

Neural network architecture and learning

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February 16, 2022

Fully connected multi-layer Neural Networks

Computing gradients and learning weights

Supervised learning setup

- ▶ Have an input $\mathbf{x} \in \mathbb{R}^d$ – a vector of d real numbers.
- ▶ And an output y (real number: regression, integer ID: classification).
- ▶ Want to learn a prediction function $f(\mathbf{x}) = y$ that will work on a new input.
- ▶ In a neural network with $L - 1$ hidden layers the function f is defined using composition of L functions,
$$f(x) = f^{(L)}[\dots f^{(1)}[x]] \in \mathbb{R}.$$
- ▶ Linear model is special case with $L = 1$ function, 0 hidden layers.
- ▶ “Deep” learning means $L \geq 3$ functions, at least 2 hidden layers.

Each function is matrix multiplication and activation

- ▶ Prediction function $f(x) = f^{(L)}[\dots f^{(1)}[x]] \in \mathbb{R}$.
- ▶ Each function $l \in \{1, \dots, L\}$ is a matrix multiplication followed by an activation function: $f^{(l)}[z] = \sigma^{(l)}[W^{(l)}z]$ where $W^{(l)} \in \mathbb{R}^{u^{(l)} \times u^{(l-1)}}$ is a weight matrix to learn, and $z \in \mathbb{R}^{u^{(l-1)}}$ is the input vector to that layer.
- ▶ If the loss function is defined in terms of a real-valued predicted score (typical, like we did in linear models), then the last activation function is fixed to the identity $\sigma^{(L)}[z] = z$.
- ▶ The other activation functions must be non-linear, e.g. logistic/sigmoid $\sigma(z) = 1/(1 + \exp(-z))$ or rectified linear units (ReLU)

$$\sigma(z) = \begin{cases} z & \text{if } z > 0, \\ 0 & \text{else.} \end{cases}$$

Non-linear activation functions

$$\sigma(z) = \begin{cases} z & \text{if } z > 0, \\ 0 & \text{else.} \end{cases}$$

$$\sigma(z) = 1/(1 + \exp(-z))$$



Network size

For binary classification with inputs $x \in \mathbb{R}^d$, the overall neural network architecture is $(u^{(0)} = d, u^{(1)}, \dots, u^{(L-1)}, u^{(L)} = 1)$, where $u^{(1)}, \dots, u^{(L-1)} \in \mathbb{Z}_+$ are positive integers (hyper-parameters that control the number of units in each hidden layer, and the size of the parameter matrices $W^{(l)}$).

- ▶ “Units” is a synonym for “features” and “variables.”
- ▶ First and last layer are “visible” others are “hidden.”
- ▶ First layer size $u^{(0)}$ is fixed to input size.
- ▶ Last layer size $u^{(L)}$ is fixed to output size.
- ▶ Number of layers and hidden layer sizes $u^{(1)}, \dots, u^{(L-1)}$ must be chosen (by you).
- ▶ No hidden layers/units means $L = 1$, linear model.
- ▶ “Deep” learning means $L \geq 3$ functions, at least 2 hidden layers.

