Proof: Law of total probability and Bayes’ theorem

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Summary

This proof sheet demonstrates that the law of total probability and Bayes’ theorem are true.

*Before reading this proof sheet, it is recommended that you read* [*Guide: Conditional probability*](../studyguides/conditionalprobability.qmd) *and* [*Guide: Law of total probability and Bayes’ theorem*](../studyguides/bayestheorem.qmd)*.*

# Proof of the law of total probability

First of all, here is a restatement of the law of total probability from [Guide: Law of total probability and Bayes’ theorem](../studyguides/bayestheorem.qmd):

|  |
| --- |
| Definition of the law of total probability |
| Suppose an event depends on several possible scenarios. These scenarios can be described by events , that are:   * **Mutually exclusive**: they cannot occur at the same time, and * **Exhaustive**: one of them must always occur.   Then, the **law of total probability** states that the probability of event is: |

The proof of the law of total probability comes directly from the definition of conditional probability given in [Guide: Conditional probability](../studyguides/conditionalprobability.qmd):

Multiplying by gives the multiplication rule (again from [Guide: Conditional probability](../studyguides/conditionalprobability.qmd)) :

As scenarios are mutually exclusive (so for all ) and exhaustive (), it follows from results in set theory (see [Guide: Operations on sets]) that:

Substituting the above expressions gives:

Which results in the law of total probability:

# Proof of Bayes’ theorem

Here is the statement of Bayes’ theorem from [Guide: Law of total probability and Bayes’ theorem](../studyguides/bayestheorem.qmd):

|  |
| --- |
| Statement of Bayes’ Theorem |
| If and are events with , then Bayes’ Theorem states:  where:   * is the probability of given , * is the probability of given , * and are the individual probabilities of and , respectively. |

Bayes’ Theorem is derived directly from the definition of conditional probability: see [Guide: Conditional probability](../studyguides/conditionalprobability.qmd). Start with the conditional probabilities of two events and :

You can rearrange by multiplying both sides by , giving the multiplication rule:

Substitute this result into equation to get:

This gives Bayes’ Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

# Further reading

[Click this link to go back to Guide: Law of total probability and Bayes’ theorem.](../studyguides/bayestheorem.qmd)

## Version history

v1.0: initial version created 04/25 by Sophie Chowgule as part of a University of St Andrews VIP project.

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