Proof: Law of total probability and Bayes' theorem

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Summary

This proof sheet demonstrates that the law of total probability and Bayes' theorem are true.

Before reading this proof sheet, it is recommended that you read Guide: Conditional probability and Guide: Law of total probability and Bayes' theorem.

Proof of the law of total probability

First of all, here is a restatement of the law of total probability from Guide: Law of total probability and Bayes' theorem:

Definition of the law of total probability

Suppose an event B depends on several possible scenarios. These scenarios can be described by events $A_1, A_2, \ldots, A_n$, that are:

- **Mutually exclusive**: they cannot occur at the same time, and
- **Exhaustive**: one of them must always occur.

Then, the **law of total probability** states that the probability of event B is:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B \mid A_i)$$

The proof of the law of total probability comes directly from the definition of conditional probability given in Guide: Conditional probability:

$$\mathbb{P}(B \mid A_i) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(A_i)}$$

Multiplying by $\mathbb{P}(A_i)$ gives the multiplication rule (again from Guide: Conditional probability)
:

$$\mathbb{P}(B \cap A_i) = \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

As scenarios $A_1, A_2, \ldots, A_n$ are mutually exclusive (so $A_i \cap A_j = \emptyset$ for all $1 \leq i \neq j \leq n$) and exhaustive ($\bigcup_{1 \leq i \leq n} A_i = B$), it follows from results in set theory (see [Guide: Operations on sets]) that:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \cdots + \mathbb{P}(B \cap A_n) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i)$$

Substituting the above expressions gives:

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(B \mid A_n)\mathbb{P}(A_n) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

Which results in the law of total probability:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B \mid A_i)$$

Proof of Bayes' theorem

Here is the statement of Bayes' theorem from Guide: Law of total probability and Bayes' theorem:

> **Statement of Bayes' Theorem**
>
> If A and B are events with $\mathbb{P}(B) > 0$, then Bayes' Theorem states:
>
> $$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$
>
> where:
>
> - $\mathbb{P}(A \mid B)$ is the probability of A given B,
> - $\mathbb{P}(B \mid A)$ is the probability of B given A,
> - $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are the individual probabilities of A and B, respectively.

Bayes' Theorem is derived directly from the definition of conditional probability: see Guide: Conditional probability. Start with the conditional probabilities of two events A and B:

$$(1) \quad \mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{and} \quad (2) \quad \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

You can rearrange (2) by multiplying both sides by $\mathbb{P}(A)$, giving the multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\mathbb{P}(A)$$

Substitute this result into equation (1) to get:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

This gives Bayes' Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

Further reading

Click this link to go back to Guide: Law of total probability and Bayes' theorem.

Version history

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