The product rule

Tom Coleman

Summary

The product rule is one of the three central techniques of differentiation, allowing you to differentiate a product of two functions. This guide introduces the product rule and explains examples of where it is used.

Before reading this guide, it is recommended that you read Guide: Introduction to differentiation and the derivative.

What is the product rule?

[MOTIVATION AND BLURB]

This guide will look at the idea of differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

The summary table of key derivatives from Guide: Introduction to differentiation and the derivative is reproduced here.

Function $f(x)$	Derivative $f'(x)$	Notes
$f(x) = c$	$f'(x) = 0$	
$f(x) = ax + b$	$f'(x) = a$	
$f(x) = ax^n$	$f'(x) = anx^{n-1}$	$n \neq 0$
$f(x) = ae^{bx}$	$f'(x) = abe^{bx}$	
$f(x) = a\sin(bx)$	$f'(x) = ab\cos(bx)$	
$f(x) = a\cos(bx)$	$f'(x) = -ab\sin(bx)$	
$f(x) = a\ln(bx)$	$f'(x) = \dfrac{a}{x}$	

Statement of the product rule

The product rule

Let $u(x)$ and $v(x)$ be two differentiable functions. Then the **product rule** says that

$$(uv)'(x) = \frac{d}{dx}\left(u(x)v(x)\right) = u(x)v'(x) + u'(x)v(x)$$

that is, the derivative of the product of $u(x)$ and $v(x)$ is equal to the product of $u(x)$ and the derivative of $v(x)$, plus the product of $v(x)$ and the derivative of $u(x)$.
This can also be written as

$$\frac{d}{dx}\left(u(x)v(x)\right) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Sometimes $f(x)$ and $g(x)$ are written instead of $u(x)$ and $v(x)$ in the statement of the product rule. The reason that $u(x)$ and $v(x)$ is used here is that sometimes the product function is called $f(x)$; and you then can't use it again!

Examples

Example 1

Find the derivative of the function $f(x) = \frac{x^2}{2} + 1$.
Here, you can use the sum and scaling rules to differentiate this function. Using the sum rule, you can see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2} + 1\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}(1)$$

Now, using the scaling rule on the first of these terms, you can get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}(1) = \frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(1)$$

Now you can use the derivatives of common functions. The function x^2 is of the form x^n, so the derivative of x^2 is $2x^{2-1} = 2x^1 = 2x$. The function 1 is a constant, so the derivative of 1 is 0. Putting these results in gives

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(1) = \frac{1}{2}(2x) + 0$$

Simplifying gives the answer: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2} + 1\right) = x$ and so the derivative of $f(x) = x^2/2 + 1$ is equal to $f'(x) = x$.
You can then say that this function $f(x) = x^2/2+1$ is increasing if $x > 0$, decreasing if $x < 0$, and is stationary exactly when $x = 0$.

You do not need to be so meticulous in your general practice when differentiating. You can differentiate functions without stating what rules you are using.

Example 2

Find the derivative of the function $f(x) = 4e^{3x} - \sqrt{x}$.
Of these two terms, $4e^{3x}$ looks like a common function, but you have not seen how to differentiate $\sqrt{x}$ yet. In fact, this is another case of x^n; in this case, $n = 1/2$. (See Guide: Laws of indices for why this is.)
You can then rewrite the function as

$$f(x) = 4e^{3x} - x^{1/2}$$

which you can then apply your knowledge of differentiation to. Using the results above, the derivative of $4e^{3x}$ is $4 \cdot 3 \cdot e^{3x} = 12e^{3x}$. The derivative of $x^{1/2}$ is $\left(\frac{1}{2}\right) x^{1/2-1} = \frac{x^{-1/2}}{2}$. Therefore, the derivative of $4e^{3x} - \sqrt{x}$ is

$$f'(x) = 12e^{3x} - \frac{x^{-1/2}}{2}$$

You could turn $x^{-1/2}$ into $1/\sqrt{x}$; however, you may be asked to differentiate this function again (particularly if you are looking into behaviour of a function), and often it is best to leave the power as a real number rather than using the square root notation.

As a rule of thumb, you should always write an nth root using a fractional power instead. Here's another example of where you should write a term involving a power of x:

Example 3

Find the derivative of the function $f(x) = \dfrac{1}{x^4} + 4\cos(x) - \sin(4x)$.
Again, the cosine and sine terms are manageable from the derivative above, it is only the $1/x^4$ term that needs a little thought. In fact, this is yet *another* case of x^n; this time when $n = -4$. (See Guide: Laws of indices for why this is.)
You can then rewrite the function as

$$f(x) = x^{-4} + 4\sin(x) - \cos(4x)$$

and use the familiar rules of differentiation above. Using the results above with $n = -4$, the derivative of x^{-4} is $-4x^{-4-1} = -4x^{-5}$. The derivative of $4\sin(x)$ is $4(\cos(x)) = 4\cos(x)$. The derivative of $\cos(4x)$ is $-4\sin(4x)$. Taking care of your signs you can write the derivative of $\dfrac{1}{x^4} + 4\cos(x) - \sin(4x)$ as

$$f'(x) = -4x^{-5} + 4\cos(x) - (-4\sin(4x)) = -4x^{-5} + 4\cos(x) + 4\sin(4x).$$

