Factorization

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Summary

Factorization is an important concept in mathematics, especially in algebra. It involves breaking down expressions into smaller parts called factors which, when multiplied together, give the original expression. This process is essential for simplifying expressions, solving equations, and gaining a deeper understanding of mathematical structures. Factorization plays a key role in various areas of mathematics and is a valuable tool for tackling problems more efficiently.

Before reading this guide, it is recommended that you read Guide: Introduction to quadratic equations. Additionally, you may find it useful to read Guide: Completing the square for factorizing with a remainder.

What is factorization?

Factorization is the process of rewriting an expression as a product of other expressions. This transformation helps when working with more complicated expressions.

Factorization is a key concept in mathematics - anywhere where you may be asked to solve equations, chances are that you will need to utilise factorization. As you have seen in Guide: Introduction to quadratic equations, quadratic expressions are important in mathematics, and these can be manipulated by using factorization to help you solve them. For example, factorizing the expression $x^2 + 5x + 6$ transforms it into $(x + 3)(x + 2)$, which could help you to find the values of x that satisfy the equation $x^2 + 5x + 6 = 0$.

This guide will introduce you to the key concepts behind factorization. First, you will learn how to identifying common factors and use these to factorizing single expressions. From there, you can use your knowledge to move towards factorizing quadratics of the form $ax^2 + bx + c$; both where $a = 1$, and $a \neq 1$.

Introducing key concepts

The first section introduces definitions that you will need throughout this guide.

Definition of term

A **term** is a string of numbers, variables, and/or exponents, multiplied or divided together. An **expression** is a combination of terms by the operations of addition and subtraction. An expression represents a value or relationship but does not have an equals sign.

Example 1

$3x + 5$ is an expression. It contains the number 3, multiplied by the variable x, and the addition operation. The terms are $3x$ and 5.

The introduction of **brackets** turns an expression into a term, as seen in the following example.

Example 2

Take the expression $t - 4$. You can turn this into a term by putting brackets around it to get $(t - 4)$. This allows you to multiply or divide everything in the bracket by the same number. For instance, $4(t-4)$ is a term consisting of the multiple of 4 and $(t - 4)$.

Next, you can ask when you can write an expression as a term. This requires the use of **factors**:

Definition of factor

A **factor** is a term (or expression) that divides exactly into another term (or expression).

Example 3

When working in whole numbers:
The factors of 6 are $1, 2, 3$ and 6.
3 is a factor of 12, because $\frac{12}{3} = 4$. Similarly, $2x$ is a factor of $6x$, because $\frac{6x}{2x} = 3$.
The term 4 and expression $t - 4$ are both factors of $4(t - 4)$.

Important

When working with fractions or decimals, any non-zero number can be a common factor.
For example, $\frac{1}{2}$ is a factor of 1 and 2 - and any other fraction or decimal number!
So, **this guide will focus on whole number factorizations.**

Definition of common factor

A **common factor** is a term or expression that divides exactly into each term of an expression.

Example 4

What is a common factor of $4x$ and 6?
Here,

$$\frac{4x}{2} = 2x \quad \text{and} \quad \frac{6}{2} = 3.$$

2 is a common factor of $4x$ and 6, because both $4x$ and 6 can be divided by 2.

Definition of highest common factor (HCF)

The **highest common factor (HCF)** is the largest factor by which two or more terms can be divided by, without leaving a remainder. It is the greatest number that divides all the given terms exactly and is divisible by all other common factors of terms involved.

Example 5

What is the HCF of 8 and 12?
The factors of 8 are $1, 2, 4$ and 8. The factors of 12 are $1, 2, 3, 4, 6$, and 12.
They both share the factors $1, 2$, and 4; so these are all common factors.
The largest of these is 4, so the HCF of 8 and 12 is 4.

Factorizing single brackets

Definition of factorization

The method of rewriting a term or expression as a single product of factors is known as **factorization**.

One approach to factorizing an expression is by identifying common factors in each term.

For example, expressions of the form $ax + ab$ can be factorized to become $a(x + b)$ where a is a common factor. To factorize an expression into its **simplest form**, you must factorize using the HCF.

When factorizing, you can take out any common factor shared by the terms in an expression. In real numbers, this means you can take out factors that appear in every term. For example, if an expression contains terms that are all divisible by a common number or variable, you can factor that out to simplify the expression.

The steps to factorize an expression fully:

1. Find all common factors for all the terms in the expression.
2. Determine the HCF of all the common factors.
3. Write a bracket, with the HCF on the outside, and the remaining terms inside.

Tip

Always check for the highest common factor before using other factorization methods!

