Numerology

Deninger Conjectures

$\exists$ category $Sch_{\mathbb{F}_1}$ and a functor

$$Sch_{\mathbb{F}_1} \xrightarrow{\otimes_{\mathbb{F}_1}\mathbb{Z}} Sch_{\mathbb{Z}}$$

with an additional functor

$$H^\bullet_{dyn} : Sch_{\mathbb{F}_1} \to \text{Graded } K\text{ Alg}$$

such that ...

Weil Conjectures

$\exists$ category $\acute{E}t$ and a functor

$$Var/\mathbb{F}_q \longrightarrow \acute{E}t$$

and a functor

$$\acute{E}t \longrightarrow H^\bullet$$

such that ...

What could the category $Sch_{\mathbb{F}_1}$ look like?

Affine space

Numerology: $\#\mathbb{P}^n(\mathbb{F}_q) = \dfrac{\mathbb{F}_q^{n+1}\setminus\{0\}}{\mathbb{F}_q\setminus\{0\}}$

$$\Rightarrow \#\mathbb{P}^n(\mathbb{F}_q) = \frac{\#\mathbb{F}_q^{n+1}\setminus\{0\}}{\#\mathbb{F}_q\setminus\{0\}} = \frac{q^{n+1}-1}{q-1}$$

$$= 1+q+q^2+\cdots+q^n$$

Another way: $\mathbb{P}^1 - pt = \mathbb{A}^1$ or $\mathbb{P}^1 = \mathbb{A}^1 + pt$

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup pt$$

$$\#\mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + q^{n-2} + \cdots + q + 1$$

$$= [n]_q \longrightarrow \boxed{\mathbb{P}^n = G(1, n+1)}$$

Recall

$G \times X \to X$ Left

$G/\mathrm{Stab}_G(x) \simeq \mathrm{Orb}_G(x)$

$x = (\langle e_1, e_2\rangle, \langle e_3, e_4, e_5\rangle)$, $e_j \in \mathbb{F}_q^5$

$\langle \bar{e}_3, \bar{e}_4, \bar{e}_5\rangle = \mathbb{F}_q^5 / \langle e_1, e_2\rangle$

$\mathrm{Orb}(x) \cong G(2,5)$

$\mathrm{Stab}(x) = P$ (parabolic subgroup)

$= \begin{bmatrix} GL_2 & 2\times 3 \\ & GL_3 \end{bmatrix} \simeq GL_2 \times GL_3 \times A^{2\times 3}$

$\Rightarrow$

$$G(2,3) \simeq \frac{GL_5}{GL_2 \times GL_3 \times A^{2\times 3}}$$

$$\# G(2,5)(\mathbb{F}_q) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

$[n]_q! = \ldots$

$G(2,5) = 2$ dim'l subspaces of 5 dim'l space.

$$\begin{array}{c} G(2,5) \\ \shortparallel \\ \mathbb{G}(1,4) \end{array}$$

$$\# G(2,5) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \frac{[5]_q!}{[2]_q! \, [3]_q!}$$

<u>why?</u> study wrt complete flag,

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_5 \rangle$$

$$G(2,5) = \frac{GL_5(\mathbb{F}_q)}{\left\{ \left[\begin{array}{c|c} 2\times 2 & \text{whatever} \\ \hline & 3\times 3 \end{array} \right] \right\}}$$

$$\left\{ \left[\begin{array}{c|c} GL_2 & \text{whatever} \\ \hline 0 & GL_3 \end{array} \right] \right\} = GL_2 \times GL_3 \times \mathbb{A}^{k(n-k)} = \text{parabolic subgroup}$$

Stabilizer of flag

q-binomial coeff $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$

$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$, $xy = qyx$

$G(k,n)(\mathbb{F}_q)$ = k element subsets of a finite set?

$$\frac{GL_n}{GL_k \times GL_{n-k} \times A^{n(n-k)}} \xrightarrow{q \to 1} \frac{S_n}{S_k \times S_{n-k}} \ ?$$

actions look nice!

