A USER'S GUIDE TO MOCHIZUKI'S COROLLARY 3.12 AND THEOREM 1.10

TAYLOR DUPUY AND ANTON HILADO

ABSTRACT. In this paper we state three inequalities due to Mochizuki in a self-contained way. There are Mochizuki's Corollary 3.12, and an inquality intermediate to Corollary 3.12 and Theorem 1.10, and Theorem 1.10 which is essentially Szpiro's conjecture for elliptic curves sitting inside initial theta data. It is our hope that the presentation here makes some of this material accessible to a wider mathematical audience and leads to improvements. Minor cleaning-up of the inequalities is made along the way.

CONTENTS

Date: April 3, 2020.

♠♠♠ TODO: [

(1) ENHANCEMENTS:
    (a) add a section on interpretations. (Are we using this enough to merit interpretations). How are they functors? What about this Species mutation business?
    (b) GOAL: PROVE MULTIRADIALITY OF $\Omega^{\mathrm{Ind}\,3}$ FOR APPROPRIATE CHOICE OF IND3
        (i) (Fix S9.3) What is the definition of multiradiality? What do we think it means? What does Fucheng say in his original paper? Does this jive with notions of interpretation we are using?
        (ii) in what structure is the log-kummer correspondence defined? Are we using this notion correctly?
        (iii) easier: What is an automorphism of the log-kummer correspondence? (§9.2)
(2) FIX COMPUTATION OF DEGREE OF q-pilot and relation to Conductor.
(3) FAITHFULLY FLAT DESCENT (low priority)
(4) FESENKO-VOSTIKOV:

(5) ARCHIMEDEAN CONTRIBUTION
(6) CONCLUDE THEOREM 1.10 variant
(7) FIX AVERAGE.
(8) CONCLUDE THEOREM 1.10
(9) Minor: fix Fesenko inequality. Check $\underline{\Omega}$ notations.
(10) Ind1.

]

1. Introduction

The goal of this paper is to give an exposition of Mochizuki's inequalities ([Moc15c, Corollary 3.12] and [Moc15d, Theorem 1.10]).

In the process of doing this we prove (relative to a version of Mochizuki's Corollary 3.12) the following.

Theorem 1.0.1. ♠♠♠ Taylor: [fill this in with whatever we end up proving] *Suppose that* (...) *is a tuple of initial theta data. We have*

$$*** \leq ***$$

Remark 1.0.2. This remark is intended to clarify the somewhat confusing constant C_Θ that appears in [Moc15c, Corollary 3.12] and [Moc15d, Theorem 1.10]. First, let us state unequivocally that Mochizuki's inequality is a statement of the form

$$-b \leq -a$$

for a and b real numbers. In particular, $-b = -\widehat{\underline{\deg}}(P_q)$ and $-a = -\widehat{\underline{\deg}}_{\mathrm{lgp}}(P_{\mathrm{hull}(U_\Theta)}) = \overline{\log \nu_{\mathbf{L}}}(\mathrm{hull}(U_\Theta))$. In the course of our volume computations, we will eventually find an upper bound A for $-a$. This will change the ratio of $-a/b$ to A/b, making it larger. The ratio A/b is what Mochizuki computes and what he calls C_Θ in Corollary 3.12 of IUT3 and in Theorem 1.10 of IUT4. He is using the following elementary statement about real numbers:

$$-b \leq -a \iff \forall C \in \mathbf{R}: \ C \geq \frac{-a}{b} \implies C \geq -1.$$

We omit the proof of this statement.

Theorem 1.0.3. ♠♠♠ Taylor: [Computations of expections]

$$\frac{1}{6}\ln(|\Delta_{E/F}^{\min}|) \leq \sum_p \left[\overline{\mathrm{diff}}_p + \left(1 - \mathbb{P}_{\mathrm{unr}}^{\frac{l+1}{4}}\right)\left(\log_p(b_p) + \log_p(\overline{e}_p)\right) + \mathcal{O}\left(\frac{1}{l^2}\right)\right]\ln(p)$$

Theorem 1.0.4. ♠♠♠ Taylor: [Expectations]

$$\begin{aligned}\overline{\log}_{\mu_L}(\mathrm{hull}(U_\Theta)) \leq [\sum_p & \mathbb{E}^2(\mathrm{ord}_p(a))(\not\Vdash_{\mathrm{bad}}) \\ & + (\mathbb{E}^2(\not\Vdash_{\mathrm{ram}}) + \mathbb{E}^2((\|\,\mathrm{diff}\,\|_1 - \|\,\mathrm{diff}\,\|_\infty)(\not\Vdash_{\mathrm{ram}})) \\ & + \mathbb{E}^2((j+1)\log(b_p) + \sum_i \log_p(e_i))(\not\Vdash_{\mathrm{ram}})]\ln(p)\end{aligned}$$

Theorem 1.0.5. ♠♠♠ Taylor: [Local Hull computations]

$$\begin{aligned}\overline{\log}_{\mu_L}(\mathrm{hull}(U_\Theta)^{(j+1)}_{p,(\underline{v}_0\cdots\underline{v}_j)}) \leq {} & [(\mathrm{ord}_p(a))\nVdash_{\mathrm{bad}} \\ & + (1 + \|\,\mathrm{diff}\,\|_1 - \|\,\mathrm{diff}\,\|_\infty + (j+1)\log(b_p) \\ & + \sum_i \log_p(e_i))\nVdash_{\mathrm{ram}}]\ln(p)\end{aligned}$$

In addition to sharpening [Moc15d, Theorem 1.10] from [Moc15c, Corollary 3.12] this paper exposits "scheme theoretically" constructions involed in Mochizuki's inequality. Model theoretically, much of this can perhaps be viewed as identifying the structures involved which are used in Mochizuki's anabelian interpretations of these structures. Among these things we do the following:

- Explain what initial theta data (and explain how to produce such data).
- Define the notion of a Mochizuki log-measure space and explain how these are not log-measures at all.
- Introduce the notion of "fake adeles" and "fake fractional ideals" as a way of understanding the Mochizuki measure spaces in [Moc15d], [Moc15c] and how they are used to encode Arakelov divisors. The space $\mathbf{L}$ ($\cong {}^{(0,1)}\mathcal{I}^{\mathbb{Q}}(\mathfrak{D}^{\Vdash}_{V(\mathbb{Q})})$ in Mochizuki's notation) is built from rings of fake adeles and houses Mochizuki's region U_Θ ([Moc15c, Theorem 3.11]) whose hull $\mathrm{hull}(U_\Theta) \subset \mathbf{L}$ appears in [Moc15c, Corollary 3.12].
- Give an explicit presentation of U_Θ (this involved the next two bullets).
- Give an exposition of the aspects of the p-adic logarithm needed for understanding Mochizuki's IUT. This clarifies [Moc15e, section 5] and [Moc15c,], which we found difficult to understand[1] This includes:
  - Introduce a language of partially defined maps useful in describing (and perhaps implicit in) Mochizuki's indeterminacies and multiradiality results.
  - Define a pre and post version of the logarithm in elementary terms.
  - Define the log-Kummer correspondence in elementary terms.
  - Introduce log-links (and explain why they don't exist on tensor products of fields!).
  - Give formulas for the log-link indeterminacy $\mathrm{Ind}\,3$ for subsets of tensor products of p-adic fields. (
  - Introduce log-shells. State and prove their basic properties (and later apply them in log-volume computations).
  - We also explain how one could concievably interpret this indeterminacy in multiple ways!
- Provide a key lemma which says that interpretation of a global number field can be used to glue together interpretations of local measure spaces to make sense of Arakelov degrees.

[1]This used some communication with Emmanuel Lepage and the unpublished manuscript [Tan18] as well as hard work of our own.

- Given explicit presentations of Ind 1 and Ind 2 using bases of p-adic vector spaces. We also explain how this new approach may allow us to improve [Moc15d, Theorem 1.10] (depending on what the l^∞-norm of a change of a certain change of basis matrix turns out to be — we don't know how to compute this).
- Give intuitive explanations for the relationship between tensor packets and differents (discriminant of number field).
- Simplify [Moc15d, Proposition 1.1, Proposition 1.2, Proposition 1.3, Proposition 1.4] concerning log volumes and p-adic measures via elementary considerations.[2].
- Eliminate the strange divisors in [Moc15d, Theorem 1.10] and introduced probability spaces for this exposition.
- Explain how there is "daylight" between [Moc15c, Corollary 3.12] and [Moc15c, Theorem 1.10].

We remark in passing that Mochizuki's Elliptic Curves in General Position paper [Moc10] is not covered here as [Mat13] gives an excellent exposition of this material.

Acknowledgements

This article is very much indebted to the expositions of [Fes15],[Yam17], [?], [?], [Tan18], [?], and [Hos15], []. Some of these are citations of talks funded by the NSF, Clay Mathematics Institute, and . ♠♠♠ Taylor: [Fix these references and make sure to hit all funding sources for talks] We have tried to include references to places where we believe the material is particularly well exposited.

The authors would like to especially thank Emmanuel Lepage, and Chung Pang Mok for sharing their insights with them. The author would also like to thank Shinichi Mochizuki, and Yuichiro Hoshi for their patience in explaining these things (often repeatedly). In addition the authors were greatly aided by conversations with Brian Conrad, Ivan Fesenko, Mohammed Saidi, Thomas Scanlon, Adam Topaz, Kiran Kedlaya, and Christelle Vincent.

The first author was also supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 291111/ MODAG. The first author also visited ICERM, while working on these projectve. In addition, productive conversations occured at conferences funded by

2. Fields, tensor packets, measures, and fake adeles

In this section we give presentations of local fields as well as describe the group of units of a local field.

2.1. Conventions for discrete valuation rings.

Let K be a finite extension of $\mathbb{Q}_p$ with uniformizer $\pi_K = \pi$. We will let $\operatorname{ord}_K : K^\times \to \mathbf{Q}$ be the valuation, normalized such that $\operatorname{ord}_K(\pi) = 1$. We will let $\operatorname{ord}_p = \operatorname{ord}_{\mathbb{Q}_p}$ denote the unique extension of the valuation to K.

[2]Some of this has already been done since Mochizuki's original version

We will have $\mathrm{ord}_p(\pi) = 1/e(K/\mathbb{Q}_p)$ where K is the uniformizer. The ring of integers will be denoted by

$$\mathcal{O}_K = \{x \in K : \mathrm{ord}_K(x) \geq 0\}.$$

and the maximal ideal will be denoted by

$$\mathfrak{m}_K = \{x \in K : \mathrm{ord}_K(x) > 0\}$$

and the residue field is $k = \mathcal{O}_K/\mathfrak{m}_K$. The inertia degree will be denoted by $f(K/\mathbb{Q}_p)$. For K/K_0 a finite extension of finite extensions of $\mathbb{Q}_p$ we will let $f(K/K_0)$ and $e(K/K_0)$ denote the relative inertia and ramification degrees. Recall that we have $[K : K_0] = e(K/K_0)f(K/K_0)$.

Also, given a finite K/K_0 where K_0 has a norm $|\cdot|_{K_0}$ there is a unique extension of this field to K. In particular the norm on $\mathbb{Q}_p$ extends to $\overline{\mathbb{Q}}_p$ and the completion of the field with respect to this norm is denoted by $\mathbb{C}_p$. We should warn the reader that non-archimedean absolute on finite extensions of $\mathbb{Q}_p$ are not unique! There is a different one, for example, that is used for the product formula.

For a field $K \subset \mathbb{C}_p$, and a real number $R > 0$ we will let

$$D_K(0, R) = \{x \in \mathbb{C}_p : |x|_p \leq R\} \cap K.$$

We will also let

$$D_K(0, R^-) = \{x \in \mathbb{C}_p : |x|_p < R\} \cap K.$$

Note that we have $D_K(0, 1^-) = \mathfrak{m}_K$ and $D_K(0, 1) = \mathcal{O}_K$.

2.2. **Conventions for completions of global fields.** Let F be a finite extension of $\mathbb{Q}$. We will let $V(F)$ denote the set of places of F (including the archimedean ones). If F is a field and v is a place of F we will let F_v denote the completion of F at v. We will use the notation $\mathrm{ord}_v = \mathrm{ord}_{F_v}$ for the valuation of F_v (which is clearly also a valuation on F). For extensions K/F and places $w \in V(K)$ and $v \in V(F)$ with $w \mapsto v$ under the standard map $V(K) \to V(F)$ then we write $w|v$ and the extension K_w/F_v is a finite extension of fields. It is a standard fact [Neu99, II, Corollary 8.4] that for a fixed $v \in V(F)$ we have

$$[K : F] = \sum_{w|v} [K_w : F_v].$$

Also, if K/F is Galois galois group G and $w \in V(K)$ and $v \in V(F)$ with $w|v$ then K_w/F_v is Galois. The Galois group is $D_{w/v} = \mathrm{Stab}_G(w)$ and is refered to as the decomposition group.

2.3. **Arakelov divisors.**

- $\deg([v]) = p_v^{f_v}$, (we set $e_v = 1$ if $v|\infty$)
- $\mathrm{div}(a) = \sum_v \log|f|_v[v]$ (note that $\|f\|_v = |f|_p^{[F_v:\mathbb{Q}_p]}$ are the normalized absolute values for the product formula and $|f|_v = |f|_\infty$ in the archimedean case
- The map from ideals to Arakelov divisors is the map extended multiplicatively from $P_v \mapsto [v]$. This means $\widehat{\mathrm{div}}(I) = \sum_v \mathrm{ord}_v(I)[v]$. We have $\widehat{\deg}(I) = \log|I|$.

Let K be a field. If v is a valuation of K then a non-archimedean absolute value can be defined by $|x| = \rho^{-\operatorname{ord}_v(x)}$ for an $\rho \in \mathbb{R}_{>1}$. Any non-archimedean absolute value will be equivalent to one of these.

Note that for infinite places of number fields there is a unique archimedean absolute value, and one can not fiddle with the powers since the triangle inequality breaks.

Recall that for any non-archimedian place v and non-archimedean absolute value $x \mapsto \|x\|_v$ that $\|\|$

Lemma 2.3.1. *Let $r_p \in \mathbb{R}_{>0}$, and define for $x \in \mathbb{Q}_p$ the non-archimedean absolute values $\|x\|_p = r_p^{-\operatorname{ord}_p(x)}$ and for $p = \infty$ let $\|x\|_\infty$ be the usual absolute value on the reals.*

If $\prod_{p \in V(\mathbb{Q})} \|x\|_p = 1$ then $r_p = p$ for all p.

Proof. ♠♠♠ Taylor: [FIXME] □

Lemma 2.3.2. ♠♠♠ Taylor: [Do this for number fields by passing to the norm]

The following proposition states that the degrees at the local places of Arakelov divisors are uniquely determined by the fact that principal divisors must have degree zero. This mean that if we have a construction for local places and we are able perform a weighted sum of these degrees as well as form principal divisors, then the weights can be uniquely determine to annihilate principal divisors coming from the global number field. This is how Mochizuki will reconstruct his local information into something global.

Lemma 2.3.3. *Let F be a number field. Let $c_v \in \mathbb{R}$ for each $v \in V(F)$. Consider a map $\widetilde{\deg} : \widehat{\operatorname{Div}}(F) \to \mathbf{R}$ defined by $\widetilde{\deg} = \sum_v c_v \widehat{\deg}_v \circ \operatorname{pr}_v$ where pr_v is a projection to Arakelov divisors supported at v and $\widehat{\deg}_v$ is the usual Arakelov degree, only restricted to divisors supported at v.*

If for every $a \in F$ we have $\widetilde{\deg}(\operatorname{div}(a)) = 0$ then $c_v = 1$ for every $v \in V(F)$ and hence $\widetilde{\deg} = \widehat{\deg}$.

Proof. ♠♠♠ Taylor: [FIXME] □

2.4. Presentations of local field extensions.

Lemma 2.4.1 ([Lan94, Proposition 23, page 26],[Ser79, Ch II, §5, Theorem 4])**.** *If $K/\mathbf{Q}_p$ is a finite extension of fields then*

$$\mathcal{O}_K = \mathbf{Z}_p[\pi, \gamma]$$

where π is a uniformizer and γ is representative of a generator of the residue field.

Alternatively, we can write

$$\mathcal{O}_K = \mathbf{Z}_{p^f}[x]/(g_\pi(x)).$$

Here π is the uniformizer of $\mathcal{O}_K$, g_π is the minimal polynomial of π (which is an Eisenstein), and p^f is the cardinality of the residue field. Here $\mathbb{Z}_{p^f} = W(\mathbf{F}_{p^f})$ is the maximial unramified extension of $\mathbb{Z}_p$ of degree and residue degree f.

Example 2.4.2.
- If $e = 1$, $\mathcal{O}_K = \mathbf{Z}_p[\gamma]$
- If $f = 1$, $\mathcal{O}_K = \mathbf{Z}_p[\pi]$.

Note that since $x^e - p \in \mathbf{Z}_p[x]$ is Eisenstein[3] this polynomial is irreducible. Also, $m_\gamma(x) \in \mathcal{O}_K[x]$ a representative of the minimal polynomial is irreducble since it is irreducible $\mod \pi$ (if we make a non-stupid lift of $\bar{m}_\gamma \in \mathbf{F}_p[x]$). We see $\mathcal{O}_K$ as

$$\mathcal{O}_K = \mathbf{Z}_p[\pi][\gamma] = \mathbf{Z}_p[x,y]/(x^e - p, m_\gamma(y)). \tag{2.1}$$

which is a domain with generators $\{x^i y^j : 0 \leq i < e, 0 \leq j < \deg(\gamma)\}$.

2.5. **Completion of a Galois extension.** Now we will show that the p-adic completion of a Galois extension is a Galois extension. We have the following:

Theorem 2.5.1. *Let K/F be any finite Galois extension of number fields with $G = \mathrm{Gal}(K/F)$. Let v be a place of F and $\underline{v}$ a place of K above F. Then G acts on $V(K)_v$ transitively. The extension $K_{\underline{v}}/F_v$ is Galois with Galois group $D_{\underline{v}/v} = Stab_G(\underline{v})$.*

Proof. We have a map $D_{\underline{v}/v} \to \mathrm{Aut}(K_{\underline{v}}/F_v)$ which is injective since $K \subseteq K_{\underline{v}}$. We have $K_{\underline{v}} \supset K_{\underline{v}}^D \supset F_v$. But $[K_{\underline{v}} : K_{\underline{v}}^D] = \#D = [K_{\underline{v}} : F_v]$ therefore we have $F_v = K_{\underline{v}}^D$. $\square$

2.6. **Unit groups of p-adic fields.** For this subset we will let $K/\mathbb{Q}_p$ be a finite field extension with residue field k. The purpose of this section is to introduce invariants which appear in the formulas for p-adic measures used later. What follows can be found in [Neu99, Chapter 2,§5,Proposition 5.3]: If π is the uniformizer of K then

$$K^\times = \pi^{\mathbf{Z}} \cdot \mu_{p^f-1}(K) \cdot U_1(K)$$

where

- $\pi^{\mathbf{Z}} = \{\pi^k | k \in \mathbf{Z}\}$
- p^f is the number of elements in the residue field $\mathcal{O}_K/\mathfrak{m}_K$
- $U_1(K)$ is the principal unit group of K, i.e. $U_1(K) = D_K(1, 1^-) = \{z \in K : |1-x| < 1\}$

For the units in the ring of integers we have the internal direct sum

$$\mathcal{O}_K^\times = \mu_{p^f-1}(K) U_1(K).$$

In what follows we will let e denote degree of this extension, which plays role in our formulas.

Remark 2.6.1. Let K be a valuation ring with residue field k of characteristic p. Let π be the uniformizer of K. If ζ satisfies $\zeta^{p^r} = 1$ then now that $\zeta^{p^r} - 1 \equiv (\zeta - 1)^{p^r} \equiv 0 \mod \pi$. Since $R/\pi = k$ is a field we have $(\zeta - 1) \in \pi R$ and hence the p-power roots of unity are contained in $1 + \pi R$.

[3]Let R be a DVR with uniformizer π. A polynomial $a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ is Eisenstein when $\mathrm{ord}(a_n) = 0$, $\mathrm{ord}(a_i) \geq 1$, $\mathrm{ord}(a_0) = 1$.

2.6.1. *Basis of principal units.* We now want to give some formulas for the principal units of a finite extension of the p-adic integers. This material follows [FV02, Chapter 1, sections 5–7] and is going to be applied in §14 where we give an explicit description of the action of Ind 1 and Ind 2 on p-adic regions involved Mochizuki's inequality.

To give a presentation of the principal units $1+\pi\mathcal{O}_K$ we need to define the map $\psi : k \to k$

$$\psi(x) = x^p + a_e(p)x. \tag{2.2}$$

We will now explain who the element $a_e(p) \in k$ is: Because any element of K can presented as an infinite linear combination of Teichmuller elements and powers of π, and p and π^e have the same valuation, we know that

$$p = \sum_{i=e}^{\infty} \tau(a_i(p))\pi^i,$$

where $a_i(p) \in k$ and $\tau(a_i(p))$ denotes the Teichmuller lift (in the notation of [FV02], $a_e(p) = \overline{\theta}_0$). This map ψ detects whether or not K has pth roots of unity in the following way:

(1) $\psi(k) = k \iff \mu_{p^\infty}(K) = 1$
(2) $[k : \psi(k)] = p \iff \zeta_p \in K^\times$

Suppose not that $\zeta_p \in K$ so that $e|p-1$. For our pth roots of unity we have $\mathrm{ord}_K(\zeta_p - 1) = e/(p-1)$ which means

$$\zeta_p = 1 + \tau(a_{e/(p-1)}(\zeta_p))\pi^{e/(p-1)} + \cdots$$

They they give a formula for ψ in terms of the Artin-Schreier $\wp : k \to k$ given by

$$\wp(x) = x^p - x$$

and the number

$$a_{e/(p-1)}(\zeta_p) = \frac{\zeta_p - 1}{\pi^{e/(p-1)}} \mod \pi \in k.$$

(in the notation of [FV02], $a_{e/(p-1)}(\zeta_p) = \overline{\theta}_1$).

The formula is

$$\psi(x) = a_{e/(p-1)}(\zeta_p)\wp(\frac{x}{a_{e/(p-1)}(\zeta_p)}). \tag{2.3}$$

This formula allows us to see trivially that $a_{e/(p-1)}(\zeta_p)$ is in the kernel of ψ.

Theorem 2.6.2. *Let $\overline{\theta}_1, \ldots, \overline{\theta}_f \in k$ be a basis for $k/\mathbb{F}_p$ and let $\theta_i = \tau(\overline{\theta}_i)$ be their Teichmuller lifts. Let $\overline{\eta}_1, \ldots, \overline{\eta}_n$ be representatives for generators of $k/\psi(k)$ and let $\eta_i = \tau(\overline{\eta}_i)$ for each $1 \le i \le n$. Any element of $\alpha \in 1+\pi\mathcal{O}_K$ can be written as*

$$\alpha = \prod_{1\le i\le pe/(p-1),(i,p)=1} \prod_{j=1}^{f} (1+\theta_j\pi^i)^{m_{ij}} \prod_{j=1}^{n} (1+\eta_j\pi^{pe/(p-1)})^{m_j}$$

where $m_{ij}, m_j \in \mathbb{Z}_p$. The right most product only occurs when $\zeta_p \in K$.

♠♠♠ Taylor: [I think this is the case when $p-1|e$.] In practice, we need to compute a collection of generators of $k/\psi(k)$.

♠♠♠ Taylor: [If the funky term is the torsion, then they will just get killed when we take the logarithm.]

Example 2.6.3. Let p and l be distinct primes. Let $K = \mathbb{Q}_p(\zeta_{p^2 l}) = \mathbb{Q}_p(\zeta_{p^2}, \zeta_l)$ and let k denote its residue field.

Let $K_0 = \mathbb{Q}_p(\zeta_l)$ and let k_0 denote its residue field. Here we have

$$k = k_0 = \mathbb{F}_p[\overline{\zeta}_l] = \mathbb{F}_p + \mathbb{F}_p\overline{\zeta}_l + \cdots + \mathbb{F}_p\overline{\zeta}_l^{l-1} \cong \mathbb{F}_{p^l}.$$

In this case we have $f = l$ and we will choose our basis of $k/\mathbb{F}_p$ to be $\{\overline{\zeta}_l^j : 0 \leq j \leq l-1\{$ so that

$$\{\theta_1, \ldots, \theta_f\} = \{1, \zeta_l, \cdots, \zeta_l^{l-1}\}.$$

We will now compute $k/\psi(k)$. To do this we need to compute $a_e(p)$ (the residue element of the eth coeffient of p when we Teichmuller-expand it in terms of the uniformizer π) in order to compute the map $\psi : k \to k$ defined by the formula

$$\psi(x) := x^p + a_e(p)x.$$

We have that $e(K/\mathbb{Q}_p) = p(p-1)$ and $\pi = 1 - \zeta_{p^2}$. We remind the reader that this follows identities with cyclotomic polynomials:

$$\Phi_{p^2}(x) = \Phi_p(x^p) = \sum_{j=0}^{p-1} (x^p)^j = \prod_{j \nmid p^2} (x - \zeta_{p^2}^j)$$

which shows there are $\phi(p^2) = p(p-1)$ primitive p^2-th roots of unity and

$$p = \Phi_{p^2}(1) = \prod_{j \nmid p^2} (1 - \zeta_{p^2}^j),$$

which after taking ord_p of both sides gives

$$1 = \sum_{j \nmid p} \mathrm{ord}_p(1 - \zeta_{p^2}^j) = p(p-1)\,\mathrm{ord}_p(1 - \zeta_{p^2}).$$

Using the identity $\Phi_{p^2}(1) = p$ we get

$$(1 - \zeta_{p^2})^{p(p-1)} \prod_{j \in (\mathbb{Z}/p^2)^\times} \frac{(1 - \zeta_{p^2}^j)}{(1 - \zeta_{p^2})} = p.$$

This shows

$$\tau(a_e(p))^{-1} = \prod_{j \in (\mathbb{Z}/p^2)^\times} \frac{1 - \zeta_{p^2}^j}{1 - \zeta_{p^2}}.$$

We will now compute the reduction modulo π of this quantity. Using that $\zeta_{p^2} = 1 - \pi$ we get

$$\begin{aligned}\frac{1-\zeta_p^{j^2}}{1-\zeta_{p^2}} &= \frac{1-(1-\pi)^j}{\pi} \\ &= \frac{1}{\pi}\left(1 - \sum_{i=0}^{j}\binom{j}{i}(-\pi)^i\right) \\ &= \frac{1}{\pi}\sum_{i=1}^{j}(-1)^{i+1}\binom{j}{i}\pi^i \\ &= \sum_{i=1}^{j}(-1)^{i+1}\binom{j}{i}\pi^{i-1} \\ &\equiv j \mod \pi.\end{aligned}$$

Hence we have

$$a_e(p)^{-1} \equiv \prod_{j\in(\mathbb{Z}/p^2)^\times} j \mod \pi \equiv 1.$$

♠♠♠ Taylor: [We should compute $-\overline{\theta}_1 = (\zeta_p - 1)/\pi^{e/(p-1)}$, note that this makes sense since $\mathrm{ord}_p(\zeta_p - 1) = 1/(p-1)$ and $\mathrm{ord}_p(\pi^{e/p-1}) = p/p(p-1)$ since this element modulo π is supposed to be in the kernel of ψ.
]

♠♠♠ Taylor: [Prove that the weird terms are p-torsion]

Example 2.6.4. Let $K = \mathbb{Q}_p(p^{1/p^2})$. ♠♠♠ Taylor: [Claim: This doesn't have pth roots of unity]. In this case $e = p^2$ and $f = 1$. Then the principal units are generated by

$$(1 + \pi^i)$$

where $1 \le i \le p^3/(p-1)$ and $(i,p) = 1$. The dimension of this group as a $\mathbb{Z}_p$-module should be equal to the degree of the extension which is p^2. ♠♠♠ Taylor: [Should we include the case $p = 5$?] ♠♠♠ Taylor: [We should check the set in general.]

