Differential Algebra Meets Derived Categories

Taylor Dupuy (joint with James Freitag and Aaron Royer)
The Hebrew University

POINT OF THE TALK:

Generality can provide useful insight.

(give strongly minimal sets in DCF0)

Theorem 1

There exist new twisted differential forms.

Theorem 2

Limits of prolongation spaces are prolongation spaces of limits (as functors).

Theorem 3

Limits of prolongation spaces don’t exist (as schemes).

Definition.

A differential form is **old** if is is the pullback of a nontrivial morphism

$$f : C \to C'$$

$$f^*(\omega') \qquad\qquad \omega' \in \Omega^1_{C'}$$

Hrushovski-Itai:

Newforms ``give rise” to equivalence classes (orthogonality classes) of strongly minimal differential algebraic subvarieties of curves.

Newforms exist.

Theorem 1

There exist new twisted differential forms.

Newforms exist.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & C' \arrow[d] \\ \mathrm{Jac}(C) \arrow[r, "F"'] & \mathrm{Jac}(C') \end{tikzcd}$$

$$H^0(C, \Omega^1_C) \cong H^0(\mathrm{Jac}(C), \Omega^1_{\mathrm{Jac}(C)})$$

enough to find newforms on the Jacobian!

$$\mathrm{Jac}(C) \xrightarrow[F]{} \mathrm{Jac}(C')$$

characterization

old form:

$$\omega = F^*\omega' \iff \omega|\ker(F) = 0$$

(only worry about quotients)

$$\mathrm{coker}(F) = \mathrm{Jac}(C)/\ker(F)$$

old forms from $\mathrm{Jac}(C)/A = V_A = \{\omega : \omega|A = 0\}$

(quotient)

proper subspaces

$$\text{old forms } = \bigcup_{A \subset J} V_A$$

countably many

DONE

Rosen:

Twisted differential new forms ``give rise'' to strongly minimal differential algebraic subvarieties.

what are these?

twisted differential forms

$$0 \to \mathcal{O}_C \to E_C \to \Omega^1_{C/K} \to 0$$

Rosen:

Twisted differential new forms ``give rise” to strongly minimal differential algebraic subvarieties.

Two parts of the Hrushovski-Itai argument:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & C' \arrow[d] \\ \mathrm{Jac}(C) \arrow[r, "F"'] & \mathrm{Jac}(C') \end{tikzcd}$$

$$H^0(C, \Omega^1_C) \cong H^0(\mathrm{Jac}(C), \Omega^1_{\mathrm{Jac}(C)})$$

$$0 \to \mathcal{O}_C \to E_C \to \Omega^1_{C/K} \to 0$$

Two parts of our argument:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & C' \arrow[d] \\ \mathrm{Jac}(C) \arrow[r, "F"'] & \mathrm{Jac}(C') \end{tikzcd}$$

$$\cancel{H^0(C, \Omega^1_C) \cong H^0(\mathrm{Jac}(C), \Omega^1_{\mathrm{Jac}(C)})}$$

$$H^0(C, E_C) \cong H^0(\mathrm{Jac}(C), E_{\mathrm{Jac}(C)})$$

Theorem
(Dupuy-Freitag-Royer)

$0 \to \mathcal{O}_C \to E_C \to \Omega^1_{C/K} \to 0$

Theorem
(Dupuy-Freitag-Royer)

$$H^0(C, E_C) \cong H^0(\mathrm{Jac}(C), E_{\mathrm{Jac}(C)})$$

Corollary.

Global twisted newforms exist on curves of genus bigger than one.

proof. Run Hrushovski-Itai argument.

Where are the derived categories?

$H^0(C, E_C) \cong H^0(\mathrm{Jac}(C), E_{\mathrm{Jac}(C)})$

Where are the derived categories?

$$0 \to V \to E \to H \to 0$$

$$0 \to \mathcal{O}_C \to E_C \to \Omega^1_{C/K} \to 0$$

$$\xi \in \mathrm{Ext}^1(H, V)$$

FACT 1: $\mathrm{Ext}^1(A, B) = \mathrm{Hom}(A, B[1])$

FACT 2:

$$V \to E \to H$$

$$E \to H \xrightarrow{\xi} V[1]$$

(FACT 3): $0 \to \mathcal{O}_C \to E_C \to \Omega^1_{C/K} \to 0$

$$\mathrm{KS}_C(D) \in \mathrm{Ext}^1(\Omega_{X/K}, \mathcal{O}_C)$$

Theorem
(Dupuy-Freitag-Royer)