$S_n \circlearrowright \{1,2\}, \{3,4,5\}$

$Stab_{S_n} = S_2 \times S_3$

~~Representation Theory~~

~~$\alpha: G_k \to V_k$~~

~~$\beta: G_j \to W$~~

$\alpha_k \otimes \alpha_j$

Quotients: $GL_5 \circlearrowright \mathbb{F}_q^5$

$\circlearrowright \langle e_1, e_2 \rangle \quad \langle e_3, e_4, e_5 \rangle$

$$\frac{GL_5(\mathbb{F}_q)}{P}$$

$$P = \left\{ \left[\begin{array}{c|c} GL_2 & 2\times 3 \\ \hline 0 & GL_3 \end{array} \right] \right\}$$

Point Counting

Flag Spaces as Quotients

$S_n \circlearrowright \{1,2,3,4,5\}$

$\circlearrowright \{1,2\}, \{3,4,5\}$

$$\frac{5!}{2! \times 3!} \simeq \frac{S_5}{S_2 \times S_3}$$

$\boxed{G/_{stab} \Leftrightarrow orb_G}$ = All actions of this type.

$\boxed{\# GL_n(\mathbb{F}_q) = ?}$

Notation for algebraic groups

$\mathbb{F}_q^n \setminus \{0\}$ $\qquad$ $\mathbb{F}_q^n / \mathbb{F}_q - \{0\}$ $\qquad \cdots \qquad$ $\mathbb{F}_q^n / \mathbb{F}_q^{n-1} - \{0\}$

Bruhat-Tits / # of $\mathbb{F}_q$ pt & Weyl Group

B = Borel Subgroup $\overset{\exists!}{=}$ 'Maximal Closed Solvable Subgroup containing T

maximal torus.

$W = N(\mathbb{Z})/T(\mathbb{Z})$

$$\boxed{\coprod_{w \in W} BwB \simeq G}$$

$= \forall w \quad BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{d_w} \quad d_w \geq 0$, $r := rk(G)$

$$\Longrightarrow \# G(\mathbb{F}_q) = \sum_{w \in W} (q-1)^r q^{d_w} = N(q)$$

$$\lim_{q \to 1} \frac{N(q)}{(q-1)^r} = \lim_{q \to 1} \sum_{w \in W} q^{d_w} = \# W$$

G algebraic group : $/\mathbb{Z}$

$\sqrt{G}$ = unique maximal normal solvable subgroup

T = Maximal torus $\cong \mathbb{G}_m^r$ $\quad r := \mathrm{rk}(G)$

split

unipotent $(g-1)^n = 0$ in $\mathbb{Z}[G]$

reductive $\sqrt{G} \cap U = 0 \implies \sqrt{G} = \mathbb{G}_m^r$

$$N = N_G(T) = \{ g \in G \mid gTg^{-1} = T \} \triangleleft G$$

$$\mathrm{Weyl}(G) := N(\mathbb{Z}) / T(\mathbb{Z})$$

<u>Defn</u>. Tits-Weyl model for G alg group $/\mathbb{Z}$

$G_{\mathbb{F}_1} \in \underbrace{Sch_{\mathbb{F}_1}}_{\text{some category}}$ $\qquad G \otimes_{\mathbb{F}_1} \mathbb{Z} = G_{\mathbb{Z}}$

Functor of

<u>OBS</u>: $G_{\mathbb{F}_1}$ group object in $Sch_{\mathbb{F}_1}$

$\Longrightarrow G_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z} = G_{\mathbb{Z}}$ group object.

$$\underbrace{Sch_{\mathbb{F}_1}(X, G_{\mathbb{F}_1})}_{\text{group}} \longrightarrow \underbrace{Sch_{\mathbb{Z}}(X \otimes_{\mathbb{F}_1} \mathbb{Z}, G_{\mathbb{Z}})}_{\text{group}} \quad \text{GRP HOM}$$

