

# Proof: Rules of differentiation

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## Summary

This proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, quotient, and chain rules - are true.

*Before reading this proof sheet, it is essential that you read [Guide: Introduction to differentiation and the derivative](#). In addition, reading [\[Guide: Introduction to limits\]](#) is useful. Further reading will be illustrated where required.*

The starting point of this proof sheet is the limit definition of the derivative of a function:

**i** Reminder of limit definition of the derivative

The **derivative of  $f(x)$  with respect to  $x$**  is defined to be the function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

## Sum and difference rules

**i** The sum and difference rules

**(sum rule)** The derivative of two functions  $f(x)$  and  $g(x)$  added together is the same as their derivatives  $f'(x)$  and  $g'(x)$  added together; that is,  $(f+g)'(x) = f'(x) + g'(x)$  or

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

**(difference rule)** The derivative of one function  $g(x)$  subtracted from another  $f(x)$  is the same as the derivative  $g'(x)$  subtracted from the derivative of  $f'(x)$ ; that is  $(f-g)'(x) = f'(x) - g'(x)$  or

$$\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

## Proof of the sum rule

The strategy here is direct; put the function  $(f+g)$  into the definition and pull the fraction apart to reveal the definitions of derivatives of  $f$  and  $g$ .

Let's start with  $f(x)$  and  $g(x)$  as two differentiable real-valued functions, with sum  $(f + g)(x) = f(x) + g(x)$ . Putting this into the limit definition of the derivative given above:

$$(f + g)'(x) = \lim_{h \rightarrow 0} \frac{(f + g)(x + h) - (f + g)(x)}{h}$$

Since  $(f + g)(x) = f(x) + g(x)$ , this becomes

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h}\end{aligned}$$

You can now split this into two fractions, one of which sets up the definition of  $f'(x)$ , and the other sets up the definition of  $g'(x)$ . So here

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right)\end{aligned}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{aligned}(f + g)'(x) &= \lim_{h \rightarrow 0} \left( \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(f + g)'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} = f'(x) + g'(x)$$

as required.

## Proof of the difference rule

Let's start with  $f(x)$  and  $g(x)$  as two differentiable real-valued functions, with difference  $(f - g)(x) = f(x) - g(x)$ . Putting this into the limit definition of the derivative given above:

$$(f - g)'(x) = \lim_{h \rightarrow 0} \frac{(f - g)(x + h) - (f - g)(x)}{h}$$

Using the fact that  $(f - g)(x) = f(x) - g(x)$ , and taking care of the signs in expansion, gives

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - (f(x) - g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - f(x) + g(x)}{h}\end{aligned}$$

You can now split this into two fractions, one of which sets up the definition of  $f'(x)$ , and the other sets up the definition of  $g'(x)$ . So here

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - g(x + h) - f(x) + g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(x + h) - f(x)}{h} - \frac{g(x + h) - g(x)}{h} \right)\end{aligned}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{aligned}(f - g)'(x) &= \lim_{h \rightarrow 0} \left( \frac{f(x + h) - f(x)}{h} - \frac{g(x + h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(f - g)'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} = f'(x) - g'(x)$$

as required.

## Scaling rule

### The scaling rule

The derivative of a function  $f(x)$  multiplied by a real number  $c$  is the same as the derivative  $f'(x)$  multiplied by  $c$ ; that is  $(cf)'(x) = cf'(x)$  or

$$\frac{d}{dx}(cf(x)) = c \frac{df}{dx}$$

## Proof of the scaling rule

This is similar to that of the sum and difference rules. Let's start with  $f(x)$  as a differentiable real-valued function, with scaling  $(cf)(x) = cf(x)$ . Putting this into the limit definition of the derivative given above:

$$(cf)'(x) = \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h}$$

Using the fact that  $(cf)(x) = cf(x)$  and factorizing out the  $c$  gives

$$\begin{aligned}(cf)'(x) &= \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h}\end{aligned}$$

Since the constant  $c$  does not depend on the variable in the limit  $h$ , you can use properties of limits (see [Guide: Introduction to limits]) to take the constant  $c$  out of the limit. This gives

$$\begin{aligned}(cf)'(x) &= \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\end{aligned}$$

and so, by the limit definition of the derivative

$$(cf)'(x) = c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

as required.

## Product rule

See [Guide: The product rule](#) for more about the product rule.

