you can exponentiate both sides with base a to get

$$a^{\log_a(x)} = x = a^b$$

and so $x = a^b$.

However, please be aware of the following:

> ⚠ **Warning**
>
> **You cannot take logarithms of a negative number!**

So if you are solving an equation involving logarithms, you need to check if the solution is *valid* by putting any answers into your original equation. This is outlined in Examples XX and XX.

Initial examples

The first four examples examine exactly where the golden rules are used.

> **ℹ Example 1**
>
> Solve $\log_{10}(5x + 7) = 1$ for x.
>
> Here, there is only one logarithm and so all logarithms are in the same base, which agrees with the first golden rule from above.
>
> Next, you can notice there is exactly one term on both sides of the equation, which agrees with the second golden rule from above.
>
> Now, you can undo the logarithm by exponentiating both sides to base 10. This gives
>
> $$\log_{10}(5x + 7) = 1$$
> $$10^{\log_{10}(5x+7)} = 10^1$$
> $$5x + 7 = 10$$
>
> This is a linear equation in x that you can solve by rearranging for x. From there, you can subtract 7 from both sides, giving $3 = 5x$ and then divide both sides by 5, giving $x = \dfrac{3}{5}$ as a final answer.

You may also come across equations which are similar to the one above, but hidden through laws of logarithms. Here are two examples which require this kind of manipulation.

ⓘ Example 2

Solve

$$\log_2(4t) - \log_2(3) = 2$$

for t.

Here, all logarithms are in the same base (2), which agrees with the first golden rule from above.

There are two terms on the left hand side, which doesn't agree with the second golden rule. You will need to use the laws of logarithms to combine the two logarithms into one term. Using Law 2 (see Factsheet: Laws of logarithms) on the left hand side of the equation gives

$$\log_2(4t) - \log_2(3) = \log_2\left(\frac{4t}{3}\right)$$

The equation then becomes

$$\log_2\left(\frac{4t}{3}\right) = 2$$

which agrees with the second golden rule as there is one term on both sides of the equation.

This means you can now undo the logarithm. Exponentiating both sides with base 2 gives

$$\log_2\left(\frac{4t}{3}\right) = 2$$
$$2^{(4t/3)} = 2^2$$
$$\frac{4t}{3} = 4$$

To finish solving, you can divide both sides by 4 and multiply both sides by 3. This gives $t = 3$.

> **ℹ Example 3**
>
> Rearrange
> $$z = 4\log_3(x) + \log_3(y)$$
> for x.
>
> Here, all logarithms are in the same base (3), which agrees with the first golden rule from above.
>
> There are two terms on the right hand side, which doesn't agree with the second golden rule. You will need to use the laws of logarithms to combine the two logarithms into one term. Since there is a $+$, Law 1 seems appropriate. However, the statement of Law 1 doesn't have coefficients in the logs; you need to deal with that 4 before you can do this.
>
> Using Law 3, you can write that $4\log_3(x) = \log_3(x^4)$ and so the equation becomes
> $$z = 4\log_3(x) + \log_3(y) = \log_3(x^4) + \log_3(y)$$
>
> It's now you can use Law 1 to combine the logarithms to get
> $$z = \log_3(x^4) + \log_3(y) = \log_3(x^4 y)$$
>
> The equation is now $z = \log_3(x^4 y)$ which agrees with the second golden rule as there is one term on both sides of the equation.
>
> This means you can now undo the logarithm. Exponentiating both sides with base 3 gives
> $$z = \log_3(x^4 y)$$
> $$3^z = 3^{\log_3(x^4 y)}$$
> $$3^z = x^4 y$$
>
> To finish solving, you can divide both sides by y to get $x^4 = 3^z/y$. You can now take fourth roots of both sides to get the final answer
> $$x = \sqrt[4]{\frac{3^z}{y}}.$$

This example will make use of the change of base formula stated above (which is Law 6 of the laws of logarithms). Here's a reminder of the change of base formula from Factsheet: Laws of logarithms.

