1 Introduction and Motivation

1.1 Big-picture context

Ramsey theory asks: how large must a complete graph be, under an edge-coloring, to *force* a monochromatic copy of a given graph? Exact Ramsey numbers are notoriously rare, yet paths form a sweet spot: they are structurally simple but nontrivial. In two colors, the Ramsey number for paths admits an exact closed form; in three and more colors, there are sharp asymptotic bounds and compelling conjectures, all with linear growth in the path length.

A quick contrast is instructive. While the classical clique numbers $R(K_r, K_s)$ grow very rapidly, the path numbers grow only linearly in n. Figure 1 and illustrates Figure 2 the threshold flavor: below a certain order one can still avoid a long monochromatic path, but once the order crosses the threshold, a long monochromatic path is unavoidable.

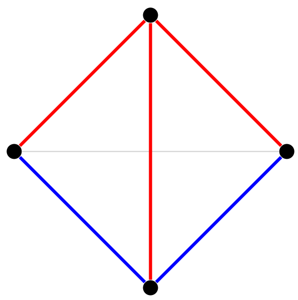

Figure 1: K_4 avoiding a mono P_4

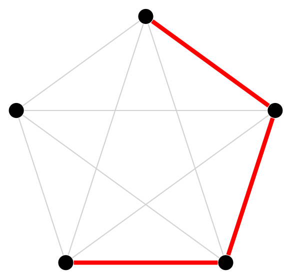

Figure 2: K_5 forces a mono P_4

1.2 Minimal setup

We write K_N for the complete graph on N vertices and P_n for a simple path on n vertices. For graphs $G_1, \dots, G_t$ the t-color Ramsey number is

$$R(G_1, \dots, G_t) = \min\{N : \text{ every } t\text{-coloring of } E(K_N) \text{ contains a mono } G_i \text{ in color } i\}.$$

This note focuses on path targets. We begin with the two-color case, then turn to three colors, and finally to the general t-color setting.

1.3 Two color exact result

The classical two-color path theorem gives a closed form.

$$R(P_n, P_m) = m + \left\lfloor \frac{n}{2} \right\rfloor - 1 \qquad (m \geq n \geq 2)$$

and, in particular, on the diagonal

$$R(P_n, P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1 = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

At a high level: the lower bound comes from a split construction; the upper bound uses a maximal-path extension argument.

1.4 Multicolor picture

In three colors the expected exact form is

$$R_3(P_n) = \begin{cases} 2n-2, & n \text{ even}, \\ 2n-1, & n \text{ odd}, \end{cases}$$

which is known to hold for all sufficiently large n. For general t colors one has linear bounds

$$(t - 1 + o(1))n \leq R_t(P_n) \leq \big(t - \tfrac{1}{2} + o(1)\big)n \qquad (n \to \infty).$$

2 Preliminaries and Notation

2.1 Graphs, color classes, and induced subgraphs

- $[N] = \{1, 2, \ldots, N\}$
- All graphs are finite and simple. For a graph G, $V(G)$ and $E(G)$ are its vertex and edge sets, respectively.
- The complete graph on N vertices is K_N, a *path* on n vertices is denoted P_n (it has $n-1$ edges)..
- For $S \subseteq V(G)$, $G[S]$ is the subgraph induced by S.
- If the edges of K_N are 2-colored (say *red* and *blue*), we denote by G_R and G_B the spanning subgraph consisting of all red and blue edges, respectively.
- For a vertex v, write $N_R(v)$ and $N_B(v)$ for its red and blue neighborhoods, and $\deg_R(v) = |N_R(v)|$, $\deg_B(v) = |N_B(v)|$.

- The complete bipartite graph with parts of size a and b is $K_{a,b}$.
- A path P in a graph is *longest* if no path has more vertices; it is *maximal* if it is not properly contained in a longer path (i.e., no single edge can be added to extend it). Note that every longest path is maximal; conversely, a maximal path need not be globally longest. In our upper-bound arguments we fix a maximal monochromatic path and exploit the constraints it imposes on vertices outside the path.

2.2 Ramsey numbers for paths

For graphs $G_1, \ldots, G_t$, the t-color Ramsey number is

$$R(G_1, \ldots, G_t) = \min\{N : \text{ every } t\text{-coloring of } E(K_N) \text{ contains a mono } G_i \text{ in color } i\}.$$

In the symmetric case $R_t(P_n) := R(\underbrace{P_n, \ldots, P_n}_{t \text{ times}})$. We will often use the two-color shorthand $R(G, H) = R_2(G, H)$. Note that $R(G, H)$ is symmetric and monotone, that is $R(G, H) = R(H, G)$ and if $G \subseteq G'$ then $R(G, H) \leq R(G', H)$.

2.3 A useful bipartite bound (exact for complete bipartite)

We record the simple extremal fact that will be used in the lower-bound construction.

