1 Introduction to q-Analogues

In enumerative combinatorics, a q-analogue of a formula, object, or quantity is a generalization involving a parameter q that returns the original concept in the limit as $q \to 1$. This process, often referred to as "quantization" transforms classical counting problems into a richer, polynomial framework. The resulting q-polynomials are not mere algebraic curiosities; they are typically generating functions for a specific combinatorial statistic (such as inversions) over the set of objects, revealing a deeper structural layer to the original problem.

The fundamental building block of q-analogues is the q-number, or q-integer, $[n]_q$, which is the q-analogue of a non-negative integer n. It is defined as:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

It is clear from the polynomial form that $\lim_{q\to 1}[n]_q = n$. From this, q-factorials are defined as $[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q$. This report will explore the q-analogues of several fundamental combinatorial numbers: the binomial coefficients, the Stirling numbers (Type A and B), the Fibonacci numbers, and the Bernoulli numbers.

2 The Gaussian (q-Binomial) Coefficient

The canonical example of a q-analogue is the Gaussian coefficient, or q-binomial coefficient, which generalizes the binomial coefficient $\binom{n}{k}$.

2.1 Formal Definition

The Gaussian coefficient, denoted $\begin{bmatrix} n \\ k \end{bmatrix}_q$, is defined for non-negative integers n and k as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}$$

This is equivalent to replacing each integer r in the formula $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1}$ with its q-analogue $q^r - 1$. By convention, $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k > n$.

2.2 The Limit as $q \to 1$

The definition of a q-analog is validated by confirming its limit.

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \lim_{q\to 1} \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}$$

We can rewrite this expression by dividing each term by $(q-1)$.

$$\lim_{q\to 1} \frac{\frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1} \cdots \frac{q^{n-k+1}-1}{q-1}}{\frac{q^k-1}{q-1} \cdot \frac{q^{k-1}-1}{q-1} \cdots \frac{q-1}{q-1}}$$

Using the property that $\lim_{q\to 1} \frac{q^r-1}{q-1} = \lim_{q\to 1}(1 + q + \cdots + q^{r-1}) = r$, the expression becomes:

$$\frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1} = \binom{n}{k}$$

This confirms the Gaussian coefficient as a valid q-analog of the binomial coefficient.

2.3 Combinatorial Interpretations

The significance of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ stems from its multiple, equivalent combinatorial interpretations.

Theorem 1 (Vector Space Interpretation). *Let V be an n-dimensional vector space over a finite field with q elements, $\mathbb{F}_q$. The number of k-dimensional subspaces of V is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. The proof proceeds by first counting the number of ordered sequences of k linearly independent vectors (which define a basis for a k-dimensional subspace) and then dividing by the number of such sequences that span the same subspace.

1. **Count ordered sequences of k independent vectors:**
   - The first vector, v_1, can be any non-zero vector. There are $q^n - 1$ choices.
   - The second vector, v_2, can be any vector not in the span of v_1. The span of v_1 contains q vectors. There are $q^n - q$ choices.
   - The third vector, v_3, can be any vector not in the span of $\{v_1, v_2\}$. This span contains q^2 vectors. There are $q^n - q^2$ choices.
   - Continuing this, the total number of ordered k-independent sequences is:

   $$(q^n - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{k-1})$$

2. **Count ordered bases for a given k-dimensional subspace:** Any given k-dimensional subspace W is itself a k-dimensional vector space. By the same logic as step 1 (with $n = k$), the number of ordered bases for W is:
   $$(q^k - 1)(q^k - q)(q^k - q^2)\dots(q^k - q^{k-1})$$

3. **Divide to find the number of subspaces:** The number of unique k-dimensional subspaces is the total number of k-independent sequences divided by the number of sequences per subspace:

   $$\text{Count} = \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})}$$

   By factoring out powers of q from each term, we get:

   $$\text{Count} = \frac{q^{\binom{k}{2}}(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{q^{\binom{k}{2}}(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$\square$

Theorem 2 (Lattice Path Interpretation). *The generating function for lattice paths from $(0,0)$ to (m,n) by area under the path (bounded by the path, the $x-axis$, and the line $x = m$) is $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$.*

Proof. Let $Q(N,k)$ be the generating function for the area under lattice paths from $(0,0)$ to $(N-k,k)$. We will show that $Q(N,k)$ satisfies the same recurrence relation as $\begin{bmatrix} N \\ k \end{bmatrix}_q$. The boundary conditions are $Q(N,0) = 1$ and $Q(N,N) = 1$, as paths along the x-axis or y-axis enclose zero area, contributing $q^0 = 1$.

