Homework 6

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December 1, 2019

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a) $a_n = a_{n-1}$

b) $a_{n+1} = 2(n+1) = 2n + 2 = a_n + 2$
$\Rightarrow a_{n+1} = a_n + 2$

c) $a_{n+1} = 2(n+1) + 3 = 2n + 3 + 2 = a_n + 2$
$\Rightarrow a_{n+1} = a_n + 2$

d) $a_{n+1} = 5^{n+1} = 5 \cdot 5^n = 5a_n$
$\Rightarrow a_{n+1} = 5a_n$

e) $a_{n+1} = (n+1)^2 = n^2 + 2n + 1 = a_n + 2\sqrt{a_n} + 1$

f) $a_{n+1} = (n+1)^2 + n + 1 = n^2 + n + 2n + 2 = a_n + 2n + 2$
$\Rightarrow a_{n+1} = a_n + 2n + 2$

g) $a_{n+1} = n + 1 + (-1)^{n+1} = n + 1 + (-1)^n \cdot (-1) = n - (-1)^n + 1 = n + (-1)^n - 2(-1)^n + 1 = a_n - 2(-1)^n + 1$
$\Rightarrow a_{n+1} = a_n - 2(-1)^n + 1$

h) $a_{n+1} = (n+1)! = (n+1) \cdot n! = (n+1) \cdot a_n$
$\Rightarrow a_{n+1} = (n+1) \cdot a_n$

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a)

$$\begin{aligned} a_n &= -a_{n-1} \\ &= (-1)^2 a_{n-2} \\ &= \dots \\ &= (-1)^n a_{n-n} \\ &= (-1)^n a_0 \\ &= (-1)^n \cdot 5 \\ \Rightarrow a_n &= (-1)^n \cdot 5 \end{aligned}$$

b)

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= a_{n-2} + 3 \cdot 2 \\ &= a_{n-3} + 3 \cdot 3 \\ &= \dots \\ &= a_{n-n} + 3 \cdot n \\ &= 3n + a_0 \\ &= 3n + 1 \\ \Rightarrow a_n &= 3n + 1 \end{aligned}$$

c)

$$\begin{aligned}
a_n &= a_{n-1} - n \\
&= a_{n-2} - (n-1) - n \\
&= a_{n-3} - (n-2) - (n-1) - n \\
&= \dots \\
&= a_{n-n} - (n + (n-1) + (n-2) \\
&+ (n-3) + (n-(n-1))) \\
&= a_0 - \frac{n(n+1)}{2} \\
\Rightarrow a_n &= \frac{-n^2 - n + 8}{2}
\end{aligned}$$

d)

$$\begin{aligned}
a_n &= 2a_{n-1} - 3 \\
&= 2(2a_{n-2} - 3) - 3 \\
&= 4a_{n-2} - 3 \cdot 3 \\
&= 4(2a_{n-3} - 3) - 3 \cdot 3 \\
&= 8a_{n-3} - 7 \cdot 3 \\
&= \dots \\
&= 2^n \cdot a_{n-n} - (2^n - 1) \cdot 3 \\
&= 2^n a_0 - (2^n - 1) \cdot 3 \\
\Rightarrow a_n &= -2^n - (2^n - 1) \cdot 3 \\
&= -4 \cdot 2^n + 3 \\
\Rightarrow a_n &= -2^{n+2} + 3
\end{aligned}$$

e)

$$\begin{aligned}
a_n &= (n+1)a_{n-1} \\
&= (n+1)na_{n-2} \\
&= (n+1)n(n-1)a_{n-3} \\
&= (n+1)n(n-1)(n-2)a_{n-4} \\
&= \dots \\
&= (n+1)n \dots (n-(n-2))a_{n-n} \\
&= (n+1)n \dots 2a_0 \\
&= 2(n+1)!
\end{aligned}$$

f)

