Optimal Airline Seat Allocation with Fare Classes Nested by Origins and Destinations

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Previous research in the optimal allocation of airline seats has followed one of two themes: marginal seat revenue or mathematical programming. Both approaches capture important elements of the revenue management problem. The marginal seat revenue approach accounts for the "nesting" of fare classes in computer reservation systems, but can only control seat inventory by bookings on legs. The mathematical programming approach will handle realistically large problems and will account for multiple origin–destination itineraries and side constraints, but it does not account for fare class nesting in the reservation systems. This paper combines both approaches by developing equations to find the optimal allocation of seats when fare classes are nested on an origin–destination itinerary and the inventory is not shared among origin–destinations. These results are applicable to seat allocation in certain reservations systems, point-of-sale control, and acceptance of groups over their entire itinerary. A special case of the analysis produces the optimal booking limits for leg-based seat allocation with nested fare classes.

INTRODUCTION

The process of "yield management" has become extremely important within the airline industry. Yield management, or more properly revenue management, consists of setting fares, setting overbooking limits, and controlling seat inventory to increase revenues. It has allowed the airlines to survive deregulation by allowing them to respond to competitors' deep discount fares on a rational basis. It also can be very profitable since it can generate *an additional* 3–10% of gross revenue, according to generally-accepted industry estimates.[1]

Seat inventory control is the process of controlling the mix of fare classes on each flight departure. This is accomplished for each flight by observing the booking patterns and regulating the availability of fare classes starting months before the date of departure. BELOBABA[2] provides a very comprehensive and very readable review of the general problem of seat inventory control and also includes analytical and experimental results. Condensed versions appear in [3] and [4].

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Reservations Systems and Real-World Constraints

Any seat inventory control system must work in tandem with the airline computer reservations system (CRS). The typical present-day seat inventory control system is an off-line system which obtains its data from the reservations system and makes recommendations for CRS settings. Thus it is important that the seat inventory control system recognize the constraints of the reservations system as well as other (internal and external) constraints.

The simplest CRS allocates seats in discrete, or separate, booking compartments (sometimes called fare class buckets) on a flight leg. (A leg is one takeoff and landing.) Each fare class has its own allocation of seats and the CRS will deny further requests when bookings reach the allocation for the fare class. (Booking class is a more accurate term than fare class because several different fare products may be containing within a booking class. However, we shall use the terms interchangeably here.) Table I shows an example of discrete fare classes. The control of seat inventory by fare class bucket may result in the refusal of higher-valued fare requests ($300) even though space is available in lower fare buckets ($150).

The concept of fare class nesting (sharing, or pool-

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TABLE I

Example of Discrete Fare Class Allocations, Number Booked and Seats Available[a]

Class	Average Fare	No. Booked	Class Allocation	Seats Available
Y	300	10	10	0
M	270	18	20	2
B	200	25	30	5
Q	150	30	40	10

[a] The number of seats available is the class allocation less the number booked in that class.

TABLE II

Example of Nested Fare Class Booking Limits, Number Booked and Seats Available[a]

Class	Average Fare	No. Booked	Booking Limit	Seats Available
Y	300	10	100	25
M	270	10	80	15
B	200	25	60	5
Q	150	30	30	0

[a] The number of seats available is the booking limit for a class less the number booked in that class and all lower classes.

ing of reservations among fare classes) was introduced to alleviate this problem. In a hierarchically nested system, the fare classes are ordered according to their "value," and a higher-valued fare class may draw seats from the inventory of a lower-valued class. In the event of an unanticipated demand, nesting avoids denying reservations requests for higher fares as long as there are seats available in a lower-value fare class. The concept of fare class allocations is replaced by booking limits: a request for a fare class will be accepted as long as the number of bookings in a class and all lower classes are less than the fare class booking limit. The highest fare class has a booking limit equal to the aircraft cabin authorization, so a request for the highest fare can always be filled if any seats are available.

Table II shows how nested fare classes and booking limits overcome the limitations of discrete fare classes. The seat availability logic in Table II is not unique. For example, some systems compute seats available as the booking limit less the total number booked in all classes.

Segment Control

Most CRSs exert control by leg bookings. Thus if a passenger wants to travel from A to C, but must travel on two legs A–B and B–C, the itinerary will be treated as two individual trips. Limited origin–destination (O&D) control of reservations is available in many CRSs by controlling segments. (A segment is an O&D itinerary on the same flight number.) If the legs A–B and B–C have the same flight number, then separate control is available for the three segments: A–B, B–C, A–C. Segment control is usually accomplished with segment control indicators (booking-on/booking-off flags), but a few reservations systems have booking limits for segments.

Additional O&D control is obtained by a CRS which allows booking on "overlapping" flights. An overlapping flight is maintained as a separate departure within the CRS, but it shares the same physical aircraft with one or more other flights. This extends the number of O&D itineraries which can be controlled simultaneously.

In addition to these characteristics of the CRS, the seat inventory control system should recognize other real-world constraints from outside the airline or within the airline. These include upper and lower bounds on available seats for certain origin–destinations, especially due to international agreements. Sale of certain fare classes may be restricted due to marketing considerations such as limiting or ensuring availability of deep-discount seats and frequent flyer redemptions. Constraints by point-of-sale can be important because of currency fluctuations and exchange rates.

Virtual Nesting

Some CRSs use virtual fare classes with nesting,[2, 8] which allows a limited form of origin–destination control. O&D fare classes in the network are assigned to classes which are used for determining availability of O&D fare classes, but these virtual classes are invisible to the reservations agent. An agent asks for available seats between an origin and destination, and an O&D fare class can only be booked if the virtual fare class to which it has been assigned has seats available on all legs of the itinerary. The virtual classes can be nested on each leg.