So our basis will be

$$\{\log(1+\pi^i) : 1 \le i \le \frac{p^3}{p-1}, (i,p) = 1\}$$

2.7. **Differents.** In this section we will give two definitions of the different and explain some basic properties of the different we will need later. The expert reader should feel free to skip this section. The standard reference for this material is [Neu99, III.2]. Readers who have never seen a different before should consult the course notes [Con] or [Sut15].

The following if the first definition of the different.

Definition 2.7.1. Let A/A_0 be a finite extension of rings. The **different** is

$$\mathrm{Diff}(A/A_0) := \mathrm{ann}_A(\Omega^1_{A/A_0}).$$

Remark 2.7.2. We will now give an alternative definition which is used in algebraic number theory and restricts $A \supset A_0$ to be a subextension of a field extension. We first need an auxillary construction. Let K_0 be a field and let $A_0 \subset K_0$ be a subdomain whose field of fractions is K_0. Let K/K_0 be a field extension and let $A \subset K$ be a subdomain whose field of fractions is K and such that $A \supset A_0$. We defiune the **complementary module** of A over A_0 is defined to be

$$C_{A/A_0} = \{x \in K : \forall y \in A, \mathrm{Tr}_{A/A_0}(xy) \in A_0\}. \tag{2.4}$$

We next can define the **different** is to be

$$\mathrm{Diff}(A/A_0) = C_{A/A_0}^{-1} \lhd A. \tag{2.5}$$

A proof that Definition 2.7.1 and the definition from equation (2.5) are equivalent can be found in [Ser79].

We will now give some elementary properties of the different. First, the different has the property the its relative norm is the relative discriminant

$$N_{A/A_0}(\mathrm{Diff}(A/A_0)) = \mathrm{Disc}(A/A_0).$$

This is useful as it relates the different to an invariant most people have experience with. Second, differents behave well in towers: if $A_2 \supset A_1 \supset A_0$ is a finite tower of finite extensions then

$$\mathrm{Diff}(A_2/A_0) = \mathrm{Diff}(A_2/A_1)\,\mathrm{Diff}(A_2/A_0).$$

2.8. **Different exponents and ramification.** Finally, differents know about the ramification of the extensions A/A_0. To state the relationship we assume A/A_0 is an extension of Dedekind domains, P is a prime of A, $P_0 = P \cap A_0$ is the prime below P. Also let $e(P/P_0)$ denotes the ramification degree of P over P_0. Also, we will let

$$d(P/P_0) := \mathrm{ord}_P \,\mathrm{Diff}(A/A_0).$$

We will call $d(P/P_0) \in \mathbf{Z}_{\geq 0}$ the *different exponent* of A over A_0 at P.

Another invariant we use is *fake different exponent* which we define as

$$\mathrm{diff}(P/p) = \mathrm{ord}_p(\mathrm{Diff}(K_P/\mathbb{Q}_p)).$$

To ensure that this behaves well in towers we define

$$\mathrm{diff}(Q/P) := \mathrm{diff}(Q/p) - \mathrm{diff}(P/p).$$

Note that the fake different exponent $\mathrm{diff}(Q/P)$ is distinct from the different exponent $d(Q/P)$.

♠♠♠ Taylor: [FIXME]

An inequality showing control of different exponents in terms of the ramification degrees is given by

$$e(P/P_0) - 1 \leq \mathrm{diff}(P/P_0) \leq e(P/P_0) - 1 + e(P/P_0)\,\mathrm{ord}_P\, e(P/P_0) \tag{2.6}$$

In the tamely ramified case $(\mathrm{ord}_P(e(P/P_0)) = 0)$, (2.6) tells us $d(P/P_0) = e(P/P_0) - 1$.

We will sometimes use terminology for valuations instead of primes. In this case for v a place above v_0 we will may use the notations $e(v/v_0), d(v/v_0)$, etc.

2.9. **Haar measures on finite extensions.**

Lemma 2.9.1. *Let $K/\mathbb{Q}_p$ be an extension of complete discretely valued fields with valuation ord_K normalized to that $\operatorname{ord}_K(\pi) = 1$ for a uniformizer π. Let $[K : \mathbf{Q}_p] = d$, and $d = fr$ where p^f is the cardinality of the residue field and e is the ramification index. For $x \in K$ and $n \in \mathrm{N}$ the following identites hold. Let μ_K be the normalized Haar measure.*

(1) $\mu_K(x\mathcal{O}_K) = |x|_p^d$.
(2) $|\mathcal{O}_K/\pi_K^n| = p^{nf}$.
(3) $\operatorname{ord}_K = e \operatorname{ord}_p$.

♠♠♠ Taylor: [Add citation][]

The following Lemma will allow us to remove some of the floors and ceilings in Mochizuki's estimates appearing at the beginning of [Moc15d].

Lemma 2.9.2. *Let K be a finite extension of $\mathbb{Q}_p$ of ramification degree e and inertia degree f. For all $n \in \mathbb{Z}$,*

$$\mu_K(D_K(0, p^{n/e})) = p^{fn}.$$

Also, for all $R > 0$,

$$\mu_K(D_K(0, R)) < R^{[K:\mathbb{Q}_p]}.$$

Proof. We will prove the first result then use it to prove the second result. Both are elementary. Let π_K be the uniformizer of K. The first follows from the sequence of equalities

$$\mu_K(D_K(0, p^{n/e})) = \mu_K(\pi_K^{-n}\mathcal{O}_K = |\pi_K^{-n}\mathcal{O}_K/\mathcal{O}_K| = |\mathcal{O}_K/\pi_K^n\mathcal{O}_K| = p^{fn}.$$

Suppose now $R > 0$. Then by discreteness of the valuation we have

$$D_K(0, R) = D_K(0, p^{m/e})$$

for some m. Also $p^{m/e} < R$. Then we have

$$\mu_K(D_K(0, R)) = \mu_K(D_K(0, p^{m/e})) = (p^{m/e})^d < R^d.$$

□

2.10. **Mochizuki log measures.**

In this section we define a notion of Mochizuki log-measure. These are actually just weighted sums of log-measures and they appear on the left hand side Mochizuki's main inequality.

Definition 2.10.1. Let V be a finite dimensional $\mathbf{Q}_p$-vector space with Haar measure μ_V. We define the normalized log measure as

$$\overline{\log \mu}_V(A) = \frac{\log(\mu_V(A))}{\dim_{\mathbf{Q}_p}(V)}.$$

Remark 2.10.2. For any $\mathbf{Q}_p$-linear endomorphism φ this measure has the property that $\mu_V(\varphi(A)) = |\det(\varphi)|_p^{\dim_{\mathbf{Q}_p}(V)}$ and hence $\overline{\log\mu}_V(\varphi(A)) = \log(|\det(\varphi)|_p) + \overline{\log\mu}_V(A)$. In the case that $\varphi : V \to V$ is given by $\varphi(v) = pv$ we get

$$\overline{\log\mu}_V(pA) = -\log(p) + \overline{\log\mu}_V(A).$$

Remark 2.10.3. Lemma 2.9.2 implies $\overline{\log\mu}_K(D_K(0,R)) < \log(R)$. We will use this frequently.

Definition 2.10.4. A **Mochizuki log-measure** on a direct sum of p-adic vector spaces $V = \bigoplus_{i=1}^m V_i$ is a weighted sum of normalized log-measures

$$\overline{\log\nu}(A) = \sum_{i=1}^m w_i \overline{\log\mu}_{V_i}(A_i)$$

where $\sum_{i=1}^m w_i = 1$.

Definition 2.10.5. Let K_0 be a finite extension of $\mathbf{Q}_p$. Let $(V, \overline{\log\nu})$ be a Mochizuki log-measure, on a K_0-vector space. We say $\overline{\log\nu}$ is **Mochizuki-normalized** provided for all $x \in K_0$

$$\overline{\log\nu}(xA) = \log|x|_p + \overline{\log\nu}(A).$$

Note that all Mochizuki log-measures are Mochizuki-normalized but not all Mochizuki log-measures are normalized log-measures. Also, neither normalized log-measures nor Mochizuki log-measures are logarithms of actual measures.

The following lemma is elementary, but used frequently, so we state it for the record.

Lemma 2.10.6 (Renormalization of Direct Sums)**.** *If $L = \bigoplus_{i=1}^m L_i$ where L_i are finit extensions of $\mathbf{Q}_p$ then the normalized log-measure on L is equal to the sum of the log-measures on L_i weighted by dimension:*

$$\overline{\log\mu}_L = \frac{1}{[L:\mathbf{Q}_p]} \sum_{i=1}^m [L_i:\mathbf{Q}_p] \overline{\log\mu}_{L_i} \circ \mathrm{pr}_{L_i}$$

Proof. We have ♠♠♠ Taylor: [CHECK THIS]

$$\begin{aligned}
\overline{\log\mu}_L(A) &= \frac{\log(\mu_L(A))}{[L:\mathbf{Q}_p]} \\
&= \frac{\log(\prod_{i=1}^m \mu_{L_i}(\mathrm{pr}_{L_i}(A)))}{[L:\mathbf{Q}_p]} \\
&= \frac{\sum_{i=1}^m \frac{L_i:\mathbf{Q}_p}{L_i:\mathbf{Q}_p} \log(\mu_{L_i}(\mathrm{pr}_{L_i}(A)))}{[L:\mathbf{Q}_p]} \\
&= \frac{\sum_{i=1}^m [L_i:\mathbf{Q}_p] \overline{\log\mu}_{L_i}(\mathrm{pr}_{L_i}(A))}{[L:\mathbf{Q}_p]}.
\end{aligned}$$

□

The above lemma is useful for defining "pretend log measures". To be more specific, given a finite extension of number fields K/K_0 and a subset $\underline{V} \in V(K)$ that bijects onto $V(K_0)$ under the natural map $V(K) \to V(K_0)$ we define a fake copy of $A_{K_0,p}$ (with its natural measure) via the formula

$$L = \bigoplus_{v|p} K_{0,v}.$$

An important property for normalized log-measure spaces is that for K a p-adic field, $a \in K$, and $A \in \mathcal{M}(K)$ (where $\mathcal{M}(K)$ denotes the collection of measurable sets) we have

$$\overline{\log \mu}_K(aA) = \log(|a|_p) + \overline{\log \mu}_K(A).$$

Note that if $\underline{K}/K$ is a finite extension and $\underline{A} \in \mathcal{M}(\underline{K})$ then

$$\overline{\log \mu}_{\underline{K}}(a\underline{A}) = \log(|a|_p + \overline{\log \mu}_{\underline{K}}(\underline{A})).$$

We will use this property to construct "fake adeles" in section **??**. Such "fake adeles" are used in Mochizuki's anableina interpretation of Arakelov divisors.

2.11. **The presentation of tensor products of fields via the Chinese remainder theorem.** In this section we study the factorization of tensor products of fields into products of fields and the image of tensor products of rings of integers in these decompositions. The results of this section will be applied in section §**??**, which are in turn used in Mochizuki's volume computations.

Let K_0 be a field. Let $\bar{K}$ be a fixed algebraic closure of K_0. Given $K_1, ..., K_n \subseteq \bar{K}$ a collection of finite finite extensions of K_0 we wish to describe how $K_1 \otimes_{K_0} K_2 \otimes_{K_0} \cdots \otimes_{K_0} K_n$ decomposes into a product of fields. To do this we need to define an equivalence relation on certain tuples of embeddings.

Definition 2.11.1. Two tuples of embeddings $(\psi_1, ..., \psi_n), (\varphi_1, ..., \varphi_n) \in \prod_{i=1}^n \mathrm{Hom}_{K_0}(K_i, \bar{K})$ are equivalent if and only if there is some $\tau \in G(\bar{K}/K_0)$ such that $(\psi_1, ..., \psi_n) = (\tau \circ \varphi_1, ..., \tau \circ \varphi_n)$.

Remark 2.11.2. Note that for any collection of finite extension $K_1, K_2, \ldots, K_n$ the set of equivalence classes of embeddings $\{(\psi_1, ..., \psi_n) : \psi_j : K_j \hookrightarrow \bar{K}\}/\sim$ is a finite set. We think of collections of embeddings as a sort-of CM type.

Given field extension $K_1, \ldots, K_n$ of K_0 and embeddings $\psi = (\psi_1, \psi_1, \ldots, \psi_n) \in \prod_{i=1}^n \mathrm{Hom}_{K_0}(K_i, \bar{K})$ we define

$$L_\psi = \psi_1(K_1)\psi_2(K_2)...\psi_n(K_n) \subset \bar{K}.$$

It turns out that a tensor product of fields is just equivalent to a direct sum of L_ψ's where ψ runs over an equivalence class of such embeddings.

Lemma 2.11.3. *Let $K_1, ..., K_n \subseteq \bar{K}$ be finite extensions of a field K_0. Their tensor product is isomorphic to the Cartesian product of fields L_ψ,*

$$K_1 \otimes_{K_0} K_2 \otimes_{K_0} \cdots \otimes_{K_0} K_n \xrightarrow{\sim} \bigoplus_{\psi} L_\psi$$

where ψ is runs over a complete set of representatives for tuples embeddings

Proof. ♠♠♠ Taylor: [CHECK THIS PROOF AGAIN. We have an injective map from the Jacobson radical. Apply Chinese remainder theorem. $(0) = \bigcap \mathfrak{m}$, $\sum \mathfrak{m} = \bigotimes K_i$, Then $\bigotimes K_i/(0) = \bigoplus_{\mathfrak{m}}((\bigotimes K_i)/\mathfrak{m})$
] It is clear that we have maps

$$\bigotimes_{i=1}^{n} K_i \to L_\psi$$

for each collection of embeddings $\psi = (\psi_1, \ldots, \psi_n)$. Moreover, it is clear that $\ker(\psi) = \ker(\psi')$ supposing $\psi = \tau\psi'$ for some $\tau \in G(\bar{K}/K_0)$. Conversely, if $\ker(\psi) = \ker(\psi')$ then $(\bigotimes_{i=1}^n K_i)/\ker(\psi) \cong (\bigotimes_{i=1}^n K_i)/\ker(\psi')$ and for any two embeddings of these fields there is an automorphism of the algebraic closure taking one to the other.

To do this we need to show that $\ker(\psi) \cap \ker(\psi') = 0$ provided ψ and ψ' are not conjugate. Suppose not. Suppose there exists ψ and ψ' not conjugate such that $\ker(\psi) \cap \ker(\psi')$ is non-trivial. Then we have a non-empty intersection of these maximal ideals. Let $f \in \ker(\psi) \cap \ker(\psi')$ an let P be a minimal prime above it. This gives a chain of ideals

$$(0) \subset (f) \subset P \subset \ker(\psi) \cap \ker(\psi') \subset \ker(\psi).$$

If this is a nontrivial inclusion of prime ideals this shows that $\bigotimes K_i$ has krull dimension greater than or equal to one. But the product of zero dimensional schemes is zero dimensional. This implies that $\ker(\psi) \cap \ker(\psi') = 0$.

$\square$

Remark 2.11.4. Let $K_1, \ldots, K_n$ be a collection of finite extensions of K_0. If $L^* = (K_1 K_2 \cdots K_n)^{\mathrm{gal}}$ is the Galois closure of the compositem then all the fields L_ψ are contained in L^*.

Remark 2.11.5. Given the isomorphism $f : K_1 \otimes_{K_0} K_2 \otimes_{K_0} \ldots \otimes_{K_0} K_n \xrightarrow{\sim} \bigoplus_\psi L_\psi$, we let f_ψ be the inverse image of a basis element for ψ so that $f_\psi^2 = f_\psi$, and $K_1 \otimes \cdots \otimes K_n = \sum_\psi (K_1 \otimes \cdots \otimes K_n) f_\psi$ This also means that $L_\psi \cong (K_1 \otimes \cdots \otimes K_n)/\operatorname{ann}_{K_1 \otimes \cdots \otimes K_n}(f_\psi)$

Remark 2.11.6. We will often write $L_1, \ldots, L_m$ instead of $L_{\psi_1}, \ldots, L_{\psi_m}$.

2.12. **Measures on tensor products of fields.** Let $K_1, \ldots, K_n$ be finite extensions of a complete discrete valuation field K_0. Write $F_1, \ldots, F_m$ for $F_{\psi_1}, \ldots, F_{\psi_m}$ where [4]

$$K_1 \otimes \cdots \otimes K_n \cong \bigoplus_{i=1}^{m} L_i =: L.$$

This L comes equipped with an additive Haar measure and it is equal to the product measure (=direct sum measure) of each of the factor measure spaces. Here we view each L_i as a measure space normalizing the measures so that $\mu_{F_i}(\mathcal{O}_{L_i}) = 1$. This means for a product of compact open subsets $A_1, \ldots, A_m$ we have

$$\mu_L(A_1 \times \cdots \times A_m) = \mu_{L_1}(A_1) \cdots \mu_{L_,}(A_m).$$

[4]When working with fields and rings we will interchange $\oplus$ and $\times$. The reader should understand that they mean the same thing for finitely many terms.

In particular observe that

$$\mu_L(\mathcal{O}_L) = \mu_{L_1\times\cdots\times L_r}(\mathcal{O}_{F_1} \times \cdots \mathcal{O}_{F_m}) = 1.$$

2.13. **Tensor packets of rings of integers, direct sums of rings of integers, and the i.** Let $K_1, \ldots, K_n$ be finite extensions of a complete discrete valuation field K_0 (think $K_0 = \mathbb{Q}_p$). Suppose that $\bigotimes_{i=1}^n K_i \cong L$ where $L = \bigoplus_\psi L_\psi$ is a product (=direct sum) of fields. Here ψ varies over a complete set of representatives of equivalence classes of tuples of embeddings $\{\psi_1, \ldots, \psi_m\}$. Here each $\psi_i = (\psi_{i1}, \ldots, \psi_{in})$ where $\psi_{ij} : K_j \to \overline{K_0}$ for $1 \leq j \leq m$. In what follows we will be interested in

$$\mu_F(\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n})$$

where we view $\mathcal{O}_{K_1} \otimes_{\mathcal{O}_{K_0}} \cdots \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_n} \subset K_1 \otimes_{K_0} \cdots \otimes_{K_0} K_n$ as a subset of L in the natural way.[5] To compute this volume is is useful to consider an the auxillary ring

$$(\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n})f_1 + \cdots + (\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n})f_m \supset \mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n}.$$

Here $f_i^2 = f_i$ are the idempotents corresponding to the summands of L. Note that $R_j := \psi_j(\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n}) \subset \mathcal{O}_{L_j}$ is not necessarily integrally closed in L_j. The ring is isomorphic to $\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n} / \operatorname{ann}_{\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n}}(f_j)$. Here observe that $f_k \in K_1 \otimes \cdots \otimes K_n$ and in general, the tensor product $\mathcal{O}_{K_1} \otimes \cdots \otimes \mathcal{O}_{K_n}$ does not break-up into a direct sum of its projections onto each factor in the tensor product of fields. That is we have

$$\left(\bigotimes_{i=1}^n K_i\right) f_1 + \cdots + \left(\bigotimes_{i=1}^n K_i\right) f_m \cong R_1 \times \cdots \times R_m, \subset \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{L_m}.$$

In summary, we have three (often strict) inclusions of rings

$$\bigotimes_{i=1}^n \mathcal{O}_{K_i} \subset R_1 \times \cdots \times R_m \subset \mathcal{O}_{L_1} \times \cdots \mathcal{O}_{L_m}. \tag{2.7}$$

The first inclusion is an abuse of notation as we really need to take the image of $\bigotimes_{i=1}^n \mathcal{O}_{K_i}$ under the isomorphism $K_1 \otimes \cdots \otimes K_n \xrightarrow{\sim} L$.

Remark 2.13.1. $|\mathcal{O}_F/R| = |\mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_r}/R_1 \times \cdots \times R_r| = |\mathcal{O}_{F_1}/R_1| \cdots |\mathcal{O}_{F_r}/R_r|$. For functions fields of curves (and possibly more generally) we know that $\operatorname{ann}_{\mathcal{O}_{F_i}}(\mathcal{O}_{F_i}/R_i) = \operatorname{Cond}_{R_i}(\mathcal{O}_{F_i})$ are related to the so-called conductor: for a domain B with integral closure B^{int} we define

$$\operatorname{Cond}_B(B^{int}) := \{r \in B : rB^{int} \subset B\}.$$

This ideal has the property that it is the largest common ideal of both B and B^{int}. It also has the property $\operatorname{Cond}_B(B^{int}) = \operatorname{ann}_{B^{int}}(B^{int}/B)$ where B^{int}/B is a quotient of abelian groups given a natural B^{int}-module structure.

[5]The image of $B \otimes B'$ in $K \otimes_F K'$ is $B \otimes_{B\cap F, B'\cap F} B'$.

2.14. **Adelic measure spaces, adelic measures, and arakelov degrees.** Let F be a number field. Define, for $j \in \mathbb{N}$,

$$\mathbb{A}_F^{\otimes j} = \prod_p \mathbb{A}_{F,p}^{\otimes j}$$

$$\mathbb{A}_{F,p} = \bigoplus_{v|p} F_v.$$

Note that $\mathbb{A}_{F,p}^{\otimes j}$ is a product of fields.

It is useful to give $\mathbb{A}_{F,p}^{\otimes j}$ various $\mathbb{A}_{F,p}$-module structures. To do this, we write

$$\mathbb{A}_{F,p}^{\otimes j} = (\bigoplus_{v|p} F_v)^{\otimes j} = \bigoplus_{(v_1,\ldots,v_j)} F_{v_1} \otimes \ldots \otimes F_{v_j}.$$

It is important to observe that

$$\dim_{\mathbf{Q}_p}(F_{v_1} \otimes \ldots \otimes F_{v_j}) = \dim_{\mathbf{Q}_p}(F_{v_1}) \ldots \dim_{\mathbf{Q}_p}(F_{v_j})$$

as the dimension of a tensor product of vector spaces as the product of the dimensions.

We observe that

$$\dim_{\mathbf{Q}_p}(F_{v_1} \otimes (\bigoplus_{(v_1,\ldots,v_j)} F_{v_2} \otimes \ldots \otimes F_{v_j}) = \dim_{\mathbf{Q}_p}(F_{v_1}) \dim_{\mathbf{Q}_p}(\mathbb{A}_F^{\otimes(j-1)})$$

$$\dim \mathbf{Q}_p(\mathbb{A}_F^{\otimes j}) = \sum_{v_1,\ldots,v_j} d_{v_1} \ldots d_{v_2} \ldots d_{v_j}$$

where $d_{v_i} = [F_{v_i} : \mathbf{Q}_p]$, and observe that we have a direct sum decomposition on $\mathbb{A}_{F,p}^{\otimes j}$ by "peeling off" the zeroth component:

$$\mathbb{A}_{F,p}^{\otimes j} = \bigoplus_{v_1} [F_{v_1} \otimes \mathbb{A}_{F,p}^{\otimes j-1}].$$

We hence define the map

$$\mathrm{peel}^0 : \mathbb{A}_{F,p} \to \mathbb{A}_{F,p}^{j}$$

$$\mathrm{peel}^0((a_v)_{v|p}) = (a_{v_1} \otimes 1_{\mathbb{A}_{F,p}^{j-1}})_{v_0|p}$$

Note that we similarly have maps peel^i for $1 \leq i \leq j$ given by

$$\mathrm{peel}^i((a_{v_i})_{v_i|p}) = (1_{\mathbb{A}_{F,p}^{\otimes i-1}} \otimes a_{v_i} \otimes 1_{\mathbb{A}_{F,p}^{\otimes j-i}})_{v_i|p})$$

For a direct sum of p-adic fields L and $A \subseteq \mathcal{M}(L)$

$$\overline{\log \mu}_L(A) = \frac{\log(\mu_L(A))}{[L : \mathbf{Q}_p]}$$

Example 2.14.1.

$$\overline{\log \mu}_{\mathbb{A}_{F,p}^{\otimes j}}(A) = \frac{\overline{\log \mu}_{\mathbb{A}_{F,p}^{\otimes j}}(A)}{\sum_{v_1,\ldots,v_j \in V(F)_p} [F_{v_1} : \mathbf{Q}_p] \ldots [F_{v_j} : \mathbf{Q}_p]}$$

We stop and make the observation that

$$\sum_{v_1 \dots v_j \in V(F)_p} [F_{v_1} : \mathbf{Q}_p] \dots [F_{v_j} : \mathbf{Q}_p] = (\sum_{v|p} [F_v : \mathbf{Q}_p])^j = [F : \mathbf{Q}_p]^j \tag{2.8}$$

This formula will come up in the future.