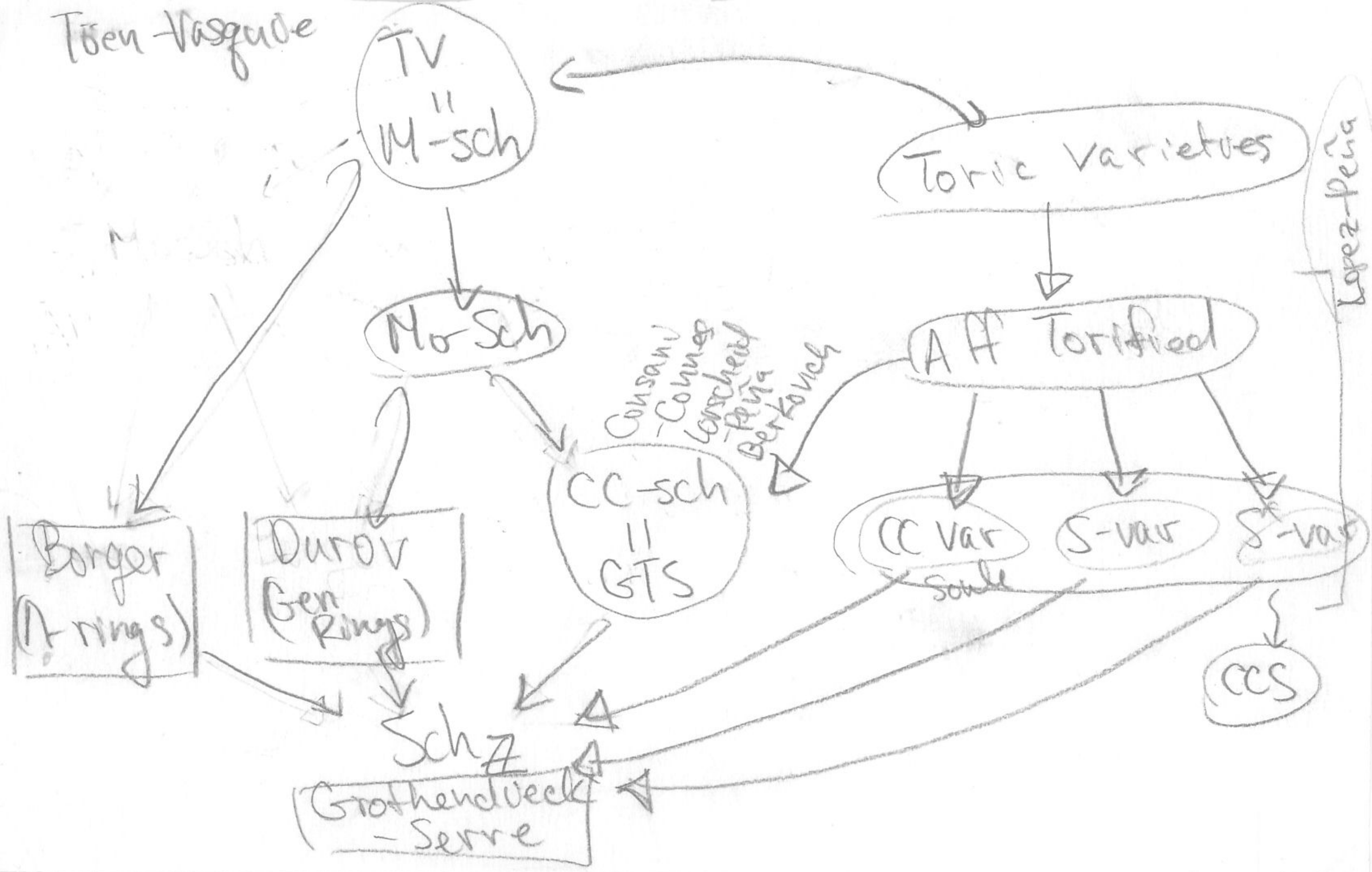
IS $\qquad\qquad$ IS

$N(\mathbb{Z})/T(\mathbb{Z}) = W \qquad\qquad G(\mathbb{Z})$

$$\boxed{G(\mathbb{F}_1) \longrightarrow G(\mathbb{Z})}$$

IMPOSES STRONG RESTRICTIONS.

Töen Vasquie $\supset$ Borger Λ-Rings $\supset$ Durov's Arakelov Geometry $\supset$ Monoidal Algebraic Geoms



Töen-Vasquez
TV = M-sch
Toric Varieties
Mo-Sch
Aff Torified
Lopez-Peña
Consani -Connes
Lorscheid -Peña
Berkovich
CC-sch = GTS
Borger (Λ-rings)
Durov (Gen Rings)
CC Var
S-var
S⁺-var
Soulé
CCS
Sch$_{\mathbb{Z}}$
Grothendieck -Serre

$\underline{X} : Ab \to$ Graded Sets

$X_{\mathbb{C}}$ aff var$/\mathbb{C}$

Natural Transformation

$e_X : \underline{X} \to X_{\mathbb{C}}$

$D \mapsto X_{\mathbb{C}}(\mathbb{C}[D])$

[22]

Chevalley:

$G_{\mathbb{Z}} \leftarrow$ Root System + Info.