Table III contains an example of virtual classes for a simple three-node, two-leg network (A–B–C) with four classes on each segment (Y, M, B, Q). In the example shown in Table III, a Q fare from A–C (Q_{AC}) cannot be booked because its virtual class (Y_3) is closed on leg B–C. Similarly, the fare class B_{AB} is available and B_{BC} is not available.

The first implementations of virtual classes assigned higher value fares to the higher ranked virtual classes, and this eliminates one objection of conventional leg-based nesting: the possibility of a lower-value itinerary in a higher class displacing a higher-value itinerary in a lower class. But, as described below, favoring higher value fare classes may *not* lead to maximum revenue, so maximum revenue cannot be guaranteed in virtually nested systems.

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TABLE III

Example of Nested Virtual Fare Classes for an A–B–C Network with Two Legs: A–B and B–C[a]

Virtual Class	O&D Classes		No. Booked	Booking Limit	Seats Available
	LegAB				
Y_0		Y_{AC}	10	100	25
Y_1	Y_{AB}	M_{AC}	10	80	15
Y_2	M_{AB}	B_{AC}	25	60	5
Y_3	B_{AB}	Q_{AC}	20	40	10
Y_4	Q_{AB}		10	10	0
	LegBC				
Y_0		Y_{AC}	5	100	40
Y_1	Y_{BC}	M_{AC}	15	75	20
Y_2	M_{BC}	B_{AC}	20	60	20
Y_3	B_{BC}	Q_{AC}	10	20	0
Y_4	Q_{BC}		10	10	0

[a] An O&D fare class is available only if its virtual class has seats available on all legs of the itinerary.

Mathematical Models

The expected revenue from any one flight is the sum of the expected revenues for each fare class. The expected revenue from each class depends on the average fare for the class and the average bookings in each fare class; the average bookings depend on the number of seats allocated to the class, and the stochastic demand for the class

$$\text{Expected Revenue} = \textstyle\sum_i f_i \mathrm{E}(b_i) \qquad (1)$$

where f_i is the average fare for fare class i and $\mathrm{E}(b_i)$ is the expected number of bookings (accepted requests) for class i.

The mathematical approaches to the seat inventory control problem follow two general themes: marginal seat revenue, and mathematical programming.

Marginal Seat Revenue

The marginal seat revenue of a fare class is the slope of the expected class revenue with respect to number of allocated seats:

$$\text{Marginal Revenue} = \mathrm{MR}_i = f_i \frac{\partial \mathrm{E}(b_i)}{\partial A_i} \qquad (2)$$

where A_i is the number of seats allocated to fare class i. Marginal revenue concepts can be applied to derive optimal allocations for non-nested fare classes. The necessary conditions for maximum revenue are achieved when the marginal revenue is the same for all classes [3].

$$\mathrm{MR}_i = \mathrm{MR}_j = \lambda \quad \text{all } i, j \qquad (3)$$

The constant λ is the LaGrange multiplier which is used to add the capacity constraint to the total expected revenue function. This set of equations is almost never solved in practice; instead, mathematical programming approaches are used to find the seat allocations for maximum revenue subject to capacity constraints.

LITTLEWOOD[5] used marginal seat revenue to argue for a simple rule for seat allocation among two fare classes in a dynamic booking context, i.e., as the booking is taking place: requests for the lower fare class should be accepted as long as the (certain) lower fare exceeded the (uncertain) marginal revenue of the higher fare class,

$$f_2 \geqslant \mathrm{MR}_1 = f_1 \frac{\partial \mathrm{E}(b_1)}{\partial N_1} \qquad (4)$$

$$f_2 \geqslant f_1 \mathrm{P}(r_1 > N_1) \qquad (5)$$

where N_1 is the number of seats reserved (protected) for fare class 1, and r_1 is the number of requests for fare class 1. Equation 5 follows from (4) only if there are no cancellations of fare class 2 after bookings are closed. Further examination of this rule by BHATIA and PAREKH[6] and RICHTER[7] showed that for two nested fare classes this allocation was optimal, not just plausible.

The extension of optimal booking limits to more than two nested fare classes has been accomplished independently by BRUMELLE and MCGILL,[17] WOLLMER,[18] and CURRY.[19] Brumelle and McGill[17] use continuous distributions, and their expressions for optimal booking limits take the form of simple probability statements which nonetheless require multidimensional numerical quadrature (integration). Curry[19] also used continuous distributions but the optimum booking limits are expressed in terms of a convolution integral. Wollmer used discrete rather than continuous distributions of passenger demand, and his results are expressed in terms of a convolution sum rather than a convolution integral.

Belobaba[2,4] has applied a marginal revenue heuristic to more than 2 nested fare classes. The so-called Expected Marginal Seat Revenue model yields formulas for the booking limits for each fare class. More details of these methods are presented later.

Marginal revenue concepts may be applied to the seat inventory control problem on virtually nested CRSs,[2] with virtual classes being considered as normal leg-based booking classes. Like all leg-based seat inventory control, virtually nested systems cannot observe constraints for specific O&Ds.

Mathematical Programming

The mathematical programming approaches have concentrated on the benefits of controlling individual O&D itineraries. Consider a two-leg flight: A–B–C. If the demand for A–B and B–C is high, and the sum of

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these two fares exceeds the A–C fare, then an O&D control system would restrict passengers from making A–C bookings in favor of A–B and B–C bookings. A leg-based seat inventory control system (nested or not) cannot discriminate between A–B and A–C bookings on the A–B leg. A virtually nested system always prefers the higher fare of the A–C itinerary, although several approaches to alleviate this problem have been explored.[8–10]

BUHR[11] considered a problem consisting of a two-leg, one-class O&D network, and this was extended by WANG.[12] GLOVER et al.[13] formulated O&D optimization as a very realistic (and very large) network flow problem, although they used deterministic rather than probabilistic passenger demand models. The aircraft manufacturers have suggested using the mathematical programming approach[14–16] (as reported in Belobaba's survey article). Even for the mathematical approaches that model demand stochastically, none account for nesting in the CRS.