Consider a path P from $(0,0)$ to $(N-k,k)$. The last step of P must be either vertical or horizontal.

1. **Last step is Vertical:** The path ends with a step from $(N-k,k-1)$ to $(N-k,k)$. This vertical step adds no area. The path preceding this step is a valid path from $(0,0)$ to $(N-k,k-1)$, counted by $Q(N-1,k-1)$. The total contribution from this case is $Q(N-1,k-1)$.

2. **Last step is Horizontal:** The path ends with a step from $(N-k-1,k)$ to $(N-k,k)$. This horizontal step, at height $y=k$, adds an area of $k\times 1=k$ units. The path preceding this step is a valid path from $(0,0)$ to $(N-k-1,k)$, counted by $Q(N-1,k)$. The total contribution from this case is $q^k\cdot Q(N-1,k)$.

Summing these disjoint cases gives the recurrence relation:

$$Q(N,k)=Q(N-1,k-1)+q^kQ(N-1,k)$$

This is a known recurrence relation for the Gaussian coefficients. Since $Q(N,k)$ and $\begin{bmatrix}N\\k\end{bmatrix}_q$ satisfy the same recurrence and boundary conditions, they must be equal. Setting $N=m+n$ and $k=m$ (or $k=n$) gives the theorem. □

Theorem 3 (Inversion Interpretation)**.** *The Gaussian coefficient* $\begin{bmatrix}n\\k\end{bmatrix}_q$ *is the generating function for the inversion statistic over the set* $\binom{S}{k}$ *of all bit strings of length* n *with* k *zeros and* $n-k$ *ones.*

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\sum_{w\in\binom{S}{k}}q^{\mathrm{inv}(w)}$$

An inversion is a pair of indices (i,j) *such that* $i<j$ *and* $b_i>b_j$*, which in this context corresponds to a 1 appearing before a 0.*

3 The q-Binomial Theorem

A fundamental identity for binomial coefficients is the Binomial Theorem, $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$. Its q-analog is a cornerstone of q-series. A common form (the "non-commutative" version) involves variables x,y such that $yx=qxy$. A second, commutative version is also central.

Theorem 4 (The q-Binomial Theorem)**.** *For any positive integer* n *and commuting variables* q,z*:*

$$\prod_{i=0}^{n-1}(1+q^iz)=\sum_{k=0}^{n}q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}_q z^k$$

Proof. The proof proceeds by induction on n.

- **Base Case ($n=1$):** The left-hand side (LHS) is $\prod_{i=0}^{0}(1+q^iz)=(1+q^0z)=1+z$. The right-hand side (RHS) is $\sum_{k=0}^{1}q^{k(k-1)/2}\begin{bmatrix}1\\k\end{bmatrix}_q z^k=q^0\begin{bmatrix}1\\0\end{bmatrix}_q z^0+q^0\begin{bmatrix}1\\1\end{bmatrix}_q z^1=1+z$. The base case holds.

- **Inductive Step:** Assume the theorem holds for $n-1$. We seek to prove it for n.

$$\prod_{i=0}^{n-1}(1+q^iz)=\left(\prod_{i=0}^{n-2}(1+q^iz)\right)(1+q^{n-1}z)$$

  Applying the inductive hypothesis to the first term:

$$\left(\sum_{k=0}^{n-1}q^{k(k-1)/2}\begin{bmatrix}n-1\\k\end{bmatrix}_q z^k\right)(1+q^{n-1}z)$$

  We expand this product and find the coefficient of z^k. This sum is the sum of two terms:

1. The z^k term from the sum, multiplied by 1:

$$q^{k(k-1)/2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q z^k$$

2. The z^{k-1} term from the sum, multiplied by $q^{n-1}z$:

$$q^{(k-1)(k-2)/2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q z^{k-1}\cdot(q^{n-1}z) = q^{(k-1)(k-2)/2+n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q z^k$$

The coefficient of z^k is:

$$C(z^k) = q^{k(k-1)/2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{(k-1)(k-2)/2+n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

We factor out the term $q^{k(k-1)/2}$. To do this, we analyze the exponent of q in the second term. The difference between these exponents is $\frac{(k-1)(k-2)}{2} + (n-1) - \frac{k(k-1)}{2} = \frac{k^2-3k+2+2n-2-(k^2-k)}{2} = \frac{-2k+2n}{2} = n-k$.

