Factorization

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Summary

Factorization is an important concept in mathematics, especially in algebra. It involves breaking down expressions into smaller parts called factors, which, when multiplied together, give the original expression. This process is essential for simplifying expressions, solving equations, and gaining a deeper understanding of mathematical structures. Factorization plays a key role in various areas of mathematics and is a valuable tool for tackling problems more efficiently.

Before reading this guide, it is recommended that you read Guide: Introduction to quadratic equations. Additionally, you may find it useful to read Guide: Completing the square

What is factorization?

Factorization is a process that involves rewriting an expression as a product of expressions. By doing this, you can break down complex algebraic equations into manageable parts, to then be solved.

As you have seen in Guide: Introduction to quadratic equations, quadratic expressions are important in mathematics, and these can be manipulated using factorization to help you solve them. For example, factorizing the expression x^2+5x+6 can transform it into $(x+3)(x+2)$, which makes it easier for you to find the values of x that satisfy the equation.

This guide will introduce you to different methods of factorization, such as identifying common factors, and factorizing quadratics of the form ax^2+bx+c, both where $a=1$, and $a \neq 1$.

Introducing key mathematical definitions

Definition of expression

An **expression** is a combination of numbers, variables, operations (such as addition or multiplication), and sometimes brackets. An expression represents a value or relationship but does not have an equals sign.

Example 1

$3x + 5$ is an expression. It contains the number 3, the variable x, and the addition operation.

Definition of term

A **term** is a single string of numbers, variables, and/or exponents, and it may also include multiplication or division. Expressions are made up of terms that are added or subtracted.

Example 2

In the expression $4x + 7$, both $4x$ and 7 are terms.

Definition of factor

A **factor** is a term that divides exactly into another term without leaving a remainder.

Example 3

The factors of 6 are $1, 2, 3$ and 6. 3 is a factor of 12, because $12 \div 3 = 4$. Similarly, $2x$ is a factor of $6x$, because $6x \div 2x = 3$.

Important

When working with fractions, any non-zero number can be a common factor. For example, $\frac{1}{2}$ is a factor of 1 and 2. However, for simplicity, this guide will focus on whole number factorizations.

Definition of common factor

A **common factor is a number**, variable or expression that divides evenly into each term of an expression.

Example 4

$$4x \div 2 = 2x$$

$$6 \div 2 = 3$$

2 is a common factor of $4x$ and 6, because both $4x$ and 6 can be divided by 2.

Definition of highest common factor (HCF)

The **highest common factor (HCF)** is the largest factor by which two or more terms can be divided without leaving a remainder. It is the greatest number that divides all the given terms evenly and is divisible by all other common factors of the numbers or expressions involved.

Example 5

What is the HCF of 8 and 12?
The factors of 8 are $1, 2, 4$ and 8.
The factors of 12 are $1, 2, 3, 4, 6,$ and 12.
They both share the factors $1, 2, and 4$, so these are all common factors.
The largest is 4, so the HCF of 8 and 12 is 4.

Factorizing single brackets

One approach to factorizing an expression is by identifying common factors in each term.

For example, expressions of the form $ax + ab$ can be factorized to become $a(x + b)$ where a is the common factor. To factorize an expression into its simplest form, you must factorize using the HCF.

When factorizing, you can take out any common factor shared by the terms in an expression. In real numbers, this means you can take out factors that appear in every term. For example, if an expression contains terms that are all divisible by a common number or variable, you can factor that out to simplify the expression.

The Steps to factorize an expression fully:

1. Find a common factor for all the numbers in the expression
2. Determine the HCF of all the common factors
3. Write a bracket, with the HCF on the outside, and the remaining terms inside.

Tip

Always check for the highest common factor before using other factorization methods!