Besides controlling O&D fare classes, the mathematical programming approach makes it easy to incorporate important side constraints such as limiting the number of passengers for a particular origin–destination, or limiting the number of passengers in a particular fare class.

Summary

Both the marginal revenue approach and the mathematical programming approach capture important elements of the revenue management problem: the marginal revenue approach accounts for CRS nesting, but only controls seat inventory by controlling leg bookings; the mathematical programming approach will handle realistically large problems and it accounts for multiple O&D itineraries and side constraints; however, it determines allocations for individual fare classes (e.g., the sizes of fare class buckets) rather than booking limits for nested fare classes.

This paper extends the previous work by finding the optimal booking limits and seat allocations when fare classes are nested on an origin–destination itinerary and the inventory is not shared among origin–destinations. A seat inventory control system using these allocations can exercise control of O&D fare classes while accounting for fare class nesting.

PROBLEM FORMULATION

CRS Nesting

The objective of this paper is not to determine the "best" method of seat inventory control and nesting. Rather, it is the analysis and development of optimal seat allocations for a particular type of seat inventory control. Yield management analysts must work within the constraints of existing inventory control systems because it is usually easier to change the optimization methodology than the inventory control system. In addition, our experience in developing optimal seat allocations for many different inventory control systems has led us to the following observations:

- There are nearly as many different methods of inventory control as there are CRS systems. (This is due to local "tuning" of purchased CRS software.)
- Small change in the inventory control *logic* or the definition of the inventory control *task* can have a major impact on the necessary conditions of optimality and optimal allocation logic.
- Therefore, optimal seat allocation logic should be derived ab initio, without blindly applying the usual methodologies.

For this analysis we assume a form of nesting that is both important and practical, one which is based on the concept of a fare class "nest." A nest is a collection of fare classes, all for the same O&D, which are ordered in a strict hierarchy. A higher ranked fare may take a seat from the inventory of any lower ranked fares in the same nest, but not from the inventory of fares outside the nest. As with leg-based nesting, seat inventory control is defined in terms of a booking limit for each fare class in the nest. The nest allocation is the number of seats allocated to all fare classes within the nest; this is the booking limit of the highest valued fare class in the nest. There are several special cases of nests.

- A fare class nest may consist of a single fare class bucket for an origin–destination.
- A fare class nest may consist of all fare classes for an origin–destination.
- Between these two extremes it is possible to have several nests for any O&D to account for various controls and constraints, such as point-of-sale control.
- A fare class nest for all bookings on a single leg (i.e., without consideration of actual O&D itineraries) represents the conventional leg-based nested seat inventory control system.

Application of O&D Nesting

The optimal allocation of seats developed in this paper can be applied when inventory is controlled with fare classes nested by O&D. There are several important applications where inventory is controlled in this manner:

Reservations Systems

At least one commercially available reservations system allows control of fare classes nested on seg-

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ments (O&D itineraries on the same flight number). In this case, a nest of fare classes consists of all fare classes on each segment (including one-leg itineraries). Although this form of inventory control may be useful for O&Ds defined by a line-of-flight, it is not practical for large O&D networks. This is a result of the small number of passengers on any one O&D itinerary; nesting, or sharing of inventory across O&Ds should be used in these cases.

Overlapping Flights

Some reservations systems maintain overlapping flights or phantom flights which use the same physical aircraft to transport passengers. In many of these systems, the inventory is not shared among overlapping flights. In this situation, a fare class nest can be assigned to each O&D for all overlapping flights. The sum of each phantom flight's authorizations on the leg with overlap is the cabin authorization for that leg.

Point-of-Sale Control

Point-of-sale control is used to limit sales from specific locations to take advantage of exchange rates, limit bad debts, etc. When the CRS allows point-of-sale control, it can be incorporated in the optimization by assigning a fare class nest to each point-of-sale; the fares within each nest are similar except for the net value to the airline. In this application, there are multiple fare class nests for the same O&D, one nest for each point-of-sale.

Group Acceptance

Passengers booked as a group will usually fly on more than one leg, and each leg can be on different days. Determining the true value of a group may be difficult because they may displace passengers on some legs, but contribute full value on others. The total value of a group may be determined by constructing an O&D network from all the legs on which the group will fly, even if they are on different days. A one-class fare class nest would be assigned to the group, and the other fare class nests would be composed of the usual fare classes. Optimizing seat allocations under these conditions will account for the total revenue impact of accepting the group.

Problem Statement

The revenue from an entire O&D network is the sum of the revenue from each O&D itinerary. (By an O&D network, we mean the collection of O&D itineraries whose seat inventory is being controlled simultaneously.) The revenue from each O&D itinerary is the sum of the revenue for all nests on that O&D. We wish to maximize the expected revenue from the complex, subject to the capacity constraints on each leg. Thus the objective is

$$\text{maximize } \textstyle\sum_k R^k(A^k) \qquad (6)$$

$$\text{subject to } \textstyle\sum_{k \in j} A^k \leqslant C^j \qquad (7)$$

where R^k is the expected revenue for the kth fare class nest, A^k is the allocation for the kth fare class nest, and C^j is the booking capacity for the jth leg in the network. The sum in Equation 7 is taken over all fare class nests traveling the jth leg.

Other constraints can be incorporated into the optimization problem statement if they can be formulated as linear inequality or equality constraints. An example of such constraints is minima/maxima on the number of passengers allowed on an O&D itinerary. We do not consider these any further because they are easily incorporated into the mathematical programming formulation.