Substituting this back, the coefficient of z^k is:

$$q^{k(k-1)/2}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right)$$

The term in the parentheses is one of the q-Pascal identities, often called the alternative recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Therefore, the coefficient of z^k simplifies to $q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q$. This is precisely the k-th term in the target sum for n. The induction is complete.

□

4 q-Stirling Numbers of Type A

Unlike the q-binomial coefficient, the "q-Stirling number" is an ambiguous term, as several distinct q-analogues exist, often related by different combinatorial statistics. We will review the most prominent definitions.

4.1 q-Stirling Numbers of the Second Kind, $S[n,k]$

The (Type A) q-Stirling numbers of the second kind, $S[n,k]$, were first defined by Carlitz. They are defined by the recurrence relation that replaces the k in the classical recurrence $S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$ with its q-integer $[k]_q$.

Definition 5. Recurrence for $S[n,k]$ The q-Stirling numbers of the second kind $S[n,k]$ satisfy $S[0,k] = \delta_{0,k}$ and, for $n \geq 1$:

$$S[n,k] = S[n-1,k-1] + [k]_q S[n-1,k]$$

Like the Gaussian coefficients, $S[n,k]$ has multiple combinatorial interpretations as a generating function.

Interpretation 1: The Inversion Statistic $S[n,k]$ is the generating function for the inversion statistic over the set $\mathcal{S}(n,k)$ of partitions of $\{1,\ldots,n\}$ into k blocks.

$$S[n,k] = \sum_{\pi \in \mathcal{S}(n,k)} q^{\mathrm{inv}(\pi)}$$

For this statistic, partitions $\pi = B_1/\ldots/B_k$ are written in **standard form**, where $\min B_1 < \min B_2 < \cdots < \min B_k$. An **inversion** is an element-block pair (b, B_j) such that $b \in B_i$ where $i < j$ (i.e., b is in an earlier block) and $b > \min B_j$.

Interpretation 2: The Non-Inversion Statistic $S[n,k]$ is also the generating function for the non-inversion, noted as nin statistic over the same set $\mathcal{S}(n,k)$.

Theorem 6 (Non-Inversion Generating Function)**.**

$$S[n,k] = \sum_{\pi \in \mathcal{S}(n,k)} q^{\mathrm{nin}(\pi)}$$

A ***non-inversion*** *is a pair* (m_i, b_j) *where* $m_i = \min B_i$, $b_j \in B_j \setminus \{\min B_j\}$, $m_i < b_j$, *and* $i < j$.

4.2 q-Stirling Numbers of the First Kind, $c[n,k]$

Analogously, the (signless, Type A) q-Stirling numbers of the first kind, $c[n,k]$, are defined by replacing the $(n-1)$ in the classical recurrence $c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$ with its q-integer $[n-1]_q$.

Definition 7. Recurrence for $c[n,k]$ The q-Stirling numbers of the first kind $c[n,k]$ satisfy $c[0,k] = \delta_{0,k}$ and, for $n \geq 1$:

$$c[n,k] = c[n-1,k-1] + [n-1]_q c[n-1,k]$$

Interpretation: The inv **Statistic** $c[n,k]$ is the generating function for an inv statistic over the set $\mathcal{C}(n,k)$ of permutations of $\{1,\ldots,n\}$ with k cycles.

$$c[n,k] = \sum_{\sigma \in \mathcal{C}(n,k)} q^{\mathrm{inv}(\sigma)}$$

For this statistic, permutations are written in **standard form**: each cycle is written with its minimal element first, and the cycles are ordered by their minimal elements. The inv statistic is then calculated on this linear arrangement, ignoring parentheses.

5 Core Properties of q-Bernoulli Numbers and Polynomials

The q-analog framework extends to the Bernoulli numbers, first introduced by L. Carlitz. These numbers do not typically follow from a simple q-analogue of the classical exponential generating function, but are instead defined recursively or through p-adic integrals.

