Proof: PMFs, PDFs, CDFs

Sophie Chowgule, Tom Coleman

Summary

Explanations as to why some PMF’s and PDF’s are valid.

*Before reading this proof sheet, it is recommended that you read* [*Guide: PMFs, PDFs, CDFs*](../studyguides/pmfspdfscdfs.qmd)*. Other recommended reading material will be said when it is needed.*

# Proof that the binomial distribution is a PMF

*Before reading this section, you may find it useful to read [Guide: The binomial theorem].*

Remember from [Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd) that the **binomial distribution** is given by the following.

|  |
| --- |
| Binomial distribution |
| where:   * the random variable measures the number of success in a set of trials   + is number of successes   + is number of trials * is the probability of success in a single trial * is the probability of failure in a single trial |

Also from [Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd), the two conditions to be a valid PMF are the following:

* **Non-negativity**: The probability assigned to each possible outcome must be greater than or equal to zero, that is:
* **Honesty condition**: The sum of probabilities of all possible outcomes of a discrete random variable must be equal to one:

First of all, every term in the PMF for the binomial distribution above is non-negative, and the product of non-negative numbers is non-negative, so for any .

The honesty condition comes about because binomial distributions follow the **binomial theorem**. The binomial theorem states that:

(See [Guide: The binomial theorem] for more.)

The number of successes ranges from (total failure) to (complete success). Therefore, the sum of all possible probabilities is:

which is the left-hand side of the binomial theorem. Using the binomial theorem with :

So, the sum of the probabilities over all possible values of equals 1, satisfying the honesty condition.

# Proof that the uniform distribution is a PDF

*Before reading this section, you may find it useful to read [Guide: Introduction to integration] and [Guide: Properties of integration].*

Remember from [Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd) that the **uniform distribition** over the interval is given by the following.

|  |
| --- |
| Uniform distribution |
| where are real numbers such that . |

Also from [Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd), the two conditions to be a valid PDF are the following:

* **Non-negativity**: The PDF must be greater than or equal to zero over its entire range of possible values:
* **Honesty condition**: The area under the entire PDF must be equal to , so:

To check if this is a valid PDF, you need to confirm that it satisfies these two key conditions.

**Non-negativity**: for all values of , as in and otherwise.

**Honesty**: To satisfy the honesty condition, the integral of the PDF over the interval must equal . Using the properties of integration, you can split the integral into three parts along the lines of the PDF:

Using the definition of on these intervals gives

Since the integral of over any limits is zero, this reduces to

Working out this integral dives

And so you can see that all uniform distributions are valid PDFs.

# Proof that the normal distribution is a PDF

*Before reading this section, you may find it useful to read [Guide: Properties of integration], [Guide: Integration by substitution], [Guide: Introduction to double integration], and [Guide: Co-ordinate changes in double integration].*

Remember from [Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd) that the **normal distribution** is given by the following.

|  |
| --- |
| Normal distribution |
| where are real numbers such that . (Here, is the mean and is the standard deviation.) |

To check if this is a valid PDF, you need to confirm that it satisfies the two key conditions.

**Non-negativity**: As an exponential function, , and as . So .

**Honesty**: Here’s the fun part.

The idea is to show that this integral , given by

is equal to . To tackle this integral, it needs to look a little nicer; you can use integration by substitution to do this (see [Guide: Integration by substitution]). Let . Then , and so . As , it follows that . Since , the integral becomes

Next, you can use the fact that is an even function to change the limits. Using the property of even function about symmetric limits (see [Guide: Properties of integration]), the integral becomes

All that you have done so far has not changed the value of the integral, so this is still equal to . Now, the choice of variables in an integral doesn’t matter, so as well. Multiplying both together gives

Now, the variables here are independent, so you can combine this into a double integral. Doing this gives

You can now change the co-ordinates to polar co-ordinates (see [Guide: Changing co-ordinates in double integrals] for more). By setting and , it follows that . The region of integration is and , which corresponds to the first quadrant of the plane; this is represented in polar co-ordinates by and . Finally, becomes by using the Jacobian. Therefore, the integral becomes

Now you can evaluate this double integral. The derivative of with respect to is ; so that means that the integral of is (you can get this result by substitution if you wanted). Using the fact that is equal to when and tends to as tends to infinity, you can get

Evaluating this final integral gives

So , implying that . But cannot be , as is a positive function and the integral of a positive function is always positive. So and therefore the normal distribution really is a PDF.

# Further reading

[Guide: PMFs, PDFs, CDFs](../studyguides/pmfspdfscdfs.qmd)

[Questions: PMFs, PDFs, CDFs](../questions/qs-pmfspdfscdfs.qmd)

## Version history and licensing

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