Final Minor Project Part A Report

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1 Introduction

Ventura Systems is an innovative Dutch door system manufacturer for public transport. Based on their plug sliding bus door, we will try to model a bus breaking and/or driving over a bump.
In part A, we have considered four basic physical scenarios:

1. Point mass attached to a spring damper system;
2. Point mass attached to a spring-damper system with collisions;
3. Two (and more) point masses attached by springs (1D);
4. Bending beam

These scenarios, described in Section 3, will form the theoretical foundation for the 2D bus door model that will be proposed in Section 4. In short, this model is a beam with one fixed end, and one end attached to both a spring and a damper.
Lastly, we will describe what steps we are planning on taking in Part B to realize and perform tests on the model described in Section 4.

2 Analytical Solution

In the following section, we present some theoretical background to very common scenarios related to spring damper systems. More precisely, we elaborate on the analytical solutions obtained from such problems.

2.1 No damping, No external force

$$m\ddot{x} + kx = 0 \tag{1}$$

In this case, we only consider Newton's law and the spring force to obtain the eigenfrequency of the spring. We guess that the solution is of the form $x = A(cos(\omega_0 t) + isin(\omega_0 t) = Ae^{i\omega_0 t}$.

$$-Am\omega_0^2 e^{i\omega_0 t} + Ake^{i\omega_0 t} = 0$$

$$Ae^{i\omega_0 t}(k - m\omega_0^2) = 0$$

$$m\omega_0^2 = k$$

Finally, the eigenfrequency is:

$$\omega_0 = \pm\sqrt{\frac{k}{m}}$$

2.2 Damping, No external force

We now consider the case with damping.

$$m\ddot{x} + \gamma\dot{x} + kx = -F_{ext} \tag{2}$$

First we solve for the case where $F_{ext} = 0$ to obtain the homogeneous solution:

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

We guess that the solution is of the form $x(t) = Fe^{i\omega t}$.

$$-m\omega^2 Fe^{i\omega t} + \gamma Fi\omega e^{i\omega t} + kFe^{i\omega t} = 0$$

$$Fe^{i\omega t}(-m\omega^2 + \gamma i\omega + k) = 0$$

We use that $\zeta = \frac{\gamma}{\sqrt{2km}}$:

$$-\omega^2 + \frac{\gamma}{m} i\omega + \frac{k}{m} = 0$$

$$-\omega^2 + \zeta 2i\omega_0\omega + w_0^2 = 0$$

$$\omega = \frac{-\zeta 2i\omega_0 - \sqrt{(\zeta 2i\omega_0)^2 + 4\omega_0^2}}{-2}$$

$$\omega = \omega_0(\zeta i + \sqrt{1-\zeta^2})$$

$$Fe^{-\omega_0\zeta t + i\omega_0\sqrt{1-\zeta^2}t}$$

$$Fe^{-\omega_0\zeta t}e^{i\omega_0\sqrt{1-\zeta^2}t}$$

$$Fe^{-\omega_0\zeta t}(sin(\omega_0\sqrt{1-\zeta^2}t) + icos(\omega_0\sqrt{1-\zeta^2}t))$$

$$e^{-\omega_0\zeta t}(c_1 sin(\omega_0\sqrt{1-\zeta^2}t) + c_2 cos(\omega_0\sqrt{1-\zeta^2}t))$$

$$x(0) = x_0 = c_2$$

$$\dot{x}(0) = v_0 = e^{-\omega_0\zeta * 0}(c_1\omega_0\sqrt{1-\zeta^2}cos(0) - \zeta\omega_0 e^{-\omega_0\zeta t}(c_1 sin(..) + c_2 cos(...))$$

$$v_0 = c_1\omega_0\sqrt{1-\zeta^2} - \zeta\omega_0 c_2$$

$$c_1 = \frac{v_0 + \zeta\omega_0 x_0}{\omega_0\sqrt{1-\zeta^2}}$$

$$x_h(t) = e^{-\omega_0\zeta t}(c_1 sin(\omega_0\sqrt{1-\zeta^2}t) + c_2 cos(\omega_0\sqrt{1-\zeta^2}t)) \tag{3}$$

Finally, the solution for the homogeneous case is

$$x_h(t) = e^{-\omega_0\zeta t}((\frac{v_0 + \zeta\omega_0 x_0}{\omega_0\sqrt{1-\zeta^2}})sin(\omega_0\sqrt{1-\zeta^2}t) + x_0 cos(\omega_0\sqrt{1-\zeta^2}t)) \tag{4}$$

3 Scenarios

Before examining the scenarios for part B of the project, it is important to consider the theoretical foundations required to build a more complex situation. Our team has investigated and simulated 4 situations, the results of which are crucial to understanding the scenario in part B.