Lemma 1 (Longest paths in $K_{a,b}$)**.** *Every path in a bipartite graph alternates between its two parts. In $K_{a,b}$,*

$$\text{(i)} \quad \nu_{\max}(K_{a,b}) \leq 2\min\{a, b\} + 1,$$

where $\nu_{\max}(G)$ denotes the maximum number of vertices *in any path of G. Moreover, equality holds in (i), and in fact:*

$$\text{(ii)} \quad \nu_{\max}(K_{a,b}) = \begin{cases} a + b, & \text{if } |a - b| \leq 1 \ \ (\textit{Hamiltonian path exists}), \\ 2\min\{a, b\} + 1, & \text{if } |a - b| \geq 2. \end{cases}$$

Proof. Any path in a bipartite graph must alternate between the two parts, so if a path uses x vertices from the smaller part and y from the larger, then $|x - y| \leq 1$, whence $x \leq \min\{a, b\}$ and $y \leq x + 1$. Thus the total number of vertices is at most $2\min\{a, b\} + 1$, proving (i). For sharpness: if $|a - b| \leq 1$ one can list vertices alternating between the parts and cover all $a + b$ vertices, yielding a Hamiltonian path. If $a \leq b - 2$, any path that alternates and starts and ends in the larger part uses exactly $a + (a + 1) = 2a + 1 = 2\min\{a, b\} + 1$ vertices; such a path exists in $K_{a,b}$ by greedily alternating through distinct vertices, so (ii) follows. □

Corollary 2. *If $a = \lfloor n/2 \rfloor - 1$ and $b = m - 1$, then $K_{a,b}$ has no red P_n (it has at most $2\lfloor n/2 \rfloor - 1 < n$ vertices per red path) and any blue path living inside a size-b clique has at most m vertices. This is the backbone of the lower-bound construction in the two-color proof.*

2.4 Two small warm-ups (used repeatedly)

Proposition 3. *$R(P_2, P_m) = m$ for $m \geq 2$.*

Proof. A red P_2 is just a red edge. In K_{m-1} color everything blue: no red edge occurs and there is no blue P_m on fewer than m vertices. In K_m, if no red edge appears then all edges are blue, which certainly contains a blue P_m. $\square$

Proposition 4. *$R(P_3, P_m) = m + 1$ for $m \geq 3$.*

Proof sketch. In K_m color edges to make blue a star from a center to the other $m - 1$ vertices and color the remaining edges red: there is no red P_3 (red is a matching) and no blue P_m (only m vertices, but a P_m needs m vertices in a single blue component with enough structure). In K_{m+1}, pick v; if v has two red neighbors we get a red P_3, otherwise $\deg_B(v) \geq m$ and there is a blue P_m from v across m neighbors. $\square$

2.5 Asymptotic notation

We use standard Landau notation. For functions $f, g : \mathbb{N} \to \mathbb{R}$ with $g(n) \neq 0$ eventually:

$$f(n) = o\big(g(n)\big) \iff \frac{f(n)}{g(n)} \to 0, \qquad f(n) = O\big(g(n)\big) \iff \exists C\, \forall n \;\; |f(n)| \leq C|g(n)|.$$

We also write $f(n) = \Theta(g(n))$ when both $f = O(g)$ and $g = O(f)$. In particular, $o(1)$ denotes any term $\varepsilon(n) \to 0$ as $n \to \infty$; thus

$$(t - \tfrac{1}{2} + o(1))\, n = (t - \tfrac{1}{2})\, n + o(n).$$

All asymptotic statements in this paper take t fixed and let $n \to \infty$; the little-o term may depend on t (one may write $o_t(1)$), but for each fixed t it vanishes as $n \to \infty$.

3 Two-Color Lower Bound

3.1 The main statement (two colors)

Theorem 5 (Gerencsér–Gyárfás)**.** *For integers $m \geq n \geq 2$,*

$$R(P_n, P_m) = m + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Equivalently, on the diagonal one has

$$R(P_n, P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1 = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

In this section we prove the *lower bound*

$$R(P_n, P_m) \;\geq\; m + \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

by constructing an explicit 2-coloring of K_{N_0} avoiding a red P_n and a blue P_m, where

$$N_0 = m + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

(The matching upper bound is proved in Section 4, completing the theorem.)

3.2 Lower bound via a split construction

Proposition 6. *Let $m \geq n \geq 2$ and put $N_0 = m + \lfloor n/2 \rfloor - 2$. There exists a red/blue coloring of K_{N_0} with* no *red P_n and* no *blue P_m.Consequently,*

$$R(P_n, P_m) \;\geq\; N_0 + 1 = m + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Proof. Partition the N_0 vertices into two parts ($A \,\dot\cup\, B$ here means $A \cup B$ and $A \cap B = \varnothing$)

$$A \,\dot\cup\, B, \qquad |A| = m - 1, \quad |B| = \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Color *all* edges *inside* A and *inside* B **blue**, and color *all* edges *between* A and B **red**. We verify that this coloring avoids both targets.

- No blue P_m: Every blue edge lies inside A or inside B; there are no blue cross-edges. Hence any blue path is contained entirely in A or entirely in B. But $|A| = m - 1$ and $|B| = \lfloor n/2 \rfloor - 1 < m$, so neither A nor B contains a blue P_m.

- No red P_n: All red edges go between A and B, so the red subgraph is the complete bipartite graph $K_{|A|,|B|} = K_{m-1,\ \lfloor n/2 \rfloor - 1}$.Any red path must alternate between A and B. By the bipartite longest-path bound from §2, the maximum *number of vertices* in a path in $K_{a,b}$ is $2\min\{a, b\} + 1$. Here $\min\{m-1,\ \lfloor n/2 \rfloor - 1\} = \lfloor n/2 \rfloor - 1$, so any red path has at most

$$2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 = 2\left\lfloor \frac{n}{2} \right\rfloor - 1$$

vertices. If n is even, this equals $n - 1$; if n is odd, it equals $n - 2$. In either case it is *strictly* less than n, so there is no red P_n.

