

Proof: Rules of differentiation

Tom Coleman

Summary

This proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, quotient, and chain rules - are true.

Before reading this proof sheet, it is essential that you read [Guide: Introduction to differentiation and the derivative](#). In addition, reading [\[Guide: Introduction to limits\]](#) is useful. Further reading will be illustrated where required.

The starting point of this proof sheet is the limit definition of the derivative of a function:

i Reminder of limit definition of the derivative

The **derivative of $f(x)$ with respect to x** is defined to be the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Sum and difference rules

i The sum and difference rules

(sum rule) The derivative of two functions $f(x)$ and $g(x)$ added together is the same as their derivatives $f'(x)$ and $g'(x)$ added together; that is, $(f+g)'(x) = f'(x) + g'(x)$ or

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

(difference rule) The derivative of one function $g(x)$ subtracted from another $f(x)$ is the same as the derivative $g'(x)$ subtracted from the derivative of $f'(x)$; that is $(f-g)'(x) = f'(x) - g'(x)$ or

$$\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

Proof of the sum rule

The strategy here is direct; put the function $(f + g)$ into the definition and pull the fraction apart to reveal the definitions of derivatives of f and g .

Let's start with $f(x)$ and $g(x)$ as two differentiable real-valued functions, with sum $(f + g)(x) = f(x) + g(x)$. Putting this into the limit definition of the derivative given above:

$$(f + g)'(x) = \lim_{h \rightarrow 0} \frac{(f + g)(x + h) - (f + g)(x)}{h}$$

Since $(f + g)(x) = f(x) + g(x)$, this becomes

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h}\end{aligned}$$

You can now split this into two fractions, one of which sets up the definition of $f'(x)$, and the other sets up the definition of $g'(x)$. So here

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right)\end{aligned}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \left(\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(f + g)'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} = f'(x) + g'(x)$$

as required.

Proof of the difference rule

Let's start with $f(x)$ and $g(x)$ as two differentiable real-valued functions, with difference $(f - g)(x) = f(x) - g(x)$. Putting this into the limit definition of the derivative given above:

$$(f - g)'(x) = \lim_{h \rightarrow 0} \frac{(f - g)(x + h) - (f - g)(x)}{h}$$

Using the fact that $(f - g)(x) = f(x) - g(x)$, and taking care of the signs in expansion, gives

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - (f(x) - g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - f(x) + g(x)}{h}\end{aligned}$$

You can now split this into two fractions, one of which sets up the definition of $f'(x)$, and the other sets up the definition of $g'(x)$. So here

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - f(x) + g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{f(x + h) - f(x)}{h} - \frac{g(x + h) - g(x)}{h} \right)\end{aligned}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \left(\frac{f(x + h) - f(x)}{h} - \frac{g(x + h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(f - g)'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} = f'(x) - g'(x)$$

as required.

Scaling rule

The scaling rule

The derivative of a function $f(x)$ multiplied by a real number c is the same as the derivative $f'(x)$ multiplied by c ; that is $(cf)'(x) = cf'(x)$ or

$$\frac{d}{dx}(cf(x)) = c \frac{df}{dx}$$

Proof of the scaling rule

This is similar to that of the sum and difference rules. Let's start with $f(x)$ as a differentiable real-valued function, with scaling $(cf)(x) = cf(x)$. Putting this into the limit definition of the derivative given above:

$$(cf)'(x) = \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h}$$

Using the fact that $(cf)(x) = cf(x)$ and factorizing out the c gives

$$\begin{aligned}(cf)'(x) &= \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h}\end{aligned}$$

Since the constant c does not depend on the variable in the limit h , you can use properties of limits (see [Guide: Introduction to limits]) to take the constant c out of the limit. This gives

$$\begin{aligned}(cf)'(x) &= \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(cf)'(x) = c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

as required.

Product rule

See [Guide: The product rule](#) for more about the product rule.

