Introduction to differentiation and the derivative

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Summary

The idea of differentiation

Before reading this guide, it is recommended that you read [Guide: Introduction to limits], [Guide: Introduction to continuity] and [Guide: Properties of functions].

What is differentiation?

Functions can be used to model many real-life processes; such as the speed of a car, the performance of a stock market over time, and the growth rate of animal populations. You can use mathematical tools examined in the first three chapters to solve some of these problems (as well as a host of others).

Sometimes it is necessary to examine the rate of change of a particular process or phenomenon. For instance, the rate of change of the speed of a car is acceleration. Studying different rates of change of such a real life problem is often useful; for instance, by looking at acceleration of a car over time, you can determine how far the car goes. Rates of change are common in studying other, real-life processes, such as: the change in value of a company in a stock exchange in economics, the change in population of a species of animal in biology, or the rate of decay of radioactive material in chemistry.

The idea of investigating the rate of change of functions and their associated applications is known as **differential calculus**. The process of **differentiation** allows you to examine these properties of rates of change of functions. Importantly, this process is *general*, allowing you to investigate the rate of change of a function at any point. Differentiating a function $y = f(x)$ gives its *derivative*, which is written as $\dfrac{dy}{dx}$ or $f'(x)$.

This guide will look at the idea of differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

Gradients of a graph

The rate of change of a straight line $y = mx + c$ is its gradient m, and this doesn't change wherever you are on the line. However, the rate of change of a function like $f(x) = x^2$ is dependent on what point you are at — when x is small, the rate of change is correspondingly small; when x is large, the rate of change is larger. You can see this in the variable 'steepness' of the curve.

[FIGURE HERE]

The idea of examining the rate of change of a function is to look at the gradients of straight lines, with endpoints as values on the graph, that approximate the function. The smaller the straight line is, the better the approximation. So let h be some small real number. Then the line with endpoints $f(x + h)$ and $f(x)$ has vertical change $f(x + h) - f(x)$, and horizontal change $(x + h) - x = h$. Therefore, the gradient of this line is

$$\frac{f(x+h) - f(x)}{h}.$$

[FIGURE HERE]

The problem is when h tends towards 0, then $f(x+h)$ and $f(x)$ get closer and closer to one another, and the length of the line gets smaller and smaller. What use is there measuring the gradient of a line if the line disappears? The idea is to look at a parallel line, which has the same gradient.

Here, this convenient parallel line is the **tangent of** $f(x)$ **at** x; this is the straight line touching the curve at the point $(x, f(x))$, but does not intersect it at that point (see [Guide: Tangents]). Therefore, you can say that the gradient of the tangent to the curve at the point x is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and this consideration defines the **derivative** of a function.

Derivatives and differentiation

Following the investigation in the previous section, the derivative of a function can now be defined.

Definition of the derivative $f'(x)$

The **derivative of $f(x)$ with respect to x** is defined to be the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of $f(x)$ with respect to x is a function in its own right. The value of the derivative $f'(x)$ at the point $x = a$, defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the gradient of the tangent to $f(x)$ at the point a.
If this limit exists, then $f(x)$ is **differentiable at $x = a$**. If $f(x)$ is differentiable at every point of its domain, then it is a **differentiable function**. The process of finding $f'(x)$ given a function $f(x)$ is called **differentiating with respect to x**.

The function $f(x)$ is often defined in terms of a second variable y. If a function is written $y = f(x)$, then there is alternative notation for writing the derivative with respect to x:

Definition of dy/dx

Let $y = f(x)$ be a function of x. Then the **derivative of y with respect to x** can be written as $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$
The derivative of y with respect to x at a point $x = a$ is written as

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = f'(a)$$

.

Warning

Although $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ *looks* like a fraction, it **isn't**. However, sometimes it *behaves* like a fraction, and you can use this to remember certain results. However, this isn't an excuse to treat it as a fraction in standard mathematical work.

This notation was used by Leibniz to represent the derivative of a function $y = f(x)$. It's useful because it codifies the derivative as a rate of change (ratio of change in y to change in x) as well as explicitly naming the variable that you are differentiating with respect to. (This will be extremely useful in [Guide: Introduction to partial differentiation].) However, it's not so useful for expressing the value of the derivative at a particular point $x = a$; for instance

$$f'(a) \quad \text{compared to} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}.$$

The important thing to consider is this:

Tip

As both notations $f'(x)$ and $\frac{\mathrm{d}f}{\mathrm{d}x}$ have their advantages over the other, **it's important to be able to use both sets of notation interchangeably**.

