Small-scale variation of convected quantities like temperature in turbulent fluid

Part 1. General discussion and the case of small conductivity

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When some external agency imposes on a fluid large-scale variations of some dynamically passive, conserved, scalar quantity θ like temperature or concentration of solute, turbulent motion of the fluid generates small-scale variations of θ. This paper describes a theoretical investigation of the form of the spectrum of θ at large wave-numbers, taking into account the two effects of convection with the fluid and molecular diffusion with diffusivity κ. Hypotheses of the kind made by Kolmogoroff for the small-scale variations of velocity in a turbulent motion at high Reynolds number are assumed to apply also to small-scale variations of θ.

Previous contributions to the problem are reviewed. These have established that the spectrum of θ varies as $n^{-\frac{5}{3}}$ (n being wave-number) at the low wave-number end of the equilibrium range, but there has been some confusion about the wave-number marking the upper end of the range of validity of this relation. The existence of a conduction 'cut-off' near $n = (\epsilon/\kappa^3)^{\frac{1}{4}}$ as put forward by Obukhoff and Corrsin is shown to hold only when $\nu \ll \kappa$, and that near $n = (\epsilon/\nu\kappa^2)^{\frac{1}{4}}$ put forward by Batchelor is shown to apply only when $\nu \gg \kappa$. In the case $\nu \ll \kappa$, the remaining problem is to determine the form of the spectrum of θ beyond the conduction cut-off; this is done in Part 2. In the case $\nu \gg \kappa$, the conduction cut-off occurs at wave-numbers much higher than $(\epsilon/\nu^3)^{\frac{1}{4}}$, which is where the energy spectrum is cut off by viscosity, and where the spectrum of θ ceases to vary as $n^{-\frac{5}{3}}$.

The form of the spectrum of θ in this latter case is determined over the range $n > (\epsilon/\nu^3)^{\frac{1}{4}}$ by analysing the effect of the velocity field, regarded as effectively a persistent uniform straining motion for these small-scale variations of θ, and of molecular diffusion on a single Fourier component of θ. The wave-number of this sinusoidal variation of θ is changed (and generally increased in magnitude) by the straining motion and the amplitude is diminished by diffusion. By supposing that the level of the spectrum of θ is kept steady at wave-numbers near $(\epsilon/\nu^3)^{\frac{1}{4}}$ by some mechanism of transfer from lower wave-numbers, the linearity of the equation for θ then allows the determination of the spectrum for $n > (\epsilon/\nu^3)^{\frac{1}{4}}$, the result being given by (4.8). The same result is obtained, using essentially the same approximation about the velocity field, from a different kind of analysis in terms of velocity and θ correlations. Finally, the relation between this work and Townsend's model of the small-scale variations of vorticity in a turbulent fluid is discussed.

1. Introduction

When the temperature of a fluid in turbulent motion is not uniform (although with such small variations that buoyancy forces are negligible), the temperature field is made random by the irregular movements of the fluid and acquires statistical properties which are directly related to those of the turbulent motion. Quite apart from its intrinsic interest as an aspect of a general study of turbulence, the distribution of temperature, and of other similarly conserved scalar physical quantities, in a turbulent fluid has a direct bearing on a number of problems in geophysics, mostly concerned with the scattering of either sound or electromagnetic waves. In these scattering problems it often happens, for extraneous reasons concerned with the choice of wavelength of the waves concerned, that the small-scale structure of the distribution of temperature (or whatever quantity is responsible for the variations in refractive index) is of particular interest. This is a fortunate circumstance, because these small-scale components may be expected, on the basis of arguments parallel to those used in Kolmogoroff's theory, to have a measure of universality and to have statistical properties which depend only weakly on the large-scale features of the distribution. It is probable that considerations of the fine structure of the distribution of quantities like temperature also have a bearing on industrial problems concerned with the mixing of one fluid in another of approximately the same density by means of turbulent motion. This paper will give a theoretical discussion of the small-scale components of quantities like temperature, 'small' being taken here to mean that the components concerned have characteristic length-scales small compared with the length-scale of the eddies containing the bulk of the kinetic energy of the turbulent motion, without regard for any of the possible applications of the results.

The dominant feature of the action of the turbulent motion on the temperature distribution is a continual reduction of the length-scale of temperature variations. The random convection of material elements of the fluid is inevitably accompanied by distortion of these elements, and, in the absence of molecular conduction, a (statistical) increase in the gradients of temperature. This process was described clearly by Obukhoff (1949), and has been analysed in terms of the way in which surfaces of constant temperature are increased in area and brought closer together (Batchelor 1952). Unless temperature variations on some definite length-scale are supplied continually by some external agency, the statistical properties of the temperature distribution cannot be exactly steady; however, the properties of the small-scale components of the temperature distribution will be approximately steady in general, because the process of convective distortion and increase of temperature gradients takes place much more quickly than the over-all decay of the temperature field. The continual increase in the magnitude of temperature gradients due to random convection will ultimately be checked by the smoothing action of thermal conduction, and no further refinement of the temperature distribution can occur; in this way, a length-scale characterizing the smallest temperature 'eddies' is determined.

There are only two properties of the quantity temperature which are relevant to these mechanical processes. One is the property of invariance of the temperature of a material element of fluid in the absence of molecular conduction; the other is that the temperature is subject to molecular conduction characterized by a diffusivity κ (of dimensions velocity by length). In these respects, temperature is no different from many other physical quantities, such as water-vapour content of air, salt concentration of water, and electron density in the ionosphere, although the numerical values of κ will be different in each case. To the extent that they do not depend on special values of κ, all the remarks and results of this paper will apply to all such dynamically passive conserved scalar quantities, the terminology associated with the typical case of temperature being employed sometimes for conciseness.

The problem to be studied is thus as follows. A quantity $\theta(x, t)$ has a distribution in the fluid which is governed by the equation

$$\frac{\partial\theta}{\partial t}+\mathbf{u}\,.\,\nabla\theta = \kappa\nabla^2\theta, \tag{1.1}$$

where $\mathbf{u}$ is the velocity of the fluid and is independent of θ. The fluid, taken as incompressible, is in turbulent motion at high Reynolds number, and the length-scale characterizing the energy-containing eddies is L. We wish to determine the statistical properties of those components of the spatial distribution of θ that have a length-scale small compared with L, and in particular to determine the wave-number spectrum of the distribution of θ. This is a familiar kind of objective in the theory of turbulence, although not one which is often achieved convincingly. It will be shown that a reasonably complete description of the spectrum of θ can be obtained, partly by the use of simple new ideas and partly by the use of old ideas in a new context, this success being made possible by the linearity in θ of all terms in (1.1). No measurements appear to be available for comparison with the theoretical results.

2. Previous work on the problem

The few published contributions that bear directly on the problem need to be described before the new work is presented. The first relevant papers seem to be those in which Obukhoff (1949; see also Yaglom 1949) and Corrsin (1951) pointed out the primary features of the effect of random convection on the spatial distribution of θ. These authors realized that the general increase of gradients of θ accompanying the irregular stirring action of the turbulence, which is a consequence of the quadratic term in (1.1), can also be thought of as a transfer between different Fourier components of the distribution of θ. If both $\mathbf{u}$ and θ are written in the form of Fourier integrals, the term $\mathbf{u}\,.\,\nabla\theta$ leads to the generation of new harmonic components of θ, and in particular to the growth of components of ever-increasing wave-number. This effective transfer from Fourier components of the θ-distribution at low wave-number to those at high wave-number is mathematically similar to that which acts on the turbulent velocity distribution, and Obukhoff and Corrsin made the plausible assumption (which will also be adopted here) that the hypotheses of Kolmogoroff's universal equilibrium theory apply equally well to the θ-distribution as to the $\mathbf{u}$-distribution. The arguments in favour of this extension of Kolmogoroff's hypotheses to apply to a temperature distribution need not be given, since they are identical in form with, and neither stronger nor weaker in rigour than, those well-known arguments concerning the velocity distribution.