Network diagram for linear model with 10 inputs/features

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

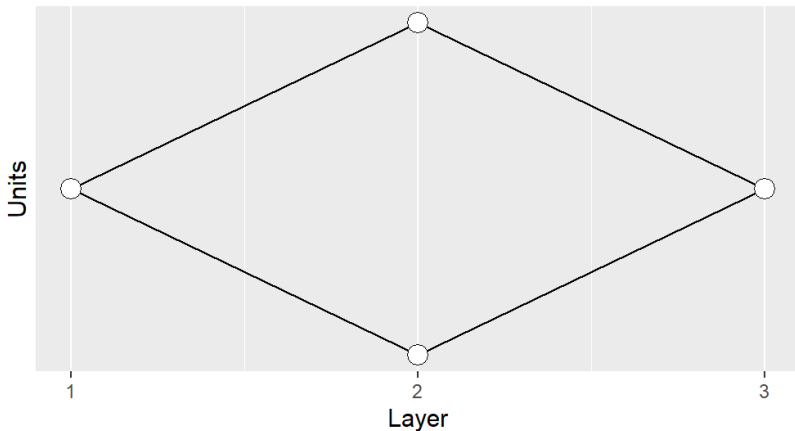
Number of units: 10,1



Network diagram for single hidden layer with 2 units

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

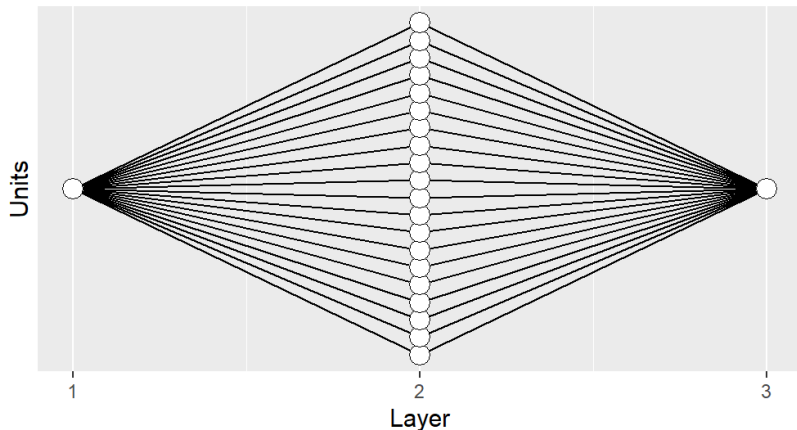
Number of units: 1,2,1



Network diagrams

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

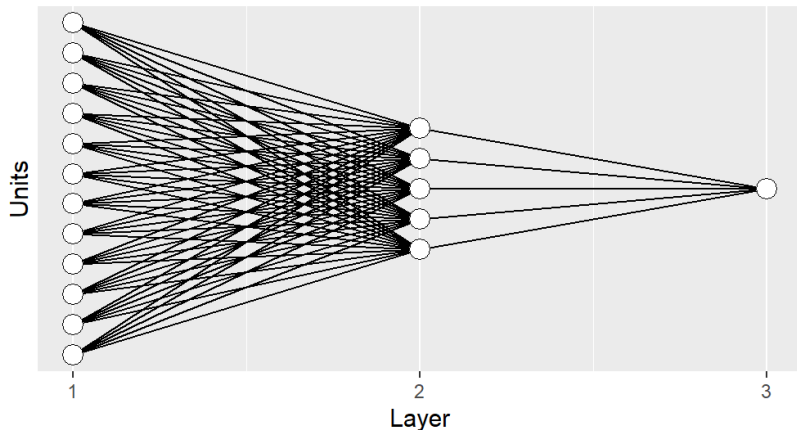
Number of units: 1,20,1



Network diagrams

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

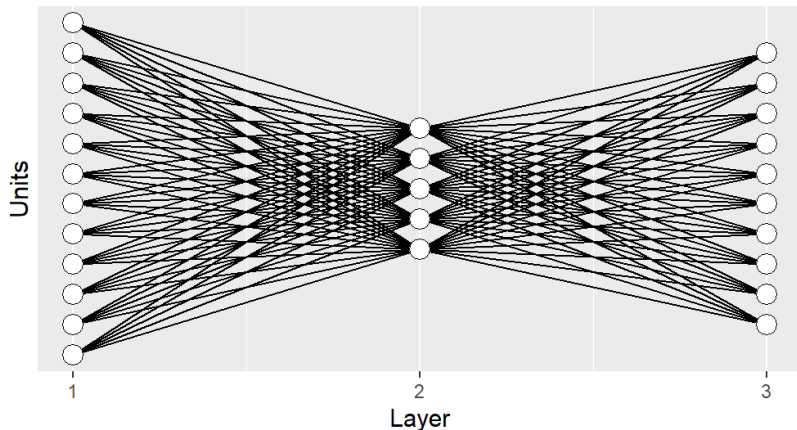
Number of units: 12,5,1



Network diagrams

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

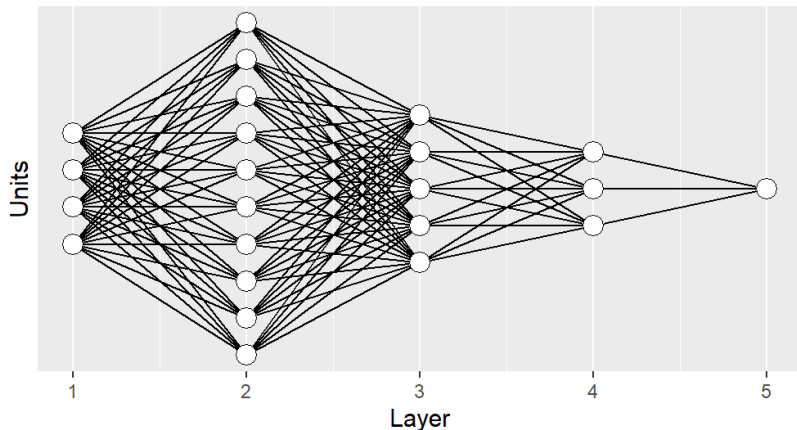
Number of units: 12,5,10



Network diagrams

Neural network diagrams show how each unit (node) is computed by applying the weights (edges) to the values of the units at the previous layer.

