Comprehensive Explanation of Stokes Vectors, Mueller Matrices, and Their Measurement

1 Stokes Vector

The Stokes vector is a mathematical representation of the polarization state of light. It consists of four components that describe the total intensity and the polarization properties of a light beam. The Stokes vector is expressed as:

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}, \tag{1}$$

where:

- S_0: Total intensity of the light.
- S_1: Difference in intensity between horizontally and vertically polarized light.
- S_2: Difference in intensity between light polarized at +45° and -45°.
- S_3: Difference in intensity between right- and left-circularly polarized light.

1.1 Total Intensity

The total intensity of light is represented by S_0, and it is calculated as the sum of intensities of all polarization components:

$$S_0 = I_{\text{horizontal}} + I_{\text{vertical}}. \tag{2}$$

1.2 Examples of Polarization States

- **Unpolarized light:**

$$\mathbf{S} = \begin{bmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

  Light has equal intensity in all polarization directions.

- **Horizontally polarized light:**

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

Light is fully polarized along the horizontal axis.

- **Right-circularly polarized light:**

$$\mathbf{S} = \begin{bmatrix} S_0 \\ 0 \\ 0 \\ S_0 \end{bmatrix} \tag{5}$$

Light is circularly polarized in the clockwise direction.

- **Elliptically polarized light:**

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \tag{6}$$

This describes a combination of linear and circular polarization.

2 Mueller Matrix

The Mueller matrix describes how a material or optical system modifies the Stokes vector of light. It is a 4x4 matrix $\mathbf{M}$ that transforms the input Stokes vector $\mathbf{S}_{\text{in}}$ into the output Stokes vector $\mathbf{S}_{\text{out}}$:

$$\mathbf{S}_{\text{out}} = \mathbf{M} \cdot \mathbf{S}_{\text{in}}. \tag{7}$$

The Mueller matrix can be decomposed into components that describe different optical effects: rotation, depolarization, dichroism, and birefringence.

2.1 Rotation Matrix

The rotation matrix $\mathbf{R}(\phi)$ describes the rotation of the material's optical axis relative to the laboratory frame and depends on the rotation angle ϕ:

$$\mathbf{R}(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\phi) & \sin(2\phi) & 0 \\ 0 & -\sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

2.2 Depolarization Matrix

The depolarization matrix $\mathbf{M}_{\text{depol}}$ describes the reduction of polarization due to scattering or mixing. It depends on the depolarization coefficients d_{ij}:

$$\mathbf{M}_{\text{depol}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{bmatrix}, \tag{9}$$

where:

- $0 \leq d_{ij} \leq 1$: Coefficients must not increase polarization.
- The Frobenius norm of d_{ij} must satisfy $\sqrt{\sum_{i,j} d_{ij}^2} \leq 1$.

2.3 Dichroism Matrix

The dichroism matrix $\mathbf{M}_{\text{dichroism}}$ represents differential absorption for linear and circular polarizations. It depends on:

- a_x, a_y: Absorption coefficients for horizontal and vertical polarization.
- a_c: Absorption coefficient for circular polarization.
- a: Average isotropic absorption.

$$\mathbf{M}_{\text{dichroism}} = \begin{bmatrix} \exp(-a) & \exp(-a_x) & 0 & \exp(-a_c) \\ \exp(-a_x) & \exp(-a_x) & 0 & 0 \\ 0 & 0 & \exp(-a_y) & 0 \\ \exp(-a_c) & 0 & 0 & \exp(-a_c) \end{bmatrix}. \tag{10}$$

2.4 Birefringence Matrix

The birefringence matrix $\mathbf{M}_{\text{birefringence}}$ describes phase shifts due to linear and circular birefringence. It depends on:

- δ: Phase shift for linear birefringence.
- θ: Phase shift for circular birefringence.

$$\mathbf{M}_{\text{birefringence}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta & \sin\delta & 0 \\ 0 & -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{11}$$

2.5 Altogether

M = R(-ϕ) $\cdot$ $\mathbf{M}$depol(dij) $\cdot$ $\mathbf{M}$dichroism(a_x, a_y, a_c, a) $\cdot$ $\mathbf{M}$birefringence(δ, θ) $\cdot$ $\mathbf{R}(\phi)$ (12)

3 Examples of Mueller Matrices

3.1 Linear Polarizer

For a linear polarizer aligned horizontally:

$$\mathbf{M}_{\text{polarizer}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

3.2 Depolarizer

For an ideal depolarizer:

$$\mathbf{M}_{\text{depolarizer}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{14}$$

3.3 Birefringent Material

For a birefringent material with a phase shift δ:

$$\mathbf{M}_{\text{birefringent}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta & \sin\delta & 0 \\ 0 & -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

4 Protocol to Measure the Mueller Matrix

1. **Input Polarization States**: Generate at least four distinct input polarization states, such as:

- Horizontal: $\mathbf{S} = [1, 1, 0, 0]$.
- Vertical: $\mathbf{S} = [1, -1, 0, 0]$.
- $+45°$: $\mathbf{S} = [1, 0, 1, 0]$.
- Right-circular: $\mathbf{S} = [1, 0, 0, 1]$.

2. **Output Analysis**: Measure the transmitted intensity for different analyzer settings (e.g., horizontal, vertical, $+45°$, circular).

3. **Construct the Matrix**: Solve for the 16 elements of the Mueller matrix using the measured intensities and known input/output polarization states.

5 When Only Linear Polarizers and Analyzers Can Be Used

A partial Mueller matrix can be determined using only linear polarizers and analyzers if:

- The material does not exhibit circular birefringence (optical rotation).
- The material does not exhibit circular dichroism.

This setup allows measurement of the top-left 3×3 submatrix, capturing linear effects like linear dichroism and linear birefring