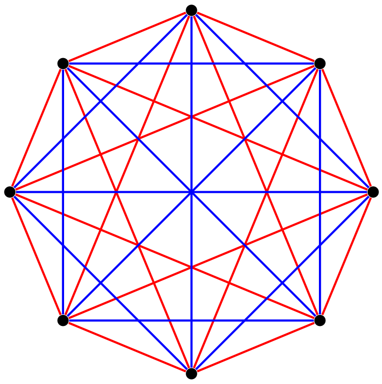

Figure 1: A 2-coloring of K_8 avoiding a monochromatic C_5: red edges form the complete bipartite graph $K_{4,4}$, blue edges form two disjoint K_4's. Each color class is bipartite, hence contains no C_5.

1 Introduction and Motivation

1.1 Big-picture context

Ramsey theory asks: how large must a complete graph be, under an edge-coloring, to force a monochromatic copy of a given graph? For cycles, this question leads to surprisingly rich behavior: parity and number of colors both matter. Even though cycles are among the simplest graph families, their Ramsey numbers show sharp asymptotic trends and exact formulas in special cases.

A quick example is instructive. Consider $R_2(C_5)$: in any red-blue coloring of K_9 one finds a monochromatic 5-cycle, but K_8 admits a coloring avoiding it (see Figure 1).

1.2 Minimal setup

We write C_n for the cycle on n vertices. The k-color Ramsey number is

$$R_k(C_n) = \min\{N : \text{ every } k\text{-coloring of } E(K_N) \text{ contains a mono. } C_n\}.$$

Our focus will be on exact formulas for two colors, and asymptotic results and bounds when $k \geq 3$. This gives a natural test ground for modern Ramsey techniques.

1.3 Two-color exact result

For $n \geq 3$, the diagonal 2-color cycle Ramsey numbers are known exactly [7, 17]:

$$R_2(C_n) = \begin{cases} 2n-1, & n \text{ odd}, \\ \frac{3n}{2}-1, & n \text{ even}. \end{cases}$$

This sharp parity dichotomy will serve as a baseline for the multicolor picture.

1.4 Multicolor picture

Three colors (diagonal):

$$R_3(C_n) = \begin{cases} 4n-3, & n \text{ odd and } n \text{ sufficiently large [14]}, \\ 2n, & n \text{ even and } n \text{ sufficiently large [1]}. \end{cases}$$

General k (diagonal):

$$R_k(C_n) = 2^{k-1}(n-1)+1 \quad \text{for } k \geq 2,\ n \text{ odd, and } n \text{ sufficiently large [12]}.$$

For even cycles, the best bounds are linear and essentially tight up to the constant [4]:

$$(k-1)n + o(n) \;\leq\; R_k(C_{2n}) \;\leq\; \Big(k - \tfrac{1}{4} + o(1)\Big)n.$$

2 Preliminaries

We recall basic definitions and results that will be used throughout the paper. All graphs are simple, finite, and undirected. For a graph G, we write $V(G)$ for its vertex set and $E(G)$ for its edge set. A *k-edge-coloring* of G is a map $\varphi : E(G) \to [k]$ assigning one of k colors to each edge.

2.1 Ramsey numbers

Given a graph H, the *Ramsey number* $R_k(H)$ is the smallest N such that every k-coloring of K_N contains a monochromatic copy of H. Equivalently,

$$R_k(H) = \min\{N : K_N \to (H)_k\},$$

where the arrow notation means that every k-edge-coloring of K_N forces a monochromatic H.

2.2 Cycles

The cycle of length n is denoted C_n, with vertex set $\{v_1, \ldots, v_n\}$ and edges $v_i v_{i+1}$ ($1 \leq i < n$) together with $v_n v_1$. Cycles are bipartite when n is even, and non-bipartite when n is odd; this parity will play a crucial role in Ramsey numbers of cycles.