> **ℹ Change of base formula**
>
> $$\log_a(x) = \frac{\log_k(x)}{\log_k(a)}$$
>
> or alternatively
>
> $$\log_k(a)\log_a(x) = \log_k(x)$$

> **i** **Example 4**
>
> Rearrange $\log_9(x + y) = \log_3(x)$ for y.
>
> Here, not all logarithms are in the same base, so you will need to use the change of base formula. You can either change everything into base 3 or everything into base 9. Since the inside of the right hand side logarithm is simpler than the left hand side, let's change that into base 9.
>
> Using the change of base rule with $a = 3$ and $k = 9$ gives
>
> $$\log_3(x) = \frac{\log_9(x)}{\log_9(3)}.$$
>
> Now, you can see that as $9^{1/2} = 3$, then $\log_9(3) = 1/2$. This means that $\log_3(x) = \frac{1}{1/2}\log_9(x) = 2\log_9(x)$. Putting this into the equation gives
>
> $$\log_9(x + y) = 2\log_9(x).$$
>
> Now all of the logarithms are in the same base, satisfying the first golden rule. However, the logarithm on the right hand side still has a coefficient, which you can incorporate into the logarithm using Law 3. Doing this, you can write that $2\log_9(x) = \log_9(x^2)$ and so the equation becomes
>
> $$\log_9(x + y) = \log_9(x^2)$$
>
> The equation now which agrees with the second golden rule as there is one term on both sides of the equation, and there are no coefficients on the logarithms.
>
> This means you can now undo the logarithm. Exponentiating both sides with base 9 gives
> $$\log_9(x + y) = \log_9(x^2)$$
> $$9^{\log_9(x+y)} = 9^{\log_9(x^2)}$$
> $$x + y = x^2$$
>
> To finish solving, you can take x from both sides to get $y = x^2 - x$ and this is your final answer.

Here's an example that combines many laws of logarithms:

Example 5

If

$$\log_8(x) + \log_2(y) = \log_2(4y + 1),$$

represent y in terms of x.

Here, not all logarithms are in the same base, so you will need to use the change of base formula. You can either change everything into base 2 or everything into base 8. Since there is only one logarithm in base 8 and two in base 2, it's less work to change everything into base 2.

Using the change of base rule with $a = 8$ and $k = 2$ gives

$$\log_8(x) = \frac{\log_2(x)}{\log_2(8)}.$$

Now, you can see that $\log_2(8) = 3$, as $2^3 = 8$. This means that $\log_8(x) = \frac{1}{3}\log_2(x)$. Putting this into the equation gives

$$\frac{1}{3}\log_2(x) + \log_2(y) = \log_2(4y + 1).$$

Now all of the logarithms are in the same base, satisfying the first golden rule. There are two terms on the left hand side, which doesn't agree with the second golden rule. You will need to use the laws of logarithms to combine the two logarithms into one term. Since there is a $+$, Law 1 seems appropriate. However, like as in Example 3, the statement of Law 1 doesn't have coefficients in the logs; you need to deal with that 4 before you can do this.

Using Law 3, you can write that $\frac{1}{3}\log_2(x) = \log_2(x^{1/3})$ and so the equation becomes

$$\log_2(x^{1/3}) + \log_2(y) = \log_2(4y + 1)$$

It's now you can use Law 1 to combine the logarithms to get

$$\log_2(x^{1/3}) + \log_2(y) = \log_2(x^{1/3}y)$$

The equation is now $\log_2(x^{1/3}y) = \log_2(4y+1)$ which agrees with the second golden rule as there is one term on both sides of the equation.

This means you can now undo the logarithm. Exponentiating both sides with base 2 gives

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$$\log_2(x^{1/3}y) = \log_2(4y + 1)$$

$$2^{\log_2(x^{1/3}y)} = 2^{\log_2(4y+1)}$$

Solving logarithmic equations

The following extra examples on solving logarithmic equations will explain the process in a more natural and less structured manner. These are designed

> **ⓘ Example 6**
>
> If $\log_3(2x - 3) = \log_9(x^2)$, what is x?
> In this case, you need to follow the first golden rule and make sure that both of your logarithms are to the same base. Since $9 = 3^2$, it makes sense to change the base of the right hand side, which is $\log_9(x^2)$. Using the change of base rule gives
>
> $$\log_9(x^2) = \frac{\log_3(x^2)}{\log_3(9)}$$
>
> Since $9 = 3^2$, it follows that $\log_3(9) = 2$ and so
>
> $$\frac{\log_3(x^2)}{\log_3(9)} = \frac{1}{2}\log_3(x^2)$$
>
> By the laws of logs and the laws of indices:
>
> $$\frac{1}{2}\log_3(x^2) = \log_3((x^2)^{1/2}) = \log_3(x)$$
>
> So the equation is now $\log_3(2x - 3) = \log_3(x)$ which is easier to deal with. Exponentiating both sides with base 3 gives
>
> $$\log_3(2x - 3) = \log_3(x)$$
> $$3^{\log_3(2x-3)} = 3^{\log_3(x)}$$
> $$2x - 3 = x$$
>
> Rearranging this equation gives $x = 3$.
> Since the equation started with logarithms, you need to check if this gives valid solutions. When $x = 3$, then both $2x - 3$ and x^2 are positive; so in this case the equation is defined and $x = 3$ is a valid solution.