Thus this coloring of K_{N_0} avoids both a red P_n and a blue P_m, proving the claim. $\square$

Remark (Parity check and tightness idea). The red longest-path bound is sharp in the bipartite graph $K_{m-1,\ \lfloor n/2\rfloor-1}$, so the obstruction really is the part-size $\lfloor n/2\rfloor - 1$ on the smaller side; increasing $|B|$ by one vertex (or, equivalently, adding one vertex to K_{N_0}) pushes the red longest path to at least $n-1$ vertices, which is exactly the threshold the upper-bound argument will exploit in Section 4.

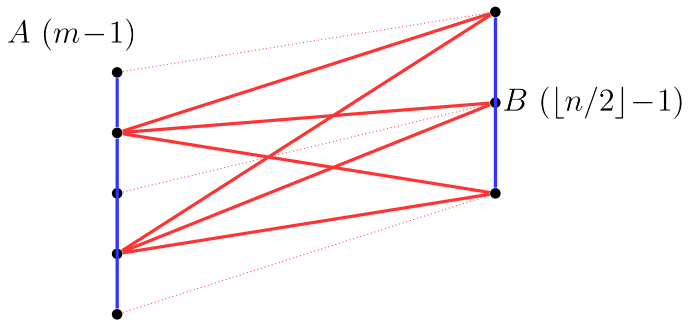

Figure 3: Lower-bound split coloring for K_{N_0} with $N_0 = m + \lfloor n/2\rfloor - 2$: blue edges lie within A and within B; all A–B edges are red.
Any blue path lives entirely inside A or B, so has at most $m-1$ vertices.
Any red path alternates across A–B and thus has at most $2\lfloor n/2\rfloor - 1 < n$ vertices.

3.3 Structure of the extremal coloring: lower-bound summary

The extremal coloring in Proposition 6 is the canonical obstruction:

- Blue is fragmented into two disjoint cliques, so it cannot host a long blue path spanning both sides.
- Red is a complete bipartite graph with the smaller side of size $\lfloor n/2\rfloor - 1$, which caps the red path length below n.

This demonstrates that the formula in the theorem cannot be improved from below. The complementary *upper bound* (showing that $K_{m+\lfloor n/2\rfloor-1}$ *always* contains a red P_n or a blue P_m) will be proved next by a maximal-path extension argument.

4 Two-Color Upper Bound

In this section we prove the matching upper bound

$$R(P_n, P_m) \ \le\ m + \left\lfloor \frac{n}{2} \right\rfloor - 1 \qquad (m \ge n \ge 2),$$

completing the proof of the Gerencsér–Gyárfás theorem together with the lower bound from Section 3.

4.1 Setup and a key maximal-path lemma

Fix integers $m \geq n \geq 2$ and put

$$N := m + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Consider an arbitrary red/blue edge-coloring of K_N. If the red graph G_R contains a red P_n we are done, so assume *no* red P_n exists. Let

$$P := v_1 v_2 \cdots v_r$$

be a longest red path (so $r \leq n-1$). Put $Y := V(K_N) \setminus V(P)$.

Lemma 7 (Maximality constraints)**.** *For every $y \in Y$:*

1. (Endpoint condition) *The edges yv_1 and yv_r are blue.*
2. (Parity condition) *For every $i \in \{1, \dots, r-1\}$, at least one of yv_i, yv_{i+1} is blue. Equivalently, for the index set $S_y := \{i \in [r] : yv_i \text{ is blue}\}$ we have $S_y \cap \{i, i+1\} \neq \emptyset$ for all i.*

Proof. (1) If yv_r were red, we could extend P to $v_1 \cdots v_r y$, contradicting maximality; similarly for v_1. (2) If both yv_i and yv_{i+1} were red for some i, then

$$v_1 \cdots v_i \, y \, v_{i+1} \cdots v_r$$

would be a longer red path, again contradicting maximality. □

For a path $P = v_1 \cdots v_r$, write $O := \{v_1, v_3, \dots\}$ and $E := \{v_2, v_4, \dots\}$ for its odd- and even-indexed vertices.

Lemma 8 (Dichotomy of outside vertices)**.** *For each $y \in Y$, either y is blue to every vertex of O or y is blue to every vertex of E. Equivalently, Y splits into $Y_O \cup Y_E$ (disjoint), where*

$$Y_O := \{y \in Y : yv \text{ is blue for all } v \in O\}, \qquad Y_E := \{y \in Y : yv \text{ is blue for all } v \in E\}.$$

Proof. By Lemma 7(2), S_y meets every consecutive pair $\{i, i+1\}$. A simple induction shows that any subset of $\{1, \dots, r\}$ meeting each consecutive pair contains either all odd indices or all even indices. Thus either $O \subseteq S_y$ or $E \subseteq S_y$, as required. □

Consequently, all edges between Y_O and O are blue, and all edges between Y_E and E are blue.