Example 6

Factorize $5x + 15$.
The terms are $5x$ and 15.
The terms have a common factor of 5.
factorizing out the 5 from both terms gives:

$$5x + 15 = 5(x) + 5(3)$$

There are no other common factors between x and 3, so 5 is the HCF. You can then factorize out the HCF, which gives the final form:

$$5x + 15 = 5(x + 3)$$

Example 7

Factorize $6x^2 + 9x$
The terms have a common factor of 3.
Factorizing out the 3 from both terms gives:

$$6x^2 + 9x = 3(2x^2) + 3(3x)$$

Now looking at $2x^2$ and $3x$, both terms also contain an x, so you can factorize x out as-well to give:

$$6x^2 + 9x = 3x(2x) + 3x(3)$$

There are no other common factors between $2x$ and 3, so the HCF is 3.
You can then factorize out the HCF, which gives the final form:

$$6x^2 + 9x = 3x(2x + 3)$$

Example 8

Factorize $x(x-4)+3(2x-8)$
Here, instead of the common factor being a single term, such as 2 or x, the common factor is the bracketed expression $(x-4)$.
factorizing out $(x-4)$ from both terms gives:

$$x(x-4)+3(2x-8)=(x-4)(x)+(3)(2(x-4))$$

$$x(x-4)+3(2x-8)=(x-4)(x)+(6)(x-4)$$

There are no further factors between x and 6, so $(x-4)$ is the HCF. You can then factorize out the HCF, which gives the final form:

$$x(x-4)+3(2x-8)=(x-4)(x+6)$$

Factorizing double brackets

Factorizing quadratic expressions involves rewriting a quadratic equation (in the form ax^2+bx+c) as a product of two bracketed expressions. This process is essential in algebra and helps simplify expressions, whilst also being able to solve quadratic equations and find their roots.

There are two main cases when factorizing quadratics that you will learn about:

1. When $a=1$
2. When $a\neq 1$

Case 1: $a=1$

When the quadratic expression has the form x^2+bx+c, you need to find two numbers that multiply to c, and add to b. This can be seen as:

$$x^2+(d+e)x+de=(x+d)(x+e)$$

However, it is best to see this explained with an example.

Example 9

Factorize $x^2 + 7x + 10$

You want to find two numbers that multiply to make 10 and add to make 7.

The positive, whole number, factors of 10 are $1, 2, 5,$ and 10.

You want to look at the factors and see which pairings multiply to make 10.These are:

$$1 \cdot 10 = 10$$

$$2 \cdot 5 = 10$$

Using this, you need to select the pairing that adds to make 7:

$$1 + 10 = 11 \neq 7$$

So, this factor pairing does not have the ability to factorize this quadratic.

$$2 + 5 = 7$$

This is true, so the correct factors are 2 and 5. You can now write the factorized form by writing two brackets and putting the variable x at the start of each one:

$$(x + ?)(x + ?)$$

Write one factor in the first bracket and the other factor in the second bracket, to give:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

Important

The order of the factors don't matter, but the signs do! (EXAMPLE)

Now consider an example with a negative bx term.

Tip

If the bx term is negative, and the $+c$ is positive, the two factors must both be negative, as two negatives multiplied together creates a positive!

Example 10

Factorize $x^2 - 5x + 6$

You want to find two numbers that multiply to make 6 and add to make -5.

Now, you need to consider all the factors, positive and negative, of 6.

The factors of 6 are $1, -1, 2, -2, 3, -3, 6$, and -6.

You want to look at the factors and see which pairings multiply to make 6 and add to -5. These are:

$$-2 \cdot -3 = 6$$

$$-2 + -3 = -5$$

So, -2 and -3 are factors of the quadratic.

Now, put them into the brackets like you practised in Example 9:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Now consider an example where the $+c$ term is negative

Tip

If the $+c$ term is negative, and the bx term is positive, one factor must be negative, and the other positive, as a positive and a negative multiplied together creates a negative!

Important

The order of the factors don't matter, but the signs do!

Example 11 is an example of this - if the signs of the factors were swapped, it would result in a different quadratic to the one you have been asked to factorize.

Example 11

Factorize $x^2 + x - 2$

Now, you need to consider all the factors, positive and negative, of -2.

The factors of 6 are $1, -1, 2$, and -2.

You want to look at the factors and see which pairings multiply to make -2 and add to 1. These are:

$$2 \cdot -1 = -2$$

$$2 + -1 = 1$$

So 2 and -1 are factors of the quadratic.

Now, put them into the brackets like you practised in Example 9:

$$x^2 + x - 1 = (x + 2)(x - 1)$$

Case 2: $a \neq 1$

For quadratic expressions of the form $ax^2 + bx + c$ they can be factorized as follows:

$$ax^2 + bx + c = (ax + b)(x + c)$$

You need to multiply a and b together and write out the factor pairs. You must then select the factor pairs that add to b, and so you find it more helpful to visualise the quadratic equation in this way:

$$dx^2 + (e + df)x + ef = (dx + e)(x + f)$$

These are the steps to factorize a quadratic:

1. Multiply a and c together
2. Find two numbers that multiply to give $a \cdot c$, and add to give b.
3. Rewrite the middle term using these two numbers.
4. factorize by grouping.

