Introduction to Matrices

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Summary

Matrices are rectangular arrays of numbers with entries arranged in rows and columns. Matrices are a very useful tool within mathematics, this guide will explain what they are and how to perform arithmatic with matrices.

What is a matrix?

You can think of a matrix as a rectangular array or table, with entries in rows and columns. Understanding matrices can make solving equations more efficient and can open the door to learning much more mathematics.

Definition a matrix

A $m \times n$ **matrix** is a rectangular array of mn entries set out in m **rows** and n **columns**. You can write it like so:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

This matrix has **dimension** $m \times n$.

The entries in a matrix are usually numbers, but they can be other mathematical objects. Any type of number may be an entry in your matrix, positive or negative, rational or irrational, real or complex. If complex numbers are unfamiliar to you, you can read more about them at **Guide: Introduction to complex numbers.** Note here that while entries can be other mathematical object, for this study guide you will just use entries within the complex numbers.

If you have read **Guide: Introduction to solving simultaneous equations** then one way of thinking of matrices is as an array encoding the coefficients of the variables of your simultaneous equations.

Matrices are a fundamental tool within linear algebra, and they have a wide range of real-life applications. They are used in computer graphics, data analysis, search engine optimization, cryptography, economics, robotics, genetics, quantum mechanics, and many more. Matrices are used anywhere where information needs to be analysed and calculated efficiently.

In this guide, you will see how you can read, write, and understand matrices, you will learn how to do arithmetic with matrices, and see some examples of special matrices.

Working with matrices

Example 1

Here are some matrices:

$$A = \begin{bmatrix} 0 & -2 \\ \pi & 5 \\ 1/3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2/5 & 0 & \sqrt{3} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \pi & -1 & 0 & 7/3 \\ 0 & 5/6 & -\sqrt{2} & -3 \\ 4/9 & 7 & -\pi & 0 \end{bmatrix}$$

Matrix A here has dimension 3×2. The entry in the 2nd row and 1st column is called $a_{2,1}$, and here that is equal to π.The entry in the 1st row and 2nd column is called $a_{1,2}$, and here that is equal to -2. You can notice her that $a_{2,1}$ is not equal to $a_{1,2}$.
Matrix B here has dimension 1×4. The entry in the 1st row and 2nd column is called $b_{1,2}$, and here that is equal to $2/5$.The entry in the 1st row and 4th column is called $b_{1,4}$, and here that is equal to $\sqrt{3}$.
Matrix C here has dimension 3×4. The entry in the 2nd row and 3rd column is called $c_{2,3}$, and here that is equal to $-\sqrt{2}$.

$$D = \begin{bmatrix} \sqrt{2} \\ 11 \\ -3/8 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} -5 & 1/2 & 0 \\ 0 & -\pi & 4/9 \\ 7 & 0 & -\sqrt{3} \\ -1 & 5 & 0 \\ 1/4 & 0 & 8 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 9 \end{bmatrix}$$

Matrix D here has dimension 3×1.
Matrix E here has dimension 5×3.
Matrix F here has dimension 1×1.

You will notice that B only has one row, you call such a matrix a **row matrix**. Similarly you would call a matrix like D, with only one column, a **column matrix**. You can also describe these as **vectors**. You can read more about vectors in **Guide: Introduction to vectors**.

Tip

The entry a_{ij} refers to the $(ij)^{th}$ entry of your matrix, that is the i^{th} row and the j^{th} column. You can write your matrix as $[a_{ij}]_{mn}$.

Example 2

So, for example the 3×2 matrix,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

could be writen as $[1]_{3,2}$

Definition a main diagonal

The entries $a_{11}, a_{22}, ..., a_{nn}$ make up the **main diagonal**. You can define the **main diagonal** like so:

$$diagA = (a_{11}, a_{22}, ..., a_{nn})$$

Example 3

For the matrices in Example 1, the main diagonals are:
$diagA = (0, 5)$
$diagB = (-1)$
$diagD = (\sqrt{2})$
$diagE = (-5, -\pi, -\sqrt{3})$

Addition and subtraction with matrices

In this section you will see how you can add and subtract matrices.