5.1 Classical Bernoulli Numbers and Polynomials

The classical Bernoulli polynomials $B_n(x)$ are defined by the exponential generating function:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}$$

The classical Bernoulli numbers B_n are the values at $x = 0$, so $B_n = B_n(0)$.

5.2 Carlitz's q-Bernoulli Numbers $\beta_n(q)$

Carlitz defined the q-Bernoulli numbers, which we denote $\beta_n(q)$, using a symbolic recurrence relation.

Definition 8 (Recursive Definition)**.** The Carlitz q-Bernoulli numbers $\beta_n(q)$ (also denoted $\beta_{n,q}$) are defined recursively by setting $\beta_{0,q} = 1$, and for $k \geq 1$:

$$q(q\beta + 1)^k - \beta_k = \delta_{k,1}$$

where $\delta_{k,1}$ is the Kronecker delta. This formula is interpreted "umbrally" by expanding the left-hand side and replacing β^j with $\beta_{j,q}$.

Definition 9 (Explicit Formula)**.** An explicit formula for $\beta_n(q)$ is:

$$\beta_n(q) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{[l+1]_q}$$

Definition 10 (Integral Representation)**.** In p-adic analysis, the q-Bernoulli numbers are represented as a q-integral (or Volkenborn integral) over the p-adic integers $\mathbb{Z}_p$:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x)$$

where $d\mu_q(x)$ is the q-analogue of the p-adic Haar distribution.

5.3 Carlitz's q-Bernoulli Polynomials $\beta_n(x, q)$

The q-Bernoulli polynomials $\beta_n(x, q)$ are the polynomial q-analogue of $B_n(x)$.

Definition 11 (Summation Formula)**.** The Carlitz q-Bernoulli polynomial is:

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{l(l-1)/2} \beta_{l,q}$$

Definition 12 (Integral Representation)**.** This definition parallels the integral for the numbers:

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y)$$

6 q-Stirling Orthogonality and the q-Bernoulli Connection

The classical Stirling numbers are algebraically defined as the change-of-basis coefficients between the monomial basis $\{t^n\}$ and the falling factorial basis. This relationship is preserved in the q-analogue and provides the crucial link to the q-Bernoulli numbers.

6.1 q-Falling Factorials and Orthogonality

We define the q-falling factorial as:

$$(t; q)_k = t(t - [1]_q)(t - [2]_q) \dots (t - [k-1]_q)$$

The q-Stirling numbers of the second kind, $S[n,k]$, satisfy **Carlitz's Identity**, which defines them as the change-of-basis coefficients:

$$t^n = \sum_{k=0}^{n} S[n,k](t;q)_k$$

The inverse relation is given by the q-Stirling numbers of the first kind:

$$(t;q)_n = \sum_{k=0}^{n} s[n,k]t^k$$

where $s[n,k] = (-1)^{n-k}c[n,k]$. This demonstrates that the matrices $S = [S[n,k]]$ and $s = [s[n,k]]$ are inverses of each other, establishing the q-orthogonality of the two kinds of q-Stirling numbers.

6.2 The q-Bernoulli Connection

This algebraic framework provides the tool to prove the fundamental relationship between Carlitz's q-Bernoulli numbers and the q-Stirling numbers.

Theorem 13 (The q-Bernoulli-Stirling Relation)**.** *The Carlitz q-Bernoulli numbers $\beta_{n,q}$ are related to the q-Stirling numbers of the second kind $S[n,k]$ by the formula:*

$$\beta_{n,q} = q\sum_{k=0}^{n} S[n,k]\frac{(-1)^k[k]!}{[k+1]_q}$$

Proof. The proof relies on the p-adic integral representation of $\beta_{n,q}$ and Carlitz's Identity.