On the basis of these hypotheses, and with the assumptions that the Reynolds number of the turbulence is sufficiently high and that there are no external sources of variations of θ on a small length-scale, the statistical properties of the small-scale components of the θ-distribution are homogeneous, isotropic and steady, irrespective of the properties of the components with length-scale of the order of L. Moreover, owing to the dependence of the distribution of θ on that of $\mathbf{u}$, the components of the θ-distribution having these properties will be defined by the same condition as the components of the $\mathbf{u}$-distribution having the same properties, namely, that their linear size is small compared with L; the two distributions have a common 'equilibrium range' of wave-numbers. It is thus possible to define a spectrum function for the (small-scale components of the) spatial distribution of θ as

$$\Delta(\mathbf{n}) = \frac{1}{8\pi^3}\int S(\mathbf{r})\, e^{-i\mathbf{n}\cdot\mathbf{r}}\, d\mathbf{r}, \tag{2.1}$$

where $\mathbf{n}$ is the vector wave-number and $S(\mathbf{r}) = \overline{\theta\theta'} = \overline{\theta(\mathbf{x})\,\theta(\mathbf{x}+\mathbf{r})}$ is the covariance of θ as a function of position ($\overline{\theta}$ being taken as zero for convenience). In view of the isotropy, Δ and S are functions of the magnitudes n and r alone, and the spectrum function giving the distribution with respect to wave-number magnitude is

$$\Gamma(n) = 4\pi n^2 \Delta(n) = \frac{2n}{\pi}\int_0^\infty S(r)\, r \sin nr\, dr; \tag{2.2}$$

it follows from the transform of (2.1) that

$$\overline{\theta^2} = \int_0^\infty \Gamma(n)\, dn.$$

The statistical properties of the small-scale components of the θ-distribution are also independent of the detailed form of the properties of the large-scale components, according to the usual hypotheses,* and are affected by these large-scale components only inasmuch as the latter determine the magnitude of the rate of transfer from large-scale to small-scale components. In order to see exactly what is being transferred between different parts of the θ-spectrum, it is necessary to note only that the quadratic term in (1.1) makes no contribution to $\partial\overline{\theta^2}/\partial t$. Thus, when one Fourier component of the θ-distribution is changed by interaction between the θ and $\mathbf{u}$ fields, other Fourier components are changed simultaneously in such a way that the sum of the (necessarily independent) contributions to $\overline{\theta^2}$ from all Fourier components remains the same. This shows that what is transferred across the θ-spectrum, and conserved while being transferred, when the turbulent motion distorts the temperature distribution, is a contribution to $\overline{\theta^2}$ from Fourier components; for lack of a suitable word, let us call it $\overline{\theta^2}$-stuff.

Following the usual line of argument of the Kolmogoroff theory, we now suppose that the diffusivity κ is so small as to make the effect of conduction

* Which, however, are now challenged by Kraichnan (1958, 1959).

negligible for some of the Fourier components (namely, those at the small wave-number end of the range) comprising the group whose statistical properties are steady and isotropic. The part of the equilibrium range of wave-numbers for which the Fourier components of the **u**-distribution are independent of viscosity is usually termed the 'inertial subrange', and an appropriate term for the part of the equilibrium range for which the Fourier components of the θ-distribution are independent of molecular diffusion is the 'convection subrange'. No actual destruction of $\overline{\theta^2}$-stuff takes place at wave-numbers in, or smaller than those in, the convection subrange; all the destruction takes place at higher wave-numbers as a result of the action of molecular diffusion, the total rate of destruction of the $\overline{\theta^2}$-stuff in unit volume of fluid being

$$2\kappa\overline{\theta\nabla^2\theta} = -2\kappa\overline{(\nabla\theta)^2} = -\chi, \quad \text{say.} \tag{2.3}$$

We assume that, as in the case of the dissipation of kinetic energy by viscosity, the value of χ is determined by statistical interaction of the Fourier components of **u** and θ each with wave-numbers of order L^{-1}, and is a 'given' quantity so far as considerations of the small-scale components alone are concerned.

Thus the mean rate at which $\overline{\theta^2}$-stuff is transferred from wave-numbers smaller than, to wave-numbers larger than, any wave-number in the convection subrange, per unit volume of the fluid, is χ, and this is one of the parameters on which the form of the θ-spectrum in the convection subrange depends. The only other parameters on which the θ-spectrum in the convection subrange can depend are those which determine the Fourier components of the velocity distribution in the equilibrium range of wave-numbers, that is, the total rate of viscous dissipation of kinetic energy per unit mass of fluid, ϵ, and the kinematic viscosity ν. Provided the Reynolds number of the turbulence is so large that viscous effects, as well as conduction effects, are unimportant at at least some of the wave-numbers in the convection subrange, dimensional requirements lead to

$$\Gamma(n) \propto \chi\epsilon^{-\frac{1}{3}}n^{-\frac{5}{3}} \tag{2.4}$$

for $n \gg L^{-1}$ and n less than some upper limit yet to be determined. Thus the θ-spectrum here has the same dependence on n as the **u**-spectrum in the inertial subrange, namely,

$$E(n) \propto \epsilon^{\frac{2}{3}}n^{-\frac{5}{3}}, \tag{2.5}$$

as was demonstrated by Obukhoff (1949) and independently by Corrsin (1951).

A relation different from (2.4) has been put forward by Inoue (1950, 1951, 1952). Inoue appears to have thought of the various components, of different linear dimensions, into which the temperature distribution is resolved as being physical entities, and of the conserved quantity which is transferred between different components as a result of the stirring motion of the fluid as being heat—or, equivalently, since the fluid is of uniform heat capacity per unit volume, temperature. A dimensional argument like that leading to (2.4), but using a quantity specifying a rate of heat transfer in place of χ, then gives $\Gamma(n)$ as proportional to $n^{-\frac{7}{3}}$. This argument of Inoue's does not seem to be sound; the 'components' of a random spatial field are not entities relating to different portions of the fluid but are (or, at any rate, should be, if the concept is to be self-consistent) different members of an appropriate set of orthogonal functions. In his discussion of a temperature field Obukhoff (1949) pointed out that, for small fluctuations about the mean, $\overline{\theta^2}$ is a measure of the deficiency of entropy of the fluid relative to a state in which the temperature is uniform with the same mean value, and that this supplies a physical interpretation of the $\overline{\theta^2}$-stuff that is transferred across the spectrum. However, all that is strictly relevant to the above argument is that the value of θ for a material particle is unchanged by convection and that the different Fourier components of the distribution of θ make independent contributions to $\overline{\theta^2}$.

The papers by Obukhoff, Corrsin and Inoue, already referred to, appear to be the only published works on the form of the θ-spectrum in the equilibrium range of wave-numbers, apart from a rather unplausible suggestion by Villars & Weisskopf (1955) that $[\overline{(\theta-\theta')^2}]^{\frac{1}{2}}$ is proportional to r and to the gradient of $\overline{\theta}$ when conduction is unimportant. (The simple mixing process on which this suggestion is based is sound enough in itself, but I think the authors have overlooked the fact that the part of the fluctuating gradient of θ due to components with length-scale larger than r is much larger than the mean gradient of θ, and that the magnitude of this fluctuating gradient determines $\overline{(\theta-\theta')^2}$.) Obukhoff and Inoue gave expressions for the θ-spectrum only in the convection subrange. Corrsin obtained (2.4), like Obukhoff, and noted further that if one supposes the transfer across both the θ-spectrum and the $\mathbf{u}$-spectrum to be representable mathematically as an eddy diffusion process in the manner suggested by von Weizsäcker and Heisenberg, with the smaller-scale components of $\mathbf{u}$ acting as the transfer agent for the larger-scale components of both θ and $\mathbf{u}$, then both the θ-spectrum function $\Gamma(n)$ and the $\mathbf{u}$-spectrum function $E(n)$ are proportional to n^{-7} at large wave-numbers beyond both the viscous and conduction cut-off wave-numbers. Since Corrsin's paper was published, the von Weizsäcker–Heisenberg hypothesis has come to be regarded, on both deductive and empirical grounds, as of doubtful value for predictions about the $\mathbf{u}$-spectrum over the part of the equilibrium range for which viscous effects are important, and consequently the above prediction about the θ-spectrum is not now convincing.

