The product rule

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Summary

The product rule is one of the three central techniques of differentiation, allowing you to differentiate any product of two differentiable functions. This guide introduces the product rule and explains examples of where it is used.

*Before reading this guide, it is recommended that you read* [*Guide: Introduction to differentiation and the derivative*](introtodifferentiation.qmd)*.*

# What is the product rule?

[MOTIVATION AND BLURB]

This guide will look at the product rule for differentiation; where it comes from, how it can be used, and how you can apply its techniques to functions that you may be familiar with.

The summary table of key derivatives from [Guide: Introduction to differentiation and the derivative](introtodifferentiation.qmd) is reproduced here for reference:

| Function | Derivative | Notes |
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# Statement of the product rule

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| The product rule |
| Let and be two differentiable functions. Then the **product rule** says that  that is, the derivative of the product of and is equal to the product of and the derivative of , plus the product of and the derivative of .  This can also be written as |

Sometimes and are written instead of and in the statement of the product rule. The reason that and is used here is that sometimes the product function is called ; and you then can’t use it again!

It does not matter which function you pick to be and which function you pick to be ; this is because .

To see why this really is the derivative of the product of two functions, please see [Proof sheet: Rules of differentiation.]

# Examples

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|  | **Example 1**  What is the derivative of ?  In this case, you have two functions multiplied together to make . The two functions are and . In order to use the product rule on , you could differentiate and beforehand and then substitute them into the product rule. Doing this gives   * For , then by the power rule. * For , then .   Putting these into the statement of the product rule gives  and this is your answer. You could also factorise the answer if you wanted to to get . |

Here’s another example of the product rule, which is included solely for a chance to remember the signs for the derivatives of and !

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|  | **Example 2**  Find the derivative of the function .  In this case, there are two options. You can either   * use the constant rule to take the out and differentiate using the product rule, or * take and and differentiate directly.   Let’s do the first of these. To use the product rule take and . Then and . Therefore, using the product rule gives  Therefore, the derivative of is |

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| Tip |
| You don’t always need the product rule to differentiate the product of two functions if that product of two functions can be expressed in a different way.  For instance, the function in Example 2 is , which by a trigonometric identity is equal to . (See [Guide: Trigonometric identities (radians)](trigonometricidentities-radians.qmd) for more.) You can then differentiate by the rules above to get . By trigonometric identities, this is also equal to , which is the same answer as in Example 2.  This shows you should explore all potential avenues before starting to use differentiation techniques. |

As a rule of thumb, you should always write an th root using a fractional power instead. Here’s another example of where you should write a term involving a power of :

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|  | **Example 3**  Find the derivative of the function .  Again, the cosine and sine terms are manageable from the derivative above, it is only the term that needs a little thought. In fact, this is yet *another* case of ; this time when . (See [Guide: Laws of indices](lawsofindices.qmd) for why this is.)  You can then rewrite the function as  and use the familiar rules of differentiation above. Using the results above with , the derivative of is . The derivative of is . The derivative of is . Taking care of your signs you can write the derivative of as |

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|  | **Example 4**  Determine the behaviour of the function when .  Here, you will first need to differentiate the function to find . Then, you will need to evaluate the derivative when to see how the function behaves.  Using your rules of differentiation as you found above, you can say that the derivative of is , and the derivative of is . Therefore, the derivative of the function is  You can evaluate the derivative at to get  and so the derivative is positive at . This implies that the function is increasing at the point .  It also means that the gradient of the tangent to the function at the point is . You can see this in the figure below. |

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|  | **Example 5**  Determine the behaviour of the function when .  Here, you will first need to differentiate the function to find . Then, you will need to evaluate the derivative when to see how the function behaves.  Using your rules of differentiation as you found above, you can say that the derivative of is , and the derivative of is . Therefore, the derivative of the function is  You can evaluate the derivative at to get  and so the derivative is positive at . This implies that the function is increasing at the point .  It also means that the gradient of the tangent to the function at the point is . You can see this in the figure below. |

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|  | **Example 6**  Determine the behaviour of the function when .  Here, you will first need to differentiate the function to find . Then, you will need to evaluate the derivative when to see how the function behaves.  Using your rules of differentiation as you found above, you can say that the derivative of is , and the derivative of is . Therefore, the derivative of the function is  You can evaluate the derivative at to get  and so the derivative is positive at . This implies that the function is increasing at the point .  It also means that the gradient of the tangent to the function at the point is . You can see this in the figure below. |

# Quick check problems

1. Answer the following questions true or false:
2. The derivative of a function at is equal to the gradient of the tangent to at .
3. If , then the function is increasing at .
4. If , then the derivative .
5. The derivative of is .
6. The derivative of is .
7. The power of in the derivative of is .
8. Differentiate the following functions with respect to .

# Further reading

[For more questions on the subject, please go to Questions: The product rule](../questions/qs-introtodifferentiation.qmd)

For more about techniques of differentiation, please see [Guide: The quotient rule], and [Guide: The chain rule].

For more about why the rules and techniques of differentiation are true, please see [Proof sheet: Rules of differentiation].

## Version history

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