$$\begin{aligned}
a_n &= 2na_{n-1} \\
&= 2n2(n-1)a_{n-2} \\
&= 2n2(n-1)2(n-2)a_{n-3} \\
&= 2n2(n-1)2(n-2)2(n-3)a_{n-4} \\
&= \dots \\
&= 2^n n \dots (n-(n-1))a_{n-n} \\
&= 2^n n! \cdot 3
\end{aligned}$$

g)

$$\begin{aligned}
a_n &= n - 1 - a_{n-1} \\
&= n - 1 - (n - 2 - a_{n-2}) \\
&= n - 1 - (n-2) + (n - 3 - a_{n-3}) \\
&= \dots \\
&= (n-1) - (n-2) + \dots \\
&+ (-1)^{n-1}(n-n) + (-1)^n a_{n-n} \\
&= -1 + 2 - 3 + \dots + (-1)^{n-1} n + (-1)^n 7 \\
&= \frac{2n - 1 + (-1)^n}{4} + (-1)^n \cdot 7
\end{aligned}$$

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In 2002: $P_0 = 6.2$ Billion.
$r = 1.3\%/year = 0.013/year$
At year n after 2002:

a)

$$\begin{aligned}
P_1 &= P_0 + P_0 \cdot r = (1.013)P_0 \\
P_2 &= P_1 + P_1 \cdot r = (1.013)P_1 \\
\Rightarrow P_n &= (1.013)P_{n-1}
\end{aligned}$$

b)

$$\begin{aligned}
P_n &= 1.013P_{n-1} \\
&= 1.013^2 P_{n-2} \\
&= 1.013^3 P_{n-3} \\
&= \dots \\
\Rightarrow P_n &= (1.013)^n P_0 \\
\Rightarrow P_n &= 1.013^n \cdot 6.2\,(Billion)
\end{aligned}$$

c) In 2022, $n = 2022 - 2002 = 20$
$\Rightarrow P_{20} = 1.013^{20} \cdot 6.2 = 8.0275$ (Billion)
The world population will be approximately 8.0275 billion people.

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a) In 1999, $S_0 = \$50000$
A year late, $S_1 = S_0 \cdot (1 + 0.05) + \$1000$
A year late, $S_2 = S_1 \cdot (1 + 0.05) + \$1000$
Then, $S_n = S_{n-1} \cdot (1.05) + \$1000$

b) In 2007, $n = 2007 - 1999 = 8$
Using result from c):
$S_8 = 1000\frac{(1.05^8-1)}{1.05-1} + 50000(1.05)^8 = 83422$
So his salary then will be about \$83422

c)

$$
\begin{aligned}
S_n &= S_{n-1} \cdot (1.05) + 1000 \\
&= (S_{n-2} \cdot (1.05) + 1000) \cdot (1.05) + 1000 \\
&= S_{n-2} \cdot (1.05)^2 + 1000 \cdot (1.05 + 1) \\
&= (S_{n-3} \cdot (1.05) + 1000) \cdot (1.05)^2 \\
&+ 1000 \cdot (1.05 + 1) \\
&= S_{n-3} \cdot (1.05)^3 + 1000 \cdot (1.05^2 + 1.05 + 1) \\
&= \dots \\
&= S_{n-n} \cdot (1.05)^n + 1000 \cdot (1.05^{n-1} + 1.05^{n-2} \\
&+ \dots + 1.05^1 + 1.05^0) \\
&= 50000(1.05)^n + 1000 \cdot (1.05^{n-1} + 1.05^{n-2} \\
&+ \dots + 1.05^1 + 1.05^0) \\
&= 1000\frac{(1.05^n - 1)}{1.05 - 1} + 50000(1.05)^n \\
&= 70000(1.05)^n - 20000
\end{aligned}
$$

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Let a_n be the number of messages transmitted in n μs.
One signal need 1 μs : a_{n-1}
One signal need 2 μs : a_{n-2}
One signal need 2 μs : a_{n-2}

For $n \geq 2 : a_n = a_{n-1} + 2a_{n-2}$

In 0 μs, only empty message can be sent.

$$\Rightarrow a_0 = 1$$

In 1 μs, only 1 message can be sent.

