Further sigma notation

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Summary

Sigma notation is used to express many additions at once. Understanding what this notation is, how it works, and how to manipulate them is a valuable skill to learn for use in almost any area of mathematics.

Properties

In this section you will learn about a few properties of sigma notation which means you'll have a toolkit to rearrange sums!

The first property you'll learn about sigma notation is *distribuitivity*. This property allows you to take constants from inside the sigma notation to outside the summation.

> **Distribuitivity**
>
> Let $a_k, a_{k+1}, \dots, a_n$ be a sequence of numbers (where k and n are integers with $k \leq n$) and C be any constant. Then
>
> $$\sum_{i=k}^{n} Ca_i = C \sum_{i=k}^{n} a_i.$$

You can see this is true by writing the entire sum out, like this:

$$\begin{aligned}
\textstyle\sum_{i=k}^{n} Ca_i &= Ca_k + Ca_{k+1} + Ca_{k+2} + \dots + Ca_n \\
&= C(a_k + a_{k+1} + a_{k+2} + \dots + a_n) \\
&= C \textstyle\sum_{i=k}^{n} a_k
\end{aligned}$$

> **Example 7**
>
> What is the value of $\sum_{n=2}^{5} 6n^2$?
> Using distributivity, $\sum_{n=2}^{5} 6n^2 = 6 \sum_{n=2}^{5} n^2$. From Example 2, you know that $\sum_{n=2}^{5} n^2 = 54$. Therefore, $6 \sum_{n=2}^{5} n^2 = 6 \times 54 = 324$.

Double sums

Sometimes, you'll want to multiply two sums together. This can be written succinctly using something called *double sums*.

Double sums

Let $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ be two sequences of numbers (where k, n, t, and m are integers with $k \leq n$ and $t \leq m$). Then the **double sum** $\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j$ is defined as

$$\begin{aligned}\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} \\ &\quad + \dots + a_{k+1} b_m + \dots + a_n b_m.\end{aligned}$$

Tip

You might find it easier to remember the above by thinking of $\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j$ as $a_1(\sum_{j=t}^{m} b_j) + a_2(\sum_{j=t}^{m} b_j) + \dots + a_n(\sum_{j=t}^{m} b_j)$.

You will now see how this relates to multiplying two sums together. Suppose that $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ are like above, and consider the product $(\sum_{i=k}^{n} a_i)(\sum_{j=t}^{m} b_j)$. Writing it all out and performing the multiplication, you get

$$\begin{aligned}\left(\sum_{i=k}^{n} a_i\right)\left(\sum_{j=t}^{m} b_j\right) &= (a_k + a_{k+1} + \dots + a_n)(b_t + b_{t+1} + \dots + b_m) \\ &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} + \\ &\quad \dots + a_{k+1} b_m + a_{k+2} b_t + \dots + a_n b_m \\ &= \sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j\end{aligned}$$

You can write this as a result:

Double sums and products of two sums

Let $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ be two sequences of numbers (where k, n, t, and m are integers with $k \leq n$ and $t \leq m$). Then

$$\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j = (\sum_{i=k}^{n} a_i)(\sum_{j=t}^{m} b_j).$$

Example 10

Write $(1+2+3+4+5+6)(2+4+6+8+10+12)$ as a double sum and as a product of two sums.

First, notice you can write out the above expression in the form $(1)(2)+(1)(4)+ ...(1)(12)+(2)(2)+(2)(4)...(3)(2)+...(6)(12)$

From the definition above you may now rewrite the expression to the double sum

$$\sum_{i=1}^{6}\sum_{j=1}^{6} i*2j$$

using the distrubitivity property this can be written as

$$2\sum_{i=1}^{6}\sum_{j=1}^{6} ij$$

This can then be written using the product of two sums rule above to

$$2\sum_{i=1}^{6} i\sum_{j=1}^{6} j$$

It is evident that the two sums are the same with different index variables this means that they can be combined to form

$$2\sum_{k=1}^{6} k^2$$

k has been used to differentiate the new sum from the ones involving i and j before but as always the choice of index variable is relatively unimportant

Quick check problems

1. What is the value of $\sum_{i=2}^{6} i$.

Answer: The value of the above is: ___.

2. Given $\sum_{j=1}^{100} i$ Identify the index of the sum.

Answer: The index is _

3. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) The sum $3+6+9+12$ can be expressed as $\sum_{i=0}^{4} 3i$ Answer: TRUE / FALSE.

(b) The sum $-1+1-1+1$ can be expressed as $\sum_{i=1}^{4} -i$ Answer: TRUE / FALSE.

(c) $\sum_{i=1}^{100} i = \sum_{i=0}^{101} i$ Answer: TRUE / FALSE.

(d) $\sum_{i=1}^{100} 6i = 6\sum_{i=0}^{100} i$ Answer: TRUE / FALSE.

(e) $\sum_{i=1}^{100} 9i + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 27i^2$ Answer: TRUE / FALSE.

(f) $\sum_{i=1}^{100} 12i - \sum_{i=1}^{100} 4i = 8\sum_{i=1}^{100} i$ Answer: TRUE / FALSE.

4. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) $\sum_{i=1}^{10}\sum_{j=2}^{6} ij$ can be expressed as $\left(\sum_{i=2}^{6} i\right)\left(\sum_{j=1}^{10} j\right)$ Answer: TRUE / FALSE.

(b) $\left(\sum_{i=1}^{5} 2i\right)\left(\sum_{j=5}^{10} 3j\right)$ can be expressed as $6\left(\sum_{i=1}^{5}\sum_{j=5}^{10} ij\right)$\$ Answer: TRUE / FALSE.

(c) The sum $(1+2+3+4+5+6)(-1-2)(3+6+9)$ can be expressed as $\sum_{i=1}^{6}\sum_{j=1}^{2}\sum_{k=1}^{3} -3ijk$\$

Answer: TRUE / FALSE.

Further reading