4.2 Building a long blue path

Let $a := |O| = \lceil r/2 \rceil$ and $b := |E| = \lfloor r/2 \rfloor$. By Lemma 8, the blue graph contains the complete bipartite subgraphs $K_{|Y_O|,a}$ and $K_{|Y_E|,b}$. By the $K_{u,v}$ longest-path fact recalled in Section 2, the longest *blue* path inside $K_{|Y_O|,a}$ has $2\min\{|Y_O|, a\} + 1$ vertices (and similarly for $K_{|Y_E|,b}$). Hence

$$L_{\max} \;\geq\; \max\Big\{ 2\min\{|Y_O|, a\} + 1,\;\; 2\min\{|Y_E|, b\} + 1 \Big\}. \tag{1}$$

We now show the right-hand side of (1) is at least m. Let $y := |Y| = N - r$. Since $Y = Y_O \cup Y_E$, we have $|Y_O| + |Y_E| = y$. Also $a + b = r$ and $\min\{|Y_O|, a\} + \min\{|Y_E|, b\} \geq \min\{y, a + b\} = \min\{y, r\}$. Therefore at least one of $\min\{|Y_O|, a\}$ or $\min\{|Y_E|, b\}$ is at least $\frac{1}{2}\min\{y, r\}$, and so by (1)

$$L_{\max} \;\geq\; 2 \cdot \frac{1}{2}\min\{y, r\} + 1 \;=\; \min\{y, r\} + 1. \tag{2}$$

We distinguish two subcases.

- *Subcase A: $y \geq r$.* Then $L_{\max} \geq r + 1$ by (2). Since $r \leq n - 1$, we have $r + 1 \geq n$. Because $m \geq n$, this gives $L_{\max} \geq m$, and a blue P_m exists.

- *Subcase B: $y < r$.* Then $L_{\max} \geq y + 1$ by (2). Compute

  $$y + 1 \;=\; (N - r) + 1 \;=\; \Big(m + \Big\lfloor \frac{n}{2} \Big\rfloor - 1 - r\Big) + 1 \;=\; m + \Big\lfloor \frac{n}{2} \Big\rfloor - r.$$

  If $L_{\max} < m$, then necessarily $\lfloor n/2 \rfloor - r \leq -1$, i.e. $r \geq \lfloor n/2 \rfloor + 1$. Combining $y < r$ with $r \geq \lfloor n/2 \rfloor + 1$ yields

  $$N \;=\; y + r \;<\; 2r \;\;\Rightarrow\;\; r \;>\; \frac{N}{2} \;=\; \frac{m + \lfloor n/2 \rfloor - 1}{2}.$$

  Since $m \geq n$, the right-hand side is at most $\frac{n}{2} + \frac{\lfloor n/2 \rfloor - 1}{2}$, which is $\geq n - 1$ (both parities). Thus $r \geq n$, contradicting the assumption that no red P_n exists.

In either subcase we obtain a contradiction to the assumption "no red P_n and no blue P_m." Therefore, every red/blue coloring of K_N contains a red P_n or a blue P_m, proving the upper bound.

4.3 Conclusion and the diagonal corollary

Combining the upper bound just proved with the lower bound from Section 3 gives

$$R(P_n, P_m) = m + \Big\lfloor \frac{n}{2} \Big\rfloor - 1 \qquad (m \geq n \geq 2).$$

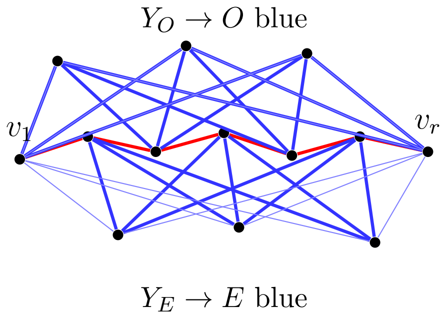

Figure 4: Maximal red path $P = v_1 \cdots v_r$ and the parity-based blue structure: every $y \in Y$ is blue to all odd or all even vertices of P. This yields long blue paths inside $K_{|Y_O|,|O|}$ or $K_{|Y_E|,|E|}$.

In particular, for $n = m$,

$$R(P_n, P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1 = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

5 Stability and Extremal Structure

At the threshold size

$$N_0 \;=\; m + \left\lfloor \frac{n}{2} \right\rfloor - 2$$

the lower-bound split coloring from Section 3 avoids both a red P_n and a blue P_m. In this section we discuss what such *extremal* colorings must look like and give a simple "stability" formulation. Throughout we assume $m \geq n \geq 2$.

5.1 Critical and saturated extremal colorings

Definition 9 (Critical and saturated)**.** A red/blue coloring of K_{N_0} is (n, m)-*critical* if it contains no red P_n and no blue P_m. It is *saturated* if, in addition, recoloring *any one* red edge to blue creates a blue P_m, and recoloring *any one* blue edge to red creates a red P_n.

The split construction (blue inside two parts A, B of sizes $m-1$ and $\lfloor n/2 \rfloor - 1$, all cross-edges red) is saturated in this sense.

5.2 Component and size constraints

Write G_B for the blue spanning subgraph and let its connected components be $C_1, \ldots, C_k$ with $|C_1| \geq \cdots \geq |C_k|$.

Proposition 10 (Blue component sizes in (n, m)-critical colorings)**.** *In any (n, m)-critical coloring of K_{N_0}:*

1. *The largest blue component has size $|C_1| = m - 1$.*
2. *The remaining vertices have total size $\sum_{i\geq 2} |C_i| = \lfloor n/2 \rfloor - 1$.*
3. *The red graph G_R induces a complete multipartite graph with parts $C_1, \ldots, C_k$ (i.e., all edges between distinct blue components are red).*

Proof sketch. (1) Since no blue P_m exists, every blue component has order at most $m-1$. If $|C_1| \leq m-2$, then $\sum_{i\geq 2} |C_i| \geq \lfloor n/2 \rfloor$ (because the total is N_0), and a longest red path in the red complete multipartite graph across the C_i's contains a $K_{|C_1|,\, \sum_{i\geq 2}|C_i|}$ as a red subgraph, hence by the $K_{a,b}$ bound (Section 2) has at least $2 \min\{|C_1|, \sum_{i\geq 2} |C_i|\} + 1 \geq 2\lfloor n/2 \rfloor + 1 > n$ vertices, a contradiction. Thus $|C_1| = m - 1$.