1. Recall Carlitz's Identity, which defines $S[n,k]$:

$$t^n = \sum_{k=0}^{n} S[n,k](t;q)_k$$

2. Recall the p-adic integral representation of $\beta_{n,q}$:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x)$$

3. Substitute the q-number $[x]_q$ for the variable t in Carlitz's Identity:

$$[x]_q^n = \sum_{k=0}^{n} S[n,k]([x]_q;q)_k$$

4. Substitute this polynomial expansion into the integral for $\beta_{n,q}$:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} \left(\sum_{k=0}^{n} S[n,k]([x]_q;q)_k\right) d\mu_q(x)$$

5. By the linearity of the integral, we can swap the finite summation and the integral:

$$\beta_{n,q} = \sum_{k=0}^{n} S[n,k]\left(\int_{\mathbb{Z}_p} ([x]_q;q)_k d\mu_q(x)\right)$$

6. The proof is reduced to evaluating the integral $I_k = \int_{\mathbb{Z}_p} ([x]_q;q)_k d\mu_q(x)$.

7. This integral I_k is a known result in p-adic q-calculus. It evaluates to:

$$I_k = q \cdot (-1)^k \frac{[k]!}{[k+1]_q}$$

8. Substituting this evaluation back into the summation yields the theorem:

$$\beta_{n,q} = \sum_{k=0}^{n} S[n,k] \left(q(-1)^k \frac{[k]!}{[k+1]_q} \right) = q \sum_{k=0}^{n} S[n,k] \frac{(-1)^k [k]!}{[k+1]_q}$$

□

7 q-Fibonacci Numbers

The q-analog framework also applies to other combinatorial sequences. A q-Fibonacci number $F_n(q)$ has been defined as a generating function for a statistic on a set of partitions counted by the n-th Fibonacci number, F_n.

7.1 Origin and Definition

The set of set partitions of $[n]$ that avoid the patterns 13/2 (a non-layered partition) and 123 (a block of size 3 or more) is denoted $\Pi_n(13/2, 123)$. These are the **layered matchings** of $[n]$ (partitions where all blocks are singletons or doubletons, and all elements of a block are contiguous). The number of such partitions is F_n (using $F_0 = 1, F_1 = 1$).

The q-Fibonacci number $F_n(q)$ is the generating function for the rb (right bigger) statistic over this set.

Definition 14 (q-Fibonacci Number)**.** For a layered matching $\pi = B_1/\ldots/B_k$, a pair (b, B_j) is **right bigger** if $b \in B_i$ with $i < j$ and $b < \max B_j$. The q-Fibonacci number is:

$$F_n(q) = \sum_{\pi \in \Pi_n(13/2,123)} q^{\mathrm{rb}(\pi)}$$

7.2 Properties of $F_n(q)$

This q-analog satisfies a q-version of the Fibonacci recurrence.

Proposition 15 (Recurrence for $F_n(q)$)**.** *The generating function $F_n(q)$ satisfies $F_0(q) = 1$, $F_1(q) = 1$, and for $n \geq 2$:*

$$F_n(q) = q^{n-1} F_{n-1}(q) + q^{n-2} F_{n-2}(q)$$

Proof. We partition the set $\Pi_n(13/2, 123)$ based on the final block of the layered matching π.

1. **The last block is a singleton $\{n\}$.** The partition is $\pi = \pi'/\{n\}$, where $\pi' \in \Pi_{n-1}(13/2, 123)$. The total rb statistic is $\mathrm{rb}(\pi) = \mathrm{rb}(\pi') + (\text{pairs involving } \{n\})$. The pairs involving the last block $\{n\}$ are $(b, \{n\})$ where $b \in B_i$ for $i < k$ and $\max\{n\} > b$. Since $\max\{n\} = n$, this is true for *all* $b \in \bigcup_{i=1}^{k-1} B_i = [n-1]$. Thus, $n-1$ new rb pairs are created. The total contribution from this case is $\sum_{\pi' \in \Pi_{n-1}} q^{\mathrm{rb}(\pi')} \cdot q^{n-1} = q^{n-1} F_{n-1}(q)$.

2. **The last block is a doubleton $\{n-1, n\}$.** The partition is $\pi = \pi'/\{n-1, n\}$, where $\pi' \in \Pi_{n-2}(13/2, 123)$. The total rb statistic is $\mathrm{rb}(\pi) = \mathrm{rb}(\pi') + (\text{pairs involving } \{n-1, n\})$. The pairs are $(b, \{n-1, n\})$ where $b \in \bigcup_{i=1}^{k-1} B_i = [n-2]$ and $\max\{n-1, n\} > b$. Since $\max = n$, this is true

for all $n-2$ elements in $[n-2]$. The total contribution from this case is $\sum_{\pi' \in \Pi_{n-2}} q^{\mathrm{rb}(\pi')} \cdot q^{n-2} = q^{n-2} F_{n-2}(q)$.