> **i Example 7**
>
> Solve $2\ln(x-4) = \ln(10/3 - x) + \ln(3)$.
>
> Here, you'll need to combine the logs on the right hand side. It also can't hurt to get rid of that 2 as well. So, by the laws of logs, you can rewrite the equation as
>
> $$2\ln(x-4) = \ln(10/3 - x) + \ln(3)$$
> $$\ln(x-4)^2 = \ln\left(3(10/3 - x)\right)$$
> $$\ln(x-4)^2 = \ln(10 - 3x)$$
>
> Exponentiating both sides with base e gives:
>
> $$\ln(x-4)^2 = \ln(10 - 3x)$$
> $$e^{\ln(x-4)^2} = e^{\ln(10-3x)}$$
> $$(x-4)^2 = (10 - 3x)$$
>
> Expanding the brackets on both sides and simplifying gives
>
> $$(x-4)^2 = (10 - 3x)$$
> $$x^2 - 8x + 16 = 10 - 3x$$
> $$x^2 - 5x + 6 = 0$$
>
> You can factorize this quadratic equation (see Guide: Factorization for more) or use the quadratic formula (see Guide: Using the quadratic formula) to get $(x-2)(x-3) = 0$, so $x = 2$ or $x = 3$.
>
> However, **neither of these are valid solutions to the equation**! The original equation is
> $$2\ln(x-4) = \ln(10/3 - x) + \ln(3)$$
>
> and so any potential solution has to be greater than 4; this is because you can't have the logarithm of a negative number. Since $x - 4$ is negative for both $x = 2$ and $x = 3$, you can say that there are **no real solutions** to this equation.

Here's another example which has.

ℹ Example 8

Solve $\log_{10}(4) = \log_{10}(x+1) + \log_{10}(x-4)$.

Using the laws of logs on the right hand side gives

$$\log_{10}(x+1) + \log_{10}(x-4) = \log_{10}(x+1)(x-4)$$

and so, exponentiating both sides to base 10:

$$\log_{10} 4 = \log_{10}(x+1)(x-4)$$
$$10^{\log_{10} 4} = 10^{\log_{10}(x+1)(x-4)}$$
$$4 = (x+1)(x-4)$$

Expanding the brackets and rearranging gives

$$4 = (x+1)(x-4)$$
$$4 = x^2 - 3x - 4$$
$$0 = x^2 - 3x - 8$$

You can use the quadratic formula to solve this quadratic equation (see Guide: Using the quadratic formula) with $a = 1$, $b = -3$ and $c = -8$ to get

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-8)}}{2(1)} = \frac{3 \pm \sqrt{41}}{2}.$$

However, you need to check if these answers are valid! You can check to see that $\sqrt{41} \approx 6.40312$, so $\sqrt{41} > 6$.

- You can say that $3 + \sqrt{41} > 9$, and so $(3 + \sqrt{41})/2 > 9/2 > 4$. As both $\log_{10}(x+1)$ and $\log_{10}(x-4)$ are defined when $x > 4$, it follows $x = (3 + \sqrt{41})/2$ is a valid solution.
- Next, you can say that $3 - \sqrt{41} < -3 < 0$. Since $\log_{10}(x-4)$ is undefined for any $x \leq 4$, it follows that $x = (3 - \sqrt{41})/2$ is not a valid solution.

So the only solution to this equation is $x = (3 + \sqrt{41})/2$.

Quick check problems

1. Solve the following logarithmic equations.

(a) $\log_3 5x + 4 = 2$

(b) $\log_{10} 5x - \log_{10} 9 = 3$

(c) $2\log_4(3) + \log_2(t + 5) = 2$

2. Solve the expression $\log_{10}(p + 6) = 2\log_{10}(p - 6)$, making sure to comment on the validity of the solutions.

Further reading

For more questions on the subject, please go to Questions: Solving equations involving logarithms.

Version history and licensing

v1.0: initial version created 08/23 by Ellie Gurini, Krish Chaudhary, Mark Toner as part of a University of St Andrews STEP project, and updated 09/25 by tdhc.