Here is the product rule, restated with  $f(x) = u(x)$  and  $g(x) = v(x)$  for visual ease in the proof that follows.

### The product rule

Let  $f(x)$  and  $g(x)$  be two differentiable functions. Then the **product rule** says that

$$(fg)'(x) = \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

that is, the derivative of the product of  $f(x)$  and  $g(x)$  is equal to the product of  $f(x)$  and the derivative of  $g(x)$ , plus the product of  $g(x)$  and the derivative of  $f(x)$ .

This can also be written as

$$\frac{d}{dx} (f(x)g(x)) = f \frac{dg}{dx} + g \frac{df}{dx}.$$

## Proof of the product rule

Here's why the product rule works. It requires a little more thought than the proof of the sum rule and the scaling rule; you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with  $f(x)$  and  $g(x)$  as two differentiable real-valued functions, with product  $(fg)(x) = f(x)g(x)$ . Putting this into the limit definition of the derivative given above:

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

Since  $(fg)(x) = f(x)g(x)$ , this becomes

$$(fg)'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now, there's no way of pulling this apart. You have to force the issue slightly by creatively adding 0. The way to do this is to add  $-f(x+h)g(x) + f(x+h)g(x)$  into the numerator, and factorize in slightly different ways. This is fine to do, as  $-f(x+h)g(x) + f(x+h)g(x) = 0$ . Doing this, and factorizing to manufacture the definitions of  $f'(x)$  and  $g'(x)$  gives:

$$\begin{aligned} (fg)'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right) \end{aligned}$$

Using properties of limits, and the fact that  $g(x)$  is constant as  $h$  varies to take it outside the limit gives

$$\begin{aligned}
(fg)'(x) &= \lim_{h \rightarrow 0} \left( f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right) \\
&= \left( \lim_{h \rightarrow 0} f(x+h) \right) \left( \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) + g(x) \left( \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right)
\end{aligned}$$

Now, as  $h$  tends to 0, it follows that  $f(x+h)$  tends to  $f(x)$ . The other two limits are the definitions of  $g'(x)$  and  $f'(x)$  respectively. Therefore, you can write that

$$\begin{aligned}
(fg)'(x) &= \left( \lim_{h \rightarrow 0} f(x+h) \right) \left( \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) + g(x) \left( \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \right) \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

which is the product rule.

## Quotient rule

See [Guide: The quotient rule](#) for more about the quotient rule.

Here is the quotient rule, restated with  $f(x) = u(x)$  and  $g(x) = v(x)$  for visual ease in the proof that follows.

### The quotient rule

Let  $f(x)$  and  $g(x)$  be two differentiable functions. Then the **quotient rule** says that

$$\left( \frac{f}{g} \right)'(x) = \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

that is, the derivative of  $u(x)$  divided by  $v(x)$  is equal to the difference of  $u'(x)v(x)$  and  $u(x)v'(x)$ , divided by the square of the function  $v(x)$ .

This can also be written as

$$\frac{d}{dx} \left( \frac{u(x)}{v(x)} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

## Proof of the quotient rule

Here's why the quotient rule works. Again, there is a step beyond algebraic manipulation where you have to manufacture the definition of the derivative in one place by using a creative addition by 0.

So, let's start with  $f(x)$  and  $g(x)$  as two differentiable real-valued functions (with  $g(x)$  not the zero function), with quotient  $(f/g)(x) = f(x)/g(x)$ . Putting this into the limit definition of the derivative gives

$$\left(\frac{f}{g}\right)'(x) = \lim_{h \rightarrow 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

You can try your best to reduce this down by cross-multiplying to get a common denominator of the numerator of the limit. Then, you can drop that denominator down to get a single fraction. Doing this:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}\end{aligned}$$

Now, the hope is to pull this apart into two separate limits. Since you have no way of cancelling the  $h$ , you could try and manufacture the definitions of the derivatives of  $f(x)$  and  $g(x)$ . You have to force the issue slightly by creatively adding 0; in this case, by adding  $-f(x)g(x) + f(x)g(x) = 0$  to the numerator. In addition, you can use properties of limits to get rid of the  $g(x)g(x+h)$  in the denominator. Doing these steps and simplifying gives:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\ &= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h}\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right) \\ &= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right)\end{aligned}$$