Summing these two disjoint cases yields the recurrence. □

These q-Fibonacci numbers are also directly related to those previously studied by Carlitz, $F_n^K(q)$, and Cigler, $F_n^C(x, y, q)$. The relationship is a transformation:

$$F_n(q) = q^{\binom{n}{2}} F_n^K(1/q)$$

This identity reveals that the q-Fibonacci numbers of Goyt and Sagan are, up to a q-shift and $q \to 1/q$ transformation, equivalent to the classical q-Fibonacci numbers of Carlitz.

7.3 A q-Fibonacci Identity

This combinatorial model lends itself to elegant proofs of q-identities. We use the generating function $F_n(x, y, q)$ which tracks singletons with x and doubletons with y. The recurrence is $F_n(x, y, q) = xq^{n-1}F_{n-1}(\dots) + yq^{n-2}F_{n-2}(\dots)$, and $F_n(1, 1, q) = F_n(q)$.

Theorem 16 (The F_{m+n} Identity)**.** *For all $m, n \geq 0$:*

$$F_{m+n}(x, y, q) = F_m(x, y, q)F_n(xq^m, yq^m, q) + yq^{m-1}F_{m-1}(x, y, q)F_{n-1}(xq^{m+1}, yq^{m+1}, q)$$

Proof. The proof is a combinatorial dissection of the set $\Pi_{m+n}(13/2, 123)$. We consider the "fault line" between elements m and $m+1$.

1. **No block is $\{m, m+1\}$.** Since π is a layered matching, this means π splits into two independent partitions: $\pi_1 \in \Pi_m(13/2, 123)$ (on set $[m]$) and π_2, a layered matching on $\{m+1, \dots, m+n\}$. The weight $\omega(\pi) = \omega(\pi_1) \cdot \omega(\pi_2)$. The sum of weights for π_1 is $F_m(x, y, q)$. For π_2, it is a layered matching of $[n]$, but every element is shifted by m. The weight of a block B_j is $wq^{\min B_j - 1}$ (where $w = x$ or y). For a block B'_j in π_2, its corresponding block B_j in Π_n has $\min B'_j = \min B_j + m$. Thus, $\omega(B'_j) = wq^{(\min B_j + m) - 1} = wq^{\min B_j - 1} \cdot q^m$. Every block in π_2 contributes an extra factor of q^m. The total weight for π_2 is $F_n(xq^m, yq^m, q)$. The total contribution from this case is $F_m(x, y, q)F_n(xq^m, yq^m, q)$.

2. **The block $\{m, m+1\}$ exists.** This block $B_j = \{m, m+1\}$ must be present. Since π is a layered matching, the partition is composed of:

   - π_1: A layered matching on $[m-1]$. Total weight: $F_{m-1}(x, y, q)$.
   - B_j: The block $\{m, m+1\}$. Its weight is $y \cdot q^{\min B_j - 1} = yq^{m-1}$.
   - π_2: A layered matching on $\{m+2, \dots, m+n\}$. This is a matching of $[n-1]$ where all elements are shifted by $m+1$. By the logic of Case 1, its total weight is $F_{n-1}(xq^{m+1}, yq^{m+1}, q)$. The total contribution from this case is the product of these three parts

$$yq^{m-1}F_{m-1}(x, y, q)F_{n-1}(xq^{m+1}, yq^{m+1}, q)$$

Summing the two disjoint cases completes the proof. □

8 An Advanced Extension: q-Stirling Numbers in Type B

Recent research has extended the q-Stirling framework to other Coxeter groups, most notably Type B, the symmetry group of the hypercube. This extension provides a striking example of the q-analog philosophy, where the integers governing the Type A recurrences are replaced by q-analogs of the "odd integers" that govern Type B combinatorics.

8.1 Type B Recurrences

The classical (signless) Type B Stirling numbers $S_B(n,k)$ and $c_B(n,k)$ satisfy recurrences that are analogous to their Type A counterparts:

- $S_B(n,k) = S_B(n-1,k-1) + (2k+1)S_B(n-1,k)$
- $c_B(n,k) = c_B(n-1,k-1) + (2n-1)c_B(n-1,k)$

The q-analogs are defined by directly quantifying these integer coefficients.