Definition of matrix addition and subtraction

The **matrix sum of** A and B can be calculated by adding corresponding entries of A and B,

$$(a+b)_{ij} = a_{ij} + b_{ij}$$

Similarly, the **matrix difference of** A and B can be calculated by subtracting corresponding entries of A and B,

$$(a-b)_{ij} = a_{ij} - b_{ij}$$

! Important

You can only add and subtract matrices if they share the same dimensions.

Example 4

Let's add two (2×2) matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

The sum $(A+B)$ is:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$

So, the sum of the two matrices is:

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Example 5

Let's subtract two 3×4 matrices.

$$A = \begin{bmatrix} 1 & -3 & 5 & -4 \\ 3 & 0 & -2 & 2 \\ -7 & 8 & 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 & 3 & -2 \\ 2 & -3 & 0 & 1 \\ -6 & -3 & 5 & 1/2 \end{bmatrix}$$

To subtract these matrices, you subtract the corresponding elements:

$$A - B = \begin{bmatrix} 1-(-1) & -3-4 & 5-3 & -4-(-2) \\ 3-2 & 0-3 & -1/2-0 & 2-1 \\ -7-(-6) & 8-(-3) & 4-5 & 3-1/2 \end{bmatrix}$$

Then you can simplify the signs,

$$A - B = \begin{bmatrix} 1+1 & -3-4 & 5-3 & -4+2 \\ 3-2 & 0-3 & -1/2 & 2-1 \\ -7+6 & 8+3 & 4-5 & 3-1/2 \end{bmatrix}$$

So, the difference of the two matrices is:

$$A - B = \begin{bmatrix} 2 & -7 & 2 & -2 \\ 1 & -3 & -1/2 & 1 \\ -1 & 11 & -1 & 5/2 \end{bmatrix}$$

You saw earlier that you can only add and subtract matrices if they share the same dimensions. You can look to this non-example to see why this is the case.

Non-Example

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4/5 & 0 & 6 \end{bmatrix} \quad \text{(a } 2 \times 3 \text{ matrix)} \quad B = \begin{bmatrix} -8 & 8 \\ 0 & 11 \end{bmatrix} \quad \text{(a } 2 \times 2 \text{ matrix)}$$

Why can you not add A and B?
Matrix addition requires that each entry in one matrix corresponds to an entry in the other matrix. But, since:

- Matrix A has dimensions 2×3, that is, 2 rows and 3 columns
- Matrix B has dimensions 2×2, that is, 2 rows and 2 columns

they do **not** have the same dimensions. Consider how you would attempt to calculate entries $ab_{1,3}$ and $ab_{2,3}$. The operation: $A + B$ is **undefined**.

Scalar multiplication

In this section you will see how you can multiply a matrix by a number or variable. You call this scalar multiplication.

Definition of Scalar multiplication with matrices

A **scalar multiplication** of a matrix A, by a number or variable k, is obtained by multiplying each entry in A by k,

$$kA = \left[k \cdot a_{ij}\right]$$

You can now see a few examples of this.

Example 6

Let's multiply a 3×2 matrix by a scalar.

$$A = \begin{bmatrix} 0 & 4/3 \\ -3 & \sqrt{2} \\ -7 & 12 \end{bmatrix} \qquad k = 3$$

To multiply this matrix by 3, you multiply each element by 3:

$$kA = \begin{bmatrix} 3 \cdot 0 & 3 \cdot 4/3 \\ 3 \cdot -3 & 3 \cdot \sqrt{2} \\ 3 \cdot -7 & 3 \cdot 12 \end{bmatrix}$$

By carrying out these multiplications in the entries here, you can arrive at,

$$kA = \begin{bmatrix} 0 & 4 \\ -9 & 3\sqrt{2} \\ -21 & 36 \end{bmatrix}$$

Example 7

Let's multiply a 1×4 matrix by a scalar.

