Introduction to differentiation and the derivative

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Summary

The idea of differentiation is everywhere in modern mathematics and in the sciences as it is related to the rate of change of an object or process. In this guide, the idea of differentiation and the derivative is introduced from first principles, its role in explaining the behaviour of functions is explained, and derivatives of common functions are introduced and used.

*Before reading this guide, it is recommended that you read [Guide: Properties of functions],* [*Guide: Laws of indices*](lawsofindices.qmd)*,* [*Guide: Logarithms*](logarithms.qmd)*, and [Guide: Tangents].*

# What is differentiation?

Functions can be used to model many real-life processes; such as the speed of a car, the performance of a stock market over time, and the growth rate of animal populations. You can use mathematical tools examined in the first three chapters to solve some of these problems (as well as a host of others).

Sometimes it is necessary to examine the rate of change of a particular process or phenomenon. For instance, the rate of change of the speed of a car is acceleration. Studying different rates of change of such a real life problem is often useful; for instance, by looking at acceleration of a car over time, you can determine how far the car goes. Rates of change are common in studying other, real-life processes, such as: the change in value of a company in a stock exchange in economics, the change in population of a species of animal in biology, or the rate of decay of radioactive material in chemistry.

The idea of investigating the rate of change of functions and their associated applications is known as **differential calculus**. The process of **differentiation** allows you to examine these properties of rates of change of functions. Importantly, this process is *general*, allowing you to investigate the rate of change of a function at any point. Differentiating a function gives its *derivative*, which is written as or .

This guide will look at the idea of differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

# Gradients of a graph

The rate of change of a straight line is its gradient , and this doesn’t change wherever you are on the line. However, the rate of change of a function like is dependent on what point you are at — when is small, the rate of change is correspondingly small; when is large, the rate of change is larger. You can see this in the variable ‘steepness’ of the curve in the figure below.

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| Figure 1: The graph of , with the variable steepness of the graph illustrated at three points , and . |

The idea of examining the rate of change of a function is to look at the gradients of straight lines, with endpoints as values on the graph, that approximate the function. The smaller the straight line is, the better the approximation. So let be some small real number. Then the line with endpoints and has vertical change , and horizontal change . Therefore, the gradient of this line is

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| Figure 2: Example of an approximation of the gradient of a function . You can expect that as gets smaller, then the gradient of the line segment on the graph gets closer to the gradient of the red line, providing a better approximation of the steepness of at a point. |

The problem is when tends towards , then and get closer and closer to one another, and the length of the line gets smaller and smaller. What use is there measuring the gradient of a line if the line disappears? The idea is to look at a parallel line, which has the same gradient.

Here, this convenient parallel line is the **tangent of at** ; this is the straight line touching the curve at the point , but does not intersect it at that point (see [Guide: Tangents]). Therefore, you can say that the gradient of the tangent to the curve at the point is

and this consideration defines the **derivative** of a function.

# Derivatives and differentiation

Following the investigation in the previous section, the derivative of a function can now be defined.

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| Definition of the derivative |
| The **derivative of with respect to**  is defined to be the function  The derivative of with respect to is a function in its own right. The value of the derivative at the point , defined by  is the gradient of the tangent to at the point .  If this limit exists, then is **differentiable at** . If is differentiable at every point of its domain, then it is a **differentiable function**. The process of finding given a function is called **differentiating with respect to** . |

The function is often defined in terms of a second variable . If a function is written , then there is alternative notation for writing the derivative with respect to :

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| Definition of |
| Let be a function of . Then the **derivative of with respect to**  can be written as  The derivative of with respect to at a point is written as  . |

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| Tip |
| Although the functions in this guide are defined in terms of the variable , you can express and differentiate functions with respect to any variable. This explains the terminology ‘with respect to ’; as this can change depending on the variable. For instance, if you are differentiating , you would say ‘with respect to ’, and you could write  in this instance. Any combination of variables is acceptable; you may see commonly, and maybe even Greek letters such as . |

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| Warning |
| Although *looks* like a fraction, it **isn’t**. However, sometimes it *behaves* like a fraction, and you can use this to remember certain results. However, this isn’t an excuse to treat it as a fraction in standard mathematical work.  What really represents is the application of an **operator**, specifically , to a function . It is correct to write the following:  For more about this, see [Guide: The differential operator and higher derivatives]. |

The notation was used by [Leibniz](https://mathshistory.st-andrews.ac.uk/Biographies/Leibniz/) to represent the derivative of a function . It’s useful because it codifies the derivative as a rate of change (ratio of change in to change in ) as well as explicitly naming the variable that you are differentiating with respect to. (This will be extremely useful in [Guide: Introduction to partial differentiation].) However, it’s not so useful for expressing the value of the derivative at a particular point ; for instance

The important thing to consider is this:

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| Tip |
| As both notations and have their advantages over the other, **it’s important to be able to use both sets of notation interchangeably**. |

Finally, the actual **value** of the derivative of a function at a point is very important, because it tells you about the behaviour of the function at that point. This is referred to as the **rate of change of at** , and is equal to the gradient of the tangent to at the point . The way in which the function changes at a point depends on the value of :

* If , then the function is **increasing** at ;
* If , then the function is **decreasing** at ;
* If , then the function is **stationary** at . The point is called a **stationary** point of the function .

# Differentiating well-known functions

You can use the limit definition of the derivative to show that the following key functions have the following derivatives:

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| List of derivatives |
| * For any number , the derivative of the constant function is . * For any number , the derivative of any function is . This includes not only whole numbers, but also fractions and other non-zero real numbers. * The derivative of the exponential function is . * The derivative of the sine function is , and the derivative of the cosine function is . (Notice the minus sign here!) * The derivative of the natural logarithm function is . |

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| Important |
| If you are going to use differentiation in your work/life, **you should remember these results.** |

Most of these results are explained in [Proof sheet: Derivatives of functions from first principles](../proofsheets/ps-introtodifferentiation.qmd).

