Determinants

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Summary

This guide introduces the determinant of a square matrix. You will learn what a determinant is, how to calculate it, and why it matters in linear algebra and geometry.

Before reading this guide, you should have a solid understanding of matrix operations, including addition, scalar multiplication, and matrix multiplication. You should also be comfortable with rows, columns, and square matrices.

What is a determinant?

The determinant of an $n \times n$ matrix A, written $det(A)$ or $|A|$, is a single number that describes important features of the linear transformation represented by the matrix. One of the most important uses of the determinant is determining whether a matrix is invertible. If $det(A) = 0$, the matrix collapses space into a lower dimension—such as flattening a 3D object onto a plane—so the transformation cannot be reversed and the matrix is not invertible. If $det(A) \neq 0$, the matrix preserves dimension, meaning no collapse occurs and the matrix is invertible.

The determinant also has a geometric interpretation: it measures how the matrix scales area in 2D or volume in 3D. For example, if $det(A) = 3$, then the transformation multiplies all areas or volumes by 3. A negative determinant additionally indicates a reflection, meaning the transformation flips orientation. Although the determinant is a single value, it provides concise information about how a matrix stretches, compresses, flips, or collapses space.

> ⚠ Warning
>
> Only **square matrices** have determinants.

Unique Ways to Calculate 2×2 and 3×3 Matrices

For small matrices, there are formulas that can make calculating determinants faster and more intuitive. A 2×2 matrix has a formula involving the products of its diagonals, while a 3×3 matrix can often be solved quickly using **Sarrus' Rule**. Although not strictly necessary, these methods save time compared to the general cofactor expansion used for larger matrices.

Determinants of a 2×2 Matrix

For a 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant is calculated using the following formula:

$$\det(A) = ad - bc.$$

> **Example (2×2)**
>
> Compute the determinant of
>
> $$A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}.$$
>
> **Steps:**
>
> 1. Multiply the main diagonal: $3 \cdot 5 = 15$
> 2. Multiply the other diagonal: $4 \cdot 2 = 8$
> 3. Subtract: $15 - 8 = 7$
>
> **Answer:**
>
> $$\det(A) = 7$$

Determinants of a 3×3 Matrix

Sarrus' Rule

Sarrus' Rule is unique method for calculating the determinant of a 3×3 matrix. It works only for 3×3 matrices and provides a systematic way to account for all the products needed. For a matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

Sarrus' Rule states that the determinant is given by summing the products of the diagonals from top-left to bottom-right and subtracting the products of the diagonals from top-right to bottom-left:

$$\det(A) = (aei + bfg + cdh) - (ceg + bdi + afh).$$

Example

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

Step 1: Multiply the diagonals from top-left to bottom-right:

$$1 \cdot 0 \cdot 8 = 0, \quad 2 \cdot 5 \cdot 6 = 60, \quad 3 \cdot 4 \cdot 7 = 84$$

Sum: $0 + 60 + 84 = 144$
Step 2: Multiply the diagonals from top-right to bottom-left:

$$3 \cdot 0 \cdot 6 = 0, \quad 2 \cdot 4 \cdot 8 = 64, \quad 1 \cdot 5 \cdot 7 = 35$$

Sum: $0 + 64 + 35 = 99$
Step 3: Subtract the second sum from the first:

$$\det(A) = 144 - 99 = 45$$

Answer:

$$\det(A) = 45$$

Cofactor Expansion for all Matrices

The cofactor expansion formula can be applied to any square matrix, whether it is 4×4, 7×7, or larger. For an $n \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

The determinant can be computed by choosing any row i or any column j to expand along. Here, a_{ij} refers to the entry in **row i and column j**. The corresponding **cofactor** is

$$C_{ij} = (-1)^{i+j}\,|M_{ij}|,$$

where $|M_{ij}|$ is the determinant of the **minor matrix** obtained by removing row i and column j. Each cofactor is then multiplied by its entry a_{ij}, and the sum of these terms gives the determinant:

$$\det(A) = \sum_j a_{ij}C_{ij} \quad \text{(expansion along row } i\text{)}.$$

The general formula for expanding along row i is:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot C_{ij}, \quad C_{ij} = a_{ij} \cdot |M_{ij}|$$

Giving you the most commonly used formula,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot |M_{ij}|$$

General Sign Pattern for n x n Matrices

For any n x n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

the sign of each entry in cofactor expansion is determined by

$$(-1)^{i+j}, \quad i = 1, \dots, n, \; j = 1, \dots, n.$$

This produces a **checkerboard pattern** of signs across the matrix. Mathematically, the pattern can be represented as:

$$\text{sign}(a_{ij}) = \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & \cdots & (-1)^{1+n} \\ (-1)^{2+1} & (-1)^{2+2} & \cdots & (-1)^{2+n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} & (-1)^{n+2} & \cdots & (-1)^{n+n} \end{bmatrix} = \begin{bmatrix} - & + & - & \cdots \\ + & - & + & \cdots \\ - & + & - & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Each entry is *positive* if $i + j$ is even, and *negative* if $i + j$ is odd.