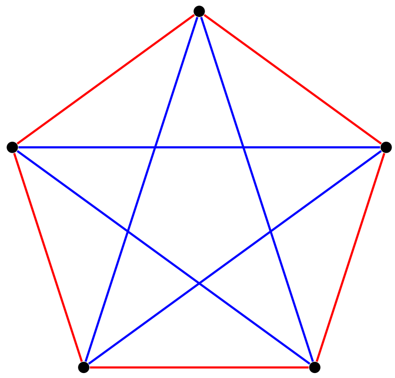

Figure 2: A 2-coloring of K_5 with no monochromatic triangle (lower bound for $R(3,3)$).

2.3 Two classical extremal results

Two theorems from extremal graph theory will be frequently invoked:

Theorem 1 (Erdős–Gallai, 1959 [6]). *Let G be a graph on n vertices with more than $\frac{1}{2}(k-1)(n-1)$ edges. Then G contains a cycle of length at least k.*

Theorem 2 (Dirac, 1952 [5]). *Every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle.*

These results provide structural control on long cycles, and are standard ingredients in many Ramsey-type arguments.

2.4 Small values

Before tackling cycles of length $n \geq 4$, it is helpful to recall the classical fact

$$R_2(C_3) = R(3,3) = 6 \quad [11].$$

Lower bound. The 5-vertex coloring in Figure 2 has no monochromatic triangle, so $R(3,3) > 5$.

Upper bound. In any red-blue coloring of K_6, pick a vertex v. Among the 5 edges from v, at least 3 share a color; assume they are red to a, b, c. If any of ab, bc, ca is red we get a red triangle; otherwise all three are blue, forming a blue triangle abc. Hence every 2-coloring of K_6 contains a monochromatic C_3, proving $R(3,3) = 6$.

3 Two-color Ramsey numbers for cycles

Theorem 3 (Faudree–Schelp; Rosta [7, 17]). *For integers $3 \le m \le n$, the two-color cycle Ramsey numbers satisfy*

$$R(C_n, C_m) = \begin{cases} 2n-1, & m \text{ odd}, \ (n,m) \neq (3,3), \\ n-1+\frac{m}{2}, & m,n \text{ even}, \ (n,m) \neq (4,4), \\ \max\{\, n-1+\frac{m}{2},\, 2m-1 \,\}, & m \text{ even}, \ n \text{ odd}. \end{cases}$$

Corollary 4 (Diagonal exact formula). *For $n \ge 3$,*

$$R_2(C_n) = \begin{cases} 2n-1, & n \text{ odd}, \\ \frac{3n}{2}-1, & n \text{ even}, \end{cases}$$

with the small exceptions $R_2(C_3) = R_2(C_4) = 6$ [11, 3].

We now give a self-contained proof of Corollary 4.

Lemma 5 (Erdős–Gallai, path version [6]). *If a graph on s vertices contains no path on ℓ vertices, then it has at most $\frac{\ell-2}{2}\, s$ edges. Equivalently, if a graph on s vertices has more than $\frac{\ell-2}{2}\, s$ edges, it contains a path on ℓ vertices.*

Lemma 6. *The longest even cycle in the complete bipartite graph $K_{a,b}$ has length of $2\min\{a,b\}$.*

Proof. Every cycle in a bipartite graph alternates between the two sides, so any cycle uses the same number of vertices from each side. Hence a cycle can use at most $\min\{a,b\}$ vertices from the smaller side and the same number from the other side, giving the claimed maximum, and $K_{a,b}$ indeed contains all even cycle lengths $4, 6, \ldots, 2\min\{a,b\}$ by taking appropriate alternating tours. □

Proof of Corollary 4. We prove the lower and upper bounds for odd and even n separately.