3.1 Point mass attached to spring damper system

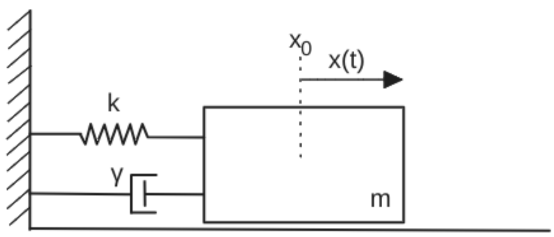

Figure 1: Spring-damper system

Let us analyze a system with a constant force applied: then equation (2) becomes:

$$m\ddot{x} + kx = F_{ext} \tag{5}$$

where F_{ext} is a constant. Then every solution of equation (5) can be written in the form: $x(t) = x_{homogeneous}(t) + x_{particular}(t)$.
The homogeneous solution x_h can be obtained from equation (3) when $\gamma = 0$ ($\zeta = \frac{\gamma}{\sqrt{2km}} = 0$).

$$u_h = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) \tag{6}$$

For the particular solution, we have $x_p = A \cdot F$ so we can plug it into equation (5). A and F are two constants so the second derivative of $\ddot{x} = 0$.

$$\begin{aligned} k(A \cdot F) &= F \\ A &= \frac{1}{k} \end{aligned} \tag{7}$$

Combining equations (6) and (7) the solution is obtained:

$$u(t) = c_1 cos(\omega_0 t) + c_2 sin(w_0 t) + \frac{F}{k} \tag{8}$$

To find c_1 and c_2 initial conditions can be plugged. ($c_1 = u_0 - \frac{F}{k}$ and $c_2 = \frac{v_0}{\omega_0}$)

Let us analyze a system with a constant force applied and a damping factor c:

$$m\ddot{u} + c\dot{+}ku = F \tag{9}$$

The solution to this problem will be in the form $x(t) = x_{homogeneous}(t) + x_{particular}(t)$. The homogeneous solution $x_h(t)$ is already found in equation (3) and the particular solution can be found following the same steps as in equation (7). The solution is then:

$$x(t) = e^{-\omega_0 \zeta t}(c_1 sin(\omega_0\sqrt{1-\zeta^2}t) + c_2 cos(\omega_0\sqrt{1-\zeta^2}t)) + \frac{F}{k} \tag{10}$$

Initial conditions can be used to find the constants c_1 and c_2. ($c_1 = u_0 - \frac{F}{K}$ and $c_2 = \frac{v_0 + \omega_0 \zeta c_1}{\omega\sqrt{1-\zeta^2}}$)

3.2 Two point masses attached by springs

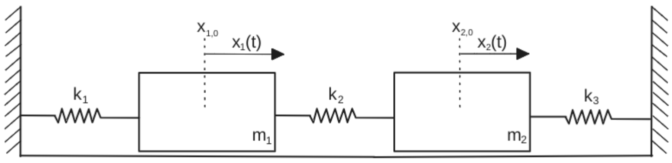

Figure 2: Two point masses attached by springs

We now consider an undamped two-spring-coupled mass system without external forces. We can describe the motion of the masses with the following equations:

$$\begin{cases} m_1\ddot{u}_1 = -(k_1 + k_2)u_1 + k_2 u_2 \\ m_2\ddot{u}_2 = k_2 u_1 - (k_2 + k_3)u_2 \end{cases}. \tag{11}$$

We rewrite this system of equations for convention into a matrix-vector system.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

To find a solution to these second-order differential equations, we make a guess for the solution vector $\vec{u}$:

$$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t} \tag{13}$$

$$\ddot{\vec{u}} = \begin{bmatrix} (i\omega)^2 a \\ (i\omega)^2 b \end{bmatrix} e^{i\omega t} \tag{14}$$

When we fill these guesses into our previously defined matrix vector system we obtain the following:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} (i\omega)^2 a \\ (i\omega)^2 b \end{bmatrix} e^{i\omega t} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} -\omega^2 m_1 & 0 \\ 0 & -\omega^2 m_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

The found solution for the coefficients of $\ddot{u}$ is the zero vector, which gives the trivial solution to this system's differential equations. This describes the motion of the masses while they are in the equilibrium state.

