Proof: The square root of 2 is irrational

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Summary

Proof by contradiction of the irrationality of $\sqrt{2}$

Before reading this proof sheet, it is recommended that you read **Overview: Number sets.**

Proof

You might remember from **Overview: Number sets** that an irrational number is a number that cannot be represented as a fraction of integers, here you can prove that $\sqrt{2}$ is irrational. This particular proof dates back to the ancient Greeks and and relies on a method of proof called proof by contradiction. In a proof by contradiction you begin by assuming that what you're tying to prove is false, then you show that from that assumption you can derive a contradiction, so your assumption must have been false.

Let's prove that $\sqrt{2}$ is irrational by contradiction.

Suppose $\sqrt{2}$ is rational. Then it can be expressed as a fraction:

$$\sqrt{2} = \frac{p}{q},$$

where p and q are integers with no common factors other than 1, meaning the fraction is in its simplest form, and $q \neq 0$.

Then you can square both sides:

$$2 = \frac{p^2}{q^2}.$$

Then multiply both sides by q^2:

$$2q^2 = p^2.$$

This implies that p^2 is even (since it is divisible by 2). Since the square of an odd number is odd, p must be even. Let $p = 2k$ for some integer k.

You can then substitute $p = 2k$ into the equation:

$$2q^2 = (2k)^2 = 4k^2.$$

Dividing both sides by 2:

$$q^2 = 2k^2.$$

This shows that q^2 is also even, which means q must be even.

Since both p and q are even, they share a common factor of 2, contradicting your assumption that p and q have no common factors other than 1.

This contradiction implies that your initial assumption was false, $\sqrt{2}$ cannot be written as a fraction of integers, and so, $\sqrt{2}$ is irrational.

Version history

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