INTERACTION BETWEEN CECH COHOMOLOGY AND GROUP COHOMOLOGY

TAYLOR DUPUY

Abstract. [Summer 2012] These are old notes and they are just being put online now. They are incomplete and most likely riddled with errors that I may never fix. There are important technical hypotheses missing that make which make certain proofs invalid as stateded below. For example we sometimes need the sheaf of abelian group to be quasi-coherent modules to make things work. If someone reads this and wants to be awesome and email me I would appreciate it. –Taylor

Abstract. [Original] Given a sheaf of groups acting on an abelian sheaf of groups we define a group cohomology theory. A special case of a theorem proves that twisted homomorphism in certain group cohomologies when paired with cocycles in cech cohomology give rise to 1-cocycles of vector bundles.

1. Cech Cohomology

Let I be some index set and let $\mathcal{U} = \{U_i : i \in I\}$ be a collection of open sets with $X = \bigcup_{i\in I} U_i$. Let $\mathcal{G}$ be a sheaf of groups on X.

A collection $(g_{ij}) \in \prod_{i,j\in I} \Gamma(U_i \cap U_j, \mathcal{G})$ with $g_{ij} = g_{ji}^{-1}$ is said to be a **cocycle** provided

$$g_{ij}g_{jk}g_{ki} = 1 \tag{1.1}$$

on $U_i \cap U_j \cap U_k$. The collection of cocycles will be denoted by $Z^1(\mathcal{U}; X, \mathcal{G})$. A collection $g_{ij} \in \prod_{i,j\in I} \Gamma(U_i \cap U_j, \mathcal{G})$ is called a **coboundary** if there exists some collection $\varepsilon_i \in \mathcal{G}(U_i)$ for each $i \in I$ such that

$$g_{ij} = \varepsilon_i \varepsilon_j^{-1}.$$

We will denote the collection of coboundaries by $B^1(\mathcal{U}; X, \mathcal{G})$. Note that if $g_{ij} = \varepsilon_i\varepsilon_j^{-1}$, then we have $g_{ji} = \varepsilon_j\varepsilon_i^{-1} = (\varepsilon_i\varepsilon_j^{-1})^{-1} = g_{ij}^{-1}$. Also for all i, j and k we have $g_{ij}g_{jk}g_{ki} = \varepsilon_i\varepsilon_j^{-1}\varepsilon_j\varepsilon_k^{-1}\varepsilon_k\varepsilon_i^{-1} = 1$. Which shows that every coboundary is a cocycle.

Next we define what is means for two cocycles to be cohomologous. Let (g_{ij}) and (h_{ij}) be cocycles with respect to some cover $\mathcal{U}$. We say that (g_{ij}) and h_{ij} are **cohomologous** and write $(g_{ij}) \sim (h_{ij})$ if and only if there exists some $(\varepsilon_i) \in \prod_{i\in I} \mathcal{G}(U_i)$ such that

$$\varepsilon_i g_{ij} = h_{ij}\varepsilon_j.$$

Proposition 1.1 (Being Cohomologous Defines an Equivalence Relation)**.** *The above is an equivalence relation.*

Proof. We'll show that the relation on $Z^1(\mathcal{U}; X, \mathcal{G})$ is reflexive, transitive and symmetric. The relation is reflexive since we can take $\varepsilon_i = 1_i \in \mathcal{G}(U_i)$. The relation is symmetric since $\varepsilon_i g_{ij} = h_{ij}\varepsilon_j \implies \varepsilon_i^{-1}h_{ij} = g_{ij}\varepsilon_j^{-1}$. Next suppose that $f_{ij} \sim g_{ij}$ and $g_{ij} \sim h_{ij}$. We can write the first relation as $f_{ij} = \varepsilon_i^{-1}g_{ij}\varepsilon_j$ and the second relation as $g_{ij} = \delta_i^{-1}h_{ij}\delta_j$ which gives us

$$\begin{aligned} f_{ij} &= \varepsilon_i^{-1}g_{ij}\varepsilon_j \\ &= \varepsilon_i^{-1}\delta_i^{-1}h_{ij}\delta_j\varepsilon_j \\ &= (\delta_i\varepsilon_i)^{-1}h_{ij}(\delta_j\varepsilon_j). \end{aligned}$$

□

The **first cohomology SET** of a sheaf of groups $\mathcal{G}$, on a topological space X, with respect to the open cover $\mathcal{U}$ is

$$\check{H}^1(\mathcal{U},\mathcal{G}) := Z^1(\mathcal{U};X,\mathcal{G})/\sim .$$

Here $\sim$ is the relation of two cocycles being cohomologous. We can make Cech Cohomology independent of the cover under consideration by taking the injective limit of the cohomology sets. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. We say that $\mathcal{U}$ is a **refinement** of $\mathcal{V}$ provided that for all $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subset V$. The notion of refinement defines a partial ordering on the collection of open covers. If $\mathcal{U}$ is a refinement of $\mathcal{V}$ we will write $\mathcal{U} \leq \mathcal{V}$.

Proposition 1.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a scheme X. If $\mathcal{U} \leq \mathcal{V}$ then for every sheaf of groups $\mathcal{F}$ on X there exists a map*

$$\varphi^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U};X,\mathcal{F}) \to H^1(\mathcal{V};X,\mathcal{F}).$$

Note that we now have a partially ordered set consisting of all open covers of X and for each open cover we have a sets $H^1(\mathcal{U},\mathcal{F})$ such that for every $\mathcal{U} \leq \mathcal{V}$ there exists a map $\varphi^{\mathcal{U}}_{\mathcal{V}}$. One can check that this collection is an injective system. Using this injective system we can now define the First Homology Set for a sheaf of groups,

$$\check{H}^1(X,\mathcal{G}) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},G).$$

Recall that $\varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},G) = \frac{\coprod_{\mathcal{U}} \check{H}^1(\mathcal{U}:XG)}{\sim}$ where $\eta \in H^1(\mathcal{U},\mathcal{F})$ is said to be similar to $\xi \in H^1(\mathcal{V},\mathcal{F})$ if and only if there exists some $\mathcal{W}$ such that $\mathcal{U} \leq \mathcal{W}$ and $\mathcal{V} \leq \mathcal{W}$ and $\varphi^{\mathcal{U}}_{\mathcal{W}}(\eta) = \varphi^{\mathcal{V}}_{\mathcal{W}}(\xi)$.