We similarly define, for $A \in \mathcal{M}(\mathbb{A}_F^{\otimes j})$,

$$\overline{\log \mu}_{\mathbb{A}_F^{\otimes j}}(A) = \sum_p \overline{\log \mu}_{\mathbb{A}_F^{\otimes j}}(\mathrm{pr}_p(A))$$

and observe that by formula 2.8 we have

$$\overline{\log \mu}_{\mathbb{A}_F^{\otimes j}}(A) = \frac{\log(\mu_{\mathbb{A}_F^{\otimes j}}(A))}{[F : \mathbf{Q}]^j}$$

so it is literally a root of an adelic log-volume.

Remark 2.14.2. Via symmetry we can define

$$\Omega_{D,p}^{(j;i)} := \mathrm{peel}^i((a_v)_{v|p}) \cdot \mathcal{O}_{\mathbb{A}_{F,p}^{\otimes j}}$$

for any $1 \leq i \leq j$. In fact, Mochizuki uses an indexing for $(j+1)$ factors with indices $\{0, \dots j\}$ and uses peel^j in this case for his Module structures. Note $\Omega_{D,p}^{j;1} = \Omega_{D,p}^j$.

Example 2.14.3. In concrete terms, say we only have two primes above p so that $V(F)_p = \{P_1, P_2\}$ then

$$\begin{aligned}\mathbb{A}_{F,p}^{(3)} = & [F_{P_1} \otimes (F_{(P_1,P_1)} \oplus F_{(P_1,P_2)} \oplus F_{(P_2,P_1)} \oplus F_{(P_2,P_2)})] \\ & \oplus [F_{P_2} \otimes (F_{(P_1,P_1)} \oplus F_{(P_1,P_2)} \oplus F_{(P_2,P_1)} \oplus F_{(P_2,P_2)})]\end{aligned}$$

and $\bigoplus_{v|p} F_{v_0} = F_{P_1} \oplus F_{P_2}$ so the map $\bigoplus_{v_0|p} F_{v_0} \to \mathbb{A}_{F,p}^{(3)}$ is given by

$$(a_{P_1}, a_{P_2}) \mapsto (a_{P_1} \otimes 1, a_{P_1} \otimes 1, a_{P_1} \otimes 1, a_{P_1} \otimes 1, a_{P_2} \otimes 1, a_{P_2} \otimes 1, a_{P_2} \otimes 1, a_{P_2} \otimes 1).$$

Lemma 2.14.4. *Let F a number field. Let $D \in \widehat{Div}(F)$. We have, for each $j \in \mathbb{N}$,*

$$\frac{\widehat{\deg}_F(D)}{[F : \mathbf{Q}]} = -\overline{\log \mu}_{\mathbb{A}_F^{\otimes j}}(\Omega_D^{(j)}).$$

Proof. It suffices to prove this for a "local" divisor

$$D = \sum_{v|p} a_v[v].$$

We have

$$
\begin{aligned}
\log \mu_{\mathbb{A}_{F,p}^{\otimes j}}(\Omega_{D,p}^{(j)}) &= \frac{1}{[F:\mathbb{Q}]^j}(\log(\mu_{\mathbb{A}_{F,p}^{\otimes j}}(\mathrm{peel}^1((a_v)_{v|p}) \cdot \mathcal{O}_{\mathbb{A}_{F,p}^{\otimes j}}))) \\
&= \frac{1}{[F:\mathbb{Q}]^j}(\log(\mu_{\mathbb{A}_{F,p}^{\otimes j}}(\mathrm{peel}^1((a_v)_{v|p}) \cdot \mathcal{O}_{\mathbb{A}_{F,p}^{\otimes j}}))) \\
&= \frac{1}{[F:\mathbb{Q}]^j}(\log(\prod_{v|p} |a_v|_p^{\dim_{\mathbb{Q}_p}(K_v \otimes \mathbb{A}_{F,p}^{\otimes j-1})}) \\
&= \frac{1}{[F:\mathbb{Q}]^j}(\sum_{v|p}[K_v:\mathbb{Q}_p][F:\mathbb{Q}]^{j-1}\log|a_v|_p) \\
&= \frac{1}{[F:\mathbb{Q}]}(-\sum_{v|p}[K_v:\mathbb{Q}_p]\operatorname{ord}_p(a_v)\log(p)) \\
&= \frac{1}{[F:\mathbb{Q}]}(-\sum_{v|p}\frac{[K_v:\mathbb{Q}_p]}{e(v/p)}\operatorname{ord}_v(a_v)\log(p)) \\
&= \frac{1}{[F:\mathbb{Q}]}(-\sum_{v|p} f(v/p)e(v/p)\operatorname{ord}_v(a_v)\log(p)) \\
&= -\frac{\widehat{\deg}_F(\sum_v a_v[v])}{[F:\mathbb{Q}]}
\end{aligned}
$$

□

2.15. **Fake adelic measure spaces, fake adelic measures, and arakelov degrees.** Consider now an extension of number fields, K/K_0, and fix a collection of places $\underline{V} \subseteq V(K)$ that bijects with $V(K_0)$ under the natural map $V(K) \to V(K_0)$.

Define,for $j \in \mathrm{N}$, the topological ring of fake adeles for the tuple $K, K_0, \underline{V}$

$$\mathbb{A}_{K,\underline{V}}^{\otimes j} = \prod_p \mathbb{A}_{K,\underline{V},p}^{\otimes j}$$

where

$$\mathbb{A}_{K,\underline{V},p}^{\otimes j} = \bigoplus_{v|p}(K_{\underline{v}})^{\otimes j}.$$

In the case $K_0 = K$, this forces $\underline{V} = V$ and we have $\mathbb{A}_{K,\underline{V}}^{\otimes j} = \mathbb{A}_K^{\otimes j} = \mathbb{A}_{K_0}^{\otimes j}$.

Definition 2.15.1. For each index i where $0 \le i \le j$ we define

$$\mathbb{A}_{K,\underline{V},p} \xrightarrow{\mathrm{peel}^i} \mathbb{A}_{K,\underline{V},p}^{\otimes j}.$$

We first define peel^1 by

$$\mathrm{peel}^1((a_{\underline{v}})_{\underline{v}|p}) = (a_{\underline{v}} \otimes 1_{\mathbb{A}_{F,v,p}^{\otimes(j-1)}})_{v|p}$$

where we have used the direct sum decomposition (which peels of the first term)

$$\mathbb{A}_{K,\underline{V},p}^{\otimes j} = \bigoplus_{v|p}[K_{\underline{v}} \otimes \mathbb{A}_{K,\underline{V},p}^{\otimes(j-1)}].$$

For $i \neq 1$ we define $\mathrm{peel}^i = \sigma_{(i1)}^{-1} \mathrm{peel}_1^0 \sigma_{(i1)}$ where $\sigma_{(i1)}$ is any automorphism of $\mathbb{A}_{K,\underline{V},p}^{\otimes j}$ which permutes the ith and 1st tensor factors.

The spaces of fake adeles will come with two log-measure - the usual log measure $\overline{\log \mu}_{\mathbb{A}_{K,\underline{V}}^{\otimes j}}$ and the "fake-out measure" $\overline{\log \nu}_{\mathbb{A}_{K,\underline{V}}^{\otimes j}}$ which allows subsets of $\mathbb{A}_{K,\underline{V}}^{\otimes j}$ to pretend as if they were subsets of $\mathbb{A}_{K_0}^{\otimes j}$. We will now explain this cryptic statement.

Definition 2.15.2. The fake adelic measure on $\mathbb{A}_{K,\underline{V}}$ is defined for $\underline{A} \in \mathcal{M}(\mathbb{A}_{K,\underline{V}})$ to be

$$\overline{\log \mu}_{\mathbb{A}_{K,\underline{V}}}(\underline{A}) = \sum_p \overline{\log \mu}_{\mathbb{A}_{K,\underline{V},p}}(\mathrm{pr}_p(\underline{A}))$$

where for $\underline{B}_p \in \mathcal{M}(\mathbb{A}_{K,\underline{V},p})$ we define

$$\overline{\log \mu}_{\mathbb{A}_{K,\underline{V},p}}(\underline{B}_p) = \frac{\sum_{\underline{v}} [K_{0,v} : \mathbb{Q}_p] \overline{\log \mu}_{K_{\underline{v}}}(\underline{B}_p)}{K_0 : \mathbb{Q}_p}.$$

Here we have substituted the weights $\overline{\log \mu}_{\mathbb{A}_{K_0},p}$ in **??** for the weights that would normally appear in $\overline{\log \mu}_{\mathbb{A}_{K,p}}$ in the renormalization formula.

For the renormalization formula we have

$$\overline{\log \mu}_{\mathbb{A}_{K_0,p}^{\otimes j}} = \frac{\sum_{\vec{v} \in V(K_0)_p^j} [K_0, \vec{v} : \mathbb{Q}_p] \overline{\log \mu}_{K_0,\vec{v}} \circ \mathrm{pr}_{\vec{v}}}{[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p}.$$

Here we have used the notation $K_0, \vec{v} = K_{0,(v_1,\ldots,v_j)} = K_{0,v_1} \otimes \ldots \otimes K_{0,v_j}$.

Motivated by this we define the fake K_0-adelic measure on $\mathbb{A}_{K,\underline{V},p}^{\otimes j}$ by

$$\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}} = \frac{\sum_{\vec{v} \in V(K_0)_p^j} [K_0, \vec{v} : \mathbb{Q}_p] \overline{\log \mu}_{K_{\vec{v}}} \circ \mathrm{pr}_{\vec{v}}}{[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p}.$$

Remark 2.15.3. (1) Since

$$[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p] = \sum_{\vec{v} \in V(K_0)_p^j} [K_{0,\vec{v}}^{\otimes j} : \mathbb{Q}_p]$$

this log-measure still has the property that for $a \in K_0$, and $\underline{A} \in \mathbb{A}_{K,\underline{V},p}^{\otimes j}$ then one has

$$\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}}(a\underline{A}) = \log(|a|_p) + \overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}}(\underline{A})$$

(2) This measure does not agree with $\overline{\log \mu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}}$. For $\underline{A}$ such that

$$\mathrm{pr}_{\underline{\vec{v}}}(A) = \mathcal{O}_{K_{\underline{\vec{v}}}}$$

for all $\underline{\vec{v}} \in \underline{V}_p^j$ not equal to some fixed $\underline{\vec{v_0}}$, we will have

$$\overline{\log \mu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}} = \frac{1}{[\mathbb{A}_{K,\underline{V},p}^{\otimes j} : \mathbb{Q}_p]} \overline{\log \mu}_{K_{\underline{\vec{v_0}}}}(\underline{A}_{\underline{\vec{v_0}}})$$

while

$$\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}} = \frac{1}{[\mathbb{A}_{K_0,\underline{V},p}^{\otimes j} : \mathbb{Q}_p]} \frac{[K_{\underline{\vec{v_0}}} : \mathbb{Q}_p]}{[K_{0,\underline{\vec{v_0}}} : \mathbb{Q}_p]} \overline{\log \mu}_{K_{\underline{\vec{v_0}}}}(\underline{A}_{\underline{\vec{v_0}}})$$

Definition 2.15.4. We define, for

$$D = \sum_{v \in V(K_0)} a_v[v] \in \widehat{\mathrm{Div}}(K_0)$$

the set

$$\underline{\Omega}_D^{(j)} = \prod_p \underline{\Omega}_{D,p}^{(j)} \in \mathcal{M}(\mathbb{A}_{K,\underline{V}}^{\otimes j})$$

where

$$\underline{\Omega}_{D,p}^{(j)} = \mathrm{peel}^0((a_{\underline{v}})_{\underline{v}|p}) \cdot \mathcal{O}_{\mathbb{A}_{K,\underline{V}}^{\otimes j}}$$

Remark 2.15.5. We defined

$$\underline{\Omega}_{D,p}^{(j;i)} = \mathrm{peel}^i((a_{\underline{v}})_{\underline{v}|p}) \cdot \mathcal{O}_{\mathbb{A}_{K,\underline{V}}^{\otimes j}}$$

and set $\underline{\Omega}_{D,p}^{(j)} = \underline{\Omega}_{D,p}^{(j;1)}$.

Lemma 2.15.6. *Let $D \in \widehat{Div}(K_0)$. We have*

$$\frac{\widehat{\deg}_{K_0}(D)}{[K_0 : \mathbb{Q}]} = -\overline{\log \nu}_{\mathbb{A}_{K,\underline{V}}^{\otimes j}}(\underline{\Omega}_D^{(j)})$$

Proof. We observe that it suffices to prove this for divisors of ther form

$$D = \sum_{v|p} a_v[v].$$

By the previous lemma ♠♠♠ Taylor: [What is this Lemma?], it suffices to show

$$\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}}(\underline{\Omega}_{D,p}^{(j)}) = \overline{\log \mu}_{\mathbb{A}_{K_0}^{\otimes j}}(\underline{\Omega}_{D,p}^{(j)}).$$

We have

$$\begin{aligned}
\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j}}(\underline{\Omega}_{D,p}^{(j)}) &= \sum_{(v_1,\ldots,v_j)} \frac{[K_{0,v_1} \otimes \ldots \otimes K_{0,v_j} : \mathbb{Q}_p]}{[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p]} \overline{\log \mu}_{K_{\underline{v}_1} \otimes \ldots \otimes K_{\underline{v}_j}}(a_v \mathcal{O}_{(\underline{v}_1,\ldots \underline{v}_j)})) \\
&= \sum_{(v_1,\ldots,v_j)} \frac{[K_{0,v_1} \otimes \ldots \otimes K_{0,\underline{v}_j} : \mathbb{Q}_p]}{[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p]} \log(|a_{v_1}|_p) \\
&= \sum_{(v_1,\ldots,v_j)} \frac{[K_{0,v_1} \otimes \ldots \otimes K_{0,v_j} : \mathbb{Q}_p]}{[\mathbb{A}_{K_0,p}^{\otimes j} : \mathbb{Q}_p]} \overline{\log \mu}_{K_{v_1} \otimes \ldots \otimes K_{v_j}}(a_v \mathcal{O}_{(v_1,\ldots v_j)}))
\end{aligned}$$

$$= \overline{\log \mu}_{\mathbb{A}_{K_0,p}^{\otimes j}}(\underline{\Omega}_{D,p}^{(j)})$$

Here we used the notations

$$\begin{aligned}
\mathcal{O}_{(\underline{v}_1,\ldots \underline{v}_j))} &= \mathcal{O}_{K_{\underline{v}_1} \otimes \ldots \otimes K_{\underline{v}_j}}) \\
\mathcal{O}_{(v_1,\ldots v_j))} &= \mathcal{O}_{K_{0,v_1} \otimes \ldots \otimes K_{0,v_j}})
\end{aligned}$$

We are also using the property that $\overline{\log \mu}_L(a\mathcal{O}_L) = \log |a|_p$ for L a direct sum of vector spaces with normalized log measure. ∎

3. Differents, rings of integers of tensor products, and tensor products of rings of integers.

The relationship between differents of p-adic fields, tensor products of rings of integers, and rings of integers of tensor products, is what allows us to obtain the discrimant terms appearing in Szpiro's inequality from [Moc15c, Corollary 3.12]. The following lemma describes this relationship.

Lemma 3.0.1 ([Moc15d, Proposition 1.1])**.** *Let $K_1, \dots, K_n$ be finite extensions of K_0. Let $L = \bigotimes_{i=1}^n K_i$ be the tensor product of these fields and write $L = \bigoplus_\psi L_\psi$, where ψ runs over representatives in $E = \left(\bigoplus_{i=1}^n \operatorname{Hom}(K_i, \overline{K_0})\right)/G_{K_0})$. Let $\mathcal{O}_L = \bigoplus_\psi \mathcal{O}_{L_\psi}$ be the ring of integers of L and view it as an $\mathcal{O}_{K_1}\otimes\cdots\otimes\mathcal{O}_{K_n}$-module via the embeddings $\psi = (\psi_1, \dots, \psi_n)$. For each $i \in \{1, \dots, n\}$ define*

$$\mathcal{D}_i = \operatorname{Diff}(K_i/K_0)\otimes\cdots\otimes\mathcal{O}_{K_i}\otimes\cdots\otimes\operatorname{Diff}(K_n/K_0).$$

and

$$\mathcal{D} = \sum_{i=1}^n \mathcal{D}_i \subset \bigotimes_{i=1}^n \mathcal{O}_{K_i}.$$

(1) For each i where $1 \leq i \leq n$ we have $\mathcal{D}_i \cdot \mathcal{O}_L \subset \bigotimes_{i=1}^n \mathcal{O}_{K_i}$.
(2) We have $\mathcal{D}\cdot\mathcal{O}_L \subset \bigotimes_{i=1}^n \mathcal{O}_{K_i}$
(3) Let $K_0 = \mathbb{Q}_p$. The polydisc $D_L(0,R) \subset \mathcal{O}_L$ is contained in $\bigotimes_{i=1}^n \mathcal{O}_{K_i}$ where

$$\log_p(R) = \max_{1\leq i\leq n} \operatorname{ord}_p(\operatorname{Diff}(K_i/\mathbb{Q}_p)) - \sum_{i=1}^n \operatorname{ord}_p(\operatorname{Diff}(K_i/\mathbb{Q}_p)).$$

Here $D_L(0,R) = \prod_\psi D_{L_\psi}(0,R) \subset \mathcal{O}_L$.

Remark 3.0.2. In the case that $K_i = K_{\underline{v}_i}$ and $m = j$, we will let $\operatorname{diff}(\underline{\vec{v}}/p) = (\operatorname{diff}(\underline{v}_0/p), \dots, \operatorname{diff}(\underline{v}_j/p))$. This allows us to write

$$p^{\|\operatorname{diff}(\underline{\vec{v}}/p)\|_1 - \|\operatorname{diff}(\underline{\vec{v}}/p)\|_\infty}\mathcal{O}_{L_{\underline{\vec{v}}}} \subset \bigotimes_{i=0}^j \mathcal{O}_{K_{\underline{v}_i}}.$$

We will make use of this notation in later log-volume computations.

Before giving a proof of this Lemma we first need to give a description of the idempotents of $L = K_1 \otimes \ldots \otimes K_n$. It turns out that clearing the denominators of these idempotents is essential to understanding the relationship between $\mathcal{O}_{K_1}\otimes\cdots\otimes\mathcal{O}_{K_n}$ and $\mathcal{O}_L$.

3.1. The following discussion gives a procedure for removing the ceiling term from the different appearing in [Moc15d]. Let $(\psi_{j1}, \dots, \psi_{jm}) \in \Phi$ be such that $L_j = \psi_{j1}(K_1)\cdots\psi_{jm}(K_m)$. Let $\operatorname{Diff}(K_i/\mathbb{Q}_p) = \pi_{K_i}^{d_i}\mathcal{O}_{K_i}$ so that $\operatorname{ord}_p(\pi_{K_i}^{d_i}) = \operatorname{diff}(K_i/\mathbb{Q}_p)$.

$$\mathcal{D}_i\mathcal{O}_L = \bigoplus_{i=1}^r \alpha_{ij}\mathcal{O}_{L_j} \tag{3.1}$$

where

$$\alpha_{ij} = \frac{1}{\psi_{ji}(\pi_{K_i})^{d_i}} \prod_{s=1}^{m} \psi_{js}(\pi_{K_s})^{d_s}.$$

We observe that $|\alpha_{ij}|_p = |\alpha_{il}|_p = R_i$ for every $1 \le j, l \le r$. This shows that

$$\mathcal{D}_i \mathcal{O}_L = D_L(0, R_i). \tag{3.2}$$

Next Mochizuki finds the smallest integer power of $p^{N_i}\mathcal{O}_L \subset D_L(0, R_i)$. This means

$$N_i = \lceil \operatorname{ord}_p(\alpha_{ij}) \rceil = \lceil \sum_{s=1}^{m} \operatorname{ord}_p(\pi_{K_s}^{d_s}) - \operatorname{ord}_p(\pi_{K_i}^{d_i}) \rceil.$$

Taking the minimum of these $N = \min\{N_i\}$ give the bound found in [Moc15d]: we have

$$p^N \mathcal{O}_L \subset \bigotimes_{i=1}^{m} K_i \tag{3.3}$$

$$N = \lceil \|(\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p)\|_1 - \|(\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p)\|_\infty \rceil.$$

We now give a procedure which allows us to remove the ceilings from this expression. We rewrite (3.1) by letting $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ir}) \in \mathcal{O}_L$ then we have $\alpha_i \mathcal{O}_L \subset \bigotimes \mathcal{O}_{K_i}$ and hence we get that $\mathcal{O}_L \subset \alpha_i^{-1} \bigotimes \mathcal{O}_{K_i}$ where $\alpha_i^{-1} = (\alpha_{i1}^{-1}, \ldots, \alpha_{ir}^{-1})$. This is the basic trick, we just perform this more generally with a slight improvement.

Observe that we want to make the region on the left hand side as large as possible so the bounds are tighter. This means that taking the union of $\mathcal{D}_i\mathcal{O}_L$ or the sum of $\mathcal{D}_i\mathcal{O}_L$ to gives us a new bound. Taking the sum gives the best result. Writing $\alpha_{ij} = \pi_{L_j}^{\operatorname{ord}_{L_j}(\alpha_{ij})}$ and doing this we find that

$$\sum_{i=1}^{m} \bigoplus_{j=1}^{r} \pi_{L_j}^{\operatorname{ord}_{L_j}(\alpha_{ij})} \mathcal{O}_{L_j} = \bigoplus_{j=1}^{r} \sum_{i=1}^{m} \pi_{L_j}^{\operatorname{ord}_{L_j}(\alpha_{ij})} \mathcal{O}_{L_j} = \bigoplus_{j=1}^{r} \beta_j \mathcal{O}_{L_j} = \beta \mathcal{O}_L$$

where $\beta_j = \pi_{L_j}^{\min\{\operatorname{ord}_{L_j}(\alpha_{ij}):1\le i\le m\}}$.

$$\min\{\operatorname{ord}_{L_j}(\alpha_{ij}) : 1 \le i \le m\} = \min\{-e(L_j/\mathbb{Q}_p)\log_p(R_i)\} = e(L_j/\mathbb{Q}_p)\min\{-\log_p(R_i)\}$$

$$-\log_p(R_i) = \|(\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p))\|_1 - \operatorname{diff}(K_i/\mathbb{Q}_p).$$

$$\min\{-\log_p(R_i)\} = \|(\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p))\|_1 - \|(\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p))\|_\infty$$

This shows $\operatorname{ord}_{L_j}(\beta_j) = e(L_j/\mathbb{Q}_p)\left(\|\operatorname{diff}\|_1 - \|\operatorname{diff}\|_\infty\right)$. Using that $e(L_j/\mathbb{Q}_p)\operatorname{ord}_p = \operatorname{ord}_{L_j}$ we find that

$$\operatorname{ord}_p(\beta_j) = \|\operatorname{diff}\|_1 - \|\operatorname{diff}\|_\infty$$

independent of j.

Here is how we can apply these numbers to our formulas: define $\beta = (\beta_1, \ldots, \beta_r) \in L$. This gives us the expressions:

Lemma 3.1.1. *Let* $L = K_1 \otimes \cdots \otimes K_m \cong \bigoplus_{j=1}^r L_j$*. Let* $\operatorname{diff} = (\operatorname{diff}(K_1/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_m/\mathbb{Q}_p)) \in \mathbb{Q}^m$*. There exists some* $\beta = (\beta_1, \ldots, \beta_r) \in L$ *with* $\operatorname{ord}_p(\beta_j) = \| \operatorname{diff} \|_1 - \| \operatorname{diff} \|_\infty$ *for each* $1 \leq j \leq r$ *such that*

$$\beta \mathcal{O}_L \subset \bigotimes_{i=1}^m \mathcal{O}_{K_i}.$$

(or equivalently $\mathcal{O}_L \subset \beta^{-1} \bigotimes_{i=1}^m \mathcal{O}_{K_i}$*).*

Also observe that since multiplication by β^{-1} is a $\mathbb{Q}_p$-linear transformation we know how it distorts p-adic volumes.[6]

-

3.2. **Idempotents in tensor packets of fields.** The following gives a description of idempotents in a tensor product of fields in a special case. We will use faithfully flat descent to apply this result to the general situation.

Lemma 3.2.1. *Let* $K_2, \ldots, K_n$ *finite extensions of a field* K_0 *generated by primitive elements with minimal polynomials* $f_2, \ldots, f_n$*. Let* K *be the galois closure of the compositum of the fields* $K_2, \ldots, K_n$*. Present* $L = K \otimes_{K_0} K_2 \otimes_{K_0} \cdots \otimes_{K_0} K_n$ *as*

$$K \otimes_{K_0} K_2 \otimes_{K_0} \cdots \otimes_{K_0} K_n \cong K[x_2, \ldots, x_n]/(f_2(x_2), \ldots, f_n(x_n)).$$

The idempotents of L *are given by*

$$\prod_{j=1}^n \frac{f_j(x_j)}{(x_j - \alpha_j) f_j'(\alpha_j)}$$

where α_j *is a root of* $f_j(x_j)$*.*

Proof. The proof is essentially the same as Example 3.3.1. We have $(f_i(x)) = (x - \alpha_{i1}) \cap \ldots \cap (x - \alpha_{id_i})$, with $(x - \alpha_{ij}) + (x - \alpha_{ik}) = K[x]$ for $j \neq k$ which means

$$K[x_2, \ldots, x_n]/(f_2(x_2), \ldots, f_n(x_n)) \cong \bigoplus K[x_2, \ldots, x_n]/(x_2 - \alpha_{2j_2}, \ldots, x_n - \alpha_{nj_n})$$

The ideals $I_{j_2,\ldots,j_n} = (x_2 - \alpha_{2j_2}, \ldots, x_n - \alpha_{nj_n})$ determine these component by the chinese remainder theorem and one can check that the elements

$$h_{j_2,\ldots,j_n} = \prod_{i=1}^n \frac{f_i(x_i)}{(x_i - \alpha_{ij_i}) f_i'(\alpha_{ij_i})}$$

satisfy $h_{j_2,\ldots,j_n} \equiv 0 \mod I_{k_2,\ldots,k_n}$ where $k \neq j$ and $h_{j_2,\ldots,j_n} \equiv 1 \mod I_{j_2,\ldots,j_n}$ and hence are the idempotents associated to the component with label $(j_2, \ldots, j_n)$. ∎

[6]For A a linear transformation of a p-adic vector space V we have $\mu_V(A \cdot \Omega) = |\det(A)|_p \mu_V(\Omega)$ for every measurable subset $\Omega \subset V$.