$$A = \begin{bmatrix} -6 & 7/3 & \pi & 12 \end{bmatrix} \qquad k = -1/2$$

To multiply this matrix by $-1/2$, you multiply each element by $-1/2$:

$$kA = \begin{bmatrix} -1/2 \cdot -6 & -1/2 \cdot 7/3 & -1/2 \cdot \pi & -1/2 \cdot 12 \end{bmatrix}$$

By carrying out these multiplications in the entries here, you can arrive at,

$$kA = \begin{bmatrix} -3 & -7/6 & -\pi/2 & -6 \end{bmatrix}$$

Now you can combine you're understanding of matrix addition and scalar multiplication to tackle questions that combine both of these skills.

Example 8

Let's calculate $A + (2 \cdot B)$, for

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$$

First, let's work out $2 \cdot B$,

$$2 \cdot B = \begin{bmatrix} 2 \cdot -1 & 2 \cdot 2 \\ 2 \cdot -1 & 2 \cdot -3 \end{bmatrix}$$

By carrying out these multiplications, you can arrive at,

$$2 \cdot B = \begin{bmatrix} -2 & 4 \\ -2 & -6 \end{bmatrix}$$

Now $A + 2 \cdot B$,

$$A + 2 \cdot B = \begin{bmatrix} -1 + (-2) & 2 + 4 \\ -1 + (-2) & -3 + (-6) \end{bmatrix}$$

By computing the sums here, you can arrive at your answer,

$$A + 2 \cdot B = \begin{bmatrix} -3 & 6 \\ -3 & -9 \end{bmatrix}$$

Matrix multiplication

Now you can look at how and when you can two multiply matrices together.

First, lets see how you multiply two 2×2 matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The product $C = AB$ is also a 2×2 matrix:

$$C = \begin{bmatrix} (a_{11}b_{11}) + (a_{12}b_{21}) & (a_{11}b_{12}) + (a_{12}b_{22}) \\ (a_{21}b_{11}) + (a_{22}b_{21}) & (a_{21}b_{12}) + (a_{22}b_{22}) \end{bmatrix}$$

Here you can see that in order to calculate the first entry of C you multiply the corresponding elements from the first row of A by the elements from the first column of B, and sum together the results.

Now you can see the definition of matrix multiplication for more general matrices.

Definition of matrix multiplication

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

The product $C = AB$ is an $m \times p$ matrix, where each entry c_{ij} is given by the summation formula:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad \text{where } 1 \leq i \leq m \text{ and } 1 \leq j \leq p$$

If sigma notation is unfamiliar to you, you can read more about it in **Guide: Introduction to sigma notation**.

Important

You can only multiply matrices A and B if A has the same number of columns as B has rows.

Warning

Matrix multiplication is **non-commutative**. This means that its not always true for matrices that $AB = BA$. In fact, in some cases AB may be defined and BA wouldn't be.

To take a better look at this unusual property, you can now see some examples of this in action.

Example 9

Lets multiply two 2×2 matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$$

Since both matrices are 2×2, their product AB is also 2×2:

$$AB = \begin{bmatrix} (1 \cdot 0) + (2 \cdot 6) & (1 \cdot 5) + (2 \cdot 7) \\ (3 \cdot 0) + (4 \cdot 6) & (3 \cdot 5) + (4 \cdot 7) \end{bmatrix}$$

By carrying out these multiplications you will have,

$$AB = \begin{bmatrix} 0 + 12 & 5 + 14 \\ 0 + 24 & 15 + 28 \end{bmatrix}$$

And by computing the sums, you will get,

$$AB = \begin{bmatrix} 12 & 19 \\ 24 & 43 \end{bmatrix}$$

Now lets compute BA:

$$BA = \begin{bmatrix} (0 \cdot 1) + (5 \cdot 3) & (0 \cdot 2) + (5 \cdot 4) \\ (6 \cdot 1) + (7 \cdot 3) & (6 \cdot 2) + (7 \cdot 4) \end{bmatrix}$$

By carrying out these multiplications you will have,

$$BA = \begin{bmatrix} 0 + 15 & 0 + 20 \\ 6 + 21 & 12 + 28 \end{bmatrix}$$

And by computing the sums, you will get,

$$BA = \begin{bmatrix} 15 & 20 \\ 27 & 40 \end{bmatrix}$$

Compare AB and BA.

$$AB = \begin{bmatrix} 12 & 19 \\ 24 & 43 \end{bmatrix}, \quad BA = \begin{bmatrix} 15 & 20 \\ 27 & 40 \end{bmatrix}$$

Since $AB \neq BA$, this example confirms that **matrix multiplication is not commutative** in general.