Here is the product rule, restated with $f(x) = u(x)$ and $g(x) = v(x)$ for visual ease in the proof that follows.

The product rule

Let $f(x)$ and $g(x)$ be two differentiable functions. Then the **product rule** says that

$$(fg)'(x) = \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

that is, the derivative of the product of $f(x)$ and $g(x)$ is equal to the product of $f(x)$ and the derivative of $g(x)$, plus the product of $g(x)$ and the derivative of $f(x)$.

This can also be written as

$$\frac{d}{dx}(f(x)g(x)) = f \frac{dg}{dx} + g \frac{df}{dx}.$$

Proof of the product rule

Here's why the product rule works. It requires a little more thought than the proof of the sum rule and the scaling rule; you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with $f(x)$ and $g(x)$ as two differentiable real-valued functions, with product $(fg)(x) = f(x)g(x)$. Putting this into the limit definition of the derivative given above:

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

Since $(fg)(x) = f(x)g(x)$, this becomes

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now, there's no way of pulling this apart. You have to force the issue slightly by creatively adding 0. The way to do this is to add $-f(x+h)g(x) + f(x+h)g(x)$ into the numerator, and factorize in slightly different ways. This is fine to do, as $-f(x+h)g(x) + f(x+h)g(x) = 0$. Doing this, and factorizing to manufacture the definitions of $f'(x)$ and $g'(x)$ gives:

$$\begin{aligned}
(fg)'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h \rightarrow 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right)
\end{aligned}$$

Using properties of limits, and the fact that $g(x)$ is constant as h varies to take it outside the limit gives

$$\begin{aligned}
(fg)'(x) &= \lim_{h \rightarrow 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right) \\
&= \left(\lim_{h \rightarrow 0} f(x+h) \right) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right)
\end{aligned}$$

Now, as h tends to 0, it follows that $f(x+h)$ tends to $f(x)$. The other two limits are the definitions of $g'(x)$ and $f'(x)$ respectively. Therefore, you can write that

$$\begin{aligned}
(fg)'(x) &= \left(\lim_{h \rightarrow 0} f(x+h) \right) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

which is the product rule.

Quotient rule

See [Guide: The quotient rule](#) for more about the quotient rule.

Here is the quotient rule, restated with $f(x) = u(x)$ and $g(x) = v(x)$ for visual ease in the proof that follows.

The quotient rule

Let $f(x)$ and $g(x)$ be two differentiable functions. Then the **quotient rule** says that

$$\left(\frac{f}{g} \right)'(x) = \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

that is, the derivative of $u(x)$ divided by $v(x)$ is equal to the difference of $u'(x)v(x)$ and $u(x)v'(x)$, divided by the square of the function $v(x)$.

This can also be written as

$$\frac{d}{dx} \left(\frac{u(x)}{v(x)} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

Proof of the quotient rule

Here's why the quotient rule works. Again, there is a step beyond algebraic manipulation where you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with $f(x)$ and $g(x)$ as two differentiable real-valued functions (with $g(x)$ not the zero function), with quotient $(f/g)(x) = f(x)/g(x)$. Putting this into the limit definition of the derivative gives

$$\left(\frac{f}{g} \right)'(x) = \lim_{h \rightarrow 0} \frac{\left(\frac{f}{g} \right)(x+h) - \left(\frac{f}{g} \right)(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

You can try your best to reduce this down by cross-multiplying to get a common denominator of the numerator of the limit. Then, you can drop that denominator down to get a single fraction. Doing this:

$$\begin{aligned} \left(\frac{f}{g} \right)'(x) &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \end{aligned}$$

Now, the hope is to pull this apart into two separate limits. Since you have no way of cancelling the h , you could try and manufacture the definitions of the derivatives of $f(x)$ and $g(x)$. You have to force the issue slightly by creatively adding 0; in this case, by adding $-f(x)g(x) + f(x)g(x) = 0$ to the numerator. In addition, you can use properties of limits to get rid of the $g(x)g(x+h)$ in the denominator. Doing these steps and simplifying gives:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right) \\
&= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right)
\end{aligned}$$