Lemma 1.3. *Let $\Phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of groups. If $(f_{ij}) \in Z^1(\mathcal{U};X,\mathcal{F})$ then $g_{ij} := \Phi_{U_{ij}}(f_{ij}) \in Z^1(\mathcal{U};X,\mathcal{G})$.*

Proof. Let $\Phi_{U_{ij}}(f_{ij}) = g_{ij}$ as above. We have

$$\begin{aligned} g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} \cdot g_{ki}|_{U_{ijk}} &= \Phi_{U_{ij}}(f_{ij})|_{U_{ijk}} \cdot \Phi_{U_{jk}}(f_{jk})|_{U_{ijk}} \cdot \Phi_{U_{ki}}(f_{ki})|_{U_{ijk}} \\ &= \Phi_{U_{ijk}}(f_{ij}|_{U_{ijk}}) \cdot \Phi_{U_{ijk}}(f_{jk}|_{U_{ijk}}) \cdot \Phi_{U_{ijk}}(f_{ki}|_{U_{ijk}}) \\ &= \Phi_{U_{ijk}}(f_{ij}|_{U_{ijk}} \cdot f_{jk}|_{U_{ijk}} \cdot f_{ki}|_{U_{ijk}}) \\ &= \Phi_{I_{ijk}}(1_{\mathcal{F}}) = 1_{\mathcal{G}} \end{aligned}$$

□

Corollary 1.4. *If $\Phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of groups on a topological space X then there exists an induced map*

$$\Phi_* : H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}).$$

Every short exact sequence of groups induces a long exact sequence on cohomlogy[1]

Lemma 1.5. *Let*

$$1 \longrightarrow \mathcal{G}' \xrightarrow{\iota} \mathcal{G} \xrightarrow{p} \mathcal{G}'' \longrightarrow 1$$

be an exact sequence of sheaves of groups on X. There is a well-defined map

$$\delta : H^0(X, \mathcal{G}'') \to H^1(X, \mathcal{G}').$$

(procedure is outlined at the end of the proof)

Proof. A global section of $\mathcal{G}''$ is a collection of $g_i'' \in \mathcal{G}''(U_i)$ such that

$$g_i''|_{U_i \cap U_j} = g_j''|_{U_i \cap U_j}.$$

For every open subset U the map $p_U : \mathcal{G}(U) \to \mathcal{G}''(U)$ is surjective. This means that there exists some $g_i \in \mathcal{G}(U_i)$ such that $p_{U_i}(g_i) = g_i''$. Since p is a sheaf homomorphism we have

$$p_{U_i}(g_i)|_{U_i \cap U_j} = p_{U_j}(g_j)|_{U_i \cap U_j} \implies p_{U_i \cap U_j}(g_i|_{U_i \cap U_j}) = p_{U_i \cap U_j}(g_j|_{U_i \cap U_j})$$

and since $p_{U_i \cap U_j} : \mathcal{G}(U_i \cap U_j) \to \mathcal{G}''(U_i \cap U_j)$ is a sheaf homomorphism we have

$$1_{\mathcal{G}''} = p_{U_i \cap U_j}(g_i|_{U_i \cap U_j} \left(g_j|_{U_i \cap U_j}\right)^{-1}).$$

Since the sequence is exact there exists some $g_{ij}'' \in \mathcal{G}'(U_i \cap U_j)$ such that $\iota_{U_i \cap U_j}(g_{ij}') = g_i|_{U_i \cap U_j} \left(g_j|_{U_i \cap U_j}\right)^{-1}$. These satisfy the cocycle condition $g_{ij}'|_{U_{ijk}} g_{jk}'|_{U_{ijk}} g_{ki}'|_{U_{ijk}} = 1_{\mathcal{G}'(U_{ijk}}$. Briefly (omiting subscripts and restrictions) we have $g_{ij}' g_{jk}' g_{ki}' = 1$ if and only if $\iota(g_{ij}' g_{jk}' g_{ki}') = 1$ since ι is injective. We have

$$\begin{aligned}
\iota(g_{ij}' g_{jk}' g_{ki}') &= \iota(\iota^{-1}(g_i g_j^{-1}) \cdot \iota^{-1}(g_j g_k^{-1}) \cdot \iota^{-1}(g_k g_i^{-1})) \\
&= g_i g_j^{-1} \cdot g_j g_k^{-1} \cdot g_k g_i^{-1} \\
&= 1,
\end{aligned}$$

which shows that the collection $(g_{ij}, U_{ij}) \in Z^1(\mathcal{U}; X, \mathcal{G}')$.

It remains to show that this cocycle gives a well-defined cohomology class. Suppose that used the surjectivity $p_{U_i} : \mathcal{G}(U_i) \to \mathcal{G}''(U_i)$ to find some other elements $\tilde{g}_i \in \mathcal{G}(U_i)$ with $\tilde{g}_i \neq g_i$ for at least one i and $p(\tilde{g}_i) = g_i''$. We need to show that the cocycle $\tilde{g}_{ij}' := \iota^{-1}(\tilde{g}_i \tilde{g}_j^{-1})$ is cohomologous to the one we just defined from the lifts (g_i).

This is true since

$$h_i \tilde{g}_{ij} = g_{ij} h_j$$

where $h_i := \iota^{-1}(g_i \tilde{g}_i^{-1})$. □

Remark 1.6 (Procedure For Defining Connecting Homomorphism). If $(g_i'')_{i \in I} \in \prod_{i \in I} \mathcal{G}''(U_i)$ defines a global section in the sense that g_i and g_j are equal on $U_i \cap U_j$ we do the following to get a cocycle in $\mathcal{G}'$,

(1) Lift g_i'' to g_i.[2]
(2) Form $\iota^{-1}(g_i g_j^{-1}) := g_{ij}'$.

[1]as far as we have the first cohomology defined

[2]This means g_i satisfies $p(g_i) = g_i''$

Using the connecting homomorphism above on relative tangent sequence:

$$0 \longrightarrow \mathcal{T}_{\mathfrak{X}/S} \longrightarrow \mathcal{T}_{\mathfrak{X}} \longrightarrow \pi^*\mathcal{T}_S \longrightarrow 0$$

where $\pi : \mathfrak{X} \to S$ is a morphism of schemes which makes the relative tangent sequence exact we get a map

$$\check{H}^0(\mathfrak{X}, \mathcal{T}_S) \to \check{H}^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}),$$

which Oort and Steenbrink use as the definition of the Kodaira-Spencer Map.[3]

2. Cech Cohomlogy for Sheaves of Abelian Groups

$$(d\xi)_{i_0 i_1 \ldots i_{p+1}} = \sum_{i=0}^{p+1} (-1)^j \xi_{i_0 \cdots \hat{i_j} \cdots i_{p+q}} \tag{2.1}$$

$$(d\xi)_J = \sum_{h=0}^{p+1} (-1)^h \xi_{J \setminus \{j_h\}} \tag{2.2}$$

$J \setminus j_h := J_h = \{j_{h1} < j_{h2} < \cdots < j_{hp}\}$

$$\begin{aligned}
(d(d\xi))_J &= \sum_{h=0}^{p+2} (-1)^h (d\xi)_{J\setminus\{j_h\}} \\
&= \sum_{h=0}^{p+2} (-1)^h \left(\sum_{k=0}^{p+1} (-1)^k \xi_{J_h \setminus \{j_{hk}\}} \right) \\
&= \sum_{h=0}^{p+2} \left(\sum_{k<h}^{p+1} (-1)^{h+k} \xi_{J\setminus\{j_h, j_k\}} + \sum_{h<k}^{p+1} (-1)^{h+k-1} \xi_{J\setminus\{j_h,j_k\}} \right) \\
&= 0
\end{aligned}$$

3. Pairings

Proposition 3.1 (Cup Product)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of commutative groups. There exists a morphism of abelian groups $H^p(X, \mathcal{F}) \times H^q(X, \mathcal{F}) \to \mathcal{H}^{p+q}(X, F \otimes G)$ given by*

$$(\eta \cup \xi)_{i_0 i_1 \cdots i_{p+q}} = \eta_{i_0 \cdots i_p} \otimes \xi_{i_p \cdots i_{p+q}} \tag{3.1}$$

Proof. We need to show that $(\eta \cup \xi)$ is a cocycle.

