

**Due: 22 Oct 2020**

**1. Rayleigh quotient.** Let  $A$  be a symmetric matrix. Consider the Rayleigh quotient

$$\rho_A(x) = \frac{x^T A x}{x^T x}, \quad x \in \mathbb{R}^d.$$

Note that since  $\rho_A(x)$  is invariant along every line passing through the origin, one can consider the function  $\phi(x) = x^T A x$  restricted to the unit sphere  $\|x\|_2^2 = 1$  instead. Prove that:

(a)  $v$  is a stationary point of  $\rho_A$  if and only if  $v$  is an eigenvector of  $A$  and  $\rho_A(v) = \lambda_v$ , the corresponding eigenvalue.

**Solution.** First, we show that  $v$  being an eigenvector of  $A$  and  $\rho_A(v) = \lambda_v$  implies  $\nabla \rho_A(v) = 0$ :

$$\begin{aligned} \nabla \rho_A(x) &= \nabla \left( \frac{x^T A x}{x^T x} \right) = \frac{x^T x (2Ax) - 2x^T A x (x)}{(x^T x)^2} \\ \nabla \rho_A(v) &= \frac{v^T v (2Av) - 2v^T A v (v)}{(v^T v)^2} = \frac{v^T v (2\lambda_v v) - 2v^T (\lambda_v v) v}{(v^T v)^2} = \frac{v^T v (2\lambda_v v - 2\lambda_v v)}{(v^T v)^2} = 0. \end{aligned}$$

Next, to show the other direction, we manipulate the expression  $\nabla \rho_A(v)$ :

$$\begin{aligned} \nabla \rho_A(v) &= \frac{v^T v (2Av) - 2(v^T A v)v}{(v^T v)^2} = 0 \\ \implies v^T v (2Av) - 2(v^T A v)v &= 0 \\ \implies v^T v (2Av) &= 2(v^T A v)v \\ \implies Av &= \frac{v^T A v}{v^T v} v. \end{aligned}$$

Let  $\lambda_v = \frac{v^T A v}{v^T v}$ . Then we have  $Av = \lambda_v v$ , i.e.  $v$  is an eigenvector of  $A$  and  $\rho_A(v) = \lambda_v$  as desired.

(b) The only local minima and maxima of  $\rho_A$  are the global minimum and maximum and all other stationary points are saddles.

**Solution.** Here, we use  $\rho_A$  instead of  $\phi_A$  for simplicity. Let  $x_M, x_m \in \mathbb{R}^d$  be the global maximizer and global minimizer such that  $\rho_A(x_m) \leq \rho_A(x) \leq \rho_A(x_M) \forall x \in \mathbb{R}^d$ . As a result, we also have  $\nabla \rho_A(x_m) = \nabla \rho_A(x_M) = 0$ . By (a), then, we have  $Ax_m = \lambda_m x_m$  and  $Ax_M = \lambda_M x_M$  for eigenvalues  $\lambda_m, \lambda_M$ . Therefore, we have that

$$\rho_A(x_m) = \lambda_m \leq \rho_A(x) \leq \lambda_M = \rho_A(x_M) \quad \forall x \in \mathbb{R}^d.$$

So we have that the global maximum and minimum exist and occur at the maximum and minimum eigenvectors.

To show that all the other stationary points (i.e., eigenvectors) are saddle points, we consider the problem

$$\begin{aligned} \phi(x) &= x^T A x \rightarrow \min \text{ subject to} \\ \|x\|_2^2 &= 1 \end{aligned}$$

which has Lagrangian  $\Lambda(\ell, x) = x^T A x - \ell(x^T x - 1)$ . Now use the *bordered Hessian*, defined for constrained optimization problems as

$$H(\ell, x) = \begin{bmatrix} 0 & \left( \frac{\partial g}{\partial x} \right)^T \\ \frac{\partial g}{\partial x} & \frac{\partial^2 \Lambda}{\partial x^2} \end{bmatrix}$$

where  $g(x) = \|x\|_2^2 = 1$  is our constraint (from Wikipedia). Plugging in  $g$  and  $\Lambda$ , we get

$$H(\ell, x) = \begin{bmatrix} 0 & 2x^T \\ 2x & 2A - 2\ell I \end{bmatrix}.$$

From line 2 of (Bindel, page 4), we see that when  $x$  is an eigenvector  $v$  of  $A$ , the optimal Lagrange multiplier is  $\ell = \rho_A(v)$ . Thus, when evaluating the Hessian at some eigenvector  $v \neq x_M, x_m$  we have

$$H(\lambda_v, v) = \begin{bmatrix} 0 & 2v^\top \\ 2v & 2A - 2\lambda_v I \end{bmatrix}.$$