To find non-trivial solutions, we know that the solution for the coefficients of $\ddot{u}$ should not be the zero vector. This implies that inverse matrix in equation 19 does not exist, or that the determinant of this matrix is equal to zero. Below, we will refer to this matrix as matrix A.

$$det(A) = det(\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix}) = 0 \tag{20}$$

$$(-\omega^2 m_1 + k_1 + k_2) * (-\omega^2 m_2 + k_2 + k_3) - k_2^2 = 0 \tag{21}$$

$$\omega^4 m_1 m_2 - \omega^2 m_1(k_2 + k_3) - \omega^2 m_2(k_1 + k2) + k_1 k_2 + k_1 k_3 + k_2 k_3 = 0 \tag{22}$$

We can find a solution for the natural frequencies $\omega_0^{(1)}$ and $\omega_0^{(2)}$ using the abc-formula.

$$\omega_0^{2(1)} = \frac{m_1(k_2 + k_3) + m_2(k_1 + k2) + \sqrt{(-m_1(k_2 + k_3) - m_2(k_1 + k_2))^2 - 4m_1 m_2(k_1 k_2 + k_1 k_3 + k_2 k_3)}}{2m_1 m_2} \tag{23}$$

$$\omega_0^{2(2)} = \frac{m_1(k_2 + k_3) + m_2(k_1 + k2) - \sqrt{(-m_1(k_2 + k_3) - m_2(k_1 + k_2))^2 - 4m_1 m_2(k_1 k_2 + k_1 k_3 + k_2 k_3)}}{2m_1 m_2} \tag{24}$$

Once the values of the natural frequencies are obtained we can derive a complete analytical solution for $\vec{u}$. This is shown in Lisette de Bruin's report [1] equations (7.16) through (7.37). The solution for has the form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = M^{-\frac{1}{2}} P \begin{bmatrix} c_1 \cos(\omega_0^{(1)} t) + c_2 \sin(\omega_0^{(1)} t) \\ c_3 \cos(\omega_0^{(2)} t) + c_4 \sin(\omega_0^{(2)} t) \end{bmatrix}$$

Matrix $M^{-\frac{1}{2}}$ is given by: $\begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 \\ 0 & \frac{1}{\sqrt{m_2}} \end{bmatrix}$

And matrix P is the matrix formed from the normalized eigenvectors of the matrix:

$$\hat{K} = M^{-\frac{1}{2}} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} M^{-\frac{1}{2}}$$

P is also called the model matrix for this system.

3.3 Point mass attached to spring damper system with collisions

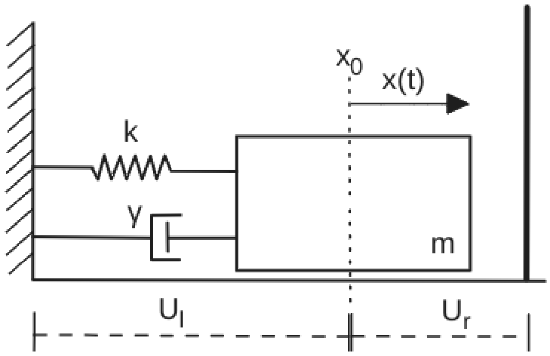

Figure 3: Spring-damper system with collisions

This scenario is very similar to figure 3.1, so the same differential equation applies to the point mass. We do however want to model elastic collisions between the object and the walls. These collisions happen when $x(t) = U_r$ or $x(t) = U_l$. Since we consider elastic collisions, we know the formulas for the velocity after impact:

$$v' = \frac{m - m_{wall}}{m + m_{wall}} \cdot v$$

The velocity of the wall is taken to be 0, hence we only have 1 term in the expression for velocity.

We can only simulate this scenario numerically, by updating the velocity of our object when the displacement reaches the position of the walls.