Let $I = \{i_0 < i_1 < \ldots < i_{p+q}\}$ and define

$$I_{\leq p} := \{i_0 < i_1 < \ldots < i_p\} \quad \text{and} \quad I_{\geq p} := \{i_p < i_{p+1} < \ldots < i_{p+q}\}.$$

this gives $\#I_{\leq p} =$ and $\#I_{\geq p} = q$ and allows us to write

$$(\eta \cup \xi)_I = \eta_{I_{\leq p}} \otimes \xi_{I_{\geq p}}.$$

[3]They actually define their map using the long exact sequence for cohomology of abelian groups in the style of Hartshorne (via the right derived functors of sheaf hom with the structure sheaf).

Now let $J = \{j_0 < j_1 < \ldots < j_{p+q+1}\}$

$$\begin{aligned}
d(\eta \cup \xi)_J &= \sum_{h=0}^{p+q} (-1)^h (\eta \cup \xi)_{J\setminus\{j_h\}} \\
&= \sum_{h=0}^{p+q} (-1)^h (\eta_{(J_h)_{\leq p}} \otimes \xi_{(J_h)_{\geq p}}) \\
&= \sum_{0\leq h\leq p} (-1)^h (\eta_{J_{\leq p+1}\setminus\{j_h\}} \otimes \xi_{J_{\geq p+1}}) + \sum_{p+1\leq h\leq q+1} (-1)^h (\eta_{J_{\leq p}} \otimes \xi_{J_{\geq p}\setminus\{j_h\}}) \\
&= (d\eta)_{J_{\leq p+1}} \otimes \xi_{J_{\geq p+1}} + \sum_{0\leq h\leq q} (-1)^{h+p+1} (\eta_{J_{\leq p}} \otimes \xi_{J_{\geq p}\setminus\{j_{h+p+1}\}}) \\
&= (d\eta)_{J_{\leq p+1}} \otimes \xi_{J_{\geq p+1}} + (-1)^{p+1} \eta_{J_{\leq p}} \otimes (d\xi)_{J_{\geq p}}
\end{aligned}$$

so that we have

$$d(\eta \cup \xi) = d\eta \cup \xi + (-1)^p \eta \cup d\xi, \tag{3.2}$$

from this is follows that is η and ξ are cocycles then so is $\eta \cup \xi$, $\square$

The dual of an $\mathcal{O}$-module is the sheaf of sets $\mathcal{F}^\vee := \underline{\mathrm{Mod}}_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$ which takes an open set U to the natural transformation of sheaves:

$$U \mapsto \mathrm{Mod}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{O}|_U).$$

In the case that $\mathcal{F}|_U = \underline{\mathcal{F}(U)}$ ($\mathcal{F}$ is quasi-coherent) we have a isomorphism of abelian groups

$$\mathcal{F}^\vee(U) \cong \mathrm{Mod}_{\mathcal{O}}(\mathcal{F}|_U, \mathcal{O}|_U) \cong \mathrm{Mod}_{\mathcal{O}(U)}(\mathcal{F}(U), \mathcal{O}(U)) = \mathcal{F}(U)^\vee.$$

Proposition 3.2. *Let $\mathcal{F}$ be an invertible $\mathcal{O}$-module. Then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}^\vee \cong \mathcal{O}$.*

Proof. This is true on trivializing open sets since for any R-module or rank 1 we have $M^* \otimes_R M \cong R$ as R-modules via the map $m^* \otimes m \mapsto m^*(m)$. $\square$

4. Group Cohomology

Let G be a group and A be an abelian group and suppose that A has the structure of a left $\mathbb{Z}[G]$-module. Equivalently we can say that there exists an left action of G on A which is group homomorphism $\rho : G \to \mathrm{Aut}(A)$.[4]. We will denote the left action of an element $g \in G$ on $a \in A$ by $g \cdot a$. Right actions will be denoted by a^g. These notations respect the associativity required for left and right actions respectively.

Now that we've discussed group actions we can move on to group cohomology. There are two ways (that I know of) to define group cohomology for a group G on A with respect to a particular action. The first way is Ext^i functors –which are the derived functors of $\mathsf{Mod}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$. This means we take an injective resolution of A, apply the hom functor $\mathsf{Mod}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ to the sequence and take quotients of images mod kernels of succesive maps. The alternative approach is to define a cochain

[4]If we first compose with the inverse then get a map to the group of automorphism this is a right action (sometimes called an antihomomorphism)

complex directly. Our **cochains** are going to be maps of sets from some cartesian power of G to A:

$$C^n(G,A) := \mathsf{Maps}(G^n, A).$$

Note that since A is an abelian group the set of cochains is a group. As usual we need maps $d : C^n(G,A) \to C^{n+1}(G,A)$. If $\varphi : G^n \to A$, then we define $(d\varphi) : G^{n+1} \to A$ as follows

$$\begin{aligned}(d\varphi)(g_1, g_2, g_3, \ldots, g_{n+1}) &= g_1\varphi(g_2, g_3, \ldots, g_{n+1}) - \varphi(g_1g_2, g_3, \ldots, g_n) + \varphi(g_1, g_2g_3, \ldots, g_{n+1}) + \cdots \\ &\quad +(-1)^n\varphi(g_1, g_2, \ldots, g_ng_{n+1}) + (-1)^{n+1}\varphi(g_1, g_2, \ldots, g_n).\end{aligned}$$

One can see that $d\varphi$ is a morphism of abelian groups if one tries. The following fact is slightly more painful to verify

Proposition 4.1. *For all n the image of the map $d : C^{n-1}(G,A) \to C^n(G,A)$ is contained in the kernel of the map $d : C^n(G,A) \to C^{n+1}(G,A)$. In other words*

$$d^2 = 0.$$

The kernel of the map $d : C^n(G,A) \to C^{n+1}(G,A)$ is the group of n-**cocycles**, and we denote it by $Z^n(G,A)$. The image of the map $d : C^{n-1}(G,A) \to C^n(G,A)$ is group of n-**coboundaries**. Because of the proposition which tells us that $d^2 = 0$ (proposition 4.1) the group of n-cocycles are contained in the group of n-coboundaries, this implies that the quotient of n-cocycles modulo n-coboundaries exists. This quotient is the nth **cohomology Group** for group cohomology:

$$H^n(G,A) := \frac{\ker(d : C^n(G,A) \to C^{n+1}(G,A))}{\mathrm{Im}(d : C^{n-1}(G,A) \to C^n(G,A))} = \frac{Z^n(G,A)}{B^n(G,A)}.$$

Remark 4.2. For arbitrary groups G and A (with A not necessarily abelian) we can still define the zeroth and first cohomology groups. Supposing our action of G on A is a right action we will have

$$Z^1(G,A) := \{\phi : G \to A \mid \phi(g_1g_1) = \phi(g_1)^{g_2}\phi(g_2)\}. \tag{4.1}$$

Two cocycles $\phi, \psi \in Z^1(G,A)$ are said to be cohomologous if and only if there exists an a in A such that for all $g \in G$,

$$a^g\phi(g) = \psi(g)a.$$

One should check that when A is abelian that this is equivalent to the previous definition given above. Now we define the **First Cohomology Set** by

$$H^1(G,A) = Z^1(G,A)/\sim.$$

We should stress that this is not a group but a pointed set[5] whose trivial element corresponds to trivial cocycles.

