DEFORMATIONS:

Family of Spaces

$$\{\mathfrak{X}_t \mid t \in T\}$$

example (Dwork Family) of K3 surfaces in $\mathbb{P}^3$

$$X^4 + Y^4 + Z^4 + W^4 = tXYZW, \quad \mathfrak{X}$$

$$X^4 + Y^4 + Z^4 + W^4 = 5XYZW, \quad \mathfrak{X}_5 \quad \text{(fiber at } t=5\text{)}$$

encode

$$\begin{array}{c} \mathfrak{X} \\ f \downarrow \text{ flat} \\ T \end{array}$$

flat ← makes properties vary nicely (dimension of fibers, etc)

$$\mathfrak{X}_t = f^{-1}(t)$$

Fiber at $t \in T$.

example

$\mathfrak{X}: \; x^2 = ty \quad \mathbb{A}^2$

exercise: $\mathfrak{X}/\mathbb{A}^1$ still flat.

Rem Blow-ups are not flat.

Defn. $A \to B$ Map of rings is flat, provided B a flat A-Module. For sheaves do it on the local rings.

(Start with this).

Affine Rings

$\mathrm{Spec}(A)$ = prime ideals $\neq A$

$V(f)$ = closed sets $= \{ \mathfrak{p} \in \mathrm{Spec}(A) \,;\, f(\mathfrak{p}) = 0 \}$ $f \in A$.

$f \equiv 0 \bmod \mathfrak{p}$

$$\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec} A \setminus V(f)) = A_f = A[\tfrac{1}{f}]$$

$\boxed{\mathrm{CRing}^{op} \xhookrightarrow[\text{faithful}]{} \text{Schemes}}$

$\boxed{\text{Thm}}$ Sheaf of modules [illegible] determined on the [illegible] and [illegible]

Graded Rings:

$S = \bigoplus_{d \geq 0} S_d$, f.g./degree 0.

$S_d S_e \subseteq S_{d+e}$

$S_+ := \bigoplus_{d>0} S_d$ "irrelevant ideal",

example

$S = \mathbb{C}[x_0, \dots, x_n]$

$S_+ = \langle x_0, \dots, x_n \rangle$

does not corresp to pt in projective space.

$\boxed{\mathrm{GrRing}^{op} \xrightarrow[\text{faithfully}]{} \text{Schemes}}$

$\mathrm{Proj}(S) = \{$ homog prime ideals not contained in irrelevant ideal $\}$, f homogeneous $\in \bigsqcup_{d \geq 0} S_d$

$V(f) = \{ \mathfrak{p} \in \mathrm{Proj}(S) : f \equiv 0 \bmod \mathfrak{p} \}$,

$$\mathcal{O}_{\mathrm{Proj}(S)}(\mathrm{Proj}(S) \setminus V(f)) = \underline{(S_{(f)})_0}$$ degree zero piece after localization

Example 2. $\mathcal{X} = \mathrm{Proj}\left(R[X,Y,Z,W] / (X^4+Y^4+Z^4+W^4 - tXYZW) \right)$

~~homogeneous~~ ~~prime~~

$T = \mathrm{Spec}(R)$, $R = \mathbb{C}[t]$, $(S_{\mathcal{X}})_0 = R$

$= \mathbb{A}^1_{\mathbb{C}}$ (affine space)

$\mathbb{C}[t] \hookrightarrow S_{\mathcal{X}}$, (non-graded way); $S_{\mathcal{X}} = \dfrac{R[X,Y,Z,W]}{(X^4 + \cdots)}$

associated graded ring

$\cancel{\left(\bigoplus_{d>0} S_d\right)}$

NOTE: $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{A}^1$, $\mathbb{P}^3 \times \mathbb{A}^1 \leftrightarrow \mathbb{C}[X,Y,Z,W] \otimes_{\mathbb{C}} \mathbb{C}[t]$

$C \to B$, $C = \bigoplus_{d\geq 0} C_d$, $B = \bigoplus_{d\geq 0} B_d$ graded rings

$\mathrm{Proj}(B) \setminus V \to \mathrm{Proj}(C)$

$V = V_+(C_+ B)$, $R_+ = \bigoplus_{d>0}$

within

$\mathbb{C}[t][X,Y,Z,W]$

$= \bigoplus_{d\geq 0} \mathbb{C}[X,Y,Z,W]_d [t]$

~~Determined sheaves of modules and proj~~

example 1: the fiber at $t=5$

$\mathcal{X}_5 : x^2 - 5y = 0$.