$$H^0(C, E_C) \cong H^0(\mathrm{Jac}(C), E_{\mathrm{Jac}(C)})$$

fill in

$\text{apply } \mathrm{Hom}(\mathcal{O}_C, -)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] & E_C \arrow[r] & \Omega_{C/K} \arrow[r, "\mathrm{KS}_{C/K}(D)"] & \mathcal{O}_C[1] \\
0 \arrow[r] \arrow[u] & j^*\mathcal{O}_{\mathrm{Jac}(C)} \arrow[r] \arrow[u] & j^*E_{\mathrm{Jac}(C)} \arrow[r] \arrow[u, "\nu"'] & j^*\Omega_{\mathrm{Jac}(C)/K} \arrow[r, "j^*\mathrm{KS}_{\mathrm{Jac}(C)/K}(D)"'] \arrow[u] & \mathcal{O}_C[1] \arrow[u]
\end{tikzcd}$$

FACT 4: You can fill in "distinguished triangles"

rotate

Theorem
(Dupuy-Freitag-Royer)

$$H^0(C, E_C) \cong H^0(\mathrm{Jac}(C), E_{\mathrm{Jac}(C)})$$

isomorphisms

isomorphism

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}_C) \arrow[r] & H^0(E_C) \arrow[r] & H^0(\Omega_{C/K}) \arrow[r] & H^1(\mathcal{O}_C) \\
0 \arrow[r] \arrow[u] & H^0(E_C) \arrow[r] \arrow[u] & H^0(j^* E_{\mathrm{Jac}(C)}) \arrow[r] \arrow[u, "\nu"'] & H^0(j^*\Omega_{\mathrm{Jac}(C)/K}) \arrow[r] \arrow[u] & H^1(\mathcal{O}_C) \arrow[u]
\end{tikzcd}$$

5 LEMMA

last step $H^0(j^* E_{\mathrm{Jac}(C)}) = H^0(E_{\mathrm{Jac}(C)})$

Theorem 2

Limits of prolongation spaces are prolongation spaces of limits (as functors).

Theorem 3

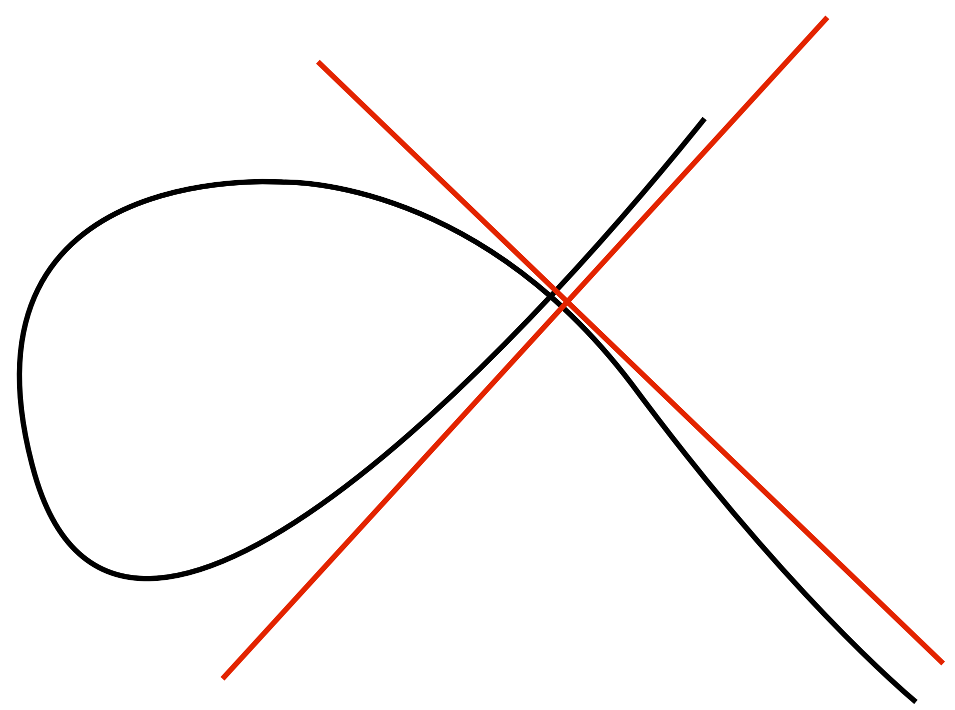
Limits of prolongation spaces don't exist (as schemes).

Motivation:

Kolchin Irreducibility:

$$X \ \text{ irreducible} \implies J^\infty(X) \ \text{ irreducible}$$

$$X = \mathrm{Spec}\ K[x]/(f(x))$$
$$J^1(X) = \mathrm{Spec}\ K[x,x']/(f(x), D(f(x)))$$
$$\vdots$$
$$J^\infty(X) = \mathrm{Spec}\ K[x,x',x'',\ldots]/(f, D(f), D^2(f), \ldots)$$



Kolchin Irreducibility:

$$X \quad \text{irreducible} \implies J^\infty(X) \quad \text{irreducible}$$

Does Kolchin Irreducibility hold for other arc spaces/infinite prolongation spaces?

$$J^1(X)(A) \cong X(A[t]/(t^2))$$

$$A \mapsto A[t]/(t^2)$$

replace

Do these arc spaces even exist?!