We would now like to look at some one cocycles in particular cases. For these examples note that $Z^1(G,A)$ consists of maps $\varphi : G \to A$ satisfying

$$\varphi(g_1g_2) = \varphi(g_1) + g_1\varphi(g_2).$$

These are called **twisted homomorphism**[6]. We will now give some examples:

[5] meaning a set with a distinguished element this is trivial

[6] Also called skew or crossed homomorphism

Derivative: Let $G = \mathrm{Aut}_R(R[x])$ and $A = R[x]^\times$. Let us identify automorphism of $R[x]$ by the polynomial that x is sent to under the automorphism. under this identification composition of polynomials. If we let our action G on A be composition:

$$(a^g) = (a \circ g)$$

then the derivative operation $D = \frac{d}{dx} : \mathrm{Aut}_R(R[X]) \to R[X]^\times$ is a cocycle. First note that the map makes sense since the derivative of any automorphism is a unit. Also since $D[(f \circ g)(x)] = D[f](g(x))D[g](x) = (D[f]^g)(x)D[g](x)$ we have that D is a cocycle.

Schwarzian Derivative: Let $G = \mathrm{Aut}_R(R[X])$ and $A = R[x]$ (with the addition being the group operation). The map

$$S[f] := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

called the Schwarzian Derivative, is actually a cocycle when viewed appropriately. The schwarzian derivative satifies the following "chain rule":[7]

$$S[f \circ g] = g'(x)^2 S[f] \circ g + S[g]$$

If we let G act on the right of A

$$(F^f)(x) := f'(x)^2 F(f(x)),$$

then S is a cocycle. Note that

$$\begin{aligned}
(F^f)^g &= (f'^2 \cdot F \circ f)^g \\
&= g'^2 \cdot (f' \circ g)^2 \cdot F \circ f \circ g \\
&= (f \circ g)'^2 \cdot F \circ (f \circ g) \\
&= F^{f \circ g}.
\end{aligned}$$

Top Degree Coefficient: Let $G_d \leq \mathrm{Aut}_{R/p^2R}(R/p^2R[x])$ be the collection of automorphism inducing polynomials of degree less than or equal to d. In general, every automophism induction polynomial of $(R/p^2R[x]$ can be written as

$$f(x) = a + bx + pF(x)$$

where b is a unit in R/p^2R and $\mathrm{ord} F \geq 2$. We need to check that the collection of automorphism inducing polynomials of degree less than or equal to d form a subgroup. First one can convince themselves that if f and h are automorphism inducing polynomials that $\deg(f \circ g) \leq \max(\deg f, \deg h)$, after this it only remains to show that if $\deg f = d$ it's inverse h has degree less than or equal to d.

$$\begin{aligned}
x &= f(h(x)) \\
&= a_f + b_f h(x) + pF_f(h(x)) \\
&= a_f + b_g(a_h + b_h x + pF_h(x)) + pF_f(a_h + b_h x) \\
&= a_f + b_f a_h + b_f b_h x + p(b_f F_h(x) + F_f(a_h + b_h x),
\end{aligned}$$

[7] proved elsewhere

if the degree of F_h were bigger than the degree of F_f it would be impossible for the term $b_f F_h(x) + F_f(a_h + b_h)$ to be zero since $\deg(b_f F_h(x)) = \deg(F_h(x))$ and $\deg F_f(a_h + b_h x) = \deg F_f$.

Consider now that map $G_d \to R/pR$ given by $f \mapsto c_d(f)/m(f)$ where $c_d(f)$ is the top degree term divided by p, reduced mod p and $m(f)$ is the coefficient of x in f. From the computation above we can see that

$$c_d(f \circ h) = b_f c_d(h) + b_h^d c_d(f) = m(f) c_d(h) + m(h)^d c_d(f)$$

when we divide this by $b_f b_h$ we get

$$c_d(f \circ h)/m(f \circ h) = m(h)^{d-1} c_d(f)/m(f) + c_d(h)/m(h)$$

which implies that $\varphi(f) = c_d(f)/m(f)$ is a cocycle under the right action of G_d on R/pR given by $A^h = m(h)^{d-1} A$.

We would next like to describe the connection between twisted homomorphisms and semi-direct products. Let G and H be groups. Recall that given any left action $\rho : G \times H \to H$ we can form a new group $H \rtimes_\rho G$[8]. As a set $H \rtimes_\rho G$ is just the cartesian product $H \times G$. The group multiplication for this group is given as follows:

$$(h_1, g_1) *_\rho (h_2, g_2) = (h_1(g_1 h_2), g_1 g_2).$$

If we have a right action then the group is $G \ltimes^\rho H$ and the group law is given by

$$(g_1, h_1) *^\rho (g_2, h_2) = (g_1 g_2, h_1^{g_2} h_2).$$

Twisted homomorphism $\varphi : G \to H$ give maps from G to $H \ltimes G$.

Proposition 4.3 (Cocycles give Maps to Semi-Direct Products)**.** *Suppose that G acts on H on the left and $\varphi : G \to H$ is a left twisted homomorphism. Then the map*

$$g \mapsto (\varphi(g), h)$$

defines a group homomorphism $G \to H \rtimes_\rho G$.[9]

Proof.

$$\begin{aligned} (\varphi(g_1 g_2), g_1 g_2) &= (\varphi(g_1) g_1 \varphi(g_2), g_1 g_2) \\ &= (\varphi(g_1), g_1) * (\varphi(g_2), g_2) \end{aligned}$$

□

Another way of saying this is that a twisted homomorphism φ splits the exact sequence via the map above

$$1 \longrightarrow H \longrightarrow H \rtimes_\rho G \longrightarrow H \longrightarrow 1 \,,$$

This falls in line with the general theory: it turns out $N = A$ is abelian that $H^2(G, A)$ (which we don't know how to define for nonabelian N) classifies group extensions of A by G with the given action. We will make this statement more

[8]There are four reasonable notations $G \ltimes_\rho H$, $G \ltimes^\rho H$, $H \rtimes_\rho G$ and $H \rtimes^\rho G$. We will reserve the superscripts for right actions and the subscripts for left actions. We could have arranged so that we would put the G on the left for left actions and the G on the right for right actions but we won't follow this because our current choice looks best in coordinates. For right actions we will use the notation $G \ltimes^\rho H$

[9]The right action case is similar with $g \mapsto (g, \varphi(g))$

precise below. Before proving this recall that a **group extension** of a group N by a group G is some group $\tilde{G}$ such that we have the exact sequence:

$$1 \longrightarrow N \xrightarrow{i} \tilde{G} \xrightarrow{p} G \longrightarrow 1 .$$

Furthermore if we prescribe an action of G on N then it must be the same as the action of congugation by G (which is well defined up to congugation)[10]. Here are some examples of group extensions

Units of $\mathbb{Z}/p^2\mathbb{Z}$: We will show that the units of $\mathbb{Z}/p^2\mathbb{Z}$ are an extension the additive group $\mathbb{Z}/p\mathbb{Z}$ by the multiplicative group of units of $\mathbb{Z}/p\mathbb{Z}$. First note that we can include $\mathbb{Z}/p\mathbb{Z}$ into $\mathbb{Z}/p^2\mathbb{Z}$ by the map $a \mapsto (1+pa)$. It is a homomorphism since

$$(1+pa)(1+pb) = (1+p(a+b)).$$

We will call this map the exponential map. Also notice that all of these units are precisely the kernel of the quotient map $(\mathbb{Z}/p^2Z)^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$, this gives us the exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\exp} (\mathbb{Z}/p^2\mathbb{Z})^\times \xrightarrow{\text{quot}} (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow 1 .$$

