Multivariate chain rule

Donald Campbell

Summary

The multivariate chain rule is used in calculus to differentiate a function when its variables depend on other variables. It shows how the change in one variable affects the whole function by considering how the intermediate variables change. It is useful in modelling systems where one quantity depends on several factors. For example, climate models use variables like temperature, humidity and ocean currents to predict weather patterns.

*Before reading this guide, it is recommended that you read* [*Guide: Introduction to partial differentiation*](introtopartialdifferentiation.qmd) *and [Guide: The chain rule].*

# What is the multivariate chain rule?

As seen in [Guide: The chain rule], the chain rule tells you how to differentiate a function with respect to another variable when depends on .

This rule can be extended to functions of multiple variables through the **multivariate chain rule**. The extended rule tells you how to compute derivatives of composite functions (functions of functions) involving multiple variables. It allows you to track how changes in **independent variables** influence a final quantity through intermediate variables called **dependent variables**.

The multivariate chain rule is used in problems involving parametric multivariate functions, such as those describing motion along curved surfaces, transformations in computer graphics, and population models where growth depends on multiple time-varying factors.

Before moving on, it is important to define what the difference is between dependent and independent variables.

# Dependent and independent variables

When a function **depends** on a specific variable, this means that the function is written with the variable in its rule. For example, the function depends on and as the function is written in terms of and .

A **multivariate (or multivariable) function** is a function that depends on more than one variable.

Consider a multivariate function where

The variables , and are referred to as **dependent variables** as they are functions that depend on (can be written in terms of) the variables and . Even though , and are referred to as dependent variables, they are actually functions of the variables and . In this case, the dependent variables can be written in functional form as , and to emphasize dependency.

The variables and are referred to as **independent variables** as they do not depend on any other variables.

Occasionally, the dependent variables in this guide may be referred to as **intermediate variables** since they connect the independent variables and to the final quantity .

# Functions of two dependent variables with one independent variable

Suppose you have a function . Here, depends on two dependent variables and , however both and are functions of a single independent variable .

This means that, even though initially looks like a function of two dependent variables and , it really depends on a single independent variable . You can write entirely in terms of .

The goal is to find .

You cannot differentiate directly with respect to because depends intermediately on both and . Instead, you need to use the multivariate chain rule.

|  |
| --- |
| Multivariate chain rule for functions of one independent variable |
| Let where is a differentiable function of and . Also let and be differentiable functions of .  The function can then be differentiated with respect to according to |

|  |
| --- |
| Important |
| The left-most derivative is written with a straight as is considered a function of only, at this stage. Here, is written using the expressions for and in terms of .  Derivatives such as and are written with a curly as is a function that depends on two variables and .  Derivatives such as and are written with a straight as is a function that depends only on a single variable . |

As changes, both dependent variables and change as they are both functions of the independent variable . This causes to change as it is a function of the variables and which are also changing.

|  |  |
| --- | --- |
|  | **Example 1**  Let where and . Use the multivariate chain rule to find .  First, calculate the derivatives required for the multivariate chain rule.  You need to substitute the derivatives into the multivariate chain rule, then substitute for and . |

|  |
| --- |
| Important |
| Make sure that your final answer is always expressed only in terms of independent variables. Your final answer should not include any intermediate variables such as and in this example. |

# Functions of two dependent variables with two independent variables

Suppose you have a function . Here, depends on two dependent variables and , however now both and are functions of two independent variables and .

This means that, even though initially looks like a function of two dependent variables and , it really depends on two independent variables and . You can write entirely in terms of and .

Since there are now two independent variables, the goal is to find both and .