(2) Follows from $N_0 = (m - 1) + (\lfloor n/2 \rfloor - 1)$.

(3) If a blue edge joined two distinct blue components, they would not be distinct. Hence every edge between distinct components is red. □

Remark. Proposition 10 leaves freedom inside components: a blue component of order $m-1$ need not be a clique, and the small side $\sum_{i\geq 2} |C_i| = \lfloor n/2 \rfloor - 1$ may split into several blue components. The next statement shows that *saturation* collapses this freedom and recovers the split pattern.

5.3 Classification under saturation

Proposition 11 (Saturated extremal colorings are split)**.** *Let a coloring of K_{N_0} be (n, m)-critical and saturated. Then, up to isomorphism and swapping colors, there is a partition $V = A \dot\cup B$ with $|A| = m - 1$ and $|B| = \lfloor n/2 \rfloor - 1$ such that:*

1. *$G_B[A]$ and $G_B[B]$ are cliques (blue complete), and there are no blue edges between A and B;*
2. *consequently, G_R is the complete bipartite graph between A and B.*

Proof sketch. By Proposition 10, let $A = C_1$ (the unique largest blue component, order $m - 1$) and $B = \bigcup_{i\geq 2} C_i$ (total order $\lfloor n/2 \rfloor - 1$).

(No blue across A–B). If a blue edge ab with $a \in A$, $b \in B$ existed, then a and b would lie in the same blue component, contradicting that A is the unique largest component of order $m - 1$ under saturation: indeed, if after adding (or revealing) ab the component containing $A \cup \{b\}$ still had no blue P_m, then one could add further blue edges inside that component without creating a blue P_m, violating saturation. Hence there are no blue A–B edges; equivalently, all A–B edges are red.

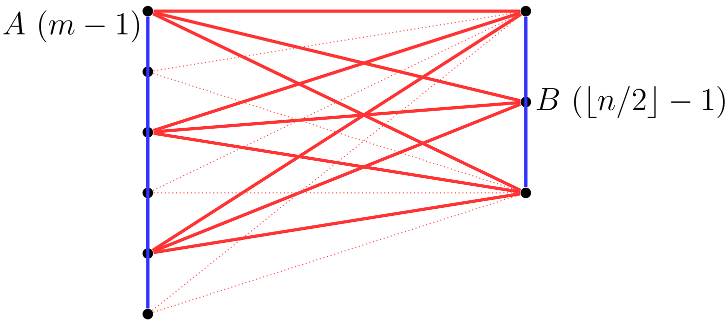

Figure 5: Saturated extremal structure at N_0: two blue cliques (A of size $m-1$ and B of size $\lfloor n/2 \rfloor - 1$), with all cross-edges red.

(Blue completeness inside A and B). Suppose two vertices $x, y \in A$ are not joined by a blue edge. By saturation, recoloring xy to blue must create a blue P_m. This forces the existence (before recoloring) of a blue path on $m-1$ vertices with x, y as endpoints, entirely inside A; thus A already contains a blue P_{m-1} whose endpoints can be chosen arbitrarily. A standard "rotation" (replace an endpoint by a neighbor) then implies A is a blue clique. An identical argument applied within B (using that any blue B–A edge is forbidden) gives that B is a blue clique. Therefore the blue graph is precisely two disjoint cliques K_{m-1} and $K_{\lfloor n/2 \rfloor - 1}$, and all cross edges are red. $\square$

5.4 A soft stability corollary

The next statement captures a "robustness" phenomenon: at N_0, extremal colorings are close to the split pattern.

Corollary 12 (Soft stability)**.** *Let a coloring of K_{N_0} be (n, m)-critical. Then there exists a partition $V = A \dot{\cup} B$ with $|A| = m-1$ and $|B| = \lfloor n/2 \rfloor - 1$ such that:*

1. *all but at most $|B|$ edges between A and B are red;*
2. *the blue graph has no edges between A and B after deleting at most $|B|$ blue edges;*
3. *in particular, by recoloring $O(|B|)$ edges one reaches the saturated split coloring of Proposition 11.*

Proof idea. Let A be the vertex set of a largest blue component (size $m-1$ by Proposition 10), and set $B = V \setminus A$. Any blue edge across A–B merges B into the largest component; one can delete at most $|B|$ such edges to separate B from A (each vertex of B participates in at most its degree many). After this, the coloring is (n, m)-critical and *blue-maximal* with respect to the bipartition, and saturating blue inside A and B (by recoloring missing blue edges) cannot create a blue P_m since the parts have sizes $m-1$ and $\lfloor n/2 \rfloor - 1$. The resulting coloring is exactly the split pattern. $\square$

6 Three Colors and Unequal Lengths

6.1 Erdős–Gallai theorem for paths

Theorem 13 (Erdős–Gallai)**.** *Let G be a graph on N vertices. If*

$$e(G) > \frac{n-2}{2}N,$$

then G contains a (simple) path on n vertices, i.e. a copy of P_n. Equivalently, any N-vertex graph with no P_n has at most $\frac{n-2}{2}N$ edges.