More generally if R is any ring which is is p-torsion free then we for any $r > s$ have a similar exact sequence

$$0 \longrightarrow R/p^rR \xrightarrow{\exp} (R/p^{r+s}R)^\times \xrightarrow{\text{quotient}} (R/p^sR)^\times \longrightarrow 1$$

where the first map is $f \mapsto 1 + p^s f$ and the second map is the canonical quotient map. Note that the first map is injectice since two elements f and g have the same image if and only if they are congruent mod p^r (which makes them the same element). We should remark that this decomposition is useful in understanding the units of $(R/p^n)[x]$, which is useful in understanding the automorphism of $(R/p^nR)[x]$,[11] which is useful for understanding the automorphisms of the p-adic completion $R[x]^{\widehat{p}}$.[12]

The Affine Linear Group: Let R be a ring. The affine linear group is the group

$$\mathrm{AL}_1(R) := (\{f(x) \in R[x] \mid f(x) = ax+b, a \in R^\times, b \in R\}, \circ) \subset \mathrm{Aut}_R(R[x]).$$

This group is an extension of the group $R^\times$ by the additive group of R. The inclusion of R into $\mathrm{AL}_1(R)$ if given by $h \mapsto \tau_h(x) := x - h$. Note that $h + k \mapsto \tau_{h+k}(x) = (\tau_h \circ \tau_k)(x)$ which shows that this is a group homomorphism. First it's clear that this map is an inclusion. We will next show you that the image of R in $\mathrm{AL}_1(R)$ is normal. First note that if $f(x) = ax+b$ then $f^{-1}(x) = a^{-1} - a^{-1}b$ so that on can compute:

$$(f \circ \tau_h \circ f^{-1})(x) = x - ah$$

which shows that h is normal and that the action of $\mathrm{AL}_1(R)$ on the image of R is given by multiplication $(a,r) \mapsto ar$. It turns out that two elements $f(x) = ax+b$ and $g(x) = cx+d$ are equivalent if and only if $a = c$ in which

[10]Warning: there exists sequences with the same N, $\tilde{G}$ and G but which have different maps which are not the same as extensions

[11]Since the derivative of every automorphism is a unit

[12]Since it the the projective limit of automorphisms of the former type.

case $(g \circ f^{-1})(x) = x + (d - b)$. This shows that each element of quotient has a representative $m_a(x) = ax$ and is isomorphic to $\mathrm{R}^\times$. This gives the sequence

$$0 \longrightarrow R \longrightarrow \mathrm{AL}_1(R) \longrightarrow R^\times \longrightarrow 1.$$

Since the exact sequence splits via the homomorphism $R^\times \to \mathrm{AL}_1(R)$ given by $a \mapsto m_a(x)$ we have that

$$\mathrm{AL}_1(R) \cong R \rtimes R^\times$$

where the isomorphism is given by $(ax+b) \mapsto (b, a)$ since $(ax+b)\circ(cx+d) = (acx + ad + b) \mapsto (ad + b, ac) = (b, a)(d, c)$.

The affine linear group $\mathrm{AL}_1(R)$ turns out be the full automorphism group $\mathrm{Aut}_R(R[x])$ when R is a noetherian integral domain. If $x \mapsto f(x)$ induces an automorphism of $R[x]$ then we have $R[x] = R[f(x)]$. This means that $R[x]/\langle f(x)\rangle \cong R[x]/\langle x\rangle \cong R$. This quotient map can be viewed as an evaluation map given by $x \mapsto c$ which implies that $x - c \in \langle f(x)\rangle$ which implies that $\langle x - c\rangle \subset \langle f(x)\rangle$. We actually have that $x - c$ and $f(x)$ are associates since $R \cong R[x]/\langle f(x)\rangle \cong \frac{R[x]/\langle x-c\rangle}{\langle f(x)\rangle/\langle x-c\rangle} \cong R/\langle f(c)\rangle$ which means that $f(c) = 0$ (which is the Noetherian hypothesis). In particular $f(x) = ax+ac$ for some unit a which show that f is affine linear. Conversely every affine linear automorphism is an automorphism.

In particular if we look at the case $R = \mathbb{Z}/p\mathbb{Z}$, the group $\mathrm{AL}_1(\mathbb{Z}/p\mathbb{Z})$ is a group extension of $\mathbb{Z}/p\mathbb{Z}$ by $(\mathbb{Z}/p\mathbb{Z})^\times$ distinct which is not isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})^\times$.[13] This group extension, by what we have said previously is the trivial group extension.

The following lemmas will be critical in simpifying cocycles we get get to the interaction of sheaf cohomology and group cohomology.

Lemma 4.4 (Actions on Quotients)**.** *Let G be a group and A be an R-module. Suppose that G acts on A and that the action is R-linear.*

(1) *For every $r \in R$ the action of G on A/rA well defined.*
(2) *There is a group homomorphism from $G \ltimes^\rho A \to G \ltimes^\rho (A/rA)$.*

Proof. The action of G on the quotient is defined by taking a representative, acting on that representative, then taking its equivalence class.

In order to show this procedure is well defined we need to show that if $x, y \in A$ and $x - y \in rA$ then for all g we have $x^g - y^g \in rA$.

Suppose that $x - y = rm$. Taking the difference of the acted on elements gives $x^g - y^g = (x - y)^g = (rz)^g = rz^g$ which is congruent to zero mod rA. This implies the action of the quotient is well-defined.

The second part is clear with $(g, a) \mapsto (g, \bar{a})$. $\square$

Here is an example of the above proposition: Let R be a p-torsion free ring. Let $G = \mathrm{Aut}_R(R[t])$ and $A = R[t]$ with the right action of G on A be given by composition

$$(F^g)(t) := F(g(t)).$$

We can reduce the image mod p and still have a well defined action of the lifted automorphism.

[13] Because one is abelian and the other isn't.

Lemma 4.5 (Actions of Quotients). *Let G and H be groups and N a normal subgroup of G.*

If for all $x \in H$ we have $x^g = x^{g'}$ whenever $g \equiv g' \mod N$, then there exists a group homomorphism

$$G \ltimes H \to (G/N) \ltimes H.$$

An application of the two reductions above could be the following. The action of $\mathrm{Aut}_A A[x]$ on $A[x]$ induces a well defined action on $A[x]/pA[x]$. The action of $\mathrm{Aut}_A(A[x])$ on $A[x]/pA[x]$ decends to an action of $\mathrm{AL}_1(A/pA)$ on $(A[x]/pA)[x]$. The first part follows from actions on quotients (lemma 4.4) and the second part of the proposition follows from proposition actions of quotients (Lemma 4.5). We will reproduce these statements in the special cases stated above:

Suppose that $F, G \in A[x]$ and that $F \equiv G \mod p$. We can write $F - G = pH$ for some $H \in A[x]$. For any $g \in \mathrm{Aut}_A A[x]$ we have

$$\begin{aligned} F(x)^g - G(x)^g &= (F(x) - G(x))^g \\ &= (pH(x))^g \\ &= pH(x)^g, \end{aligned}$$

which shows that the the action on equivalence classes is well defined mod p.