The position is thus that the form of the θ-spectrum in that part of the equilibrium range in which neither viscous forces nor conduction effects are important is reasonably well established (assuming, as is done here, that the Kolmogoroff theory in general is reasonably well established), but that the form at larger wave-numbers is not known.

The range of validity of (2.4)

Obukhoff (1949) and Corrsin (1951) have also said something about the magnitude of the wave-number marking the upper end of the range of validity of the relation (2.4) for the θ-spectrum. Obukhoff remarked that the relation (2.4), for the stated range of wave-numbers, is equivalent to a relation

$$\overline{(\theta-\theta')^2} \propto \chi\epsilon^{-\frac{1}{3}}r^{\frac{2}{3}}, \tag{2.6}$$

valid for $r \ll L$ and for r bounded below in some way as yet unknown, and that when r is sufficiently close to zero there is available the exact relation (see (2.3))

$$\overline{(\theta-\theta')^2} = \frac{1}{6}\frac{\chi}{\kappa}r^2. \tag{2.7}$$

Equations (2.7) and (2.6) are asymptotic relations valid for 'small' and 'large' values of r respectively (the 'large' values being subject to the restriction $r \ll L$). Obukhoff argued that the two ranges of r concerned will be contiguous, in which case the dividing value of r will be given approximately by the solution of

$$\frac{1}{6}\frac{\chi}{\kappa}r^2 = \chi\epsilon^{-\frac{1}{3}}r^{\frac{2}{3}}$$

(the constant of proportionality in (2.6) being assumed as usual to be of order unity). Thus Obukhoff's conclusion is that (2.6) is valid for $(\kappa^3/\epsilon)^{\frac{1}{4}} \ll r \ll L$, or, equivalently, that the relation (2.4) for the θ-spectrum is valid for

$$L^{-1} \ll n \ll (\epsilon/\kappa^3)^{\frac{1}{4}}. \tag{2.8}$$

Corrsin's argument is apparently different in form, but leads to the same conclusion. He supposed that the relation (2.4) ceases to be valid when n is so large that the effect of molecular diffusion becomes important, and that this will happen when the Péclet number appropriate to the Fourier components of the θ-distribution with wave-number n becomes of order unity. The Péclet number in general is a measure of the ratio of convection to conduction effects, and Corrsin assumes this measure to be given by $E^{\frac{1}{2}}/(n^{\frac{1}{2}}\kappa)$, with the $\mathbf{u}$-spectrum function $E(n)$ having the inertial subrange form (2.5). The conclusion is that conduction effects render (2.4) invalid when n is of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$, in agreement with Obukhoff.

The inertial subrange, within which the relation (2.5) for the $\mathbf{u}$-spectrum holds, is known to be specified by $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$. Obukhoff and Corrsin thus hold that the variations of the θ and $\mathbf{u}$-spectra as $n^{-\frac{5}{3}}$ are cut off, in the sense that at higher wave-numbers the two spectra begin to fall off more rapidly as a result of molecular diffusion being important, at wave-numbers whose ratio is given, to order of magnitude, by $(\nu/\kappa)^{\frac{3}{4}}$. The two arguments from which this result was found seem acceptable when ν and κ are of the same order of magnitude, or when $\nu \ll \kappa$, but I do not think they can be expected to hold when $\nu \gg \kappa$. When $\nu \gg \kappa$, the effect of viscosity on the $\mathbf{u}$-spectrum becomes important at wave-numbers lower than those at which conduction first affects the θ-spectrum, and so the θ-spectrum is dominated by convection processes at wave-numbers up to and beyond the largest wave-number for which (2.5) is valid. There is a limitation in Obukhoff's argument inasmuch as he assumes that (2.4) is valid for *all* wave-numbers less than that at which conduction becomes important (expressed as contiguity of the ranges for which (2.6) and (2.7) are valid); this is not likely to be a valid assumption when $\nu \gg \kappa$, because the neglect of ν in the dimensional argument on which (2.4) was based is not then permissible for the higher wave-numbers in this range. Corrsin's argument is limited in effectively the same way, in that he employs (2.5) to evaluate the Péclet number appropriate to Fourier components for wave-numbers which, when $\nu \gg \kappa$, are so large as to be beyond the range for which the inertial subrange relation (2.5) is valid.

In another paper which is relevant in this connexion (Batchelor 1952), the effect of molecular diffusion on the distribution of θ was regarded as a kind of perturbation of the effect of convection. When κ is zero, the surfaces of constant θ move as material surfaces and convective extension of these surfaces, with consequent decrease of their distance apart, leads to an increase in $\overline{(\nabla\theta)^2}$ at an (asymptotic) rate of order $\overline{(\boldsymbol{\omega}^2)}^{\frac{1}{2}}\,\overline{(\nabla\theta)^2}$ (where $\overline{(\boldsymbol{\omega}^2)}^{\frac{1}{2}}$ is the root-mean-square vorticity and is a measure of the mean rate of extension of material lines). If now a small molecular diffusivity is introduced, and if the effect of convection is not changed in form by the existence of the conduction, the value of $\overline{(\nabla\theta)^2}$ can be stationary only if the two terms in the expression for the rate of change of $\overline{(\nabla\theta)^2}$ (obtained in the usual way from (1.1)), namely,

$$\overline{(\boldsymbol{\omega}^2)}^{\frac{1}{2}}\,\overline{(\nabla\theta)^2} \quad \text{and} \quad \kappa\overline{(\nabla\theta).\nabla(\nabla^2\theta)},$$

are of the same order. On assuming that the values of these two weighted integrals of the θ-spectrum are determined by the wave-number at which the spectrum begins to fall off very rapidly as a result of conduction effects, we find that this wave-number must be of order $\overline{(\boldsymbol{\omega}^2)}^{\frac{1}{4}}/\kappa^{\frac{1}{2}}$, that is, of order $(\epsilon/\nu\kappa^2)^{\frac{1}{4}}$.

There is an apparent disagreement between this result and that obtained by Obukhoff and Corrsin, and at the time when my own paper was written I thought the conflict was real. However, I see now*—and the work to be described in the following sections will amplify the explanation—that each result is correct in its own context. Obukhoff and Corrsin found that conduction effects cut off the θ-spectrum at a wave-number of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$; in finding this expression for the cut-off wave-number they assumed that one or other of the relations (2.4) and (2.5) is valid up to this cut-off wave-number, and, as shown above, this is likely to be permissible only when $\nu \ll \kappa$ or when ν/κ is of order unity. My work, on the other hand, assumes that the mechanics of the convection process is not changed by the existence of conduction and, in particular, that the distance between surfaces of constant θ is decreased by convection at a rate which is of the same order as that for material surfaces; this will be valid, when applied to a consideration of particular Fourier components of the θ-distribution, only when the wave-numbers concerned lie beyond the range in which the stretching effect of the velocity field lies, that is, only when the conduction cut-off of the θ-spectrum lies well beyond the viscous cut-off of the $\mathbf{u}$-spectrum, that is, finally, only when $\kappa \ll \nu$. The two different expressions for the wave-number at which the conduction cut-off occurs reduce to the same form when ν/κ is of order unity.

The position, as now seen after this discussion of previous work, is briefly as follows. With the usual proviso that the Reynolds number of the turbulence is large, it may be expected that the θ-spectrum has a form depending only on ϵ, χ, ν and κ in the equilibrium range specified by the condition $n \gg L^{-1}$. Provided κ and ν are both so small that some of the Fourier components of θ and $\mathbf{u}$ in this range are unaffected by conduction and viscous forces respectively, the

* Helped by some valuable discussions with Mr I. D. Howells.