CLAIM

$\mathcal{X}_5 \cong \operatorname{Spec}(A \otimes_{\mathbb{C}[t],\sigma} \mathbb{C})$, where

$$\begin{array}{ccc} \mathbb{C}[t] & \xrightarrow{\sigma} & \mathbb{C} \\ \Psi & & \Psi \\ t & \longmapsto & 5 \end{array}$$

$$A = \frac{\mathbb{C}[t][x,y]}{\langle x^2 - ty \rangle}$$

Specialization map

idea: $A \otimes_R B = \_\_\_\_\_$

Let A = above, $B = \mathbb{C}$, $R = \mathbb{C}[t]$ and given B $\mathbb{C}[t]$-algebra structure induced by σ.

Exercise: $$\frac{\mathbb{C}[t][x,y]}{\langle x^2 - ty \rangle} \otimes_{\mathbb{C}[t],\sigma} \mathbb{C} \cong \frac{\mathbb{C}[x,y]}{\langle x^2 - 5y \rangle}$$

~~Fibers, via Base Change~~

④

$5 \in \mathbb{A}^1(\mathbb{C})$,

$$\begin{array}{ccc} \mathbb{C}[t] & \xrightarrow{\sigma} & \mathbb{C} \\ \Psi & & \Psi \\ t & \longmapsto & 5 \end{array}$$

specialization map

$A, B \in \mathrm{CRing}_R$,

$$\boxed{A \otimes_R B = \frac{\text{Free Ring on Symbols } a \otimes_R b,\ a \in A,\ b \in B}{ra \otimes_R b = a \otimes_R br,\ \text{distributive laws, etc...}}}$$

$A = \dfrac{\mathbb{C}[t][x,y]}{\langle x^2 - ty \rangle}$, $\quad \mathbb{C}[t] \subset A$, as expected

$B = \mathbb{C}$, $\quad \mathbb{C}[t] \xrightarrow{\sigma} B$, via $t \overset{\sigma}{\longmapsto} 5$

$R = \mathbb{C}[t]$,

$$\frac{\mathbb{C}[t][x,y]}{\langle x^2 - ty \rangle} \otimes_{\mathbb{C}[t],\sigma} \mathbb{C} \cong \frac{\mathbb{C}[x,y]}{\langle x^2 - 5y \rangle}$$

$\mathcal{X} = \operatorname{Spec}(A)$ $\quad t = $ max ideal corresp $\langle t-5\rangle$

$\downarrow$

$T = \operatorname{Spec}(R)$ $\qquad f^{-1}(5) = \mathcal{X}_5 = \operatorname{Spec}(A \otimes_R \mathbb{C})$

Specialization Map in general; given $t \in T$ specialization map

~~Generally~~ $\mathcal{O}_{T,t} \longrightarrow k(t)$

Residue Field

$$\boxed{f^{-1}(t) := \mathcal{X} \otimes_T k(t)}$$

Rem: t is not always a closed pt.

Tr deg depends on $\dim(\overline{\{t\}})$

$\mathcal{X}$ as in example 1.

Example: $\mathcal{X} \longrightarrow T = \mathbb{A}^1$, $\quad (0) \in \operatorname{Spec}(\mathbb{C}[t])$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \;$ generic pt.

$\qquad\qquad\qquad\qquad\qquad\qquad \eta \quad \overline{\{\eta\}} = \operatorname{Spec}(\mathbb{C}[t])$

variety over $\mathbb{C}(t)$?

$k(\eta) = \mathbb{C}(T) = \mathbb{C}(t)$: $\mathcal{X}_\eta \Rightarrow \dfrac{\mathbb{C}(t)(y)[x]}{\langle x^2 - ty\rangle}$ $\quad x^2 - ty$

$\mathfrak{U} = \{U_i\}$ open cover.

$U_{ij} := U_i \cap U_j$

btw: $\operatorname{Spec}(A) \cap \operatorname{Spec}(B) \simeq \operatorname{Spec}(A \otimes_R B)$ if A & B localizations of R.