Assumptions

It is necessary to make assumptions to proceed with the analysis and these assumptions will be reviewed later. As with the previous research, we assume that:

1. Lower-valued fare classes book before higher-valued fare classes.
2. The fare classes within a nest are ordered by fare value (highest ranked class has highest fare value).
3. There are no cancellations of bookings.
4. Demand among fare classes is independent (demand in one class does not contain information about demand in other fare classes).
5. A denied request is revenue lost to the airline, i.e., we do not consider that a passenger who is denied a request will buy a higher value ticket (passenger sell-up) or take another flight on the same airline.

Actually, Assumptions 1 and 2 can be replaced by the assumption that fare classes book one at a time. The optimization conditions will determine booking limits regardless of whether or not lower-valued classes book before higher-valued classes.

Solution

The assumption of no cancellations allows us to consider the problem of allocating all nonbooked reservations spaces. The total number of authorized reservations spaces may be greater than the actual capacity of the legs because of overbooking. Overbooking is used to account for cancellations between the time of the optimization and the day of departure, and no-shows (reservations which are booked but for which no passenger shows up).

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The "Appendix" shows that the expected revenue of a fare class nest is a convex function of the nest allocation. It follows that the expected revenue of the entire network, Equation 6, is a separably convex function of the nest allocations. Thus the maximum revenue can be found with linear programming algorithms after the nonlinear terms in Equation 6 have been approximated with piecewise-linear functions.

The solution follows these steps:

1. Compute a piecewise linear approximation to the expected revenue function $R^k(A^k)$ for each fare class nest k and all values of the nest allocations, A^k. Equivalently, and more easily, compute a stepwise approximation to the *slope* of the expected revenue functions as a function of A^k.
2. Use linear programming to find the fare class nest allocations A^k which maximize the expected revenue of the network (Equation 6) subject to the constraints of Equation 7. This is accomplished by assigning a linear programming decision variable to each step in the stepwise approximations to the slope.
3. Compute the optimum booking limits within each fare class from the allocation A^k for that nest.

In the above process, the slope of the revenue function depends on the booking limits used. Since these are the optimum booking limits, the next two sections address each of the following points in turn:

- Find the *optimal booking limits* for a given nest allocation. This ensures maximum revenue, whatever the allocation for the nest.
- Find the *optimal nest allocations*, consistent with the capacity/authorization constraints.

OPTIMAL BOOKING LIMITS

DERIVATIONS of the optimal booking limits have appeared elsewhere.[17,18] The derivation described here (and in the "Appendix") was developed independently. It is included because it contains unique information required for the O&D allocation problem: expressions for the slope of the expected revenue function with respect to the size of the fare class nest (cabin size), and a proof that the expected revenue function is convex.

Booking Limits

Figure 1 shows the notation to be used in describing a nest of n fare classes:

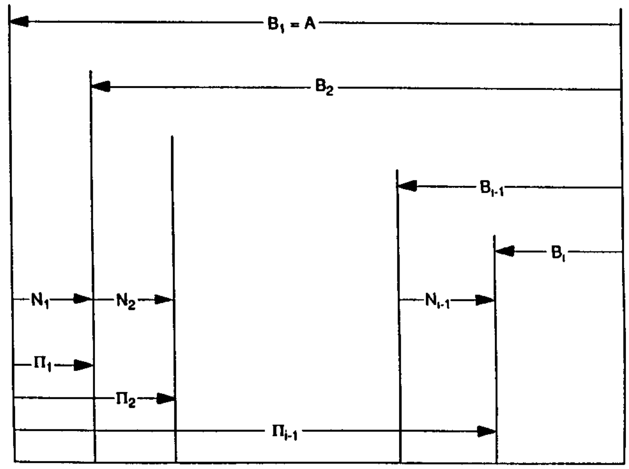

Fig. 1. Relationship among booking limits (B), protection levels (Π), nested protection levels (N), and nest allocation (A).

B_i = Booking limit, or number of seats available, for class i. Class i is open as long as the number of bookings in class i and lower classes remain less than this limit.

Π_i = Protection limit for classes 1 through i. This many seats are protected from classes $i+1, \ldots, n$.

N_i = Nested level protection for class i. This many seats are protected for class i from all lower classes.

A = Allocation for the nest. This is the maximum number of seats that may be booked by fare classes in the nest.

The following relationships are valid among the variables defined above.

$$B_1 = A$$

$$A = N_1 + N_2 + \cdots + N_{n-1} + B_n$$

$$\Pi_i = N_1 + N_2 + \cdots + N_i$$

$$\Pi_i = A - B_{i+1}$$

Maximum Expected Revenue

Let $R_i(\Pi_{i-1}, A)$ denote the expected revenue for i classes. This is a function of Π_{i-1} the protection level for classes $1, 2, \ldots, i-1$, and the nest allocation A. It also depends on $R_{i-1}(\Pi_{i-2}, A)$, the expected revenue for $i-1$ classes. $R_i(\Pi_{i-1}, A)$ exists only when $A > \Pi_{i-1}$, i.e., when the nest allocation is greater than the protection level for classes $1, 2, \ldots, i-1$. Thus the total expected revenue is given by

$$R(A) = R_i(\Pi_{i-1}, A), \quad \Pi_{i-1} \leqslant A < \Pi_i, \quad i = 1, \ldots, n \tag{8}$$

The total revenue for the fare class nest $R(A)$ and the individual R_i functions are shown in Figure 2 for a three-class example. R_1 is the expected revenue when (only) class 1 books; R_2 is the expected reve-

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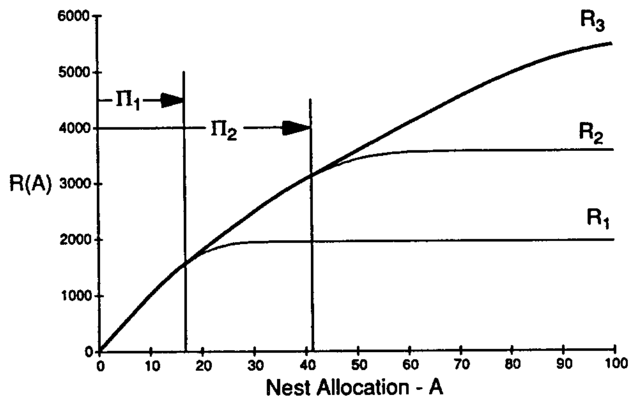

Fig. 2. Expected revenue versus fare class nest allocation A. R_i is the expected revenue for fare classes $1, \ldots, i$.

