Implicit differentiation

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Summary

Implicit differentiation allows you to find the derivatives of functions $y = f(x)$ that are 'implicitly defined' as some expression like $g(x, y) = 0$. This guide introduces the idea of an implicitly defined function in one variable, and the way to find the derivative of such a function.

Before reading this guide, it is recommended that you read Guide: Introduction to differentiation and the derivative and [Guide: The differential operator and higher derivatives]. This guide will also use the rules of differentiation outlined in Guide: The product rule, Guide: The quotient rule, and especially Guide: The chain rule.

What is implicit differentatiation?

You've seen in various guides on differentiation that you can differentiate sums and differences (Guide: Introduction to differentiation and the derivative), products (Guide: The product rule), quotients (Guide: The quotient rule) and compositions Guide: The chain rule of functions.

However, there are some expressions that may be harder to differentiate with respect to x. For instance, suppose you were asked to differentiate the equation of a circle

$$(x - 1)^2 + (y - 4)^2 = 36$$

This expression can't be rearranged to get a value for y in terms of x, as you can't undo the square on $y - 4$ without taking both the positive and negative terms. But this is a figure in 2D space, so it will have tangents. Can you find equations of these using differentiation?

The answer is yes, using **implicit differentiation**, which can be used on **implicitly defined** functions. Implicitly defined functions are found where one variable (say y) depends on another (say x) in a way that cannot be expressly defined as some relationship $y = f(x)$ between the two, but can be written as a *multivariate function* $g(x, y)$ equal to a constant 0. This situation is common in the sciences and economics; any closed curve in 2D space could be expressed as an implicitly defined function.

This guide introduces you to the concept of implicit differentiation. This involves applying a differential operator to an implicitly defined function, and rearranging to find the expression for the derivative.

Explicit and implicit functions

An **explicit function** expresses one variable directly in terms of the other variables. The method to compute the dependent variable from the independent variables is consistent and often requires rearranging the equation for the dependent variable of interest.

Explicit function

An explicit function takes the form $y = f(x)$.
It is a function that can be rearranged into a form where y is distinctly written as a function of only x.

Tip

A handy way to think of this functional form is as $y =$ terms involving only x.

Example 1

An example of an explicit function is the function $y = 2x + 3$. The function has been written in the form $y = f(x)$ and you can directly compute y for any value of x. Another example of an explicit function is the function $x^2 = \dfrac{3y+2}{y-1}$ which can be rearranged into its explicit form $y = \dfrac{x^2+2}{x^2-3}$.

An **implicit function** defines a relationship between variables without necessarily expressing one variable directly in terms of the others. For example, unlike explicit functions where you can write y as a function of x, implicit functions may not allow you to isolate y. In some cases, isolating y algebraically might be very difficult or impossible, or doing so may give multiple values of y for a single x meaning the result is not a function.

Implicit function

An implicit function takes the form $f(x, y) = 0$.
It is a function where the variables are so entangled that either you cannot rearrange the equation to write y distinctly as a function of x, or the rearranged form would violate the definition of a function (for example, by producing multiple y values for a single x). In such cases, you need to treat the relationship implicitly.

Tip

A handy way to think of this functional form is as terms involving both x and $y = 0$, where solving explicitly for y either leads to multiple values or requires techniques (like implicit differentiation) that do not involve isolating y algebraically.

Example 2

An example of an implicit function is the equation $x^3 + y^3 - 3xy - 7 = 0$. The equation has been written in the functional form $f(x, y) = 0$. Rearranging it explicitly for y is not straightforward, and in some cases may not even be possible using standard algebraic techniques.

Another example of an implicit function is $x^2 + y^2 - 25 = 0$ which is the equation of a circle of radius 5. This has a solution $y = \pm\sqrt{25 - x^2}$, which is not a function, as a function cannot take on two values of y for the same value of x. So, even though you can rearrange the equation, the result is not an explicit function. Therefore, when written in the form $x^2 + y^2 - 25 = 0$, it has been written implicitly.