Microcosm Principle: everything can be categorified

Category Objects + Morphisms

C

x, y $C(x,y)$

$x \in C$ $= \mathrm{Hom}_C(x,y)$

Class   Set

$x \xrightarrow{f} y$

composition

$x \xrightarrow{f} y,\ y \xrightarrow{g} z$. $x \xrightarrow{g \circ f} z$

associative

unit $id_x \in C(x,x)$

$f \circ id_x = \cancel{id_x \circ f} = f,$ $x \xrightarrow{f} y$

$id_x \circ g = g$, $z \xrightarrow{g} x$

Monoid: Set M

- $m: M \times M \to M$ assoc — Morphisms
- unit elt, $1 \in M$

$m \cdot 1 = 1 \cdot m = m.$

Rephrase unit ~~diag~~ diagrammatically.

Terminal Ob: $T \in C$

$\forall x \in C, \exists\ x \longrightarrow T$

Sets: $T = \{pt\}$

Ab: $T = 0$

unit for monoids:

$\varepsilon : T \longrightarrow M$

$$\begin{array}{ccccc} T \times M & \xrightarrow{\varepsilon \times 1} & M \times M & \xleftarrow{1 \times \varepsilon} & M \times T \\ & {\scriptstyle p_2} \searrow & \downarrow{\scriptstyle m} & \swarrow {\scriptstyle p_1} & \\ & & M & & \end{array}$$

2nd View: "Category"

Monoid = Category w/one object.

Γ = Cat w/one object

$G \in \Gamma$ unique object in Γ,

CLAIM: $\Gamma(G,G) \in$ Monoids

$f, g \in \Gamma(G,G)$,

$G \xrightarrow{f} G \xrightarrow{g} G$ $\quad f \circ g$

associativity & unit properties follow.

Example of "weird" Cat

Category of Categories = Cat

Ob: Categories

Mor: Functors.

FUNCTOR: $F: C \to D$

$$\begin{array}{ccc} x & & F(x) \\ f \downarrow & \longrightarrow & F(f) \downarrow \\ y & & F(y) \\ \text{in } C & & \text{in } D \end{array}$$

Monoids = Category of Categories w/one object + Functors

Enriched Category :

D category

C is D-enriched if

$$C(x,y) \in D$$

"Morphisms are more than just sets"

Examples ~~D = Abelian~~

Ab - enriched category is a category where

$$C(x,y) \in Ab.$$

Eg. Mod_R

~~Group Category~~

Tensor Categories

Sets	$\times$	$\{pt\}$
Sets	$\sqcup$	$\emptyset$
Ab	$\otimes$	$\mathbb{Z}$
Ab	$\oplus$	0
Rings	$\otimes_{\mathbb{Z}}$	$\mathbb{Z}$
Rings_R	$\otimes_R$	R

Cartesian ~~Category~~.

$\otimes : C \times C \longrightarrow C$ (Functor) $\boxed{???}$

I unit object

unitors:

$L : I \otimes C \longrightarrow C$ (Functors)

$R : C \otimes I \longrightarrow C$

associator: $\forall a, b, c \in C$

$\alpha_{abc} : (a \otimes b) \otimes c \longrightarrow a \otimes (b \otimes c)$

+ unit axioms

+ left/right associator axioms

$A \xrightarrow{f} C$

$B \xrightarrow{g} C$

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & C \times C \arrow[d] \\ A \otimes B \arrow[r] & C \otimes C \end{tikzcd}$$

Monoid C = Tensor Category

① **C-Monoid** Object $R \in C$

1) multiplication $\mu : R \otimes R \to R$

2) unit $\varepsilon : I_C \to R$

s.t. – μ assoc (pent)