Check the value of this quadratic form at another eigenpair  $(\lambda_u, u)$  and use the fact that the eigenvectors form an orthonormal basis:

$$\begin{aligned} [\lambda_u \quad u^\top] H(\lambda_v, v) \begin{bmatrix} \lambda_u \\ u \end{bmatrix} &= [\lambda_u \quad u^\top] \begin{bmatrix} 0 & 2v^\top \\ 2v & 2A - 2\lambda_v I \end{bmatrix} \begin{bmatrix} \lambda_u \\ u \end{bmatrix} \\ &= [\lambda_u \quad u^\top] \begin{bmatrix} 2v^\top u \\ 2\lambda_u v + 2Au - 2\lambda_v u \end{bmatrix} \\ &= 2\lambda_u v^\top u + 2\lambda_u u^\top v + 2u^\top Au - 2\lambda_v u^\top u \\ &= 2\lambda_u - 2\lambda_v. \end{aligned}$$

Therefore, if we take  $u$  to be an eigenvector such that  $\lambda_u > \lambda_v$ , this quantity will be positive, while if  $\lambda_u < \lambda_v$ , the quantity is negative. Hence the Hessian is an indefinite form—neither positive- nor negative-semidefinite—at every eigenvector that is not  $x_m$  or  $x_M$ ; as a result, these eigenvectors are all saddle points.

**2. Eckart-Young-Mirsky.** Prove the Eckart-Young-Mirsky Theorem for any Ky-Fan norm, i.e., if  $A = U\Sigma V^\top$  is the SVD of  $A$  and  $M$  is any matrix of the size of  $A$  such that  $\text{rank } M \leq k$ , then

$$\|A - M\| \geq \|A - U_k \Sigma_k V_k^\top\|$$

for any Ky-Fan norm  $\|\cdot\|$ .

**Solution.** The definition of a Ky Fan norm is as follows:

$$\|A\|_p = \sum_{j=1}^{\min(p,r)} |\sigma_j|$$

where  $r = \text{rank } A$  and  $\sigma_1 \geq \dots \geq \sigma_n \geq 0$  are the singular values of  $A \in \mathbb{R}^{n \times d}$ . Let  $C := A - M$  and let the singular values of  $M$  and  $C$  be  $\mu_1 \geq \dots \geq \mu_n \geq 0$  and  $\gamma_1 \geq \dots \geq \gamma_n \geq 0$ , respectively.

Our approach follows (Dax 2013). First, let  $\text{rank } M = k$ . We will apply the Right and Left Courant-Fischer Minimax Theorems to show that  $\gamma_1 \geq \sigma_{k+1}$ . Recall that the Right Courant-Fischer Minimax Theorem states that

$$\begin{aligned} \sigma_j &= \max_{V_j} \min\{\|Av\|_2 \mid v \in V_j, \|v\|_2 = 1\} \quad \text{and} \\ \sigma_j &= \min_{V_{j^*}} \max\{\|Av\|_2 \mid v \in V_{j^*}, \|v\|_2 = 1\} \end{aligned}$$

where  $j^* = d + 1 - j$  and  $V_j$  is an arbitrary  $j$ -dimensional subspace of  $\mathbb{R}^n$  (Dax 2013). Similarly, the Left Courant-Fischer Minimax Theorem says

$$\begin{aligned} \sigma_i &= \max_{U_i} \min\{\|A^\top u\|_2 \mid u \in U_i, \|u\|_2 = 1\} \quad \text{and} \\ \sigma_i &= \min_{U_{i^*}} \max\{\|A^\top u\|_2 \mid u \in U_{i^*}, \|u\|_2 = 1\} \end{aligned}$$

where  $i^* = n + 1 - i$  and  $U_i$  is an arbitrary  $i$ -dimensional subspace of  $\mathbb{R}^d$  (Dax 2013). To prove the earlier inequality, take  $V_{j^*} = \text{nul } M$ . Then  $j^* = d - k$  and  $j = k + 1$  from the definition of  $j^*$  in the theorem. As a result, we have