Number of units: 4,10,5,3,1



Units in each layer

We can write the units at each layer as $h^{(0)}, h^{(1)}, \dots, h^{(L-1)}, h^{(L)}$ where

- ▶ $h^{(0)} = x \in \mathbb{R}^d$ is an input feature vector,
- ▶ and $h^{(L)} \in \mathbb{R}$ is the predicted output.

For each layer $l \in \{1, \dots, L\}$ we have:

$$h^{(l)} = f^{(l)} \left[h^{(l-1)} \right] = \sigma^{(l)} \left[W^{(l)} h^{(l-1)} \right].$$

Total number of parameters to learn is $\sum_{l=1}^L u^{(l)} u^{(l-1)}$.

Quiz: how many parameters in a neural network for $d = 10$ inputs/features with one hidden layer with $u = 100$ units? (one output unit, ten output units)

Fully connected multi-layer Neural Networks

Computing gradients and learning weights

Gradient descent learning

Basic idea of gradient descent learning algorithm is to iteratively update weights $\mathbf{W} = [W^{(1)}, \dots, W^{(L)}]$ to improve predictions on the subtrain set.

- ▶ Need to define a loss function $\mathcal{L}(\mathbf{W})$ which is differentiable, and takes small values for good predictions.
- ▶ Typically for regression we use the mean squared error, and for binary classification we use the mean logistic loss (sometimes called cross entropy).
- ▶ The mean loss $\mathcal{L}(\mathbf{W})$ is averaged over all N observations or batches i :

$$\mathcal{L}(\mathbf{W}) = \frac{1}{N} \sum_{i=1}^N \mathcal{L}(\mathbf{W}, \mathbf{x}_i, \mathbf{y}_i)$$

- ▶ The mean full gradient $\nabla \mathcal{L}(\mathbf{W})$ is a function which tells us the local direction where the loss is most increasing:

$$\nabla \mathcal{L}(\mathbf{W}) = \frac{1}{N} \sum_{i=1}^N \nabla \mathcal{L}(\mathbf{W}, \mathbf{x}_i, \mathbf{y}_i)$$

Loss functions



Gradient descent animations

<https://yihui.org/animation/example/grad-desc/>

$$z = x^2 + 3\sin(y)$$



Basic full gradient descent algorithm

- ▶ Initialize weights \mathbf{W}_0 at some random values near zero (more complicated initializations possible).
- ▶ Since we want to decrease the loss, we take a step $\alpha > 0$ in the opposite direction of the mean full gradient,

$$\mathbf{W}_t = \mathbf{W}_{t-1} - \alpha \nabla \mathcal{L}(\mathbf{W}_{t-1})$$

- ▶ This is the **full** gradient method (same as we did for linear models): batch size = n = subtrain set size, so 1 step per epoch/iteration.

Stochastic gradient descent algorithm

- ▶ Initialize weights \mathbf{W} at some random values near zero (more complicated initializations possible).
- ▶ for each epoch t from 1 to max epochs:
- ▶ for each batch i from 1 to n :
- ▶ Let $\mathcal{L}(\mathbf{W}, \mathbf{X}_i, \mathbf{y}_i)$ be the loss with respect to the single observation in batch i .

$$\mathbf{W} \leftarrow \mathbf{W} - \alpha \nabla \mathcal{L}(\mathbf{W}, \mathbf{X}_i, \mathbf{y}_i)$$

- ▶ This is the **stochastic** gradient method: batch size = 1, so there are n steps per epoch.