– ε, μ left right inverse laws

$f : S \to T$

- $$\begin{tikzcd} S \otimes S \arrow[r, "\mu_S"] \arrow[d, "f \otimes f"'] & S \arrow[d, "f"] \\ T \otimes T \arrow[r, "\mu_T"'] & T \end{tikzcd}$$ respects mult

- $$\begin{tikzcd} I \arrow[r] \arrow[d] & S \arrow[d] \\ I \arrow[r] & T \end{tikzcd}$$ respects unit

decategorify

Equiv: $\operatorname{Hom}(x,y) \in C$ (abelian cat)

$\boxed{\text{C-enriched category w/ 1 object } \Gamma(\bullet,\bullet) \in C}$

Collection of C-Monoid Cats:

Mor = Functors

Ob = C-Monoid Cats.

$B \in$ C-Monoid Cats, Object $= \{b\}$

$B(b,b) =$ C-Monoid

C-Monoid Cats $\simeq$ C-Monoids.

R Monoid $\rightsquigarrow$ B_R Associated C-Monoid Cat.

example

$\mathcal{C}$ = category of sets

$\otimes = \times$

Set-Monoid

$$\mu : S \times S \to S$$
$$\varepsilon : \{pt\} \to S$$

+ assoc
+ left/right inverse laws,

__example__ Ab enriched

$R \in$ (Objects of Ab)

1) $m: R \times R \to R$ defined by

$m(a,b) = \mu(a \otimes b)$

2) $1_R = \varepsilon(1_{\mathbb{Z}})$

$\mu: R \otimes R \to R$, morphism of abelian groups,

$\mu(a\otimes b + c\otimes d) = \mu(a\otimes b) + \mu(c\otimes d)$

$m(a, b+d) = m(a,b) + m(a,d)$ __Left Right Distr__

unit axioms $\rightsquigarrow$ unit axioms of monad.

Ring

(Need to show is a ~~Ran~~ functorial))

$R \cong$ Ab-Enriched Category with 1 object.

$\Uparrow$

CRing — 2-category of Ab-Enriched categories w/ one object

Ab $\otimes$?

Sets $\sqcup$?

other categories?

- Naturally a zero in groups? (No)
- Naturally a zero in Set-Monoids?

__Yes__

Left Group Action

$$\rho : G \times S \to S$$

$$\rho(g,s) = g(s)$$

$$g_1(g_2 s) = (g_1 g_2)(s)$$

$$I \cdot s = s$$

Right Action

$$\rho : S \times G \to S$$

$$\rho(s,g) := s^g$$

$$(s^{g_1})^{g_2} = s^{g_1 g_2}$$

$$s^I = s,$$

$$\text{Grps} \cong [\text{Cats w/ one object in Sets}']$$

$$= \text{Sets}'\text{-Monoid.}$$

ACTION:

$$\underbrace{\rho : \underbrace{\Gamma}_{\text{Cat w/ one obj}} \longrightarrow \text{Sets}}_{\text{Functor.}}$$

REP'N

$$\rho : \underbrace{\Gamma \longrightarrow \text{Vect}}_{\text{Functor to category of Modules}}$$

GO CRAZY GENERALIZING!!

Representations of C-Monoids

$$\underbrace{\Gamma \xrightarrow{\quad \rho \quad} D}_{\text{Functor}}$$

C-enriched category w/ one object

any category in the world

(Rings)
(Monoids)
(Groups)
⋮
GO CRAZY!

Sets – action
Vect – vector spaces
⋮

Rings are Ab-Monoids
R-Modules are ?

$\rho : R \otimes M \to M$
satisfies axioms of an
<u>action</u>

<u>Traditional Point of View</u>

What is a module?

R Ring
M Abelian Group

$\boxed{R \times M \to M \quad \text{mult map}}$

+ Axioms.

$\Downarrow$

$R \otimes M \to M$
In abelian groups

<u>Abstract Defn</u>

C Tensor Cat
R C-Monoid
<u>Left R-Mod</u>

($C = Ab$ $\Rightarrow R$ a ring)

$M \in C$

w/ $\rho : R \otimes M \to M$
that satisfy axioms of a group action.