3.3. **Proof of Lemma 3.0.1.**

(1) By symmetry, we only need to prove the claim for $i = 1$. We will first apply a series of reductions using faithfully flat descent.

reduction - general case to K_i/K_0 totally ramified: ♠♠♠ Taylor: [We apply the functor

$$- \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$$

where $\mathcal{O}_{K_i} = \mathbf{Z}_{p^{r_i}}[x]/(f_i(x))$ using **??** and let $r = \max_{1 \leq i \leq n}\{r_i\}$, so that all the the rings are base extended leaving only a ramified extension.

]

reduction -K_i/K_0 totally ramified to K_1 galois hull of $K_2 \cdots K_n$: ♠♠♠ Taylor: [We apply the functor

$$- \otimes_{\mathcal{O}_{K_1}} \mathcal{O}_K$$

where K is the galois hull of the compositum of the fields i.e. $K = (K_1 K_2 \cdots K_n)^{\text{gal}}$. We view the inclusions

$$I \cdot B \subset A \tag{3.4}$$

is an inclusion of A_1-modules. To reduce our problem to the case where K_1 is the galois hull of the other fields we need to show $(I \cdot B) \otimes_{A_1} A_1' = (I \otimes_{A_1} A_1') \cdot (B \otimes_{A_1} A_1')$ since the summands of $B \otimes_A A'$ are easy to understand (the generators of $I \otimes_A A'$ stay the same by flatness)] ♠♠♠ Taylor: [All the ring extensions are finite free as modules and so are flat...]

K_1 galois/ K_i ramified case: Consider the case when $K_2, \ldots, K_n$ are totally ramified over K_0 and K_1 is the galois closure of the composition of the other extensions, i.e. $K_1 = (K_2 \cdots K_n)^{\text{gal}}$. The idea will be to use an explicit description of idempotents of L. It just so happens that their denominators come from generators of differents and multiplication by these tensor product of different ideals will clear the denominators making the elements integral. We will need to set up the notation to carry this computation out.

In our setup we have

$$\mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2} \otimes \cdots \otimes \mathcal{O}_{K_n} \cong \frac{\mathcal{O}_{K_1}[x_2, \ldots, x_n]}{(f_2(x_2), \ldots, f_n(x_n))}$$

where each f_i is Eisenstein and has a complete set of roots in $\mathcal{O}_{K_1}$ i.e.

$$f_i(x) = \prod_{j=1}^{e(K_i/K_0)} (x - \alpha_{ij}), \quad i \geq 2.$$

This presentation is important and we will use it throughout this subproof. Since K_1 is the galois closure of the compositum of the other fields we may write

$$L \cong \bigoplus_{\psi = (\psi_1, \ldots, \psi_n)} \psi_1(K_1)\psi_2(K_2) \cdots \psi_n(K_n) = \bigoplus_{\psi} K_1.$$

Here ψ runs over an equivalence class of embeddings. Let's write $\alpha_{i1} =: \alpha_i$ for $i = 2, \ldots, n$ and choose these elements to be the uniformizers of $\mathcal{O}_{K_i}$ for $2 \leq i \leq n$ (in fact, any of our elements can be chosen if we wanted to). Given this notation we may write

$$\mathrm{Diff}(K_i/K_0) = (\alpha_i^{d_i}) = (f_i'(\alpha_i)) \subset \mathcal{O}_{K_i}. \tag{3.5}$$

As a final bit of notation we will choose a specific set of representatives $E = \{\psi_1, \ldots, \psi_m\}$ of embeddings. We will apply the following notation for the components of ψ_j:

$$\psi_j = (\psi_{j1}, \psi_{j2}, \ldots, \psi_{jn}) \in E \subset \bigoplus_{i=1}^{n} \mathrm{Hom}(K_i, \overline{K_0}),$$

for $j = 1, \ldots, m$.
Given all of the notation above the idempotents g_{ψ_j} of $L = K_1[x_2, \ldots, x_n]/(f_2(x_2), \ldots, f_n(x_n)) = \bigoplus Lg_{\psi_j}$ are presented as

$$g_{\psi_j} = \prod_{i=2}^{n} \frac{f_i(x_i)}{(x_i - \psi_{ji}(\alpha_i))f_i'(\psi_{ji}(\alpha_i))} \in L.$$

Note that for each embedding $\psi \in E$ we have

$$\left(\prod_{i=2}^{n} \alpha_i^{d_i}\right) g_{\psi_j} = \left(\prod_{i=2}^{n} \frac{f_i(x_i)}{(x_i - \psi_{ji}(\alpha_i))}\right) \cdot u_{\psi_j}$$

for some $u_{\psi_j} \in \mathcal{O}_{K_1}^{\times}$ (this is a product of units, one from each $\mathcal{O}_{K_i}^{\times}$). This follows from (3.5)
We can now show $\mathcal{D}_1\mathcal{O}_L \subset \bigotimes_{i=1}^{n} \mathcal{O}_{K_i}$. We use the presentations:

$$\mathcal{D}_1 = (\alpha_2^{d_2} \cdots \alpha_n^{d_n}) \subset \mathcal{O}_{K_1}[x_2, \ldots, x_n]/(f_2(x_2), \ldots, f_n(x_n)),$$
$$\mathcal{O}_L = \mathcal{O}_K g_{\psi_1} + \mathcal{O}_{K_1} g_{\psi_2} + \cdots + \mathcal{O}_K g_{\psi_m} \subset K_1[x_2, \ldots, x_n]/(f_2(x_2), \ldots, f_n(x_n)).$$

It is enough to show for a general element $a_1 g_{\psi_1} + \cdots + a_m g_{\psi_m} \in \mathcal{O}_L$ with $a_i \in \mathcal{O}_{K_i}$, that $(\alpha_1^{d_1} \cdots \alpha_n^{d_n}) \cdot (a_1 g_{\psi_1} + \cdots + a_m g_{\psi_m}) \in \bigotimes_{i=1}^{n} \mathcal{O}_{K_i}$. Indeed, this follows precisely from the fact that

$$a_j(\alpha_1^{d_1} \cdots \alpha_n^{d_n} g_{\psi_j}) = a_j(\prod_{i=2}^{n} \frac{f_i(x_i)}{x_i - \psi_{ji}(\alpha_i)}) u_{\psi_j} \in \mathcal{O}_{K_1}[x_2, \ldots, x_n] = \bigotimes_{i=1}^{n} \mathcal{O}_{K_i}.$$

(2) Since the sum of submodules is a submodule we have $\sum_{i=1}^{n} \mathcal{D}_i\mathcal{O}_L \subset \otimes_{i=1}^{n} \bigotimes_{i=1}^{n} \mathcal{O}_{K_i}$. Also, $(\sum_{i=1}^{n} \mathcal{D}_i)\mathcal{O}_L \subset (\sum_{i=1}^{m} \mathcal{D}_i)\mathcal{O}_L$. This is again a general fact about modules and ideals. □

(3) Write $\mathrm{Diff}(K_i/\mathbb{Q}_p) = (\beta_i)$. Then the ideal $\mathcal{D}$ has generators in the ψ-component (where we write $\psi = (\psi_1, \ldots, \psi_n)$) given by

$$\psi(\mathcal{D}) = \left(\frac{\psi_1(\beta_1)\psi_2(\beta_2)\cdots\psi_n(\beta_n)}{\psi_1(\beta_1)}, \frac{\psi_1(\beta_1)\psi_2(\beta_2)\cdots\psi_n(\beta_n)}{\psi_2(\beta_2)}, \ldots, \frac{\psi_1(\beta_1)\psi_2(\beta_2)\cdots\psi_n(\beta_n)}{\psi_n(\beta_n)}\right),$$

and since $\mathcal{O}_{L_\psi}$ is a PID with uniformizer π_ψ we will have $\psi(\mathcal{D}) = (\pi_\psi^{c_\psi})$. Here the integer c_ψ is the minimum ord_{L_ψ}-valuation of each of the generators. Equivalently, this corresponds to the radius

$$R = \max_{1\le i\le n}\left|\frac{\prod_{j=1}^n \psi_j(\beta_j)}{\psi_i(\beta_i)}\right|_p = \frac{\prod_{j=1}^n |\psi_j(\beta_j)|_p}{\min_{1\le i\le n}|\psi_i(\beta_j)|_p}$$

which gives

$$\log_p(R) = \max_{1\le i\le n}\mathrm{diff}(K_i/\mathbb{Q}_p) - \sum_{i=1}^n \mathrm{diff}(K_i/\mathbb{Q}_p).$$

□

Example 3.3.1 (Tensor product of a Galois field with itself)**.** Let $K/\mathbf{Q}_p$ be Galois with Galois group G. We view it as a subfield of $\mathbf{Q}_p$. This means that for all $\tau \in \mathrm{Aut}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$, $\tau(K) = K$. Let

$$\Phi = \{(\psi_1.\psi_2)|\psi_i : K \to \bar{\mathbf{Q}}_p\}/\mathrm{isom}.$$

Note that we have a bijection between $\{\psi : K \mapsto \bar{\mathbf{Q}}_p\}$ and G so the set of equivalence classes can be put in bijection of embeddings are in bijection with G as

$$\Phi \cong \{(\mathrm{id}, \sigma) : \sigma \in G\}.$$

This tells us

$$K \otimes_{\mathbf{Q}_p} K \cong \bigoplus_{\sigma\in G} K, \tag{3.6}$$

where the map in (3.6) is given by $(a\otimes b) \mapsto (a\sigma(b))_{\sigma\in G}$, extended linearly. We wish to write this direct sum as an internal direct sum. To do this we now that for each $\sigma \in G$, there exists some idempotent $e_\sigma \in K \otimes_{\mathbf{Q}_p} K$ with e_σ^2 so that

$$K \otimes_{\mathbf{Q}_p} K = \sum_{\sigma\in G}(K \otimes_{\mathbf{Q}_p} K)e_\sigma$$

as Modules. The sum here is direct.

4. Mochizuki's first inequality: Corollary 3.12 of IUT 3

4.1. **Tate parameters of semistable elliptic curves.** In this section we explain how Tate parameters are related to the minimal discriminant. Recall that good and multiplicative reduction types are stable under base change while additive reduction can change ([Sil09, VII.5]) — when an elliptic curve over a number field has good or multiplicative reduction at all of its bad places it is called semi-stable. We also remind the reader that potentially good reduction is equivalent to integral j-invariant (loc. cit.).

Now consider E an elliptic curve over a complete non-archimedean field. If $|j_E| > 1$ then E has a unique Tate parameter $q = q_E$ and Tate uniformization $E_q \cong E$ defined over the algebraic closure [Sil13, V. Lemma 5.1]. This implies, every elliptic curve without potentially good reduction has a unique Tate uniformization over the algebraic closure. In fact, one can write down an explicit formula for q_E as a rational function in j_E with $\mathbb{Q}_p$-coefficients.

Lemma 4.1.1. *Let E be an elliptic curve over a complete discretely valued field K with valuation v.*

(1) If E has multiplicative reduction then $\operatorname{ord}_v(\Delta^{\min}) = \operatorname{ord}_v(q_E)$.
(2) Every Tate curve E_q is a minimal Weierstrass model.

Proof. (1) From Ogg's formula [Sil09, 2009] we have $f_v = \operatorname{ord}_v(\Delta_v^{\min}) + 1 - m_v$ where m_v is the number of irreducible components of the special fiber of $\mathcal{E} = \operatorname{Ner}(E/F_v)$. In the case that E_v has multiplicative reduction, one has

$$\operatorname{ord}_v(\Delta_v^{\min}) = m_v.$$

We can compute the number of connected components of the special fiber:

$$\mathcal{E}_k/\mathcal{E}_k^0 \cong E_q(K)/(E_q)_0(K) \cong \frac{K^\times/q^{\mathbf{Z}}}{R^\times/q^{\mathbf{Z}}} \cong \mathbf{Z}/\operatorname{ord}_v(q).$$

The first isomorphism is a Theorem of Néron-Kodaira [Sil09, C.15], and the second isomorphism follows from properties of the Tate uniformization [Sil09, C 14.1.c] — the last isomorphism in the sequence above uses ord_v. This implies $m_v = \operatorname{ord}_v(q_E)$.

(2) (See [Sil13, Chapter 5, section 3-5]) We have $|\Delta| = |q|$. This uses the formal q-series expansion of $\Delta = q\prod_{n\geq 1}(1-q^n)^{24}$.

□

In the case that E/K has split multiplicative reduction, how does the order of the minimal discriminant change as we take field extensions?

Lemma 4.1.2. *Let $K \supset F$ be an extension of number fields. Let E/F be a semi-stable elliptic curve with everywhere multiplicative reduction. We have $[K:F]\ln\Delta_{E/F}^{\min} = \ln\Delta_{E/K}^{\min}$. In particular $\ln|\Delta_{E/F}^{\min}|/[F:\mathbb{Q}]$ is stable under base change if E is semi-stable.*

Proof Idea. As we have seen $\operatorname{ord}_v(\Delta_{E/K}^{\min}) = \operatorname{ord}_v(q_E)$. If we base change to K'/K with ramification degree $e(v'/v)$ the formula $\operatorname{ord}_v = e(v'/v)\operatorname{ord}_{v'}$ shows that the minimal discriminant just changes by the ramification degree. A second way to recall that the minimal regular model $\mathcal{C}$ of an elliptic curve with split multiplicative reduction has a special fiber which is a circle of $\operatorname{ord}_v(\Delta^{\min})$ many $\mathbb{P}^1$'s. When going to a ramified base extension the local equations $xy = \pi$ turn into local equations $xy = (\pi')^e$ which are no longer regular. Indeed now we need to blow-up e-times to get a regular model. This accounts for the increase in the minimal discriminant by e again.

The rest of the formula follows from a computation using the fact that norms of ideals are products of norms of local ideals, or when stated logarithmically that

$$\ln|\Delta_{E/F}^{\min}| = \sum_p \sum_{w|p} f(w/p)\operatorname{ord}_q(q_w)\ln(p).$$

Here we used that $\Delta_{E_w}^{\min} = q_w\mathcal{O}_{F_w}$. Rewriting the sum as

$$\sum_p \sum_{v\in V(F_0)_p} \sum_{w\in V(F)_v} f(w/v)f(v/p)e(w/v)\operatorname{ord}_v(q_w)\ln(p) = \sum_p\sum_{v|p}\sum_{w|v}[F_w : F_{0,v}]f(v_p)\operatorname{ord}_v(q_v)\ln(p)$$

with the identity $\sum_{w|v}[F_w : F_{0,v}] = [F : F_0]$ gives the result.

$\square$

As seen in §4.3, an elliptic curve over $F_{0,v}$ may have multiplicative reduction but not split multiplicative reduction unless $\gamma(E/K)$ has a trivial square class. When passing to extensions F_{w_1} and F_{w_2} on obtains Tate uniformizations for $E_{F_{w_1}}$ and $E_{F_{w_2}}$ with Tate parameters q_{w_1} and q_{w_2} (Mochizuki even writes $q_{\underline{v}}$ for the Tate parameter of an elliptic curve in initial theta data at $K_{\underline{v}}$). What is the relationship between $q_{w_1}\in F_{w_1}$ and $q_{w_2}\in F_{w_2}$? Are the conjugate? Do they have the same p-adic absolute value? The answer to this is simple: they are equal.

By the appendix of [Sil13], if v is a place where $|j_E|_v > 1$ then over the algebraic closure there is some $q_v = q_{E_v}$ such that $q_{E_v}\in\mathbb{Q}_p(j_E)$. Note that $j_{E_v} = j_E$. Also, the curves base extended curves E_{w_1} and E_{w_2} are isomorphic over $\overline{F_{0,v}}$ so we have

$$j_E = j_{E_v} = j_{E_{w_1}} = j_{E_{w_2}} \in F_0.$$

This implies

$$q_v = q_{w_1} = q_{w_2} \in \mathbb{Q}_p(j_E).$$

In particular $\operatorname{ord}_v(q_{w_1}) = \operatorname{ord}_v(q_{w_2})$.

Note that while although E_v may not have split multiplicative reduction, it is the case that $E_{q_v}/\mathbb{Q}_p(j_E)$ still exists and is potentially isomorphic to E_v.

4.2. **Theta pilot objects as a divisors.** The following gives a down-to-earth description of the objects anabelianly interpreted in [Moc15c, Definition ???].

Lemma 4.2.1. *Let E be an elliptic curve with semi-stable reduction and field of moduli F_0. Fix initial theta data*

$$(\overline{F}/F, l, E_F, \underline{C}_K, \underline{V}, \underline{V}_{mod}^{\mathrm{bad}}, \underline{\epsilon})$$

and let $P_q\in\widehat{\operatorname{Div}}(F_0)$ and $P_\Theta\in\widehat{\operatorname{Div}}_{\mathrm{lgp}}(F_0)$ be the theta pilot divisors for the theta data. We have

$$\underline{\widehat{\deg}}(P_q) = \frac{1}{2l}\frac{\log\operatorname{Nm}(\Delta_{E/F_0}^{\min})}{[F_0:\mathbb{Q}]}$$

and the degrees of the theta and q pilot divisors are related by

$$\underline{\widehat{\deg}}_{\mathrm{lgp}}(P_\Theta) = \frac{l(l+1)}{12}\underline{\widehat{\deg}}_{F_0}(P_q).$$

Proof. Let $g : V(K) \to V(F_0)$ be the natural map induced by the inclusion of fields $F_0 \subset K$. We have

$$
\begin{aligned}
g^*(\sum_{v\in V(F_0),\ \mathrm{bad}} \mathrm{ord}_v(q_{\underline{v}})[v]) &= \sum_{v\in V(F_0),\mathrm{bad}} \mathrm{ord}_v(q_{\underline{v}}) \left(\sum_{w\mapsto v} e(w/v)[w]\right) \\
&= \sum_{v\in V(F_0),\mathrm{bad}} \frac{1}{e(\underline{v}/v)} \mathrm{ord}_{\underline{v}}(q_{\underline{v}}) \left(\sum_{w\mapsto v} e(\underline{v}/v)[w]\right) \\
&= \sum_{v\in V(F_0),\mathrm{bad}} \sum_{w\mapsto v} \mathrm{ord}_{\underline{v}}(q_{\underline{v}})[w] \\
&= \sum_{v\in V(F_0),\mathrm{bad}} \sum_{w\mapsto v} \mathrm{ord}_w(q_w)[w] \\
&= \sum_{w\in V(K),\mathrm{bad}} \mathrm{ord}_w(q_w)[w] \\
&= \mathrm{div}(\Delta^{\min}_{E_K/K}).
\end{aligned}
$$

By stability of the minimal discriminant for semi-stable elliptic curves (Lemma 4.1.2) we get $\underline{\widehat{\deg}}(\mathrm{div}(\Delta^{\min}_{E_K/K})) = \frac{1}{[K:\mathbb{Q}]}\log \mathrm{Nm}\, \Delta^{\min}_{E_K/K} = \frac{1}{[F:\mathbb{Q}]}\log \mathrm{Nm}\, \Delta^{\min}_{E/F}$.

The second assertion follows from

$$
\begin{aligned}
\underline{\widehat{\deg}}_{\mathrm{lgp}}(P_\Theta) &= \frac{2}{l-1}\sum_{j=1}^{(l-1)/2} \underline{\widehat{\deg}}_{F_0}((P_\Theta)_j) \\
&= \frac{2}{l-1}\sum_{j=1}^{(l-1)/2} j^2 \underline{\widehat{\deg}}_{F_0}(P_q)) \\
&= \underline{\widehat{\deg}}_{F_0}(P_q)\left[\frac{2}{l-1}\cdot\frac{l(l^2-1)}{24}\right] \\
&= \frac{l(l+2)}{12}\underline{\widehat{\deg}}F_0(P_q).
\end{aligned}
$$

□

4.3. **Initial theta data.** We will now explain how to produce a tuple of initial theta data:

$$(\overline{F}/F, l, E_F, \underline{C}_K, \underline{V}, \underline{V}^{\mathrm{bad}}_{mod}, \underline{\epsilon}).$$

This is [Moc15a, Definition 3.1]. First we fix F a number field and an algebraic closure $\overline{F}$. We fix a rational prime number $l \in \mathbb{Z}$. We take E to be an elliptic curve defined over F with field of moduli $F_0 = \mathbb{Q}(j_E)$ ($= F_{mod}$ in Mochizuki's notation), such that F/F_0 is Galois.

We assume that E/F has split multiplicative reduction at the bad places of $V(F)$ [7] and that F contains the 15-torsion of E and $\sqrt{-1}$. [8] We also assume that the image of $\rho_l : G_F \to \mathrm{Aut}(E[l](\overline{F}))$ contains $\mathrm{SL}_2(\mathbb{F}_l)$.[9]

Let K be a the number field $K = F(E[l](\overline{F}))$ i.e. the field obtained by adjoining the l-torsion of E to F. [10] Next we fix a section s of the natural map

$$\begin{tikzcd} V(K) \arrow[r] & V(F_0) \arrow[l, bend right, "s"'] \end{tikzcd}$$

and define $\underline{V} = s(V(F_0))$.[11]

We let $V_{mod}^{bad} \subset V(F_{mod})$ be a non-empty collection of places exclusing places that divide 2 such that for every $w|v \in V_{mod}^{bad}$, E has bad reduction.[12]

We give two descriptions of the pair $(\underline{C}_K, \underline{\epsilon})$. First let us recall that $X_0(l)$ is a modular curve defined over $\mathbb{Z}[1/l]$ where for a field L, the open subset $Y_0(l)(L) \subset X_0(l)(L)$ parametrizes pairs (A, M) where A is an elliptic curve defined over L and $M \subset A[l]$ is a subgroup isomorphic to $\mathbb{Z}/l$. Let us also recall that the Fricke involution is an involution of algebraic schemes $X_0(l) \to X_0(l)$ which on $Y_0(l)$ has the moduli interpretation

$$(A, M) \mapsto (A/M, A[l]/M) =: (\underline{A}, \underline{M}).$$

Let us first $X = E \setminus \{o\}$ where o is the identity of E which we consider as a log-scheme. Then Mochzuki defines $\underline{X} \to X$ to be a double cover of X obtained as the dual Fricke isogeny, and one takes $\underline{C}$ to be the log-stack $\underline{C}_K = [\underline{X}_K/\pm 1]$. Then if $\underline{a}$ is a nonzero cusp of $\underline{X}$ we take $\underline{\epsilon}$ to be the image of $\underline{a}$ in $\underline{C}_K$ coming from the map $\underline{X}_K \to \underline{C}_K$.

Remark 4.3.1. A given in IUT1, Mochizuki actually gives a slightly more complicated definition than what was just discussed. He actually defines $\underline{C}_K$ to be the fiber product of log algebraic stacks $\underline{C}_K = X_K \times_{C_K} \underline{X}_K$ where C_K is defined to be the terminal object of the category $\mathrm{Loc}(X_K)$, the category of DM-stacks isogenous to X_K.

Also, in his notation Q is a subgroup of $\underline{E}_K$ not of E_K.

Lemma 4.3.2. *The following information is equivalence*

(1) The algebraic stack with the image of a non-zero cusp $(\underline{C}_K, \underline{\epsilon}_K)$*.*
(2) A pair $(E_K, Q) \in Y_0(l)(K)$*.*

♠♠♠ Taylor: [We need to double check that we aren't cheating here]

[7]The term "stable reduction" is used in [Moc15a, Definition 3.1], but Mochizuki works with curves having semi-stable reduction in the sense of [Sil13, pg 388] (meaning good or multiplicative reduction at the bad places of an elliptic curve).

[8]The estimates [Moc15d, Theorem 1.10] require 5-torsion as well. We observe that this is not baked into [Moc15a, Definition 3.1]. We may as well require F to contain the 30-torsion as well.

[9]Since the image of CM elliptic curves is abelian, this rules out CM elliptic curves.

[10]In the anabelian interpretation we work with decomposition groups of l-torsion points of elliptic curves and it will be convenient that their residue fields have the same field of definition as the base field.

[11]This section is used, for example, in producing "fake adeles".

[12]This is just stipulating that we need some bad places away from 2.

Proof. Given such a subgroup, call it Q, one can form E/Q and then take the dual isogeny of $E \to E/Q =: \underline{E}$ to get $\underline{E} \to E$. Quotienting $\underline{E}$ by the involution gives the Deligne-Mumford stack $\underline{C}$ (whose coarse space is just $\mathbb{P}^1$).

Conversely, suppose we are given $\underline{C}_K$. Since it is isogenous to X_K there is a map to C_K and hence we have the diagram of algebraic stacks

$$\begin{tikzcd} & X_K \arrow[d] \\ \underline{C}_K \arrow[r] & C_K \end{tikzcd},$$

hence we can take the fiber product to obtain the isogeny $\underline{X}_K \to X_K$ which we know is a morphism of algebraic schemes. Taking the map of the compactifications (since maps of curves are determined by maps of their fields) we get a map $\underline{E}_K \to E_K$. The dual isogeny is then $E_K \to \underline{E}_K$ with kernel Q. The image of the non-zero cusp $\underline{\epsilon}$ is just and element of $(\underline{Q} \setminus \{\underline{o}\})/\setminus \pm 1\} \cong \mathbb{F}_l^\times/\{\pm 1\}$ which in Mochizuki's notation is $\mathbb{F}_l^{\circledast}$. ☐

♠♠♠ Taylor: [V_{mod}^{bad} is a non-empty collection of places away from 2 over which E/F has bad reduction (so no places of V_{mod} with potentially good reduction)] ♠♠♠ Taylor: [

- C_K is assumes to be an F-core.
- This means it descends to F_{mod} among other things.
- l is prime to $\mathrm{ord}_v(q_{\underline{v}})$
- Requirement on the section: Over $K_{\underline{v}}$ we have the extension

$$(1 \to \mu_l \to E[l] \to \mathbf{Z}/l \to 0, \text{generator of } \mathbf{Z}/l).$$

From initial theta data we have

$$(1 \to M \to E_K[l] \to Q \to 1, \underline{\epsilon}).$$

The base change of the global data by the $K_{\underline{v}}$ needs to coincide with the local data.

- requirement:[?, Definition 2.5.i] $\underline{\epsilon}_{\underline{v}}$ is the generator of the quotient $\hat{\mathbb{Z}}$ up to sign.