For the second statement, suppose that $f, g \in \mathrm{Aut}_A(A[x])$ are congruent mod p and that $F(x)$ represents some class in $A[x]/pA[x]$. First now that $f - g \equiv = 0$ mod p implies that $f' - g' = 0 \mod p$. Using the above we have

$$\begin{aligned} \overline{F(x)^g} &= \overline{g'(x)F(g(x))} \\ &= \overline{g'(x)^2}\bar{F}(\overline{g(x)}) \\ &= \overline{f'(x)^2}\bar{F}(\overline{f(x)}) \\ &= \overline{F(x)^f}, \end{aligned}$$

which shows that automorphisms induced by polynomials which are equivalent mod p define the same map on equivalence classes mod p. Since $\mathrm{AL}_1(A/pA) \cong \mathrm{Aut}_A(A[x]) \mod p$ we are done.

5. Group Cohomology for Sheaves of Groups

Let $\mathcal{G}$ be a sheaf of groups on X and $\mathcal{A}$ be a sheaf of abelian groups on X. A **left action** of $\mathcal{G}$ on $\mathcal{A}$ a collection of left actions of groups

$$\rho_U : \mathcal{G}(U) \times \mathcal{A}(U) \to \mathcal{A}(U),$$

for all inclusions of open sets of X, $U' \subset U$ we have

$$\begin{tikzcd} \mathcal{G}(U) \times \mathcal{A}(U) \arrow[r, "\rho_U"] \arrow[d, "\mathrm{res}\times\mathrm{res}"'] & \mathcal{A}(U) \arrow[d, "\mathrm{res}"] \\ \mathcal{G}(U') \times \mathcal{A}(U') \arrow[r, "\rho_{U'}"] & \mathcal{A}(U') \end{tikzcd}.$$

When given a left action ρ of $\mathcal{G}$ on $\mathcal{A}$ we will denote the left action of $g \in \mathcal{G}(U)$ on $A \in \mathcal{A}(U)$ by $g \cdot a$. Similarly, given a right action and some g and a we will use the notation A^g.

Multiplicative Group Acting on the Structure Sheaf: If we let X be a scheme, let $\mathcal{A} = \mathcal{O}_X$ and let $\mathcal{G} = \mathcal{O}_X^\times$ then we can define an action of $\mathcal{G}$ on $\mathcal{A}$ by $(m, f) \mapsto mf$.

Sheaf of Automorphisms acting On a Sheaf in the Obvious Way: Let X be a scheme, let $\mathcal{A} = \mathcal{O}_{X\times\mathbb{A}^1}$ and $\mathcal{G} = \underline{\mathrm{Aut}}(\mathbb{A}^1)$ then we can define an action. Note that for every open set we have

$$\mathcal{A}(U) = \mathcal{O}_{X\times A^1}(U) \approx \mathcal{O}_X(U)[t]$$

where t is the parameter on $\mathbb{A}^1$. The sheaf $\underline{\mathrm{Aut}}(\mathbb{A}^1)$ assigns to each open subset of X the collection of morphism

$$\begin{tikzcd} U \times \mathbb{A}^1 \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_1"] & U \times \mathbb{A}^1 \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd} .$$

In terms of rings these are maps $\varphi^* \in \mathrm{Aut}_{\mathcal{O}(U)}(\mathcal{O}(U)[t])$. Identifying $\underline{\mathrm{Aut}}\mathbb{A}^1(U)$ with a subset of polynomials in $\mathcal{O}_X(U)[t]$ where the multiplication law is composition we have $g \in \underline{\mathrm{Aut}}\mathbb{A}^1(U)$ acting on $F(t) \in \mathcal{O}_{X\times\mathbb{A}^1}(U) \approx \mathcal{O}_X(U)[t]$ via

$$(F, f) \mapsto F \circ f.$$

We can "twist" this action after multiplying by the derivative of f:

$$(F^f)(x) := f'(x)^n F(f(x))$$

defines another action of the $\underline{\mathrm{Aut}}(\mathbb{A}^1)$ on $\mathcal{O}_{X\times A^1}$.

Also we should mention that semi-direct products exist in the category of sheaves. Given a left action ρ of $\mathcal{G}$ on $\mathcal{A}$ we will form the presheaf $\mathcal{A} \rtimes_\rho \mathcal{G}$ by letting $(\mathcal{A} \rtimes_\rho \mathcal{G})(U) = \mathcal{A}(U) \rtimes_{\rho_U} \mathcal{G}(U)$. To check that this is a sheaf we just need to check the sheaf axiom.[14]ince the product of sheafs of sets is a sheaf, and $\mathcal{A} \rtimes_\rho \mathcal{G}$ is a product as a sheaf of sets, it satisfies the sheaf axiom.

Affine Linear Group: Let X be a scheme and let $\mathcal{A} = \mathcal{O}$ under addition and $\mathcal{G} = \mathcal{O}^\times$. Let the left action of $\mathcal{O}^\times$ on $\mathcal{O}$ be the standard multiplication action $(f, m) \mapsto mf$. The sheaf $\mathcal{O} \rtimes \mathcal{O}^\times$ with the standard multiplication action is a semidirect product of sheaves.

In ordinary group cohomology of a group G acting on A,[15] we take the n cochains to be the maps of sets from the nth cartesian power of the group G to the group A:

$$C^n(G, A) = \mathsf{Maps}(G^n, A).$$

These sets have the structure of an abelian group themselves since A is an abelian group. Next, we define a maps $d : C^n(G, A) \to C^{n+1}(G, A)$ which single out the cocycles $Z^n(G, A) = \ker(d : C^n(G, A) \to C^{n+1}(G, A)$ and the coboundaries $B^n(G, A) = \mathrm{Im}(d : C^{n-1}(G, A) \to C^n(G, A))$. We then get to define our cohomology groups

$$H^i(G, A) = \frac{\ker(d : C^n \to C^{n+1})}{\mathrm{Im}(d : C^{n-1} \to C^n)} = \frac{Z^n(G, A)}{B^n(G, A)}$$

in virtue of the fact that $d^2 = 0$ (which is not immediately obvious).

[14] S

[15] say on the left

To define a group cohomology of a sheaf of groups $\mathcal{G}$ on a sheaf of abelian group $\mathcal{A}$ with respect to an action ρ we proceed in a similar fashion. Our cochains are going to be morphism of sheaves *of sets*

$$C^n(\mathcal{G},\mathcal{A}) := \mathsf{Nat}(\mathcal{G}^n,\mathcal{A}).$$

From the viewpoint of category theory, presheaves are just contravariant functors from $\mathsf{Top}(X)$[16] to Sets. Morphisms of presheaves (and hence of sheaves) are natural transformations of these functors (hence to notation "Nat"). Our maps $d : C^n(\mathcal{G},\mathcal{A}) \to C^{n+1}(\mathcal{G},\mathcal{A})$ are going to be defined similarly: If $\Phi : \mathcal{G}^{n-1} \to \mathcal{A}$, then for all $U \subset X$ and all $g_1,\ldots,g_n \in \mathcal{G}(U)$ we define

$$\begin{aligned}(d\Phi)_U(g_1,g_2,g_3,\ldots,g_n) \quad &= \quad g_1\Phi_U(g_2,g_3,\ldots,g_n) - \Phi_U(g_1g_2,g_3,\ldots,g_n) + \Phi_U(g_1,g_2g_3,\ldots,g_n)\\ &\quad + \cdots + (-1)^n\Phi_U(g_1,g_2,\ldots,g_{n-1}g_n).\end{aligned}$$