C any category

Ob: $F, G : C \to C$ functors

Mor: Natural Trans $\Phi \in \mathrm{Nat}(F, G)$

$$\begin{tikzcd} F(x) \arrow[r, "\Phi_x"] \arrow[d] & G(x) \arrow[d] \\ F(y) \arrow[r, "\Phi_y"] & G(y) \end{tikzcd}$$

C^C

Endo Functor Category

$X^Y = \{F : Y \to X\}$

$2^X = \{f : X \to \{0,1\}\}$ (power set)

TENSOR PRODUCT:

$F \otimes G := F \circ G$

UNIT: Id_C = Identity Functor

(Category of Monoid objects In Endo Functor Category) $\cong$ [C^C-enriched categories w/ one object] $\cong$ C-Monads

$\mathcal{C}$-Monads $\underset{\text{Natural Equiv}}{\simeq}$ $\mathcal{C}^{\mathcal{C}}$-monoids

Decategorify

Ψ

Σ

$\mathrm{Mod}_\Sigma \simeq \mathrm{Mod}_{F(\Sigma)}$

SU

$\mathcal{C}$	Tensor Cat	$\rightsquigarrow$ $\mathcal{C}$-Monoids
R	$\mathcal{C}$-Monoid	$\rightsquigarrow$ R-Modules
Σ	$\mathcal{C}$-Monad	$\rightsquigarrow$ Σ-Module

DUROV'S APPROACH

$$\begin{array}{c} \text{CRing} \\ \Psi \\ R \end{array} \longrightarrow \underbrace{\left[\begin{array}{c} \text{Sets} \xrightarrow{F_R} \text{Sets} \\ X \longmapsto R^X \end{array}\right]}_{\text{Monad}}$$

$$\begin{array}{ll} R \times R \xrightarrow{m} R & \text{mult} \\ R \xrightarrow{\varepsilon} R & \text{unit} \end{array}$$

+ AXIOMS $\rightsquigarrow$ F_R monad on category of Sets.

Special Properties

"commutative"

commutes w/ filtered limits.

AFTER GENERALIZING RINGS

- localization / primes $|A|\backslash\mathbb{P}$ multiplicative
- pre sh
- sheaves
- ringed spaces
- locally ringed spaces
- schemes
- scheme w/ zero

Advantages:

$\mathbb{Z}_\infty = [-1, 1]$

Tropical Geometry?

Problems:

Singular at ∞

Monoidal at Heart

Can't Recover Arakelov Theory.

<u>MONAD</u>: $T : C \to C$

nat $\begin{cases} \eta : Id_C \to T \\ \mu : T^2 \to T \end{cases}$

$\mu \circ T\mu = \mu \circ \mu T \qquad T^3 \to T$

$\mu \circ T\eta = \mu \circ \eta T = 1_T,$

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "T(\eta_X)"'] \arrow[dr] & T^2 \arrow[d] \\ T^2 \arrow[r] & T \end{tikzcd}$$

$\Sigma : C \to C$ endo functor

$\mu : \Sigma^2 \to \Sigma$ ~~mult~~ ~~nat~~ ~~trans~~

$\varepsilon : Id_C \to \Sigma$ ident.

~~MORPHISM~~

$\Sigma_1 \xrightarrow{\Phi} \Sigma_2$ nat trans

$f : X \to Y$ in C

$\Phi_Y \circ \Sigma_1(f) = \Sigma_2(f) \circ \Phi_X$ (natural trans)

$\Phi_X \circ \varepsilon_{1,X} = \varepsilon_{2,X}$

$$\begin{tikzcd} T^3(X) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[d, "T(\eta_X)"'] & T^2(X) \arrow[d, "\mu(X)"] \\ T^2(X) \arrow[r, "\mu(X)"'] & T(X) \end{tikzcd}$$

ASSOC $\qquad$ Unit.