θ-spectrum has the form (2.4) for $n \gg L^{-1}$ and n small compared with some wave-number which depends on the ratio ν/κ. When $\nu \ll \kappa$, the convection subrange is not as extensive as the inertial subrange and is specified by $L^{-1} \ll n \ll (\epsilon/\kappa^3)^{\frac{1}{4}}$. When ν/κ is of order unity, the convection and inertial subranges may be expected to be of comparable extent and to be specified by $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$. In both these cases the θ-spectrum falls off more rapidly than as $n^{-\frac{5}{3}}$ at the end of the convection subrange as a consequence of conduction becoming important. When $\nu \gg \kappa$, convection effects dominate the θ-spectrum at wave-numbers beyond the inertial subrange and conduction does not become important until wave-numbers of order $(\epsilon/\nu\kappa^2)^{\frac{1}{4}}$ are reached; thus in this case there are two distinct parts to the convection subrange. In the part at the lower wave-number end, in convection subrange A say, specified by $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$, the relations (2.5) and (2.4) hold; at the higher wave-number end, in convection subrange B specified by

$$(\epsilon/\nu^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu\kappa^2)^{\frac{1}{4}},$$

neither (2.5) nor (2.4) holds because viscosity has an effect on the **u**-spectrum.

Part of the purpose of this paper is to review past work, to clear up some real and apparent conflicts and to assess the conditions under which the available results may be expected to be valid; this has now been done. The other intention is to complete the picture just described by obtaining expressions for the θ-spectrum in ranges where none is available.

3. The cases $\nu \ll \kappa$ and ν/κ is of order unity

When $\nu \ll \kappa$, the θ-spectrum has the form (2.4) over the entire convection subrange, and begins to fall off more rapidly at wave-numbers near $(\epsilon/\kappa^3)^{\frac{1}{4}}$ as a result of the effect of conduction. The remaining problem here is to determine the shape of the θ-spectrum in the neighbourhood of, and beyond, the conduction cut-off wave-number $(\epsilon/\kappa^3)^{\frac{1}{4}}$, and in particular to ascertain whether the cut-off is sharp. For many purposes it will be sufficient to know merely that the θ-spectrum begins to fall off more rapidly than as $n^{-\frac{5}{3}}$ when n is of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$, but for other purposes (for example, the calculation of high-order integral moments of the θ-spectrum) more precise information about the spectrum will be useful. In Part 2 of this paper a specific mechanism for the effect of the velocity distribution on the Fourier components of θ is proposed, and from it the unknown form of the θ-spectrum at wave-numbers beyond the conduction cut-off is determined; no further reference to the case $\nu \ll \kappa$ will be made here.

When κ is not very different from ν, the convection and inertial subranges both terminate at wave-numbers near $(\epsilon/\nu^3)^{\frac{1}{4}}$, and it is known that the **u**-spectrum subsequently falls off sharply owing to the effect of viscosity. The exact forms of the two spectra in the neighbourhood of this cut-off wave-number are not known, and may be different. It is possible that some exact relation between the two spectra exists for the special case $\nu = \kappa$; however, it does not matter much if the precise form of $\Gamma(n)$ near $n = (\epsilon/\nu^3)^{\frac{1}{4}}$ is not known, since the coincidence of conduction and viscosity effects in this neighbourhood makes it virtually certain that the cut-off of the θ-spectrum is sharp (and, in all probability, sharp enough to make integral moments of $\Gamma(n)$ of all orders converge).

4. The case $\nu \gg \kappa$: Lagrangian analysis in terms of Fourier components

When $\nu \gg \kappa$, the convection subrange is more extensive than the inertial subrange, and, as explained in §2, consists of two distinct parts. In the part defined by $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$, the result (2.4) holds. In the part defined by $(\epsilon/\nu^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu\kappa^2)^{\frac{1}{4}}$, which is a range of wave-numbers lying beyond the viscous cut-off of the **u**-spectrum, the shape of the θ-spectrum is affected indirectly by viscosity, in a manner to be established here. It turns out to be possible to do more than this and to find the form of $\Gamma(n)$ for all $n \gg (\epsilon/\nu^3)^{\frac{1}{4}}$, that is, for a range embracing the convection subrange B and higher wave-numbers at which conduction effects are important.

The essential fact underlying the analysis that follows is that the spatial rate of change of the fluid velocity is approximately uniform over regions with linear dimensions not much smaller than $(\nu^3/\epsilon)^{\frac{1}{4}}$. In the inertial subrange the spectrum of $\partial u_i/\partial x_j$ rises slowly (as $n^{\frac{1}{3}}$), reaching a maximum near the wave-number $(\epsilon/\nu^3)^{\frac{1}{4}}$, and falls off sharply at higher wave-numbers, so that wave-numbers less than $(\epsilon/\nu^3)^{\frac{1}{4}}$ make a dominant contribution to the area under this spectrum curve. As more direct evidence, it can readily be calculated from the equation for the balance of mean-square vorticity, using the hypotheses of Kolmogoroff's theory and with an empirical value of about $-0{\cdot}3$ for the skewness factor of $\partial u_1/\partial x_1$, that

$$\overline{\left(\frac{\partial u_i'}{\partial x_j'} - \frac{\partial u_i}{\partial x_j}\right)^2} \Big/ \overline{\left(\frac{\partial u_k}{\partial x_l}\right)^2} \approx 0{\cdot}03 \frac{r^2}{(\nu^3/\epsilon)^{\frac{1}{2}}} \tag{4.1}$$

(the repeated indices on the left being summed) for small values of r; this shows that r does not need to be much less than $(\nu^3/\epsilon)^{\frac{1}{4}}$ for the velocity gradient to be approximately uniform over a region of linear dimensions r. If now we imagine a material element of fluid of linear dimensions somewhat less than $(\nu^3/\epsilon)^{\frac{1}{4}}$ to be chosen and followed in its motion, the internal distortion of this element, and of any distribution of θ contained in it, will be at any instant approximately a pure straining motion.

Theoretical work does not yet seem to have thrown any light on the degree of persistence of this straining motion, but Townsend (1951*b*) has made some valuable inferences from observations of the rate at which the temperature of small hot fluid elements decreases owing to the combined effect of convective distortion and conduction. He finds that the local straining motion is remarkably persistent, and that the time-scale of change of the principal rates of strain and of change of the directions, relative to the fluid, of the principal axes of the straining motion, is large compared with $(\nu/\epsilon)^{\frac{1}{2}}$ (largeness here presumably implying variation as some positive fractional power of the Reynolds number of the turbulence), this latter quantity being the only one with the dimensions of time which can be formed from the parameters determining the equilibrium range of the **u**-spectrum. That the principal axes of the rate of strain rotate only slowly relative to the fluid is also suggested by pictures of the position of portions of marked fluid at different instants, like those for two-dimensional motion published by Welander (1955); on the whole, the marked fluid is drawn out into long thin streaks, of ever-increasing length, which do not show the small-scale wriggles and rapid variations in curvature that would result from local rotation of the principal axes of rate of strain relative to the fluid and from consequent local rotation of a small part of the streak relative to the remainder. (It is possible that this persistence of the stretching of a material line is no more than a reflexion of the fact that material lines tend to set themselves in the direction of the greatest local principal rate of strain (Batchelor 1952), and that if the principal axis of greatest rate of strain should rotate relative to the fluid, the material line with which it coincided initially would automatically turn and tend to align itself with the new direction of the principal axis of greatest rate of strain.) The conclusion, to be adopted here, is that the effect of convection on the spatial distribution of θ within a material element of suitably small size is approximately the same as that of a pure straining motion of constant magnitude and form relative to the fluid, so far as temporal changes of the distribution of θ on a time scale of order $(\nu/\epsilon)^{\frac{1}{2}}$ are concerned.

This picture of the convective distortion of small elements of the fluid has been used by Townsend (1951*a*) in a theory of the form of the **u**-spectrum at very large wave-numbers, and by Batchelor (1952; see also the review by Batchelor & Townsend 1956) in a discussion of the way in which material line elements in the fluid are extended and material surface elements are increased in area. The use to which it will be put in this and the following section has links with both of these earlier investigations, although there are also some new features.