Implicit differentiation

Here is the statement of the product rule:

The product rule

Let $u(x)$ and $v(x)$ be two differentiable functions. Then the **product rule** says that

$$(uv)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right) = u'(x)v(x) + u(x)v(x)$$

that is, the derivative of the product of $u(x)$ and $v(x)$ is equal to the product of $v(x)$ and the derivative of $u(x)$, plus the product of $u(x)$ and the derivative of $v(x)$.

This can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Tip

The discovery of the product rule is often credited to Gottfried Leibniz (link to Mactutor biography, external site), one of the co-founders of calculus (along with Isaac Newton).

Sometimes $f(x)$ and $g(x)$ are written instead of $u(x)$ and $v(x)$ in the statement of the product rule. The reason that $u(x)$ and $v(x)$ is used here is that sometimes the product function is called $f(x)$; and you then can't use it again!

It does not matter which function you pick to be $u(x)$ and which function you pick to be $v(x)$; this is because $u(x)v(x) = v(x)u(x)$.

To see why this really is the derivative of the product of two functions, please see [Proof sheet: Rules of differentiation.]

Examples

Example 1

What is the derivative of $y = x^3e^x$?
In this case, you have two functions multiplied together to make y. The two functions are $u(x) = x^3$ and $v(x) = e^x$. In order to use the product rule on y, you could differentiate $u(x)$ and $v(x)$ beforehand and then substitute them into the product rule. Doing this gives

- For $u(x) = x^3$, then $u'(x) = 3x^2$ by the power rule.
- For $v(x) = e^x$, then $v'(x) = e^x$.

Putting these into the statement of the product rule gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= u'(x)v(x) + u(x)v'(x)\\ &= (3x^2)(e^x) + (x^3)(e^x)\\ &= 3x^2e^x + x^3e^x\end{aligned}$$

and this is your answer. You could also factorize the answer if you wanted to get $y'(x) = (x^3 + 3x^2)e^x$.

You don't need to be so rigorous in your own working. Here's another example of using the product rule.

Example 2

What is the derivative of $f(z) = e^{3z}z^{1/3}$ with respect to z?
You can use the product rule on $f(z)$ to differentiate with respect to z. Set $u(z) = e^{3z}$ and $v(z) = z^{1/3}$. Differentiating both with respect to z gives $u'(z) = 3e^{3z}$ and $v'(x) = \frac{1}{3}z^{-2/3}$. Putting these into the statement of the product rule gives

$$\begin{aligned} f'(z) &= u'(z)v(z) + u(z)v'(z) \\ &= (3e^{3z})(z^{1/3}) + (e^{3z})\left(\frac{1}{3}z^{-2/3}\right) \\ &= e^{3z}\left(3z^{1/3} + \frac{1}{3}z^{-2/3}\right) \end{aligned}$$

and this is your answer. Here, the answer has been factorized given the common e^{3z} term.

Here's another example of the product rule, which is included solely for a chance to remember the signs for the derivatives of $\sin(x)$ and $\cos(x)$!

Example 3

Find the derivative of the function $f(x) = 2\sin(x)\cos(x)$.
In this case, there are two options. You can either

- use the constant rule to take the 2 out and differentiate $\sin(x)\cos(x)$ using the product rule, or
- take $u(x) = 2\sin(x)$ and $v(x) = \cos(x)$ and differentiate directly.

Let's do the first of these. To use the product rule, take $u(x) = \sin(x)$ and $v(x) = \cos(x)$. Then $u'(x) = \cos(x)$ and $v'(x) = -\sin(x)$. Therefore, using the product rule gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)\cos(x)) &= u'(x)v(x) + u(x)v'(x) \\ &= (\cos(x))(\cos(x)) + (\sin(x))(-\sin(x)) \\ &= \cos^2(x) - \sin^2(x) \end{aligned}$$

Therefore, the derivative of $f(x) = 2\sin(x)\cos(x)$ is

$$f'(x) = 2\left(\cos^2(x) - \sin^2(x)\right).$$

Tip

You don't always need the product rule to differentiate the product of two functions if that product of two functions can be expressed in a different way.