$\underline{i=1}$: $\check{C}(\mathfrak{U};X,G) \subset \prod_{i=1}^{n} G(U_{ij})$

$$\| \quad \{(g_{ij}) \mid g_{ij}\, g_{jk}\, g_{ki} = I\}$$

- $H^1(\mathfrak{U};G) := \check{C}(\mathfrak{U};X,G)/\sim$

$(g_{ij}) \sim (h_{ij}) \iff \exists\, g_i \in \prod_i G(U_i)$

$$g_i\, g_{ij}\, g_j^{-1} = h_{ij}.$$

Finally,

$$H^1(X,G) := \varinjlim H(\mathfrak{U};X,G) = \frac{\coprod_{\mathfrak{U}} H(\mathfrak{U};X,G)}{\sim}$$

$[g_{ij}]_{\mathfrak{U}} \sim [h_{ij}]_{\mathfrak{U}'}$

$\iff$ exist refinement $\mathfrak{U}''$ of $\mathfrak{U}$ & $\mathfrak{U}'$ s.t.

$[g_{ij}]_{\mathfrak{U}''} = [h_{ij}]_{\mathfrak{U}'''}$

En $\underline{i=2}$: Gerbes

$\underline{i=3}$: OPEN PROBLEM.

ČECH COHOM.

X top / site

G sheaf of group

~~$\mathcal{U} = \{U_i\}$ cover of X~~

$i=0$: $G(X) = H^0(X,G)$ = global §

$\check{C}^1(\mathcal{U}, X, G) \subset \prod_{i,j=1}^{n} G(U_{ij})$, $U_{ij} = U_i \cap U_j$, $g_{ij} g_{jk} g_{ki} = 1$

$(g_{ij}) \sim (h_{ij}) \iff \exists\, g_i \in \prod_{i=1}^{n} G(U_i)$, $g_{ij} = g_{ji}^{-1}$

$g_i g_{ij} g_j^{-1} = h_{ij}$.

$\boxed{\check{C}^1/\sim \; = \check{H}^1}$

$H^1(X,G) = \varinjlim H^1(\mathcal{U}, X, G) = \coprod H^1(\mathcal{U},G)/\sim$, $[g_{ij}]_{\mathcal{U}} \sim [h_{ij}]_{\mathcal{V}} \iff$ have common refinements.

OPEN PROBLEM:

Construct H^3 for G non-abelian.

$H^0(X,G) = ?$

Tangent sheaf: X scheme, $T_X(\mathcal{U}) := \{\delta : \mathcal{O}(\mathcal{U}) \to \mathcal{O}(\mathcal{U})\}$ derivation

$T_{X/R}$ R-linear derivations.

~~Application~~

~~Prop~~ ~~Slogan~~. B = universal family (i.e. a moduli space) $\Rightarrow$ KS an isom.

example: $M_g(\mathbb{C}) \cong \{X/\mathbb{C}$ curve of genus $g\}$

$h^0(X, \Omega_X) = g$ (holomorphic 1-forms.

$\Rightarrow \dim(M_g) = 3g-3.$

~~Rem, Families~~

GEOMETRIC KODAIRA-SPENCER MAP

$\mathcal{M}_g(R) \leftrightarrow \{X/R \text{ genus } g\}/\text{iso}$ moduli space of ...

$\dim(\mathcal{M}_g) = 3g-3$

$\mathcal{M}_g$ irreducible & smooth

DEFNS

$$\mathcal{X} \xrightarrow{\pi} B$$

$\delta_b \in T_{B,b}$

$\delta \in T_B(U), \quad U \ni b$

$\delta_i \in T_X(U_i), \quad \bigcup_i U_i \supset \pi^{-1}(U)$

$\pi_* \delta_i = \delta$

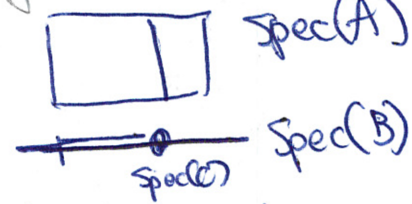
$KS(\delta) := [\delta_i - \delta_j] \in \check{H}^1(\mathcal{X}_b, T_{\mathcal{X}_b})$

$$\boxed{KS : T_{B,b} \longrightarrow H^1(\mathcal{X}_b, T_{\mathcal{X}_b})}$$

OPEN, When B is "universal" (a moduli space, then gives an isom)

$\boxed{\text{OPEN PROBLEM}}$: • Construct nice families of moduli spaces for Calabi-Yau 3-folds (or any threefolds).