Morphism cont

$$\begin{tikzcd} \Sigma^3(X) \arrow[r, "\Sigma(\mu_X)"] \arrow[d, "\mu_{\Sigma(X)}"'] & \Sigma^2(X) \arrow[d, "\mu_X"] \\ \Sigma^2(X) \arrow[r, "\mu_X"'] & \Sigma(X) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{id}_C\Sigma(X) \arrow[r, "\varepsilon_{\Sigma(X)}"] \arrow[dr, "\mathrm{id}_{\Sigma(X)}"'] & \Sigma^2(X) \arrow[d, "\mu_X"] & \Sigma\,\mathrm{id}_C(X) \arrow[l, "\Sigma(\varepsilon_X)"'] \arrow[dl, "\mathrm{id}_{\Sigma(X)}"] \\ & \Sigma(X) & \end{tikzcd}$$

$$\Phi_Y \circ \Sigma_1(f) = \Sigma_2(f) \circ \Phi_X$$ (natural trans)

$$\boxed{\Phi : \Sigma_1 \longrightarrow \Sigma_2}$$

$$\Phi_X \circ \varepsilon_{1,X} = \varepsilon_{2,X}$$

$$\mu_{2,X} \circ \Sigma_2(\Phi_X) \circ \Phi_{\Sigma_1(X)} = \Phi_X \circ \mu_{2,X}$$

$$= \mu_{2,X} \circ \Phi_{\Sigma_2(X)} \circ \Sigma_1(\Phi_X)$$

$$\begin{tikzcd}
\Sigma_1^2(X) \arrow[rr, "\mu_{1,X}"] \arrow[d, "\Sigma_1(\Phi_X)"'] \arrow[dr, "\Phi_{\Sigma_1(X)}"] & & \Sigma_1(X) \arrow[dd] \\
\Sigma_1\Sigma_2(X) \arrow[d, "\Phi_{\Sigma_2(X)}"'] & \Sigma_2\Sigma_1(X) \arrow[dl, "\Sigma_2(\Phi_X)"] & \\
\Sigma_2^2(X) \arrow[rr, "\mu_{2,X}"'] & & \Sigma_2(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma_1(X) \arrow[d, "\Phi_X"'] \\
\Sigma_2(X)
\end{tikzcd}
\qquad \overset{\Sigma_1}{\rightsquigarrow} \qquad
\begin{tikzcd}
\Sigma_1^2(X) \arrow[d, "\Sigma_1(\Phi_X)"] \\
\Sigma_1(\Sigma_2(X))
\end{tikzcd}$$

$$X = \Sigma_1(X) \rightsquigarrow \qquad
\begin{tikzcd}
\Sigma_1^2(X) \arrow[d, "\Phi_{\Sigma_1(X)}"] \\
\Sigma_2\Sigma_1(X)
\end{tikzcd}$$

$$\Sigma_1\Sigma_2(X) \xrightarrow{\Phi_{\Sigma_2(X)}} \Sigma_2^2(X)$$

$$\Sigma_2\Sigma_1(X) \xrightarrow{\Sigma_2(\Phi_X)} \Sigma_2^2(X)$$

NATURAL WAYS TO GET THERE

Adjoint Pairs

$F: C \to D$ $\quad C(c, G(d)) \cong D(F(c), d)$
$G: D \to C$

C	D	$F: C \to D$	$G: D \to C$
Sets	Groups	Free	Forget
Sets	Monoids	Free	Forget

$\Sigma = G \circ F$

$\eta: id_C \to G \circ F$

$\xi: F \circ G \to id_C$

$\varepsilon = \xi$

$\mu = F \otimes \eta \otimes G$

Arakelov Theory 50's

How to compactify $\mathrm{Spec}(\mathbb{Z})$?

$X/\mathbb{C}$ curve,

$p \in X(\mathbb{C}) \rightsquigarrow$ **VALUATION**

$\mathrm{ord}_p : \underbrace{k(X)}_{\text{function field of } X}^{\times} \longrightarrow \mathbb{Z}$ group hom

$\dfrac{\mathbb{C}(x)[y]}{\langle y^2 - x^3 - 1 \rangle}$

$\mathrm{ord}_p(f) =$ order of Laurent series at p.

Valuations $\rightsquigarrow$ Absolute Values

v

$|f| := c^{-v(f)}$ $\quad c > 1$

$$\deg(f) = \sum_{P \in X} \operatorname{ord}_P(f) = 0$$

$$\prod_{P \in X} |f|_P = 1, \quad \text{(Product Formula)}$$

$$|x|_p = p^{-\operatorname{ord}(x)}, \quad \boxed{\begin{array}{l} x \in \mathbb{Z} \\ x \equiv 0 \mod p^n \\ x \not\equiv 0 \mod p^{n+1} \end{array}}$$

Arakelov's Idea:

$$\prod_{p \in \operatorname{Spec}(\mathbb{Z})} |x|_p = \frac{1}{|x|_{\mathbb{R}}}$$

$|\cdot|_{\mathbb{R}} \Leftrightarrow$ "infinite place" ??

