Proof: Rules of differentiation

Tom Coleman

Summary

The proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, chain, and quotient rules - are true.

Before reading this proof sheet, it is essential that you read Guide: Introduction to differentiation and the derivative. In addition, reading [Guide: Introduction to limits] is useful. Further reading will be illustrated where required.

The starting point of this proof sheet is the limit definition of the derivative of a function:

> **Reminder of limit definition of the derivative**
>
> The **derivative of $f(x)$ with respect to x** is defined to be the function
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

To find the derivative of a function $f(x)$, replace the function f in the right-hand side of the above limit, and then evaluate the limit. Care has to be taken to remove the h in the denominator of the fraction, as you cannot evaluate the limit as h tends to 0 with it still there! (Remember, dividing by 0 can be extremely hazardous to your health.)

From here, the proof sheet will start with the proof of the derivative of x^2, which will then be expanded using the binomial theorem to find the derivative of x^n. Then, the derivative of e^x is found. Finally, the derivatives of $\sin(x)$ and $\cos(x)$ are demonstrated.

Sum and difference rules

Scaling rules

Product rule

Here is the product rule, restated with $f(x) = u(x)$ and $g(x) = v(x)$ for visual ease in the proof that follows.

The product rule

Let $f(x)$ and $g(x)$ be two differentiable functions. Then the **product rule** says that

$$(fg)'(x) = \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

that is, the derivative of the product of $f(x)$ and $g(x)$ is equal to the product of $f(x)$ and the derivative of $g(x)$, plus the product of $g(x)$ and the derivative of $f(x)$.
This can also be written as

$$\frac{d}{dx}(f(x)g(x)) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}.$$

Proof of the product rule

Here's why the product rule works. It's not quite as straightforward as the proof of the sum rule and the constant rule; you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with $f(x)$ and $g(x)$ as two differentiable real-valued functions, with product $(fg)(x) = f(x)g(x)$. Putting this into the limit definition of the derivative given above:

$$(fg)'(x) = \lim_{h\to 0}\frac{(fg)(x+h) - (fg)(x)}{h}$$

Since $(fg)(x) = f(x)g(x)$, this becomes

$$(fg)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Quotient rule

Before this proof, you may find it useful to read [Guide: Laws of indices]

There is an important limit that needs to be used to find the derivative of e^x, which is given by:

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

The proof that this is true relies on some *real analysis* and the fact that the constant e can be defined by $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Derivative of e^x

The derivative of $f(x) = e^x$ is $f'(x) = e^x$.

Again, let's use the limit definition of the derivative with $f(x) = e^x$. This gives:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

Typically, the goal has been to remove the h in the denominator; however, it is not possible in this case, as expanding e^{x+h} doesn't give a h that you can get rid of while factorizing. In this case, what you can do is try and reduce to a known limit. In particular, the known limit involving e^h is $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ and so you may be able to manufacture this limit by manipulation of the expression above. Using the laws of indices, you can write that

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

Factorising out e^x gives

$$\lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} \left(e^x \left(\frac{e^h - 1}{h} \right) \right)$$

Now, e^x is constant as h varies, and so you can take this out of the limit to get:

$$\lim_{h \to 0} \left(e^x \left(\frac{e^h - 1}{h} \right) \right) = e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right)$$

You can now evaluate the limit to get:

$$e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = e^x(1) = e^x$$

It follows that $f'(x) = e^x$, as required.

Derivative of $\sin(x)$ and $\cos(x)$

Before reading this section, it may be useful to read Guide: Trigonometric identities (radians).

As with the derivative of e^x in the previous section, there are two important limits involving $\cos$ and $\sin$ that are needed before continuing. These are:

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos(h) - 1}{h}.$$

To show these are true, you need techniques from *real analysis*, which is the study of functions on real numbers.

Derivative of $\sin(x)$

> **Derivative of $\sin(x)$**
>
> The derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$.

Starting off with the limit definition of differentiation and $f(x) = \sin(x)$ gives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

As in the case of e^x, expanding $\sin(x+h)$ won't give a single h to cancel with! However, it could be that you can reduce this expression to the limits given above. Using the sum rule for $\sin$ (see Guide: Trigonometric identities (radians)), you can write that

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

The bones of the two given limits are here, and now it's time to uncover them. Factorising out $\sin(x)$ and $\cos(x)$ gives

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} = \lim_{h \to 0} \left(\sin(x) \left(\frac{\cos(h) - 1}{h} \right) + \cos(x) \left(\frac{\sin(h)}{h} \right) \right)$$

Using the two limits given above, and the fact that the values of $\sin(x)$ and $\cos(x)$ are independent of h:

$$\lim_{h \to 0} \left(\sin(x) \left(\frac{\cos(h) - 1}{h} \right) + \cos(x) \left(\frac{\sin(h)}{h} \right) \right) = \sin(x)(0) + \cos(x)(1) = \cos(x)$$

and so $f'(x) = \cos(x)$ is the derivative of $\sin(x)$.

Derivative of $\cos(x)$

> **Derivative of $\cos(x)$**
>
> The derivative of $f(x) = \cos(x)$ is $f'(x) = -\cos(x)$.

Starting off with the limit definition of differentiation and $f(x) = \cos(x)$ gives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

As in the case of e^x, expanding $\cos(x+h)$ won't give a single h to cancel with! However, it could be that you can reduce this expression to the limits given above. Using the sum rule for $\cos$ (see Guide: Trigonometric identities (radians)), you can write that

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

The bones of the two given limits are here, and now it's time to uncover them. Factorizing out $\cos(x)$ and $\sin(x)$ gives

$$\lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} = \lim_{h\to 0} \left(\cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right)$$

Using the two limits given above, and the fact that the values of $\sin(x)$ and $\cos(x)$ are independent of h:

$$\lim_{h\to 0} \left(\cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right) = \cos(x)(0) - \sin(x)(1) = -\sin(x)$$

and so $f'(x) = -\sin(x)$ is the derivative of $\cos(x)$.

Further reading

Click this link to go back to Guide: Introduction to differentiation and the derivative.

Click this link to go back to Guide: The product rule

For questions on differentiation and the derivative, please go to Questions: Introduction to differentiation and the derivative.

Version history

v1.0: initial version created in 04/25 by tdhc.