You cannot differentiate directly with respect to or because depends intermediately on both and . Instead, you need to use the multivariate chain rule.

|  |
| --- |
| Multivariate chain rule for functions of two independent variables |
| Let where is a differentiable function of and . Also let and be differentiable functions of and .  The function can then be differentiated with respect to and with |

|  |
| --- |
| Important |
| The left-most derivatives and are written with a curly as is considered a function of both and at this stage. Here, is written using the expressions for and in terms of and .  Notice that the , , and derivatives are now written with a curly instead of a straight as and are functions that now depend on two independent variables and . This means the derivatives are now partial derivatives instead of ordinary derivatives. |

As and change, both dependent variables and change as they are both functions of the independent variables and . This causes to change as it is a function of variables and which are also changing.

|  |  |
| --- | --- |
|  | **Example 2**  Let where and . Use the multivariate chain rule to find and .  First, calculate the derivatives required for the multivariate chain rule.  You need to substitute the partial derivatives into the multivariate chain rule, then substitute for and .  Firstly, calculate the partial derivative of with respect to :  Now, calculate the partial derivative of with respect to : |

# Generalized multivariate chain rule

Suppose you have a function . Here, depends on dependent variables where each of these variables is a function of independent variables .

The previous cases investigated were

where:

* Case 1 corresponds to the chain rule for one dependent variable and one independent variable - see [Guide: The chain rule].
* Case 2 corresponds to the chain rule for two dependent variables and one independent variable.
* Case 3 corresponds to the chain rule for two dependent variables and two independent variables.

The multivariate chain rule can be generalized as follows.

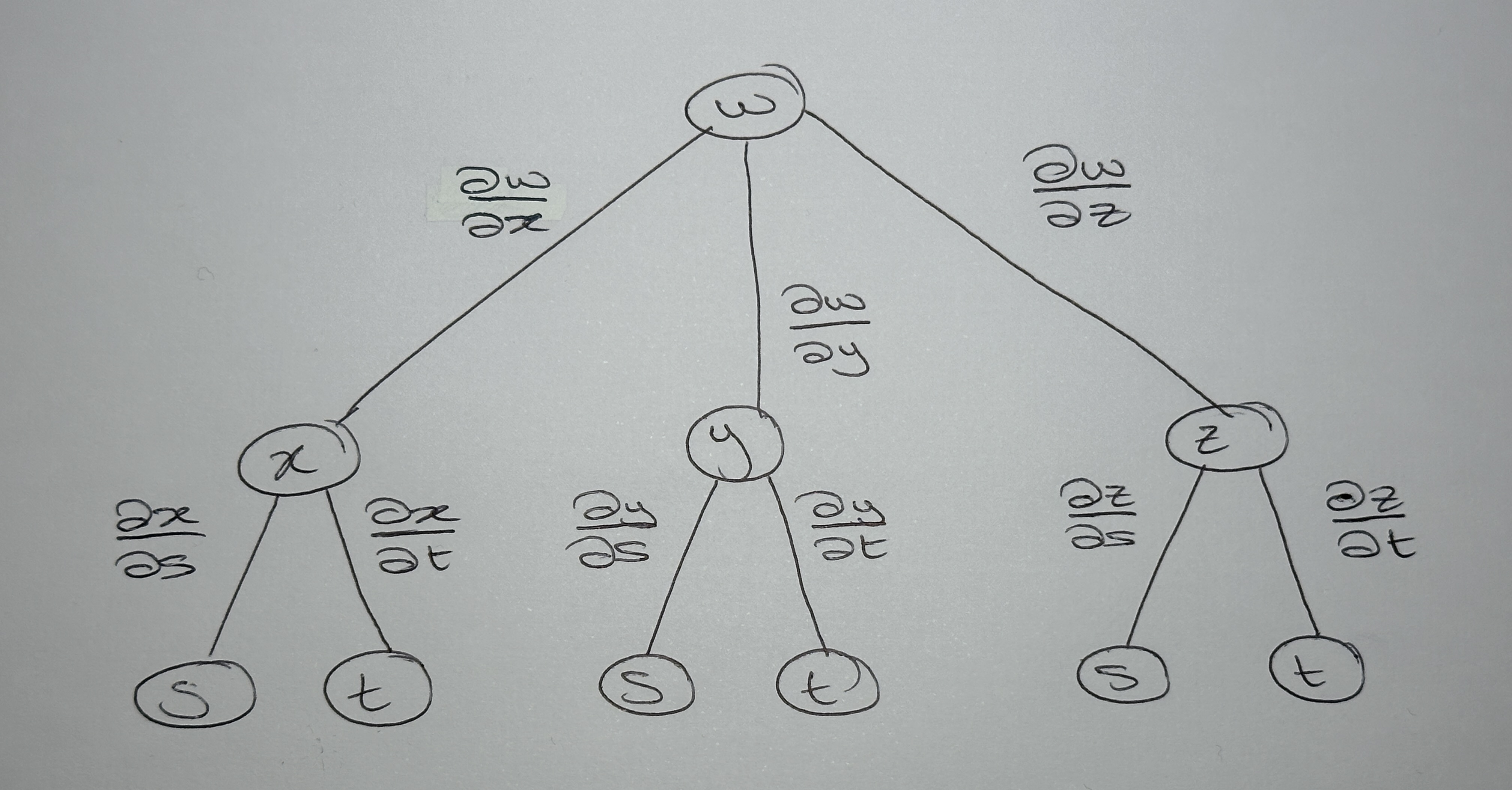
|  |
| --- |
| Generalized multivariate chain rule |
| Let where is a differentiable function of . Also let the following dependent variables be differentiable functions of .  The function can then be differentiated with respect to with |