- Lower Bound:
  - Odd n: Let $n = 2t+1$. Consider $K_{2n-2} = K_{4t}$ with the following coloring. Partition the vertex set into two parts A and B with $|A| = |B| = n-1 = 2t$. Color edges inside A and inside B red; color all edges across A and B blue. The red graph is $K_{2t} \cup K_{2t}$, which has no red C_n since each component has only $2t$ vertices. The blue graph is $K_{2t,2t}$, which is bipartite and therefore has no odd cycle, in particular no C_n. Thus $R_2(C_n) \ge 2n-1$ for odd n.

- Even n: Let $n = 2t \geq 6$. Consider $K_{\frac{3n}{2}-2} = K_{3t-2}$. Partition the vertex set into $A \cup B$ with $|A| = n-1 = 2t-1$ and $|B| = \frac{n}{2} - 1 = t-1$. Color all edges inside A red, all edges inside B red, and all edges across A and B blue. The red graph is $K_{2t-1} \cup K_{t-1}$, which contains no red C_n since each component has fewer than n vertices. By Lemma 6, the longest even cycle in the blue $K_{2t-1,t-1}$ has length $2\min\{2t-1, t-1\} = 2(t-1) = n-2$, so there is no blue C_n. Thus $R_2(C_n) \geq \frac{3n}{2} - 1$ for even n.

- Upper Bound:

  - Odd n: Let $n = 2t+1 \geq 5$ and suppose $K_{2n-1} = K_{4t+1}$ is 2-colored red/blue with no monochromatic C_n. Fix a vertex v and let X be the set of red neighbors of v, Y the set of blue neighbors, so $|X| + |Y| = 4t$.

    If $|X| \geq n = 2t+1$, consider the red subgraph $G_R[X]$ induced by X. If $G_R[X]$ contains a red path on n vertices (a P_n), then together with the two red edges from v to the path's endpoints we obtain a red C_n, a contradiction. Hence $G_R[X]$ is P_n-free, so by Lemma 5,

$$e_R(X) \;\leq\; \frac{n-2}{2}\,|X| \;=\; (t - \tfrac{1}{2})\,|X|.$$

    Consequently the number of *blue* edges inside X is

$$e_B(X) \;=\; \binom{|X|}{2} - e_R(X) \;\geq\; \binom{|X|}{2} - (t - \tfrac{1}{2})|X| \;=\; \frac{(|X| - 1 - 2t)\,|X|}{2}.$$

    Since $|X| \geq 2t+1$, we have $|X| - 1 - 2t \geq 0$, so $e_B(X) \geq 0$; moreover, when $|X| \geq 2t+2$ we get $e_B(X) \geq \frac{|X|}{2} > 0$, hence $G_B[X]$ is nonempty.

    We now claim that $G_B[X]$ contains a blue path on $n-1 = 2t$ vertices. If $|X| = 2t+1$, the average blue degree inside X is at least

$$\frac{2e_B(X)}{|X|} \;\geq\; \frac{|X| - 1 - 2t}{1} \;=\; 0,$$

    and if $|X| \geq 2t+2$, the average blue degree is at least 1. In either case, consider a longest blue path P in $G_B[X]$ with endpoints a, b. By maximality, every neighbor of a (in $G_B[X]$) lies on P, and similarly for b. If $|P| \geq 2t$ (i.e., P has $2t$ vertices), then using v and two blue edges from v to suitable vertices of P we can close a blue odd cycle of length $2t+1 = n$ (the path contributes an even number of internal edges). Thus $|P| \leq 2t-1$.

    Remove $V(P)$ from X; each removed vertex kills at most one potential increase in the average blue degree. But the remaining graph still has at least one blue edge unless $|X| = 2t+1$ and $e_B(X) = 0$. Hence the only way to avoid creating a blue C_n is that $|X| = 2t+1$ and $G_B[X]$ is edgeless, i.e., $G_B[X]$ is an independent set. In that extremal subcase, $G_R[X]$ is a clique, so $G_R[X]$ contains a red Hamiltonian path on $2t+1$ vertices; then together with v we obtain a red C_{2t+1}, a contradiction. Therefore the assumption $|X| \geq n$ fails.