⇝ Hilb Scheme Nasty many irred components
(thought looking at Betti #s in ~~[illegible]~~ graded results would fix enough params to give nice moduli of curves on $\mathbb{P}^4$ but NO)



⇝ Stacks used to prove irred.

Rem: Relative tangent sequence "short exact sequences from long exact sequences", ~~someth~~

$$0 \to f^*\Omega_{B/C} \to \Omega_{X/C} \to \Omega_{X/B} \to 0$$

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0, \quad C \to B \xrightarrow{f} A \text{ ring hom}$$

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(K) \end{array}$$

- K a field with a derivation $\delta : K \to K$

(Think: generic fiber of previous construction)

- $X = \bigcup_i U_i$ affine open

$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i)$, $\delta_i|_K = \delta$

$(K \subseteq \mathcal{O}(U_i))$

$$\boxed{KS(\delta) : \mathrm{Der}(K) \to H^1(X, T_{X/K})}$$

<u>GENERIC GEOMETRIC</u>
<u>KODAIRA-SPENCER</u>

DELIGNE-ILLUSIE

R, p torsion free ring.

(e.g. $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}}_p)$)

(D) Lift of Frob: $B \in$ Rings$_R$

$\phi: R \to B$, $\phi(a) \equiv a^p$ mod p

(P) Lifts of Frobenius & p-ders are in bijection,

$$\begin{cases} \dfrac{\phi(a) - a^p}{p} = \delta(a) \\ \phi(a) = a^p + p\,\delta(a) \end{cases}$$

X/R, scheme, R p-tors free,

DI_0: $\{p\text{-der}_R\} \to \boxed{H^1(X_0, F^* T_{X_0})}$

$\delta \in p\text{-der}(R)$

construction: $X = \bigcup U_i$

$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i)$, $\delta_i|_R = \delta$.

$\delta_i - \delta_j$ mod p : $\mathcal{O}(U_{ij})/p \to \mathcal{O}(U_{ij})/p$

$\quad \| $

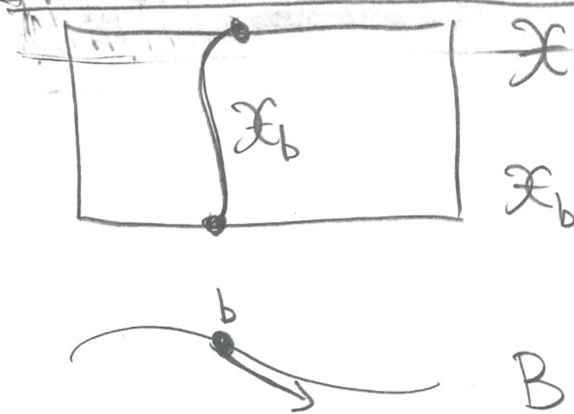
$\quad D_{ij}$

$D_{ij}(x+y) = D_{ij}(x) + D_{ij}(y)$

$D_{ij}(xy) = D_{ij}(x) y^p + x^p D_{ij}(y)$

$\Rightarrow \boxed{[\delta_i - \delta_j] \in H^1(X_0, F_{X_0}^* T_{X_0})}$

INFINITESIMAL DEFORMATION



$\mathcal{X}_b = \mathcal{X} \otimes k(b)$

$\delta_b \in T_{B,b} \longleftrightarrow \mathrm{Der}(\mathcal{O}_{B,b}, \mathcal{O}_{B,b})$

$\rightsquigarrow \mathrm{Der}(\mathcal{O}_{B,b}, k(b)[\varepsilon])$

$\rightsquigarrow \exists v \in B(k(b)[\varepsilon])$

or

$\mathrm{Spec}(k[b][\varepsilon]) \xrightarrow{\mathrm{Spec}(v)} B.$

$\mathcal{X}/B$ a flat family $\rightsquigarrow$ a $\tilde{\mathcal{X}}_{b,v}$ by base change,

$$\begin{tikzcd} \tilde{\mathcal{X}}_{b,v} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(k(b)[\varepsilon]) \arrow[r] & B \end{tikzcd}$$