One should check that for all $\Phi \in \mathsf{Nat}(\mathcal{G}^{n-1},\mathcal{A})$ that $(d\Phi) \in \mathsf{Nat}(\mathcal{G}^{n+1},\mathcal{A})$. By this we mean that you should convince yourself that for all $U' \subset U$ inclusions of open sets the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}(U)\times\cdots\times\mathcal{G}(U) \arrow[r, "(d\Phi)_U"] \arrow[d, "\mathrm{res}^n"] & \mathcal{A}(U) \arrow[d, "\mathrm{res}"] \\
\mathcal{G}(U')\times\cdots\times\mathcal{G}(U') \arrow[r, "(d\Phi)_{U'}"] & \mathcal{A}(U')
\end{tikzcd}.$$

Again we now proceed in the fashion as with ordinary group cohomology to define cocycles and coboundaries. Since for each n, $C^n(\mathcal{G},\mathcal{A})$ is an abelian group and each d is a morphism of abelian groups the coboundaries and cocycles are already defined: The cocycles are those natural transformation[17] whose image under d is indentically zero and the coboundaries are the image of d as before. This is all taking place in the category of abelian groups so we don't need to worry about sheafifying the image or anything. Again we can define

$$\begin{aligned}Z^n(\mathcal{G},\mathcal{A}) &= \ker(d : C^n(\mathcal{G},\mathcal{A}) \to C^{n+1}(\mathcal{G},\mathcal{A})),\\ B^n(\mathcal{G},\mathcal{A}) &= \mathrm{Im}(d : C^{n-1}(\mathcal{G},\mathcal{A}) \to C^n(\mathcal{G},\mathcal{A})).\end{aligned}$$

The proof that $d^2 = 0$ follows from ordinary group cohomology fact and the property that $\mathcal{A}$ is a sheaf,[18] and we get to define our cohomology groups

$$H^i(\mathcal{G},\mathcal{A}) := \frac{Z^i(\mathcal{G},\mathcal{A})}{B^i(\mathcal{G},\mathcal{A})}.$$

Twisted homomorphisms will play a particularly important role in group cohomology.

Derivatives: We will now copy the derivations example in the ordinary group cohomology section for sheaves of groups. Let X be a scheme and where

[16]The category whose objects are open subsets of X and whose morphisms are inclusions

[17]Morphism of sheaves of sets

[18]On open subsets d acts like the d from group cohomology which means that the d^2 will equal zero on open subsets. Since $\mathcal{A}$ is a sheaf, the local triviality will patch showing that d^2 is in fact the zero map.

$\pi : X \times \mathbb{A}^1 \to X$. We will let $\mathcal{G} = \underline{\mathrm{Aut}}\mathbb{A}^1$ and $\mathcal{A} = (\pi_* O_{X\times\mathbb{A}^1})^\times$. As we have seen before $\mathcal{G}(U) = \underline{\mathrm{Aut}}\mathbb{A}^1_X(U)$ is the collection of maps φ where

$$\begin{tikzcd} U \times \mathbb{A}^1 \arrow[r, "\varphi"] \arrow[d] & U \times \mathbb{A}^1 \arrow[dl] \\ U & \end{tikzcd}.$$

They can be identified with subsets of $\mathcal{O}(U)[T]$ consisting of polynoimals which induct automorphisms $\varphi^* \in \mathrm{Aut}_{\mathcal{O}(U)}(\mathcal{O}(U)[T])$. Here T is the pullback of the parameter on $\mathbb{A}^1$. We should also mention that $\mathcal{A}(U) = \pi_*\mathcal{O}_{X\times\mathbb{A}^1}(U)^\times \cong \mathcal{O}_X(U)[T]^\times$.

We want to show that $D : \mathcal{G} \to \mathcal{A}$ defined by

$$D[g](T) = g'(T)$$

is a cocycle in group cohomology with respect to the right group action

$$(F(t), g(t)) \mapsto (F^g)(t) := g'(t)F(g(t)).$$

First we should check that D is a well-defined morphism of sheaves of sets: Suppose that $U' \subset U$ is an inclusion of open subset of X. We need to show that the following diagram commutes

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, "D_U"] \arrow[d, "\mathrm{res}"'] & \mathcal{A}(U) \arrow[d, "\mathrm{res}"'] \\ \mathcal{G}(U') \arrow[r, "D_{U'}"] & \mathcal{A}(U') \end{tikzcd}$$

In our particular case, this means the following

$$\begin{tikzcd} \underline{\mathrm{Aut}}\mathbb{A}^1(U) \arrow[r, "D_U"] \arrow[d, "\mathrm{res}^U_{U'}"] & \mathcal{O}(U)[T]^\times \arrow[d, "\mathrm{res}^U_{U'}"] \\ \underline{\mathrm{Aut}}\mathbb{A}^1(U') \arrow[r, "D_{U'}"] & \mathcal{O}(U')[T]^\times \end{tikzcd}. \tag{5.1}$$

Note that we can identify $\underline{\mathrm{Aut}}\mathbb{A}^1(U)$ and $\mathcal{O}(U')[T]^\times$ both as with a subsets of $\mathcal{O}(U)[t]$. In both these cases we have the restiction maps given by

$$a_0 + a_1T + \cdot + a_nT^n \mapsto \mathrm{res}^U_{U'}(a_0) + \mathrm{res}^U_{U'}(a_1)T + \cdots + \mathrm{res}^U_{U'}(a_n)T^n$$

where $\mathrm{res}^U_{U'} : \mathcal{O}(U) \to \mathcal{O}(U')$ is the restriction map for the structure sheaf. The commutivity of the diagram is just

$$D_{U'} \circ \mathrm{res}^U_{U'} = \mathrm{res}^U_{U'} \circ D_U$$

where $\mathrm{res}^U_{U'}$ denotes the restriction map on polynomials $\mathcal{O}(U)[T] \to \mathcal{O}(U')(T)$ induced by the restiction map $\mathcal{O}(U) \to \mathcal{O}(U')$ on the structure sheaf. The commutivity we needed in equation 5.1 is now clear since the operation $\mathrm{res}^U_{U'}$ does nothing to the indeterminates T and the operation D does nothing to the coefficients.

It remains to check that check that D is twisted homomorphism in group cohomology. This means that for every U, $D_U : \underline{\mathrm{Aut}}\mathbb{A}^1(U) \to \mathcal{O}(U)[T]^\times$ is a cocycle in $Z^1(\mathrm{Aut}_{\mathcal{O}(U)}(\mathcal{O}(U)[T]), \mathcal{O}(U)[T])$. This follows from the analysis of the cocycle in group cohomology on page-6

Of particular importance in this theory are the quotient lemmas (Lemmas 4.4 and 4.5 of the section on group cohomology (section 4) and lemma the tells us that cocycles in group cohomology give us maps to the semidirect products (Lemma 4.3 of the section on group cohomology (section 4).

Lemma 5.1 (Semi-Direct Products Induce Maps)**.** *Let X be a topological space. Let $\mathcal{G}$ and $\mathcal{A}$ be sheaves of groups and let $\rho : \mathcal{G} \to \mathcal{A}$ be a left action. Every $\Phi \in Z^1(\mathcal{G}, \mathcal{A})_\rho$ induces a sheaf homomorphism $\mathcal{G} \to \mathcal{A} \rtimes_\rho \mathcal{G}$ via*

$$g \mapsto (\Phi(g), g).$$

This is what is going to happen: we will take some complicated cocycle in group cohomology then apply a cocycle in group cohomology in to get a 1-cocycle in group cohomology of the twisted cocycle. After having this we will apply a series of reductions like the quotients of action and actions of quotients. After all this is said and done we will get sections of familiar line bundles (to which we can apply trace maps to) to get sections of line bundles.

