Further sigma notation

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Summary

Sigma notation is used to express many additions at once. Understanding what this notation is, how it works, and how to manipulate them is a valuable skill to learn for use in almost any area of mathematics.

Properties

In this section you will learn about a few properties of sigma notation which means you'll have a toolkit to rearrange sums!

The first property you'll learn about sigma notation is *distribuitivity*. This property allows you to take constants from inside the sigma notation to outside the summation.

i Distribuitivity

Let a_k, a_{k+1}, \dots, a_n be a sequence of numbers (where k and n are integers with $k \leq n$) and C be any constant. Then

$$\sum_{i=k}^{n} Ca_i = C \sum_{i=k}^{n} a_i.$$

You can see this is true by writing the entire sum out, like this:

$$\begin{array}{rcl} \sum_{i=k}^{n} Ca_{i} & = & Ca_{k} + Ca_{k+1} + Ca_{k+2} + \ldots + Ca_{n} \\ & = & C(a_{k} + a_{k+1} + a_{k+2} + \ldots + a_{n}) \\ & = & C\sum_{i=k}^{n} a_{k} \end{array}$$

i Example 7

What is the value of $\sum_{n=2}^5 6n^2$? Using distributivity, $\sum_{n=2}^5 6n^2 = 6\sum_{n=2}^5 n^2$. From Example 2, you know that $\sum_{n=2}^5 n^2 = 54$. Therefore, $6\sum_{n=2}^5 n^2 = 6\times 54 = 324$.

Double sums

Sometimes, you'll want to multiply two sums together. This can be written succinctly using something called *double sums*.

i Double sums

Let $a_k, a_{k+1}, \ldots, a_n$ and $b_t, b_{t+1}, \ldots, b_m$ be two sequences of numbers (where k, n, t, and m are integers with $k \leq n$ and $t \leq m$). Then the **double sum** $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$ is defined as

$$\begin{array}{lcl} \sum_{i=k}^n \sum_{j=t}^m a_i b_j & = & a_k b_t + a_k b_{t+1} + \ldots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} \\ & & + \ldots + a_{k+1} b_m + \ldots + a_n b_m. \end{array}$$

Tip

You might find it easier to remember the above by thinking of $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$ as $a_1(\sum_{j=t}^m b_j) + a_2(\sum_{j=t}^m b_j) + \ldots + a_n(\sum_{j=t}^m b_j).$

You will now see how this relates to multiplying two sums together. Suppose that $a_k, a_{k+1}, \ldots, a_n$ and $b_t, b_{t+1}, \ldots, b_m$ are like above, and consider the product $(\sum_{i=k}^n a_i)(\sum_{j=t}^m b_j)$. Writing it all out and performing the multiplication, you get

$$\begin{split} \left(\sum_{i=k}^{n}a_{i}\right)\left(\sum_{j=t}^{m}b_{j}\right) &= (a_{k}+a_{k+1}+\ldots+a_{n})(b_{t}+b_{t+1}+\ldots+b_{m}) \\ &= a_{k}b_{t}+a_{k}b_{t+1}+\ldots+a_{k}b_{m}+a_{k+1}b_{t}+a_{k+1}b_{t+1}+ \\ &\qquad \qquad \ldots+a_{k+1}b_{m}+a_{k+2}b_{t}+\ldots+a_{n}b_{m} \\ &= \sum_{i=k}^{n}\sum_{j=t}^{m}a_{i}b_{j} \end{split}$$

You can write this as a result:

Double sums and products of two sums

Let $a_k, a_{k+1}, \ldots, a_n$ and $b_t, b_{t+1}, \ldots, b_m$ be two sequences of numbers (where k, n, t, and m are integers with $k \leq n$ and $t \leq m$). Then

$$\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j = (\sum_{i=k}^{n} a_i)(\sum_{j=t}^{m} b_j).$$

i Example 10

Write (1+2+3+4+5+6)(2+4+6+8+10+12) as a double sum and as a product of two sums.

First, notice you can write out the above expression in the form (1)(2) + (1)(4) + ...(1)(12) + (2)(2) + (2)(4)...(3)(2) + ...(6)(12)

From the definition above you may now rewrite the expression to the double sum

$$\sum_{i=1}^{6} \sum_{j=1}^{6} i * 2j$$

using the distrubitivity property this can be written as

$$2\sum_{i=1}^{6}\sum_{j=1}^{6}ij$$

This can then be written using the product of two sums rule above to

$$2\sum_{i=1}^{6} i \sum_{j=1}^{6} j$$

It is evident that the two sums are the same with different index variables this means that they can be combined to form

$$2\sum_{k=1}^{6}k^2$$

k has been used to differentiate the new sum from the ones involving i and j before but as always the choice of index variable is relatively unimportant

Quick check problems

1. What is the value of $\sum_{i=2}^{6} i$.

Answer: The value of the above is: ____.

2. Given $\sum_{j=1}^{100} i$ Identify the index of the sum.

Answer: The index is _

3. You are given several statements below based on the properties of sums. Identify whether they are true or false.

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- (a) The sum 3+6+9+12 can be expressed as $\sum_{i=0}^4 3i$ Answer: TRUE / FALSE.
- (b) The sum -1+1-1+1 can be expressed as $\sum_{i=1}^4 -i$ Answer: TRUE / FALSE.
- (c) $\sum_{i=1}^{100} i = \sum_{i=0}^{101} i$ Answer: TRUE / FALSE.
- (d) $\sum_{i=1}^{100} 6i = 6\sum_{i=0}^{100} i$ Answer: TRUE / FALSE.
- (e) $\sum_{i=1}^{100} 9i + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 27i^2$ Answer: TRUE / FALSE.
- (f) $\sum_{i=1}^{100} 12i \sum_{i=1}^{100} 4i = 8 \sum_{i=1}^{100} i$ Answer: TRUE / FALSE.
- 4. You are given several statements below based on the properties of sums. Identify whether they are true or false.
- (a) $\sum_{i=1}^{10} \sum_{j=2}^6 ij$ can be expressed as $\left(\sum_{i=2}^6 i\right) \left(\sum_{j=1}^{10} j\right)$ Answer: TRUE / FALSE.
- (b) $\left(\sum_{i=1}^5 2i\right) \left(\sum_{j=5}^{10} 3j\right)$ can be expressed as $6\left(\sum_{i=1}^5 \sum_{j=5}^{10} ij\right)$ \$ Answer: TRUE / FALSE.
- (c) The sum (1+2+3+4+5+6)(-1-2)(3+6+9) can be expressed as $\sum_{i=1}^6 \sum_{j=1}^2 \sum_{k=1}^3 -3ijk$ \$ Answer: TRUE / FALSE.

Further reading