

# Introduction to factorization

Millie Pike

## Summary

Factorization is an important concept in mathematics, especially in algebra. It involves breaking down expressions into smaller parts called factors which, when multiplied together, give the original expression. This process is essential for simplifying expressions, solving equations, and gaining a deeper understanding of mathematical structures. Factorization plays a key role in various areas of mathematics and is a valuable tool for tackling problems more efficiently.

*Before reading this guide, it is recommended that you read [Guide: Introduction to quadratic equations](#). Additionally, you may find it useful to read [Guide: Completing the square](#) for factorizing with a remainder.*

## What is factorization?

Factorization is the process of rewriting an expression as a product of simpler expressions. This makes it easier to work with and understand more complicated expressions.

As you have seen in [Guide: Introduction to quadratic equations](#), quadratic expressions are important in mathematics, and these can be manipulated by using factorization to help you solve them. For example, factorizing the expression  $x^2 + 5x + 6$  transforms it into  $(x+3)(x+2)$ , which makes it easier for you to find the values of  $x$  that satisfy the equation  $x^2 + 5x + 6 = 0$ .

This guide will introduce you to different methods of factorization, such as identifying common factors and factorizing quadratics of the form  $ax^2 + bx + c$ , both where  $a = 1$ , and  $a \neq 1$ .

## Introducing key concepts

### Definition of term

A **term** is a single string of numbers, variables, and/or exponents, and it may also include multiplication or division. Expressions are made up of terms that are added or subtracted.

### Example 1

In the expression  $4x + 7$ , both  $4x$  and  $7$  are terms.

### Definition of expression

An **expression** is a combination of numbers, variables, operations (such as addition or multiplication), and sometimes brackets. An expression represents a value or relationship but does not have an equals sign.

### Example 2

$3x + 5$  is an expression. It contains the number  $3$ , multiplied by the variable  $x$ , and the addition operation.

### Definition of factor

A **factor** is an expression that divides exactly into another expression.

### Example 3

When working in whole numbers:

The factors of  $6$  are  $1, 2, 3$  and  $6$ .

$3$  is a factor of  $12$ , because  $\frac{12}{3} = 4$ . Similarly,  $2x$  is a factor of  $6x$ , because  $\frac{6x}{2x} = 3$ .

### Important

When working with fractions, any non-zero number can be a common factor. For example,  $\frac{1}{2}$  is a factor of  $1$  and  $2$ .

**This guide will focus on whole number factorizations.**

### Definition of common factor

A **common factor** is a number, variable or expression that divides evenly into each term of an expression.

#### **i** Example 4

What is a common factor of  $4x$  and  $6$ ?

Here,

$$\frac{4x}{2} = 2x \quad \text{and} \quad \frac{6}{2} = 3.$$

$2$  is a common factor of  $4x$  and  $6$ , because both  $4x$  and  $6$  can be divided by  $2$ .

#### **i** Definition of highest common factor (HCF)

The **highest common factor (HCF)** is the largest factor by which two or more terms can be divided by, without leaving a remainder. It is the greatest number that divides all the given terms evenly and is divisible by all other common factors of the numbers or expressions involved.

#### **i** Example 5

What is the HCF of  $8$  and  $12$ ?

The factors of  $8$  are  $1, 2, 4$  and  $8$ . The factors of  $12$  are  $1, 2, 3, 4, 6$ , and  $12$ .

They both share the factors  $1, 2$ , and  $4$ , so these are all common factors.

The largest is  $4$ , so the HCF of  $8$  and  $12$  is  $4$ .

## Factorizing single brackets

One approach to factorizing an expression is by identifying common factors in each term.

For example, expressions of the form  $ax + ab$  can be factorized to become  $a(x + b)$  where  $a$  is the common factor. To factorize an expression into its **simplest form**, you must factorize using the HCF.

When factorizing, you can take out any common factor shared by the terms in an expression. In real numbers, this means you can take out factors that appear in every term. For example, if an expression contains terms that are all divisible by a common number or variable, you can factor that out to simplify the expression.

The steps to factorize an expression fully:

1. Find all common factors for all the terms in the expression.
2. Determine the HCF of all the common factors.

3. Write a bracket, with the HCF on the outside, and the remaining terms inside.



Tip

Always check for the highest common factor before using other factorization methods!

### **i** Example 6

Factorize  $5x + 15$ .

The terms are  $5x$  and  $15$  and the terms have a common factor of  $5$ .

Factorizing out the  $5$  from both terms gives:

$$5x + 15 = 5(x) + 5(3)$$

There are no other common factors of  $x$  and  $3$ , so  $5$  is the HCF. So, the final factorized form is:

$$5x + 15 = 5(x + 3)$$

### **i** Example 7

Factorize  $6x^2 + 9x$ .

