Conditional probability

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Summary

Conditional probability describes how the likelihood of an event changes when new information is introduced. It is a key concept for understanding relationships between events and making better predictions in uncertain scenarios. This guide outlines how to calculate conditional probabilities, apply the multiplication rule, test for independence, and distinguish between independent and disjoint events.

*Before reading this guide, it is highly recommended that you read [Guide: Introduction to probability].*

# Introduction

Conditional probability is a fundamental concept in probability theory. It describes how the likelihood of an event changes when additional information becomes available. In practice, it is used to update probabilities based on the outcome of related events and is widely applied in areas such as data analysis, forecasting, and decision-making.

It also forms the basis for several key results in statistics, including the multiplication rule, the Law of Total Probability, and Bayes’ Theorem. This guide covers the definition of conditional probability, methods for calculating it, ways to test for independence, and highlights the distinction between independent and disjoint events.

# Conditional probability

In everyday life, the likelihood of an event often changes as new information becomes available. For example, if you’re planning to leave the house and notice dark clouds gathering, you might assume it’s more likely to rain and decide to bring an umbrella. This shift in expectation can be understood using **conditional probability**.

**Conditional probability** is the probability of an event occurring given that another event has already occurred. It helps quantify how one event influences the likelihood of another. More formally:

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| Definition of conditional probability |
| If and are events, then the **conditional probability** of occurring given that has occurred, written as , is defined as:  where:   * is the probability that both and occur. * is the probability that occurs, regardless of whether occurs. |

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| Note |
| The notation is read as **“the probability of given ”**, where the vertical bar “” means “given”. |

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| Note |
| The notation is read as **“the probability of and ”**, where “” denotes the **intersection**, the set of outcomes where both events occur.  For a more detailed explanation on the set notations used here, see [Guide: Operations on Sets]. |

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| Warning |
| Conditional probability is only defined when , meaning that the event must have a chance of occurring. Otherwise, it doesn’t make sense to ask about the probability of given if has no chance of occurring. This is also reflected in the formula for conditional probability since dividing by zero would make undefined. |

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|  | **Example 1**  Suppose you roll a fair six-sided die. The possible outcomes are . You are told that the number is greater than 2, so what is the probability that the number is even?  Knowing that the result is greater than , the possible outcomes are now and among these outcomes, the even numbers are and .  To calculate the conditional probability:  First, find the individual probabilities:   * is the probability of rolling an even number above , meaning either a or a , so: * is the probability that the roll is greater than 2, which includes the outcomes and , so:   Substituting these into the formula gives:  So, given that the roll is greater than 2, the probability that it is even is . |

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|  | **Example 2**  Now suppose you draw a card at random from a standard deck of 52 cards. You are told that it is a face card, that is a Jack, Queen, or King. What is the probability that the card is a King?  Since you know the card is a face card, the possible outcomes are now limited to the four Jacks, four Queens, and four Kings, with one King in each suit.  Using the conditional probability formula:  First, find the individual probabilities:   * is the probability that the card is a King. Since there are Kings in the deck: * is the probability that the card is any face card. There are face cards in total, so:   Substituting these into the conditional probability formula gives:  So, given that the card is a face card, the probability that it is a King is . |

# Multiplication rule

Sometimes you want to find the probability that two events both occur. For example, suppose you draw two cards from a deck, one after the other. You might know the probability that the first card is a Queen, and you might also know the probability that the second card is a Queen given that the first card was a Queen. What if you want to find the probability that both cards are Queens?

The **multiplication rule** allows you to calculate the probability that two events both occur, using a conditional probability and the probability of one of the events. It is especially useful when you have partial information about two related events.

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| Multiplication rule |
| If and are events, then the probability that both occur, , is given by:  where:   * is the probability that occurs, regardless of whether occurs. * is the probability that occurs given that has occurred. |

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| Tip |
| The multiplication rule also works the other way around:  Both forms are correct, so choose the one that matches the information you have. |

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| Note |
| The multiplication rule comes from rearranging the definition of conditional probability:  To get , multiply both sides by :  The terms cancel out on the right-hand side, which gives the multiplication rule: |

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|  | **Example 3**  Suppose you draw two cards from a standard deck of cards, one after the other, without replacement. What is the probability that both cards drawn are queens?  To solve this, use the multiplication rule:  First, calculate the probability that the first card is a queen and the probability that the second card is a queen given the first was a queen. There are queens in a deck of cards, so:  If the first card was a queen, there are now queens remaining in a card deck:  Substituting these into the multiplication rule gives:  So, the probability of drawing two queens without replacement is approximately . |

# Checking for independence using conditional probability

Conditional probability can also be used to check whether two events are **independent**. If two events and are independent, then knowing that has occurred does not change the probability of occurring. In other words:

This is equivalent to:

If either of these conditions do not hold, then the probability of is affected by the occurrence of , which means and are **dependent** events.

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| Tip |
| For a more detailed explanation of independent and dependent events, see [Guide: Introduction to Probability]. |

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|  | **Example 4**  Imagine you roll a fair six-sided die as in example 1. Let be the event that the die shows an even number, and let be the event the die shows a number greater than .  First, find:   * , the probability the die shows an even number ( or ): * Now find , the probability the number is even given that it is greater than . The outcomes greater than are and and among the two, only is even. So:   Since:  the events and are independent. |

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|  | **Example 5**  Suppose you draw a card at random from a standard deck. Let be the event that the card is a Queen, and let be the event that the card is a face card, a Jack, Queen, or King.  First, find the individual probabilities:   * is the probability the card is a Queen. As there are Queens in the deck: * is the probability the card is a face card. As there are face cards in total: * is the probability the card is both a Queen and a face card. Since all Queens are face cards, this is equal to the probability of drawing a Queen:   Now check whether the multiplication rule holds:  But:  And since:  the events and are **dependent**. |

# Disjoint events

When working with probability, it’s important to understand the difference between **disjoint** events and **independent** events. These terms are often confused, but they describe fundamentally different relationships between events.

* **Disjoint** events (also called **mutually exclusive** events) are events that cannot occur at the same time. If and are disjoint, then:

For example, flipping a coin once cannot result in both heads and tails.

* **Independent** events are events where the outcome of one does not affect the probability of the other. If and are independent, then:

For example, flipping a coin and rolling a die are independent events because the result of the coin flip does not change the probability of the die roll.

| Feature | Disjoint Events | Independent Events |
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| Can both events occur at the same time? | No | Yes |
| Formula |  |  |
| Relationship between events | If one happens, the other cannot | One happening does not change the probability of the other |

Table 1: Table comparing the key differences between disjoint and independent events.

## Quick Check Problems

1. Are the following statements true or false?
2. Conditional probability is only defined when the probability of the given event is greater than zero.
3. If , then and are independent.
4. Disjoint events are always independent.
5. A card is drawn at random from a standard deck of 52 cards.
6. Given that the card is red, what is the probability that it is a heart?
7. Given that the card is a face card, what is the probability that it is a King?
8. A bag contains 3 green sweets and 2 yellow sweets. Two are picked without replacement.  
   What is the probability that both are green?
9. In a study:  
   , , and   
   Are events and independent or dependent?
10. For each pair of events, decide whether they are disjoint, independent, both, or neither.
11. = “rolling an even number”, = “rolling a number greater than 4” (on a fair six-sided die)
12. = “drawing a Queen”, = “drawing a face card” (from a standard deck)

# Further reading

[For more questions on the subject, please go to Questions: Conditional Probability.]

## Version history and licensing

v1.0: initial version created 05/25 by Sophie Chowgule as part of a University of St Andrews VIP project.

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