From here, you can combine two functions into one and differentiate. The following results say this.

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| Rules for differentiation |
| Let and be differentiable functions, and be a non-zero real number. Then:   * (**sum rule**) The derivative of two functions and added together is the same as their derivatives and added together; that is, or * (**difference rule**) The derivative of one function subtracted from another is the same as the derivative subtracted from the derivative of ; that is or * (**scaling rule**) The derivative of a function multiplied by a real number is the same as the derivative multiplied by ; that is or |

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| Important |
| These differentiation rules do not tell you how to differentiate all possible combinations of functions. For instance:   * you need the **product rule** to differentiate the product of two functions * you need the **quotient rule** (and ) to differentiate the quotient of two functions * you need the **chain rule** to differentiate the composition of two functions   These techniques are important enough to merit their own guides. For more about these, see [Guide: The product rule](productrule.qmd), [Guide: The quotient rule](quotientrule.qmd), [Guide: The chain rule](chainrule.qmd). For more on where these results come from, see [Proof sheet: Rules of differentiation.](../proofsheets/ps-rulesofdifferentiation.qmd) |

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| Tip |
| The sum rule and scaling rule says that is a **linear operator**; see [Guide: The differential operator and higher derivatives] for more. |

Specifically, using the scaling and chain rule (see [Guide: The chain rule](chainrule.qmd)) can be used to show the following results, which are incredibly useful for any work involving differentiation.

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| Generalized list of derivatives |
| Let be real numbers where . Then:   * The derivative of a straight line is . * The derivative of a power is . * The derivative of the exponential function is . * The derivative of the sine function is , and the derivative of the cosine function is . (Notice the minus sign here!) * The derivative of the natural logarithm function is .   So if there is a constant inside the argument of the function, you should multiply by that constant when you differentiate. |

## Examples

Here’s some examples of differentiation in action.

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|  | **Example 1**  Find the derivative of the function with respect to .  Here, you can use the sum and scaling rules to differentiate this function. Using the sum rule, you can see that  Now, using the scaling rule on the first of these terms, you can get  Now you can use the derivatives of common functions. The function is of the form , so the derivative of is . The function is a constant, so the derivative of is . Putting these results in gives  Simplifying gives the answer: and so the derivative of is equal to .  You can then say that this function is increasing if , decreasing if , and is stationary exactly when . |

You do not need to be so meticulous in your general practice when differentiating. You can differentiate functions without stating what rules you are using.

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|  | **Example 2**  Differentiate with respect to .  Of these two terms, looks like a common function, but you have not seen how to differentiate yet. In fact, this is another case of ; in this case, . (See [Guide: Laws of indices](lawsofindices.qmd) for why this is.)  You can then rewrite the function as  which you can then apply your knowledge of differentiation to. Using the results above, the derivative of is . The derivative of is . Therefore, the derivative of is  You could turn into ; however, you may be asked to differentiate this function again (particularly if you are looking into behaviour of a function), and often it is best to leave the power as a real number rather than using the square root notation. |

As a rule of thumb, you should always write an th root using a fractional power instead. Here’s another example of where you should write a term involving a power of .

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|  | **Example 3**  Find the derivative of the function with respect to .  Again, the cosine and sine terms are manageable from the derivative above, it is only the term that needs a little thought. In fact, this is yet *another* case of ; this time when . (See [Guide: Laws of indices](lawsofindices.qmd) for why this is.)  You can then rewrite the function as  and use the familiar rules of differentiation above. Using the results above with , the derivative of is . The derivative of is . The derivative of is . Taking care of your signs you can write the derivative of as |

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|  | **Example 4**  Determine the behaviour of the function when .  Here, you will first need to differentiate the function to find . Then, you will need to evaluate the derivative when to see how the function behaves.  Using your rules of differentiation as you found above, you can say that the derivative of is , and the derivative of is . Therefore, the derivative of the function is  You can evaluate the derivative at to get  and so the derivative is positive at . This implies that the function is increasing at the point .  It also means that the gradient of the tangent to the function at the point is . You can see this in the figure below. |

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| Figure 3: The graph of , with the tangent to the graph at illustrated, demonstrating that the function is increasing at . |

## Summary

Here’s a table of derivatives that you should remember going into any further reading on differentiation. Here, are any real numbers.

| Function | Derivative | Notes |
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# Quick check problems

1. Answer the following questions true or false:
2. The derivative of a function at is equal to the gradient of the tangent to at .
3. If , then the function is increasing at .
4. If , then the derivative .
5. The derivative of is .
6. The derivative of is .
7. The power of in the derivative of is .
8. Differentiate the following functions with respect to .

# Further reading

[For more questions on the subject, please go to Questions: Introduction to differentiation and the derivative.](../questions/qs-introtodifferentiation.qmd)

For more about techniques of differentiation, please see [Guide: The product rule](productrule.qmd), [Guide: The quotient rule](quotientrule.qmd), and [Guide: The chain rule](chainrule.qmd).

For more about the differential operator, and its application to finding derivatives of derivatives, please see [Guide: The differential operator and higher derivatives].

For more about where the derivatives in the above table come from, please see [Proof sheet: Derivatives of functions from first principles](../proofsheets/ps-introtodifferentiation.qmd) and [Proof sheet: Derivatives of other common functions]. For more about why the rules of differentiation are true, please see [Proof sheet: Rules of differentiation](../proofsheets/ps-rulesofdifferentiation.qmd).

## Version history

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* v1.1 updated version with section on differentiating with respect to different variables 05/25 by tdhc.

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