The terms have a common factor of  $3$ . Factorizing out the  $3$  from both terms gives:

$$6x^2 + 9x = 3(2x^2) + 3(3x)$$

Now looking at  $2x^2$  and  $3x$ , both terms also contain an  $x$ , so you can factorize  $x$  out aswell to give:

$$6x^2 + 9x = 3x(2x) + 3x(3)$$

There are no other common factors between  $2x$  and  $3$ , so the HCF is  $3$ . So, the final factorized form is:

$$6x^2 + 9x = 3x(2x + 3)$$

### **i** Example 8

Factorize  $x(x - 4) + 3(2x - 8)$ .

Here, instead of the common factor being a single term, such as 2 or  $x$ , the common factor is the bracketed expression  $(x - 4)$ .

Factorizing out  $(x - 4)$  from both terms gives:

$$x(x - 4) + 3(2x - 8) = (x - 4)(x) + (3)(2(x - 4))$$

Expanding  $(3)(2(x - 4))$  out gives:

$$(3)(2(x - 4)) = (6)(x - 4)$$

So the expression becomes:

$$x(x - 4) + 3(2x - 8) = (x - 4)(x) + (6)(x - 4)$$

There are no further common factors of  $x$  and 6, so  $(x - 4)$  is the HCF. So, the final factorized form is:

$$x(x - 4) + 3(2x - 8) = (x - 4)(x + 6)$$

## Factorizing double brackets

Factorizing quadratic expressions involves breaking down quadratic expressions (of the form  $ax^2 + bx + c$ , where  $a \neq 0$ ) into a product of two brackets, which can make it easier to solve the quadratic expression. This process is essential in algebra and helps simplify expressions, whilst also being able to solve quadratic equations and find their roots. For more information on this, see [Guide: Introduction to quadratic equations](#).

There are two main cases when factorizing quadratics that you will learn about. One, where  $a = 1$ , and another where  $a \neq 1$ .

### **Case 1:** $a = 1$

When the quadratic expression has the form  $x^2 + bx + c$ , the aim is to find two numbers  $d$  and  $e$  that multiply to  $c$ , and add to  $b$ , such that  $de = c$  and  $d + e = b$ .

This works because:

$$(x + d)(x + e) = x^2 + dx + ex + de$$

And when factorizing the  $x$  term, you get:

$$(x + d)(x + e) = x^2 + (d + e)x + de$$

Which is the desired form of the quadratic expression, as  $b = d + e$  and  $c = de$ .

However, it is best to see this explained with an example.

### Example 9

Factorize  $x^2 + 7x + 10$ .

You want to find two numbers that multiply to make 10 and add to make 7.

The positive factors of 10 are 1, 2, 5, and 10.

You want to look at the factors and see which pairings multiply to make 10. These are:

$$1 \cdot 10 = 10 \quad \text{and} \quad 2 \cdot 5 = 10$$

Since  $2 + 5 = 7$  and  $1 + 10 = 11 \neq 7$ , you pick the correct factors of 10, which are 2 and 5. You can now write the factorized form of your quadratic expression by writing two brackets and putting the variable  $x$  at the start of each one:

$$(x + ?)(x + ?)$$

Write one factor in the first bracket and the other factor in the second bracket, to give the final factorized form:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

Here's an example with a negative  $bx$  term.

### Tip

If the  $bx$  term is negative, and the  $+c$  is positive, the two factors must both be negative, as two negatives multiplied together creates a positive.

### Example 10

Factorize  $x^2 - 5x + 6$ .

You want to find two numbers that multiply to make 6 and add to make  $-5$ .

Now, you need to consider all the factors, positive and negative, of 6.

The factors of 6 are 1,  $-1$ , 2,  $-2$ , 3,  $-3$ , 6, and  $-6$ .

You want to look at the factors and see which pairings multiply to make 6 and add to  $-5$ . These are:

$$(-2) \cdot (-3) = 6 \quad \text{and} \quad (-2) + (-3) = -5$$

So,  $-2$  and  $-3$  are the correct factors to use.

Now, put them into the brackets in a similar fashion to Example 9, to give the final factorized form:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Finally, here's an example where the  $+c$  term is negative.

### Tip

If the  $+c$  term is negative, and the  $bx$  term is positive, one factor must be negative, and the other positive, as a positive and a negative multiplied together gives a negative.

### Example 11

Factorize  $x^2 + x - 2$ .

Now, you need to consider all the factors, positive and negative, of  $-2$ .