By symmetry, we must also have $|Y| < n$ (the same argument with colors swapped), so $|X| \leq n-1$ and $|Y| \leq n-1$. But $|X| + |Y| = 4t = 2n-2$, forcing $|X| = |Y| = n-1$. Run the previous argument again with $|X| = n-1$ replaced by $|X| = n$ after adding any one vertex from Y (it is red- or blue-adjacent to v); the same path-extension reasoning yields a monochromatic C_n. This contradiction proves $R_2(C_n) \leq 2n-1$ for odd n.

– Even n: Let $n = 2t \geq 6$ and suppose $K_{\frac{3n}{2}-1} = K_{3t-1}$ is 2-colored with no monochromatic C_n. Fix a vertex v and let X be the red neighborhood of v, Y the blue neighborhood, so $|X| + |Y| = 3t-2$.

If $|X| \geq t+1$, consider $G_R[X]$. If $G_R[X]$ contains a red path on n vertices (P_{2t}), then with v we get a red C_{2t}; thus $G_R[X]$ is P_{2t}-free. By Lemma 5 with $\ell = 2t$, we have $e_R(X) \leq (t-1)|X|$. Hence

$$e_B(X) \;=\; \binom{|X|}{2} - e_R(X) \;\geq\; \binom{|X|}{2} - (t-1)|X| \;=\; \frac{(|X|-2t+1)|X|}{2}.$$

If $|X| \geq 2t-1$, then $e_B(X) \geq \frac{|X|}{2} > 0$, so $G_B[X]$ contains a blue edge. Take a longest blue path P in $G_B[X]$. If $|P| \geq t$ (i.e., at least t vertices), then we may close a blue cycle of length $2t$ by joining v to the endpoints of P (the path contributes $t-1$ internal edges, producing an even cycle of length $2 + (t-1) + (t-1) = 2t$). Thus $|P| \leq t-1$, so every component of $G_B[X]$ has at most $t-1$ vertices.

Now examine Y. We have $|Y| = 3t-2-|X|$; if $|X| \leq 2t-2$, then $|Y| \geq t$. If $|Y| \geq 2t-1$, the same argument (switching colors) gives a blue C_{2t}—contradiction. Hence $|X| \geq t+1$ forces $t \leq |X| \leq 2t-2$ and $t \leq |Y| \leq 2t-2$.

Pick $x \in X$ maximizing the number of blue neighbors in X. Since the average blue degree inside X is $\frac{2e_B(X)}{|X|} \geq |X| - 2t + 1$, we have

$$d_B^X(x) \;\geq\; |X| - 2t + 1.$$

Also x has at least $|Y| - (t-1)$ blue neighbors in Y (otherwise a simple path extension between X and Y via v yields a blue cycle C_{2t}). Summing and using $|X| + |Y| = 3t-2$ gives

$$d_B(x) \;\geq\; (|X| - 2t + 1) + (|Y| - (t-1)) \;=\; (3t-2) - 3(t-1) \;=\; t+1.$$

Apply the same argument to a blue neighbor $y \in Y$ with many blue neighbors; chaining longest blue paths between X and Y through v yields a blue even cycle of length at least $2t$, contradicting our assumption. Therefore $R_2(C_n) \leq \frac{3n}{2} - 1$ for even n.

□

4 Three-color Ramsey numbers for cycles

When moving from two to three colors, the picture changes drastically. Unlike the two-color case, where exact formulas are known for all n, in the three-color case the behavior depends strongly on the parity of n, and the proofs require more sophisticated tools such as Szemerédi's Regularity Lemma and stability arguments.

4.1 Odd cycles

The odd case was the first to be settled. Łuczak [15] proved that asymptotically

$$R_3(C_n) = (4 + o(1))n \quad \text{as } n \to \infty,\ n \text{ odd}.$$

A few years later, Kohayakawa, Simonovits, and Skokan [14] strengthened this to an exact result for sufficiently large odd n.