Proof. Write $k := n-2$. We prove the contrapositive: if G has no P_{k+2}, then $e(G) \leq \frac{k}{2}N$.

Step 1 (Peeling to a $(k+1)$-core or empty). Repeatedly delete any vertex of degree at most k. This process halts with a (possibly empty) induced subgraph H in which every vertex has degree at least $k+1$, or deletes all vertices.

Step 2 (If a $(k+1)$-core remains, we find a P_{k+2}). Suppose H is nonempty. Let v_1 be any vertex of H and greedily extend a path $v_1v_2\cdots$ by always choosing a new neighbor of the current endpoint. At the moment the path has t vertices, the current endpoint has at least $k+1$ neighbors in H, of which at most $t-1$ already lie on the path. If $t \leq k+1$, then $k+1-(t-1) \geq 1$, so the path can be extended. Hence we obtain a path on $k+2$ vertices in H (and thus in G), contradicting the assumption that G has no P_{k+2}.

Therefore the deletion process must remove *all* vertices.

Step 3 (Edge count bound). Let $d_1, d_2, \ldots, d_N$ be the degrees of the vertices at their deletion moments. Each time we delete a vertex, we remove exactly d_i edges, and no edge is removed twice; hence $e(G) = \sum_{i=1}^{N} d_i$. Since each $d_i \leq k$, we obtain

$$e(G) = \sum_{i=1}^{N} d_i \leq kN = 2 \cdot \frac{k}{2}N.$$

But Step 2 showed that the case "all vertices deleted" is the only possibility when G has no P_{k+2}. Combining this with the assumption "no P_{k+2}" gives the sharper bound

$$e(G) \leq \frac{k}{2}N,$$

as follows. If $e(G) > \frac{k}{2}N$, then the *average* degree satisfies $\overline{d} = 2e(G)/N > k$. In that case, at the very first step there must exist a vertex of degree at least $k+1$, so the deletion process cannot delete *all* vertices without at some stage producing a nonempty subgraph whose minimum degree is $\geq k+1$. By Step 2 such a subgraph contains a P_{k+2}, contradiction. Therefore $e(G) \leq \frac{k}{2}N$, proving the claim. $\square$

Remark. Two takeaways often used in practice:

- If the average degree $2e(G)/N$ exceeds k, then G contains a subgraph with minimum degree at least $k+1$ (the $(k+1)$-core); this is the "peeling" argument above.
- A graph with minimum degree $\geq k+1$ contains a P_{k+2} by the greedy extension: an endpoint of any maximal path has at least one neighbor outside the path until the path has $k+2$ vertices.

We now turn to three colors. Recall the symmetric notation $R_3(P_n) = R(P_n, P_n, P_n)$.

6.2 Statement: the Faudree–Schelp formula and its status

Conjecture 14 (Faudree–Schelp)**.** *For all $n \geq 2$,*

$$R_3(P_n) = \begin{cases} 2n-2, & n \text{ even}, \\ 2n-1, & n \text{ odd}. \end{cases}$$

This is known to hold for all *sufficiently large* n by a theorem of Gyárfás, Ruszinkó, Sárközy and Szemerédi (2007; with a 2008 corrigendum). Small values agree with the formula in all cases verified to date, and no counterexample is known. In particular, asymptotically $R_3(P_n) = (2 + o(1))\, n$.

6.3 Lower bounds: constructions at (2n-2)/(2n-1)

Proposition 15 (Lower bounds)**.** *For each $n \geq 2$ there exists a 3-coloring of*

$$K_{2n-3} \quad \text{if } n \text{ is even}, \qquad \text{and of} \qquad K_{2n-2} \quad \text{if } n \text{ is odd},$$

that contains no monochromatic P_n. Hence

$$R_3(P_n) \;\geq\; \begin{cases} 2n-2, & n \text{ even}, \\ 2n-1, & n \text{ odd}. \end{cases}$$

Construction idea (Faudree–Schelp). Partition the vertex set into three blocks whose sizes depend on the parity of n. Inside each block, color all edges with a distinct color (say red in A, blue in B, green in C) so that no block alone hosts a P_n (i.e., each block has size at most $n-1$). Color cross-edges *sparingly* in each color so that every monochromatic color class is a disjoint union of a bounded number of cliques and stars; in particular, ensure that in the green class the cross-edges form only vertex-disjoint stars rather than a dense complete bipartite piece. With this, any monochromatic path either stays inside one block (limited to $\leq n-1$ vertices) or alternates across a star structure (limited to $O(1)$ extra vertices). A parity tweak (moving one vertex between blocks) yields the claimed orders $2n-3$ (even n) and $2n-2$ (odd n). □

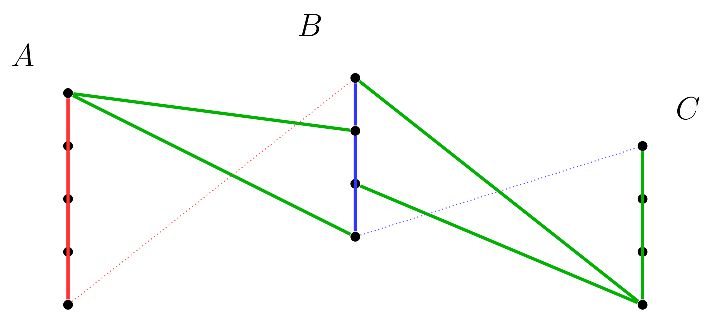

Figure 6: Schematic of a three-color extremal pattern (not to scale). Each color class decomposes into small components (cliques or stars), preventing any color from hosting a P_n.

