

# Proof: Law of total probability and Bayes' theorem

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## Summary

This proof sheet demonstrates that the law of total probability and Bayes' theorem are true.

*Before reading this proof sheet, it is recommended that you read [Guide: Conditional probability](#) and [Guide: Law of total probability and Bayes' theorem](#).*

## Proof of the law of total probability

First of all, here is a restatement of the law of total probability from [Guide: Law of total probability and Bayes' theorem](#):

### **i** Definition of the law of total probability

Suppose an event  $B$  depends on several possible scenarios. These scenarios can be described by events  $A_1, A_2, \dots, A_n$ , that are:

- **Mutually exclusive:** they cannot occur at the same time, and
- **Exhaustive:** one of them must always occur.

Then, the **law of total probability** states that the probability of event  $B$  is:

$$\mathbb{P}(B) = \sum_{i=1}^n \mathbb{P}(A_i) \mathbb{P}(B \mid A_i)$$

The proof of the law of total probability comes directly from the definition of conditional probability given in [Guide: Conditional probability](#):

$$\mathbb{P}(B \mid A_i) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(A_i)}$$

Multiplying by  $\mathbb{P}(A_i)$  gives the multiplication rule (again from [Guide: Conditional probability](#)):

$$\mathbb{P}(B \cap A_i) = \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

As scenarios  $A_1, A_2, \dots, A_n$  are mutually exclusive (so  $A_i \cap A_j = \emptyset$  for all  $1 \leq i \neq j \leq n$ ) and exhaustive ( $\bigcup_{1 \leq i \leq n} A_i = B$ ), it follows from results in set theory (see [Guide: Operations on sets]) that:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \dots + \mathbb{P}(B \cap A_n) = \sum_{i=1}^n \mathbb{P}(B \cap A_i)$$

Substituting the above expressions gives:

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \dots + \mathbb{P}(B \mid A_n)\mathbb{P}(A_n) = \sum_{i=1}^n \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

Which results in the law of total probability:

$$\mathbb{P}(B) = \sum_{i=1}^n \mathbb{P}(A_i)\mathbb{P}(B \mid A_i)$$

## Proof of Bayes' theorem

Here is the statement of Bayes' theorem from [Guide: Law of total probability and Bayes' theorem](#):

### Statement of Bayes' Theorem

If  $A$  and  $B$  are events with  $\mathbb{P}(B) > 0$ , then Bayes' Theorem states:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

where:

- $\mathbb{P}(A \mid B)$  is the probability of  $A$  given  $B$ ,
- $\mathbb{P}(B \mid A)$  is the probability of  $B$  given  $A$ ,
- $\mathbb{P}(A)$  and  $\mathbb{P}(B)$  are the individual probabilities of  $A$  and  $B$ , respectively.

Bayes' Theorem is derived directly from the definition of conditional probability: see [Guide: Conditional probability](#). Start with the conditional probabilities of two events  $A$  and  $B$ :

$$(1) \quad \mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{and} \quad (2) \quad \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

You can rearrange (2) by multiplying both sides by  $\mathbb{P}(A)$ , giving the multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\mathbb{P}(A)$$

Substitute this result into equation (1) to get:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

This gives Bayes' Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

## Further reading

[Click this link to go back to Guide: Law of total probability and Bayes' theorem.](#)

## Version history

v1.0: initial version created 04/25 by Sophie Chowgule as part of a University of St Andrews VIP project.

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