|  |
| --- |
| Important |
| A particular case exists where the variables each depend on only one independent variable . In this case, a straight should be used instead of a curly .  Similarly, if there is only one dependent variable , then a straight should be used instead of a curly for the derivatives of with respect to . |

|  |  |
| --- | --- |
|  | **Example 3**  Consider the case with dependent variables and independent variables.  Let where  Use the multivariate chain rule to find and .  Firstly, calculate the derivatives required for the multivariate chain rule.  You need to substitute the partial derivatives into the multivariate chain rule, then substitute for , and .  You can calculate the partial derivative of with respect to as:  You can calculate the partial derivative of with respect to as: |

# Tree representation of the multivariate chain rule

There is a useful way for determining the required form of the multivariate chain rule without needing to remember the generalized rule above. The following diagram is a **tree representation** of the dependencies between variables in a multivariate function.



Tree diagram for determining the form of the multivariate chain rule in Example 3.

At the top of the tree is the function , which depends on three intermediate variables , and . Each of these three intermediate variables depend on independent variables and . The tree shows all possible paths from to the independent variables and .

Each path represents a **chain of partial differentiation** (a product of partial derivatives) when applying the chain rule. All paths are combined by summing these chains at the end to find the complete partial derivative of with respect to the relevant independent variable.

To find , follow all paths leading from to .

You can perform a similar method to find .

|  |
| --- |
| Important |
| Remember that if a variable has only a single dependency on another variable (no separate branch), a straight should be used in the derivative instead of a curly . |

# Quick check problems

1. Which of the following correctly describes a dependent variable?
2. Consider the function where and .
3. How many dependent variables and independent variables are there in this function?
4. State the required form of the multivariate chain rule for calculating .
5. Find the partial derivative of with respect to given by .
6. Find the full derivative of with respect to given by .
7. Find the partial derivative of with respect to given by .
8. Find the full derivative of with respect to given by .
9. Use the multivariate chain rule to compute the full derivative of with respect to given by , expressing your answer in terms of only.

# Further reading

[For more questions on the subject, please go to Questions: Multivariate chain rule.](../questions/qs-multivariatechainrule.qmd)

[For a way to differentiate implicit functions of more than one variable, please see Guide: Multivariate implicit differentiation.](multivariateimplicitdifferentiation.qmd)

## Version history

v1.0: initial version created 05/25 by Donald Campbell as part of a University of St Andrews VIP project.

[This work is licensed under CC BY-NC-SA 4.0.](https://creativecommons.org/licenses/by-nc-sa/4.0/?ref=chooser-v1)