There is one other thing that you should know before proceeding. What happens if you try and differentiate a function $f(x)$ on the variable x with respect to a *different* variable t? Well, as the derivative with respect to t measures the rate of change of a function with respect to t, and since $f(x)$ does not change as t changes, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(x)) = 0$$

So differentiating a function of some variable with respect to a different variable results in nothing. This is why it's important to specify the variable you are differentiating with respect to, and it will be particularly important when it comes to differentiating functions of **more than one** variable: see [Guide: Introduction to partial differentiation].

Differentiating well known functions

You can use the limit definition of the derivative to show that the following key functions have the following derivatives:

List of derivatives

- For any number a, the derivative of the constant function $f(x) = a$ is $f'(x) = 0$.
- For any number $n \neq 0$, the derivative of any function $f(x) = x^n$ is $f'(x) = nx^{n-1}$.
- The derivative of the exponential function $f(x) = e^x$ is $f'(x) = e^x$.
- The derivative of the sine function $f(x) = \sin(x)$ is $f'(x) = \cos(x)$, and the derivative of the cosine function $g(x) = \cos(x)$ is $g'(x) = -\sin(x)$. (Notice the minus sign here!)

- The derivative of the natural logarithm function $f(x) = \ln(x)$ is $f'(x) = \dfrac{1}{x}$.

All of these results are explained in [Proof sheet: Derivatives of functions from first principles].

Example 2

Here, the equation $y^4 - 10y^2 + 25 = 0$ may look like a quartic equation, but it is actually a quadratic equation. Using the laws of indices, you can rewrite the equation as $(y^2)^2 - 10y^2 + 25 = 0$. Therefore, the variable of the quadratic equation is y^2, and the coefficients are $a = 1$, $b = -10$, $c = 25$.

Example 3

You are given the equation $-e^{2x} + 4e^x - 5 = 0$. Using the laws of indices, you can rewrite the equation as $-(e^x)^2 + 4e^x - 5 = 0$. The variable of the quadratic equation is e^x, and the coefficients are $a = -1$, $b = 4$, $c = -5$. This is not the only solution to the coefficients; since the right-hand side is equal to 0, you can multiply the equation through by -1 to get $(e^x)^2 - 4e^x + 5 = 0$, which gives $a = 1$, $b = -4$ and $c = 5$. Both solutions are equally valid.

Example 4

You are given the equation $t + 1 = \frac{4}{t-3}$. This really is a quadratic equation! You can multiply both sides by $t - 3$ to get

$$(t+1)(t-3) = 4$$

You can then expand the brackets to get

$$t^2 + t + 3t + 3 = 4$$

and so $t^2 + 4t + 3 = 4$. Finally, you are able to subtract 4 from both sides to get $t^2 + 4t - 1 = 0$. It follows that the variable of the quadratic equation is t, and the coefficients are $a = 1$, $b = 4$, $c = -1$.

Solving a quadratic equation

To solve the quadratic equation, you could use one of three methods:

- You could **factorise** the quadratic equation $ax^2 + bx + c = 0$ into linear equations $(mx+n)(px+q)$, then work out the roots when each of these linear equations is zero. See (Guide: Factorisation) for more.

- You could **complete the square** in order to reduce the quadratic equation $ax^2 + bx + c = 0$ into the form $(x + b/2a)^2 = d$, and then solve from there (not forgetting the negative root). See Guide: Completing the square for more.

- You could **use the quadratic formula**; for a quadratic equation $ax^2 + bx + c = 0$, the two roots to the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Each method is equally valid, but some may involve more work than others. It is up to you to decide which method is best for each quadratic you encounter; but it is thoroughly advised that if you are not sure which method is best, then the quadratic formula is the one to choose. See Guide: Using the quadratic formula for more.

The discriminant

What the roots of the quadratic formula look like are determined by the term $b^2 - 4ac$; this term has a special name.

> **i** The discriminant
>
> The term $D = b^2 - 4ac$ is known as the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

There are then three separate cases for solutions to quadratic equations.