- The top-left corner a_{11} is always $+$, producing the alternating pattern across rows and columns.

- This formula works for any matrix size, whether $n = 3$, $n = 4$, $n = 100$, or larger.

This is the mathematical reason the cofactor expansion alternates signs: it ensures that the determinant calculation correctly accounts for orientation and scaling in the associated linear transformation.

! Understanding Minor Matrices

In a square matrix, the minor of an element is the determinant of the smaller matrix formed by deleting the row and column that element sits in. Click on any number in the matrix below to see its minor matrix. The highlighted row and column show which parts of the original matrix are removed. Notice how different elements lead to different minor matrices, and think about how these minors contribute to calculating the determinant of the full matrix.

Click on a number in the matrix to see its minor:

Selected Minor:

How to Create the Formula Step by Step

1. **Pick a row or column** (say row i).
2. For each entry a_{ij} in that row, **calculate its minor** M_{ij} by removing row i and column j.
3. Multiply each minor by its **entry** a_{ij} and the **sign factor** $(-1)^{i+j}$.
4. **Sum all these products** to obtain the determinant.

Using this method, you can systematically construct the determinant formula for any square matrix, from 2×2 up to 100×100 and beyond.

> Tip
>
> Use Zeros to Simplify Cofactor Expansion
> When performing a cofactor expansion, **choosing a row or column with many zeros** can greatly simplify your calculations. This is because any entry that is zero contributes nothing to the sum:
>
> $$(-1)^{i+j} \cdot 0 \cdot |M_{ij}| = 0.$$
>
> By expanding along a row or column with zeros, you reduce the number of minors you need to calculate, saving time and reducing the chance of mistakes. Always scan the matrix for a row or column with the most zeros before starting your expansion.

> Example: Computing a 4×4 Determinant Step by Step
>
> Consider the matrix:
>
> $$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{bmatrix}$$
>
> We will compute $det(A)$ by **expansion along column 2** (because it has several zeros, which simplifies calculations).
>
> **Step 1: Cofactor formula**
>
> For any entry a_{ij}:

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the **minor matrix** obtained by removing row i and column j.
Expansion along column j:

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij})$$

Step 2: Identify nonzero entries in column 2

Column 2:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Only **row 3** contributes, because all other entries are zero:

$$\det(A) = a_{32} \cdot C_{32} = 1 \cdot C_{32}$$

Step 3: Compute cofactor C_{32}

$$C_{32} = (-1)^{3+2} \det(M_{32}) = -\det(M_{32})$$

The minor matrix M_{32} (remove row 3, column 2):

$$M_{32} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

Step 4: Compute $det(M_{32})$ using cofactor expansion along row 2

$$\det(M_{32}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Row 2 entries:

- $a_{21} = 3$
- $a_{22} = 0$
- $a_{23} = 5$

Step 5: Compute minors and cofactors

Entry $a_{21} = 3$

- Minor M_{21} (remove row 2, column 1):

$$M_{21} = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$$

- Cofactor:

$$C_{21} = (-1)^{2+1} \det(M_{21}) = -(2 \cdot 0 - 5 \cdot (-1)) = -5$$

Entry $a_{22} = 0$

- Minor M_{22} (remove row 2, column 2):

$$M_{22} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

- Cofactor:

$$C_{22} = (-1)^{2+2} \det(M_{22}) = 1 \cdot (1 \cdot 0 - (-1) \cdot 1) = 1$$

- Contribution to determinant:

$$a_{22}C_{22} = 0 \cdot 1 = 0$$

Entry $a_{23} = 5$

- Minor M_{23} (remove row 2, column 3):

$$M_{23} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$$

- Cofactor:

$$C_{23} = (-1)^{2+3} \det(M_{23}) = -(1 \cdot 5 - 2 \cdot 1) = -3$$

Step 6: Combine to get $det(M_{32})$

$\det(M_{32}) = a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23} = 3(-5)+0(1)+5(-3) = -15+0-15 = -30$