$$\Rightarrow a_1 = 1$$

Using characteristic equation:
Let $a_n = r^2, a_{n-1} = r, an - 2 = 1$
$\Rightarrow r = 2$ or $r = -1$
Solution recurrence relation:
$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

From initial conditions, we have:

$$1 = a_0 = \alpha_1 + \alpha_2$$
$$1 = a_1 = 2\alpha_1 - \alpha_2$$

Then:

$$\alpha_1 = \frac{2}{3}$$
$$\alpha_2 = \frac{1}{3}$$

The solution to the recurrence relation is:

$$a_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n$$

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a) The recurrence relation is simply:

$$L_n = \frac{1}{2}(L_{n-1} + L_{n-2})$$

b) The characteristic equation:

$$r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \Rightarrow r = -\frac{1}{2} \text{ or } r = 1$$

The general solution is:

$$a_n = \left(-\frac{1}{2}\right)^n \alpha_1 + 1^n \alpha_2$$

We have:

$$L_1 = 100000 = \frac{1}{2}\alpha_1 + \alpha_2$$

$$L_2 = 300000 = \left(\frac{1}{2}\right)^3 \alpha_1 + \alpha_2$$

Then:

$$\left\{\alpha_1 = \frac{8000000}{3} \text{ and } \alpha_2 = \frac{700000}{3}\right\}$$

Then:

$$L_n = \left(\frac{800000}{3}\right)\left(\frac{-1}{2}\right)^n + \left(\frac{700000}{3}\right)$$

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If r_0 is the only root of $r^2 - c_1 r - c_2 = 0$
Then: $r^2 - c_1 r - c_2 = (r - r_0)^2 = 0$
$\Leftrightarrow r^2 - c_1 r - c_2 = r^2 - 2rr_0 + r_0^2$
$\Leftrightarrow c_2 = -r_0^2$ and $c_1 = 2r_0$
Prove the solution:

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= 2r_0 a_{n-1} - r_0^2 a_{n-2} \\ &= 2r_0(\alpha_1 r_0^{n-1} + \alpha_2 n r_0^{n-1}) \\ &- r_0^2(\alpha_1 r_0^{n-2} + \alpha_2 n r_0^{n-2}) \\ &= \alpha_1 r_0^n + \alpha_2 n r_0^n \\ &= a_n \end{aligned}$$

Thus the sequence a_n is a solution of the recurrence relation.

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We have for $n \geq 3$

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

The characteristic equation is:

$$r^3 - 2r^2 - r + 2 = 0 \Rightarrow r = -1; r = 1; r = 2$$

Then the solution is:

$$a_n = (-1)^n \alpha_1 + \alpha_2 + 2^n \alpha_3$$

We also have:

$$a_0 = 3 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 6 = -\alpha_1 + \alpha_2 + 2\alpha_3$$

$$a_2 = 0 = \alpha_1 + \alpha_2 + 4\alpha_3$$

Then: $\alpha_0 = -2; \alpha_1 = 6; \alpha_2 = -1$ The solution is:

$$a_n = -2 \cdot (-1)^n + 6 - 2^n$$

HanoiTower.cpp

```
#include <chrono>
#include <iomanip>
#include <iostream>
using namespace std;

int count = 0;

void move(int num_of_disks, int from, int mid, int to) {
  if (num_of_disks > 0) {
    move((num_of_disks - 1), from, to, mid);
    count++;
```

```
    cout << setw(4) << count << ". Move a disk from peg " << from << " to peg "
         << to << "\n";
    move((num_of_disks - 1), mid, from, to);
  }
}

int main() {
  cout << "Input number of disks: " << endl;
  int n;
  cin >> n;

  auto start = std::chrono::high_resolution_clock::now();
  move(n, 1, 2, 3);
  auto stop = std::chrono::high_resolution_clock::now();

  auto duration =
      std::chrono::duration_cast<std::chrono::microseconds>(stop - start);

  cout << endl
       << "It takes " << count << " moves in " << duration.count()
       << " microseconds." << endl;
  return 0;
}
```