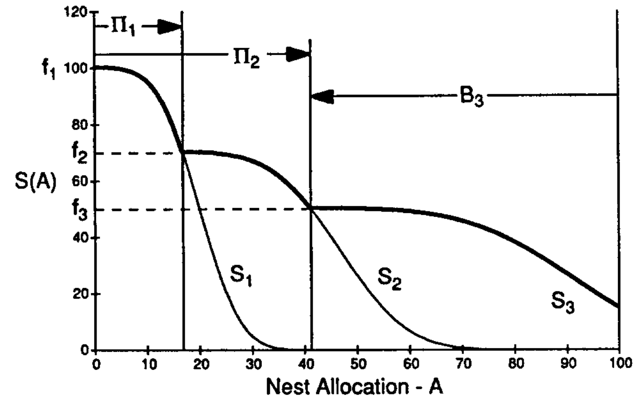

Fig. 3. Slope of expected revenue versus fare class nest allocation A. S_i is the slope of the expected revenue for fare classes $1, \ldots, i$.

nue when classes 1 and 2 book; R_3 is the expected revenue when classes 1, 2 and 3 book. Class 2 is not allowed to book until there are at least Π_1 seats protected for class 1; similarly, class 3 is not allowed to book until there are at least Π_2 seats protected for class 2.

The "Appendix" derives necessary conditions for the optimal booking limits for a nest of fare classes (or classes nested on a leg). The optimal protection level Π_{i-1} is determined from the following equation:

$$\frac{\partial R_{i-1}}{\partial A}(\Pi_{i-2}, \Pi_{i-1}) \equiv S_{i-1}(\Pi_{i-2}, \Pi_{i-1}) = f_i \quad (9)$$

In this equation S_{i-1} is the slope of the expected revenue function with respect to the nest allocation for $i - 1$ fare classes. Figure 3 shows how these protection levels are determined. S_1 is the revenue slope when (only) class 1 books; S_2 is the revenue slope when classes 1 and 2 book; S_3 is the revenue slope when classes 1, 2 and 3 book. The optimality conditions state that seats should be protected for class 1 as long as the revenue slope is greater than f_2, the fare of the next lower class; seats should be protected for classes 1 and 2 as long as the slope of the revenue from class 1 *and* class 2 is greater than f_3, the fare from the next lower class.

From the revenue perspective, it can be seen in Figure 2 that the revenue growth continues to decrease as the allocation is increased. When the rate of revenue growth for all active classes becomes less than the fare of the next lower fare class, then the next lower class is allowed to book, thus maintaining the revenue growth.

OPTIMAL NEST ALLOCATIONS

THE SOLUTION to the problem of finding the optimal booking limits has an analytical flavor in the sense that we derived the necessary conditions for optimality (Equation 9). In contrast, the solution to finding the optimal nest allocations has a numerical flavor since we will show how the optimal nest allocations may be found with mathematical programming. This numerical approach is treated in two parts: fare class buckets (a one class nest) and fare class nests.

Fare Class Buckets

For the special case when all fare class nests are individual fare classes and there is no nesting, the problem is the same one solved by Glover et al.[13] with deterministic demand, and by others with probabilistic demands.[14-16] Our approach, first proposed by D'SYLVA,[16] is to use a piecewise linear approximation of the expected revenue function. The expected revenue function is separably convex since the slope of the expected revenue for each fare class never increases as the fare class allocation increases. The slope never increases because it is proportional to $P(r > A)$. Importantly, the solution to this seat allocation problem is integer-valued because this is a form of the transportation problem which has integer solutions if the input parameters are integers.

Figure 4 shows the expected revenue curve, $R(A)$ and its slope $S(A)$ for a single fare class bucket, a typical distribution (e.g., truncated Gaussian), and a piecewise linear approximation. Experimentation has shown that 5–10 such pieces are adequate for each single-class nest. A decision variable is allocated to each linear piece, and a numerical solution can be found quite efficiently using special purpose algorithms.

Fare Class Nests

The very same techniques and computer code used for fare class buckets can be used to optimize a network of O&D fare classes if the expected revenue of the nest, $R(A)$, is convex. The "Appendix" derives S_i,

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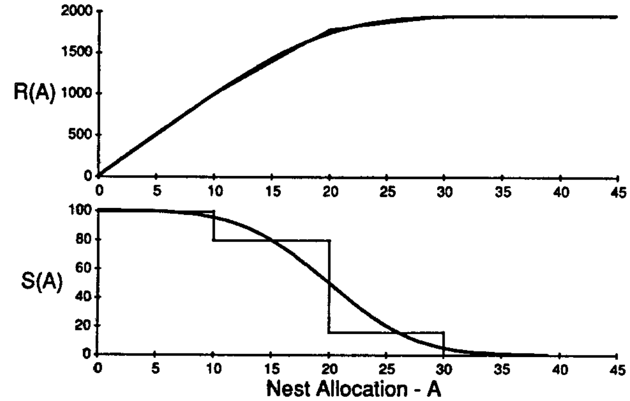

Fig. 4. Expected revenue R(A) and its slope S(A) versus allocation for a fare class bucket. Also shown is a piecewise linear approximation.