The basic idea of the investigation is to make use of the linearity of the equation (1.1) for θ and to examine the effect of both convection and conduction on each Fourier component of the θ-distribution, the Fourier analysis being carried out with respect to axes which move with the fluid locally in translation and rotation and which in effect are distorted with the fluid locally. We consider any material element of fluid with linear dimensions somewhat less than $(\nu^3/\epsilon)^{\frac{1}{4}}$ and resolve the instantaneous distribution of θ within this element into its Fourier components. This material element moves in translation and rotation and is subjected to a pure straining distortion, and at the end of a finite time it will have a different position and orientation and will have experienced a finite pure strain. Only the straining of the element affects the distribution of θ within the element, and this straining will be regarded as representative of the effect of fluid convection on the Fourier components of θ with wave-numbers large compared with $(\epsilon/\nu^3)^{\frac{1}{4}}$. Of course, the way in which the distribution of θ is continued beyond the material element and joins up with other material elements which are being translated, rotated and strained also has an influence on Fourier components of θ, but this influence may be expected to be important only for components with wave-numbers of order $(\epsilon/\nu^3)^{\frac{1}{4}}$ or less (since $(\nu^3/\epsilon)^{\frac{1}{4}}$ is the characteristic length-scale for spatial variations of the rotational and straining motions).

Consider first the changes in an initially sinusoidal variation of θ throughout the material element. We choose Cartesian axes which translate with the element and which are always principal axes of the rate of strain of the element. According to the approximation explained above, these axes are fixed in the fluid, and the principal rates of strain α, β, γ are constant, for time intervals at least as large as $(\nu/\epsilon)^{\frac{1}{2}}$. Then, the distribution of θ is governed by the equation

$$\frac{\partial\theta}{\partial t}+\alpha x\frac{\partial\theta}{\partial x}+\beta y\frac{\partial\theta}{\partial y}+\gamma z\frac{\partial\theta}{\partial z} = \kappa\nabla^2\theta, \tag{4.2}$$

and the initial condition is $\theta = A_0 \sin(\mathbf{l}.\mathbf{x})$ at $t = 0$. This equation is satisfied by

$$\theta(\mathbf{x}, t) = A(t)\sin[\mathbf{m}(t).\mathbf{x}], \tag{4.3}$$

with
$$\frac{dA}{dt} = -\kappa m^2 A,$$

$$\frac{dm_1}{dt} = -\alpha m_1, \quad \frac{dm_2}{dt} = -\beta m_2, \quad \frac{dm_3}{dt} = -\gamma m_3,$$

where m_1, m_2, m_3 are components of $\mathbf{m}$. Thus the solution is

$$\theta(\mathbf{x}, t) = A_0 \exp\left[\frac{\kappa}{2\alpha}(m_1^2-l_1^2)+\frac{\kappa}{2\beta}(m_2^2-l_2^2)+\frac{\kappa}{2\gamma}(m_3^2-l_3^2)\right]\sin(\mathbf{m}.\mathbf{x}), \tag{4.4}$$

in which

$$m_1 = l_1 e^{-\alpha t}, \quad m_2 = l_2 e^{-\beta t}, \quad m_3 = l_3 e^{-\gamma t}. \tag{4.5}$$

The planes of constant θ are turned so that the direction of their normal approaches asymptotically the direction of the greatest rate of contraction in the fluid, and, if $\alpha > \beta > \gamma$ (which implies $\alpha > 0$, $\gamma < 0$, since $\alpha+\beta+\gamma = 0$ by the continuity equation) we have

$$m^2 = m_1^2+m_2^2+m_3^2 \to l_3^2 e^{-2\gamma t}, \quad \theta \to A_0 \exp\left(\frac{\kappa m^2}{2\gamma}\right)\sin(\mathbf{m}.\mathbf{x}) \tag{4.6}$$

(provided $l_3 \neq 0$) as $t \to \infty$. The duration of time for which the principal axes of strain remain fixed relative to the fluid and α, β, γ remain constant is limited, but these asymptotic relations become quite accurate long before $t = 10/|\gamma|$ (except for certain special choices of l_2, l_3 and β/γ) and since $|\gamma|$ is of order $(\epsilon/\nu)^{\frac{1}{2}}$ the use of the asymptotic relations as being *typical* of what is happening to Fourier components of θ is consistent with the approximation described above.

All Fourier components of the initial distribution of θ in the material element will be changed in this way, and the distribution of θ at a time t subsequent to the initial instant can be obtained by superimposing the changed Fourier components. The distribution of θ tends towards a one-dimensional form with variation only in the direction of the principal axis of least rate of strain. Gradients of θ in the z-direction are made steeper by the convection process (by crests of the distribution being squeezed together), but are simultaneously made more gradual by conduction effects, and ultimately all the variation of θ is erased by conduction. However, it will be noticed that the smaller the value of κ, the longer is the time required for erasure of the variations of θ and the greater are the gradients of θ which are built up in the meantime.

This information about the way in which the convective distortion converts a Fourier component of certain wave-number into one of larger wave-number, the magnitude of the coefficient being diminished meanwhile by conduction, can now be used to find an expression for the steady θ-spectrum at wave-numbers large compared with $(\epsilon/\nu^3)^{\frac{1}{4}}$. This is made possible by the fact that the convective modulation of Fourier components of the θ-distribution is entirely one-way as soon as the process of alignment of the wave-number vectors is nearly complete; $\overline{\theta^2}$-stuff is transferred wholly to components of larger wave-number. In the neighbourhood of the small wave-number end of the range in which the above approximations about the effect of the velocity field on Fourier components of θ are valid, this end being given by wave-number magnitude n_0 say, we may suppose that the level of the θ-spectrum is kept constant by a continual supply of $\overline{\theta^2}$-stuff from lower wave-numbers (the exact mechanism of this supply being irrelevant at the moment). The Fourier components with wave-numbers larger than n_0 are derived from those at smaller wave-number by the straining process, and the asymptotic relations (4.6) show that the $\overline{\theta^2}$-stuff which is spread over the wave-number range dn', at a wave-number (magnitude) n' at which the alignment of the wave-number vectors is nearly complete, is spread over the range $n\,dn'/n'$ at a later stage at which the Fourier component with wave-number n' has been distorted to that with wave-number n, and that during the time required for this change this amount of $\overline{\theta^2}$-stuff is reduced, at those places in the fluid where the least principal rate of strain has the value γ, by the factor $\exp\,[\kappa(n^2-n'^2)/\gamma]$ owing to the action of conduction. The principal rates of strain will not be uniform throughout the fluid, so we are obliged to assume that γ is an effective average value of the least principal rate of strain. The constant level of the θ-spectrum at wave-numbers well above n_0 is then given by

$$\Gamma(n)\,\frac{n}{n'}\,dn' = \Gamma(n')\,dn'\exp\left[\frac{\kappa}{\gamma}(n^2-n'^2)\right],$$

that is,

$$\Gamma(n) \propto \frac{1}{n}\exp\left(\frac{\kappa}{\gamma}n^2\right); \qquad (4.7)^*$$

the θ-spectrum is also isotropic at these wave-numbers in view of the isotropic distribution of the straining motion.

The dimensional factors in the constant of proportionality in (4.7) can be obtained by noticing that when $n \ll (-\gamma/\kappa)^{\frac{1}{2}}$—and such values of n can exist when κ/ν is sufficiently small, notwithstanding the existing restrictions on n—the exponential factor is approximately constant and $\Gamma(n) \propto n^{-1}$. At these values of n the effects of conduction are unimportant and the decrease of Γ with n is due entirely to the spreading of $\overline{\theta^2}$-stuff over a wider wave-number range by convective straining. The only parameters relevant to the form of $\Gamma(n)$ are then χ

* For simplicity this relation has been obtained from a consideration of the changes occurring in one Fourier component whose wave-number vector becomes aligned with the direction of the greatest rate of compression. There are, of course, some wave-number vectors which take a long time to become aligned (namely, those for which l_3 is small in (4.5)), and it is desirable to verify that the relation (4.7) is not affected by these untypical Fourier components. It is readily established, in fact, that if a continuous isotropic θ-spectrum with most of the $\overline{\theta^2}$-stuff at wave-numbers with magnitudes near n_0 is maintained by transfer from smaller wave-numbers, the steady spectrum produced at much larger wave-numbers by combined convective straining and conduction has the form (4.7), except in the particular case in which $\beta = \gamma$, when the form is different at wave-numbers of order $(-\gamma/\kappa)^{\frac{1}{2}}$.