Example 6

Factorize the expression $5x + 15$.
The terms in this expression are $5x$ and 15, and they have a common factor of 5.
Factorizing out the 5 from both terms gives:

$$5x + 15 = 5(x) + 5(3).$$

There are no other common factors of x and 3, so 5 is the HCF. The final factorized form is:

$$5x + 15 = 5(x + 3).$$

Example 7

Fully factorize $6x^2 + 9x$.
The terms of this expression are $6x^2$ and $9x$. They have a common factor of 3.
Factorizing out the 3 from both terms gives:

$$6x^2 + 9x = 3(2x^2) + 3(3x).$$

Now looking at $2x^2$ and $3x$, both terms also contain an x, so you can factorize x out to give:

$$6x^2 + 9x = 3x(2x) + 3x(3).$$

There are no other common factors between $2x$ and 3, so the HCF of $6x^2$ and $9x$ is $3x$. So, the final, fully factorized form is:

$$6x^2 + 9x = 3x(2x + 3).$$

Example 8

Factorize $x(x-4)+3(2x-8)$.
Here, the common factor of both terms $x(x-4)$ and $3(2x-8)$, the common factor is the bracketed expression $(x-4)$. This is because $(x-4)$ divides $(2x-8)$ exactly. So, you can factorize $(x-4)$ from both terms to get:

$$x(x-4)+3(2x-8)=(x-4)(x)+(3)(2(x-4)).$$

You can tidy up the second of these terms to get:

$$(3)(2(x-4))=(6)(x-4).$$

The expression then becomes:

$$x(x-4)+3(2x-8)=(x-4)(x)+(6)(x-4).$$

There are no further common factors of x and 6, so $(x-4)$ is the HCF. So the final factorized form is:

$$x(x-4)+3(2x-8)=(x-4)(x+6).$$

Factorizing double brackets

Factorizing quadratic expressions involves breaking down quadratic expressions (of the form ax^2+bx+c, where $a \neq 0$) into a product of two brackets, which can make it easier to solve the quadratic expression. This process is essential in algebra, and is used to help simplify expressions, whilst also being able to solve quadratic equations and find their roots. For more information on this, see Guide: Introduction to quadratic equations.

There are two main cases when factorizing quadratics that you will learn about. One where $a=1$, and another where $a \neq 1$.

Case 1: $a=1$

When the quadratic expression has the form x^2+bx+c, the aim is to find two numbers d and e that multiply to c, and add to b, such that $de=c$ and $d+e=b$. If you can do this, then you can rewrite the expression as

$$x^2 + bx + c = x^2 + (d+e)x + de.$$

You can expand the brackets to see that

$$x^2 + (d+e)x + de = x^2 + dx + ex + de.$$

x is a common factor of the first two terms, and e is a common factor of the second two terms, you can factorize using the techniques above to get

$$x^2 + dx + ex + de = x(x+d) + e(x+d)$$

Since $(x+d)$ is a common factor of both of the terms $x(x+d)$ and $e(x+d)$, you can factorize again to get

$$x^2 + bx + c = x(x+d) + e(x+d) = (x+d)(x+e).$$

This is the desired form of the quadratic expression! It is best to see this process explained with an example.

Example 9

Factorize $x^2 + 7x + 10$ into a product of two bracketed expressions.
You want to find two numbers that multiply to make 10 and add to make 7.
The positive factors of 10 are $1, 2, 5,$ and 10.
You want to look at the factors and see which pairings multiply to make 10. These are:

$$1 \cdot 10 = 10 \quad \text{and} \quad 2 \cdot 5 = 10.$$

Since $2+5 = 7$ and $1 + 10 = 11 \neq 7$, you pick the correct factors of 10, which are 2 and 5. You can now write the factorized form of your quadratic expression by writing two brackets and putting the variable x at the start of each one:

$$(x + ?)(x + ?)$$

Write one factor in the first bracket and the other factor in the second bracket, to give the final factorized form:

$$x^2 + 7x + 10 = (x+2)(x+5).$$

Here's an example with a negative bx term.

Tip

If the bx term is negative, and the $+c$ is positive, the two factors must both be negative, as two negatives multiplied together creates a positive.

Example 10

Factorize $x^2 - 5x + 6$.
You want to find two numbers that multiply to make 6 and add to make -5.
Now, you need to consider all the factors, positive and negative, of 6.
The factors of 6 are $1, -1, 2, -2, 3, -3, 6$, and -6.
You want to look at the factors and see which pairings multiply to make 6 and add to -5. These are:

$$(-2) \cdot (-3) = 6 \quad \text{and} \quad (-2) + (-3) = -5.$$

So, -2 and -3 are the correct factors to use.
Now, put them into the brackets in a similar fashion to Example 9, to give the final factorized form:

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

Finally, here's an example where the $+c$ term is negative.

Tip

If the $+c$ term is negative, and the bx term is positive, one factor must be negative, and the other positive, as a positive and a negative multiplied together gives a negative.

