Answers: Law of total probability and Bayes' theorem

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Summary

Answers to questions relating to the guide on the law of total probability and Bayes' theorem.

These are the answers to Questions: Law of total probability and Bayes' theorem.

Please attempt the questions before reading these answers.

Q1

1.1.

You know:

- $\mathbb{P}(\text{Ward A}) = 0.4$
- $\mathbb{P}(\text{Recover} \mid \text{Ward A}) = 0.8$
- $\mathbb{P}(\text{Ward B}) = 0.6$
- $\mathbb{P}(\text{Recover} \mid \text{Ward B}) = 0.6$

Using the law of total probability:

$$\mathbb{P}(\text{Recover}) = \left(\frac{4}{10}\right)\left(\frac{8}{10}\right) + \left(\frac{6}{10}\right)\left(\frac{6}{10}\right) = 0.32 + 0.36 = 0.68$$

So the probability that a randomly chosen patient recovers is 0.68.

1.2.

You know:

- $\mathbb{P}(\text{Veg}) = 0.5$, $\mathbb{P}(\text{Finish} \mid \text{Veg}) = 0.9$

- $\mathbb{P}(\text{Chicken}) = 0.3$, $\mathbb{P}(\text{Finish} \mid \text{Chicken}) = 0.7$
- $\mathbb{P}(\text{Fish}) = 0.2$, $\mathbb{P}(\text{Finish} \mid \text{Fish}) = 0.8$

Using the law of total probability:

$$\mathbb{P}(\text{Finish}) = (0.5)(0.9) + (0.3)(0.7) + (0.2)(0.8) = 0.45 + 0.21 + 0.16 = 0.82$$

So the probability that a randomly chosen student finishes their lunch is 0.82.

1.3.

You know:

- $\mathbb{P}(F_1) = 0.2$, $\mathbb{P}(\text{Defective} \mid F_1) = 0.05$
- $\mathbb{P}(F_2) = 0.3$, $\mathbb{P}(\text{Defective} \mid F_2) = 0.02$
- $\mathbb{P}(F_3) = 0.5$, $\mathbb{P}(\text{Defective} \mid F_3) = 0.01$

Using the law of total probability:

$$\mathbb{P}(\text{Defective}) = (0.2)(0.05) + (0.3)(0.02) + (0.5)(0.01) = 0.01 + 0.006 + 0.005 = 0.021$$

So the probability that a randomly selected product is defective is 0.021.

1.4.

You know:

- $\mathbb{P}(\text{Home}) = 0.5$, $\mathbb{P}(\text{Complete} \mid \text{Home}) = 0.7$
- $\mathbb{P}(\text{Library}) = 0.3$, $\mathbb{P}(\text{Complete} \mid \text{Library}) = 0.9$
- $\mathbb{P}(\text{Café}) = 0.2$, $\mathbb{P}(\text{Complete} \mid \text{Café}) = 0.6$

Using the law of total probability:

$$\mathbb{P}(\text{Complete}) = (0.5)(0.7) + (0.3)(0.9) + (0.2)(0.6) = 0.35 + 0.27 + 0.12 = 0.74$$

So the probability that the student completes their homework is 0.74.

Q2

2.1.

You know:

- $\mathbb{P}(D) = 0.02$
- $\mathbb{P}(\text{Pos} \mid D) = 0.95$
- $\mathbb{P}(\text{Pos} \mid \neg D) = 0.1$ (where $\neg D$ means the person does not have the disease)
- $\mathbb{P}(\neg D) = 0.98$

Using the law of total probability:

$$\mathbb{P}(\text{Pos}) = (0.02)(0.95) + (0.98)(0.1) = 0.019 + 0.098 = 0.117$$

Now applying Bayes' theorem:

$$\mathbb{P}(D \mid \text{Pos}) = \frac{(0.95)(0.02)}{0.117} \approx 0.162$$

So the probability that the person has the disease, given that they test positive, is approximately 0.162. Not a very good test!

2.2.

You know:

- $\mathbb{P}(\text{Rain}) = 0.4$
- $\mathbb{P}(\text{Dry}) = 0.6$
- $\mathbb{P}(F \mid \text{Rain}) = 0.8$
- $\mathbb{P}(F \mid \text{Dry}) = 0.1$

Using the law of total probability:

$$\mathbb{P}(F) = (0.4)(0.8) + (0.6)(0.1) = 0.32 + 0.06 = 0.38$$

Then applying Bayes' theorem gives:

$$\mathbb{P}(\text{Rain} \mid F) = \frac{(0.8)(0.4)}{0.38} \approx 0.842$$

So the probability that it actually rains in St Andrews, given that the forecast predicts rain, is approximately 0.842.

2.3.

You know:

- $\mathbb{P}(A) = 0.7$
- $\mathbb{P}(B) = 0.3$
- $\mathbb{P}(F \mid A) = 0.02$
- $\mathbb{P}(F \mid B) = 0.05$

Using the law of total probability:

$$\mathbb{P}(F) = (0.7)(0.02) + (0.3)(0.05) = 0.014 + 0.015 = 0.029$$

Then applying Bayes' theorem gives:

$$\mathbb{P}(B \mid F) = \frac{(0.05)(0.3)}{0.029} \approx 0.517$$

So the probability that the broken biscuit came from Machine B, given that it is broken, is approximately 0.517.

2.4.

You know:

- $\mathbb{P}(\text{Red}) = 0.4$

- $\mathbb{P}(\text{Blue}) = 0.6$
- $\mathbb{P}(W \mid \text{Red}) = 0.3$
- $\mathbb{P}(W \mid \text{Blue}) = 0.7$

Using the law of total probability:

$$\mathbb{P}(W) = (0.4)(0.3) + (0.6)(0.7) = 0.12 + 0.42 = 0.54$$

Then applying Bayes' theorem gives:

$$\mathbb{P}(\text{Red} \mid W) = \frac{(0.3)(0.4)}{0.54} \approx 0.222$$

So the probability that the sweet is red, given that it has a wrapper, is approximately 0.222.

Version history and licensing

v1.0: initial version created 05/25 by Sophie Chowgule as part of a University of St Andrews VIP project.