Now, factorizing this expression, using the properties of limits) and moving  $g(x)$  and  $-f(x)$  (notice that this needs to be done to ensure the correct definition of the derivative) out of the

limits where appropriate gives

$$\begin{aligned}
 \left(\frac{f}{g}\right)'(x) &= \left(\lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h}\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right) \\
 &= \left(\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{h}\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right) \\
 &= \left(\left(\lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x))}{h}\right) + \left(\lim_{h \rightarrow 0} \frac{-f(x)(g(x+h) - g(x))}{h}\right)\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right) \\
 &= \left(g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\right) - f(x) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}\right)\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right)
 \end{aligned}$$

Now, as  $h$  tends to 0, it follows that  $g(x+h)$  tends to  $g(x)$ , implying that the final limit tends to  $1/(g(x))^2$ . The other two limits are precisely the definitions of  $f'(x)$  and  $g'(x)$ .

Therefore, you can write that

$$\begin{aligned}
 \left(\frac{f}{g}\right)'(x) &= \left(g(x) \left(\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}\right) - f(x) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}\right)\right) \left(\lim_{h \rightarrow 0} \frac{1}{g(x)g(x+h)}\right) \\
 &= (g(x)f'(x) - g'(x)f(x)) \cdot \frac{1}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
 \end{aligned}$$

which is the quotient rule.

## Chain rule

See [Guide: The chain rule](#) for more about the chain rule.

Here is the chain rule, restated with  $f(x) = u(x)$  and  $g(x) = v(x)$  for visual ease in the proof that follows.

### The chain rule

Let  $f(x)$  and  $g(x)$  be two differentiable functions. Then the **chain rule** says that

$$(f \circ g)'(x) = \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x)$$

that is, the derivative of  $f(x)$  composed with  $g(x)$  with respect to  $x$  is equal to the product of the derivative of  $f$  with respect to  $g$  and the derivative of  $g$  with respect to  $x$ .

This can also be written as

$$\frac{d}{dx} (f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx}.$$



## Proof of the chain rule

Here's why the chain rule can be used. The idea is to take the limit definition of  $(f \circ g)'(x)$  and split the limit into the product of the two derivatives  $f'(g(x))$  and  $g'(x)$ . It requires more thought than the proofs of the product and chain rule, primarily due to the reliance on definitions of differentiation and the fact that it isn't a creative addition of 0 that splits the derivative, but a creative multiplication by 1 instead.

## Alternative definition of derivative

Proving the chain rule requires the restatement of the limit definition of a derivative at a point  $a$ . Here are the two definitions side by side.

### **i** Limit definition of the derivative (1)

The **derivative of  $f(x)$  with respect to  $x$  at the point  $a$**  is defined to be

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

### **i** Limit definition of the derivative (2)

The **derivative of  $f(x)$  with respect to  $x$  at the point  $a$**  is defined to be

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

(See [Guide: Introduction to differentiability] for more.) To see that these are equal, start with definition (1). Here,  $h$  is the variable as the limit depends on  $h$ . Now, rescale the limit by setting  $h = x - a$  (see [Guide: Properties of limits] for more). This gives

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x-a \rightarrow 0} \frac{f(a+(x-a)) - f(a)}{x-a}. \end{aligned}$$

As  $x - a \rightarrow 0$ , it follows that  $x \rightarrow a$ ; in addition,  $a + (x - a) = x$ . So the limit becomes

$$f'(a) = \lim_{x-a \rightarrow 0} \frac{f(a+(x-a)) - f(a)}{x-a} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}$$

so the definitions are the same at a point. Since a function  $f$  is differentiable on an interval

$I$  of real numbers if and only if  $f'(a)$  exists for all  $a$  in  $I$ , it follows that you can use this definition for a differentiable function.

## Intuition

The idea is to start with the second limit definition of the derivative above and put the function  $(f \circ g)(x) = f(g(x))$  into the definition to get:

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Now, you would want to generate the derivative of  $f$  with respect to  $g(x)$  at  $a$  and the derivative of  $g$  with respect to  $x$  at  $a$ . To do this, you can notice that the  $x - a$  is already there for the derivative of  $g$ . You can multiply top and bottom of the fraction by  $g(x) - g(a)$ . Since  $\frac{g(x) - g(a)}{g(x) - g(a)} = 1$ , this does not change the value of the limit. This gives

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \left( \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{(g(x) - g(a))}{(g(x) - g(a))} \right).$$