]

We remark that once we have fixed a semi-stable elliptic curve E/F_0 (satisfying mild hypotheses on the galois represenation and reduction) the rest of the initial theta data can be obtained. $(\underline{C}_K, \underline{V}, V_{mod}^{bad}, \underline{\epsilon})$, you can get from the a choice of elliptic curve with the hypotheses as in the example.

For purposes of exposition we take its Weierstrass model to be

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

For a place w of F we can consider the local minimal model E at w which we denote by E_w and is defined over F_w. Associated to this model we have an invariant (c.f. [Sil13, pg 439])

$$\gamma(E_w/F_w) = -c_{4,w}/c_{6,w}$$

The elliptic curve E_w has split multiplicative reduction if and only if $\gamma(E_w/F_w)$ is a square in F_w. To be explicit, $\gamma(E_w/F_w)$ is defined in terms of the coefficients on the local minimal Weierstrass model at $w \in V(F)$ given by

$$y^2 + a_{1,w}xy + a_{3,w}y = x^3 + a_{2,w}x^2 + a_{4,w}x + a_{6,w},$$

and

$$\begin{aligned}
b_{2,w} &= a_{1,w}^2 + 4a_{2,w},\\
b_{4,w} &= 2a_{1,w}^2 + 4a_{2,w},\\
b_{6,w} &= a_{3,w}^2 + 4a_{6,w},\\
b_{8,w} &= a_{2,w}^2 a_{6,w} + 4a_{3,w}a_{6,w} - a_{1,w}a_{3,w}a_{4,4},\\
c_{4,w} &= b_{2,w}^2 - 24b_{4,w},\\
c_{6,w} &= b_{2,w}^2 + 36b_{2,w}b_{4,w} - 216b_{6,w}.
\end{aligned}$$

If X/F has a global minimal Weierstrass model (which means we may write $a_{i,w} = a_i \in F$) if and only if its "Weierstrass class" (a certain ideal class left over from the minimal discriminant after changes of coordinates) is equal to one F in the class group of F [Sil13, pg 244]. In this case we can always enlarge our field by adjoining the square root of $\gamma(E/F)$ to make sure the multiplicative reduction is split. Finally, as a last remark, we recall that Tate parameters are absolute invariants, so if E/F_w has Tate parameter q_w and $\underline{w} \in \underline{V}$ is a lift of w then $E_{K_{\underline{w}}}$ has the same Tate parameter.

To find an elliptic curve with initial theta data one can take an elliptic curve A over $\mathbb{Q}$ such that

- A has multiplicative reduction at its bad places.
- The Galois representation $G_{\mathbb{Q}} \to \mathrm{Aut}(A[l](\overline{Q}))$ is surjective.

The following provides such an example.

Example 4.3.3. The elliptic curve E with Cremona label `E11a1` (LMFDB label `11.a1`) given by

$$E : y^2 + y = x^3 - x^2 - 7820x - 263580$$

is defined over $\mathbb{Q}$.

- $c_4(E) = 496 = 2^4 \cdot 31$
- $c_6(E) = 20008 = 2^3 \cdot 41 \cdot 61$
- $j_E = -1 \cdot 2^{12} \cdot 11^{-5} \cdot 31^3$

Non-empty Collection of Places of Bad Reduction V_{mod}^{bad}: There is only one place of bad reduction. That is at $p = 11$ and it has split multiplicative reduction.

Minimal Weierstrass Model at $p = 11$: The local Weierstrass minimal model at $p = 11$ of E is

$$E_{11}^{\min} : y^2 + y = x^3 - x^2 - 10x - 20.$$

As E is defined over $\mathbb{Q}$ which has class number 1 we have that $E_{11}^{\min} = E^{\min}$ is a global minimal model.

- Minimal discriminant valuation: 5
- Conductor exponent: 1
- Kodaira Symbol: $I5$
- Tamagawa Number: $|\mathcal{E}/\mathcal{E}^0| = |\mathbf{Z}/q| = \mathrm{ord}(q) = 5$

Note that if we didn't have split multiplicative reduction here we would

The Tate Uniformization at $p = 11$: The Tate model of this curve is

$$E_{q_{11}} : y^2 + xy = x^3 + s_4(q_{11})x + s_6(q_{11})$$

where q_{11} is the Tate parameter. We have computed these out to $O(11^{25})$ using `Sage`:

$$\begin{aligned} q_{11} &= 0.0, 0, 0, 0, 10, 2, 6, 6, 5, 4, 4, 1, 4, 1, 0, 5, 9, 9, 3, 3, 1, 3, 4, \ldots \\ s_4 &= 0.0, 0, 0, 0, 5, 7, 1, 0, 5, 9, 1, 9, 2, 10, 2, 0, 1, 6, 2, 6, 4, 10, 10, \ldots \\ s_6 &= 0.0, 0, 0, 0, 1, 8, 4, 4, 5, 5, 10, 2, 1, 8, 2, 7, 10, 9, 6, 3, 3, 8, 5, \ldots \end{aligned}$$

Here the decimal is the beginning of the integral digits and commas separate 11-adic digits.

We will make use of the the uniformization map

$$\overline{\mathbb{Q}}_{11}^{\times}/q_{11}^{\mathbf{Z}} \xrightarrow{\varphi} E(\overline{\mathbb{Q}}_{11}).$$

In particular for each n if we let ζ_n be a primitive nth root of unity and $Q_n = q^{1/n}$ be a choice of nth root then

$$P_{1,n} = \varphi(\zeta_n), P_{2,n} = \varphi(Q_n)$$

give a

Weierstrass Uniformization: ♠♠♠ Taylor: [To get an isomorphism

$$\langle \overline{\omega_1/n}, \overline{\omega_2/n} \rangle = \frac{1}{n}\Lambda/\Lambda \cong E[n](\overline{\mathbb{Q}}) \cong Q_n^{\mathbf{Z}}\zeta_n^{\mathbf{Z}}/q^{\mathbf{Z}} = \langle \overline{Q_n}, \overline{\zeta_n} \rangle$$

We can

- Look at 2-torsion points.
- Look at $\tau = \omega_1/\omega_2$ or whatever combination gives us something in the upper-half plane.

]

Field F: We will let $F = \mathbb{Q}(\sqrt{-1}, E(\overline{\mathbb{Q}})[6])$. To obtain $E(\overline{\mathbb{Q}})[6]$ numerically in $\overline{\mathbb{Q}}$ we fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{11}$ and Here we observe that $\varphi = f \circ \phi$ where $f : E_{q_{11}} \to E$ is the morphism defined over $\mathbb{Q}_{11}$ (since E has split multiplicative reduction over $\mathbf{Q}_{11}$ — with a global minimal Weierstrass model defined over $\mathbf{Q}$.) that $E(\overline{\mathbb{Q}}_{11})[6] \cong \zeta^{\mathbf{Z}}Q^{\mathbf{Z}}/q_{11}^{\mathbf{Z}}$ where $Q = q_{11}^{1/6}$ is some choice of 6th root of q and ζ is a primitive lth root of unity.

Condition on l: The prime $l = 13$ satisfies the divisibility hypotheses of initial theta data. We have seen that $\mathrm{ord}(q) = 5$, the only bad place is $p = 11$, and that the order of $[F : \mathbb{Q}(j_E)]$ (here $\mathbb{Q}(j_E) = \mathbb{Q}$) can be computed by viewing F as a succession of prime torsion extensions. The order of the r-torsion extension divides $|\,\mathrm{GL}_2(\mathbf{F}_r)|$ for a given prime r. The relevant orders for 6-torsion are given below:

r	$\lvert GL_2(\mathbf{F}_r)\rvert = (r^2-1)(r^2-1-(r-1))$
2	$6 = 2 \cdot 3$
3	$48 = 2^4 \cdot 3$

Surjectivity of ρ_l: We require that $\rho_l : G_F \to \mathrm{Aut}(E[l])$ contains a copy of $\mathrm{SL}_2(\mathbf{F}_l)$ relative to some basis. We checked using sage that l Galois represenation for $l > 5$

Computation of the Field K: The field $K = F(E_K[11](\overline{F}))$ There are two approaches to computing these things. One may use the Weierstrass uniformization and the Tate uniformization.

The generator $\underline{\epsilon}$: We use $q^{1/l}$

These values can be gleaned from the Tate uniformization

$$(\overline{\mathbf{Q}}_{11}^{\times}/q^{\mathbf{Z}})[6] = (\zeta_6)^{\mathbf{Z}}(q^{1/6})^{\mathbf{Z}}/q^{\mathbf{Z}}$$

So the torsion field $\mathbf{Q}_{11}(\zeta_6, q_{11}^{1/6})$.

`http://doc.sagemath.org/html/en/reference/padics/sage/rings/padics/padic_extension_generic.html`

`http://sporadic.stanford.edu/reference/curves/sage/schemes/elliptic_curves/ell_tate_curve.html`

Remark 4.3.4. Can we just find an elliptic curve over $\mathbb{Q}$ with lots of $\mathbb{Q}$-torsion points? No. Recall that Mazur's theorem says that if E is an elliptic curve defined over $\mathbb{Q}$ then the torsion then the possible torsion subgroups of $E(\mathbb{Q})$ are $C_1, C_2, \ldots, C_{10}, C_{12}, C_2 \oplus C_4, C_2 \oplus C_4, C_2 \oplus C_6, C_2 \oplus C_8$. Here C_n denotes the cyclic group of order n. This preclude finding some E appearing in initial theta data with lots of rational torsion.

4.4. **Gaussian monoids and theta pilot objects.** In [Moc15c, Definition 3.8] tells us how q and Theta pilot objects are "determined by generators of monoids Ψ_{gau}". The aim of this subsection is to clarify this definition in a "scheme theoretic" sense.

4.5. **Statement of Corollary 3.12.** ♠♠♠ Taylor: [This subsection needs to be redone] To give the statements of this construction we need the following Mochizuki log-measure spaces:

$$\mathbf{L} = \prod_p \mathbf{L}_p, \qquad \overline{\log \nu}_{\mathbf{L}}(B) = \sum_p \overline{\log \nu}_{\mathbf{L}_p}(\mathrm{pr}_p(B)),$$

$$\mathbf{L}_p = \bigoplus_{j=1}^{(l-1)/2} \mathbb{A}_{K,\underline{V},p}^{\otimes j+1}, \qquad \overline{\log \nu}_{\mathbf{L}_p}(B) = \mathbf{E}\left(\overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j+1}}(\mathrm{pr}_{p,j}(B)) | 1 \le j \le \frac{l-1}{2}\right),$$

$$\mathbb{A}_{K,\underline{V},p}^{\otimes j+1} = (\bigoplus_{\underline{v}|p} K_{\underline{v}})^{\otimes j+1}, \qquad \overline{\log \nu}_{\mathbb{A}_{K,\underline{V},p}^{\otimes j+1}}(B) = \mathbf{E}(\overline{\log \mu}_{L_{\vec{v}}}(\mathrm{pr}_{\vec{v}}(B)) | \vec{v} \in V(F_0)_p^{j+1}).$$

Here we have used the notation

$$\mathbf{E}(f(x) : x \in S) := \sum_{s \in S} w_s f(s),$$

for any finite set probability space $(S, \{w_s\}_{s\in S})$ and any function f on S. The number w_s are weights.

When averaging a subset of integers, the probability distribution is uniform and when averaging places of fields the probabilities are weighted according to the dimension of as $\mathbb{Q}_p$-vector spaces of the tensor product of the corresponding fields. Also, the various pr denote the obvious projection operators. It is important to note that $\mathbb{A}_{K,\underline{V},p}^{\otimes j+1}$ decomposes as a direct sum of fields in accordance with Lemma 2.11.3. We have

$$\mathbb{A}_{K,\underline{V},p}^{\otimes j+1} = (\bigoplus_{\underline{v}|p} K_{\underline{v}})^{\otimes j+1} = \bigoplus_{(\underline{v}_0,\ldots,\underline{v}_j)\in \underline{V}_p^{j+1}} K_{\underline{v}_0} \otimes \cdots \otimes K_{\underline{v}_j} = \bigoplus_{(\underline{v}_0,\ldots,\underline{v}_j)\in \underline{V}_p^{j+1}} L_{(\underline{v}_0,\ldots,\underline{v}_j)},$$

where for each $(\underline{v}_0,\ldots,\underline{v}_j) \in \underline{V}_p^{j+1}$ we have the further decomposition into finite extensions of $\mathbb{Q}_p$,

$$L_{(\underline{v}_0,\ldots,\underline{v}_j)} = \bigoplus_{\psi\in\Phi} L_\psi.$$

Here Φ is a choice of complete set of representatives of embeddings $\psi = (\psi_0, \ldots, \psi_j)$ where $\psi_i : K_{\underline{v}_i} \to \overline{\mathbb{Q}_p}$.

Remark 4.5.1. In IUT proper, the space $\mathbf{L}$ corresponds to the spaces ♠♠♠ Taylor: [FIXME] where the inner part indicates where the interpretation is taking place.

The space $\mathbb{A}_{K,\underline{V},p}$ corresponds to ♠♠♠ Taylor: [FIXME] and the space $L_{\underline{\vec{v}}}$ correspond to ♠♠♠ Taylor: [FIXME].

Theorem 4.5.2 (Mochizuki's Corollary 3.12)**.** *Fix initial theta data*

$$(\overline{F}/F, l, E_F, \underline{C}_K, \underline{V}, \underline{V}_{mod}^{\mathrm{bad}}, \underline{\epsilon}).$$

Let $P_q \in \widehat{\mathrm{Div}}(F_0)$ and $P_\Theta \in \widehat{\mathrm{Div}}_{\mathrm{lgp}}(F_0)$ be the theta a q pilot divisors associated to the initial theta data. The following inequality holds:

$$-\widehat{\deg}_{F_0}(P_q) \leq -\widehat{\deg}_{\mathrm{lgp}}(P_{\mathrm{hull}(U_\Theta)}). \tag{4.1}$$

♠♠♠ Taylor: [I think we may need to put the ratio of the two degrees here. It would be $12/l(l+1)$ on the right hand side.] *Here, $U_\Theta \subset \mathbf{L}$ is the region*

$$U_\Theta := \langle \mathrm{Ind1}, \mathrm{Ind2}\rangle \cdot (\Omega_{P_\Theta})^{\mathrm{Ind3}} \subset \mathbf{L}.$$

Some unravelling of this inequality is in order.

First, the set $\Omega_{P_\Theta} \subset \mathbf{L}$ is a region associated to $P_\Theta \in \widehat{\mathrm{Div}}(F_0)$. In this section we will explain that objects of $\widehat{\mathrm{Div}}(F_0)$ are tuples of divisors and that to each such lgp-Arakelov divisor we can associate a region in $\mathbf{L}$ whose associated Mochizuki log-volume is the negative of its Arakelov degree. We have ♠♠♠ Taylor: [FIXME]

$$\mathrm{pr}_{p,j}(\Omega_{P_\Theta}) = \bigoplus_{(\underline{v}_0,\ldots,\underline{v}_j)\in \underline{V}^{j+1}} q_{\underline{v}_j}^{j^2/2l} \mathcal{O}_{(\underline{v}_0,\ldots,\underline{v}_j)} \subset \mathbf{L}_{p,j},$$

Again, to every object $P \in \mathrm{Div}_{\mathrm{lgp}}(F_0)$ there is a region Ω_P such that $-\widehat{\deg}_{\mathrm{lgp}}(P) = \overline{\log \nu_{\mathbf{L}}}(\Omega_P)$. Similarly, to every $\Omega \in \mathcal{M}(\mathring{A}^{\otimes i}_{K,\underline{V},p})$ there is a $P_\Omega \in \widehat{\mathrm{Div}}(F_0)$ such that $-\widehat{\deg}_{F_0}(P_\Omega) = \overline{\log \nu}_{\mathring{A}^{\otimes i}_{K,\underline{V},p}}(\Omega)$.

The "indeterminacies" Ind1, Ind2, Ind3 are artifacts of the anabelian interpretation of Ω_{P_Θ} in a structure which is essentially an infinite collection of local galois groups of fields and global arithmetic fundamental groups of hyperbolic curves.

The first notation $\langle \mathrm{Ind1}, \mathrm{Ind2} \rangle$ denotes a subgroup of $\mathbb{Q}_p$-vector space automorphisms in $\prod_{\underline{v} \in \underline{V}} \mathrm{Aut}_{\mathbb{Q}_p}(L_{\underline{v}} : \mathcal{I}_{\underline{v}})$. The group $\mathrm{Aut}_{\mathbb{Q}_p}(L_{\underline{v}} : \mathcal{I}_{\underline{v}})$ is the collection of automorphism of $L_{\underline{v}}$ which fix the log shell $\mathcal{I}_{\underline{v}}$ set-wise (or equivalently fix $\log(\mathcal{O}^{\times}_{K_{\underline{v}}})$).[13]

The first indeterminacy, Ind1, the the "procession normalization" indeterminacy.

The second indeterminacy Ind2 are the isometry indeterminacies and has to do with the fact that in algebraically closed p-adic fields, the galois invariant images of the logarithm are the not the same as images of galois invariants. These are an artifact that in IUT proper our measure spaces are interpreted from interpretations of log-shells in certain topological group (by perfecting–i.e. tensoring certain discrete subgroups up to $\mathbb{Q}$) The bullet $\cdot$ is meant to indicate that there is an action of $\mathrm{Aut}_{\mathbb{Q}}(L_{\underline{v}}/\mathcal{I}_{\underline{v}})$ on $\mathbf{L}$. We remark that this representation is actually a product of the representations on each of the factors $\mathbb{A}^{\otimes j+1}_{K,\underline{V},p}$ (in fact, in the anabelian interpretation, the galois acting on each of the factors of $\mathbf{L}$ are "synchronized"—this is important for reconstructing the product rule, which determines how the local places interact.).

The indeterminacy ind 3 is the log-link indeterminacy which "permutes $\mathcal{O}_{K_{\underline{v}}}$ and $\log(\mathcal{O}^{\times}_{K_{\underline{v}}})$" for every $\underline{v} \in \underline{V}$. This is not a literal permutation action but rather a non-commutativity of diagrams. This action is responsible for our having to consider log-shells.

In the next several sections we explain how to associate to a Arakelov divisors various Fractional ideal interpretations. In IUT proper, the constructions below correspond the the LGP versions of an lgp divisor.

5. Remarks on Mochizuki's interpretation of the Theta pilot divisor

♠♠♠ Taylor: [This needs to have the correct form at the good primes. This will get rid of the other term]

♠♠♠ Anton: [This is a more or less word-for-word transcription of the handwritten notes. I will edit this later, especially the references and also the formatting.]

This is [Moc15c, Definition 3.8]. This will clarify why locally we are looking at regions of the form

$$(\star\star\star) \qquad q^{j^2}_{\underline{v}_j} \mathcal{I}_{(\underline{v}_0 \cdots \underline{v}_j)}$$

where only the Tate parameter appears. ♠♠♠ Taylor: [This, we explained. We actually don't consider a region like this but it is forced on us. And, actually, we don't consider this but a theta pilot object.] ♠♠♠ Taylor: [The region here needs to be fixed.]

[13] ♠♠♠ Taylor: [Insert proof of equivalence here.]

The goal of this section is to exposit the construction of the theta pilot object P_Θ together with its embeddings.

$$P_\Theta \in \prod_{j\in \mathbf{F}_l^*} (\mathcal{F}_{\mathrm{mod}}^{\circledast})_j \text{ or } P_\Theta^{\mathbf{R}} \in \prod_{j\in \mathbf{F}_l^*} (\mathcal{F}_{\mathrm{mod}}^{\circledast\mathbf{R}})_j .$$

Here the Frobenioid $\mathcal{F}_{\mathrm{mod}}^{\circledast}$ has one object in its base category so it is completely determined by the monoid of Arakelov divisors associated to the field F_{mod}. To give a proper exposition we require three things:

(1) An explanation of the local embeddings

$$\mathcal{C}_{\mathrm{lgp}}^{\Vdash} \hookrightarrow \prod_{j\in \mathbf{F}_l^*} (\mathcal{F}_{\mathrm{mod}}^{\circledast\mathbf{R}})_j .$$

This is [Moc15b, Remark 4.8.i]

(2) The monoids $\Psi_{\mathcal{F}_{\mathrm{LGP},\underline{v}}}(\mathrm{HT})$; elements of local fractional ideals generating monoids. This is [Moc15c, Proposition 3.4.ii] .

(3) The definition of the localizations $(\rho_{\underline{\mathrm{lgp}},\underline{v}})_{\underline{v}\in\underline{V}}$. This is supposed to be [**?**, Prop. 5.3] but this reference is impossible so we refer to [Moc15a].

‘

6. Logarithms, log-shells, and log-links

The logarithm is ubiquitous in Mochizuki's theory.

6.1. **A review of the p-adic logarithm: basic properties.** A good reference for the statements I give in this section is [Kob12, Chapter 4]. The p-adic logarithm and exponential are initially defined as series

$$\begin{aligned}
\log(x) &:= -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} \in \mathbf{Q}[[1-x]] \\
\exp(x) &:= \sum_{n=0}^{\infty} \frac{x^n}{n!} \in \mathbf{Q}[[x]].
\end{aligned}$$

These power series then define p-adic analytic functions on subdomains of $\mathbf{C}_p$. Recall that for a series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbf{C}_p[[x]]$ the radius of convergence r is given by the formula $1/r = \limsup_{n\to\infty} |a_n|^{1/n}$.

Lemma 6.1.1. *(1) The disc of convergence of* $\log$ *is* $D(1,1^-) \subset \mathbf{C}_p$.
(2) The disc of convergence of $\exp$ *is* $D(0, p^{-1/(p-1)-})$.

Proof. (1) In the case of the logarithm we have $|a_n|^{1/n} = |1/n|^{1/n} = |p|^{\mathrm{ord}_p(n)/n}$ and achieves its supremum along the subsequence $n = p^m$. This implies $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{m\to\infty} |\frac{1}{p^m}|^{1/p^m} = \lim_{m\to\infty} p^{\mathrm{ord}_p(p^m)/p^m} = 1$. The series diverges for $|x| > 1$.

(2) (Sketch) In the case of the p-adic exponential we have

$$\begin{aligned}\mathrm{ord}_p(n!) &= \frac{n - S_n}{p-1} \\ S_n &= (\text{sum of digits of } n \text{ in base } p\ )\end{aligned}$$

This leads to the computation $r = 1/p^{1/(p-1)}$.

□

For applications it is of particular interest to us to understand the image of sets like $\mathcal{O}_K^\times \subset \mathbf{C}_p$ under the p-adic logarithm where $K/\mathbf{Q}_p$ is some finite extension (think very ramified).

Warning 6.1.2. If $x \in D(1, 1^-) = \mathrm{domain}(\sum_{n=1}^\infty \frac{(1-x)^n}{n}) \subset \mathbf{C}_p$ it is not necessarily the case that $\log(x) \in D(0, p^{-1/(p-1)-}) = \mathrm{domain}(\sum_{n=0}^\infty \frac{x^n}{n!})$!!! This means there are points in the domain of the p-adic logarithm which can't be applied.

If we restrict the domain of the logarithm we do have inverses.

Lemma 6.1.3. *The functions*

$$\begin{aligned}\exp : D(0, p^{-1/(p-1)-}) \to D(1, p^{-1/(p-1)-}) \\ \log : D(1, p^{-1/(p-1)-}) \to D(0, p^{-1/(p-1)-})\end{aligned}$$

are mutually inverse to each other on the indicated domains in $\mathbf{C}_p$. *We also have* $\exp(a+b) = \exp(a)\exp(b)$ *and* $\log(ab) = \log(a) + \log(b)$ *on these domains.*

The logarithm is extended to domains outside their radius of convergence by demanding that $\log(ab) = \log(a) + \log(b)$.

6.2. **The size of the p-adic logarithm of a number.** The following example illustrates the how uniformizers of very ramified extensions can produce "weird images" under the p-adic logarithm: One of the goals of this section is to establish some bounds on $\log(\mathcal{O}_K^\times)$ (both in terms of it's p-adic absolute value, and in terms of $\mathcal{O}_K$-modules which contain it). The difficulty here will be when K a ramified extension.

Remark 6.2.1. Given a power series $\sum_{n=1}^\infty a_n$ where a_n live is a field K which is complete with respect to a nonarchimedean absolute value $|-|_K$ we can estimate the absolute value of the series by taking the supremum of the absolute values of the coefficients. In other words $|\sum_{n=1}^\infty a_n| < \sup_n |a_n|$.

To see this, note that if $A = \sum_{n=0}^\infty a_n$ is a convergent p-adic series we can take partial sums $S_N = \sum_{n=0}^N a_n$ and with tail $T_N = \sum_{n>N} a_n$ We have $A = S_N + T_N$ and since T_N is nearly zero we have $|A| = |S_N|$ for n sufficiently large. We can then obtain an upper bound on A by taking $\sup |a_n|$. If the absolute values of the coefficients are distinct then $|A| = \sup_n |a_n|$.

Lemma 6.2.2. *Let* $a \in \mathbf{C}_p$ *have* $|a|_p < 1$. *We have the following estimates:*

(1) $|\log(1+a)|_p \leq \frac{c_p}{\mathrm{ord}_p(a)}$, *where* $c_p = (\exp(1)\ln(p))^{-1}$.

(2) $|\log(1+a)|_p \leq b_p \frac{|a|_p}{\mathrm{ord}_p(a)}$, *where* $b_p = \frac{1}{\ln(p)e^{\ln(p)^2}}$.

Remark 6.2.3. When $\operatorname{ord}_p(a) > 1/\ln(p)^2$ the estimate proportional to $|a|/\operatorname{ord}_p(a)$ is better. Otherwise, the other one performs better. We will be interested in applying this to $\operatorname{ord}_p(1+\pi)$ where π is the uniformizer of some finite extension of $\mathbb{Q}_p$. This means the second estimate is only useful controlling the size of $\log(\mathcal{O}_K^\times)$ for fields with bounded ramification.

♠♠♠ Taylor: [Are we using the correct estimate?]

Proof. To get an upper bound on $|-\log(1-a)| = |\sum_{n\geq 1} \frac{a^n}{n}|$ for $|a| < 1$ it suffices to compute $\max |a^n/n|$. Equivalently, we can compute the minimum of $\operatorname{ord}_p(a^n/n)$.