$$\begin{aligned} \gamma_1 &= \max\{\|(A - M)v\|_2 \mid v \in \mathbb{R}^d, \|v\|_2 = 1\} && \text{(definition of singular values of } C) \\ &\geq \max\{\|(A - M)v\|_2 \mid v \in \text{nul } M, \|v\|_2 = 1\} && (\text{nul } M \subset \mathbb{R}^d) \\ &= \max\{\|Av\|_2 \mid v \in \text{nul } M, \|v\|_2 = 1\} && (v \in \text{nul } M \implies Mv = 0) \\ &\geq \sigma_j = \sigma_{k+1} && \text{(Right Courant-Fischer Minimax Theorem)} \end{aligned}$$

and so this Lemma is proven.

The next building block we'll need is Weyl's Theorem. Under our current setup, the theorem states that  $\mu_i + \gamma_j \geq \sigma_{i+j-1}$ , given that  $\sigma_\ell = 0$  whenever  $\ell > d$ . To prove it, let  $M_{i-1}$  be a rank  $i-1$  truncated SVD of  $M$  and  $C_{j-1}$  be a rank  $j-1$  truncated SVD of  $C$ . As a result, the largest singular value of  $M - M_{i-1}$  is  $\mu_i$  and the largest singular value of  $C - C_{j-1}$  is  $\gamma_j$ :

$$\begin{aligned}\mu_i &= \max\{\|(M - M_{i-1})v\|_2 \mid v \in \mathbb{R}^d, \|v\|_2 = 1\} \quad \text{and} \\ \gamma_j &= \max\{\|(C - C_{j-1})v\|_2 \mid v \in \mathbb{R}^d, \|v\|_2 = 1\}.\end{aligned}$$

If we define  $F = M_{i-1} + C_{j-1} \in \mathbb{R}^{n \times d}$ , then  $\text{rank } F \leq i + j - 2$  and  $A - F = (M - M_{i-1}) + (C - C_{j-1})$ . Therefore, combining the two definitions for  $\mu_i$  and  $\gamma_j$  from above, we get

$$\mu_i + \gamma_j \geq \max\{\|(A - F)v\|_2 \mid v \in \mathbb{R}^d, \|v\|_2 = 1\} \geq \sigma_{i+j-1}$$

where the last inequality is a result from our previous Lemma (plug in the matrices in question). So Weyl's Theorem is proven.

However, we haven't used all the assumptions yet—in particular,  $\text{rank } M \leq k$ , which implies that  $\mu_i = 0$  for  $i = k + 1, \dots, d$ . If we make the substitution  $i = k + 1$  in the statement of Weyl's Theorem, we get

$$\mu_i + \gamma_j \geq \sigma_{i+j-1} \implies \gamma_j \geq \sigma_{j+k}$$

from basic arithmetic. Simply put, if  $\text{rank } M \leq k$ , then the singular values of  $C$  majorize the singular values of  $A - U_k \Sigma_k V_k^T$ .

From here, our job is mostly done. Since the singular values of  $A - M$  majorize the singular values of  $A - U_k \Sigma_k V_k^T$  when  $\text{rank } M \leq k$ , we have that necessarily

$$\sum_{j=1}^k |\gamma_j| \geq \sum_{j=1}^k |\sigma_{j+k}| \implies \|A - M\|_k \geq \|A - U_k \Sigma_k V_k^T\|_k$$

where the norms in the second inequality are the  $k$ th Ky Fan norms. Finally, recall that the Ky Fan Dominance Theorem states  $\|X\|_k \geq \|Y\|_k \iff \|X\| \geq \|Y\|$  for any orthogonally invariant norm  $\|\cdot\|$  (Li and Mathias 1998). Therefore, by the Ky Fan Dominance Theorem,

$$\|A - M\|_k \geq \|A - U_k \Sigma_k V_k^T\|_k \implies \|A - M\| \geq \|A - U_k \Sigma_k V_k^T\|$$

for any orthogonally invariant norm  $\|\cdot\|$  and the proof is complete.

References used:

Dax, A. (2013). From eigenvalues to singular vales: a review. *Advances in Pure Mathematics*, 3, pp. 8-24.

Li, C. and Mathias, R. (1998). Generalizations of Ky Fan's dominance theorem. *Siam Journal of Matrix Analysis and Applications*, 19(1), pp. 99-106.