Example 10

Let A be a 2×3 matrix and B be a 3×2.

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ -2 & 5 \\ 1 & -2 \end{bmatrix}$$

Since A is 2×3 and B is 3×2, the product AB is a 2×2 matrix:

$$AB = \begin{bmatrix} (-2 \cdot 3) + (1 \cdot -2) + (3 \cdot 1) & (-2 \cdot -1) + (1 \cdot 5) + (3 \cdot -2) \\ (4 \cdot 3) + (-5 \cdot -2) + (2 \cdot 1) & (4 \cdot -1) + (-5 \cdot 5) + (2 \cdot -2) \end{bmatrix}$$

By carrying out these multiplications you will have,

$$AB = \begin{bmatrix} -6 - 2 + 3 & 2 + 5 - 6 \\ 12 + 10 + 2 & -4 - 25 - 4 \end{bmatrix}$$

And by computing the sums, you will get,

$$AB = \begin{bmatrix} -5 & 1 \\ 24 & -33 \end{bmatrix}$$

Since B is 3×2 and A is 2×3, the product BA is a 3×3 matrix:

$$BA = \begin{bmatrix} (3 \cdot -2) + (-1 \cdot 4) & (3 \cdot 1) + (-1 \cdot -5) & (3 \cdot 3) + (-1 \cdot 2) \\ (-2 \cdot -2) + (5 \cdot 4) & (-2 \cdot 1) + (5 \cdot -5) & (-2 \cdot 3) + (5 \cdot 2) \\ (1 \cdot -2) + (-2 \cdot 4) & (1 \cdot 1) + (-2 \cdot -5) & (1 \cdot 3) + (-2 \cdot 2) \end{bmatrix}$$

By carrying out these multiplications you will have,

$$BA = \begin{bmatrix} -6 - 4 & 3 + 5 & 9 - 2 \\ 4 + 20 & -2 - 25 & -6 + 10 \\ -2 - 8 & 1 + 10 & 3 - 4 \end{bmatrix}$$

And by computing the sums, you will get,

$$BA = \begin{bmatrix} -10 & 8 & 7 \\ 24 & -27 & 4 \\ -10 & 11 & -1 \end{bmatrix}$$

Here you can see that not only are then entries in AB not equal to those in BA, AB is of different dimension to BA.

Example 11

Now, lets let try using a 2×3 matrix and a 3×1 matrix.

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix} B = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

You can compute AB since, in order to multiply matrices, you require our first matrix to have the same number of columns as our second matrix has rows. Here A has 3 columns, and B has 3 rows. Lets calculate AB,

$$AB = \begin{bmatrix} 0 \cdot 1 + 2 \cdot -2 + -1 \cdot 0 \\ 4 \cdot 1 + 0 \cdot -2 + 3 \cdot 0 \end{bmatrix}$$

By carrying out these multiplications you will have,

$$AB = \begin{bmatrix} 0 + -4 + 0 \\ 4 + 0 + 0 \end{bmatrix}$$

And by computing these sums, you will get,

$$AB = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

So here AB is the 2×1 matrix given above. However, in this example, BA is undefined, as you require our first matrix to have the same number of columns as our second matrix has rows, B here only has 1 column, where as A has 2 rows.

Let's begin to explore how matrices can be linked to simultaneous equations.