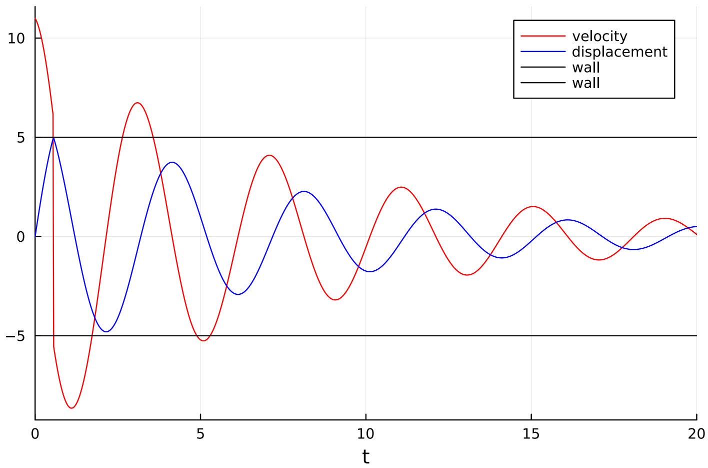

Figure 4: Numerical solution, with $m = 40, k = 100, y = 10$

3.4 Beam deflection

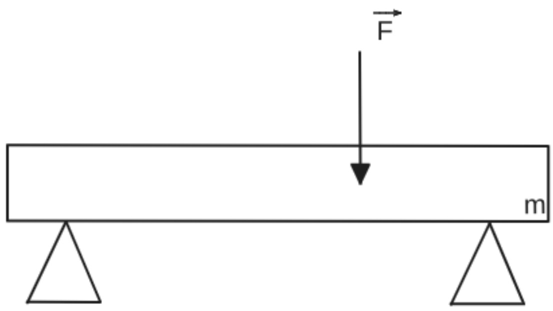

Figure 5: Beam deflection

The equation that models steady state beam deflection is given by the Euler-Bernoulli static beam theory [2]:

$$\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 y}{\partial x^2}) = q(x) \tag{25}$$

We take E, I to be constants.

$$EI\Delta^2 y = q(x) \tag{26}$$

Using central difference, we can approximate the second derivative by:

$$\Delta y_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2) \tag{27}$$

We can then write the system as:

$$EI\Delta^2 \mathbf{y} = A^2 \mathbf{y} = \mathbf{q}(x) \tag{28}$$

where A is the system matrix, $\mathbf{y}$ is the vector with all the solutions at the nodes, and vector $\mathbf{q}$(x) describes how the load is distributed.

For a simply supported beam, n nodes are considered with a Dirichlet boundary condition. So we require that at the ends y'' and y' are zero, and y is fixed. The system matrix A has dimensions $n \times n$, solution vector $\mathbf{y}$ and the load vector $\mathbf{q}$ have n elements:

$$A = \frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \mathbf{y}(x) = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_4 \\ y_n \end{bmatrix}, \quad \mathbf{q}(x) = \begin{bmatrix} 0 \\ q_1(x) \\ q_2(x) \\ q_3(x) \\ \vdots \\ q_n(x) \\ 0 \end{bmatrix} \tag{29}$$

For the analytical solution, we have:

$$y(x) = \begin{cases} \frac{Pbx(L^2-b^2-x^2)}{6LEI}, & 0 \leq x \leq a \\ \frac{Pbx(L^2-b^2-x^2)}{6LEI} + \frac{(x-a)^3}{6LEI}, & a \leq x \leq L \end{cases} \tag{30}$$

where a is the point where the load is applied.

If we compare the analytical solution with our numerical solution, we observe that indeed they are the same.

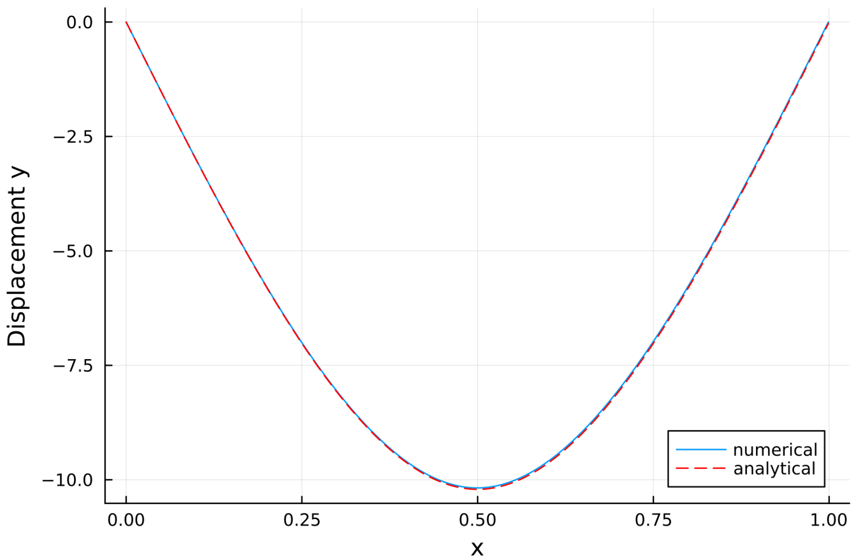

Figure 6: 1D simulation of a simply supported beam with a force of 490 N applied in the middle of the beam.