$\bar{\mathcal{X}}_{b,v}$ = an Infinitesimal Deformation of $\mathcal{X}_b$

<u>NOTE</u>: $k[b](\varepsilon) \twoheadrightarrow k(b)$

$\tilde{\mathcal{X}}_{b,v} \otimes k(b) = \mathcal{X}_b$

Local Deformations

A = Artin Local Ring
(ε.g $k[\varepsilon]$, $\mathbb{Z}/p^2$)

given, X/K, K = (res value of A)

$$\mathrm{Def}_X(A) = \{\mathcal{X}/A,\ \mathcal{X} \otimes_A K \simeq X\}/\mathrm{iso}$$

Deformation functor of X, the deformation functor is representable

THM
Serre Tate: $\mathrm{Def}_X(A)$, X, curve of genus g.

$\simeq \mathrm{CRing}(\hat{\mathcal{O}}_{M_g,[X]}, A)$ (local moduli)

$A = k[\varepsilon]$, get the tangent space at $\boxed{T_{[X]}M}$

COR

$$\boxed{\mathrm{Def}_X(k[\varepsilon]) \underset{KS}{\cong} H^1(X, T_X)}$$

=

Infinitesimals $A = k[\varepsilon]$

Witt finitesimal $A = \mathbb{Z}/p^2$

or $A = \hat{\mathbb{Z}}_p^{ur}/p^2$

Generally: given
$X_0/\overline{\mathbb{F}}_p$
X/A, $X \otimes_A \overline{\mathbb{F}}_p = X_0$
$\Rightarrow X$ is infinitesimal lift of X_0

Connection to Descent:

~~HM~~ (Buium, ~~...~~ 90s)

$X|K$, $\delta : K \to K$, $K = \overline{K}$

$KS(\delta) = 0$

$\Longleftrightarrow \exists X' | K^{\delta}$

$X' \otimes_{K^{\delta}} K \simeq X$. (Descent to constants)

$K^{\delta} = \{ x \in K \mid \delta(x) = 0 \}$

(transcendental descent)

~~Ideal~~ Almost True: "X can be defined by eqns w/ coeff in K^{δ}"

BORGER: $\mathrm{Sch}_{\mathbb{F}_1}$ $\left[X/\mathbb{Z}, + \phi_p : X \to X \text{ commuting lifts of Frob} \right]$

$= X_{\mathbb{F}_1}$

"$\mathbb{F}_1$-model" $X|R$ $\quad R = \widehat{\mathbb{Z}_p^{ur}}$

Fact $DI_0(X) = 0 \Longleftrightarrow$ X_1 admits lift of p-Frob ϕ

If $\exists X_{\mathbb{F}_1}$

$X_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z} = X$ (forget full functor)

$\Longrightarrow DI_0(\delta) = 0$

Before Serre-Tate?

Problem: • Classify

$$\mathrm{Def}_X(\widehat{\mathbb{Z}}_p^{ur}/p^2)$$

• WARNING: Some $X_0/\overline{\mathbb{F}}_p$ do not admit lifts!!

$\boxed{\text{DELIGNE-ILLUSIE}}$ CONS

X/R

$$DI_0: \begin{Bmatrix} p\text{-der} \\ \text{on } R \end{Bmatrix} \longrightarrow H^1(X_0, F^* T_{X_0})$$

$$(F^* T_{X_0})(U) = \{D : \mathcal{O}(U) \to \mathcal{O}(U)\}$$

$$\begin{cases} D(x+y) = D(x) + D(y) \\ D(xy) = D(x)y^p + x^p D(y) \end{cases}$$

semi-linear maps.

(Compare to "Generic" "KS Construction)

Recent Work: X/R curve "provided exists A_n str"

$$DI: \begin{Bmatrix} p\text{-der} \\ \text{on } R \end{Bmatrix} \longrightarrow H^1(X, L)$$

$$L \otimes_{R_0} R_0 = F^* T_{X_0}$$

conj: • DI exists for any X

• DI_n exists for general X_n/R_n

• DI_n classifies lifts.

