

# Proof: Rules of differentiation

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## Summary

The proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, chain, and quotient rules - are true.

*Before reading this proof sheet, it is essential that you read [Guide: Introduction to differentiation and the derivative](#). In addition, reading [\[Guide: Introduction to limits\]](#) is useful. Further reading will be illustrated where required.*

The starting point of this proof sheet is the limit definition of the derivative of a function:

**i** Reminder of limit definition of the derivative

The **derivative of  $f(x)$  with respect to  $x$**  is defined to be the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

To find the derivative of a function  $f(x)$ , replace the function  $f$  in the right-hand side of the above limit, and then evaluate the limit. Care has to be taken to remove the  $h$  in the denominator of the fraction, as you cannot evaluate the limit as  $h$  tends to 0 with it still there! (Remember, dividing by 0 can be extremely hazardous to your health.)

From here, the proof sheet will start with the proof of the derivative of  $x^2$ , which will then be expanded using the binomial theorem to find the derivative of  $x^n$ . Then, the derivative of  $e^x$  is found. Finally, the derivatives of  $\sin(x)$  and  $\cos(x)$  are demonstrated.

## Sum and difference rules

## Scaling rules

## Product rule

Here is the product rule, restated with  $f(x) = u(x)$  and  $g(x) = v(x)$  for visual ease in the proof that follows.

### **i** The product rule

Let  $f(x)$  and  $g(x)$  be two differentiable functions. Then the **product rule** says that

$$(fg)'(x) = \frac{d}{dx} (f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

that is, the derivative of the product of  $f(x)$  and  $g(x)$  is equal to the product of  $f(x)$  and the derivative of  $g(x)$ , plus the product of  $g(x)$  and the derivative of  $f(x)$ .

This can also be written as

$$\frac{d}{dx} (f(x)g(x)) = f \frac{dg}{dx} + g \frac{df}{dx}.$$

## Proof of the product rule

Here's why the product rule works. It's not quite as straightforward as the proof of the sum rule and the constant rule; you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with  $f(x)$  and  $g(x)$  as two differentiable real-valued functions, with product  $(fg)(x) = f(x)g(x)$ . Putting this into the limit definition of the derivative given above:

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

Since  $(fg)(x) = f(x)g(x)$ , this becomes

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

## Quotient rule

*Before this proof, you may find it useful to read [Guide: Laws of indices]*

There is an important limit that needs to be used to find the derivative of  $e^x$ , which is given by:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

The proof that this is true relies on some *real analysis* and the fact that the constant  $e$  can be defined by  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ .

### **i** Derivative of $e^x$

The derivative of  $f(x) = e^x$  is  $f'(x) = e^x$ .

Again, let's use the limit definition of the derivative with  $f(x) = e^x$ . This gives:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

Typically, the goal has been to remove the  $h$  in the denominator; however, it is not possible in this case, as expanding  $e^{x+h}$  doesn't give a  $h$  that you can get rid of while factorizing. In this case, what you can do is try and reduce to a known limit. In particular, the known limit involving  $e^h$  is  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$  and so you may be able to manufacture this limit by manipulation of the expression above. Using the laws of indices, you can write that

$$\lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h}$$

Factorising out  $e^x$  gives

$$\lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \rightarrow 0} \left( e^x \left( \frac{e^h - 1}{h} \right) \right)$$

Now,  $e^x$  is constant as  $h$  varies, and so you can take this out of the limit to get:

$$\lim_{h \rightarrow 0} \left( e^x \left( \frac{e^h - 1}{h} \right) \right) = e^x \lim_{h \rightarrow 0} \left( \frac{e^h - 1}{h} \right)$$

You can now evaluate the limit to get:

$$e^x \lim_{h \rightarrow 0} \left( \frac{e^h - 1}{h} \right) = e^x(1) = e^x$$

It follows that  $f'(x) = e^x$ , as required.

## **Derivative of $\sin(x)$ and $\cos(x)$**

*Before reading this section, it may be useful to read [Guide: Trigonometric identities \(radians\)](#).*

As with the derivative of  $e^x$  in the previous section, there are two important limits involving  $\cos$  and  $\sin$  that are needed before continuing. These are:

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h}.$$

To show these are true, you need techniques from *real analysis*, which is the study of functions on real numbers.

## Derivative of $\sin(x)$

### **i** Derivative of $\sin(x)$

The derivative of  $f(x) = \sin(x)$  is  $f'(x) = \cos(x)$ .

Starting off with the limit definition of differentiation and  $f(x) = \sin(x)$  gives

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

As in the case of  $e^x$ , expanding  $\sin(x+h)$  won't give a single  $h$  to cancel with! However, it could be that you can reduce this expression to the limits given above. Using the sum rule for  $\sin$  (see [Guide: Trigonometric identities \(radians\)](#)), you can write that

$$\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

The bones of the two given limits are here, and now it's time to uncover them. Factorising out  $\sin(x)$  and  $\cos(x)$  gives

$$\lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} = \lim_{h \rightarrow 0} \left( \sin(x) \left( \frac{\cos(h) - 1}{h} \right) + \cos(x) \left( \frac{\sin(h)}{h} \right) \right)$$

Using the two limits given above, and the fact that the values of  $\sin(x)$  and  $\cos(x)$  are independent of  $h$ :

$$\lim_{h \rightarrow 0} \left( \sin(x) \left( \frac{\cos(h) - 1}{h} \right) + \cos(x) \left( \frac{\sin(h)}{h} \right) \right) = \sin(x)(0) + \cos(x)(1) = \cos(x)$$

and so  $f'(x) = \cos(x)$  is the derivative of  $\sin(x)$ .

## Derivative of $\cos(x)$

### **i** Derivative of $\cos(x)$

The derivative of  $f(x) = \cos(x)$  is  $f'(x) = -\cos(x)$ .

Starting off with the limit definition of differentiation and  $f(x) = \cos(x)$  gives

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h}$$

As in the case of  $e^x$ , expanding  $\cos(x+h)$  won't give a single  $h$  to cancel with! However, it could be that you can reduce this expression to the limits given above. Using the sum rule for  $\cos$  (see [Guide: Trigonometric identities \(radians\)](#)), you can write that

$$\lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

The bones of the two given limits are here, and now it's time to uncover them. Factorizing out  $\cos(x)$  and  $\sin(x)$  gives

$$\lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} = \lim_{h \rightarrow 0} \left( \cos(x) \left( \frac{\cos(h) - 1}{h} \right) - \sin(x) \left( \frac{\sin(h)}{h} \right) \right)$$

Using the two limits given above, and the fact that the values of  $\sin(x)$  and  $\cos(x)$  are independent of  $h$ :

$$\lim_{h \rightarrow 0} \left( \cos(x) \left( \frac{\cos(h) - 1}{h} \right) - \sin(x) \left( \frac{\sin(h)}{h} \right) \right) = \cos(x)(0) - \sin(x)(1) = -\sin(x)$$

and so  $f'(x) = -\sin(x)$  is the derivative of  $\cos(x)$ .

## Further reading

[Click this link to go back to Guide: Introduction to differentiation and the derivative.](#)

[Click this link to go back to Guide: The product rule](#)

For questions on differentiation and the derivative, please go to [Questions: Introduction to differentiation and the derivative.](#)

## Version history

v1.0: initial version created in 04/25 by tdhc.

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