We find these two lower bounds by using

$$\operatorname{ord}_p(a^n/n) = n\operatorname{ord}_p(a) - \operatorname{ord}_p(n) \geq n\operatorname{ord}_p(a) - \log_p(n)$$

and

$$\operatorname{ord}_p(a^n/n) = p^m \operatorname{ord}_p(a) - m.$$

along the sequence $n = p^m$.

In the first case we minimize the function

$$f(x) = xc - \log_p(x)$$

using calculus. The function has global minimum $x_0 = 1/c\ln(p)$ which gives

$$f(x) \geq f(x_0) = \frac{1}{\ln(p)} + \log_p(c\ln(p)).$$

plugging in this result into the absolute values gives our result.

In the second case we can minimize the function

$$f(x) = p^x c - x.$$

It is minimized at $x_0 = -\log_p(c\ln(p))$ which gives

$$f(x) \geq f(x_0) = c\ln(p) + \log_p(c\ln(p)).$$

Note that when $c = 1/\ln(p)^2$ the two estimates are equal. In particular if $c > 1/\ln(p)^2$ then $c\ln(p) > 1/\ln(p)$. If $c < 1/\ln(p)^2$ then $c\ln(p) < 1/\ln(p)$. ☐

The following upper bounds on the domain of the p-adic logarithm are used when taking tensor packets of log-shells.

Lemma 6.2.4. *Let $K/\mathbb{Q}_p$ be a finite extension of $\mathbf{Q}_p$ of ramification degree e and uniformizer π. If $e < p-1$ then $\log(\mathcal{O}_K^\times) = \pi\mathcal{O}_K$. In general*

$$\log(\mathcal{O}_K^\times) \subset D_K(0, c_p e(K/\mathbb{Q}_p))$$

where $c_p = (\exp(1)\ln(p))^{-1}$. Here $\exp(1) = 2.71828182\ldots$ is Euler's constant. Note in particular that if K is unramified then $\mathcal{I}_K = \mathcal{O}_K$.

Proof. We will prove the first statement for $e < p-1$ and then prove the second statement.

We first show the inclusion $\log(\mathcal{O}_K^\times) \subset \pi\mathcal{O}_K$. From the explicit description of one has $\mathcal{O}_K^\times \cong (\mathcal{O}_K^\times)_{\mathrm{tors}} \cdot (1+\pi\mathcal{O}_K)$. ♠♠♠ Taylor: [FIXME]

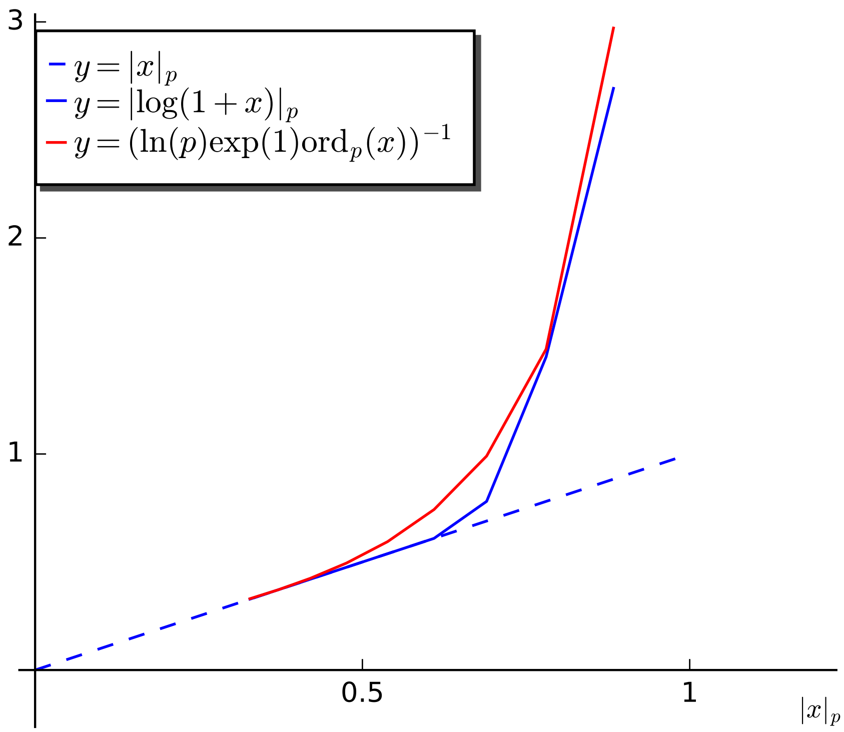

FIGURE 1. A plot of $|\log(1+x)|_p$ and its upper bound as a function of $|x|_p$. One can see that for $|x| < r_p$ one has $|\log(1+x)|_p = |x|_p$ beyond this point is where the inequality kicks in. Here we have used $e(K/\mathbb{Q}_p) = 13$ and $p = 5$.

We now show the inclusion $\pi\mathcal{O}_K \subset \log(\mathcal{O}_K^\times)$. Since $\pi\mathcal{O}_K \subset D(0, |p|^{1/(p-1)-})$, for all $\pi a \in \pi\mathcal{O}_K$ the series $\exp(\pi a)$ makes sense, an element of K and has the property that $\log(\exp(\pi a)) = \pi a$.

(2) This follows directly from Lemma 6.2.2.1 with π being the uniformizer of $\mathcal{O}_K$: $p^{-b} = \frac{1}{e\ln(p)\,\mathrm{ord}_p(\pi)}$.

□

Remark 6.2.5. In [Moc15d, Proposition 1.2] Mochizuki states gives a similar bound with

$$\log(\mathcal{O}_K^\times) \subset p^{-b}\mathcal{O}_K$$

where he takes $b = \lceil \lfloor 1 + \frac{\log(e/(p-1))}{\log(p)} \rfloor - \frac{1}{e} \rceil$. Figure 6.2 shows our bounds in relation to some numerical data for the p-adic logarithm.

♠♠♠ Anton: [This replaces the previous discussion on the iterates of the logarithm. We used to have the result that iterating the logarithm results in the empty set, but this result is wrong.]

6.3. log-shells.

For anabelian reasons it makes sense to define log shells.

Definition 6.3.1. Let K be a finite extension of $\mathbf{Q}_p$. The **log-shell** of K is the set

$$\mathcal{I} = \mathcal{I}(K) := \frac{1}{2p}\log(\mathcal{O}_K^\times) \subset K.$$

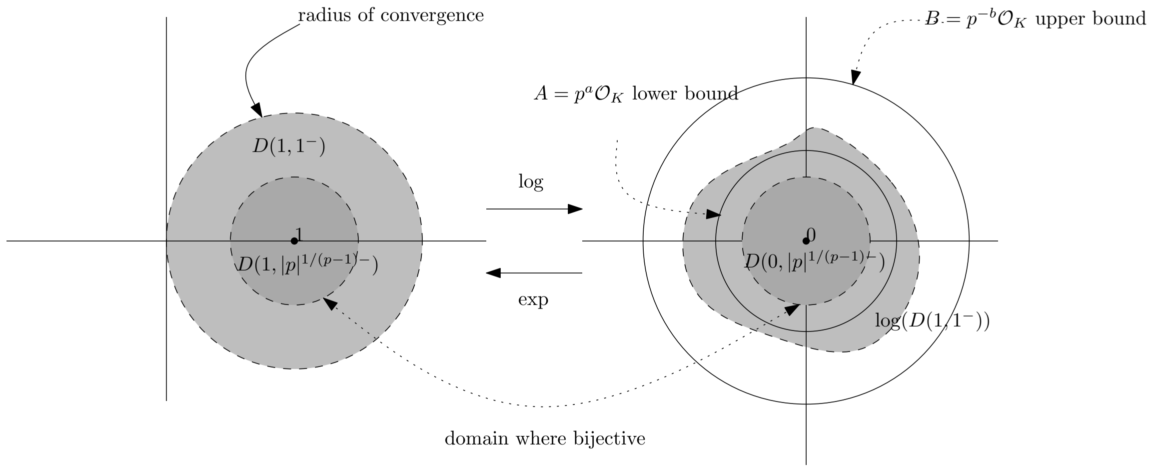

FIGURE 2. Illustration of domains in the p-adic logarithm. ♠♠♠ Taylor: [This needs to be fixed. Really.]

The following property is important for log-links.

Lemma 6.3.2. *For K an algebraic extension of $\mathbb{Q}_p$ we have $\mathcal{O}_K$ and $\log(\mathcal{O}_K^\times)$ contained in $\mathcal{I}_K$.*

Proof. It is clear that $\log(\mathcal{O}_K^\times) \subset \mathcal{I}_K$. Conversely, we have $\log_K(1+2p\mathcal{O}_K) = 2p\mathcal{O}_K$ since $\mathrm{ord}_p(2p) > 1/(p-1)$ (this is the condition for an element to be in the domain of bijectivity). This implies

$$\mathcal{I}_K \supset \frac{1}{2p}\log(1+2p\mathcal{O}_K) \supset \frac{1}{2p}(2p\mathcal{O}_K) = \mathcal{O}_K.$$

□

This property is important in resolving the so-called log-link indeterminacy which is one of the main indeterminacies in Mochizuki's inequality. We will also be interested in p-adic measures of $\mathbb{Z}_p$-tensor products of log shells

$$\mathcal{I}_{K_1} \otimes \cdots \otimes \mathcal{I}_{K_n} \subset K_1 \otimes \cdots \otimes K_n$$

for a fixed collection $K_1, \ldots, K_n$ of finite extensions of $\mathbb{Q}_p$.

6.4. **The p-adic measure of sets under the image of the logarithm, and tensor products of those sets.**

Lemma 6.4.1. *Let $K/\mathbf{Q}_p$ be a finite extension of degree d, ramification index e and residue degree f. Let $p^f = \#\mathcal{O}_K/m_K$ and let $p^m = \#\mu_{p^\infty}(\mathcal{O}_K)$.*

(1) ([Moc15e, Proposition 5.4.iii]) If $A \subset \mathcal{O}_K^\times$ is a subset where $A \to \log(A)$ is bijective then $\mu_K(A) = \mu_K(\log(A))$.
(2) ([Moc15e, Proposition 5.8.iii]) $\mu_K(\log(\mathcal{O}_K^\times)) = p^{-f-m}$
(3) Suppose $p \neq 2$. We have $\mu_K(\mathcal{I}) = p^{d-f-m}$.

Proof. (1) (c.f. [Moc15e, Proposition 5.7.i.c]) Due to the disconnectedness of the topology of K every set may be written as a disjoint union $A = \coprod_{i\in I}(a_i + m_K^{n_i})$, so by additivity of the Haar measure vol is suffices to show this for sets of the form $a + m_K^n$ with n sufficiently large. But on these sets we have $\log(x + m_K^n) = \log(x) + m_K^n$ and equality follows.

(2) We use the equality

$$\log(\mathcal{O}_K^\times) = \log(\mu_{p^f-1}(\mathcal{O}_K)\cdot(1+\pi\mathcal{O}_K)) = \log(1+\pi\mathcal{O}_K)$$

We will decompose $1+\pi\mathcal{O}_K$ into c_N cosets of $1+\pi^N\mathcal{O}_K$ and use the fact that the for large N we have $\log(1+\pi^N\mathcal{O}_K) = \pi^N\mathcal{O}_K$: Let

$$Q_N = (1+\pi\mathcal{O}_K)/(1+\pi^N\mathcal{O}_K).$$

Observe that this set is in bijection with $\mathcal{O}_K/\pi^{N-1}$ which has cardinality $p^{(N-1)f}$. We will use this fact later.

Let $\{q_1, q_2, \ldots, q_{c_N}\} \subset 1+\pi\mathcal{O}_K$ be a set of coset representatives for Q_N, i.e. $1+\pi\mathcal{O}_K = \bigcup_{j=1}^{c_N} q_j(1+\pi^N\mathcal{O}_K)$. Taking p-adic logs gives

$$\log(1+\pi\mathcal{O}_K) = \bigcup_{j=1}^{p^{(N-1)f}} \left(\log(q_j) + \log(1+\pi^N\mathcal{O}_K)\right).$$

Label the representatives $\{q_1,\ldots,q_{c_N}\}$ so that $\mu_{p^\infty}(\mathcal{O}_K) = \{q_1,\ldots,q_{p^m}\}$ we get

$$\begin{aligned}\log(1+\pi\mathcal{O}_K) =&\bigcup_{j=1}^{p^m}(\log(q_j)+\log(1+\pi^N\mathcal{O}_K)) \cup \bigcup_{j=p^m+1}^{c_N}(\log(q_j)+\log(1+\pi^N\mathcal{O}_K))\\ =&\log(1+\pi^N\mathcal{O}_K)\cup\bigcup_{j=p^m+1}^{c_N}(\log(q_j)+\log(1+\pi^N\mathcal{O}_K))\\ =&\pi^N\mathcal{O}_K\cup\bigcup_{j=p^m+1}^{c_N}(\log(q_j)+\pi^N\mathcal{O}_K)\\ =&\bigcup_{q\in Q_N/\mu_{p^\infty}(\mathcal{O}_K)}(\log(q)+\pi^N\mathcal{O}_K).\end{aligned}$$

Taking volumes gives

$$\begin{aligned}\mu_K(\log(1+\pi\mathcal{O}_K)) &= \mu_K\left(\bigcup_{q\in Q_N/\mu_{p^\infty}(\mathcal{O}_K)}(\log(q)+\pi^N\mathcal{O}_K)\right)\\ &= \frac{\#Q_N}{\#\mu_{p^\infty}(\mathcal{O}_K)}\mu_K(\pi^N\mathcal{O}_K)\\ &= \frac{p^{(N-1)f}}{p^m}p^{-Nf}\\ &= p^{-f-m}\end{aligned}$$

On the last line we used that $\#\mu_{p^\infty}(\mathcal{O}_K) = p^m$, $\mu_K(\pi^N \mathcal{O}_K) = |\pi^N| = p^{-N/e}$ and $\#Q_N = p^{(N-1)f}$.

(3) We have $\mu_K(\mathcal{I}) = |2p|_K^d \mu_K(\log(\mathcal{O}_K^\times) = p^d p^{-m-f}$. The volume of $\log(\mathcal{O}_K^\times)$ was given in Lemma 6.4.1.

□

Lemma 6.4.2. *For $r > e/(p-1)$ we have $\mu_{p^\infty}(K) \cap (1+\pi^r \mathcal{O}_K) = 1$.*

Proof. The function $\log(x)$ is a bijection from $D(1, |p|^{1/(p-1)}) \to D(0, |p|^{1/(p-1)})$. Since p-power roots of unity are torsion they can't be in this domain. Note that

$$|\pi^r| < |p|^{1/(p-1)} \iff |p|^r = |\pi^{re}| < |p|^{e/(p-1)} \iff r > e/(p-1).$$

□

6.5. **$\mathcal{O}_K$-module structures of log-shells.** The ideas here are discussed in [Moc15e, Remark 5.8.1] In Mochizuki's reconstruction inequality, one of the steps is take a reconstructed p-adic region and obtain a module from it. This means we need to ask the following question about finite extensions K of $\mathbb{Q}_p$: When is $\log(\mathcal{O}_K^\times) \subset \mathcal{O}_K$ an $\mathcal{O}_K$ submodule? When is it a $\mathbf{Z}_p$-module?

♠♠♠ Taylor: [This question as there are certain regions which can be reconstructed in an anabelian way, but which aren't represented by Frobenioid elements.] We will address these questions now. For $\log(\mathcal{O}_K^\times) = \log(1+\pi\mathcal{O}_K)$ to be an $\mathcal{O}_K$-module we need: for all $a \in \mathcal{O}_K$ and all $b \in 1+\pi\mathcal{O}_K$ we need

$$a \cdot \log(b) = \log(b^a).$$

To find this we need to determine when $b^a = (1+\pi b_0)^a$ makes sense. This amounts to studying the binomial series $B_{a,p}(x) = \sum_{n=0}^\infty \frac{a(a-1)\cdots(a-n+1)}{n!} x^n \in \mathbf{Q}(a)[[x]]$.

Lemma 6.5.1. *The expression*

$$b^a := \sum_{n=0}^\infty \frac{a(a-1)\cdots(a-n+1)}{n!}(b-1)^n$$

converges for $a, b \in \mathbf{C}_p$ such that

(1) $|a| > 1$ and $|b-1| < p^{-1/(p-1)}/|a|$
(2) $|a| \leq 1$ and $|b-1| < p^{-1/(p-1)}$
(3) $a \in \mathbf{Z}_p$, and $|b| < 1$.

Proof. (1) Since $|a-i| = |a|$ we have $|n\text{th term}| = |ax|^n/|n!|$ which implies $B_{a,p}(x)$ has a disc of convergence $D(0, p^{-1/(p-1)}/|a|^-)$ (just like the exponential series).

(2) $|a-i| \leq 1$ which implies $|\frac{a(a-1)\cdots(a-n+1)}{n!}x^n| \leq |x^n/n!|$ so the disc of convergence contains the one for the exponential series $D(0, p^{-1/(p-1)})$.

(3) (Sketch) One shows that $B_{a,p} \in \mathbf{Z}_p[[x]]$. Then generally the region of convergence contains $D(0, 1^-)$.

□

We observe that the second and third bounds in Lemma 6.5.1 are not optimal.

Lemma 6.5.2. *Let K be a finite extension of $\mathbf{Q}_p$.*

(1) $\log(\mathcal{O}_K^\times)$ is a $\mathbf{Z}_p$-module.

(2) If $e < p-1$ then $\log(\mathcal{O}_K)$ has the natural structure of a $\mathcal{O}_K$-module.

Proof. (1) Since $\log(\mathcal{O}_K^\times) = \log(1+\pi\mathcal{O}_K)$ we need to show that for all $1+\pi b_0 \in 1+\pi\mathcal{O}_K$ and all $a \in \mathbf{Z}_p$ that $(1+\pi b_0)^a$ makes sense. This is exactly what Lemma 6.5.1 says.

(2) This follows from the identity $\log(\mathcal{O}_K^\times) = \pi\mathcal{O}_K$. Alternatively, one may argue by $a \cdot \log(1+\pi b_0) = \log((1+\pi b_0)^a)$ making sense by since $|\pi b_0| < p^{-1/(p-1)}$ via the bound on the radius of convergence given in Lemma 6.5.1.

□

6.6. **Hulls.** Note that in particular $\log(\mathcal{O}_{\mathbf{C}_p}^\times)$ is not a $\mathcal{O}_{\mathbf{C}_p}$-module while it is a $\mathbf{Z}_p$-module — so the arithmetic of the ring of constants of module structure matters. In particular if K has large ramification degree one does not in general know that $\log(\mathcal{O}_K^\times)$ is an $\mathcal{O}_K$-module.

Also hulls are needed to compare mono-analytic adelic regions to etale-like adelic regions via an abstract isomorphism of Frobenioids. The particular Frobenioids in question encode fractional ideals hence it is necessary to turn our mono-analytic region into a fractional ideal which can be passed to the other side of the theta link.

Definition 6.6.1. Let $L = \prod_{i\in I} L_i$ where for each $i \in I$ and L_i is a finite extension of $\mathbf{Q}_{p_i}$ for some rational prime p_i. Suppose that L_i has uniformizer α_i. Let $\Omega \subset L$ be a compact subset. We define the **hull** of Ω to be

$$\mathrm{hull}(\Omega) = \bigcap \{\prod_{i\in I} p_i^{n_i}\mathcal{O}_{L_i} : n_i \in \mathbf{Z} \text{ and } \prod_{i\in I} p^{n_i}\mathcal{O}_{K_i} \supset \Omega\}$$

Define the **minimal module** to be

$$\mathrm{hull}^*(\Omega) = \bigcap \{\prod_{i\in I} \alpha_i^{n_i^*}\mathcal{O}_{L_i} : n_i^* \in \mathbf{Z} \text{ and } \prod_{i\in I} \alpha_i^{n_i^*}\mathcal{O}_{L_i} \supset \Omega\}.$$

Lemma 6.6.2. *Let $L = K_1 \otimes \cdots \otimes K_n$ be a tensor product of finite extensions of a complete discretely valued field K_0. Suppose that $L \cong L_1 \oplus \cdots \oplus L_m$ with each L_i a field and considered with its l^∞-norm. Let α_i be the uniformizer of L_i for $1 \le i \le m$.*

(1) $\pi_1^{a_1}\mathcal{O}_{K_1} \otimes ... \otimes \pi_n^{a_n}\mathcal{O}_{K_n} \subset \{y \in L : \|y\|_\infty < p^{-\sum_{i=1}^n a_i/e_i}\}$

(2) $\pi_1^{a_1}\mathcal{O}_{K_1} \otimes ... \otimes \pi_n^{a_n}\mathcal{O}_{K_n} \subset \bigoplus_{i=1}^m \alpha_i^{b_i}\mathcal{O}_{L_i}$ where $b_j = \lfloor e(L_j/\mathbb{Q}_p)\sum_{i=1}^n \frac{a_i}{e(K_i/\mathbb{Q}_p)}\rfloor$

Proof. We will show that each elementary term $x_1 \otimes \cdots \otimes x_n \in \pi_1^{a_1}\mathcal{O}_{K_1} \otimes \cdots \otimes \pi_n^{a_n}\mathcal{O}_{K_n}$ is contained in $D = \{y \in \mathbb{C}_p : \|y\|_\infty < p^{-\sum_{i=1}^n a_i/r_i}\}$ Since D is closed under addition, this will prove the general result.

Recall from Lemma 2.11.3 that $K_1 \otimes \cdots \otimes K_n = \bigoplus_{j=1}^m L_j$ where each factor L_j is the compositum $\psi_1(K_1)\cdots\psi_n(K_n)$ for some collection of embeddings $\psi_i : K_i \to \overline{K_0}$ for $1 \le i \le n$. Hence, it suffices to show that for each collection of embeddings ψ_i the image of $x_1 \otimes \cdots \otimes x_n$ in this compositum satisfies this inequality. If we take the order with respect to p, we get

$$\mathrm{ord}_p(\psi_1(x_1)\cdots\psi_n(x_n)) = \sum_{i=1}^n \mathrm{ord}_p(\psi_i(x_i)) \ge \sum_{i=1}^n \mathrm{ord}_p(\pi_i^{a_i}) = \sum_{i=1}^n \frac{a_i}{e(K_i/K_0)}.$$

This second item just comes considering $D \cap L$; The number b_j is the smallest integer such that

$$|\alpha_j^{b_j}|_p < p^{-\sum_{i=1}^n a_i/e(K_i/K_0)}.$$

This gives $b_j = \lfloor e(L_j/\mathbb{Q}_p) \sum_{i=1}^n a_i/e(K_i/K_0) \rfloor$. □

6.7. **Log-links.** The present subsection exposits the notion of log-linked fields. This notion was introduced in [Moc15e, Definition 5.4.ii] (which appears again in [Moc15c, 2nd page of introduction]).

Remark 6.7.1. In [Moc15c] it appears as if Mochizuki defines log-links starting with monoids. By Lemma 6.8.1 one can upgrade the monoid to a field.

Let field $K/\mathbb{Q}_p$ an algebraic extension viewed as an Ind-topological ring. We will defined a new field $K_{\log}$ with a new addition and multiplication which will be called the log-linked copy of K. To do this we first observe that the logarithm

$$(\mathcal{O}_K^\times, *_K) \to (\log_K(\mathcal{O}_K^\times), +_K)$$

can be factored as in the following way:

(6.1)
$$\begin{tikzcd}
(\mathcal{O}_K^\times, *_K) \arrow[rr, "\log_K"] \arrow[dr, "\log_K^{\mathrm{pre}}"'] & & (\log_K(\mathcal{O}_K^\times), +_K) \ . \\
& (\mathcal{O}_K^{\times\mu}, *_K) \arrow[ur, "\log_K^{\mathrm{post}}"'] &
\end{tikzcd}$$

Here $\mathcal{O}_K^{\times\mu} = \mathcal{O}_K^\times/(\text{torsion})$. We can extend the diagram in the following way:

$$\begin{tikzcd}
(\mathcal{O}_K^\times, *_K) \arrow[rr, "\log_K"] \arrow[dr, "\log_K^{\mathrm{pre}}"'] & & (\log_K(\mathcal{O}_K^\times), +_K) \arrow[r] & (\log_K(\mathcal{O}_K^\times) \otimes \mathbf{Q}, +_K) \arrow[r, "\sim"] & (K, +_K) \ . \\
& (\mathcal{O}_K^{\times\mu}, *_K) \arrow[ur, "\log_K^{\mathrm{post}}"'] \arrow[dr] & & & \\
& & (\mathcal{O}_K^{\times\,\mathrm{pf}}, *_K) \arrow[uur, "\log_K^{\mathrm{post}} \otimes \mathbb{Q}"'] & &
\end{tikzcd}$$

In this diagram $\mathcal{O}_K^{\times\,\mathrm{pf}} = (\mathcal{O}_K^\times, *) \otimes_{\mathbf{Z},+} (\mathbf{Q}, +)$, it is the so-called perfection of $\mathcal{O}_K^\times$ (note that this kills torsion). By abuse of notation we will use $\log^{\mathrm{post}}$ to also denote $\log^{\mathrm{post}} \otimes \mathbb{Q}$, and identify $(\log(\mathcal{O}_K^\times) \otimes \mathbb{Q}, +_K)$ with $(K, +_K)$ under the natural isomorphism between the two. In this notation we just have a map

$$\log^{\mathrm{post}} : (\mathcal{O}_K^{\times\,\mathrm{pf}}, *_K) \to (K, +_K).$$

We can now construct an isomorphic copy of the field $(K, +, *)$. Here we just pull back everything using the galois-equivariant group homomorphism $\log_K^{\mathrm{post}}$.

Definition 6.7.2. Given an algebraic extension K of $\mathbb{Q}_p$ we define the **log-linked copy** of K to be the field

$$(K_{\log}, +_{\log}, *_{\log}) = (\mathcal{O}_K^{\times\,\mathrm{pf}}, *_K, \log_K^{\mathrm{post}\,*}(*_K)).$$

That is, $K_{\log}$ is the set $\mathcal{O}_K^{\times\,\mathrm{pf}}$ with the field structure pulled back from the bijection $\log_K^{\mathrm{post}}$.