3.5 Julia Solver

Our team has implemented the scenarios above in Julia using the `DifferentialEquations.js` package. It allows the user to create ODEs, define events in the simulation, and choose different solvers. It is worth discussing the different solvers that Julia offers. A solver represents a time integration algorithm. Julia provides implementations of the Forward-Euler method, the Trapezoidal method, an optimized version of RK4 named `Tsit5`, and many others. By default, `Tsit5` algorithm is used, which has an adaptive time-step. This is beneficial for the equations we aim to compute as we don't have to fix a time step. In methods like Forward-Euler choosing too big of a time-step will influence the stability of the equation. Moreover, the `Tsit5` provides a better local error of $O(h^5)$, compared to $O(h^2)$ of the Forward-Euler.

4 Final scenario

In the following section, we depict the scenario we aim to model in part B of the project. We will describe the elements of the diagram and how they relate to the previously mentioned scenarios.

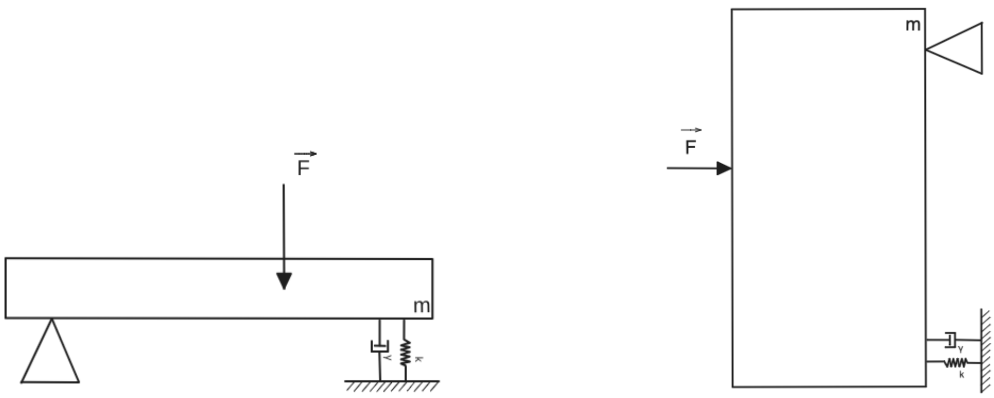

Figure 7: Simplified diagram for part B

The bus door can be modeled as a beam, attached to a fixed support on one side and a suspension on the other. The force F will act parallel to the door's glass pane as viewed from outside the bus. Figure 7 shows how this setup would look like. The suspension is modeled as a spring-damped system.

4.1 Boundary-value problem

There are 2 consequences of applying force $\vec{F}$ on the system:

- The beam will deflect. This effect is modeled by the Euler-Bernoulli beam theory, which is discussed in Scenario 3.4. However, for this scenario, we need to use the dynamic-beam equation.
- The end that is attached to the spring-damped system will oscillate according to the equations in Scenario 3.1.

We write the boundary-value problem as follows:

$$\begin{aligned}
\Omega &= (0, L) && \text{(L = length of the beam)} \\
\partial\Omega_1 &= \{0\} \\
\partial\Omega_2 &= \{L\} \\
EI\frac{\partial^4}{\partial x^4}w &= -\mu\frac{\partial^2 w}{\partial t^2} + q(x,t) \\
w(x,t) &= 0 && (x \in \partial\Omega_1, \forall t) \\
m\frac{\partial^2}{\partial t^2}w + \gamma\frac{\partial}{\partial t}w + kw &= 0 && (x \in \partial\Omega_2, \forall t) \\
w(x,0) &= 0
\end{aligned}$$

To model this complex scenario numerically, we will represent the beam as a uniform mesh such as in Scenario 3.4. Because of the boundary condition on $\partial\Omega_2$, we will have to build the matrix for the Laplacian operator differently. Assuming that this matrix is A, the discretized equation will be:

$$EI \cdot A^2 w_t = -\mu w_t'' + q_t \qquad \text{(Note that } EI \text{ is assumed to be constant)}$$

With the equation expressed in this form a time integration scheme will be used to get a numerical solution. Julia's default solver should work well in this case.