This is best seen with an example.

Example 12

Factorize $2x^2 + 5x + 3$

You need to multiply 2 and 3 together;

$$2 \cdot 3 = 6$$

Write out the factor pairs of 6 that add to make 5:

The factor pairs of 6 are $(1, 6)$ and $(2, 3)$

The factor pair that multiplies to give 6 and adds to give 5 is $(2, 3)$.

$$2 + 3 = 5$$

$$2 \cdot 3 = 6$$

Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier.

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3 = 2x(x + 1) + 3(x + 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization! You can remove the $(x + 1)$ common factor, to give the result:

$$2x(x + 1) + 3(x + 1) = (2x + 3)(x + 1)$$

And this gives you your final factorized answer:

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

Example 13

Factorize $3x^2 - 2x - 5$

You need to multiply 3 and -5 together:

$$3 \cdot -5 = -15$$

Write out the factor pairs of -15 that add to make -2

The factor pairs of -15 are $(1, -15), (-1, 15), (3, -5)$, and $(-3, 5)$.

The factor pair that multiplies to give -15 and adds to give -2 is $(3, -5)$.

$$3 + -5 = -2$$

$$3 \cdot -5 = -15$$

Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier.

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5 = 3x(x + 1) - 5(x + 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization! You can remove the $(x + 1)$ common factor, to give the result:

$$3x(x + 1) - 5(x + 1) = (3x - 5)(x + 1)$$

And this is your final factorized answer:

$$3x^2 - 2x - 5 = (3x + 5)(x + 1)$$

Example 14

Factorize $4x^2 + 4x - 3$
You need to multiply 4 and -3 together, which gives -12.
Write out the number pair that multiplies to make -12, and that adds to make 4:

$$(-2, 6)$$

Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier.

$$4x^2 + 4x - 3 = 4x^2 - 2x + 6x - 3$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:
Note - Remember to use the HCF!

$$4x^2 + 4x - 3 = 4x^2 - 2x + 6x - 3 = 2x(2x - 1) + 3(2x - 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization! You can remove the $(2x + 3)$ common factor, to give the result:

$$2x(2x - 1) + 3(2x - 1) = (2x - 1)(2x + 3)$$

And this is your final factorized answer:

$$4x^2 + 4x - 3 = 2x(2x - 1) + 3(2x - 1) = (2x - 1)(2x + 3)$$

Using factorization to solve quadratic equations

By factorizing, you can break down a quadratic expression into simplified expressions, making it easier to find values that satisfy the equation.

Once a quadratic expression is factorized, you can solve it by setting each bracket equal to zero.

Example 15

Remember from Example 10;

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Setting this equal to zero;

$$(x - 2)(x - 3) = 0$$

Now, set each individual bracket equal to zero.

$$(x - 2) = 0$$

$$(x - 3) = 0$$

Now you can solve each one for x:

$$x = 2$$

$$x = 3$$

Once solved, you can see the solutions of $x^2 - 5x + 6$ are $x = 2$ and $x = 3$.

Using tools like Desmos, you can verify your solutions by graphing the quadratic function and identifying the points where it intersects with the x axis.

Seen below is an example....

(quadratic desmos, where interesections with x axis are highlighted and stated?)

Quick check problems

You are given four expressions below. Match the correct expression to the correct equivalent factorized expression.

(a) $5(2 - x)$
(b) $5(x - 2)$
(c) $10(1 - x)$
(d) $5(2 + x)$

(A) $10 + 5x =$

(B) $-10 + 5x =$
(C) $10 - 5x =$
(D) $10 - 10x =$

You are given three expressions below. Factorize them into their simplest form.

(1) $4x^2y + 6xy^2$

(2) $x^2 + 6x + 9$

(3) $4x^2 - x - 3$

Find the solutions of the quadratic equation below.

(4) $2x^2 + 3x - 2 = 0$

Further reading

For more questions on the subject, please go to Questions: Completing the square.

Version history

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