Do these arc spaces even exist?!

YES

$$\lim X \circ R_n \cong X \circ R$$

(as functors)

NO

The functors are not schemes

Do these arc spaces even exist?!

YES

$$\lim X \circ R_n \cong X \circ R$$

(as functors)

Ingredients:

X scheme

$\lim_n R_n = R$ limit of ring schemes

A ring

example

$R_n(A) = A[t]/(t^{n+1})$

$R(A) = A[[t]]$

$$X \text{ quasiprojective} \quad \Longrightarrow \quad \lim_n X(R_n(A)) = X(R(A))$$

What goes into this?

Brandenberg-Chiravisu: $\mathrm{Sch}(S, X) \cong \mathrm{Fun}^L_\otimes(\mathrm{Vect}(S), \mathrm{Vect}(X))$

by pullback

Stupid Fact: $\lim_n \mathrm{Vect}(S_n) = \mathrm{Vect}(S)$

Do these arc spaces even exist?!

YES

$$\lim X \circ R_n \cong X \circ R$$

(as functors)

What goes into this?

$$\mathsf{Sch}(S, X) \cong \mathrm{Fun}^L_{\otimes}(\mathrm{Vect}(S), \mathrm{Vect}(X))$$

$$\mathsf{Sch}(S, X) \cong \mathrm{Fun}^L_{\otimes}(\mathrm{QCoh}(S), \mathrm{QCoh}(X))$$

$$\mathsf{Sch}(S, X) \cong \mathrm{Fun}^L_{\otimes}(D(S), D(X))$$

$$\mathsf{Sch}(S, X) \cong \mathrm{Fun}_{\otimes}(D_{perf}(S), D_{perf}(X))$$

Do these arc spaces even exist?!

NO

The functors are not schemes

Example. not a scheme! $J^{\infty}(X/B, \sigma) = \prod_{n \geq 0} X^{\sigma^n}$

Moosa-Scanlon jet space

Lemma.

$(Y_i)_{i \in \mathbf{N}}$ quasicompact schemes

$\prod_{i \geq 0}^{\infty} Y_i$ a scheme $\Longrightarrow$ all but finitely many dudes are affine

Proof. Descent in fpqc topology.

Proof. Descent in fpqc topology.

faithfully flat + quasi compact

$$f : X \to Y$$

$$\mathrm{Spec}(A_X) \to \mathrm{Spec}(A_Y)$$

flat + surjective

$$\exists (Y_i \to Y)_{i \in I} \text{ affine open, } \forall i \in I,$$
$$f^{-1}(Y_i) \text{ quasicompact}$$

GLUE!

$$\begin{tikzcd} & X \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

fpqc covering

Theorem.

affine/closed affine/closed

Mathoverflow: Bradenberg, Elencwajg, Moret-Bailey, Wise

Proof. Descent in fpqc topology.

suppose $\prod_{i \geq 0}^{\infty} Y_i$ a scheme (over field)

Step 1: Opens are messed up.

affine when U is

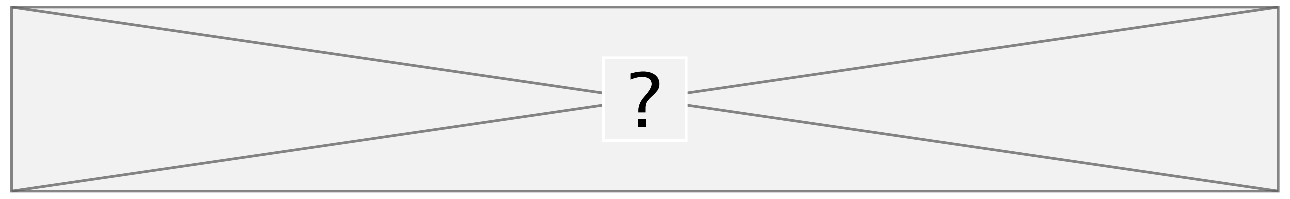
Step 2: Separatedness of factors.

