Introduction to complex numbers

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Summary

Complex numbers are an extremely important concept in mathematics, as you can express a solution to any algebraic equation using complex numbers. This guide introduces the idea of a complex number in the form , as well as the concepts of real and imaginary parts, complex conjugates, and Argand diagrams.

# What is a complex number?

**Real numbers**, which consist of all possible decimal numbers, can do lots of things very well. For instance, you can measure all possible lengths using real numbers, which you cannot do using rational numbers with whole numbers and non-zero. You can also solve any equation for , with non-zero real numbers.

But real numbers cannot do everything. Consider the equation

Since the square of any real number is greater than or equal to , it follows that this equation has **no** solutions which are real numbers. So you will have to look elsewhere for the solution to such an equation; perhaps defining a new kind of number. Setting to be a number such that solves this issue; this is known as the **imaginary unit**.

It turns out that introducing this imaginary unit is enough to express solutions to **all** equations involving an term, known as **quadratic equations**; for more on these, see [Guide: Introduction to quadratic equations](introtoquadratics.qmd). By using this imaginary unit as the basis for a new number system containing the real numbers, you can solve all possible quadratic equations. This number system is known as the **complex numbers**.

Complex numbers appear almost everywhere in mathematics. Aside from being used to solve any quadratic equation, they have important applications in the theory of numbers more generally, in higher forms of algebra, the study of functions, the theory of electromagnetism, solving differential equations, and even measuring the motion of tornadoes!

This guide will focus on introducing the concept of a complex number, with the exact definition of complex numbers together with some initial examples of complex numbers. Then, the concepts of real and imaginary parts of complex numbers are explored, the complex conjugate is defined, and the pictorial representation of a complex number via Argand diagrams is explained.

# Initial definitions of complex numbers

The key concept that separates complex numbers from real numbers is the existence of the imaginary unit, otherwise known as ;

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| Definition of the imaginary unit |
| The **imaginary unit** is defined by the fact that . |

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| Important |
| Sometimes the imaginary unit is written by instead of . This is often used in contexts where means something else, such as in engineering and in the programming language Python. |

Now, you can use the imaginary unit to define a **complex number**:

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| Definition of a complex number |
| Any number where and are real numbers is called a **complex number**. |

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| Tip |
| * The set of all complex numbers is called . * Sometimes a complex number can be written as . This particularly happens when is a square root such as , or the output of a function such as . |

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|  | **Example 1**  Here are some complex numbers: , , , . Some more complex numbers are and . |

You can notice that in Example 1, and have and respectively. These are examples of some special kinds of complex numbers:

* If , then is known as **real number**. So every real number is also a complex number.
* If , then is known as a **purely imaginary** number.

# Square roots of negative numbers

You will notice that is defined by the fact that . It is very important to say that is **not** the only solution to the equation . Considering and using the laws of indices (see [Guide: Laws of indices](lawsofindices.qmd)) gives

and so is also a solution to .

Purely imaginary numbers are used to express solutions to the equation , where is a positive real number.

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| Solutions to |
| Let be a positive real number. Then the solutions to are given by |

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|  | **Example 2**  The two solutions of are and .  The two solutions of are and . |

# Real and imaginary parts

The real numbers that form the two separate parts of the complex number have special names. These real numbers can be recovered from the complex number by taking **real and imaginary parts**:

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| Definition of real and imaginary parts |
| If is a complex number, then:   * the **real part** of is defined to be , * the **imaginary part** of is defined to be , the *coefficient* of in .   Sometimes, the real part of a complex number is written as , and the imaginary part as . |

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| Warning |
| The real and imaginary parts of a complex number are **real numbers**, so your answer should not involve the imaginary unit. |

In this language, a complex number is real if and only if ; a complex number is purely imaginary if and only if .

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|  | **Example 3**  In Example 1, you saw six complex numbers to . Here, their real and imaginary parts are given.   * The real part of is , and the imaginary part of is . * The real part of is , and the imaginary part of is . * The real part of is , and the imaginary part of is . * The real part of is , and the imaginary part of is . * The real part of is , as . The imaginary part of is . * The real part of is . The imaginary part of is , as . |

# The complex conjugate

You saw in the previous section on square roots of negative numbers that the solutions to are and , where is a positive number. As it turns out, this is an example of a more general phenomenon known as the **complex conjugate**.

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| Definition of the complex conjugate |
| If is a complex number, then the **complex conjugate** (or ) of is the complex number . |

So the complex conjugate of is the complex number with the same real part and with the sign of the imaginary part switched from either plus to minus or from minus to plus. Let’s take a look at some examples.

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|  | **Example 4**  In Example 1, you saw six complex numbers to . Here, their complex conjugates are given.   * The complex conjugate of is . * The complex conjugate of is . * The complex conjugate of is . * The complex conjugate of is . * The complex conjugate of is . * The complex conjugate of is . Notice that taking the complex conjugate of any real number gives that same real number again! |

You can see from Examples 2 and 3 (and the definition of complex conjugate) that if then , and if then .

# Argand diagrams

Because every complex number is writable as for , you can express every complex number as a point in two-dimensional space. Such a graph is known as an **Argand diagram**.

An Argand diagram has two axes called the **real** and **imaginary** axes. The real axis takes the place of the -axis and the imaginary axis takes the place of the -axis. See [Figure 1](#fig-1) for more.

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| Figure 1: A blank Argand diagram, with axes labelled Re and Im for real and imaginary parts respectively. |

To draw a complex number on an Argand diagram, you can plot the complex number at the point . See [Figure 2](#fig-2) for a representation of on an Argand diagram.

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| Figure 2: An Argand diagram with the complex number displayed. |

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|  | **Example 5**  In Example 1, you saw six complex numbers to , and Example 3 demonstrated the real and imaginary parts of these.   * The real part of is , and the imaginary part of is , so this corresponds to the point . * The real part of is , and the imaginary part of is . This corresponds to the point . * The real part of is , and the imaginary part of is . This corresponds to the point on the plane. * The real part of is , and the imaginary part of is . This corresponds to the point . * The real part of is , as . The imaginary part of is . This corresponds to a point , on the -axis. * The real part of is . The imaginary part of is , as . This corresponds to on the -axis.   See [Figure 3](#fig-3) for an Argand diagram. |

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| Figure 3: An Argand diagram with the six complex numbers in Example 5. |

# Quick check problems

1. Give the real and imaginary parts of the following complex numbers: , , , , and .
2. You are given three statements below. Decide whether they are true or false.
3. The complex conjugate of any purely imaginary number is a real number.
4. The complex conjugate of is for all real numbers .
5. Any real number is also a complex number.

# Further reading

[For more questions on the subject, please go to Questions: Introduction to complex numbers.](../questions/qs-introtocomplexnumbers.qmd)

[For how to add, subtract, multiply, and divide complex numbers, please go to Guide: Arithmetic on complex numbers.](arithmeticoncomplexnumbers.qmd)

## Version history

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