

# Introduction to algebraic fractions

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## Summary

Algebraic fractions extend the idea of numerical fractions to expressions that include variables. This guide explains what algebraic fractions are and why certain values are not allowed in the denominator. You will learn how to rewrite algebraic fractions in equivalent forms and how to simplify them by factorizing and cancelling common factors. These skills are essential for later topics such as solving algebraic equations, manipulating formulas, and working with rational functions.

*Before reading this guide, it is recommended that you read [Guide: Introduction to numerical fractions](#), [Guide: Arithmetic on numerical fractions](#) and [Guide: Factorization](#).*

## What is an algebraic fraction?

In [Guide: Introduction to numerical fractions](#), you learned about numerical fractions, which are fractions where both the numerator and denominator are numbers:

$$\frac{3}{4} \quad \frac{15}{7} \quad \frac{23}{100}$$

These ideas can be extended to **algebraic fractions**. These look similar, but instead of including only numbers, the numerator or denominator (or both) can include letters and expressions:

$$\frac{x}{5} \quad \frac{2}{x^2} \quad \frac{x+3}{x-2}$$

By the end of this guide, you will be able to find equivalent fractions for algebraic fractions and simplify them in a similar way to numerical fractions. The main difference is that, because the denominator may contain algebra, you must be careful about restrictions.

## Restrictions on the denominator

Every algebraic fraction has a denominator, and one of the most important rules in mathematics is that **you can never divide by zero**.

If the denominator of a fraction becomes zero for a certain value of a variable, then the fraction is said to be **undefined** at that point. This means that the expression has no value at that point.

When working with algebraic fractions, you need to state any values of the variable that make the denominator zero. These are called the **restricted values**.

### i Restrictions on the denominator

Let  $f(x)$  and  $g(x)$  be functions of the variable  $x$ .

For any algebraic fraction

$$\frac{f(x)}{g(x)}$$

the fraction is defined only when

$$g(x) \neq 0$$

The **restricted values** are the values of  $x$  that make  $g(x) = 0$ . At these points, the fraction is said to be **undefined**.

### i Example 1

For what values of  $x$  is the following fraction undefined?

$$\frac{2}{x}$$

Here, the denominator is  $x$ . The fraction is undefined when  $x = 0$ , so the restriction is

$$x \neq 0$$

### i Example 2

For what values of  $x$  is the following fraction undefined?

$$\frac{1}{x - 3}$$

Here, the denominator is  $x - 3$ . The fraction is undefined when  $x - 3 = 0$ , which occurs when  $x = 3$ , so the restriction is

$$x \neq 3$$

### i Example 3

For what values of  $x$  is the following fraction undefined?

$$\frac{x+2}{(x-1)(x+4)}$$

Here the denominator is  $(x-1)(x+4)$ . The fraction is undefined when  $(x-1)(x+4) = 0$ , which occurs when either  $x = 1$  or  $x = -4$ , so the restrictions are

$$x \neq 1 \text{ and } x \neq -4$$

Restrictions of multiple values can also be written in set notation. In this example, you can also write  $x \notin \{1, -4\}$ . The symbol  $\in$  means “is a member of” and the symbol  $\notin$  means “is not a member of”. This means that  $x \notin \{1, -4\}$  reads as “ $x$  is not any of 1 and  $-4$ ”.

## Equivalent algebraic fractions

Similarly to numerical fractions, two algebraic fractions can look different but still represent the same value. If you multiply or divide both the numerator and the denominator by the same non-zero expression, the fraction’s value does not change. In this case, what you have done is written it in a new form.

Equivalent algebraic fractions are important because they let you rewrite fractions with different denominators when adding or subtracting.

### i Equivalent algebraic fractions

Two algebraic fractions are equivalent if they have the same value, even though the expressions may look different.

You can create an equivalent algebraic fraction by:

- Multiplying the numerator and denominator by the same expression (not equal to zero).
- Dividing the numerator and denominator by the same common factor.

A simplified fraction is an equivalent fraction of one of its unsimplified forms.

### i Example 4

Below are some examples of equivalent algebraic fractions.