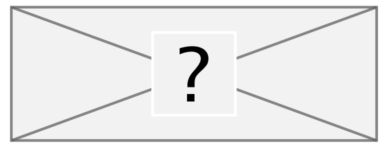
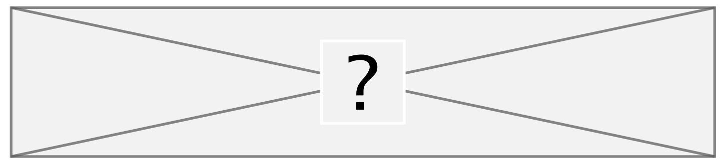
Step 3: Affineness of factors.

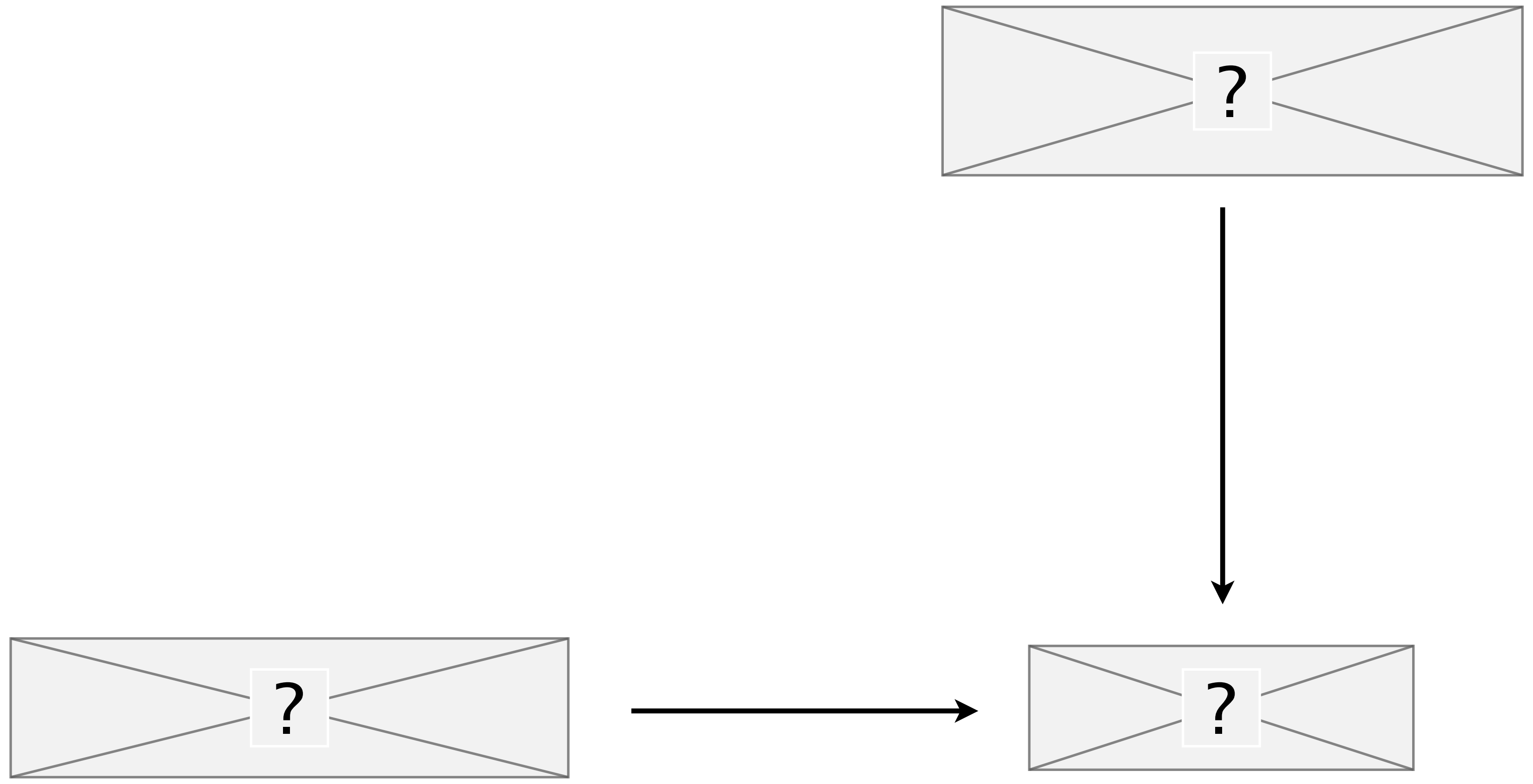
Mathoverflow: Bradenberg, Elencwajg, Moret-Bailey, Wise

Step 1: Opens are messed up.

suppose $\prod_{i\geq 0}^{\infty} Y_i$ a scheme

suppose  affine open

suppose  closed point and  closed point



$\{y\} \times \prod_{i\in I\setminus I'} Y_i = (\prod_{i\in I\setminus I'} Y_i)_{\kappa(y)}$ affine

apply fpqc descent

Thank you!