Introduction to matrices

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Summary

Matrices are rectangular arrays of mathematical objects with entries arranged in rows and columns. Matrices are a very useful concept within mathematics, you'll see them used for solving simultaneous equations and much more, this guide will explain what they are and how to perform arithmetic with matrices.

What is a matrix?

A matrix is a rectangular array or table, with entries in rows and columns. Understanding matrices can make solving equations more efficient and can open the door to learning much more mathematics.

If you have read [Guide: Introduction to solving simultaneous equations] then one way of thinking of matrices is as an array encoding the coefficients of the variables of your simultaneous equations.

Matrices are a fundamental tool within linear algebra, and they have a wide range of real-life applications. They are used in computer graphics, data analysis, search engine optimization, cryptography, economics, robotics, genetics, quantum mechanics, and many more areas of study. Matrices are used anywhere where information needs to be analyzed and calculated efficiently.

In this guide, you will see how you can: read, write, and understand matrices; learn how to do addition, subtraction and scalar multiplication with matrices; and be able to identify some special matrices.

Working with matrices

Definition of a matrix

A $m \times n$ **matrix** A is a rectangular array of mn many numbers (called **entries**) set out in m **rows** and n **columns**. You can write it like so:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

This matrix has **dimension** $m \times n$.

The notation a_{ij} refers to the ij^{th} entry of your matrix, that is the number in the i^{th} **row and the** j^{th} **column**. An alternative way of writing the above matrix is $A = [a_{ij}]$. You can specify the ij^{th} entry of a particular matrix A by writing $[A]_{ij} = a_{ij}$.

Tip

The entries in a matrix are usually numbers, but they can be other mathematical objects. Any type of number could be an entry in a matrix, positive or negative, rational or irrational, real or complex. (If complex numbers are unfamiliar to you, you can read more about them at Guide: Introduction to complex numbers.

Note that while entries can be other mathematical objects, for this study guide you will exclusively use entries within the real numbers.

Example 1

Here are some matrices:

$$A = \begin{bmatrix} 0 & -2 \\ \pi & 5 \\ 1/3 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 2/5 & 0 & \sqrt{3} \end{bmatrix}$$

As A has three rows and two columns, it has dimension 3×2. The entry in the 2nd row and 1st column is called $a_{21} = [A]_{21}$, and here that is equal to π. The entry in the 1st row and 2nd column is called $a_{12} = [A]_{21}$, and here that is equal to -2. You can notice here that a_{21} is **not equal** to a_{12}.
Matrix B here has dimension 1×4, as it has one row and four columns. The entry in the 1st row and 2nd column is called $b_{12} = [B]_{12}$, and here that is equal to $2/5$. The entry in the 1st row and 4th column is called $b_{14} = [B]_{12}$, and here that is equal to $\sqrt{3}$.

$$C = \begin{bmatrix} \sqrt{2} \\ 11 \\ -3/8 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} -5 & 1/2 & 0 \\ 0 & -\pi & 4/9 \\ 7 & 0 & -\sqrt{3} \\ -1 & 5 & 0 \\ 1/4 & 0 & 8 \end{bmatrix}$$

Matrix C has dimension 3×1, as it has three rows and one column.
As D has five rows and three columns, the matrix D has dimension 5×3.

You can notice that B only has one row; you can call such a matrix a **row matrix**. Similarly, you would call a matrix like C, with only one column, a **column matrix**. These are commonly known as **vectors**. You can read more about vectors in [Guide: Introduction to vectors].

Example 2

For example the 3×2 matrix, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ could be written as $[1]_{32}$.

Special matrices

There are special matrices

Zero matrix

Identity matrix

Diagonal matrix

i Definition of a main diagonal

For a matrix $A = [a_{ij}]_{mn}$, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ make up the **main diagonal** of A. You can define the **main diagonal** like so:

$$\text{diag}A = (a_{11}, a_{22}, \ldots, a_{nn})$$

i **Example 3**

For the matrices in Example 1, the main diagonals are:
$\text{diag}A = (0, 5)$
$\text{diag}B = (-1)$
$\text{diag}C = (\sqrt{2})$
$\text{diag}D = (-5, -\pi, -\sqrt{3})$

Upper and lower triangular matrices

Addition and subtraction with matrices

In this section, you will see when and how you can add and subtract matrices.

! Important

You can only add and subtract matrices if they share the same dimensions.