Example 12

Lets explore the equation $AX = B$ for

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 7 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$AX = \begin{bmatrix} -2 \cdot x + 1 \cdot y \\ 3 \cdot x + 7 \cdot y \end{bmatrix}$$

By carrying out these multiplications you will have,

$$AX = \begin{bmatrix} -2x + y \\ 3x + 7y \end{bmatrix}$$

and if $AX = B$, then

$$\begin{bmatrix} -2x + y \\ 3x + 7y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Here you can see, the equation $AX = B$ is equivalent to the following set simultaneous equations:

$$-2x + y = 4$$

$$3x + 7y = 2$$

So you could solve for x and y in $AX = B$, using tools you can learn in **Guide: Introduction to solving simultaneous equations**.

In some cases, it may be most efficient for you to solve a system of simultaneous equations using matrices, you can read more about this in **Guide: Introduction to Gaussian elimination**.

Now you've seen matrix addition and subtraction, scalar multiplication and matrix multiplication you can state some key properties that hold through arithmetic with matrices. Note here again that for this study guide you are assuming that matrices have entries within the complex numbers, without that assumption the following theorem is not always true.

Properties of matrix arithmetic

For any three matrices, A, B, C, the following four properties hold, where the operations make sense.

1. Matrix addition is commutative, that is $A + B = B + A$

2. Matrix addition is associative, that is $(A+B)+C = A+(B+C)$
3. Matrix multiplication is associative, that is $(AB)C = A(BC)$
4. The distributive property holds for matrices, that is $A(B+C) = AB+AC$ and $(A+B)C = AC+BC$

You can see a proof of this in **Proof: Properties of matrix arithmetic**.

Special matrices

The case where $n = m$ is a very important one.

Definition a square matrix

A **square matrix** is a matrix with $n = m$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Example 13

Here are some square matrices:

$$\begin{bmatrix} -3 & 0 \\ 7/4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5/6 & 7 \\ -2 & 4/3 & 8 & 0 \\ 9/2 & -1 & 0 & -3 \\ 0 & 6 & 7/5 & 2 \end{bmatrix} \begin{bmatrix} 5/2 \end{bmatrix} \begin{bmatrix} 0 & -1/3 & 5 \\ 4 & 2/7 & -6 \\ -2 & 3 & 0 \end{bmatrix}$$

Now you can define a special kind of square matrix.

Definition of a diagonal matrix

A **diagonal matrix** is a square matrix in which all non-diagonal entries are 0.

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix can be defined entirely in terms of its main diagonal.

Here you can define one important example of a diagonal matrix.

Definition of a identity matrix

An **identity matrix** is a diagonal matrix where the main diagonal consists only of ones.

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix is important as it leaves matrices unchanged under multiplication, that is for a $m \times n$ matrix A

$$A \cdot 1_n = 1_n \cdot A = A$$

.

There is have one final, special type of matrix for you to define in this section.

Definition of a zero matrix

A **zero matrix** is a matrix where all entries are 0.

$$0_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The zero matrix is important as it leaves matrices unchanged under addition, that is for a $m \times n$ matrix A

$$A + 0_{mn} = 0_{mn} + A = A$$

.

> **Warning**
>
> You should note here that a zero matrix does not have to be square.

Quick check problems

1. Give the dimensions of the following matrices:

$$A = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 4 & -2 \\ 1/3 & -5 & 6 \\ 8 & -3/7 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & -6 \\ -4 & 2/3 \\ 0 & 5 \\ 7/8 & -9 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad D = \left[\begin{array}{cccc} 5/6 & -1 & 3 & \\ -2 & 0 & 4 & - \end{array}\right.$$

2. You are given three statements below. Decide whether they are true or false.

(a) If A is a 2×3 matrix, and B is a 3×2 matrix, then AB is a 3×3 matrix.

(b) For the matrix B in question 1, the entry $b_{12} = 1/3$.

(c) You can only add or subtract matrices that share the same dimensions.

3. Multiply the following matrices,

$$E = \begin{bmatrix} 3 & 0 \\ 4 & -2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$

, give the first entry, ef_{11}, of the product EF.

Further reading

For more questions on this topic, please go to **Questions: Introduction to matrices.**

Version history

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