Proof: Rules of differentiation

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Summary

The proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, chain, and quotient rules - are true.

*Before reading this proof sheet, it is essential that you read* [*Guide: Introduction to differentiation and the derivative*](../studyguides/introtodifferentiation.qmd)*. In addition, reading [Guide: Introduction to limits] is useful. Further reading will be illustrated where required.*

The starting point of this proof sheet is the limit definition of the derivative of a function:

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| --- |
| Reminder of limit definition of the derivative |
| The **derivative of with respect to**  is defined to be the function |

To find the derivative of a function , replace the function in the right-hand side of the above limit, and then evaluate the limit. Care has to be taken to remove the in the denominator of the fraction, as you cannot evaluate the limit as tends to with it still there! (Remember, dividing by can be extremely hazardous to your health.)

From here, the proof sheet will start with the proof of the derivative of , which will then be expanded using the binomial theorem to find the derivative of . Then, the derivative of is found. Finally, the derivatives of and are demonstrated.

# Sum and difference rules

# Scaling rules

# Product rule

Here is the product rule, restated with and for visual ease in the proof that follows.

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| --- |
| The product rule |
| Let and be two differentiable functions. Then the **product rule** says that  that is, the derivative of the product of and is equal to the product of and the derivative of , plus the product of and the derivative of .  This can also be written as |

## Proof of the product rule

Here’s why the product rule works. It’s not quite as straightforward as the proof of the sum rule and the constant rule; you have to manufacture the definition of the derivative in one place by using a creative addition by .

So, let’s start with and as two differentiable real-valued functions, with product . Putting this into the limit definition of the derivative given above:

Since , this becomes

# Quotient rule

*Before this proof, you may find it useful to read [Guide: Laws of indices]*

There is an important limit that needs to be used to find the derivative of , which is given by:

The proof that this is true relies on some *real analysis* and the fact that the constant can be defined by .

|  |
| --- |
| Derivative of |
| The derivative of is . |

Again, let’s use the limit definition of the derivative with . This gives:

Typically, the goal has been to remove the in the denominator; however, it is not possible in this case, as expanding doesn’t give a that you can get rid of while factorizing. In this case, what you can do is try and reduce to a known limit. In particular, the known limit involving is and so you may be able to manufacture this limit by manipulation of the expression above. Using the laws of indices, you can write that

Factorising out gives

Now, is constant as varies, and so you can take this out of the limit to get:

You can now evaluate the limit to get:

It follows that , as required.

# Derivative of and

*Before reading this section, it may be useful to read* [*Guide: Trigonometric identities (radians)*](../studyguides/trigonometricidentities-radians.qmd)*.*

As with the derivative of in the previous section, there are two important limits involving and that are needed before continuing. These are:

To show these are true, you need techniques from *real analysis*, which is the study of functions on real numbers.

### Derivative of

|  |
| --- |
| Derivative of |
| The derivative of is . |

Starting off with the limit definition of differentiation and gives

As in the case of , expanding won’t give a single to cancel with! However, it could be that you can reduce this expression to the limits given above. Using the sum rule for (see [Guide: Trigonometric identities (radians)](../studyguides/trigonometricidentities-radians.qmd)), you can write that

The bones of the two given limits are here, and now it’s time to uncover them. Factorising out and gives

Using the two limits given above, and the fact that the values of and are independent of :

and so is the derivative of .

### Derivative of

|  |
| --- |
| Derivative of |
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# Further reading

[Click this link to go back to Guide: Introduction to differentiation and the derivative.](../studyguides/introtodifferentiation.qmd)

[Click this link to go back to Guide: The product rule](../studyguides/productrule.qmd)

[For questions on differentiation and the derivative, please go to Questions: Introduction to differentation and the derivative.](../questions/qs-introtodifferentiation.qmd)

## Version history

v1.0: initial version created in 04/25 by tdhc.

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