Theorem 7 (Kohayakawa–Simonovits–Skokan [14]). *For every odd n sufficiently large,*

$$R_3(C_n) = 4n - 3.$$

- Lower Bound construction: Consider K_{4n-4}. Partition the vertex set into four classes A, B, C, D of size $n - 1$ each. Color all edges between consecutive classes $A-B$, $B-C$, $C-D$, $D-A$ with the first color; all edges $A-C$, $B-D$ with the second color; and all edges inside classes with the third color. Each color class is bipartite, so no odd cycle of length n can appear. Thus $R_3(C_n) \geq 4n - 3$.

- Upper Bound method: The proof that $R_3(C_n) \leq 4n - 3$ is considerably deeper. Łuczak's *connected matching method* reduces the problem via the Regularity Lemma to a colored reduced graph with large minimum degree. In such a graph one can find a monochromatic connected matching of size n, and by "lifting" the matching back through the regular pairs, one obtains a monochromatic odd cycle of length n. Kohayakawa–Simonovits–Skokan provided a precise stability analysis to eliminate small exceptional cases and complete the exact formula.

n odd	5	7	9	11
$R_3(C_n)$	17	23	31	41

Table 1: Small diagonal values for odd cycles.

n even	6	8	10	12
$R_3(C_n)$	12	16	20	24

Table 2: Small diagonal values for even cycles.

4.2 Even cycles

The even cycle case turned out to be more subtle. In 2007, Figaj and Łuczak [8] showed that

$$R_3(C_{2n}) = (2 + o(1))n.$$

This was later sharpened by Benevides and Skokan [1], who determined the exact value for sufficiently large n.

Theorem 8 (Benevides–Skokan [1]). *For every even n sufficiently large,*

$$R_3(C_n) = 2n.$$

Lower bound construction. Take K_{2n-1} and partition the vertex set into two sets A and B with $|A| = n$ and $|B| = n - 1$. Color all edges between A and B with the first color. This produces $K_{n,n-1}$, which contains no cycle longer than $2(n-1) < 2n$. Edges inside A are colored with the second color, and edges inside B with the third color. Thus there is no monochromatic C_n.

Upper bound method. As in the odd case, one applies the Regularity Lemma and the connected matching method. The key observation is that in any 3-coloring of K_{2n}, one of the colors induces a subgraph with sufficiently large minimum degree. Stability arguments show that this subgraph must contain a connected matching with n edges, which in turn yields a monochromatic C_{2n} in the original graph.

4.3 Summary

For three colors, the diagonal Ramsey numbers for cycles are now known exactly in both the odd and even cases (for sufficiently large n):

$$R_3(C_n) = \begin{cases} 4n - 3, & n \text{ odd}, \\ 2n, & n \text{ even}. \end{cases}$$

These results illustrate the power of the Regularity Lemma and stability arguments in modern Ramsey theory, and serve as a foundation for the general multicolor case.

5 General multicolor Ramsey numbers for cycles

When the number of colors k is allowed to grow, the landscape becomes even richer. For odd cycles, a long-standing conjecture of Bondy and Erdős asserted that the Ramsey numbers grow exactly as $2^{k-1}(n-1)+1$. This was finally resolved by Jenssen and Skokan. For even cycles, the best known results are sharp up to a constant factor.