Arakelov Divisors

$D = \sum n_{\mathfrak{p}} \mathfrak{p} + \sum \alpha_\sigma \sigma$, $\quad \sigma : K \hookrightarrow \mathbb{C}$
$\alpha_\sigma \in \mathbb{R}$

$$\widehat{\deg}(D) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \# \mathcal{O}_K/\mathfrak{p} + \sum_\sigma \alpha_\sigma$$

$$\widehat{\mathrm{ord}}_y(f) = \begin{cases} \mathrm{mult}_y(f), & y \nmid 1 \\ -\log |f|_y, & y \mid 1 \end{cases}, \quad (f) = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_r^{e_r}, \quad \mathrm{ord}_{\mathfrak{p}_j}(f) = e_j.$$

$$\prod_y |f|_y = 1, \quad \boxed{(f) = \sum_y \mathrm{ord}_y(f)\, y} \quad \text{(Tate's Thesis)}$$

<u>Tate</u>: $\mathrm{Pic}^0(B)$

prove: $\mathcal{O}_K^\times$ finitely gen

$\# \mathrm{Cl}(\mathcal{O}_K) < \infty$

$\# C(K) < \infty$ $\quad C/K$

<u>Parshin - Arakelov</u>: (K fn field)

$\forall A$, $\# \{ P \in C(K) : h(P) \le A \} < \infty$

$\exists b$

$\rightsquigarrow$ $\forall P \in C(K)$, $h(P) \le b \Rightarrow \# C(K) < \infty$.

$C/K \rightsquigarrow \mathcal{X}/B$,

$\mathcal{X}_K \simeq C$, $\quad C(K) \simeq \{ P : B \to \mathcal{X} \}$

(1950)

$K(B) = K$,

$\rightsquigarrow$ Intersection theory

$\rightsquigarrow$ Bounds

Arakelov: An Intersection theory for divisors on an arithmetic surface 1974.

Arakelov:

⇝ Divisor class group

⇝ Intersection theory

Arakelov's Idea: C/K, $[K:\mathbb{Q}] < \infty$

$\mathcal{X} \to B = \mathrm{Spec}(\mathcal{O}_K)$

$\mathcal{X}_\eta \simeq C$

$\mathcal{X}_{\mathfrak{p}} = C_{\mathfrak{p}}$ when def

$C(K) \leftrightarrow \{ B \xrightarrow{s} \mathcal{X} \}$

Int Theory?

$\overline{\mathcal{X}} = ?$

$\overline{\mathcal{X}} = \mathcal{X} + C(\mathbb{C})$ base changed along σ

$\sigma : K \hookrightarrow \mathbb{C}$

F_σ of $\mathcal{X}$ = fibers at ∞

$D = D_{\text{finite}} + D_\infty$

D_{finite} = Weil

$D_\infty = \sum_\sigma \alpha_\sigma F_\sigma$, $\alpha_\sigma \in \mathbb{R}$

Arakelov: • $\exists$ natural symmetric bilinear pairing.

• factors through div of rational fns.

$P, Q : B \to \mathcal{X}$ distinct §'s

$\langle P, Q \rangle_{\text{fin}} + \langle P, Q \rangle_\infty$

usual weird

$$-\sum_{\sigma: K \hookrightarrow \mathbb{C}} \log G(P_\sigma, Q_\sigma),$$

$\boxed{G = \text{Distance Fn on } X^\sigma}$ $= \text{Arakelov-Green fn}$

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Write down G axioms

$\rightsquigarrow$ unique

No approach (No canonical class ineq)

- R.R.
- Hodge-Index
- Noether Formula

(FALTINGS)

$$a + b = c \qquad \le \sqrt[3]{abc}^{\,1-\varepsilon}$$

Canonical Class Ineq $\Longrightarrow$ ABC

$\Uparrow$

Szpiro

- Differential Geom at ∞
- Can classes of an Arithmetic surface?,
- Relation to other inv?