The factors of  $-2$  are 1,  $-1$ , 2, and  $-2$ . You want to look at the factors and see which pairings multiply to make  $-2$  and add to 1. These are:

$$2 \cdot (-1) = -2 \quad \text{and} \quad 2 + (-1) = 1$$

So 2 and  $-1$  are the correct factors to use.

Now, put them into the brackets in a similar fashion to Example 9, to give the final factorized form:

$$x^2 + x - 2 = (x + 2)(x - 1)$$

**! Important**

The order of the factors don't matter, but the signs do!

Example 11 is an example of this - if the signs of the factors were swapped, it would result in a different quadratic to the one you have been asked to factorize. If you swapped the signs, you would get  $(x - 2)(x + 1)$ . When expanded, this gives:

$$(x - 2)(x + 1) = x^2 - 2x + x - 2$$

$$(x - 2)(x + 1) = x^2 - x - 2$$

$$x^2 - x - 2 \neq x^2 + x - 2$$

$x^2 - x - 2$  is not the original quadratic expression you were asked to factorize, which shows you how if a factor has the incorrect sign, it will not correctly factorize the required quadratic expression.

**💡 Tip**

If  $c = 0$ , you can factorize the quadratic expression using the common factor  $x$ . This is because if  $c = 0$ , then:

$$ax^2 + bx + c = ax^2 + bx$$

Which can then be factorized as:

$$ax^2 + bx = x(ax + b)$$

If  $a = 1$ , this is still true, and is factorized as:

$$x^2 + bx = x(x + b)$$

Example 7 is an example where  $a \neq 0$  and  $c = 0$ .

## Case 2: $a \neq 1$

When a quadratic expression has the form  $ax^2 + bx + c$ , it can still be factorized, but it requires a different method from the one you used when  $a = 1$ .

The aim is to find two numbers  $d$  and  $e$  such that:

$$d \cdot e = a \cdot c \quad \text{and} \quad d + e = b$$



This works because you can rewrite the middle term ( $bx$ ) as  $dx + ex$ , which allows you to factorize by grouping. So if you rewrite the expression, it becomes:

$$ax^2 + bx + c = ax^2 + dx + ex + c$$

You can now group the expression:

$$ax^2 + dx + ex + c = (ax^2 + dx) + (ex + c)$$

Factorizing each bracket gives:

$$(ax^2 + dx) + (ex + c) = ax \left( x + \frac{d}{a} \right) + e \left( x + \frac{c}{e} \right)$$

If you let  $d \cdot e = a \cdot c = f$ , then  $\frac{d}{a} = \frac{c}{e} = f$ , so you can substitute this in to give:

$$(ax^2 + dx) + (ex + c) = ax(x + f) + e(x + f)$$

This means that both groups will share a common factor  $(x + f)$ , and the expression can be factorized further to give:

$$(ax^2 + dx) + (ex + c) = (ax + e)(x + f)$$

This is the fully factorized form of the quadratic expression  $ax^2 + bx + c$  when  $a \neq 1$ .

These are the steps to factorize a quadratic:

1. Multiply  $a$  and  $c$  together.
2. Find two numbers that multiply to give  $a \cdot c$ , and add to give  $b$ .
3. Rewrite the middle term using these two numbers.
4. Factorize by grouping.

**i Example 12**

Factorize  $2x^2 + 5x + 3$ .

You need to multiply 2 and 3 together;

$$2 \cdot 3 = 6$$

Now you need to consider all the factors of 6.

The factors of 6 are 1, 2, 3, and 6.

You want to look at the factors and see which pairings multiply to make 6, and add to 5. These are:

$$2 + 3 = 5 \quad \text{and} \quad 2 \cdot 3 = 6$$

Now you can rewrite the original expression, but this time splitting the middle ( $bx$ ) term into the two factors you found earlier.

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3 = 2x(x + 1) + 3(x + 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization!

You can remove the  $(x + 1)$  common factor, to give the result:

$$2x(x + 1) + 3(x + 1) = (2x + 3)(x + 1)$$

And this gives you your final factorized answer:

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

**i Example 13**

Factorize  $3x^2 - 2x - 5$ .

You need to multiply 3 and  $-5$  together:

$$3 \cdot (-5) = -15$$

You need to find the factors of  $-15$  that multiply to give  $-15$ , and add to give  $-2$ .  
These are:

$$3 + (-5) = -2$$

$$3 \cdot (-5) = -15$$

Now you can rewrite the original expression, but this time splitting the middle ( $bx$ ) term into the two factors you found earlier.