the revenue slope for i active fare classes, in terms of S_{i-1}, the revenue slope for $i - 1$ active fare classes.

$$S(A) = S_i(\Pi_{i-1}, A) \qquad \Pi_{i-1} \leq A < \Pi_i, \quad i = 1, \ldots, n \tag{10}$$

$$S_i(\Pi_{i-1}, A) = f_i P(r_i > A - \Pi_{i-1}) + \int_0^{A-\Pi_{i-1}} dr_i\, p_i(r_i) S_{i-1}(\Pi_{i-2}, A - r_i) \tag{11a}$$

$$S_1(\Pi_0, A) = f_1 \int_A^{\infty} p_1(r_1)\, dr, \quad \Pi_0 = 0 \tag{11b}$$

where $p_i(r_i)$ is the probability density function of requests for class i. The "Appendix" also shows that the revenue function R_i is a convex function of the nest allocation, that is, the slope S_i never increases with increasing number of fare classes and nest allocation. Thus the nest allocations which maximize revenue for the entire O&D network can be found with the same techniques used for fare class buckets: piecewise linear approximations to the expected revenue of the nest.

DISCUSSION

Optimal Booking Limits

Different but equivalent versions of the optimal booking limits have been derived by Brumelle and McGill,[17] and by Wollmer.[18] Brumelle and McGill[17] use continuous distributions and derive necessary conditions similar to Equation 9. Their expressions for optimal booking limits take the form of simple probability statements which nonetheless require multidimensional numerical quadrature (integration). Wollmer used discrete rather than continuous distributions of passenger demand, and his results are expressed in terms of a convolution sum rather than a convolution integral as in Equation 11.

The optimal booking limit for the two-class case is the same for all three derivations

$$f_2 = S_1(0, \Pi_1) = f_1 P(r_1 > \Pi_1). \tag{12}$$

This limit also agrees with that published by Littlewood[5] and Belobaba.[2,4] However, when there are more than two fare classes, the optimal booking limits do not agree with the booking limits of the expected marginal seat revenue (EMSR) model.[2,4] The difference lies in the computation of the protection level of the higher fares. EMSR "protects" higher fare seats from each lower fare class using the two-class formulas:

$$\Pi_{i-1} = \textstyle\sum_{j=1}^{i-1} S_i^j \quad i = 2, \ldots, n \tag{13}$$

where S_i^j satisfies

$$f_i = f_j P(r_j > S_i^j), \qquad j < i, \quad i = 2, \ldots, n. \tag{14}$$

The protection level for class $i - 1$ is the sum of seats protected for classes $1, \ldots, i - 1$ against class i, assuming that only two fare classes are active at any one time. These computations are done for all possible pairs of fare classes.

The accuracy of the EMSR approximation has been evaluated over limited range of variables. Brumelle and McGill[17] considered only three classes with fare ratios of {1.0, 0.7 to 0.9, 0.6 to 0.8}, which is not as low as the 0.30 of many discount fares. The demand distributions were Gaussian with coefficient of variations of 0.4. Wollmer used 5 fare classes with fare ratios of {1.0, 0.540, 0.508, 0.495, 0.333} and Gaussian demands with a coefficient of variation of 0.33. Both investigators found that the revenue penalty of the EMSR algorithm rarely exceeded 0.5%, but the booking limits may be substantially different from the optimal settings, suggesting a very flat revenue function. Lastly, BELOBABA et al.[22] have reported similar results with 5 fare classes and more disparate fare ratios.

Revenue Slope

The revenue slope is required as an input to the linear program to find optimal fare nest allocations. A sketch of the slope functions for a nest are shown in Figure 3, discussed previously. This figure, and Equation 11 for S_i show how the slopes are solved from low to high allocations (and from higher fare classes to low), and how the slopes and optimal booking limits are computed at the same time:

1. Find S_1 the revenue slope for fare class 1 for the nest allocation ranging between 0 and a practical

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upper limit (determined by capacity limits or numerically small answers relative to fare f_2)

$$S_1(0, A) = f_1 \mathrm{P}(r_1 > A) \quad 0 \leq A < \infty. \tag{15}$$

2. Find Π_1, the protection level for fare class 1, from the necessary condition

$$f_2 = S_1(0, \Pi_1) = f_1 \int_{\Pi_1}^{\infty} p_1(r_1)\, dr_1 \tag{16}$$

Repeat the following steps for $i = 2, 3, \ldots, n-1$.

3. Find the revenue slope for i fare classes, S_i, from Equation 11; compute this as the allocation A ranges from Π_{i-1} to some practical upper limit.
4. Find the next higher protection level Π_i at the nest allocation where

$$f_{i+1} = S_i(\Pi_{i-1}, \Pi_i), \quad i = 2, \ldots, n-1. \tag{17}$$

In other words, the slope functions are solved in a "boot strap" manner, i.e., S_1 and Π_1 are found for the first class; then S_2 and Π_2 are found from S_1 and Π_1; and so on.

Several approaches are possible for the numerical solution of the revenue slope. LaPlace transforms can be used to find each expected S_i. Numerical quadrature is a viable option; we have found that a dr_i of one standard deviation provides a sufficiently accurate approximation to the exact dr_i of 1 seat or request. Lastly, the EMSR approximation could be used to determine the booking limits. Further approximations to the expected revenue slope function S_i will also be required, taking care to ensure convexity of the expected revenue function.

Review of the Assumptions

At this point we examine the impact and importance of the assumptions which were necessary to obtain the results derived in this paper.

Fare Class Hierarchy

It was assumed that the hierarchy of nesting in the CRS was the same as the hierarchy of fare values. Although this is likely when considering gross fares, it may not be true when using net fares which are the proper fares to use for optimization. Fare classes should not be protected from lower-ranked fares which have a higher value, i.e., these classes should have the same booking limit. If the lower-ranked classes book first then it is possible, and desirable, for all of the allocated seats to be taken by the lower-ranked but higher-valued class.