Rem. * 2000s Buium-Hurlburt

- E elliptic curve
- A abelian variety

Proof Strategy

$A = \frac{R[x,y]}{(f)}$

$$\Lambda_p \odot X = J^1(X)$$

$u_1 = D(f_x)^{-1} \dot{x} \ \delta f$

$u_2 = D(f_y)^{-1} \dot{y} \ \delta f$

$$x: f(x,y) = 0 \qquad \delta f = \frac{\phi(f) - f^p}{p}$$

$$\frac{f^\phi(x^p + p\dot{x}, y^p + p\dot{y}) - f(x,y)^p}{p} = 0$$

$$= r + f^\phi_x(x^p,y^p)\dot{x} + f^\phi_y(x^p,y^p)\dot{y}$$

$$+ \frac{p}{2}\Big[f^\phi_{xx}(x^p,y^p)\dot{x}^2 + 2f^\phi_{xy}(x^p,y^p)\dot{x}\dot{y} + f^\phi_{yy}(x^p,y^p)\dot{y}^2 \Big] = 0$$

$$\boxed{\begin{aligned} \delta(x) &= \frac{\phi(x) - x^p}{p} \\ \phi(x) &= x^p + p\,\delta(x) \end{aligned}}$$

$\delta(x)$ a p-derivation.

$$r = \frac{f^\phi(x^p,y^p) - f(x,y)^p}{p}$$

$\in \hat{\mathcal{O}}(U)$

$$\hat{\mathcal{O}}(J^1(U)) = \mathcal{O}(U)[\dot{x}]^\wedge = \mathcal{O}(U)[\dot{y}]^\wedge$$

(Think $\Omega_{R[x,y]/(f)} \cong A\,dx \cong A\,dy$)

$$\mathcal{O}(J^1(U)) \xleftarrow{\sim} \mathcal{O}(U)[T]^\wedge = \varprojlim \mathcal{O}(U)[T]/p^n$$

$\dot{x} \mapsto T$	ψ_1^*
or	
$\dot{y} \mapsto T$	ψ_2^*

$$(\psi_{21})^* = (\psi_2^{-1} \circ \psi_1)^* = \psi_1^* \circ (\psi_2^{-1})^*, \quad T \mapsto \dot{y} = \psi_{21}(\dot{x}) \mapsto \psi_{21}(T)$$

$J^1(U) \xrightarrow[\psi_i]{\sim} \hat{U} \times \hat{\mathbb{A}}^1$, (trivialization on the geom side)

~~C[...]~~ $\psi_1 \circ \psi_2^{-1} : U \times \mathbb{A}^1 \longrightarrow U \times \mathbb{A}^1$

$$\begin{tikzcd} U \times \mathbb{A}^1 \arrow[rr] \arrow[dr] & & U \times \mathbb{A}^1 \arrow[dl] \\ & U & \end{tikzcd}$$

~~Ⓟ X/R a curve $J^1(X) \to X$ admits trivializing cover $\{U_i\}$ $J^1(U) \xrightarrow{\psi_i} U \times \mathbb{A}^1$, such that~~

solve for $\dot{x}$ in terms of $\dot{y}$ & convert everything to t's.

$$\psi_{ij}(T) = a_{ij} + b_{ij} T + p\, c_{ij} T^2 \mod p^2$$

example: $X = \mathrm{Spec}(A)$, $A = R[x,y]/(f(x,y))$

x U_1 f_y inv

y U_2 f_x inv

Recall

$$r + f_x^p \dot{x} + f_y^p \dot{y} \equiv 0 \mod p$$

$$\Rightarrow f_y^p \dot{y} = -r - f_x^p \dot{x}$$

$$\Rightarrow \dot{y} = \frac{-r - f_x^p \dot{x}}{f_y^p}$$

$$\boxed{\psi_{21}(T) = \frac{-r - f_x^p T}{f_y^p}} \Rightarrow a_{21} = \frac{-r}{f_y^p}, \quad b_{21} = \frac{-f_x^p}{f_y^p}$$

generally, $X \subset \bigcup_i U_i$

$$J^1(U_i) \xrightarrow{\psi_i} U_i \times \mathbb{A}^1$$

$$\psi_i \circ \psi_j^{-1} := \psi_{ij} \in H^1(X, \underline{\mathrm{Aut}}(\mathbb{A}^1))$$