4.2 Constants and forces

Setting proper constants is crucial to having a realistic scenario. The constants that the model needs are:

- m: Mass of the door.
- γ: Damping constant
- k: Spring constant.
- μ: Density of the door.
- E: Elastic modulus (measures an object's resistance to being deformed elastically).
- I: Second area moment.

Some of these constants have been established in other reports about Ventura's doors. Choosing I, will depend on the shape of the door.

There is also a discussion to be made about how the external force should look like. In the case of the bus braking we will have a force that is uniform in space:

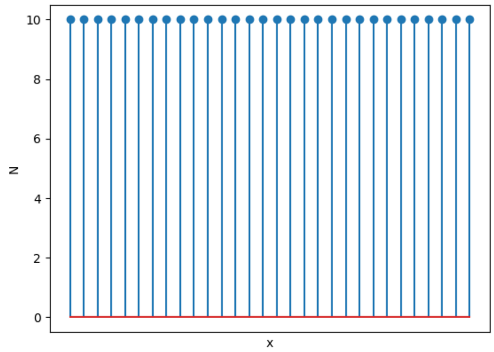

Figure 8: Uniform force on every mesh point

This is a simplified way to treat the deceleration force. In reality, there is weight transfer involved, which will alter the component of the force that acts on each mesh point. In our case, the force will be expressed as $F_{dec} = m \cdot a$, where m is the mass of the beam. The deceleration can vary in time so we will denote its value at a given time instance as a_t. This gives the function of load as:

$$q_t(x) = -m \cdot a_t$$

For realistic choices of m and a, the door might not deform very much. If we want to observe if and how bigger deformations can occur, the force needs to be stronger. That can be achieved with a pulse function:

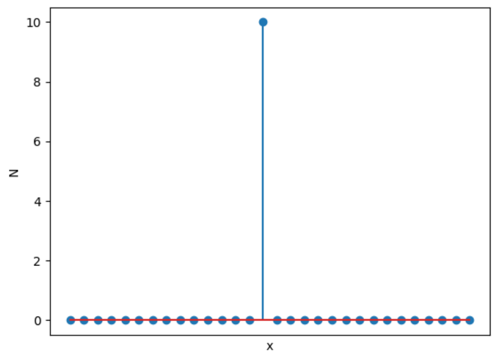

Figure 9: Force is applied at a single point

These kinds of forces can result from a collision with a heavy object. In that case, the force will not be constant in time. It will only act at the beginning of the simulation (moment of impact)

$$q_t(x) = \begin{cases} F, & x = x_{target}, t = 0 \\ 0, & t > 0 \end{cases}$$

Another advantage of this force expression is that we can select x_{target} and observe how deformations occur depending on where the object hits.

4.3 Extensions for final scenario

The current scenario considers the beam as a 1D object. In this way, the equations and boundary conditions are easy to write. While this setup already gives good insights into how the door will vibrate our team devised some extensions, that can better approximate the door as it is in real life:

1. **$E \cdot I$ not constant**. With this change, the door can be more elastic in some places of the mesh. This is also the case in real life, where the door has a rigid frame with a glass pane inside. Making $E \cdot I$ non-constant means that we cannot take it out of the second-order spatial derivative, therefore a different Laplacian operator has to be assembled.

2. **Modelling the door as a 2D element**. This change adds complexity to the discretization process, but it creates a more accurate picture of what happens in real- life. If the door were modeled as a 2D rectangle, the support elements would look different. The top edge of the door would be completely fixed, while only the bottom-right part would be attached to the spring-damper system.

3. **Modelling collisions between the beam and door frame**. Similar to how collisions occur in Scenario 3.3, if the force is strong enough, the door will change velocity sharply when hitting the frame.

References

[1] Lisette de Bruin. *Mathematical Model of Ventura's Bus DoorSystem.* Delft University of Technology, 2020.

[2] Wikipedia. *Euler–Bernoulli beam theory — Wikipedia, The Free Encyclopedia.* https://en.wikipedia.org/w/index.php?title=EulerBernoulli%20beam%20theory&oldid=1159872044#Static_beam_equation. 2023.