Proof: The square root of 2 is irrational

Jessica Taberner

Summary

Proof by contradiction of the irrationality of

Before reading this proof sheet, it is recommended that you read [Overview: Number sets].

# is irrational

You can remember from [Overview: Number sets] that an irrational number is a number that cannot be represented as a fraction of integers, where . Here you can prove that is irrational. This particular proof dates back to the ancient Greeks and relies on a method of proof called **proof by contradiction**. In a proof by contradiction you begin by assuming that what you’re trying to prove is false, then you show that from that assumption you can derive a contradiction, so your assumption must have been false.

Let’s prove that is irrational by contradiction.

Suppose is rational. Then it can be expressed as a fraction:

where and are integers with no common factors other than , meaning the fraction is in its simplest form, and .

Then you can square both sides:

and multiply both sides by :

This implies that is even (since it is divisible by ). Since the square of an odd number is odd, must be even. Let for some integer .

You can then substitute into the equation:

Dividing both sides by :

This shows that is also even, which means must be even.

Since both and are even, they share a common factor of , contradicting the assumption that and have no common factors other than .

This contradiction implies that your initial assumption was false, cannot be written as a fraction of integers, and so, is irrational.

# is irrational

You can extend this proof to show that is irrational for any prime .

Suppose is rational for some prime . Then it can be expressed as a fraction:

where and are integers with no common factors other than , meaning the fraction is in its simplest form, and .

Squaring both sides:

Multiplying both sides by :

This implies that is divisible by . So must also be divisible by (since is prime). Let for some integer .

Substituting into the equation:

Dividing both sides by :

This implies that is also divisible by , so is divisible by .

So, both and are divisible by , contradicting the assumption that is in its simplest form.

Therefore, the assumption must be false, and is irrational for any prime number .

# Further reading

For more on this topic, please go to [Overview: Number sets].

## Version history

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