Definition of matrix addition and subtraction

Let A and B be matrices of the same dimension. The **matrix sum of** A and B can be calculated by adding corresponding entries of A and B,

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

Similarly, the **matrix difference of** A and B can be calculated by subtracting corresponding entries of A and B,

$$(A-B)_{ij} = a_{ij} - b_{ij}$$

Example 4

Let A and B be the following two 2×2 matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

The sum $A+B$ is:

$$A+B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

The difference $A-B$ is:

$$A-B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Example 5

Now let A and B be two 3×4 matrices.

$$A = \begin{bmatrix} 1 & -3 & 5 & -4 \\ 3 & 0 & -2 & 2 \\ -7 & 8 & 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 3 & -2 \\ 2 & -3 & 0 & 1 \\ -6 & -3 & 5 & 1/2 \end{bmatrix}$$

To subtract these matrices, you subtract the corresponding elements:

$$A - B = \begin{bmatrix} 1-(-1) & -3-4 & 5-3 & -4-(-2) \\ 3-2 & 0-3 & -1/2-0 & 2-1 \\ -7-(-6) & 8-(-3) & 4-5 & 3-1/2 \end{bmatrix}$$

Then you can simplify the signs,

$$A - B = \begin{bmatrix} 1+1 & -3-4 & 5-3 & -4+2 \\ 3-2 & 0-3 & -1/2 & 2-1 \\ -7+6 & 8+3 & 4-5 & 3-1/2 \end{bmatrix} = \begin{bmatrix} 2 & -7 & 2 & -2 \\ 1 & -3 & -1/2 & 1 \\ -1 & 11 & -1 & 5/2 \end{bmatrix}$$

You saw earlier that you can only add and subtract matrices if they share the same dimensions. You can look to this non-example to see why this is the case.

Non-example

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4/5 & 0 & 6 \end{bmatrix} \quad \text{(a } 2 \times 3 \text{ matrix)} \quad B = \begin{bmatrix} -8 & 8 \\ 0 & 11 \end{bmatrix} \quad \text{(a } 2 \times 2 \text{ matrix)}$$

Why can you not add A and B?
Matrix addition requires that each entry in one matrix corresponds to an entry in the other matrix. But, since:

- Matrix A has dimensions 2×3, that is, 2 rows and 3 columns
- Matrix B has dimensions 2×2, that is, 2 rows and 2 columns

they do **not** have the same dimensions. How you would attempt to calculate entries $(A + B)_{13}$ and $(B - A)_{23}$?

Scalar multiplication

Sometimes you will multiply a matrix by a number or variable. This is called **scalar multiplication**.

i Definition of scalar multipication with matrices

A **scalar multiplication** of a matrix A, by a number or variable k, is obtained by multiplying each entry in A by k,

$$kA = [k \cdot a_{ij}]$$

You can now see a few examples of this.

i **Example 6**

Let's multiply a 3×2 matrix by a scalar.

$$A = \begin{bmatrix} 0 & 4/3 \\ -3 & \sqrt{2} \\ -7 & 12 \end{bmatrix} \qquad k = 3$$

To multiply this matrix by 3, you multiply each element by 3:

$$3A = \begin{bmatrix} 3 \cdot 0 & 3 \cdot 4/3 \\ 3 \cdot -3 & 3 \cdot \sqrt{2} \\ 3 \cdot -7 & 3 \cdot 12 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -9 & 3\sqrt{2} \\ -21 & 36 \end{bmatrix}$$

i **Example 7**

Let's multiply a 1×4 matrix by a scalar.

$$A = \begin{bmatrix} -6 & 7/3 & \pi & 12 \end{bmatrix} \qquad k = -1/2$$

To multiply this matrix by $-1/2$, you can multiply each element by $-1/2$:

$$-1/2A = \begin{bmatrix} -1/2 \cdot -6 & -1/2 \cdot 7/3 & -1/2 \cdot \pi & -1/2 \cdot 12 \end{bmatrix} = \begin{bmatrix} -3 & -7/6 & -\pi/2 & -6 \end{bmatrix}$$

Now you can combine your understanding of matrix addition and scalar multiplication to tackle questions that combine both of these skills.

Example 8

Let's calculate $A + 2B$, for

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$$

First, let's work out $2B$,

$$2B = \begin{bmatrix} 2 \cdot -1 & 2 \cdot 2 \\ 2 \cdot -1 & 2 \cdot -3 \end{bmatrix}$$

By carrying out these multiplications, you can arrive at,

$$2B = \begin{bmatrix} -2 & 4 \\ -2 & -6 \end{bmatrix}$$

Now $A + 2B$,

$$A + 2B = \begin{bmatrix} -1 + (-2) & 2 + 4 \\ -1 + (-2) & -3 + (-6) \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -3 & -9 \end{bmatrix}$$

Quick check problems

1. Give the dimensions of the following matrices:

$$A = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 4 & -2 \\ 1/3 & -5 & 6 \\ 8 & -3/7 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & -6 \\ -4 & 2/3 \\ 0 & 5 \\ 7/8 & -9 \\ 0 & -2 \end{bmatrix}$$

2. You are given three statements below. Decide whether they are true or false.

(a) If A is a 2×3 matrix, and B is a 3×2 matrix, then AB is a 3×3 matrix.

(b) For the matrix B in Q1, the entry $b_{12} = 1/3$.

(c) You can only add or subtract matrices that share the same dimensions.

3. Multiply the following matrices,

$$D = \begin{bmatrix} 3 & 0 \\ 4 & -2 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}.$$

   Give the first entry, de_{11}, of the product DE.

Further reading

For more questions on this topic, please go to [Questions: Introduction to matrices].

For more on this topic, please go to [Guide: Introduction to Gaussian elimination].

Version history

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