Definition 17 (Type B q-Stirling Recurrences)**.** The **Type B q-Stirling number of the second kind**, $S_B[n,k]$, is defined by $S_B[0,k] = \delta_{0,k}$ and:

$$S_B[n,k] = S_B[n-1,k-1] + [2k+1]_q S_B[n-1,k]$$

The (**signless**) **Type B q-Stirling number of the first kind**, $c_B[n,k]$, is defined by $c_B[0,k] = \delta_{0,k}$ and:

$$c_B[n,k] = c_B[n-1,k-1] + [2n-1]_q c_B[n-1,k]$$

Theorem 18 (Interpretation of $S_B[n,k]$)**.** *$S_B[n,k]$ is the generating function for the* inv *statistic over $S_B(\langle n\rangle,k)$, the set of **signed partitions** of $\langle n\rangle = \{-n,\dots,n\}$ with k paired blocks. A signed partition is $\rho = S_0/S_1/\dots/S_{2k}$ where $S_0 = -S_0$ (the zero block) and $S_{2i} = -S_{2i-1}$ for $i \geq 1$. In standard form, $m_i = \min|S_i|$ and $m_0 < m_2 < \dots < m_{2k}$. An **inversion** is a pair (s, S_j) where $s \in S_i$ for $i < j$ and $s \geq m_j$.*

$$S_B[n,k] = \sum_{\rho \in S_B(\langle n\rangle,k)} q^{\mathrm{inv}(\rho)}$$

Proof. We show the sum satisfies the recurrence for $S_B[n,k]$.

1. **$\{-n\}/\{n\}$ is a block pair.** This partition ρ is formed from $\rho' \in S_B(\langle n-1\rangle, k-1)$. In standard form, $S_{2k} = \{n\}$ and $m_{2k} = n$. The element n does not satisfy $i < j$ for any j, so it creates 0 inversions. The element $-n$ is negative and, by definition, cannot cause an inversion. The contribution is $S_B[n-1,k-1]$.

2. **$\pm n$ are added to an existing partition $\rho' \in S_B(\langle n-1\rangle, k)$.** ρ' has $2k+1$ blocks $(S_0,\dots,S_{2k})$. We insert n into one of these blocks, S_i', which forces $-n$ into the partner block.

    - The element $-n$ never creates an inversion, as the statistic requires $s \geq m_j$, implying s is positive.
    - The element n, when placed in S_i', is positive and $n \geq m_j$ for all j. It creates inversions (n, S_j) for all j such that $i < j$. The number of such blocks is $2k - i$.
    - The total q-factor for inserting n into S_i' is q^{2k-i}.
    - We can insert n into any of the $2k+1$ blocks, for $i = 0, 1, \dots, 2k$.
    - Summing the contributions: $\sum_{i=0}^{2k} q^{2k-i} = q^{2k} + q^{2k-1} + \dots + q^0 = [2k+1]_q$.

- The total contribution from this case is $[2k+1]_q \cdot S_B[n-1,k]$. Summing the two cases gives $S_B[n,k] = S_B[n-1,k-1] + [2k+1]_q S_B[n-1,k]$, which matches the recurrence.

□

Theorem 19 (Interpretation of $c_B[n,k]$). *$c_B[n,k]$ is the generating function for the* inv *statistic over $c_B(\langle n\rangle', k)$, the set of **signed permutations** of $\langle n\rangle' = \langle n\rangle\backslash\{0\}$ with k paired cycles. A signed permutation π satisfies $\pi(-i) = -\pi(i)$. Its cycles are either paired (c and $-c$) or unpaired (containing i and $-i$). In standard form, cycles are written as a word $w = w_1 \dots w_{2n}$. An **inversion** is a pair (i,j) such that $i < j$ and $w_i > |w_j|$.*

$$c_B[n,k] = \sum_{\pi \in c_B(\langle n\rangle', k)} q^{\mathrm{inv}(\pi)}$$

Proof. We show the sum satisfies the recurrence for $c_B[n,k]$.

1. **π contains the paired cycle $(-n)(n)$.** This permutation π is formed from $\pi' \in c_B(\langle n-1\rangle', k-1)$. In standard form, this pair is last, $w = \dots, -n, n$. Neither n nor $-n$ creates inversions. The contribution is $c_B[n-1,k-1]$.