and γ, and the dimensional part of the proportionality constant is thus χ/γ. Furthermore, the numerical part must have the value -1 in order that (4.7) should be consistent with the identity

$$\int_0^\infty n^2\Gamma(n)\,dn = \frac{\chi}{2\kappa}.$$

Several estimates of the average value of the least principal rate of strain in the fluid have been made (Batchelor & Townsend 1956), and they are all in the neighbourhood of $-0{\cdot}5(\epsilon/\nu)^{\frac{1}{2}}$. The final expression for the θ-spectrum in the wave-number range $n \gg (\epsilon/\nu^3)^{\frac{1}{4}}$ is therefore

$$\Gamma(n) = -\frac{\chi}{\gamma n}\exp\left(\frac{\kappa n^2}{\gamma}\right), \quad \text{with} \quad \gamma \approx -0{\cdot}5\left(\frac{\epsilon}{\nu}\right)^{\frac{1}{2}}. \tag{4.8}$$

According to this relation, in the wave-number range designated earlier as convection subrange B and specified by $(\epsilon/\nu^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu\kappa^2)^{\frac{1}{4}}$, the θ-spectrum is given by

$$\Gamma(n) \approx -\frac{\chi}{\gamma n}. \tag{4.9}$$

It is worth noting that this latter relation could have been predicted (apart from a numerical constant) on dimensional grounds right at the beginning, provided the thesis adopted here, that the primary effect of the convection on variations of θ on a length-scale small compared with $(\nu^3/\epsilon)^{\frac{1}{4}}$ is a uniform straining at a rate of order $(\epsilon/\nu)^{\frac{1}{2}}$, be granted. The parameters in the constant of proportionality in (4.9) are consistent with the need for the relation (4.9) to join smoothly on to the relation (2.4) at lower wave-numbers; for the join occurs at values of n near $(\epsilon/\nu^3)^{\frac{1}{4}}$, when both relations show Γ to be of order $\chi\nu^{\frac{5}{4}}\epsilon^{-\frac{3}{4}}$.

The relation (4.9) describes a comparatively slow rate of decrease of θ as n increases, even slower than that holding at smaller wave-numbers in convection subrange A. The rate of decrease is so slow that $\int_0^n \Gamma(n)\,dn$ does not converge, as $n \to \infty$, for a fluid such that $\kappa = 0$—which is to be interpreted as meaning that, in a fluid with a very small value of κ/ν, a statistically steady state for the small-scale components of the θ-distribution can be set up only if there is a sufficiently large reservoir of $\overline{\theta^2}$-stuff in the large-scale components of θ and if sufficient time is available. The amount of $\overline{\theta^2}$-stuff in convection subrange B, as given by (4.9), is of order

$$-\frac{\chi}{2\gamma}\log\frac{\nu}{\kappa}, \tag{4.10}$$

and this may be larger than the amount in the large-scale components of θ without violating in any way the assumptions on which the analysis is based (although $\overline{\theta^2}$ would then not be a measure of the $\overline{\theta^2}$-stuff associated with Fourier components with wave-numbers of order L^{-1}). The time required for a stationary state to be set up over convection subrange B can be estimated from the time required for a Fourier component of θ with wave-number of order $(\epsilon/\nu^3)^{\frac{1}{4}}$ to be deformed by the straining process into one with wave-number of order $(\epsilon/\nu\kappa^2)^{\frac{1}{2}}$ and is

$$-\frac{1}{\gamma}\log\frac{\nu}{\kappa}$$

(in agreement with (4.10) in view of the definition of χ), and this may be larger than the time scale of the large-scale components of θ again without inconsistency.

The relation (4.9) can also be interpreted in terms of the formation of steep spatial gradients of θ in the fluid. If $|\nabla\theta|$ became infinite (or as near infinite as the small conductivity allowed) at a finite number of points in unit volume of the fluid as a result of the effect of convection, $\Gamma(n)$ would vary as n^{-2}. The divergence of $\overline{(\nabla\theta)^2}$ as given by (4.9) is stronger than this, corresponding to the fact that convection actually steepens the gradient of θ, and does so persistently, throughout typical material elements of fluid; large values of $|\nabla\theta|$ thus appear, not at isolated points, but over a finite fraction of the whole fluid.

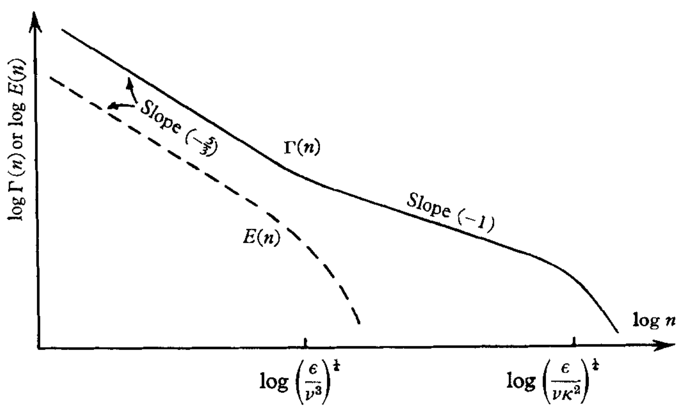

FIGURE 1. Spectra of θ and $\mathbf{u}$ in the equilibrium range of wave-numbers for the case $\nu \gg \kappa$.

The conduction cut-off given by (4.8) is sharp, and occurs, as expected, at wave-numbers of order $(\epsilon/\nu\kappa^2)^{\frac{1}{4}}$. The available information about $\Gamma(n)$ in the equilibrium range in the case $\nu \gg \kappa$ is shown schematically in figure 1. Finally, it is worth noting that these results about the form of $\Gamma(n)$ beyond the viscous cut-off wave-number do not require the Reynolds number of the turbulence to be so large that an inertial subrange exists. Whatever the Reynolds number of the turbulence, the distortion of sufficiently small material elements of fluid will be approximately a pure straining motion. Thus the first part of the relation (4.8) will still hold, although the estimate of the straining rate γ may not be accurate at low Reynolds number.

5. The case $\nu \gg \kappa$: Eulerian analysis in terms of correlations

Inasmuch as the analysis presented in the preceding section contains some novel features and may not carry immediate conviction, it may be useful to show how the same results can be obtained in a quite different way from essentially the same assumptions about the effect of the fluid motion on small-scale features of the θ-distribution. This time the analysis involves mean values of products of θ and $\mathbf{u}$, and is rather more direct, although the character of the action of convection on the θ-distribution is not revealed so explicitly.

The expression for rate of change of the temperature covariance is readily found from (1.1) to be

$$\frac{\partial \overline{\theta\theta'}}{\partial t} = \frac{\partial}{\partial r_i}(\overline{u_i\theta\theta'} - \overline{u_i'\theta\theta'}) + 2\kappa\nabla^2\overline{\theta\theta'}, \tag{5.1}$$

where θ' is written for $\theta(\mathbf{x}+\mathbf{r}, t)$, and statistical homogeneity of θ and $\mathbf{u}$ has been assumed (as is appropriate for the small-scale features of their spatial distributions). When the distance r between the points to which θ, $\mathbf{u}$ and θ', $\mathbf{u}'$ refer is sufficiently small, we have

$$u_i' \approx u_i + r_j\frac{\partial u_i}{\partial x_j}, \tag{5.2}$$

the restriction on r being that it should be somewhat less than $(\nu^3/\epsilon)^{\frac{1}{4}}$ as established in the paragraph containing (4.1). Moreover, in accordance with the description of the convection process given in the preceding section for this same case $\nu \gg \kappa$, there are some values or r, at the upper end of this range, for which $\theta' - \theta$ is not linear in r because the θ-distribution has a finer structure than the $\mathbf{u}$-distribution. We shall therefore employ the approximation (5.2) in (5.1), without introducing a similar approximation for $\theta' - \theta$. Also, we shall approximate the left-hand side of (5.1) by $-\chi$, on the understanding (and in view of the results obtained in §4, there is need for care in the wording here) that we are investigating a distribution of θ which is stationary so far as the small-scale components are concerned and that the rate at which $\overline{\theta^2}$ and $\overline{\theta\theta'}$ are decreasing is due entirely to a decrease of $\overline{\theta^2}$-stuff in the large-scale components with length-scale L at a rate χ (or equivalently that $\overline{\theta^2}$-stuff is being supplied to these large-scale components at a rate χ in a case in which the whole of the θ-distribution is statistically stationary). Equation (5.1) then becomes

$$\begin{aligned} -\chi &= -r_j\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial \theta\theta'}{\partial r_i}} + 2\kappa\nabla^2\overline{\theta\theta'} \\ &= \tfrac{1}{2}\overline{(\mathbf{r}\,.\,\nabla\mathbf{u})\,.\,\nabla(\theta-\theta')^2} - \kappa\nabla^2\overline{(\theta-\theta')^2}. \end{aligned} \tag{5.3}$$

where ∇ is everywhere a gradient with respect to $\mathbf{r}$ alone, $\mathbf{x}$ being held constant where necessary.