Example 11

Factorize $x^2 + x - 2$.

Now, you need to consider all the factors, positive and negative, of -2.

The factors of -2 are $1, -1, 2,$ and -2. You want to look at the factors and see which pairings multiply to make -2 and add to 1. These are:

$$2 \cdot (-1) = -2 \quad \text{and} \quad 2 + (-1) = 1.$$

So 2 and -1 are the correct factors to use.

Now, put them into the brackets in a similar fashion to Example 9, to give the final factorized form:

$$x^2 + x - 1 = (x + 2)(x - 1).$$

Important

The order of the factors don't matter, but the signs do!

Example 11 is an example of this - if the signs of the factors were swapped, it would result in a different quadratic to the one you have been asked to factorize. If you swapped the signs, you would get $(x - 2)(x + 1)$. When expanded, this gives:

$$(x - 2)(x + 1) = x^2 - 2x + x - 2$$

$$= x^2 - x - 2$$

$$x^2 - x - 2 \neq x^2 + x - 2.$$

$x^2 - x - 2$ is not the original quadratic expression you were asked to factorize, which shows you how if a factor has the incorrect sign, the required quadratic expression will not be factorized correctly.

Tip

If $c = 0$, you can factorize the quadratic expression using the common factor x. This is because if $c = 0$, then:

$$ax^2 + bx + c = ax^2 + bx.$$

Which can then be factorized as:

$$ax^2 + bx = x(ax + b).$$

If $a = 1$, this is still true, and is factorized as:

$$x^2 + bx = x(x + b).$$

Example 7 is an example where $a \neq 0$ and $c = 0$.

Case 2: $a \neq 1$

When a quadratic expression has the form $ax^2 + bx + c$, it can still be factorized, but it requires a different method from the one you used when $a = 1$.

The aim is to find two numbers d and e such that:

$$d \cdot e = a \cdot c \quad \text{and} \quad d + e = b.$$

This works because you can rewrite the middle term (bx) as $dx + ex$, which allows you to factorize by grouping. So if you rewrite the expression, it becomes:

$$ax^2 + bx + c = ax^2 + dx + ex + c.$$

You can now group the expression:

$$ax^2 + dx + ex + c = (ax^2 + dx) + (ex + c).$$

Factorizing each bracket gives:

$$(ax^2 + dx) + (ex + c) = ax\left(x + \frac{d}{a}\right) + e\left(x + \frac{c}{e}\right).$$

Since $d \cdot e = a \cdot c$ from the start of your working, dividing both sides by ae gives $\frac{d}{a} = \frac{c}{e}$. You can define this as a value f, where $f = \frac{d}{a} = \frac{c}{e}$, and substitute this in to give:

$$(ax^2 + dx) + (ex + c) = ax(x + f) + e(x + f).$$

This means that both groups will share the common factor $(x + f)$, and the expression can be factorized further to give:

$$(ax^2 + dx) + (ex + c) = (ax + e)(x + f).$$

This is the fully factorized form of the quadratic expression $ax^2 + bx + c$ when $a \neq 1$.

These are the steps to factorize a quadratic of the form $ax^2 + bx + c$:

1. Multiply a and c together.
2. Find two numbers that multiply to give $a \cdot c$, and add to give b.
3. Rewrite the middle term using these two numbers.
4. Factorize by grouping as above.

You can see that this method extends Case 1 (where $a = 1$). If $a = 1$, then in this case $a \cdot c = c$ and this method then reduces to the above. Let's see an example

Example 12

Factorize $2x^2 + 5x + 3$.
You can follow the steps above.

1. First, you need to multiply 2 and 3 together: $2 \cdot 3 = 6$.

2. Now you need to consider all the factors of 6, which are $1, 2, 3$, and 6. You want to look at the factors and see which pairings multiply to make 6, and add to 5. These are:
$$2 + 3 = 5 \quad \text{and} \quad 2 \cdot 3 = 6.$$

3. Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier:
$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3.$$

4. If you now look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3 = 2x(x + 1) + 3(x + 1).$$

Now, this is similar to the type of factorizing you did in single bracket factorization! You can now take out the the $(x + 1)$ common factor, to give the result:

$$2x(x + 1) + 3(x + 1) = (2x + 3)(x + 1).$$

And this gives you your final factorized answer:

$$2x^2 + 5x + 3 = (2x + 3)(x + 1).$$

You don't have to use the steps prescriptively in every example; it was only for instructive purposes.

