Solving exponential equations

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Summary

This guide applies the laws of indices to solve equations involving powers; a key skill in mathematics and many areas of science.

*It is highly recommended that you read* [*Guide: Laws of indices*](lawsofindices.qmd) *before reading this guide. In addition,* [*Guide: Logarithms*](logarithms.qmd) *is recommended but not required for some of the material in this guide.*

In [Guide: Laws of indices](lawsofindices.qmd), you learned about the different ways that you can manipulate expressions involving powers (like and ) and th roots (such as and ). In general mathematical life however, this knowledge will not be enough; you will need to apply these laws of indices to help you solve equations.

Solving equations involving indices is a key skill in areas whenever exponential growth or decay plays a role: economics, physics (particularly nuclear physics), chemistry, biology, and many more.

Before you get started however, it is worth reiterating the recommendation above:

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| Important |
| **This guide assumes an excellent knowledge of the laws of indices. Please make sure that you have read** [**Guide: Laws of indices**](lawsofindices.qmd) **before continuing.**  The numbering of the laws will follow the numbering in this guide and on [Factsheet: Laws of indices](../factsheets/f-lawsofindices.qmd).  In addition, an initial understanding of logarithms will be required later in the guide. While this understanding is partially explained in this guide, you may want to familiarize yourself with [Guide: Logarithms](logarithms.qmd) before attempting Examples 7, 8, 9, and 10. |

# Discussion of techniques

In general, the golden rule of solving equations involving indices is the following:

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| Tip |
| **Make sure both sides are the in the same base before simplifying**. This is because if is a positive number not equal to and , then you can immediately say that and solve the equation from there. |

However, you may be wondering; why on earth is this true? How can you ignore the base in this case? Or, more pertinently, *what if you can’t write the equation in a form where both sides have the same base*?

What you can do in this case is isolate the variable on one side of the equation, and have a constant on the other (see [Guide: Introduction to rearranging equations](introtorearrange.qmd) for more); perhaps it looks like . From there, you can take **logarithms** to base of both sides.

Since for all with and all real numbers :

it follows that

implies that

solving the equation! What this means is that logarithms undo exponentiation and exponentiation undoes logarithms.

In particular, this is why implies that . Taking logarithms to base of both sides of gives ; using the above result then yields .

However, please be aware of the following:

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| Warning |
| **You cannot take logarithms of a negative number!** So an expression like is **not** defined. |

This is particularly pertinent in Example 9 below.

In this guide, logarithms are only used to undo exponentiation to solve equations; there will not be any applications of the laws of logarithms nor the change of base rule. (For more on these, see [Guide: Logarithms](logarithms.qmd).)

# Examples

## Initial examples

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|  | **Example 1**  Solve .  You can start squaring both sides of the equation to get . Using Law 3, ; so then you get the answer . To check if the answer is correct, you can substitute back into the equation: . |

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|  | **Example 2**  Solve .  Using Law 5, can be re-expressed as . Multiplying both sides by gives you . Then dividing both sides by gives you . Remembering you can have positive and negative roots, you get or as the two solutions to this equation. |

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|  | **Example 3**  Solve .  You can notice here that the two sides of the equation have different bases; so you need to write these in the same base. As , the equation can be rewritten as: . Then using Law 3, you can write . Since the bases of both sides are equal, you can say that . Rearranging gives . |

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|  | **Example 4**  Solve .  If you look at all of the indices in the question, the denominator is and that is a hint of what you need to multiply by. Multiply by on both sides of the equation; using Law 1, this gives Using Law 4, you can further simplify this expression to get , and so . This is a quadratic equation; factorizing this equation gives you , then you can get and as the two possible solutions to this equation. |

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|  | **Example 5**  Solve .  Here, you will need to condense the left-hand side into a single base. Using Law 1, you can write that  Next, use Law 7 to combine and into a single base;  Therefore, . Since , it follows that . (Or indeed, you could have taken logarithms of both sides to base .) |

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|  | **Example 6**  Solve .  First of all, you can use Law 7 to write . Using Law 10, you can then write the left hand side of this equation as  You can now work on ; using Law 7 and Law 10, you can write  Therefore, the initial equation becomes  Dividing both sides by (thereby using Laws 2 and 6) and cancelling the gives  Squaring both sides to undo the square root gives , and so . |

## Examples involving logarithms

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|  | **Example 7**  Solve .  Taking the logarithm of both sides to base gives you . Using the definition of logarithms, you can express the equation as . Rearranging the equation gives you and this is your final answer. |

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| Tip |
| Although here was treated as some constant, it is actually a very important constant called *Euler’s number*.  In addition, there is a special name for ; this is the **natural logarithm** of , often written .  To see more about Euler’s number and natural logarithms, please read [Guide: Logarithms](logarithms.qmd). |

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|  | **Example 8**  Given the equation , solve for .  First of all, you can notice that as is not a power of (or vice versa) getting these in the same base is difficult. What you can do instead is use the laws of indices to get an expression of the form and then take logarithms. Using Law 1, you can write that  Dividing both sides by and using Law 8 gives  and so you are left with . Taking logarithms of both sides to base gives ; and this is your final answer! |

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|  | **Example 9**  Given the equation , you are asked to solve for .  Start by letting . Then, you can rewrite the equation given as using Law 3, which is the same as writing  Recognizing that this is a quadratic equation (see [Guide: Introduction to quadratic equations](introtoquadratics.qmd) for more), you can use the quadratic formula or otherwise to show that  As , this means that or . By taking the logarithm of both sides to base , you can show that  Remember from above that the logarithm of a negative number is not defined, so is the only viable solution. |

The final example uses many of the laws of indices!

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|  | **Example 10**  Solve .  This example is similar to Example 8, only with a few extra steps. Once again, as is not a power of or vice versa, the strategy is to write the equation in the form and then take logarithms of both sides.  First of all, use Law 1 to write and Law 3 to write . Since , the equation becomes  Dividing both sides by gives  Using Law 8 and Law 5, you can write  Taking logarithms to base on both sides gives , which is your final answer. |

# Quick check problems

1. Solve for .
2. Solve for .
3. Solve for .

# Further reading

For more questions on the subject, please go to [Questions: Solving exponential equations](../questions/qs-solvingeqsindices.qmd).

## Version history and licensing

v1.0: initial version created 08/23 by Ritwik Anand, Zheng Chen, and Zoë Gemmell as part of a University of St Andrews STEP project.

* v1.1: edited 04/24 by tdhc.

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