CONJ

$$\boxed{\psi_{ij}(T) = a_{ij} + b_{ij} T + p c_{ij} T^2}$$

CONJ $\exists A_n$, s.t. ... $A_n = \psi_{ij}(T) = a$ [illegible] p-alphas of [illegible]

should exist in general

$$[\psi_{ij}(T)] \in H^1(X_0, AL_1) \mod p$$

$AL_1 \cong \mathcal{O} \rtimes \mathcal{O}^\times$, (Left)

$$(a + bT) \circ (c + dT) = (a + bc + bdT)$$

$$(a,b)(c,d) = (a + bc, bd)$$

$\psi_{ij} \sim (a_{ij}, b_{ij})$ cocycle.

Invertible ...

D CLAIM: If $\varphi_i : \mathcal{O}(U_i) \to L(U_i)$, $\varphi_i(v) = v_i$

s.t. $\varphi_{ij}(1) = b_{ij}$ $\quad b_{ij} v_i = v_j$

$$[L] = [b_{ij}] \in H^1(X, \mathcal{O}^\times)$$ X

note: $[L^\vee] = \left[\frac{1}{b_{ij}}\right]$ Picard Group

Cocycle cond

$(a_{ij}, b_{ij})(a_{jk}, b_{jk})(a_{ki}, b_{ki})$ $\quad b_{ij} = \varphi_{ij}$

$= (a_{ij} + b_{ij} a_{jk}, b_{ij} b_{jk})(a_{ki}, b_{ki})$

$= (a_{ij} + b_{ij} a_{jk} + b_{ij} b_{jk} a_{ki}, b_{ij} b_{jk} b_{ki})$

$= (0, 1)$

$$\begin{cases} a_{ij} + b_{ij} a_{jk} + b_{ij} b_{jk} a_{ki} = 0 \\ b_{ij} b_{jk} b_{ki} = 1 \end{cases}$$

P

$$\begin{array}{ccc} H^1(X, \mathcal{O} \rtimes \mathcal{O}^\times) & \xrightarrow{\pi_{left}} & H^1(X, \mathcal{O}^\times) \\ \downarrow & & \downarrow \\ (a_{ij}, b_{ij}) & \longmapsto & b_{ij} \end{array}$$

$$\boxed{\pi_{left}^{-1}([b_{ij}]) \underset{i_{left}}{\cong} L}$$ Left Embedding

pf idea

$(a_{ij}, b_{ij}) \longmapsto \varphi_i(a_{ij}) := s_{ij}$
$= a_{ij} v_i$

$s_{ij} + s_{jk} + s_{ki}$

$= a_{ij} v_i + a_{jk} v_j + a_{ki} v_k$

$= a_{ij} v_i + a_{jk} b_{ij} v_i + a_{ki} b_{ik} v_i$

$= (a_{ij} + a_{jk} b_{ij} + a_{ki} b_{ij} b_{jk}) v_i$

$= 0.$ //

Prop: X/R a curve,

1) $\Pi([\psi_{ij} \bmod p]) = [F^* T_{X_0}]$

2) $\tau_{Left}([\psi_{ij} \bmod p]) = DI_0(S)$

Conjecture: This holds true in general, X/R any dimension & smooth. (singular?)

Idea: $f(x,y) = 0$

$$f_x\,dx + f_y\,dy = 0$$

$$\varphi_1 : \mathcal{O}(U_1) \to \Omega(U_1)$$

$$\begin{cases} dx = v_1 \\ dy = v_2 \end{cases}$$

Then, $b_{21} v_2 = v_1$

$$\boxed{\frac{f_x}{f_y}\,dx = dy}$$

$\Rightarrow b_{21} = f_x / f_y$

$\Rightarrow$ believable.

General Strategy

- ψ_{ij} some A_n-structure,
- slap a cocycle on (group cocycle)
- use dual pairing

Apply τ_x and use the trivialization to win.

$$\boxed{\begin{aligned} &H^q(X, L \otimes \Omega^p_{X/k}) = 0, \quad p+q > d, \\ &H^q(X, L^{-1} \otimes \Omega^p_{X/k}) = 0, \quad p+q < d \end{aligned}}$$

Kodaira-Vanishing