Lower-Class Books First

This assumption was investigated by MAYER[20] and TITZE and GRIESSHABER[21] for the two-class case. Simulations showed that the results are robust to deviations in the assumption because early booking by higher (more valuable) classes produces more revenue and reduces the impact of errors in seat allocation.

No Cancellations

Cancellation of current and future bookings can be implicitly incorporated in this formulation by using an appropriate overbooking level for each leg. This overbooking level may change each time the optimization is performed, since it will be a function of days-left before departure. Cancellations cannot be explicitly incorporated without a reformulation of the problem.

Independent Class Demand

Optimal allocations are much more complex if class demands are not independent. For example, if requests for class n alter the probability density function of class 1 requests, then the slope S_1 should not be calculated until the class n requests have been received. It is only then that the probability density function of r_1 is known. But the booking limit B_n cannot be computed without knowing S_1.

An iterative approach may be possible, but seems unwieldy and unnecessary. Seat inventory control systems are applied dynamically, i.e., the static problem outlined above is solved several times during the booking process. This is usually done in two separate procedures: forecasting remaining demand and optimization. If class demands are dependent then actual booking information in one class can be used to improve the forecast of remaining demand in other classes.

To the best of our knowledge, all present-day, practical seat inventory control systems use the two-step process of reforecasting demand and then applying the (static) formulas using the revised forecasts. This captures the first order effect of dependent class demand in a practical, convenient manner.

Passenger Sell-Up

Passengers who are denied a reservation request may purchase a higher valued fare, and this action, called a sell-up, can be important: if *some* of the denied requests will purchase higher fares, then reducing the booking limit may generate sell-ups and possibly more revenue with fewer passengers. This type of passenger behavior was not considered here. Belobaba[2,4] has examined the possibility of a one class vertical shift into the lowest class which is available for booking. That formulation does not reflect the influence of booking limits on requests for higher fare classes. In general, the *distribution* of requests in higher classes

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should depend on the booking limit of lower classes. For example, the probability density function of requests $p_{i-1}(r_{i-1})$ becomes $p_{i-1}(r_{i-1}/r_i, B_i)$. Taking partial derivatives with respect to B_i yields additional terms in Equation (A-2) of the form

$$\int_0^{B_i} dr_{i-1} \frac{\partial}{\partial B_i} p_{i-1}(r_{i-1} \mid r_i, B_i). \qquad (18)$$

These are very complex expressions, even for simple sell-up models. Thus other approaches should be used when passenger sell-ups are important, approaches which make realistic assumptions about passenger choice behavior. Modeling passenger choice behavior for sell-ups is difficult; little has been published in the open literature, and it appears to be the subject of much present-day research effort.

SUMMARY

PREVIOUS RESEARCH in the optimal allocation of airline seats has followed one of two themes: marginal seat revenue or mathematical programming. Both the marginal seat revenue approach and the mathematical programming approach capture important elements of the revenue management problem: the marginal revenue approach accounts for CRS fare class nesting, but only controls seat inventory by controlling leg bookings. The mathematical programming approach handles realistically large problems and can control seat inventory on multiple O&D itineraries with other side constraints, but it does not account for fare class nesting in the reservations systems.

This paper has combined both approaches by developing equations for maximum revenue on an O&D network of itineraries when the fare classes are nested by origins–destinations and inventory is not shared among origins–destinations (fare class nests). These results are applicable to certain reservations systems, point-of-sale control, control of overlapping flights, and analysis of a group of passengers over their entire itinerary.

These new results were also applied to an old problem: determining booking limits for a CRS with leg-based nesting of fare classes. The results were consistent with the other, independently derived, formulations.[17,18]

In spite of the recent analytical advances in optimal booking limits, the optimal allocation of O&D seats with shared inventory has not yet been derived. This is an important and unsolved problem. (The allocations for nonshared inventory are provided by the mathematical programming formulations.) Even the form of the optimal solution would be valuable information. It may be, for example, that booking limits are not the proper architecture at all. Or if booking limits are appropriate, how should the inventory be shared among different O&D fare classes? Needless to say, the airlines have not waited for these results and are proceeding with virtual nesting and other heuristic approaches until analytical and simulation results provide something better.

APPENDIX

THE OBJECTIVE of this Appendix is to determine the optimal booking limits and the revenue slope of a "nest" of fare classes in which the fare classes are nested in a strict hierarchy. These results can be directly applied to leg-based nesting systems or as part of the process of maximizing the revenue of a network that has fares nested by origins–destinations. The work presented here was first reported in [19]. Independent derivations using alternative formulations may be found in Brumelle and McGill[17] and Wollmer.[18]

Optimal Booking Limits

Optimal Revenue

The notation for booking limits and fare class protection levels are displayed in Figure 1. It is assumed that the classes book one at a time. Since the fare requests in each class are independent, we may find the expected revenue for i classes, R_i, in terms of the revenue for class i, *plus* the expected revenue of the remaining $i-1$ classes; however, these $i-1$ classes are subject to an allocation limit which is the actual allocation A less the number of passengers who have booked in class i

$$\begin{aligned} &R_i(\Pi_{i-1}, A) \\ &= \int_0^{A-\Pi_{i-1}} dr_i p_i(r_i)[f_i r_i + R_{i-1}(\Pi_{i-2}, A - r_i)] \\ &\quad + [f_i(A - \Pi_{i-1}) + R_{i-1}(\Pi_{i-2}, \Pi_{i-1})] \int_{A-\Pi_{i-1}}^{\infty} dr_i p_i(r_i) \\ &\qquad \Pi_0 \equiv 0, \quad R_0 \equiv 0 \quad \text{(A-1)} \end{aligned}$$

where $p_i(r_i)$ is the probability density function of class-i requests, and $R_{i-1}(\Pi_{i-2}, A - r_i)$ is the expected revenue of the remaining $i-1$ classes. Each of the two terms contains the expected revenue for fare class i plus the expected revenue from the remaining $i-1$ classes. Note that the number of seats available to the $i-1$ classes has been reduced by the already-received requests for class i. The first term is the expected revenue when the number of requests for class i are less than the booking limit $B_i = A - \Pi_{i-1}$; the second term is the expected revenue when the number of requests for class i equals or exceeds the booking limit B_i.