5.1 Odd cycles

Theorem 9 (Jenssen–Skokan [12]). *For every fixed $k \geq 2$ and all sufficiently large odd n,*

$$R_k(C_n) = 2^{k-1}(n-1) + 1.$$

- Lower Bound construction: The extremal coloring uses the $(k-1)$-dimensional hypercube Q_{k-1}. Label the 2^{k-1} vertices with binary strings of length $k-1$, and assign a vertex class of size $n-1$ to each string. Edges between two classes are colored according to the coordinate in which their binary strings differ. Each color class is then bipartite, so contains no odd cycle, and the total number of vertices is $2^{k-1}(n-1)$, proving the lower bound.
- Upper Bound method: The proof uses a refinement of the connected matching method. Starting from a k-coloring of $K_{2^{k-1}(n-1)+1}$, regularity and stability arguments ensure that one color must induce a dense, well-connected subgraph. One shows that such a subgraph necessarily contains a connected matching on n vertices, which yields a monochromatic C_n. This matches the lower bound exactly.

5.2 Even cycles

For even cycles the situation is more complicated. Although the order of magnitude was known earlier, the exact constant factor remains unresolved. The best known bounds are due to Davies, Jenssen, and Roberts.

Theorem 10 (Davies–Jenssen–Roberts [4]). *For each fixed $k \geq 3$ and all sufficiently large n,*

$$(k-1)n + o(n) \;\leq\; R_k(C_{2n}) \;\leq\; \Big(k - \tfrac{1}{4} + o(1)\Big)n.$$

- Lower Bound construction: The $(k-1)$-partite Turán graph provides the basic example: split the vertex set into $k-1$ equal parts of size $n-1$, and color edges inside each part with one color, while edges across parts are colored differently. This prevents a monochromatic C_{2n}, giving the lower bound.

- Upper Bound method: The upper bound comes from combining the connected matching approach with a stability refinement of the Erdős–Gallai theorem. In particular, a color class with slightly more edges than $(k-1)(n-1)$ must already contain a connected matching on n edges, forcing a cycle of length $2n$. The refined analysis reduces the additive constant to 1/4, which remains the current best.

5.3 Diagonal case: structure, thresholds, and extremal colourings

For odd cycles the picture is completely understood. For each fixed $k \geq 2$ and sufficiently large odd n one has

$$R_k(C_n) = 2^{k-1}(n-1) + 1 \qquad [12].$$

The extremal colouring for the lower bound uses the $(k-1)$-dimensional hypercube Q_{k-1}. Label the 2^{k-1} vertices with binary strings of length $k-1$, and assign to each a class of size $n-1$. If two strings x, y differ first in coordinate i, then all edges between V_x and V_y are given colour i. Each colour class is bipartite (since it can be split by the i-th bit), so it contains no odd cycle. This avoids a monochromatic C_n on $2^{k-1}(n-1)$ vertices and forces the lower bound. The upper bound, proved in [12], combines the Regularity Lemma with the connected matching method: in any k-colouring of $K_{2^{k-1}(n-1)+1}$ one finds a dense monochromatic subgraph that must contain a connected matching on n vertices, which lifts to a monochromatic C_n.

For even cycles the situation is more subtle. For each fixed $k \geq 3$ and large n,

$$(k-1)n + o(n) \;\leq\; R_k(C_{2n}) \;\leq\; \big(k - \tfrac{1}{4} + o(1)\big)n$$

by the work of Davies, Jenssen, and Roberts [4]. The lower bound arises from multipartite colourings such as the construction of Sun, Yang, Xu and Li [18], which partitions the vertex set into $k-1$ equal parts of size about n and colours edges so that each colour class is bipartite or too small to contain C_{2n}. The upper bound uses the connected matching method together with a stability refinement of the Erdős–Gallai theorem: a colour class with slightly more than $(k-1)(n-1)$ edges must contain a connected matching on n edges, which lifts to a cycle of length $2n$. The refined analysis improves the constant to $k - \frac{1}{4}$, which is the best known.