- If $D = b^2 - 4ac$ is positive, then $\sqrt{D}$ is a real number and the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{D}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{D}}{2a}$$

These two roots are both real numbers and distinct from each other. You can observe this behaviour on a graph in Figure 1; the parabola crosses the x-axis in two places.

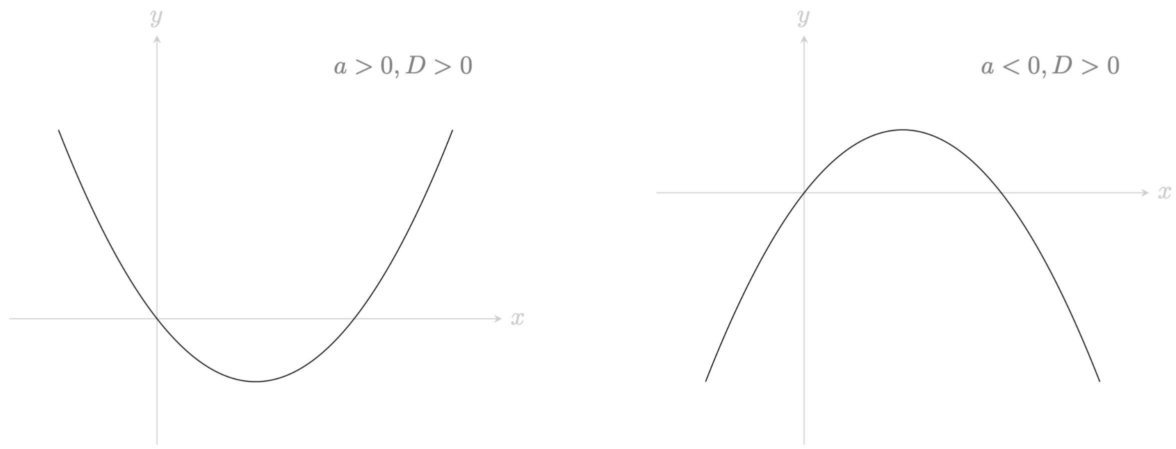

Figure 1: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D > 0$.

- If $D = b^2 - 4ac = 0$ is zero, then $\sqrt{D} = 0$. In this case, the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b}{2a} \qquad \text{and} \qquad x = \frac{-b}{2a}$$

  These two roots are given by the same real number. To be sure that you express both roots, you can write '$x = -b/2a$ twice'. You can observe this behaviour on a graph in Figure 2; the parabola touches the x-axis in exactly one place.

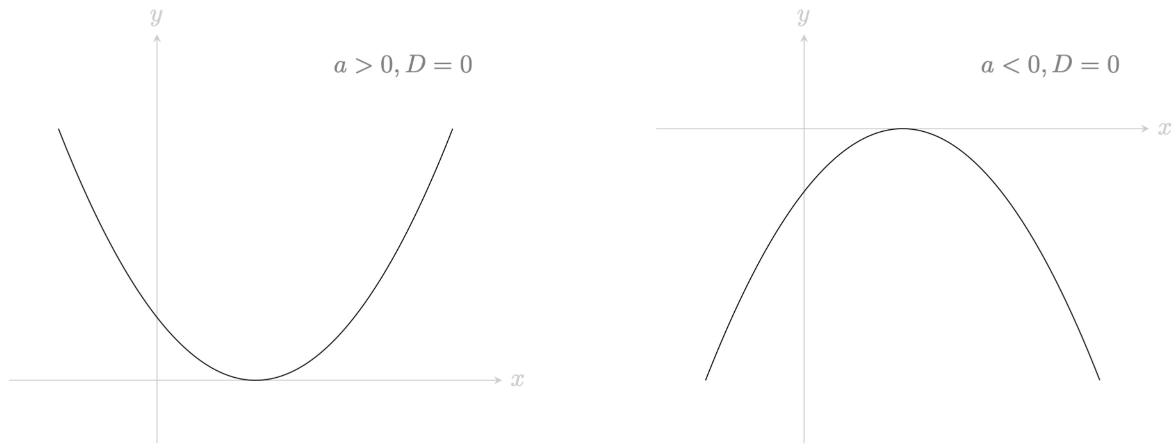

Figure 2: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D = 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D = 0$.