Multiplying by a constant:

$$\frac{1}{x} = \frac{1 \cdot 3}{x \cdot 3} = \frac{3}{3x} \quad \text{if } x \neq 0$$

Dividing by a common factor:

$$\frac{6x^2}{9x} = \frac{6x^2 \div 3x}{9x \div 3x} = \frac{2x}{3} \quad \text{if } x \neq 0$$

Multiplying by a variable expression:

$$\frac{2}{x} = \frac{2(x+1)}{x(x+1)} \quad \text{if } x \neq 0 \text{ and } x \neq -1$$

Writing a whole number as an algebraic fraction:

$$2 = \frac{2}{1} = \frac{2x}{x} = \frac{2x(x+1)}{x(x+1)} \quad \text{if } x \neq 0 \text{ and } x \neq -1$$

## Simplifying algebraic fractions

Similarly to numerical fractions, algebraic fractions can also be written in a simplified form. Simplifying makes expressions more convenient to work with, and it helps when comparing fractions or solving equations.

### i Simplifying algebraic fractions

To simplify an algebraic fraction:

1. Factorize the numerator and the denominator completely.
2. State the restrictions on the denominator, if any exist.
3. Cancel any common factors that appear in both the numerator and the denominator.
4. Write the final simplified result.

### ! Important

Between the numerator and the denominator, you can only cancel factors, not terms.

- A factor is part of a multiplication. In  $x(x + 3)$ , both  $x$  and  $(x + 3)$  are factors.
- A term is part of an addition or subtraction. In  $x + 3$ , both  $x$  and  $3$  are terms.

This is why, in the simplifying process, you cannot cancel the  $x$  terms by writing

$$\frac{x+3}{x+2} = \frac{x+3}{x+2} = \frac{3}{2}$$

### i Example 5

Simplify  $\frac{6x}{9x}$ .

Both the numerator and denominator contain  $x$  as a factor. As long as  $x \neq 0$ , it can be cancelled out to reduce the algebraic fraction to a numerical fraction.

$$\frac{6x}{9x} = \frac{6}{9} = \frac{2}{3} \quad \text{if } x \neq 0$$

### i Example 6

Simplify  $\frac{x^2 + 3x}{x}$ .

The numerator can be factorized to give a common factor of  $x$ :

$$x^2 + 3x = x(x + 3)$$

As long as  $x \neq 0$ , the common factor of  $x$  can be cancelled out to give  $x + 3$ .

$$\frac{x^2 + 3x}{x} = \frac{x(x + 3)}{x} = x + 3 \quad \text{if } x \neq 0$$

### i Example 7

Simplify  $\frac{x^2 - 9}{x^2 - 3x}$ .

The numerator can be factorized as it is a difference of two squares:

$$x^2 - 9 = (x + 3)(x - 3)$$

The denominator can be factorized to give a common factor of  $x$ :

$$x^2 - 3x = x(x - 3)$$

As long as  $x \neq 3$ , the common factor of  $x - 3$  can be cancelled out to give  $\frac{x+3}{x}$ .

$$\frac{x^2 - 9}{x^2 - 3x} = \frac{(x + 3)(x - 3)}{x(x - 3)} = \frac{x + 3}{x} \quad \text{if } x \neq 0 \text{ and } x \neq 3$$

## Quick check problems

1. For which value of  $x$  is the fraction  $\frac{3}{x-2}$  undefined?

- (a)  $x = 0$
- (b)  $x = 2$
- (c)  $x = -2$
- (d) There are no restricted values

2. Which of the following fractions is equivalent to  $\frac{4}{x}$ ?

- (a)  $\frac{4x}{x^2}$
- (b)  $\frac{8}{4x}$
- (c)  $\frac{4}{x + 1}$
- (d)  $\frac{x}{4}$

3. Simplify the fraction  $\frac{6x^2}{3x}$ , assuming  $x \neq 0$ .

4. For which value(s) of  $x$  is the fraction  $\frac{x+5}{x(x-4)}$  undefined?

- (a)  $x = 0$  only
- (b)  $x = 4$  only
- (c)  $x = 0$  or  $x = 4$
- (d) There are no restricted values

## Further reading

For more questions on the subject, please go to [Questions: Introduction to algebraic fractions](#).

To learn how to perform arithmetic on algebraic fractions, please see [Guide: Arithmetic on algebraic fractions](#).

## Version history

v1.0: initial version created 12/25 by Donald Campbell as part of a University of St Andrews VIP project.

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