You can now pull this limit apart to attempt to make the two definitions of  $f'(g(a))$  and  $g'(a)$ . Using the properties of limits to do this gives

$$\begin{aligned} (f \circ g)'(a) &= \lim_{x \rightarrow a} \left( \frac{f(g(x)) - f(g(a))}{x - a} \cdot \frac{(g(x) - g(a))}{(g(x) - g(a))} \right) \\ &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \rightarrow a} \frac{(g(x) - g(a))}{x - a}. \end{aligned}$$

The second of these terms is  $g'(a)$ , which is what you want. The first of these terms **would** be the definition of  $f'(g(a))$ ... if the limit was  $g(x) \rightarrow g(a)$  rather than  $x \rightarrow a$ . Here is the problem, because you **cannot guarantee** the behaviour of  $g(x) - g(a)$  as  $x$  gets closer to  $a$ ; it could be that  $g(x) - g(a) = 0$ , which is a big problem. In fact, it could be that as  $x$  gets closer to  $a$ , then  $g(x) - g(a)$  could be 0 in infinitely many different places. This needs to be rectified.

## Overcoming the technicality

The idea is to 'fill in' the places where  $g(x) - g(a) = 0$ , by defining the value of the function  $\frac{f(g(x)) - f(g(a))}{g(x) - g(a)}$  at these points. You can define the function

$$\phi(y) = \begin{cases} \frac{f(y) - f(g(a))}{y - g(a)} & \text{if } y \neq g(a) \\ f'(g(a)) & \text{if } y = g(a) \end{cases}$$

You can notice here that  $f'(g(a))$  is already defined as  $f$  is a differentiable function, meaning that  $f'(y)$  exists for all  $y$ .

Now, consider the expression

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a}.$$

The idea is to prove that

$$\frac{f(g(x)) - f(g(a))}{x - a} = \phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a}$$

for all  $x$ . This way, you can evaluate the limit of the right hand side instead of the left hand side. However, this does depend on whether or not  $g(x) = g(a)$ .

- If  $g(x) \neq g(a)$ , then  $g(x) - g(a) \neq 0$ . You can use the first part of the definition to say that

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{(g(x) - g(a))}{x - a}$$

Since  $g(x) - g(a) \neq 0$ , you can cancel these to get

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a}.$$

- If  $g(x) = g(a)$  then  $g(x) - g(a) = 0$  and also  $f(g(x)) - f(g(a)) = 0$ . This implies that

$$\phi(g(x)) \cdot \frac{(g(x) - g(a))}{x - a} = \phi(g(x)) \cdot 0 = 0.$$

So they really are equal. Using this expression, together with the properties of limits gives

$$\begin{aligned} (f \circ g)'(a) &= \left( \frac{f(g(x)) - f(g(a))}{x - a} \right) \\ &= \lim_{x \rightarrow a} \phi(g(x)) \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \end{aligned}$$

The idea is then to prove that these two limits exist; as then  $(f \circ g)'(a)$  would exist. The second of these limits is precisely the definition of  $g'(a)$ , so let's focus on the limit of  $\phi(g(x))$  as  $x$  tends to  $a$ . If this function  $\phi \circ g$  is continuous at  $a$  (see [Guide: Introduction to continuity]) then this limit exists and is equal to  $\phi(g(a))$ . The function  $\phi$  is defined whenever  $f$  is. Since  $f$  is differentiable, then it is continuous at every point, including  $g(a)$ ; therefore,  $\phi$  is continuous at  $g(a)$ . Since  $g$  is differentiable at  $a$ , then  $g$  is continuous at  $a$ . Therefore, by properties of continuous functions (see [Guide: Introduction to continuity]),  $\phi \circ g$  is continuous at  $a$ . It follows that

$$\lim_{x \rightarrow a} \phi(g(x)) = \phi(g(a)) = f'(g(a))$$

by definition and so

$$\begin{aligned} (f \circ g)'(a) &= \lim_{x \rightarrow a} \phi(g(x)) \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \cdot g'(a) \end{aligned}$$

and this is the chain rule!

## Further reading

[Click this link to go back to Guide: Introduction to differentiation and the derivative.](#)

[Click this link to go back to Guide: The product rule.](#)

[Click this link to go back to Guide: The quotient rule.](#)

[Click this link to go back to Guide: The chain rule](#)

For questions on differentiation and the derivative, please go to [Questions: Introduction to differentiation and the derivative.](#)

## Version history

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