- If $D = b^2 - 4ac$ is negative, then $\sqrt{D}$ is not a real number. In this case, the two roots of the quadratic equation are **complex numbers**. You can express the two roots of the quadratic equation by

$$x = \frac{-b + i\sqrt{-D}}{2a} \qquad \text{and} \qquad x = \frac{-b - i\sqrt{-D}}{2a}$$

where i is the imaginary unit (so $i^2 = -1$; see Guide: Introduction to complex numbers). In a graph, the parabola does not cross the x-axis at all; this indicates that there are no real solutions to this quadratic equation. See Figure 3 for a picture.

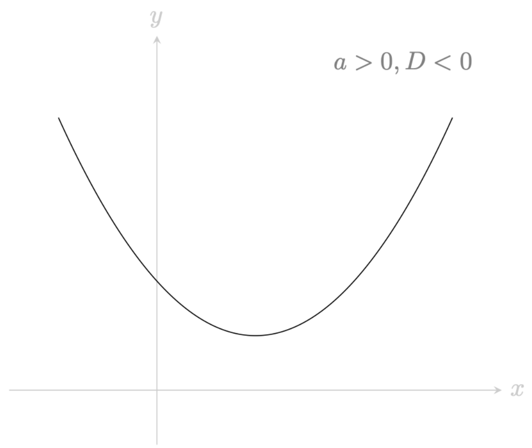

Figure 3: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D < 0$.

Warning

You can use the discriminant to check how many roots a quadratic equation has in the variable given to you. However, this is at most a maximum number of solutions. Conditions on that variable may also reduce the number of valid solutions, particularly if you have real valued functions. For instance, since $e^x > 0$ for all real number x, there are no solutions in x if you find $e^x = -1$.

Here's some examples of using the discriminant and known properties of functions to rule out solutions.

Example 5

In Example 1, you identified the coefficients of $2x^2+4x-8=0$ as $a=2, b=4, c=-8$. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(4)^2-4(2)(-8)=16+64=80.$$

Since $D=80$, you can say that this quadratic equation has two distinct real roots in $x=r_1$ and $x=r_2$.

Example 6

In Example 2, you identified the coefficients of $y^4-10y^2+25=0$ as $a=1, b=-10, c=25$, and the variable as y^2. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(-10)^2-4(1)(25)=100-100=0.$$

Since $D=0$, you can say that this quadratic equation has at most one real root r in terms of y^2.
Whether or not the equation itself has real solutions in y depends on whether r is positive or negative! You cannot take the square root of a negative number, so if r is negative the equation has no real solutions. If r is positive, then the equation has two real roots in y; that is, $y=\pm\sqrt{r}$.

Example 7

In Example 3, you identified the coefficients of $-e^{2x}+4e^x-5=0$ as $a=-1, b=4, c=-5$, and the variable as e^x. Using these, you can work out the value of the discriminant $D=b^2-4ac$ as

$$D=(4)^2-4(-1)(-5)=16-20=-4.$$

Since $D=-4$, you can say that this quadratic equation has complex roots.
This equation therefore has no real solutions in x. This is because e^x is real for any real x; if e^x is complex, it follows that x cannot be real.

Example 8

In Example 4, you rearranged the equation $t + 1 = \frac{4}{t-3}$ to $t^2 + 4t - 1 = 0$, and therefore identified the coefficients as $a = 1, b = 4, c = -1$, and the variable as t. Using these, you can work out the value of the discriminant $D = b^2 - 4ac$ as

$$D = (4)^2 - 4(1)(-1) = 16 + 4 = 20.$$

Since $D = 20$, you can say that this quadratic equation has two distinct real roots in t.

Quick check problems

1. What is the discriminant of the quadratic equation $x^2 - x - 1 = 0$?
2. You are given the quadratic equation $4h^2 - h + 101 = 0$. Identify the variable, and the coefficients a, b, c.
3. You are given three statements below. Decide whether they are true or false.

(a) The quadratic equation $m^2 + 4m + 4 = 0$ has two distinct real roots.

(b) The quadratic equation $m^2 - 4m - 4 = 0$ has exactly one real root.

(c) The quadratic equation $4m^2 + 4m + 4 = 0$ has no real roots.

Further reading

For more questions on the subject, please go to Questions: Introduction to quadratic equations.

For a way to solve quadratic equations, please see Guide: Using the quadratic formula.

Version history

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