Step 7: Compute C_{32} and final determinant

$$C_{32} = (-1)^{3+2}\det(M_{32}) = -(-30) = 30$$

$$\det(A) = a_{32} \cdot C_{32} = 1 \cdot 30 = 30$$

Final Answer:

$$\det(A) = 30$$

Triangular or Diagonal Matrices

If a matrix is **upper triangular**, **lower triangular**, or **diagonal**:

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}, L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix}, D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}$$

the determinant can be calculated by multiplying the diagonal entries of the matrix:

$$\det(A) = (a_{11})(a_{22})(\dots)(a_{nn})$$

Example

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -4 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-4) \cdot 5 = -40$$

Tip: Using Row Operations to Simplify Determinants

For any square matrix, you can use **row operations** to simplify it before calculating the determinant. This approach is often much faster than expanding large matrices, such as a 10×10 matrix, and can make the calculation much more manageable.

Important: Row operations **affect the determinant**, so you must account for these changes:

- **Swap two rows** $\rightarrow$ determinant is multiplied by -1
- **Multiply a row by a constant** k $\rightarrow$ determinant is multiplied by k
- **Add a multiple of one row to another** $\rightarrow$ determinant **does not change**

Procedure:

1. Use row operations to reduce the matrix to **upper triangular form**.
2. Multiply the **diagonal entries** to get the determinant.
3. Adjust the result for any **row swaps** or **row scalings** you performed.

This method works for any square matrix, saving time and reducing errors when calculating determinants of larger matrices.

Example: 8×8 Matrix with Row Swap and Row Operations

Consider the matrix:

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Properties and Rules of Determinants

Here is a comprehensive list of key properties and rules that apply to determinants:

1. **Square matrices only**
   - Determinants are defined **only for square matrices**.
2. **Determinant of the transpose**

$$\det(A^T) = \det(A)$$

3. **Multiplicative property**

$$\det(AB) = \det(A) \cdot \det(B)$$

4. **Effect of row operations**
   - Swapping two rows $\rightarrow$ **flips the sign** of the determinant
   - Multiplying a row by a scalar $k \rightarrow$ determinant is **scaled by** k
   - Adding a multiple of one row to another $\rightarrow$ determinant **does not change**
5. **Determinant of an identity matrix**

$$\det(I_n) = 1$$

6. **Determinant of a triangular or diagonal matrix**

$$\det(A) = \text{product of the diagonal entries}$$

7. **Invertibility criterion**
   - If $det(A) \neq 0 \rightarrow A$ is **invertible**
   - If $det(A) = 0 \rightarrow A$ is **singular** (not invertible)

8. **Determinant of a scalar multiple**

$$\det(kA) = k^n \det(A) \quad \text{for an } n \times n \text{ matrix}$$

9. **Determinant of the inverse**

$$\det(A^{-1}) = \frac{1}{\det(A)} \quad \text{if } A \text{ is invertible}$$

10. **Zero row or column**

    - If a row or column consists entirely of zeros $\rightarrow$ determinant is 0

11. **Equal rows or columns**

    - If two rows or two columns are identical $\rightarrow$ determinant is 0

12. **Linear combination in a row or column**

    - Determinant is linear with respect to each row or column individually

$$\det(\text{row } i + \text{row } j) = \det(\text{row } i) + \det(\text{row } j)$$

These properties are extremely useful for simplifying determinant calculations, checking invertibility, and understanding the behaviour of matrices in linear transformations.

Quick Check Problems

1. Compute the determinant of the following 2×2 matrix: $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

2. Compute the determinant of the following 2×2 matrix: $\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$

3. Compute the determinant of the following 3×3 matrix using Sarrus' Rule: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

4. Compute the determinant of the following 3×3 matrix using Sarrus' Rule: $\begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 5 & 2 \end{bmatrix}$

5. Which row would be the quickest to use for cofactor expansion? $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix}$

6. Compute the determinant of the following triangular matrix: $\begin{bmatrix} 3 & 5 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

7. Compute the determinant using row operations: $\begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

8. Use determinant properties to find $det(B)$ if $B = 3A$ and $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

9. If swapping two rows of a matrix C gives $det(C') = -12$, what is $det(C)$?

10. If a 3×3 matrix D has two identical rows, what is $det(D)$?

Further reading

For more questions on this topic, please go to [Questions: Determinants].

Version history

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