6. Interaction Between Cech and Group Cohomology

The fact that cocycles in group cohomology give maps to semi-direct products (Lemma **??**) and the morphisms of sheaves lemma gives maps on cohomology (Lemma 1.4) give us a pairing

$$H^1(\mathcal{G}, \mathcal{A}) \times \check{H}^1(X, \mathcal{G}) \mapsto \check{H}^1(X, \mathcal{A} \rtimes \mathcal{G}).$$

Proposition 6.1. *The map described above is well defined.*

Proof. Given a cocycle in $Z^1(\mathcal{G}, \mathcal{A})$ we get a map from $\mathcal{G} \to \mathcal{G} \ltimes \mathcal{A}$. The morphism of sheaves of groups obviously defines a morphism of on cohomology (Proposition 1.3).we just need to check that this is well defined on the level of cocycles. □

Lemma 6.2 (Twisted Cocycle Condition)**.** *Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{A}$ be sheaf of abelian groups that have an action. (g_{ij}, A_{ij}) is a for $\mathcal{G} \ltimes \mathcal{A}$ if and only if (g_{ij}) is a cocycle for $\mathcal{G}$ and*

$$A_{ij}^{g_{jk}g_{ki}} + A_{jk}^{g_{ki}} + A_{ki} = 0 \tag{6.1}$$

Remark 6.3. The condition for left twisted cocycles is similar: If $G_{ij} = (A_{ij}, g_{ij}) \in \mathcal{A}(U_{ij}) \rtimes \mathcal{G}(U_{ij})$ is a left twisted cocycle the one can derive

$$A_{ij} + g_{ij}A_{jk} + g_{ij}g_{jk}A_{ki} = 0. \tag{6.2}$$

Proof. For any sheaf of groups $\mathcal{H}$ the cocycle condition is $h_{ij}h_{jk}h_{ki} = 1$. This is what we want to show for $\mathcal{H} = \mathcal{G} \ltimes A$ and with $h_{ij} = (g_{ij}, A_{ij})$ satisfying the hypotheses of the lemma:

$$\begin{aligned} h_{ij}h_{jk}h_{ki} &= (g_{ij}, A_{ij}) * (g_{jk}, A_{jk}) * (g_{ki}, A_{ki}) \\ &= (g_{ij}g_{jk}, A_{ij}^{g_{jk}} + A_{jk}) * (g_{ki}, A_{ki}) \\ &= (g_{ij}g_{jk}g_{ki}, A_{ij}^{g_{jk}g_{ki}} + A_{jk}^{g_{ki}} + A_{ki}) \\ &= (1, 0). \end{aligned}$$

□

Lemma 6.4 (Getting Sections)**.** *Suppose that $\pi : E \to X$ is a morphism of schemes with a trivializing cover $\{U_i : i \in I\}$ with isomorphisms $\varphi_i : \Gamma(U_i, E) \to \mathcal{A}(U_i)$ where $\mathcal{A}$ is a sheaf groups. Furthermore suppose that there is a group action of $\mathcal{G}$ on $\mathcal{A}$ such that*

$$(\varphi_j \circ \varphi_i^{-1})(A) = A^{g_{ij}}. \tag{6.3}$$

Then $Z^1(U; X, E) \leftrightarrow Z^1(\mathcal{U}; \mathcal{G} \ltimes^\rho \mathcal{A})$ via $(A_{ij}, g_{ij}) \mapsto \varphi_j^{-1}(A_{ij})$.

Equation 6.3 really says that E is just glued together by a cocycle for $\mathcal{G}$. The condition for left actions is slightly different. We want the transition maps of E to be satisfy $\varphi_i \circ \varphi_j^{-1}(f) = g_{ij} f$ and embed the twisted cocycles via $\varphi_i(s_{ij}) = A_{ij}$.

Proof. Suppose that $(s_{ij}) \in Z^1(\mathcal{U}, \mathcal{A})$ is a cech cocycle. By definition this implies

$$s_{ij} + s_{jk} + s_{ki} = 0.$$

Now consider $A_{ij} := \varphi_j(s_{ij}) \in \mathcal{A}(U_{ij})$. We have

$$\varphi_i(s_{ij}) + \varphi_i(s_{jk}) + \varphi_i(s_{ki}) = 0.$$

but notice

$$\begin{aligned}
\varphi_i(s_{ki}) &= A_{ki}, \\
\varphi_i(s_{jk}) &= (\varphi_i \circ \varphi_k^{-1} \circ \varphi_k)(s_{jk}) \\
&= (\varphi_i \circ \varphi_k^{-1})(A_{jk}) \\
&= A_{jk}^{g_{ki}}, \\
\varphi(s_{ij}) &= (\varphi_i \circ \varphi k^{-1} \circ \varphi_k \circ \varphi_j^{-1} \circ \varphi_j)(s_{ij}) \\
&= ((\varphi_i \circ \varphi_k^{-1}) \circ (\varphi_k \circ \varphi_j^{-1}))(A_{ij}) \\
&= A_{ij}^{g_{jk} g_{ki}}.
\end{aligned}$$

Putting this all together we have

$$0 = A_{ij}^{g_{jk} g_{ki}} + A_{jk}^{g_{ki}} + A_{ki},$$

which is the cocycle condition for $\mathcal{G} \ltimes \mathcal{A}$.

Conversely, if we are given a cocycle (g_{ij}, A_{ij}) for $\mathcal{G} \ltimes \mathcal{A}$ then we can get a cocycle for E. Here we let $s_{ij} := \varphi_j^{-1}(A_{ij})$ as expected. A similar type of manipulation yields $s_{ij} + s_{jk} + s_{ki} = 0$. □

We will now explain this more explicitly,

Invertible Sheaves: Let X be a schem. Suppose we are given $\eta \in H^1(X, \mathrm{AL}_1(\mathcal{O}))$. Since $\mathrm{AL}_1(\mathcal{O}) \cong \mathcal{O} \ltimes \mathcal{O}^\times$ the cocycle can be viewed as a twisted cocycle $f_{ij} = a_0(f_{ij}) + a_1(f_{ij})(t) \mapsto (a_0(f_{ij}), a_1(f_{ij}))$.[19] The multiplicative part $a_1(f_{ij}) = m_{ij}$ Defines an invertible sheaf $\mathcal{L}$ on X with trivializations $\varphi_i : \mathcal{L}(U_i) \to \mathcal{O}(U_i)$, such that for all $f \in \mathcal{O}(U_{ij})$ we have $(\varphi_i \circ \varphi_j^{-1})(f) = m_{ij}.f$ By the correspondence between left twisted cocycles and sections of group schemes we have $s_{ij} := \varphi_i^{-1}(a_0(f_{ij})) \in \mathcal{L}(U_{ij})$ defining a cocycle of $Z^1(C, \mathcal{L})$.

[19]Notice that we have chosen to make this isomophism with a left action so that we need follow the correspondence for left actions.

Vector Bundles: Slightly more generally, if X is a scheme $\mathcal{G} = \mathrm{GL}_n$ and $\mathcal{A} = \mathcal{O}^n$ then the condition in the sections lemma (Lemma 6.4) tells us that twisted cocycles $(M_{ij}, \vec{v}_{ij})$ define cocycles of the vector bundle,

$$\vec{v}_{ij} \mapsto s_{ij} = \varphi_j^{-1}(\vec{v}_{ij}) \in \check{H}^1(X, E).$$