Batch (stochastic) gradient descent algorithm

- ▶ Input: batch size b .
- ▶ Initialize weights \mathbf{W} at some random values near zero (more complicated initializations possible).
- ▶ for each epoch t from 1 to max epochs:
- ▶ for each batch i from 1 to $\lceil n/b \rceil$:
- ▶ Let $\mathcal{L}(\mathbf{W}, \mathbf{X}_i, \mathbf{y}_i)$ be the mean loss with respect to the b observations in batch i .

$$\mathbf{W} \leftarrow \mathbf{W} - \alpha \nabla \mathcal{L}(\mathbf{W}, \mathbf{X}_i, \mathbf{y}_i)$$

- ▶ This is the **(mini)batch** stochastic gradient method: batch size = b , so there are $\lceil n/b \rceil$ steps per epoch.

Forward propagation

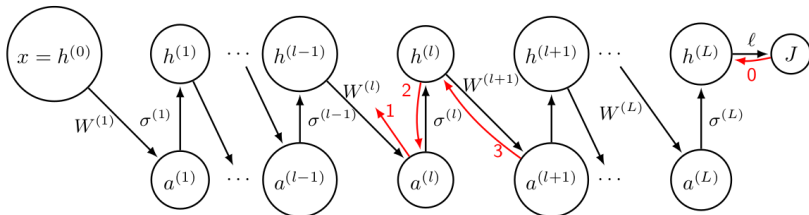
- ▶ Forward propagation is the computation of hidden units $h^{(1)}, \dots, h^{(L)}$ given the inputs x and current parameters $W^{(1)}, \dots, W^{(L)}$.
- ▶ Start from input, apply weights and activation in each layer until predicted output is computed.
- ▶ In the code this should be a for loop from first to last layer.

Back propagation

Back propagation is the computation of gradients given current parameters and hidden units.

- ▶ Start from loss function, compute gradient, send it to last layer, use chain rule, send gradient to previous layer, finally end up at first layer.
- ▶ Result is gradients with respect to all weights in all layers.
- ▶ Deep learning libraries like torch/keras do this using automatic differentiation based on your definition of the forward method and the loss function.
- ▶ This week in class we will code the gradient computation from scratch to see how it works.
- ▶ In the code this should be a for loop from last layer to first layer.

Computation graph



For each layer $l \in \{1, \dots, L\}$ we have:

$$\begin{aligned} a^{(l)} &= W^{(l)} h^{(l-1)}, \\ h^{(l)} &= \sigma^{(l)} [a^{(l)}]. \end{aligned}$$

There are essentially four rules for computing gradients during backpropagation (0-3).

Backprop rules

The rules 0–3 for backprop (from loss backwards):

Rule 0 computes $\nabla_{h^{(L)}} J$, which depends on the choice of the loss function ℓ .

Rule 1 computes $\nabla_{W^{(l)}} J$ using $\nabla_{a^{(l)}} J$, for any $l \in \{1, \dots, L\}$

$$\nabla_{w_k^{(l)}} J = (\nabla_{a^{(l)}} J) \left(h^{(l-1)} \right)^T \quad (1)$$

Rule 2 computes $\nabla_{a^{(l)}} J$ using $\nabla_{h^{(l)}} J$, for any $l \in \{1, \dots, L\}$.

$$\nabla_{a^{(l)}} J = (\nabla_{h^{(l)}} J) \odot \left(\nabla_{a^{(l)}} h^{(l)} \right) \quad (2)$$

Rule 3 computes $\nabla_{h^{(l)}} J$ using $\nabla_{a^{(l+1)}} J$, for any $l \in \{1, \dots, L-1\}$.

$$\nabla_{h^{(l)}} J = (\nabla_{a^{(l+1)}} J) \left(W^{(l+1)} \right)^T \quad (3)$$

Computation exercises (gradient descent learning)

Now assume we have used backpropagation to compute gradients with respect to four observations i :

$$\nabla_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{X}_i, \mathbf{y}_i) = \begin{cases} [-1, 1] & i = 1 \\ [-2, 2] & i = 2 \\ [-3, 2] & i = 3 \\ [-1, 2] & i = 4 \end{cases}$$

Starting at current weights $\mathbf{v} = [-2, 1]$ and using gradient descent with step size $\alpha = 0.5$, (show your work!)

1. For the full gradient method, there is one step. What is the new weight vector \mathbf{v} after that step?
2. For a batch size of 2, there are two steps. Assume batch 1 is observations $i = 1, 2$ and batch 2 is observations $i = 3, 4$. What is the new weight vector \mathbf{v} after the batch 1 step? After the batch 2 step?
3. For the stochastic gradient method, there are four steps $i = 1, 2, 3, 4$. What is \mathbf{v} after each of those steps?