For odd cycles the asymptotic formula is exact, but the threshold $n_0(k)$ beyond which it holds is unknown in general (for $k = 3$, only "sufficiently large" n is proved [14]). For even cycles the leading constant remains open: the true value lies between $k-1$ and

k	2	3	4	5
Odd n	$2n-1$	$4n-3$	$8n-7$	$16n-15$
Even n	$\frac{3n}{2}-1$	$2n$	$[3n-O(1), 3.75n+o(n)]$	$[4n-O(1), 4.75n+o(n)]$

Table 3: Diagonal multicolour Ramsey numbers of cycles. Odd cycles are known exactly for large n; even cycles are determined only up to the stated bounds.

$k-1/4$.

5.4 Summary

For general k, the diagonal cycle Ramsey numbers are understood as:

$$R_k(C_n) = \begin{cases} 2^{k-1}(n-1)+1, & n \text{ odd, } n \text{ sufficiently large,} \\ (k-1)n+o(n) \le R_k(C_{2n}) \le (k-\frac{1}{4}+o(1))n, & n \text{ even.} \end{cases}$$

Thus the odd cycle case is completely solved, while the even cycle case remains open: the true constant factor between $k-1$ and $k-1/4$ is not yet determined.

6 Variants and related problems

Beyond the diagonal case, a number of related problems have been studied. They broaden the scope of cycle Ramsey theory and often involve different techniques.

6.1 Off-diagonal cycle Ramsey numbers

Given m, n, the two-colour off-diagonal Ramsey number $R(C_m, C_n)$ is fully determined by the formula of Faudree and Schelp [7], extending the diagonal case. For more than two colours, only scattered results are known. Exact small cases have been settled computationally and are summarised in Radziszowski's survey [16]. Improving the bounds for $R(C_m, C_n)$ when both cycles are long but of different parity remains open.

6.2 Cycles versus cliques

A well-studied mixed problem is $R(C_\ell, K_n)$. Erdős, Faudree, Rousseau and Schelp conjectured that

$$R(C_\ell, K_n) = (\ell-1)(n-1)+1$$

whenever ℓ is sufficiently large relative to n. This was proved by Keevash, Long and Skokan [13] for $\ell \gtrsim \frac{\log n}{\log\log n}$. Determining the smallest ℓ for which the formula holds is an active line of research.

6.3 Gallai–Ramsey numbers for cycles

In a *Gallai colouring* (one with no rainbow triangle), the structure is more rigid, and exact formulas for cycles can be obtained. For odd cycles C_{2n+1} the Gallai–Ramsey number is known to equal $R_k(C_{2n+1}) = 2n + 1$ for all k, while for even cycles explicit linear formulas have also been determined [9, 10]. These results provide sharp contrasts with the standard multicolour case.

6.4 Bipartite Ramsey numbers of cycles

A related notion is the bipartite Ramsey number $br(C_{2m}, C_{2n})$, the smallest N such that any two-colouring of $K_{N,N}$ contains a monochromatic C_{2m} or C_{2n}. Faudree and Schelp initiated this line, and recent work has determined $br(C_{2n}, C_{2n})$ exactly for large n [2]. Multicolour bipartite cycle Ramsey numbers remain widely open.

7 Open problems and future directions

Although major progress has been made, especially for odd cycles, many natural questions remain open.

- **Even cycles in many colours.** The current bounds $(k - 1)n \leq R_k(C_{2n}) \leq (k - \frac{1}{4} + o(1))n$ leave a wide gap. Closing this constant factor is one of the central open problems in multicolour Ramsey theory.
- **Thresholds for odd cycles.** The formula $R_k(C_n) = 2^{k-1}(n - 1) + 1$ holds for all sufficiently large odd n, but the minimal threshold $n_0(k)$ is not known.
- **Small exact values.** For $k = 3, 4$ and moderate cycle lengths, many values are only bounded. Computational methods could help resolve these cases.
- **Variants.** Gallai–Ramsey numbers and bipartite Ramsey numbers for cycles are still poorly understood in the multicolour setting.

These problems highlight the continuing depth of Ramsey theory: even for the simple family of cycles, exact answers are scarce, and progress often requires new ideas in extremal graph theory.

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