For instance, the function in Example 3 is $f(x) = 2\sin(x)\cos(x)$, which by a trigonometric identity is equal to $f(x) = \sin(2x)$. (See Guide: Trigonometric identities (radians) for more.) You can then differentiate $\sin(2x)$ by the rules above to get $2\cos(2x)$. By trigonometric identities, this is also equal to $2\left(\cos^2(x) - \sin^2(x)\right)$, which is the same answer as in Example 3.

This shows you should explore all potential avenues before starting to use differentiation techniques.

Here's an example which combines two separate differentiation rules; both the sum rule and the product rule.

Example 4

Find the derivative of the function $f(x) = e^x\sqrt{x} + x^3$.
Here, you could use the sum rule for differentiation to write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\sqrt{x}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)$$

The first term needs the product rule to differentiate. To help with this, you can rewrite $e^x\sqrt{x}$ as $e^x x^{1/2}$. To use the product rule, take $u(x) = e^x$ and $v(x) = x^{1/2}$. Then $u'(x) = e^x$ and $v'(x) = \frac{1}{2}x^{-1/2}$. Therefore, using the product rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\sqrt{x}\right) &= u'(x)v(x) + u(x)v'(x)\\ &= (x^{1/2})(e^x) + \left(\frac{1}{2}x^{-1/2}\right)e^x\\ &= e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)\end{aligned}$$

The second of these terms is differentiable using the power rule, and you can write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2$$

Therefore, your final answer is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\sqrt{x}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\\ &= e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) + 3x^2\end{aligned}$$

and this is your final answer. You can tidy this up if you want to, but it's not necessary.

The next example demands that the function be rephrased in order to use the product rule.

Example 5

Find the derivative of $y = \dfrac{\ln(x)}{x^5}$.

Although this looks like one function divided by another, you can use the laws of indices (see Guide: Laws of indices to write as a product of two functions instead. Doing this gives

$$y = \frac{\ln(x)}{x^5} = \ln(x) \cdot x^{-5}.$$

Now you can use the product rule on this function. Take $u(x) = \ln(x)$ and $v(x) = x^{-5}$. Then $u'(x) = \dfrac{1}{x} = x^{-1}$ and $v'(x) = -5x^{-6}$. Therefore, using the product rule gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= u'(x)v(x) + u(x)v'(x) \\ &= \left(x^{-1}\right)\left(x^{-5}\right) + (\ln(x))(-5x^{-6}) \\ &= x^{-6} - 5x^{-6}\ln(x) + = \frac{1}{x^6}\left(1 - 5\ln(x)\right)\end{aligned}$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^6}\left(1 - 5\ln(x)\right).$$

Important

This technique does **not** work for all functions in a denominator without the use of other rules. For instance, consider the function

$$f(x) = \frac{x^5}{\ln(x)}$$

There are two ways you can differentiate this function with respect to x:

- use the **quotient rule** which deals with functions of the form $\dfrac{u(x)}{v(x)}$ - see Guide: The quotient rule (where this function is Example 3) for more, or
- write $f(x) = x^5 \cdot (\ln(x))^{-1}$, then use the product rule together with the **chain rule** to differentiate the composite function $(\ln(x))^{-1}$ (see Guide: The chain rule for more).

In particular, you cannot only use the product rule to differentiate $f(x)$ in this case.

Here's an example that again could use a rewrite before differentiating. This will save you a whole other application of the product rule!