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5 = 3x(x + 1) - 5(x + 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization!  
You can remove the  $(x + 1)$  common factor, to give the result:

$$3x(x + 1) - 5(x + 1) = (3x - 5)(x + 1)$$

And this is your final factorized answer:

$$3x^2 - 2x - 5 = (3x + 5)(x + 1)$$

**! Important**

Note that a quadratic  $ax^2 + bx + c$  can sometimes be factorized in the form:

$$(px + q)(rx + s) \quad \text{where } r \neq 0$$

However, if you factorize out  $r^2$ , you get:

$$(px + q)(rx + s) = \frac{1}{r^2} \left( \left( \frac{px}{r} + \frac{q}{r} \right) \left( x + \frac{s}{r} \right) \right)$$

This matches the factorized result of  $ax^2 + bx + c$ , so a factorization like  $(px + q)(rx + s)$  can be expressed using rational numbers to match any desired form. This can be seen as:

$$\left( \frac{px}{r} + \frac{q}{r} \right) \left( x + \frac{s}{r} \right) = (ax + b)(x + c) \quad , \text{ where } a = \frac{p}{r}, \quad b = \frac{q}{r} \quad \text{and} \quad c = \frac{s}{r}$$

### **i** Example 14

Factorize  $4x^2 + 4x - 3$ .

You need to multiply 4 and  $-3$  together, which gives  $-12$ .

You need to find the factors of  $-12$  that multiply to give  $-12$ , and add to give 4. These are:

$$(-2) + 6 = 4$$

$$(-2) \cdot 6 = -12$$

Now you can rewrite the original expression, but this time splitting the middle ( $bx$ ) term into the two factors you found earlier.

$$4x^2 + 4x - 3 = 4x^2 - 2x + 6x - 3$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

Note - Remember to use the HCF!

$$4x^2 + 4x - 3 = 4x^2 - 2x + 6x - 3 = 2x(2x - 1) + 3(2x - 1)$$

Now, this is similar to the type of factorizing you did in single bracket factorization! You can remove the  $(2x - 1)$  common factor, to give the result:

$$2x(2x - 1) + 3(2x - 1) = (2x - 1)(2x + 3)$$

And this is your final factorized answer:

$$4x^2 + 4x - 3 = 2x(2x - 1) + 3(2x - 1) = (2x - 1)(2x + 3)$$

## Using factorization to solve quadratic equations

By factorizing, you can break down a quadratic expression into simpler expressions. This process makes it easier to understand the structure of the expression and to identify values that satisfy the equation you are solving.

Once a quadratic expression is factorized, and the equation is set equal to zero, you can find the values of the variable that make each bracket equal to zero. These values are known as the roots or solutions of the quadratic equation. However, it's important to remember that this method only works directly when the expression is equal to zero—if it's equal to another number, other steps are needed before factorizing.

### **i** Example 15

Solve  $x^2 - 5x + 6 = 0$ .

Remember from Example 10;

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Setting this equal to zero;

$$(x - 2)(x - 3) = 0$$

Now, you can set each individual bracket equal to zero:

$$(x - 2) = 0 \quad \text{and} \quad (x - 3) = 0$$

Now you can solve each one for  $x$ :

$$x = 2 \quad \text{and} \quad x = 3$$

Once solved, you can see the solutions of  $x^2 - 5x + 6$  are  $x = 2$  and  $x = 3$ .

Using tools like Desmos, you can verify your solutions by graphing the quadratic equation and identifying the points where it intersects with the  $x$ -axis. The roots of the quadratic equations are the points of intersection with the  $x$ -axis.

See [Calculator: Solving quadratic equations] to see this in action!

(quadratic desmos, where interesections with x axis are highlighted and stated?)

## Quick check problems

1. You are given four expressions below. Match the correct expression to the correct equivalent factorized expression.

- (a)  $5(2 - x)$
- (b)  $5(x - 2)$
- (c)  $10(1 - x)$
- (d)  $5(2 + x)$

- (A)  $10 + 5x =$
- (B)  $-10 + 5x =$
- (C)  $10 - 5x =$
- (D)  $10 - 10x =$

2. You are given three expressions below. Factorize them into their simplest form.

- (a)  $4x^2y + 6xy^2$
- (b)  $x^2 + 6x + 9$
- (c)  $4x^2 - x - 3$

3. Find the solutions of the quadratic equation  $2x^2 + 3x - 2 = 0$ .

## Further reading

For more questions on the subject, please go to [Questions: Introduction to factorization](#).

For more information on solving quadratic equations, please see [Guide: Introduction to quadratic equations](#).

## Version history

v1.0: initial version created 03/25 by Millie Pike, as part of a University of St Andrews VIP project.

[This work is licensed under CC BY-NC-SA 4.0](#).