§13.9:21

$$\begin{cases} y^{(4)} - \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \\ y''(0) = 0 \\ y''(1) = 0 \end{cases}$$

ANS: $\lambda_n = (n\pi)^4, \quad n = 1, 2, 3, \ldots$

$y_n = \sin(n\pi t)$.

We get $\alpha^4 - \lambda = 0$ so $\alpha = \lambda^{1/4}, i\lambda^{1/4}, -\lambda^{1/4}, -i\lambda^{1/4}$.

Let $m = \lambda^{1/4}$ and write general soln

$$y = c_1 e^{mt} + c_2 e^{imt} + c_3 e^{-mt} + c_4 e^{-imt}.$$

[Note: $\{\cos(mt), \sin(mt), \cosh(mt), \sinh(mt)\}$ also a basis.]

$$\begin{cases} y(0) = 0 \\ y''(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 + c_3 + c_4 = 0 & (1) \\ m^2 c_1 - m^2 c_2 + m^2 c_3 - m^2 c_4 = 0 & (2) \end{cases}$$

(2): so $m^2 = 0$ or $c_1 + c_3 - (c_2 + c_4) = 0$

Suppose $m \neq 0 \Rightarrow c_1 + c_3 = c_2 + c_4$

$\Rightarrow 2(c_1 + c_3) = 0 \Rightarrow c_1 = -c_3$

$\Rightarrow c_2 = -c_4$

$$y(t) = c_1\left(e^{mt} + e^{-mt}\right) + c_2\left(e^{imt} - e^{-imt}\right)$$

$$= \alpha \sinh(mt) + \beta \sin(mt).$$

$$\begin{cases} y''(0)=0, & (3) \\ y''(1)=0, & (4) \end{cases}$$

(3) $m^2\alpha \sinh(mt) - \beta m^2 \sin(mt)\Big|_{t=0} = m^2\alpha\left(\frac{e+\frac{1}{e}}{2}\right)$

$$= 0$$

$\Rightarrow m^2 = 0$ or $\alpha = 0 \Rightarrow \alpha = 0.$

(9) $\quad y(t) = \beta \sin(mt)$

(4) $y''(t) = -\beta m^2 \sin(mt)$

$y''(1) = 0 \Rightarrow \sin(m) = 0 \Rightarrow m = n\pi$

$\therefore \quad m_n = n\pi, \quad m_n^4 = (n\pi)^4 = \lambda_n$

$$\begin{cases} \lambda_n = n\pi^4, & n = 1, 2, 3, \dots \\ y_n = \sin(n\pi t). \end{cases}$$