Now, factorizing this expression, using the properties of limits) and moving $g(x)$ and $-f(x)$ (notice that this needs to be done to ensure the correct definition of the derivative) out of the limits where appropriate gives

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right) \\
&= \left(\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right) \\
&= \left(\left(\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x))}{h} \right) + \left(\lim_{h \rightarrow 0} \frac{-f(x)(g(x+h) - g(x))}{h} \right) \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right) \\
&= \left(g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) - f(x) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right)
\end{aligned}$$

Now, as h tends to 0, it follows that $g(x+h)$ tends to $g(x)$, implying that the final limit tends to $1/(g(x))^2$. The other two limits are precisely the definitions of $f'(x)$ and $g'(x)$.

Therefore, you can write that

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \left(g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) - f(x) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)} \right) \\
&= (g(x)f'(x) - g'(x)f(x)) \cdot \frac{1}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

which is the quotient rule.

Chain rule

See [Guide: The chain rule](#) for more about the chain rule.

Here is the chain rule, restated with $f(x) = u(x)$ and $g(x) = v(x)$ for visual ease in the proof that follows.

i The chain rule

Let $f(x)$ and $g(x)$ be two differentiable functions. Then the **chain rule** says that

$$(f \circ g)'(x) = \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x)$$

that is, the derivative of $f(x)$ composed with $g(x)$ with respect to x is equal to the product of the derivative of f with respect to g and the derivative of g with respect to x .

This can also be written as

$$\frac{d}{dx} (f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Proof of the chain rule

Here's why the chain rule can be used. The idea is to take the limit definition of $(f \circ g)'(x)$ and split the limit into the product of the two derivatives $f'(g(x))$ and $g'(x)$. It requires more thought than the proofs of the product and chain rule, primarily due to the reliance on definitions of differentiation and the fact that it isn't a creative addition of 0 that splits the derivative, but a creative multiplication by 1 instead.

Alternative definition of derivative

Proving the chain rule requires the restatement of the limit definition of a derivative at a point a . Here are the two definitions side by side.

i Limit definition of the derivative (1)

The **derivative of $f(x)$ with respect to x at the point a** is defined to be

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

i Limit definition of the derivative (2)

The **derivative of $f(x)$ with respect to x at the point a** is defined to be

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

(See [Guide: Introduction to differentiability] for more.) To see that these are equal, start with definition (1). Here, h is the variable as the limit depends on h . Now, rescale the limit by setting $h = x - a$ (see [Guide: Properties of limits] for more). This gives

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x-a \rightarrow 0} \frac{f(a+(x-a)) - f(a)}{x-a}. \end{aligned}$$

As $x - a \rightarrow 0$, it follows that $x \rightarrow a$; in addition, $a + (x - a) = x$. So the limit becomes

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

so the definitions are the same at a point. Since a function f is differentiable on an interval I of real numbers if and only if $f'(a)$ exists for all a in I , it follows that you can use this definition for a differentiable function.

Intuition

The idea is to start with the second limit definition of the derivative above and put the function $(f \circ g)(x) = f(g(x))$ into the definition to get:

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Now, you would want to generate the derivative of f with respect to $g(x)$ at a and the derivative of g with respect to x at a . To do this, you can notice that the $x - a$ is already there for the derivative of g . You can multiply top and bottom of the fraction by $g(x) - g(a)$. Since $\frac{g(x) - g(a)}{g(x) - g(a)} = 1$, this does not change the value of the limit. This gives

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \left(\frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)} \right).$$

