The product rule

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Summary

The product rule is one of the three central techniques of differentiation, allowing you to differentiate a product of two functions. This guide introduces the product rule and explains examples of where it is used.

*Before reading this guide, it is recommended that you read* [*Guide: Introduction to differentiation and the derivative*](introtodifferentiation.qmd)*.*

# What is the product rule?

[MOTIVATION AND BLURB]

This guide will look at the idea of differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

The summary table of key derivatives from [Guide: Introduction to differentiation and the derivative](introtodifferentiation.qmd) is reproduced here.

| Function | Derivative | Notes |
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# Statement of the product rule

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| The product rule |
| Let and be two differentiable functions. Then the **product rule** says that  that is, the derivative of the product of and is equal to the product of and the derivative of , plus the product of and the derivative of .  This can also be written as |

Sometimes and are written instead of and in the statement of the product rule. The reason that and is used here is that sometimes the product function is called ; and you then can’t use it again!

# Examples

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|  | **Example 1**  Find the derivative of the function .  Here, you can use the sum and scaling rules to differentiate this function. Using the sum rule, you can see that  Now, using the scaling rule on the first of these terms, you can get  Now you can use the derivatives of common functions. The function is of the form , so the derivative of is . The function is a constant, so the derivative of is . Putting these results in gives  Simplifying gives the answer: and so the derivative of is equal to .  You can then say that this function is increasing if , decreasing if , and is stationary exactly when . |

You do not need to be so meticulous in your general practice when differentiating. You can differentiate functions without stating what rules you are using.

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|  | **Example 2**  Find the derivative of the function .  Of these two terms, looks like a common function, but you have not seen how to differentiate yet. In fact, this is another case of ; in this case, . (See [Guide: Laws of indices](lawsofindices.qmd) for why this is.)  You can then rewrite the function as  which you can then apply your knowledge of differentiation to. Using the results above, the derivative of is . The derivative of is . Therefore, the derivative of is  You could turn into ; however, you may be asked to differentiate this function again (particularly if you are looking into behaviour of a function), and often it is best to leave the power as a real number rather than using the square root notation. |

As a rule of thumb, you should always write an th root using a fractional power instead. Here’s another example of where you should write a term involving a power of :

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|  | **Example 3**  Find the derivative of the function .  Again, the cosine and sine terms are manageable from the derivative above, it is only the term that needs a little thought. In fact, this is yet *another* case of ; this time when . (See [Guide: Laws of indices](lawsofindices.qmd) for why this is.)  You can then rewrite the function as  and use the familiar rules of differentiation above. Using the results above with , the derivative of is . The derivative of is . The derivative of is . Taking care of your signs you can write the derivative of as |

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|  | **Example 4**  Determine the behaviour of the function when .  Here, you will first need to differentiate the function to find . Then, you will need to evaluate the derivative when to see how the function behaves.  Using your rules of differentiation as you found above, you can say that the derivative of is , and the derivative of is . Therefore, the derivative of the function is  You can evaluate the derivative at to get  and so the derivative is positive at . This implies that the function is increasing at the point .  It also means that the gradient of the tangent to the function at the point is . You can see this in the figure below. |

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| Figure 1: The graph of , with the tangent to the graph at illustrated, demonstrating that the function is increasing at . |

## Summary

Here’s a table of derivatives that you should remember going into any further reading on differentiation. Here, are any real numbers.

| Function | Derivative | Notes |
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# Quick check problems

1. Answer the following questions true or false:
2. The derivative of a function at is equal to the gradient of the tangent to at .
3. If , then the function is increasing at .
4. If , then the derivative .
5. The derivative of is .
6. The derivative of is .
7. The power of in the derivative of is .
8. Differentiate the following functions with respect to .

# Further reading

[For more questions on the subject, please go to Questions: The product rule](../questions/qs-introtodifferentiation.qmd)

For more about techniques of differentiation, please see [Guide: The quotient rule], and [Guide: The chain rule].

For more about why the rules and techniques of differentiation are true, please see [Proof sheet: Rules of differentiation].

## Version history

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