Remark 6.7.3. (1) To be explicit about the multiplication we have

$$x *_{\log} y = (\log^{\mathrm{post}})^{-1}(\log^{\mathrm{post}}(x)\log^{\mathrm{post}}(y)).$$

(2) The map $\mathcal{O}_K^\times \xrightarrow{\log^{\mathrm{pre}}} K_{\log}$ is not a field isomorphism!
(3) We define $\mathcal{O}_{\log,K} \subset K_{\log}$ to be $\log^{\mathrm{post}\,-1}(\mathcal{O}_K)$ where $\log^{\mathrm{post}} : K_{\log} \to K$ is an isomorphism of fields.

Remark 6.7.4. The diagram (6.1) is oddly reminiscent of diagram used to define the relative Frobenius. For a ring A over a field k of characteristic p one has a factorization of the absolute Frobenius via the relative Frobenius which fits into the following diagram:

$$\begin{tikzcd} A \arrow[rr, "x\mapsto x^p"] \arrow[dr, "x\mapsto x^{(p)}"'] & & A \\ & A^{(p)} \arrow[ur, "x^{(p)}\mapsto x^p"'] & \end{tikzcd},$$

Mochizuki views the log-link as analogous to this construction. In this analogy $\log_K^{\mathrm{post}}$ plays the role of the relative Frobenius. Observe that $\log_K^{\mathrm{post}}$ does nothing to torsion, which can be thought of as an $\mathbf{F}_1$-base.

Remark 6.7.5. The pre-logarithm on an algebraically closed field $\overline{k}$ is introduced in [Moc15e, Definition 3.1] under the notation $\log_{\overline{k}}$.

Given the construction of the log-link we now have two versions of log-shells we can consider which are "linked".

Definition 6.7.6. Let K be an algebraic extension of $\mathbb{Q}_p$. We Define the **pre-log shell** as

$$\mathcal{I}_K^{\mathrm{pre}} = (\mathcal{O}_K^{\times\mu})^{1/p}$$

Here we note that the construction of $\mathcal{I}_K^{\mathrm{pre}}$ takes place in $K_{\log}$, i.e.

$$(\mathcal{O}_K^{\times\mu})^{1/p} := \{x \in \mathcal{O}_K^{\times\,\mathrm{pf}} : x^p \in \mathcal{O}_K^{\times\mu}\}.$$

For the proposition we note that $K_{\log}$ itself is a topological ring, so it has its own power series functions and in particular, its own p-adic logarithm.

Lemma 6.7.7. *(1)* $\log^{\mathrm{pre}}(\mathcal{O}_K^\times) = \log_{K_{\log}}(\mathcal{O}_{K_{\log}}^\times)$
(2) $\mathcal{I}_K^{\mathrm{pre}} = \mathcal{I}_{K_{\log}}$.

Proof. We have an isomorphism of topological field $\log^{\mathrm{post}} : K_{\log} \to K$. This means

$$\log^{\mathrm{post}\,-1}(\log_K(\mathcal{O}_K^\times)) = \log_{K_{\log}}(\mathcal{O}_{K_{\log}}^\times).$$

On the other hand

$$\log^{\mathrm{post}\,-1}(\log_K(\mathcal{O}_K^\times)) = \mathcal{O}_K^{\times\mu} = \log^{\mathrm{pre}}(\mathcal{O}_K^\times).$$

This proves

$$\log^{\mathrm{pre}}(\mathcal{O}_K^\times) = \log_{K_{\log}}(\mathcal{O}_{K_{\log}}^\times).$$

The second assertion follows from the first given the interchange between addition and multiplication. □

6.8. **Mochizuki-Kummer maps.** ♠♠♠ Taylor: [Explain Mochizuki's way of doing this].

In this section we wish to define the so-called log-Kummer correspondence, a notion which is introduced in [Moc15c,] for the purpose of Mochizuki's Multiradiality Theorem. Let K be a finite extension of $\mathbb{Q}_p$.

At various stages we need the hypotheses of a curve being "strictly Belyi type" for anabelian purposes. Any *isogeny* between curves/schemes/stacks over K is a finite etale to finite etale correspondence. Let Z be a hyperbolic orbicurve (Deligne-Mumford stack). We say that it is *strictly Belyi type* if it is defined over a number field an isogenous to $\mathbb{P}^1_F \backslash \{0, 1, \infty\}$ where F is some number field.

We wish to the "Kummer map" construction cited in [Moc15c] during the construction of the log-kummer correspondence. There, the construction is given in terms of Frobenioids but as these are bi-interpretable with Monoid representations $(\Pi, M) \in [\Pi_Z, \mathcal{O}^*_{\overline{K}}]$ we work with these. This is implicit in Mochizuki's statements but can't explicitly give a reference to where this is stated.

The relevant construction is stated in [Moc15c, Proposition 3.5.i]. This leads to the following "reference chase":

- One finds the appropriate statement in [Moc15e, Proposition 3.2.iii]. This is formulated in the case-by-case language of **TM**-pairs.
- Mochizuki asserts here that the constructions in [Moc15e, Corollary 1.10 (c),(h)] can be performed when replacing cyclotomes $\mu_{\widehat{\mathbf{Z}}}(\Pi)$ with $\mu_{\widehat{\mathbf{Z}}}(G)$ or $\mu_{\widehat{\mathbf{Z}}}(M)$ (all versions of the Tate-module).
- In [Moc15e, Corollary 1.10 (c),(h)], Mochizuki cites interpretations performed in a number of other papers:
  - Corollary 1.10.a: [Moc04, Proposition 1.2.1]
  - Corollary 1.10.c = [Moc05, Theorem 4.3] + [Moc04, Lemma 2.5.i, ii]
  -
  - [**?**,]
  -
- Finally, within these final three older papers a number of constructions are performed. These amount to "classical" constructions say, relying on Serre's Chapter in group cohomology, the fundamental theorem of projective geometry and the theory of Weights found in Freitag.
- We finally remark that there are two ways of over interpreting the field k of Z if Z is strictly Belyi type.

An explanation of [Moc15e, Corollary 1.10] can be found in [Min16, slide 17]. See also [?]. We remark that there is nowhere which completely spells out the details of [Moc15e, Corollary 1.10] in a fashion that the authors can understand. In fact, the statements are spread over [Moc05], [Moc04]

We follow the discussion in [Moc15e, Definition 3.1, Proposition 3.2]

Theorem 6.8.1 (Frobenioid Interpretation)**.** *Let K be a finite extension of $\mathbb{Q}_p$. Let Z be a orbicurve defined over K. There is an interpretation*

$$[\Pi_Z, \mathcal{O}^*_{\overline{K}}] \to [\Pi_Z, \overline{K}]$$

$$(\Pi, M) \mapsto (\Pi, \overline{K}(M, \Pi))$$

where $\overline{K}(M,\Pi)$ is an ind-topological field and $\mathcal{O}^*_{\overline{K}(M,\Pi)}$ naturally identified with M as an ind-topological monoid. ♠♠♠ Taylor: [This last part about identifying M with $\mathcal{O}^*_{\overline{K}(M,\Pi)}$ needs to be checked.]

Proof. As explained in [Moc15e, Proposition 3.2.iii], the natural isomorphism $M \cong \mathcal{O}^*_{\overline{K}(M,\Pi)}$ follows from [Moc15c, Corollary 1.10 (h)]. Here $\overline{K}^{\mathrm{frob}}(\Pi, M)$ is defined to be the set

$$\mathrm{im}(M \to {}_\infty H^1(\Pi, T(M))) \cup \{0\}$$

with an interpreted addition and multiplication. □

We will use the notation $\overline{K}^{\mathrm{frob}}(\Pi, M)$ for the interpretation in Theorem 6.8.1 and call it the *Frobenius-like interpretation.*

Theorem 6.8.2 (Ètale Interpretation)**.** *Let K be a p-adic field. Let Z/K be an hyperbolic orbicurve of strictly Belyi type. There is an interpretation*

$$[\Pi_Z] \to [\Pi_Z, \overline{K}(Z)].$$

where $\overline{K}$ is understood to have its ind-topological field structure.

Proof. As explained in [Moc15e, Proposition 3.2.ii], the Key Lemma here is [Moc15e, Corollary 1.10 (c)] ♠♠♠ Taylor: [We need to construct $\overline{\mathcal{O}}^*(\Pi)$ and use the cyclotome $\mu(\Pi) = H^2(\Pi, \widehat{Z})$] □

We will use the notation $\overline{K}^{\mathrm{et}}(\Pi)$ for the interpreation in Theorem 6.8.2 and call it the *étale-like interpretation.*

Theorem 6.8.3 (Kummer Map)**.** *Let K be a p-adic field. Let Z/K be of strictly Belyi type. For $(M, \Pi) \in [\mathcal{O}^*_{\overline{K}}, \Pi_Z]$ there exists Π-equivariant isomorphisms*

$$\mathrm{kum} : \overline{K}^{\mathrm{frob}}(M, \Pi) \to \overline{K}^{\mathrm{et}}(\Pi),$$

of ind-topological fields which are functorial in (M, Π).

Proof. ♠♠♠ Taylor: [The proof is given by a change in cyclotomes, and I think, cyclotomic rigidity. I also think we need a cyclotomic rigidity.] □

Remark 6.8.4. The final version of Mochizuki's Kummer map is given in [Moc15c, Proposition 3.5.i] and is stated as an isomorphism in $[\pi_1^{temp}(\underline{\underline{X}}_{\underline{v}}), \mathcal{O}_{\overline{K_{\underline{v}}}}]\ \Psi_{\mathcal{F}_{\mathrm{LGP},\underline{v}}} \xrightarrow{\sim} \Psi_{\mathrm{LGP},\underline{v}}$. it can be enriched to an isomorphism of fields. The definition of $\mathcal{F}_{\mathrm{LGP}}$ is given in [Moc15c, Proposition 3.4.ii], which when you look-up boots you to the Kummer map [Moc15b, Corollary 4.6.i]. Here the Kummer map is map between

$$\Psi_{\mathrm{cns}}(\mathfrak{F}) \to \Psi_{\mathrm{cns}}(\mathfrak{D})$$

which themselves are collections maps of monoids $(\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})) \to (\Psi_{\mathrm{cns}}(\mathcal{D}_{\underline{v}}))$ constructed from prime strips $\mathfrak{F} = (\mathcal{F}_{\underline{v}})$ and $\mathfrak{D} = (\mathcal{D}_{\underline{v}})$. To understand these local maps we are asked to consult [Moc15b, Proposition 3.3.ii] and it seems to be that [Moc15b, Proposition 3.4.ii] is actually the relevant definition — which defines maps of these constant monoids by passage to monotheta environments.

6.9. Let K_i be finite extensions of $\mathbb{Q}_p$ for $1 \leq i \leq n$ of degree d_i. Let $L = K_1 \otimes \cdots \otimes K_n$ with its canonical Haar measure which is the product of Haar measures of its direct summands. In [Moc15c, Proposition 3.1.ii] and [Moc15c, Proposition 3.1.i] Mochizuki defined two distinct lattices inside (an interpreted copy of) L. The lattice $\log(\mathcal{O}_{K_1}^\times) \otimes \cdots \otimes \log(\mathcal{O}_{K_m}^\times)$ may be interpreted within absolute galois groups of p-adic fields (using local class field theory) while the lattices $\mathcal{O}_L$ may be interpreted within fundamental groups using the anabelian interpreations in [Moc15e, §1].

In [Moc15c, Proposition 3.9 (i) and (ii)] Mochizuki then considers the comparison between measures normalized on the interpreted Mochizuki measure spaces (there are 4 interpretations, and for each of these three variants of the Mochizuki measure space he considers) according to their values on theses lattices. The morphisms of these interpretated measure spaces is predicated on the fact that knowing the volume of one of these lattices automatically gives the volume of the other.

We have

$$\mu_L(\bigotimes \log(\mathcal{O}_{K_i}^\times) = \left(|Q_1|^{1/d_1} \cdots |Q_n|^{1/d_n}\right)^{d_1 \cdots d_n} \mu_L(\mathcal{O}_{K_1} \otimes \cdots \mathcal{O}_{K_n})$$

The proof of this is a standard trick in Haar measures by considering subgroups of finite index and using translation invariance. We illustrate this in the case $n = 2$. For $n = 2$ we will show

$$\mu_L(\log(\mathcal{O}_{K_1}^\times) \otimes \log(\mathcal{O}_{K_2}^\times)) = \mu_{K_1}(\log(\mathcal{O}_{K_1}^\times))\mu_{K_2}(\log(\mathcal{O}_{K_2}^\times))\mu_L(\mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2}).$$

Let $Q_i = \mathcal{O}_{K_i} / \log(\mathcal{O}_{K_i}^\times)$ for $i = 1, 2$. From the exact sequence

$$0 \to \log(\mathcal{O}_{K_2}^\times) \to \mathcal{O}_{K_2} \to Q_2 \to 0$$

we have, tensoring with the flat $\mathbf{Z}_p$-module $\log(\mathcal{O}_{K_1}^\times)$,

$$0 \to \log(\mathcal{O}_{K_1}^\times) \otimes \log(\mathcal{O}_{K_2}^\times) \to \log(\mathcal{O}_{K_1}^\times) \otimes \mathcal{O}_{K_2} \to \log(\mathcal{O}_{K_1}^\times) \otimes Q_2 \to 0.$$

Since $\log(\mathcal{O}_{K_1}^\times) \cong \mathbf{Z}_p^{d_1}$ as a $\mathbf{Z}_p$-module, where $d_1 = [K_1 : \mathbf{Q}_p]$, we have

$$\log(\mathcal{O}_{K_1}^\times) \otimes Q_2 \cong \mathbf{Z}_p^{d_1} \otimes Q_2 \cong Q_2^{d_1}.$$

Taking volumes, this gives us

$$\mu_L(\log(\mathcal{O}_{K_1}^\times) \otimes \mathcal{O}_{K_2}) = |Q_2|^{d_1} \mu_L(\log(\mathcal{O}_{K_1}) \otimes \log(\mathcal{O}_{K_2})).$$

In order to compute $\mu_L(\log(\mathcal{O}_{K_1}) \otimes \log(\mathcal{O}_{K_2}))$, we use of the exact sequence

$$0 \to \log(\mathcal{O}_{K_1}^\times) \to \mathcal{O}_{K_1} \to Q_1 \to 0$$

and we tensor with $\mathcal{O}_{K_2}$:

$$0 \to \log(\mathcal{O}_{K_1}^\times) \otimes \mathcal{O}_{K_2} \to \mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2} \to Q_1 \otimes \mathcal{O}_{K_2} \to 0$$

This gives us

$$\mu_L(\mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2}) = |Q_1|^{d_2} \mu_L(\log(\mathcal{O}_{K_1}) \otimes \mathcal{O}_{K_2}).$$

Combined with the earlier result we have

$$\mu_L(\mathcal{O}_{K_1} \otimes \mathcal{O}_{K_2}) = |Q_1|^{d_2} |Q_2|^{d_1} \mu_L(\log(\mathcal{O}_{K_1}) \otimes \log(\mathcal{O}_{K_2})).$$

One can do this with other lattices as well. For example

$$\mu_L(\bigotimes_{i=1}^n \mathcal{O}_{K_i}) = |\mathcal{O}_L / \bigotimes_{i=1}^n \mathcal{O}_{K_i}|^{-1}.$$

More generally this works by computing the index of one lattice in another. Point: to compute the volume of $\bigotimes_{i=1}^n \mathcal{O}_{K_i}$ is suffices to compute $|\mathcal{O}_L / \bigotimes_{i=1}^n \mathcal{O}_{K_i}|$.

7. Archimedean Considerations

We wish to address the following

- What are log-links? [Moc15e, Definition 5.4.v]
- What are log shells? [Moc15e, Definition 4.1.iv] gives the definition of pre-log shells.
- How are log shells reconstructed from Aut holomorphic spaces? [Moc15e, Proposition 4.2.i,ii]
- What are the archimedean contributions of $\overline{\ln \nu}(\text{hull}(U))$? This is [Moc15d, Proposition 1.5, Proof of Theorem 1.10 step vii] and [Moc15c, Proposition 3.9.iii]

We are going to omit Archimedean Kummer structures. A finer theory of Aut holomorphic spaces allegedly has something to do with non-Siegel zeros for zeta functions associated to certain Dirichlet characters.

7.1. Fix initial theta data $(F, E, C_F, l, \underline{V}, V_{mod}^{bad}, \underline{\epsilon})$. For all $\underline{v} \in \underline{V}$ with $\underline{v}|\infty$ we have $K_{\underline{v}} \cong \mathbf{C}$.

By construction $\sqrt{-1} \in F$ and hence $\sqrt{-1} \in K$. Since $K_{\underline{v}} \cong \mathbf{R}$ or $\mathbf{C}$ and we have $\sqrt{-1} \in K_{\underline{v}}$ we have $K_{\underline{v}} \cong \mathbf{C}$.

Hence to define log-links it suffices to do so for $\mathbf{C}$.

8. Holomorphic Aut Spaces

8.1. The subject of holomorphic aut-spaces (or what Mochizuki call Aut holomorphic spaces) is the subject of [Moc15e, §2, first part].

In what follows we will regard a Riemann Surface as a pair $X = (X_0, \mathcal{O}_X)$ consisting of a topological space X_0 and a sheaf of holomorphic functions $\mathcal{O}_X$. We will let RS denote the category of Riemann surfaces. We will let $\overline{\mathsf{RS}}$ denote the category of Riemann surfaces which holomorphic and anti-holomorphic morphisms.

In what follows if C is a category we let $\mathrm{Isom}_{\mathsf{C}}(X,Y) \subset \mathsf{C}(X,Y)$ denote the collection of isomorphisms and let $\mathrm{Aut}_{\mathsf{C}}(X)$ denote automorphisms in the category C. We use the following alternative notations:

- For RS, $\mathrm{Isom}_{\mathbf{C}} = \mathrm{Isom}_{\mathsf{RS}}$ and $\mathrm{Aut}_{\mathbf{C}} = \mathrm{Aut}_{\mathsf{RS}}$
- For $\overline{\mathsf{RS}}$, $\overline{\mathrm{Isom}}_{\mathbf{C}} = \mathrm{Isom}_{\overline{\mathsf{RS}}}$ and $\overline{\mathrm{Aut}}_{\mathbf{C}} = \mathrm{Aut}_{\overline{\mathsf{RS}}}$.

For a topological space X_0 we let $\mathrm{Aut}_{\mathrm{top}}(X_0)$ denotes the group of automorphisms of topological spaces. When the topology is relevant we give $\mathrm{Aut}_{\mathrm{top}}(X_0)$ the compact open topology.[14] We will also let $\mathrm{Open}(X_0)$ denote the collection of open subsets of X_0. There will be no need to regard it as a category.

8.2. We will now define the category of aut spaces AS. Later we will be interested in the category of holomorphic aut structures.

Definition 8.2.1. The category AS is spectified by the following:

- (Objects) An *aut structure* is a pair $X = (X_0, G_X)$ consisting of a topological space X together with a collection of subgroups $G_X(U) \subset \mathrm{Aut}_{\mathrm{top}}(U)$ for each open subset U of X_0. Here $\mathrm{Aut}_{\mathrm{top}}(U)$ is the collection of automorphisms in the category of topological spaces. Also we give $G_X(U) \subset \mathrm{Aut}_{\mathrm{top}}(U)$ the structure of a topological subspace using the compact open topology.
- (Morphisms) A morphism $X = (X_0, G_X) \to Y = (Y_0, G_Y)$ is a local homeomorphism of topological spaces $\phi_0 : X_0 \to Y_0$ such that if $U \subset X_0$ is a connected open such that $U \cong \phi_0(U)$ then the map
$$G_X(U) \to G_Y(\phi_0(U))$$
$$\psi \mapsto \phi_0 \psi \phi_0^{-1}$$
is an isomorphism of topological groups.

For the purposes of a developing the theory, one can define the notion of a local aut structure where everything is a relative to a certain basis of open sets. It will be useful to make use of this construction to prove certain theorems.

What follows is needed for a version of [Moc15e, 2.1.i] where Mochizuki shows it is sufficient to define holomorphic aut structures on certain open sets (he called theses Aut holomorphic structures and omits the notion of a general aut structure entirely).

[14] Recall that if A and B are topological spaces then the compact open topology on $\mathsf{Top}(A,B)$ is generated by open sets $M(K,U) = \{f : f(K) \subset U\}$ where $K \subset A$ is compact and $U \subset B$ is open.

Definition 8.2.2. Let B be a topological space. A basis of open sets $\mathcal{U} \subset \mathrm{Open}(B)$ will be called a *refined* if for all $U' \in \mathrm{Open}(B)$ and all $U \in \mathcal{U}$ if $U' \subset U$ then $U' \in \mathcal{U}$.

Mochizuki defines all his morphisms of aut holomorphic spaces in terms of refined bases calls a "local structure".

Definition 8.2.3. A *refined aut structure* is a collection $(X_0, \mathcal{U}, G|_{\mathcal{U}})$ where $\mathcal{U}$ is a refined basis, $G : \mathcal{U} \to \mathsf{Grp}$ is a function where $G(U) \subset \mathrm{Aut}_{\mathrm{top}}(U)$ for each $U \in \mathcal{U}$.

A *refined morphism* $X = (X_0, \mathcal{U}_X, G_X|_{\mathcal{U}}) \to Y = (Y_0, \mathcal{V}, G_Y|_{\mathcal{U}_Y})$ is a morphism of topological spaces $\phi_0 : X_0 \to Y_0$ such that

(1) (Local Homeomorphism) For all $x \in X_0$ there exist some $U \in \mathcal{U}_X$ such that $U \ni x_0$ such that $\phi_0|_{U_0} : U_0 \to \phi_0(U_0)$ is a homeomorphism and $\phi_0(U_0) \in \mathcal{U}_Y$.

(2) (Local Group Isomorphism) If $U \in \mathcal{U}_X$ is such that $\phi_0|_U$ is a local homeomorphisms with $\phi_0(U) \in \mathcal{U}_Y$ then the map
$$G_X(U) \to G_Y(\phi_0(U))$$
$$\psi \mapsto \phi_0 \psi \phi_0^{-1}$$
is an isomorphism of topological groups.

We will denote the category of refined aut structures by AS^*.

Note that every aut structure $X = (X_0, G_X)$ gives a refined aut structure $(X_0, \mathcal{U}_X, G_X)$ if we take $\mathcal{U}_X$ to be the collection of connected open subsets of X_0.

8.3. The point of aut-structures is that they can be quite strong.

Definition 8.3.1. The *holomorphic aut structure* associated to $\Sigma = (\Sigma_0, \mathcal{O}_\Sigma) \in \mathsf{RS}$ is the pair $\Sigma^{\mathrm{aut}} = (\Sigma_0, A_\Sigma)$ where Σ_0 is the underlying topological space and $A_X : \mathrm{Open}(\Sigma_0) \to \mathsf{Grp}$ is the function defined for $U \in \mathrm{Open}(\Sigma_0)$ by
$$A_\Sigma(U) = \{ f \in \mathrm{Aut}_{\mathrm{top}}(U) : f \text{ holomorphic } \}.$$

We will let $\mathsf{AS}^{\mathrm{hol}}$ and $\mathsf{AS}^{\mathrm{hol}*}$ denote the full subcategory of aut structures and refined aut structures whose objects are holomorphic at structures.

8.4. It turns out that this structure is just as strong as the original holomorphic structure on Σ^{top}. That is, one can detect holomorphic structures from its holomorphic aut-structure and conversely to every holomorphic aut-struture there is an aut structure.

Lemma 8.4.1 (Extension Lemma, [Moc15e, Corollary 2.3])**.** *Let Σ and Γ be Riemann surfaces with refined bases $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\Sigma^{\mathrm{aut}(\mathcal{U})}$ and $\Gamma^{\mathrm{aut}(\mathcal{V})}$ denote the associated refined aut structures. Every morphism of refined aut structures $\Sigma^{\mathrm{aut}(\mathcal{U})} \to \Gamma^{\mathrm{aut}(\mathcal{V})}$ extends uniquely to a morphism of holomorphic aut structures $\Sigma^{\mathrm{aut}} \to \Gamma^{\mathrm{aut}}$.*

We omit the proof.

8.5. **Automorphisms of the Unit Disc.** We are after [Moc15e, Proposition 2.2] which is about commensurable terminality of $\overline{\mathrm{Aut}}_{\mathbf{C}}(D) in\, \mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ where D is the unit disc.

Lemma 8.5.1. *Let $G_1 \subset G$ be a open subgroup of a topological group G. If G_1 is connected then* $\mathrm{Comm}_G(G_1) = N_G(G_1)$.[15]

Proof. Let $\alpha \in \mathrm{Comm}_G(G_1)$. And define $G_\alpha = \alpha G_1 \alpha^{-1} \cap G_1$. Consider the quotient map $q_\alpha : G_1 \to G_1/G_\alpha$. Since $[G : G_\alpha] < \infty$, the target of q_α is a finite set with the discrete topology. Also $q_\alpha^{-1}(1G_\alpha) = G_\alpha$. This implies that G_α is a closed subgroup of G_1. Since $G_\alpha = G_1 \cap \alpha G_1 \alpha^{-1}$, which is the intersection of two opens, it is an open subgroup of G_1. Since G_α is both open and closed and G_1 is connected this implies that $G_\alpha = G_1$ and hence $\alpha \in N_G(G_1)$. □

Lemma 8.5.2. *Let $\Sigma = (\Sigma_0, \mathcal{O}_\Sigma)$ be a Riemann surface. Give $\mathrm{Aut}_{\mathrm{top}}(\Sigma_0)$ the compact open topology and regard $\overline{\mathrm{Aut}}_{\mathbf{C}}(\Sigma), \mathrm{Aut}_{\mathbf{C}}(\Sigma) \subset \mathrm{Aut}_{\mathrm{top}}(\Sigma_0)$ as topological subspaces by given them the subspace topology. $\mathrm{Aut}_{\mathbf{C}}(\Sigma)$ and $\overline{\mathrm{Aut}}_{\mathbf{C}}(\Sigma)$ are closed subspaces of $\mathrm{Aut}_{\mathrm{top}}(\Sigma^{\mathrm{top}})$.*

Proof. A set if closed if and only if it contains limits of convergent sequences. Recall Montel's Theorem: Let $f_n : D \to \mathbf{C}$ be a sequence of holomorphic functions. If $f_n \to f$ uniformly on all $K \subset D$ compact then f is holomorphic.

This implies that $\mathrm{Aut}_{\mathbf{C}}(\Sigma) \subset \mathrm{Aut}_{\mathrm{top}}(\Sigma^{\mathrm{top}})$ is closed.