Example 4

Determine the behaviour of the function $f(x) = 2\ln(3x) - x$ when $x = 1$.
Here, you will first need to differentiate the function $f(x)$ to find $f'(x)$. Then, you will need to evaluate the derivative $f'(x)$ when $x = 1$ to see how the function behaves.
Using your rules of differentiation as you found above, you can say that the derivative of $2\ln(3x)$ is $2/x$, and the derivative of x is 1. Therefore, the derivative of the function $f(x) = 2\ln(3x) - x$ is

$$f'(x) = \frac{2}{x} - 1.$$

You can evaluate the derivative $f'(x)$ at $x = 1$ to get

$$f'(1) = \frac{2}{1} - 1 = 2 - 1 = 1$$

and so the derivative is positive at $x = 1$. This implies that the function $f(x) = 2\ln(3x) - x$ is increasing at the point $x = 1$.
It also means that the gradient of the tangent to the function $f(x)$ at the point $(1, 2\ln(3) - 1)$ is 1. You can see this in the figure below.

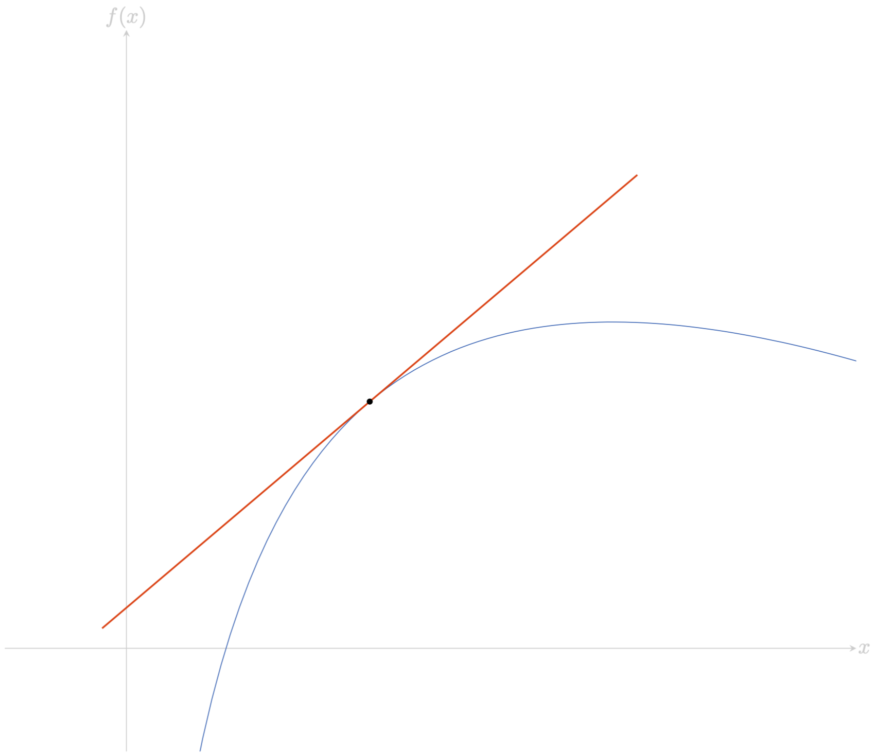

Figure 1: The graph of $f(x) = 2\ln(3x) - x$, with the tangent to the graph at $(1, 2\ln(3) - 1)$ illustrated, demonstrating that the function is increasing at $x = 1$.

Summary

Here's a table of derivatives that you should remember going into any further reading on differentiation. Here, a, b, c, n are any real numbers.

Function $f(x)$	Derivative $f'(x)$	Notes
$f(x) = c$	$f'(x) = 0$	
$f(x) = ax + b$	$f'(x) = a$	
$f(x) = ax^n$	$f'(x) = anx^{n-1}$	$n \neq 0$
$f(x) = ae^{bx}$	$f'(x) = abe^{bx}$	
$f(x) = a\sin(bx)$	$f'(x) = ab\cos(bx)$	
$f(x) = a\cos(bx)$	$f'(x) = -ab\sin(bx)$	
$f(x) = a\ln(bx)$	$f'(x) = \frac{a}{x}$	

Quick check problems

1. Answer the following questions true or false:

(a) The derivative of a function at $x = a$ is equal to the gradient of the tangent to $f(x)$ at $x = a$.

(b) If $f'(a) < 0$, then the function is increasing at $x = a$.

(c) If $f(x) = c$, then the derivative $f'(x) = c - 1$.

(d) The derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$.

(e) The derivative of $f(x) = \dfrac{1}{x}$ is $f'(x) = \ln(x)$.

(f) The power of x in the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$ is $-3/2$.

2. Differentiate the following functions with respect to x.

(a) $f(x) = 3x^7 - 14x$

(b) $f(x) = -4\cos(3x)$

(c) $f(x) = -15\sin(x) + e^{8x}$

Further reading

For more questions on the subject, please go to Questions: The product rule

For more about techniques of differentiation, please see [Guide: The quotient rule], and [Guide: The chain rule].

For more about why the rules and techniques of differentiation are true, please see [Proof sheet: Rules of differentiation].

Version history

v1.0: initial version created 04/25 by tdhc.