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Taking the derivative with respect to Π_{i-1}, and using the identity $B_i = A - \Pi_{i-1}$ yields

$$\begin{aligned}\frac{\partial R_i(\Pi_{i-1}, A)}{\partial \Pi_{i-1}} &= -p_i(B_i)[f_i B_i + R_{i-1}(\Pi_{i-2}, \Pi_{i-1})] \\ &\quad + p_i(B_i)[f_i B_i + R_{i-1}(\Pi_{i-2}, \Pi_{i-1})] \\ &\quad + \left[-f_i + \frac{\partial R_{i-1}(\Pi_{i-2}, \Pi_{i-1})}{\partial A}\right] \int_{B_i}^{\infty} dr_i\, p_i(r_i)\end{aligned} \tag{A-2}$$

$$= \left[-f_i + \frac{\partial R_{i-1}(\Pi_{i-2}, \Pi_{i-1})}{\partial A}\right] \mathrm{P}(r_i > B_i) \tag{A-3}$$

The above results make extensive use of the identity for the derivative of an integral whose limits are a function of the variable. Equation A-3 leads directly to the necessary condition to determine the protection limit for maximum revenue:

$$\frac{\partial R_{i-1}(\Pi_{i-2}, \Pi_{i-1})}{\partial A} = f_i \tag{A-4}$$

In other words, the protection limit for class $i - 1$, Π_{i-1}, is increased from Π_{i-2} as long as the expected revenue slope for $i - 1$ classes equals or exceeds the certain revenue of a single request in class i.

Revenue Slope

The derivative of the expected nest revenue with respect to the nest allocation A is required for the numerical solution of optimal nest allocations. For notational convenience, define

$$S(A) = \frac{\partial R_i(\Pi_{i-1}, A)}{\partial A} \equiv S_i(\Pi_{i-1}, A), \qquad \Pi_{i-1} \leq A < \Pi_i, \quad i = 1, \ldots, n \tag{A-5}$$

where

$$\begin{aligned}\frac{\partial R_i(\Pi_{i-1}, A)}{\partial A} &= +p_i(B_i)[f_i B_i + R_{i-1}(\Pi_{i-2}, \Pi_{i-1})] \\ &\quad + \int_0^{B_i} dr_i\, p_i(r_i) \frac{\partial R_{i-1}(\Pi_{i-2}, A - r_i)}{\partial A} \\ &\quad - p_i(B_i)[f_i B_i + R_{i-1}(\Pi_{i-2}, \Pi_{i-1})] \\ &\quad + f_i \int_{B_i}^{\infty} dr_i\, p_i(r_i).\end{aligned} \tag{A-6}$$

This may be further simplified to give

$$\frac{\partial R_i(\Pi_{i-1}, A)}{\partial A} = f_i \int_{B_i}^{\infty} dr_i p(r_i)\, dr_i + \int_0^{B_i} dr_i\, p_i(r_i) \frac{\partial R_{i-1}(\Pi_{i-2}, A - r_i)}{\partial A} \tag{A-7}$$

or, in terms of the slope notation,

$$S_i(\Pi_{i-1}, A) = f_i \mathrm{P}(r_i > B_i) + \int_0^{B_i} dr_i\, p_i(r_i) S_{i-1}(\Pi_{i-2}, A - r_i). \tag{A-8}$$

Convexity

The convexity of R_i can be examined by taking the derivative of Equation A-7:

$$\frac{\partial^2 R_i}{\partial A^2} = -p_i(B_i)\left[f_i - \frac{\partial R_{i-1}(\Pi_{i-2}, \Pi_{i-1})}{\partial A}\right] + \int_0^{B_i} dr_i\, p_i(r_i) \frac{\partial^2 R_{i-1}}{\partial A^2}(\Pi_{i-2}, A - r_i). \tag{A-9}$$

The first term is zero because it satisfies the necessary condition of optimality for protection levels. Therefore, the second derivative of R_i becomes

$$\frac{\partial^2 R_i}{\partial A^2} = \int_0^{B_i} dr_i\, p_i(r_i) \frac{\partial^2 R_{i-1}(\Pi_{i-2}, A - r_i)}{\partial A^2}. \tag{A-10}$$

This is solved recursively for each fare class starting with the first fare class:

$$R_1(0, A) = f_i \int_0^{A} dr_1\, p_1(r_1) r_1 + f_1 A_1 \int_A^{\infty} dr_1\, p_1(r_1) \tag{A-11}$$

$$\frac{\partial R_1(0, A)}{\partial A} = f_1 \int_A^{\infty} dr_1\, p_1(r_1) \tag{A-12}$$

$$\frac{\partial^2 R_1(0, A)}{\partial A^2} = -f_1 p_1(A). \tag{A-13}$$

These equations show that the expected revenue for a fare class nest is convex with respect to the allocation A: (A-13) shows that R_1 is convex everywhere; (A-10) shows that R_2 will be convex everywhere because R_1 is convex everywhere and $p_2(r_2) \geq 0$; by induction, the expected revenue for all numbers of active fare classes will be convex.

Thus it follows that the expected revenue of the entire O&D network, Equation 6, is separably convex

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in the nest allocations because the expected revenue of each fare class nest is a convex function of its own allocation.

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