The next step in the argument is to approximate to the first term on the right-hand side of (5.3), with the help of hypotheses about the persistence of the extension of material surfaces as in §4. The effect of the uniform straining motion which exists everywhere in the neighbourhood of a material point in the fluid is to turn the local surfaces of constant θ so that the directions of their normals approach that of the least principal rate of strain. Provided the material surfaces on which θ is constant continue to be extended—that is, provided the angle between the normal to the surfaces of constant θ and the direction of the least principal rate of strain continues to be small—for a time long compared with $(\nu/\epsilon)^{\frac{1}{2}}$ (which is the time characteristic of the straining motion and which is therefore a measure of the time required for approximate alignment of the normals to the surfaces of constant θ), as is believed to be so on the basis of the evidence discussed in the preceding section, the direction of $\nabla(\theta-\theta')^2$ at each point of the fluid and at all times will tend to be aligned in the direction of the local least principal rate of strain. Of the two contributions to $\mathbf{r}.\nabla\mathbf{u}$, one from rigid rotation of the fluid about the point $\mathbf{x}$ and one from the pure straining motion, only the latter is related statistically to the distribution of θ. The contribution to $\mathbf{r}.\nabla\mathbf{u}$ from the pure straining motion has a component in the direction of the least principal rate of strain equal to $r\gamma\cos\phi$, where ϕ is the angle between $\mathbf{r}$ and the principal axis of least rate of strain.

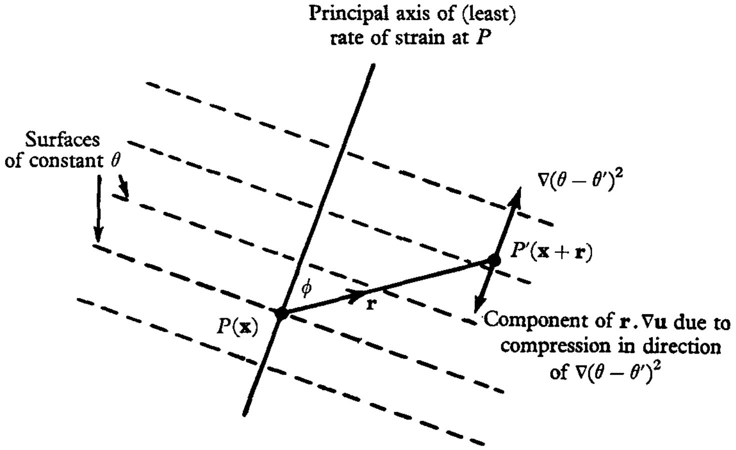

FIGURE 2. To illustrate the uniform straining motion near P and its effect on the distribution of θ.

Thus, if we assume that the process of orientation of $\nabla(\theta-\theta')^2$ along a principal axis of rate of strain is complete at all times and positions, as illustrated in figure 2, and that the fluctuations in γ are a negligible fraction of the mean value, we have

$$\overline{(\mathbf{r}.\nabla\mathbf{u}).\nabla(\theta-\theta')^2} = r\gamma\cos\phi|\nabla\overline{(\theta-\theta')^2}| = \gamma r\frac{\partial\overline{(\theta-\theta')^2}}{\partial r}. \tag{5.4}$$

These two assumptions are over-simplifications, so that, as in §4, γ must be regarded as an effective average value of the least principal rate of strain. The equation (5.3) for the covariance of θ now becomes

$$-\chi = \tfrac{1}{2}\gamma r\frac{\partial\overline{(\theta-\theta')^2}}{\partial r} - \kappa\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\overline{(\theta-\theta')^2}, \tag{5.5}$$

that is,
$$\frac{\chi}{\kappa}r^2\exp\left(-\frac{\gamma r^2}{4\kappa}\right) = \frac{\partial}{\partial r}\left[r^2\frac{\partial\overline{(\theta-\theta')^2}}{\partial r}\exp\left(-\frac{\gamma r^2}{4\kappa}\right)\right].$$

One integration gives

$$\frac{\partial\overline{(\theta-\theta')^2}}{\partial r} = \frac{\chi}{\kappa r^2}\exp\left(\frac{\gamma r^2}{4\kappa}\right)\int_0^r r^2\exp\left(-\frac{\gamma r^2}{4\kappa}\right)dr,$$

and, from a second,

$$\overline{(\theta-\theta')^2} = \frac{2\chi}{\gamma}\int_0^r \left[-\frac{1}{r}+\frac{1}{r^2}\exp\left(\frac{\gamma r^2}{4\kappa}\right)\int_0^r \exp\left(-\frac{\gamma r^2}{4\kappa}\right)dr\right]dr. \tag{5.6}$$

When $r^2 \ll 4\kappa/|\gamma|$, this relation reduces to

$$\overline{(\theta-\theta')^2} \approx \frac{\chi}{6\kappa}r^2, \tag{5.7}$$

as is required by the definition of χ (see (2.3)). At the other extreme, when $r^2 \gg 4\kappa/|\gamma|$ (although r must continue to be suitably small), we have

$$\overline{(\theta-\theta')^2} \sim -\frac{\chi}{\gamma}\log\left(-\frac{\gamma r^2}{\kappa}\right). \tag{5.8}$$

As in §4, γ can be replaced by its estimated value $-0{\cdot}5(\epsilon/\nu)^{\frac{1}{2}}$, in which case the asymptotic relation (5.8) becomes

$$\overline{(\theta-\theta')^2} \sim \chi\left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}\log\left(\frac{\epsilon r^4}{\nu\kappa^2}\right), \tag{5.9}$$

valid for $(\nu^3/\epsilon)^{\frac{1}{4}} > r \gg (\nu\kappa^2/\epsilon)^{\frac{1}{4}}$. This logarithmic form for $\overline{(\theta-\theta')^2}$ lies between the parabolic form (5.7) in the immediate neighbourhood of the origin and a variation as $r^{\frac{2}{3}}$ at values of r such that $L \gg r \gg (\nu^3/\epsilon)^{\frac{1}{4}}$.

It remains to show that the result (5.8) is effectively the same as (4.8). This can be done by beginning with the identity (see (2.1))

$$\begin{aligned}\overline{(\theta-\theta')^2} &= \frac{1}{2\pi}\int\frac{\Gamma(n)}{n^2}(1-e^{-i\mathbf{n}.\mathbf{r}})\,d\mathbf{n}\\ &= 2\int_0^\infty \Gamma(n)\left(1-\frac{\sin nr}{nr}\right)dn.\end{aligned}$$

On substituting for $\Gamma(n)$ from (4.8), we find

$$\begin{aligned}\frac{\partial\overline{(\theta-\theta')^2}}{\partial r} &= \frac{2\chi}{\gamma}\int_0^\infty \exp\left(\frac{\kappa n^2}{\gamma}\right)\left(\frac{\cos nr}{nr}-\frac{\sin nr}{n^2r^2}\right)dn\\ &= \frac{2\chi}{\gamma}\left[-\frac{1}{r}-\frac{2\kappa}{\gamma r^2}\int_0^\infty \exp\left(\frac{\kappa n^2}{\gamma}\right)\sin nr\,dn\right]\\ &= \frac{2\chi}{\gamma}\left[-\frac{1}{r}+\frac{i\kappa}{\gamma r^2}\exp\left(\frac{\gamma r^2}{4\kappa}\right)\int_0^\infty\left\{\exp\frac{\kappa}{\gamma}\left(n+\frac{ir\gamma}{2\kappa}\right)^2-\exp\frac{\kappa}{\gamma}\left(n-\frac{ir\gamma}{2\kappa}\right)^2\right\}dn\right]\\ &= \frac{2\chi}{\gamma}\left[-\frac{1}{r}+\frac{1}{r^2}\exp\left(\frac{\gamma r^2}{4\kappa}\right)\int_0^r \exp\left(-\frac{\gamma r^2}{4\kappa}\right)dr\right],\end{aligned}$$

thus reproducing (5.6).