Example 13

Factorize $3x^2 - 2x - 5$.
You need to multiply 3 and -5 together:

$$3 \cdot (-5) = -15.$$

You need to find the factors of -15 that multiply to give -15, and add to give -2. These are:

$$3 + (-5) = -2 \quad \text{and} \quad 3 \cdot (-5) = -15.$$

Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier:

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5.$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping:

$$3x^2 - 2x - 5 = 3x^2 + 3x - 5x - 5 = 3x(x+1) - 5(x+1).$$

You can now remove the $(x+1)$ common factor, to give the result:

$$3x(x+1) - 5(x+1) = (3x-5)(x+1).$$

And you can write your final factorized answer:

$$3x^2 - 2x - 5 = (3x+5)(x+1).$$

Important

Note that a quadratic $ax^2 + bx + c$ can sometimes be factorized in the form:

$$(px+q)(rx+s) \quad \text{where} \quad p, r \neq 0.$$

However, if you factorize out r from each term you get:

$$(px+q)\,(rx+s) = \frac{1}{r^2}\left(\left(\frac{px}{r} + \frac{q}{r}\right)\left(x + \frac{s}{r}\right)\right).$$

This matches the factorized result of ax^2+bx+c, so a factorization like $(px+q)(rx+s)$ can be expressed using rational numbers to match any desired form. This can be seen as:

$$\left(\frac{px}{r}+\frac{q}{r}\right)\left(x+\frac{s}{r}\right)=(dx+e)\,(x+f)\,,\quad \text{where}\quad d=\frac{p}{r},\quad e=\frac{q}{r}\quad \text{and}\quad f=\frac{s}{r}.$$

Example 14

Factorize $4x^2+4x-3$.
First, you need to multiply 4 and -3 together:

$$4\cdot(-3)=-12.$$

You need to find the factors of -12 that multiply to give -12, and add to give 4. These are:

$$(-2)+6=4\quad \text{and}\quad (-2)\cdot 6=-12.$$

Now you can rewrite the original expression, but this time splitting the middle (bx) term into the two factors you found earlier.

$$4x^2+4x-3=4x^2-2x+6x-3.$$

If you look at the expression in two halves, there is a common factor for each half, and you can factorize by grouping. Remember to use the HCF here so you can factorize fully!

$$4x^2+4x-3=4x^2-2x+6x-3=2x(2x-1)+3(2x-1).$$

Finally, you can remove the $(2x+3)$ common factor, to give the result:

$$2x(2x-1)+3(2x-1)=(2x-1)(2x+3).$$

And this is your final factorized answer:

$$4x^2+4x-3=2x(2x-1)+3(2x-1)=(2x-1)(2x+3).$$

Using factorization to solve quadratic equations

By factorizing, you can potentially break down a quadratic expression into simpler expressions. This process makes it easier to understand the structure of the expression and to identify values that satisfy the equation you are solving.

Once a quadratic expression is factorized, and the equation is set equal to zero, you can find the values of the variable that make each bracket equal to zero. These values are known as the **roots**, **zeroes**, or **solutions** of the quadratic equation. This works because $(x+d)(x+e) = 0$ if and only if $(x+d) = 0$ or $(x+e) = 0$. This happens precisely when $x = -d$ or when $x = -e$.

However, it's important to remember that this method only works directly when the expression is equal to zero—if it's equal to another number, other steps are needed before factorizing.

> **Example 15**
>
> Solve $x^2 - 5x + 6 = 0$.
> You worked out from Example 10 that
>
> $$x^2 - 5x + 6 = (x-2)(x-3).$$
>
> Setting this equal to zero gives $(x-2)(x-3) = 0$.
> Now, you can set each individual bracket equal to zero:
>
> $$(x-2) = 0 \quad \text{and} \quad (x-3) = 0$$
>
> and you can solve each one for x:
>
> $$x = 2 \quad \text{and} \quad x = 3.$$
>
> Therefore the solutions of $x^2 - 5x + 6$ are $x = 2$ and $x = 3$.

Using tools like Desmos, you can verify your solutions by graphing the quadratic equation and identifying the points where it intersects with the x-axis. The roots of the quadratic equations are the points of intersection with the x-axis. See Calculator: Solving quadratic equations for a full calculator.

Quick check problems

1. You are given four expressions below. Match the correct expression to the correct equivalent factorized expression.

(a) $5(2-x)$
(b) $5(x-2)$
(c) $10(1-x)$
(d) $5(2+x)$

(A) $10 + 5x =$
(B) $-10 + 5x =$
(C) $10 - 5x =$
(D) $10 - 10x =$

2. You are given three expressions below. Factorize them into their simplest form.

(a) $4x^2y + 6xy^2$

(b) $x^2 + 6x + 9$

(c) $4x^2 - x - 3$

3. Find the solutions of the quadratic equation $2x^2 + 3x - 2 = 0$.

Further reading

For more questions on the subject, please go to Questions: Introduction to factorization.

For more information on solving quadratic equations, please see Guide: Introduction to quadratic equations and Guide: Using the quadratic formula.

Version history

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