Remark. The gist is that each color class is *fragmented* into small components (cliques or stars), so no color supports a long path. Unlike the two-color lower bound (one bipartite color + one clique color), here each color must be kept sparse in its own way to raise the total vertex count to $\approx 2n$.

6.4 A warm-up upper bound via Erdős–Gallai

The exact upper bound $\frac{2n-2}{2n-1}$ for large n is deep. As a warm-up, we record a simple (looser) bound using the Erdős–Gallai extremal theorem for paths.

Theorem 16 (Easy upper bound)**.** *For all $n \geq 2$, $R_3(P_n) \leq 3n - 4$.*

Proof. In any 3-coloring of K_N, some color class has at least $\binom{N}{2}/3$ edges. By Erdős–Gallai, a graph on N vertices with more than $\frac{n-2}{2}N$ edges contains a P_n. Thus it suffices that

$$\frac{1}{3}\binom{N}{2} > \frac{n-2}{2}N \iff \frac{N-1}{3} > n-2 \iff N > 3n-5.$$

Hence $R_3(P_n) \leq 3n - 4$. $\square$

This bound already shows linear growth. The sharp upper bound $R_3(P_n) \leq 2n - 2$ (even n) and $\leq 2n - 1$ (odd n) for all large n requires sophisticated structure theory (regularity-type arguments plus stability).

6.5 Unequal path lengths: a brief overview

When the three targets have different lengths, exact values are mostly open. There are sharp results in *extremal regimes*; e.g., when one target length overwhelmingly dominates the others, the problem reduces to a two-color forcing mechanism and one obtains explicit formulas. At moderate imbalances, best-known results are bounds; techniques combine two-color path arguments with careful control of the third color class to prevent long alternating paths.

6.6 How the sharp upper bound works (high-level sketch)

We briefly indicate the ideas behind the asymptotically sharp upper bound (for $R_3(P_n)$):

1. **Regularity + reduced graph.** Apply a regularity partition to the 3-colored K_N and pass to a dense "reduced" graph where each edge is colored by the majority color between the corresponding clusters.

2. **Connected matchings.** Show that in any 3-coloring of a sufficiently dense reduced graph, one color contains a large *connected matching*.

3. **Lifting.** A connected matching in the reduced graph lifts (via regular pairs) to a long monochromatic path in the original graph.

4. **Stability.** If a long path is not found, the coloring must look very close to a specific extremal pattern (fragmented color classes), which in turn allows an iterative extension to reach the threshold.

The parity difference (even n vs. odd n) is handled by keeping track of endpoints during the lifting step, mirroring the ± 1 phenomenon already visible in two colors.

6.7 Summary for three colors

- Lower bounds at $2n-2/2n-1$ follow from explicit fragmented colorings (Proposition 15).

- A simple counting proof gives $R_3(P_n) \leq 3n-4$ (Theorem 16).

- The sharp upper bound $R_3(P_n) = 2n-2$ (even n) and $= 2n-1$ (odd n) holds for all sufficiently large n via regularity + connected matchings + stability.

- For unequal lengths (n, m, k), exact values are rare; best results are in extreme imbalances or as tight bounds.

7 General Multi Colors: Bounds and Asymptotics

Throughout this section $t \geq 2$ is fixed and $n \to \infty$.

7.1 Headline bounds

There exist absolute functions $\varepsilon_t(n) \to 0$ (for each fixed t) such that

$$\left(t - 1 + \varepsilon_t^-(n)\right) n \;\leq\; R_t(P_n) \;\leq\; \left(t - \tfrac{1}{2} + \varepsilon_t^+(n)\right) n. \tag{3}$$

The lower bound is achieved by probabilistic constructions; the upper bound is due to modern "connected matching + lifting" arguments (refining regularity-based methods). In particular, $R_t(P_n) = \Theta(n)$ for every fixed t.

7.2 Upper bound for all t via Erdős–Gallai

The following bound is elementary and already shows linear growth with an explicit constant.

Theorem 17 (Easy t-color upper bound)**.** *For all integers $t \geq 2$ and $n \geq 2$,*

$$R_t(P_n) \ \leq \ t(n-2)+2.$$

Proof. Consider any t-coloring of K_N. Some color class has at least $\binom{N}{2}/t$ edges. By Theorem 13 (Erdős–Gallai), if that many edges exceed $\frac{n-2}{2}N$, then that color already contains a P_n. Thus it suffices that

$$\frac{1}{t}\binom{N}{2} \ > \ \frac{n-2}{2}N \iff \frac{N-1}{t} \ > \ n-2 \iff N \ > \ t(n-2)+1.$$

Hence $R_t(P_n) \leq t(n-2)+2$. □

Remark. For $t = 3$ this recovers $R_3(P_n) \leq 3n-4$ (Section 6). Theorem 17 is typically far from sharp, but it provides a clean baseline and a quick sanity check for computations.