6. Comments on a different model of small-scale variations of θ in the case $\nu \gg \kappa$

In his theory of the form of the energy spectrum at very large wave-numbers, Townsend (1951*a*) made use of the same notion that the action of the whole flow field on small-scale variations of any quantity—vorticity, in his case—is primarily to impose a uniform persistent straining motion. The action of a steady uniform rate of strain on weak variation of vorticity is to increase the gradient of the perturbation vorticity in the direction of the least principal rate of strain and to amplify the component of vorticity in the direction of the greatest principal rate of strain; this latter effect due to stretching of vortex lines is absent in the case of scalar quantities like temperature, but there is otherwise a fairly close analogy between the two cases. Townsend made a further assumption in his work on vorticity, and this has not been employed in the preceding discussion of the θ-spectrum for the case $\nu \gg \kappa$. Since the reasons for not using this extra assumption are not self-evident, and since they throw an interesting light on the nature of the assumption, a brief comparison of the foregoing results with the form they would take with an additional assumption like Townsend's will be given.

Guided by the observation that small-scale variations of vorticity seem to have an uneven spatial distribution, some parts of the fluid being relatively free from such variations, Townsend put an intermittent variation into his model by assuming that small-scale variations of vorticity exist mainly as isolated steady vortex sheets or 'line vortices' of small thickness. Each of these sheets or lines is steady under the opposing actions of molecular diffusion and stretching of vortex lines (the occurrence of two positive principal rates of strain giving rise to a vortex sheet, and one positive principal rate of strain to a line vortex, the former being the more probable), and the variations of vorticity on length scales small compared with $(\nu^3/\epsilon)^{\frac{1}{4}}$ were supposed to occur in the form of a random distribution of such sheets or lines. The sheets or lines were assumed to be separated by distances large compared with their thickness (which is of order $(\nu/|\gamma|)^{\frac{1}{2}}$), so that amalgamation of sheets or lines which are swept together by the straining motion happens only infrequently. The steady distribution of vorticity in a sheet or line can readily be calculated in terms of the principal rates of strain and so the spectrum of vorticity, and thence of velocity, can be determined at large wave-numbers. There is an arbitrary multiplicative constant in the resulting vorticity spectrum, representing the product of the number of sheets or lines per unit volume and their strength; the value of this constant is determined by the way in which inertia forces generate vorticity perturbations on a larger length-scale and lies outside the scope of the theory.

In exactly the same way one could assume that variations of θ on a small scale occur as randomly distributed, isolated, thin layers in which the distribution of θ is steady under the combined actions of molecular diffusion and uniform straining. Just as a uniform straining motion with one negative principal rate of strain (γ) converts (asymptotically) an arbitrary transition between two regions of uniform (and different) perturbation velocity into a steady vortex sheet of thickness of order $(\nu/|\gamma|)^{\frac{1}{2}}$, so it converts an arbitrary transition between two regions of uniform θ into a layer in which the steady distribution of θ is given by

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial y} = 0, \quad \frac{\partial\theta}{\partial z} = \left(-\frac{\gamma}{2\pi\kappa}\right)^{\frac{1}{2}} A \exp\left(\frac{\gamma z^2}{2\kappa}\right), \tag{6.1}$$

A being a constant measuring the size of the jump in θ across the layer. The spectrum of θ for an array of such layers throughout the fluid, random with respect to both position and orientation, then follows as

$$\Gamma(n) = \frac{\chi}{(-\pi\gamma\kappa)^{\frac{1}{2}}}\frac{1}{n^2}\exp\left(\frac{\kappa n^2}{\gamma}\right), \tag{6.2}$$

where the multiplicative constant equal to the product of A^2 and the total layer area in unit volume of fluid has been determined from the requirement that the total rate of destruction of $\overline{\theta^2}$-stuff in unit volume of fluid be χ. (Note that this method of determining the constant is consistent with the model when $\kappa \ll \nu$, because the length-scale on which destruction of $\overline{\theta^2}$-stuff takes place is much smaller than the size of regions of the fluid over which the rate of strain is uniform. Such a determination of the multiplicative constant is not available in the case of the vorticity spectrum.) This result for the θ-spectrum should be compared with (4.8), which was obtained by considering the changes in all Fourier components of θ due to the actions of distortion and molecular diffusion and without assuming the existence of isolated layers of rapid change of θ.

It is not difficult to see that (6.2) cannot be correct. The spectrum function given by (6.2) is of order $\chi\nu^{\frac{7}{4}}/\epsilon^{\frac{3}{4}}\kappa^{\frac{1}{2}}$ at the wave-number $(\epsilon/\nu^3)^{\frac{1}{4}}$ marking the transition from convection subrange A to convection subrange B, and this is different by a factor $(\nu/\kappa)^{\frac{1}{2}}$ from the order of Γ (at the same wave-number) as determined by the relation (2.4) valid in convection subrange A. Another way of stating this difficulty in joining the relation (6.2) to the relation valid at smaller wave-numbers is to remark that, according to (6.2), the value of Γ at wave-numbers not near the conduction cut-off (that is, for $n \ll (-\gamma/\kappa)^{\frac{1}{2}}$) increases indefinitely as $\kappa \to 0$. The relation (6.2) has this behaviour because the steady rate of destruction of $\overline{\theta^2}$-stuff per unit area of a single layer across which there is a jump in θ is proportional to $\kappa^{\frac{1}{2}}$ (the gradients of θ increase, as $\kappa \to 0$, in such a way as to keep the local rate of destruction per unit volume of fluid in a layer constant, but the thickness of the layer decreases as $\kappa^{\frac{1}{2}}$), and the same average total rate of destruction of $\overline{\theta^2}$-stuff per unit volume of fluid can be achieved, as $\kappa \to 0$, only by an increase in the size of the jump in θ across the sheet or in the number of sheets in unit volume. A dependence of either of these quantities on κ is not possible, in fact, because the conditions leading to the formation of sheets are supposed to be generated by purely convective effects at wave-numbers smaller than those at which conduction is important.

The essential difference between the two theoretical models seems to be that in that leading to (6.2) the time-dependent effects accompanying the continual reduction in distance between neighbouring sheets and their ultimate amalgamation are ignored, as would be justified if the sheets were usually so far apart as to effect the form of Γ only at values of n of order $(\epsilon/\nu^3)^{\frac{1}{4}}$ (and this assumption, as seen, is open to the objection that the size of the jump in θ or the density of the layers must be supposed to increase as $\kappa \to 0$ in order to give the right total rate of destruction of $\overline{\theta^2}$-stuff), whereas in that leading to (4.8) the distribution of θ is always unsteady and the typical form of θ-variation is one in which neighbouring crests are continually approaching each other and 'amalgamating'. It might be thought that since the model leading to (6.2) incorporates explicitly an intermittent small-scale variation of θ, objections to the spectral form (6.2) are equivalent to objections to an intermittent type of variation. This is not so, however; the model leading to (4.8) is consistent with an intermittent variation of θ provided that the region in which $\nabla\theta$ fluctuates should not be a vanishingly small fraction of the total volume as $\kappa \to 0$, the existence of intermittency of this kind having no influence on the result (4.8). It is now pertinent to inquire whether Townsend's assumption that small-scale variations of vorticity occur mainly in the form of isolated sheets or lines of concentrated vorticity was necessary, but that is another story.

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