You can now pull this limit apart to attempt to make the two definitions of $f'(g(a))$ and $g'(a)$. Using the properties of limits to do this gives

$$\begin{aligned} (f \circ g)'(a) &= \lim_{x \rightarrow a} \left(\frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{g(x) - g(a)}{g(x) - g(a)} \right) \\ &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}. \end{aligned}$$

The second of these terms is $g'(a)$, which is what you want. The first of these terms **would**

be the definition of $f'(g(a))$... if the limit was $g(x) \rightarrow g(a)$ rather than $x \rightarrow a$. Here is the problem, because you **cannot guarantee** the behaviour of $g(x) - g(a)$ as x gets closer to a ; it could be that $g(x) - g(a) = 0$, which is a big problem. In fact, it could be that as x gets closer to a , then $g(x) - g(a)$ could be 0 in infinitely many different places. This needs to be rectified.

Overcoming the technicality

The idea is to 'fill in' the places where $g(x) - g(a) = 0$, by defining the value of the function $\frac{f(g(x)) - f(g(a))}{g(x) - g(a)}$ at these points. You can define the function

$$\phi(y) = \begin{cases} \frac{f(y) - f(g(a))}{y - g(a)} & \text{if } y \neq g(a) \\ f'(g(a)) & \text{if } y = g(a) \end{cases}$$

You can notice here that $f'(g(a))$ is already defined as f is a differentiable function, meaning that $f'(y)$ exists for all y .

Now, consider the expression

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a}.$$

The idea is to prove that

$$\frac{f(g(x)) - f(g(a))}{x - a} = \phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a}$$

for all x . This way, you can evaluate the limit of the right hand side instead of the left hand side. However, this does depend on whether or not $g(x) = g(a)$.

- If $g(x) \neq g(a)$, then $g(x) - g(a) \neq 0$. You can use the first part of the definition to say that

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{(g(x) - g(a))}{x - a}$$

Since $g(x) - g(a) \neq 0$, you can cancel these to get

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a}.$$

- If $g(x) = g(a)$ then $g(x) - g(a) = 0$ and also $f(g(x)) - f(g(a)) = 0$. This implies that

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \phi(g(x)) \cdot 0 = 0.$$

So they really are equal. Using this expression, together with the properties of limits gives

$$\begin{aligned}(f \circ g)'(a) &= \left(\frac{f(g(x)) - f(g(a))}{x - a} \right) \\ &= \lim_{x \rightarrow a} \phi(g(x)) \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}\end{aligned}$$

The idea is then to prove that these two limits exist; as then $(f \circ g)'(a)$ would exist. The second of these limits is precisely the definition of $g'(a)$, so let's focus on the limit of $\phi(g(x))$ as x tends to a . If this function $\phi \circ g$ is continuous at a (see [Guide: Introduction to continuity]) then this limit exists and is equal to $\phi(g(a))$. The function ϕ is defined whenever f is. Since f is differentiable, then it is continuous at every point, including $g(a)$; therefore, ϕ is continuous at $g(a)$. Since g is differentiable at a , then g is continuous at a . Therefore, by properties of continuous functions (see [Guide: Introduction to continuity]), $\phi \circ g$ is continuous at a . It follows that

$$\lim_{x \rightarrow a} \phi(g(x)) = \phi(g(a)) = f'(g(a))$$

by definition and so

$$\begin{aligned}(f \circ g)'(a) &= \lim_{x \rightarrow a} \phi(g(x)) \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \cdot g'(a)\end{aligned}$$

and this is the chain rule!

Further reading

[Click this link to go back to Guide: Introduction to differentiation and the derivative.](#)

[Click this link to go back to Guide: The product rule.](#)

[Click this link to go back to Guide: The quotient rule.](#)

[Click this link to go back to Guide: The chain rule](#)

For questions on differentiation and the derivative, please go to [Questions: Introduction to differentiation and the derivative.](#)

Version history

v1.0: initial version created in 05/25 by tdhc.

[This work is licensed under CC BY-NC-SA 4.0.](#)