2. **$\pm n$ are not in fixed points.** π is formed from $\pi' \in c_B(\langle n-1\rangle', k)$ (which has $2n-2$ elements in its word form) in one of two ways:

   - We add the *unpaired* cycle $(n, -n)$. In standard form, this is $w = \dots, n, -n$. n creates 0 inversions. This single construction contributes $q^0 \cdot c_B[n-1,k]$.
   - We insert n and $-n$ into existing cycles of π'. We can insert n just *before* any of the $2n-2$ elements w'_i in the word for π'. The position of $-n$ is then forced. As before, $-n$ creates no inversions.
   - If n is inserted before the element that is i-th from the right (where $i \in \{1, \dots, 2n-2\}$), it creates i inversions (with the i elements w'_j to its right, as $n > |w'_j|$ for all j).
   - The total contribution from these $2n-2$ insertions is $(q^1 + q^2 + \dots + q^{2n-2}) c_B[n-1,k]$.
   - Summing subcases (a) and (b): $(q^0 + q^1 + \dots + q^{2n-2}) c_B[n-1,k] = [2n-1]_q c_B[n-1,k]$.

   Summing the two main cases gives $c_B[n,k] = c_B[n-1,k-1] + [2n-1]_q c_B[n-1,k]$, matching the recurrence.

□

8.2 Algebraic Connections

The structural analogy between Type A and Type B q-Stirling numbers is deepest in their relationship to symmetric polynomials. The Type A numbers are known to be specializations of elementary (e_k) and homogeneous (h_k) symmetric polynomials. The Type B numbers obey the same identities, but with a different specialization.

Theorem 20 (Symmetric Polynomial Relations). *Let $e_k(x_1, \dots, x_n)$ and $h_k(x_1, \dots, x_n)$ be the elementary and complete homogeneous symmetric polynomials, respectively.*

(a) $c_B[n,k] = e_{n-k}([2]_q, [3]_q, \dots, [2n-1]_q)$

(b) $S_B[n,k] = h_{n-k}([2]_q, [3]_q, \dots, [2k+1]_q)$

Proof. We prove this by showing the right-hand sides satisfy the recurrences from Definition 8.1.

- **Proof of (a):** The elementary symmetric polynomials satisfy the recurrence $e_j(x_1, \ldots, x_n) = e_j(x_1, \ldots, x_{n-1}) + x_n e_{j-1}(x_1, \ldots, x_{n-1})$. Let $C(n, k) = e_{n-k}(x_1, \ldots, x_n)$ with $x_i = [2i-1]_q$.

$$C(n, k) = e_{n-k}(x_1, \ldots, x_{n-1}) + x_n e_{n-k-1}(x_1, \ldots, x_{n-1})$$

  Substituting $j = n - k$ and $x_n = [2n-1]_q$:

$$C(n, k) = C(n-1, k) + [2n-1]_q C(n-1, k-1)$$

  This is precisely the recurrence for $c_B[n, k]$.

- **Proof of (b):** The homogeneous symmetric polynomials satisfy the recurrence $h_j(x_1, \ldots, x_m) = h_j(x_1, \ldots, x_{m-1}) + x_m h_{j-1}(x_1, \ldots, x_m)$. Let $H(n, k) = h_{n-k}(x_1, \ldots, x_{k+1})$ with $x_i = [2i-1]_q$. Here, $j = n - k$, $m = k + 1$, and $x_m = x_{k+1} = [2(k+1)-1]_q = [2k+1]_q$.

$$H(n, k) = h_{n-k}(x_1, \ldots, x_k) + [2k+1]_q h_{n-k-1}(x_1, \ldots, x_{k+1})$$

  The first term is $h_{(n-1)-(k-1)}(x_1, \ldots, x_{(k-1)+1}) = H(n-1, k-1)$. The second term is $[2k+1]_q \cdot h_{(n-1)-k}(x_1, \ldots, x_{k+1}) = [2k+1]_q H(n-1, k)$.

$$H(n, k) = H(n-1, k-1) + [2k+1]_q H(n-1, k)$$

  This is precisely the recurrence for $S_B[n, k]$.

□