Example 6

Find the derivative of $f(t) = 4t^4 \cos(4t) + e^{-4t} \cos(4t)$ with respect to t.
It seems here that you will have to use the product rule twice; once on the $4t^4 \cos(4t)$ term, and once on the $e^{-4t} \cos(4t)$ term. However, you could notice here that $\cos(4t)$ is common to both terms, and so you can factorize. This gives

$$f(t) = 4t^4 \cos(4t) + e^{-4t} \cos(4t) = (4t^4 + e^{-4t}) \cos(4t).$$

You can now differentiate $f(t)$ using one application of the product rule. Set $u(t) = 4t^4 + e^{-4t}$ and $v(t) = \cos(4t)$; it follows that $u'(t) = 16t^3 - 4e^{-4t}$ and $v'(t) = -4\sin(4t)$. Putting these into the statement of the product rule gives

$$\begin{aligned} f'(t) &= u'(t)v(t) + u(t)v'(t) \\ &= (16t^3 - 4e^{-4t})(\cos(4t)) + (4t^4 + e^{-4t})(-4\sin(4t)) \\ &= (16t^3 - 4e^{-4t})\cos(4t) - 4(4t^4 + e^{-4t})\sin(4t) \end{aligned}$$

and this is a final answer. You could factorize this in any number of different ways; but the amount of times using the product rule can be minimized!

Warning

If you are differentiating a function that has terms in brackets multiplied together, **don't expand the brackets!** This not only leads to extra work (and more opportunities for mistakes), but it will also result in using the product rule more times than is necessary.

Finally, here's an example where a second application of the product rule is unavoidable.

Example 7

What is the derivative of $y = e^x \sin(2x)\cos(4x)$?

In this case, y is a product of **three** distinct functions. The only option is to take either $u(x)$ or $v(x)$ as a product of two functions, and then use the product rule to find that derivative. Let's take $u(x) = e^x$ and $v(x) = \sin(2x)\cos(4x)$. (The other three potential choices are perfectly acceptable!)

Now, you can differentiate both of these with respect to x:

- when $u(x) = e^x$, then $u'(x) = e^x$ (thanks to the indestructibility of e^x).
- Now, when $v(x) = \sin(2x)\cos(4x)$, what is $v'(x)$? You're going to have to use the product rule to find out! In this case, you're going to have to pick different names for the functions, since $u(x)$ and $v(x)$ have been used already. (The product rule still works fine even with different names.) In this case - since f has not yet been used - you can set $f(x) = \sin(2x)$ and $g(x) = \cos(4x)$. Therefore $f'(x) = 2\cos(2x)$ and $g'(x) = -4\sin(4x)$.

  You can use the product rule to say that

$$\begin{aligned}
v'(x) &= f'(x)g(x) + f(x)g'(x) \\
&= (2\cos(2x))\cos(4x) + \sin(2x)(-4\sin(4x)) \\
&= 2\cos(2x)\cos(4x) - 4\sin(2x)\sin(4x)
\end{aligned}$$

Finally, putting $u'(x)$ and $v'(x)$ into the product rule for y and simplifying gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= u'(x)v(x) + u(x)v'(x) \\
&= (e^x)(\sin(2x)\cos(4x)) + (e^x)(2\cos(2x)\cos(4x) - 4\sin(2x)\sin(4x)) + \\
&= e^x(\sin(2x)\cos(4x) + 2\cos(2x)\cos(4x) - 4\sin(2x)\sin(4x))
\end{aligned}$$

and this (finally) is your answer.

Quick check problems

Using the product rule, match the six functions to their derivatives with respect to x. One of the derivatives given does not match a function; you should circle the odd derivative out.

Function	Derivative
$3x^2\cos(2x)$	$2x\cos(3x) - 3x^2\sin(3x)$
$2x^3\sin(3x)$	$3x^2\cos(3x) + 2x\sin(3x)$
$x^2\cos(3x)$	$6x^2\cos(2x) + 6x\sin(2x)$
$3x^3\cos(2x)$	$6x\sin(2x) + 6x^2\cos(2x)$
$x^2\sin(3x)$	$6x^3\cos(3x) + 6x\sin(3x)$
$3x^2\sin(2x)$	$6x\cos(2x) - 6x^2\sin(2x)$
	$9x^2\cos(2x) - 6x^3\sin(2x)$

Further reading

For more questions on the subject, please go to Questions: The product rule.

For more about techniques of differentiation, please see Guide: The quotient rule, and Guide: The chain rule.

For more about why the rules and techniques of differentiation are true, please see Proof sheet: Rules of differentiation.

Version history

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