7.3 Sharper upper bounds: connected matchings and lifting (high-level)

The current best general upper bound improves the linear coefficient from t down to $t-\frac{1}{2}$:

Theorem 18 (Upper bound with constant improvement)**.** *For each fixed $t \geq 2$,*

$$R_t(P_n) \ \leq \ \big(t - \tfrac{1}{2} + o(1)\big)\, n \qquad (n \to \infty).$$

Idea. At a high level:

1. **Regularity & reduced graph.** Apply a Szemerédi-type regularity partition and color each pair of parts by the *majority* color, obtaining a dense t-colored reduced graph.

2. **Connected matching lemma.** Show that in any such reduced graph, one color contains a *connected matching* large enough to cover roughly $(t-\frac{1}{2})n$ vertices after lifting.

3. **Lifting paths.** Each regular pair along the matching yields long monochromatic paths; connectivity lets you concatenate these into a single path of length $\geq n$ in the original graph.

4. **Stability.** If the connected matching is too small, the coloring must resemble a fragmented extremal template; one then bootstraps a path by local adjustments.

The details require careful counting and the standard slicing/cleaning of regular pairs; we omit them here. □

7.4 Lower bounds: fragmented colorings and randomness (high-level)

On the lower side, one has asymptotically matching order:

Theorem 19 (Probabilistic lower bound)**.** *For each fixed $t \geq 3$,*

$$R_t(P_n) \;\geq\; (t-1+o(1))\,n \qquad (n \to \infty).$$

Idea. Construct t-colorings on $N \leq (t-1-\varepsilon)n$ vertices with no monochromatic P_n. Random edge-colorings with carefully tuned color probabilities (or pseudorandom explicit constructions) produce, with high probability, color classes whose component structure is fragmented: each monochromatic graph has all components of order $o(n)$ (and bounded average degree), which blocks the appearance of a path on n vertices. A second-moment or sprinkling argument plus union bounds show existence; derandomization is possible. □

Remark. For $t = 3$, the lower bound specializes to $R_3(P_n) \geq 2n - O(1)$, consistent with the exact values/parities in Section 6. For $t = 2$, the story is exceptional: $R_2(P_n) = n + \lfloor n/2 \rfloor - 1 = (1.5 + o(1))n$.

7.5 Conjecture and outlook

The prevailing conjecture is that the lower asymptotic constant is tight for all $t \geq 3$:

Conjecture 20. *For every fixed $t \geq 3$,*

$$\lim_{n\to\infty} \frac{R_t(P_n)}{n} \;=\; t-1.$$

Closing the gap between Theorems 18 and 19 would either: require pushing the connected-matching method further (perhaps with refined absorption), or discovering new constructions that force larger thresholds.

7.6 Summary for multi colors

- **Easy upper bound:** $R_t(P_n) \le t(n-2)+2$ by averaging + Erdős–Gallai.
- **Best known upper bound:** $R_t(P_n) \le (t - \frac{1}{2} + o(1))n$.
- **Lower bound:** $R_t(P_n) \ge (t-1+o(1))n$ via (pseudo)random fragmented colorings.
- **Open:** Determine the exact asymptotic constant; conjecturally $t-1$ for all $t \ge 3$.

8 Concluding Remarks and Open Problems

This article established the exact two-colour formula for paths, presented the three-colour picture (including the Faudree–Schelp conjecture and its asymptotic resolution), and surveyed the multicolour bounds with best-known constants. A recurring theme is that *linear* growth persists despite additional colours: structure (maximal paths, parity constraints, connected matchings) counters the combinatorial explosion one might expect.

- Two Colors: The Gerencsér–Gyárfás formula

  $$R(P_n, P_m) = m + \left\lfloor \frac{n}{2} \right\rfloor - 1$$

  is tight, with an extremal split colouring that is essentially unique at the threshold (Section 5). The maximal-path method is robust and extends to related problems (e.g. even cycles with minor adjustments).

- Three Colors: The Faudree–Schelp formula

  $$R_3(P_n) = \begin{cases} 2n-2, & n \text{ even}, \\ 2n-1, & n \text{ odd}, \end{cases}$$

  holds for all sufficiently large n. Small cases match the formula where verified; no counterexamples are known.

- General t: For fixed $t \ge 2$,

  $$(t-1+o(1))\,n \;\le\; R_t(P_n) \;\le\; \left(t - \tfrac{1}{2} + o(1)\right) n.$$

  The upper bound relies on regularity, connected matchings and lifting; the lower bound follows from fragmented/pseudorandom colourings that keep all monochromatic components small.

Open problems

1. **Exact $R_3(P_n)$ for all n.** Complete the proof of the Faudree–Schelp formula for the remaining finite cases.

2. **Asymptotic constant for t colours.** Determine $\lim_{n\to\infty} R_t(P_n)/n$ for each fixed t. Is it $t-1$ for all $t \geq 3$?

3. **Improve the upper constant.** Push $(t-\frac{1}{2})$ closer to $(t-1)$ uniformly in t, or show new barriers for connected-matching methods.

4. **Finite-n bounds.** Obtain explicit (effective) versions of the $o(1)$ terms with reasonable rates in n.

5. **Unequal multicolour paths.** Sharpen bounds for $R(P_{n_1}, \ldots, P_{n_t})$ with unbalanced lengths.

6. **Directed and oriented paths.** Extend techniques to tournaments and oriented complete graphs.

7. **Even cycles and trees.** Parallel improvements for C_{2k} and for bounded-degree trees under t colours.

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