In general if $G_2 \subset G_1 \subset G$ are inclusions of topological groups and G_2 is closed and $[G_2 : G_1] < \infty$ then G_1 is closed. This is because for each $g \in G$ the left translation-by-g map $t_g : G \to G$ is a homeomorphism. This means gG_2 is closed. Since $G_1 = \bigcup_{i=1}^n g_i G_2$ is a finite union of closed sets, we have that G_1 is closed.

Now $\overline{\mathrm{Aut}}_{\mathbf{C}}$ differs from $\mathrm{Aut}_{\mathbf{C}}$ by complex conjugation so we are done. □

8.6. We now go on to characterize how normalizing elements of $\mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ act on $\mathrm{Aut}_{\mathbf{C}}(D)$ by conjugation. We will use that $\mathrm{Aut}_{\mathbf{C}}(D) \cong \mathrm{PSL}_2(\mathbf{R})$ and characterize these conjugation actions. We first reduce the action to saying that $\mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ is really the same as a conjugation action by $\mathrm{PGL}_2(\mathbf{C})$. We then go on to characterize the conjugation action of $\mathrm{PGL}_2(\mathbf{C})$ on $\mathrm{PSL}_2(\mathbf{R})$. It will turn out that such an action will be the same as the conjugation action by $\mathrm{PGL}_2^{\pm}(\mathbf{R}) \supset \mathrm{PSL}_2(\mathbf{R})$. This in turn is isomorphic to $\overline{\mathrm{Aut}}_{\mathbf{C}}(D)$ hence the normalizer of $\mathrm{Aut}_{\mathbf{C}}(D)$ in $\mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ will be $\overline{\mathrm{Aut}}_{\mathbf{C}}(D)$. This then (I think) will be normally terminal.

Lemma 8.6.1. *Consider the map*

$$\mathrm{conj} : \mathrm{Aut}_{\mathrm{top}}(D_0) \to \mathrm{Aut}(\mathrm{PSL}_2(\mathbf{R}))$$

given $\mathrm{PSL}_2(\mathbf{R}) \cong \mathrm{Aut}_{\mathbf{C}}(D) \subset \mathrm{Aut}_{\mathrm{top}}(D_0)$ and acting by conjugation. For each $\alpha \in \mathrm{Aut}_{\mathrm{top}}(D_0)$ (a potentially very large group) there exists some $\beta \in \mathrm{PSL}_2(\mathbf{C}) \supset \mathrm{PSL}_2(\mathbf{R})$ (a much smaller and more managable group) such that

$$\mathrm{conj}_\alpha = \mathrm{conj}_\beta$$

[15]The commensurator of G_1 is the collection of $\alpha \in G$ such that $\alpha G_1 \alpha^{-1} \cap G_1 \subset G_1$ is finite index in G_1. Note that $N_G(G_1) \subset \mathrm{Comm}_G(G_1)$ so commensurable terminality implies normal terminality.

as automorphism of $\mathrm{PSL}_2(\mathbf{R})$*. Furthermore, we can take* β *so that*

$$\mathrm{ad}_\beta(\mathrm{sl}_2(\mathbf{R})) = \mathrm{sl}_2(\mathbf{R}).$$

For the second statement we are using that $\mathrm{sl}_2(\mathbf{R}) = \mathrm{Lie}(\mathrm{PGL}_2(\mathbf{R}))$ and $\mathrm{Lie}(\mathrm{PGL}_2(\mathbf{C})) = \mathrm{sl}_2(\mathbf{C}) \cong \mathrm{sl}_2(\mathbf{R})_{\mathbf{C}}$. We view $\mathrm{sl}_2(\mathbf{R})$ as the sub Lie algebra of $\mathrm{sl}_2(\mathbf{C})$ fixed under complex conjugation.

Proof Idea. Look at the Borel subalgebra of $\mathrm{sl}_2(\mathbf{C})$ and action of α on it. $\square$

Lemma 8.6.2. $\overline{\mathrm{Aut}}_{\mathbf{C}}(D) \cong \mathrm{PSL}_2^{\pm}(\mathbf{R})$ *where* $\mathrm{PSL}_2^{\pm}(\mathbf{R}) = \mathrm{GL}_2(\mathbf{R})/\mathbf{R}^\times_{>0}$.

Lemma 8.6.3 (Cartan's Lemma: Serre, Ch V, §9, Theorem 2)**.** *If* G_1 *and* G_2 *are Lie groups and any isomorphism of topological groups is automatically an isomorphism of Lie groups.*

Theorem 8.6.4. *Let* $D \in \overline{\mathsf{RS}}$ *be the open unit disc. The collection of holomorphic and anti-holomorphic automorphisms of* D *is normally terminal in the group of topological automorphisms if* D^{top}*.*

Proof.
- Let $\alpha \in \mathrm{Comm}_{\mathrm{Aut}_{\mathrm{top}}(D_0)}(\mathrm{Aut}_{\mathbf{C}}(D))$, then $\alpha\,\mathrm{Aut}_{\mathbf{C}}(D)\alpha^{-1} = \mathrm{Aut}_{\mathbf{C}}(D)$.
- This means $\mathrm{conj}_\alpha : \mathrm{Aut}_{\mathbf{C}}(D) \to \mathrm{Aut}_{\mathbf{C}}(D)$ is an automorphism of topological groups.
- By Cartan's Lemma, conj_α is an automorphism of Lie groups automatically.
- (second reduction) On one hand: let $\beta \in \mathrm{PGL}_2(\mathbf{C})$ and suppose it has the property that for all $x \in \mathrm{sl}_2(\mathbf{R})$, $\mathrm{ad}_\beta(x) \in \mathrm{sl}_2(\mathbf{R})$. Then $\beta \in \mathrm{PGL}_2^{\pm}(\mathbf{R}) = \mathrm{GL}_2(\mathbf{R})/\mathbf{R}^\times$.
- On the other hand: let $\beta \in \mathrm{PGL}_2(\mathbf{C})$. If you consider the action of maximal compact subgroups of $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{Aut}_{\mathbf{C}}(D)$ we find $\beta \in \overline{\mathrm{Aut}}_{\mathbf{C}}(D)$. This allows us to conclude $\overline{\mathrm{Aut}}_{\mathbf{C}}(D) \cong \mathrm{PGL}_2^{\pm}(\mathbf{R})$.

$\square$

8.7. We did all of the work above for the following conclusion.

Theorem 8.7.1 (Grothendieck Conjecture of Aut-Holomorphic Spaces)**.** *Let* Σ *and* Γ *be Riemann surfaces isomorphic to the unit disc.*

$$\overline{\mathrm{Isom}}_{\mathbf{C}}(\Sigma, \Gamma) \cong \mathrm{Isom}_{\mathsf{AS}}(\Sigma^{\mathrm{aut}}, \Gamma^{\mathrm{aut}}).$$

That is every holomorphic or anti-holomorphism isomorphism of Riemann surfaces arises unique from an isomorphism of (holomorphic) aut spaces.

Proof.
- Let D be the complex unit disc with its holomorphic structure. Given $f \in \mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ if $f\overline{\mathrm{Aut}}_{\mathbf{C}}(D)f^{-1} \subset \mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$ then $f \in \overline{\mathrm{Aut}}_{\mathbf{C}}(D)$.
- Both $\overline{\mathrm{Aut}}_{\mathbf{C}}(D)$ and $\mathrm{Aut}_{\mathsf{AS}}(D^{\mathrm{aut}})$ are subspaces of $\mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})$. Clearly $\overline{\mathrm{Aut}}_{\mathbf{C}}(D) \subset \mathrm{Aut}_{\mathsf{AS}}(D^{\mathrm{aut}})$.
- If $\phi \in \mathrm{Aut}_{\mathsf{AS}}(D^{\mathrm{aut}})$ then it is a homeomorphisms and $\phi|_U$ induces $\mathrm{Aut}_{\mathbf{C}}(U) \to \mathrm{Aut}_{\mathbf{C}}(U)$ given by $\psi \mapsto \phi_U\psi\phi_U^{-1}$ for all U. Choosing $U = D$ shows that $\phi \in N_{\mathrm{Aut}_{\mathrm{top}}(D^{\mathrm{top}})}(\mathrm{Aut}_{\mathbf{C}}(D)) = \overline{\mathrm{Aut}}_{\mathbf{C}}(D)$.

$\square$

9. Archimedean Log Links

We follow [Moc15c, Definition 1.1.ii] filling in details from [Moc15e] as necessary.

- Start with $M \cong \mathcal{O}_{\mathbf{C}}^* = D_{\mathbf{C}}(0,1)^*$ the punctured unit disc in the complex plane which is a topological monoid.
- We take the groupification, to get a copy of $\mathbf{C}^\times = (\mathcal{O}_{\mathbf{C}}^*)^{\mathrm{gp}}$. We take the pointed universal cover of this $\widetilde{\mathbf{C}^\times}$.
- This has the structure of an holomorphic aut space U.
- Theorem: $(\mathcal{O}_{\mathbf{C}}^*)^{\mathrm{gp}} \cup \{0\}$ interprets a topological field structure. Call this $\mathbf{k}(M,U)$. It is isomorphic to $\mathbf{C}$.
- We are now allowed to make formal power series. This gives us a notiona of a logarithm.
- This gives a field structure on the universal cover. Explicitly, we get a field structure on $\widetilde{\mathbf{C}^\times}$ by pulling back the field structure field $\exp^{\mathrm{pre}}$. Call this $\mathbf{C}_{\exp}$.
- We then get to define $\mathcal{O}_{\mathbf{C}_{\exp}} \subset \mathbf{C}_{\exp}$.
- Note: Mochizuki writes $\log(\mathcal{F}_{\underline{v}}) \to \mathbb{F}_{\underline{v}}$ where the map from the new monoid to the old one is backwards from the p-adic case: $\mathbf{C}_{\exp} \xrightarrow{\exp^{\mathrm{post}}} \mathbf{C}^\times$
- Definition: the pre-exp-shell is the compact topological quotient of $D_{\mathbf{C}}(0,1)/\partial D_{\mathbf{C}}(0,1)$, where $\partial D_{\mathbf{C}}(0,1)$ is interpreted via $\partial D_{\mathbf{C}}(0,1) = D_{\mathbf{C}}(0,1)^\times$, the subgroup of invertible elements. (Mochizuki calls this the pre-log-shell) It is important to know that this is a topological subquotient of $\widetilde{\mathbf{C}^\times}$.
- The exp-shell is $\mathcal{I}^* \cdot U(1)$.
- Here is how we factor the exponential

$$\begin{tikzcd} \mathbf{C}_{\exp} \arrow[r, equal] & \widetilde{\mathbf{C}^\times} \arrow[d, "\exp^{\mathrm{post}}"'] & \overline{\mathbf{C}^\times} \arrow[l, "\exp^{\mathrm{pre}}"'] \arrow[dl, "\exp"] \\ & \mathbf{C}^\times & \end{tikzcd}$$

  The map exp is a genuine exponential given in terms of power series.
- Theorem: This topological group $(\mathbf{C}^\times, *)$ together with its holomorphic aut structure interpret a field $\overline{\mathbf{C}^\times}$. We have $\overline{\mathbf{C}^\times} = \mathbf{C}^\times \cup \{0\}$ as sets. The addition is interpreted.
- This allows us to define the principal branch of the logarithm, $\mathrm{Log} : \mathcal{O}_{\overline{\mathbf{C}^\times}} \to \overline{\mathbf{C}^\times}$.
- $\mathrm{Log}^{\mathrm{pre}}$ and $\log^{\mathrm{post}}$ are induced. We define $\log^{\mathrm{post}} := (\exp^{\mathrm{pre}})^{-1}$ and we define $\mathrm{Log}^{\mathrm{pre}}$ to be the unique map fitting into the diagram

$$\begin{tikzcd} \mathbf{C}_{\exp} \arrow[r, "\log^{\mathrm{post}}"] & (\overline{\mathbf{C}^\times}, +) \\ \mathcal{O}_{\overline{\mathbf{C}^\times}} \arrow[u, "\mathrm{Log}^{\mathrm{pre}}"] \arrow[ur, "z \mapsto \mathrm{Log}(1-z)"'] & \end{tikzcd}$$

- [Moc15c, Remark 1.2.2] Upper semi-compatibility: $D(0,\pi) \supset \mathcal{O}_{\mathbf{C}}, \exp(\mathcal{O}_{\mathbf{C}})$. The second equality hold just because $e \leq \pi$.

10. Archimedean Log Links

10.1. We defined the holomorphic aut-space $\mathbf{C}_{\log}$ to be the universal cover of $\mathbf{C}^\times$.

$$\mathbf{C}_{\log} := \widetilde{\mathbf{C}^\times}$$

as a topological space. Since $\mathbf{C}^\times$ is a complex lie group, so is $\mathbf{C}_{\log}$ and hence we can give $\mathbf{C}_{\log}$ the structure of a holomorphic aut-space.

10.2.

Definition 10.2.1. The *log-shell* of $\mathbf{C}$ is just the closed disc of radius π in the complex plane.

$$\mathcal{I}_{\mathbf{C}} := D_{\mathbf{C}}(0, \pi).$$

10.3.

Lemma 10.3.1. *$\mathcal{I}_{\mathbf{C}}^{\mathrm{pre}}$ is definable in $\mathbf{C}_{\log}$. Here we are viewing $\mathbf{C}_{\log}$ as a holomorphic aut-space topological group with a galois action* [Moc15e, denoted $\mathbb{TH}\boxtimes$ pg 198]

We will first interpret $\mathcal{I}^* \subset \mathbf{C}$, the vertical line segment in the complex plane from $-i\pi$ to $i\pi$ using only the holomorphic aut-structure of $\mathbf{C}_{\log} := \widetilde{\mathbf{C}^\times}$. Explicitly:

$$\mathcal{I}^* = \{ti\pi \in \mathbf{C} : t \in [-1, 1]\}.$$

Definition 10.3.2. A line segment to be the closure of a connected pre-compact open subset of a one parameter subgroup.

Definition 10.3.3. We then define $\mathcal{I}^{\mathrm{pre}\,*}$ to be the unique line segment of $\mathbf{C}_{\log}$ such that

- $\mathcal{I}^{*\,\mathrm{pre}}$ is preserved by the involution ± 1.
- The endpoints of $\mathcal{I}^{*\,\mathrm{pre}}$ differ by elements of the kernel of $\mathrm{pr} : \mathbf{C}_{\log} \to \mathbf{C}^\times$.

In [Moc15e, Definition 5.4.v] Mochizuki defines (in other notation)

$$\mathcal{I}_{\mathbf{C}}^{\mathrm{pre}} := \overline{\mathrm{Aut}(\mathbf{C}_{\log})_{\mathrm{tors}} \cdot \mathcal{I}^{*\,\mathrm{pre}}}.$$

Here $\mathrm{Aut}(\mathbf{C}_{\log})$ is an automorphism of holomorphic aut-spaces.

It turns out that $\mathbf{C}$ has a topological group structure and $\mathrm{Aut}_{\mathrm{top}}(\mathbf{C}) \cong \mathrm{GL}_2(\mathbf{R})$. Any continuous morphism of Lie groups is automatically compatible with the manifold structure.

♠♠♠ Taylor: [We need to apply [Moc15e, Proposition 2.2] somewhere in here. This is the statement that the Aut holomorphic structure determines the holomorphic structure].

We will prove that all automorphisms of finite order are multiplication by a root of unity. Let $\widetilde{\mathbf{C}^\times} \cong (\mathbf{C}, +)$ as a complex Lie group. Let $f \in \mathrm{Aut}(\mathbf{C}, +)$. Then $f(z) = az$ for some a. Since f is finite order we have $f^n(z) = a^n z = z$ for some n and all z. This implies that $a^n = 1$ which means that a is a root of unity.

11. Archimedean Log-Volume Estimates

The following estimates come from [Moc15d, Proposition 1.5, Proof of Theorem 1.10 Step vii].

Over the Archimedean places we need to consider the summands of $H = \mathrm{hull}(U_\Theta)$ living in

$$\mathbb{L}_\infty = \bigoplus_{j=1}^{(l-1)/2} \mathbb{L}_{\infty,j} = \bigoplus_{j} \bigoplus_{\vec{v}\in V(F_0)_\infty^{j+1}} L_{\underline{\vec{v}}}$$

where

$$L_{\underline{\vec{v}}} = L_{(\underline{v}_0,\dots,\underline{v}_j)} = K_{\underline{v}_0} \otimes \cdots \otimes K_{\underline{v}_j},$$

and $K_{\underline{v}_j} \cong \mathbf{C}$ for $1 \leq i \leq j$. We denote the corresponding components of H by H_∞, $H_{\infty,j}$, and $H_{(\underline{v}_0,\dots,\underline{v}_j)}$ so that

$$H_\infty = \bigoplus_{j=1}^{(l-1)/2} H_{\infty,j} = \bigoplus_{j=1}^{(l-1)/2} \bigoplus_{\vec{v}\in V(F_0)_\infty^{j+1}} H_{\underline{v}_0,\dots,\underline{v}_j}.$$

We again compute Mochizuki log-volumes by

(1) Find an upper bound on $H_{\underline{\vec{v}}}$.
(2) Averaging that result over $\vec{v}$ to gen an upper bound for $\overline{\ln \nu}_{\mathbb{L}_{\infty,j}}(H_{\infty,j})$.
(3) Averaging then that result to get an upper bound for $\overline{\ln \nu}_{\mathbb{L}_\infty}(H_\infty)$.

11.1.

- For V and W inner product spaces we define

11.2. We follow [Moc15c, Remark 3.9.1.ii] on the discussion of the metrics.

- Let V and W be two vector spaces with a Hermitian metric.
- The tensor product metric is induced by $\langle v_1\otimes w_1, v_2\otimes w_2\rangle_{V\otimes W} = \langle v_1, v_2\rangle\langle w_1, w_2\rangle$.
- The direct sum metric is induced by $\langle (v_1, w_1), (v_2, w_2)\rangle_{V\oplus W} = \langle v_1, v_1\rangle_V + \langle w_1, w_2\rangle_W$.
- As topological rings we have $\mathbf{C}\otimes\mathbf{C} \cong \mathbf{C}\oplus\mathbf{C}$ by the Chinese Remainder Theorem. As rings with metrics, the isomorphism above is not isometric. Here is how the metrics differ: if $z, w \in \mathbf{C}\otimes\mathbf{C}$ we have
$$2d_{\mathbf{C}\otimes\mathbf{C}}(z,w) = d_{\mathbf{C}\oplus\mathbf{C}}(z,w).$$
- In terms of norms $\sqrt{2}\|z\|_{\mathbf{C}\otimes\mathbf{C}} = \|z\|_{\mathbf{C}\oplus\mathbf{C}}$

More generally we have $\mathbf{C}^{\otimes n} \cong \mathbf{C}^{\oplus 2^{n-1}}$. This can be seen inductively. The inductive stem is that $\mathbf{C}^{\otimes n} = \mathbf{C}\otimes\mathbf{C}^{\oplus 2^{n-2}} = (\mathbf{C}\otimes\mathbf{C})^{\oplus 2^{n-2}} = (\mathbf{C}^{\oplus 2})^{\oplus 2^{n-2}} = \mathbf{C}^{\oplus 2^{n-1}}$.

Lemma 11.2.1. *Let $K_1,\dots,K_m$ be metrized fields isomorphic to $\mathbf{C}$.*

(1) We have
$$K_1 \otimes_{\mathbf{R}} \cdots \otimes K_m \cong \prod_{k=1}^{r} L_j =: L$$
where $L_j \cong \mathbf{C}$ and $r = 2^{m-1}$.

(2) If $z_1 \otimes \cdots \otimes z_m \in K_1 \otimes \cdots \otimes K_m$ satisfies $|z_i|_{K_i} \leq \lambda$ then for each $1 \leq j \leq r$, $|\operatorname{pr}_{L_j}(z_1 \otimes \cdots \otimes z_m)|_{L_j} \leq \lambda^m$.

(3) If we let $\mathcal{O}_L \subset L$ be the product of the unit balls in the direct sum topology, i.e. $\mathcal{O}_L = \bigoplus_{i=1}^r \mathcal{O}_{L_i}$ where $\mathcal{O}_{L_i} = \{x \in L_i : |x| \leq 1\}$ and if $\varphi \in \prod_{i=1}^{'} \operatorname{Aut}(K_i/\mathbf{R})$ act on each tensor factor independently i.e. $(\varphi_1, \ldots, \varphi_m)(z_1 \otimes \cdots \otimes z_m) = \varphi(z_1) \otimes \cdots \otimes \varphi(z_m)$, we find that $\mathcal{O}_L$ is preserved by such φ. In otherwords $\varphi(\mathcal{O}_L) \subset \mathcal{O}_L$.

11.3.

Lemma 11.3.1. *(1) For each $\vec{v} \in V(F_0)_\infty^{j+1}$, $H_{\vec{v}} \subset \pi^{j+1}\mathcal{O}_{L_{\underline{\vec{v}}}}$.[16]*
(2) $\overline{\ln \nu}_{\mathbb{L}_{\infty,j}}(H_{\infty,j}) \leq (j+1)\ln(\pi)$
(3) $\overline{\ln \nu}_{\mathbb{L}_\infty}(H_\infty) \leq \frac{l+5}{4}\ln(\pi)$

Proof. (1) This follows from invariance of $\mathcal{O}_{L_{\underline{\vec{v}}}}$ and scaling.
(2) The average of a constant is a constant.
(3) $\frac{2}{l-1}\sum_{j=1}^{(l-1)/2}(j+1) = \frac{l+5}{4}$

□

12. Definition of log-link indeterminacy

Since we need to work with abstract version of the log-link indeterminacy we give some definitions and notation. The purpose of this section is to define ind 3 rigourously for regions $\Omega \subset K_1 \otimes \cdots \otimes K_m$.

12.1. **Partially defined maps.** The category $\mathbf{Sets}^p$ will denote the category of sets with partially defined maps $f : S \dashrightarrow T$. Here, by a partially defined map of sets, we simply mean a function $f : S' \to T$ where $S' \subset S$. We will call S' the domain of f and write $\operatorname{dom}(f) = S'$.

Example 12.1.1. Our primary example of a partially defined map is the p-adic logarithm $\log_K : K \dashrightarrow K$ with $\operatorname{dom}(\log_K) = \mathcal{O}_K^\times \subset K$.

We will make the following convention for images: if $f : S \dashrightarrow T$ is a partially defined map of sets and $A \subset S$ then

$$f(A) := f(A \cap \operatorname{dom}(f)).$$

Observe that $\mathbf{Sets}^p$ is indeed a category and that

$$\operatorname{dom}(g \circ f) = g^{-1}\operatorname{dom}(g(\operatorname{dom}(g)) \cap \operatorname{dom}(f)).$$

Example 12.1.2. Of course, one can extend the p-adic logarithm to a map defined on all of K, but we don't consider this here. In our notation then $\log_K^2(K) = \log_K(\mathcal{O}_K^\times \cap \log_K(\mathcal{O}_K^\times))$.

[16] Recall that $\mathcal{O}_{L_{\underline{\vec{v}}}}$ is just a product of unit balls in the direct sum topology.

12.2. **Equivariant partially defined maps of Monoid-sets.** We will also make use of a category of M-sets (monoid-sets with partially defined maps) $\mathbf{MSets}^p$ which we now describe. The objects are pairs (M, S) consisting of a monoid M acting on a set S via $M \times S \to S$. The morphisms

$$(M_1, S_1) \xrightarrow{(f,g)} (M_2, S_2)$$

are pairs (f, g) consisting of a map of monoids $f : M_1 \to M_2$ and a partially defined map of sets $g : S_1 \dashrightarrow S_2$ such that

$$g(m_1 \cdot s_1) = f(m_1) \cdot g(s_1)$$

whenever $m_1 \cdots s_1$ and s_1 are in the domain of g. As in the case of G-sets we will say $g : S_1 \dashrightarrow S_2$ fitting into a morphisms (f, g) of $\mathbf{MSets}^p$ **equivariant** and we will often (by abuse of notation) say $g : S_1 \dashrightarrow S_2$ is equivariant to denote that it is a morphism in $\mathbf{MSets}^p$.

Finally, a map $(f, g) : (M_1, S_1) \to (M_2, S_2)$ is **vacuously equivariant** if

$$(\forall m_1 \in M_1, \forall s_1 \in S_1)(m_1 \cdot s_1 \in \mathrm{dom}(g) \wedge s_1 \in \mathrm{dom}(g))$$

is false. In applications to the log-kummer correspondence this is the case.

12.3. **Free abelian groups.** We will need to make use of a Free abelian group construction in order to define ind 3. Since there are several inequivalent options we explain this definition now. In what follows

$$\mathrm{Free} : \mathbf{Sets}^p \to \mathrm{Ab}$$

is the functor associated to every set S the free abelian group on S, i.e.

$$\mathrm{Free}(S) = \mathbf{Z}S = \{\sum_i n_i \underline{s_i} : n_i \in \mathbf{Z}, s_i \in S\},$$

and associated to every partially defined map $f : S \dashrightarrow T$ the map of abelian groups $\overline{p}^\circ(f) : \mathbb{Z}S \to \mathbb{Z}T$ given by extending linearly the map

$$\mathrm{Free}(f)(\underline{s}) = \begin{cases} 0, & s \notin \mathrm{dom}(f) \\ \underline{s}, & s \in \mathrm{dom}(f) \end{cases}$$

Note here that when we are viewing $s \in S$ as an element of $\mathbb{Z}S$ we use the underline notation $\underline{s}$.

Remark 12.3.1. Alternatively one could have defined a notion $\mathrm{Free}^p : \mathbf{Sets}^p \to \mathrm{Ab}^p$ where Ab^p uses the notion of partially defined group homomorphism. Here we says that $f : A \dashrightarrow B$ is a partially defined group homomorphism (written multiplicatively) if it is a partially defined map of sets such that for all $a_1, a_2 \in A$ if $a_1, a_2, a_1a_2 \in \mathrm{dom}(f)$ then $f(a_1a_2) = f(a_1)f(a_2)$. The rule for composition of partially defined maps guarantees that this is indeed a category. The obvious functor Free^p then assigns S to $\mathbb{Z}S$ and turns $f : S \dashrightarrow T$ with domain S' to the partially defined group homomorphism $\mathbb{Z}S \dashrightarrow \mathbb{Z}T$ defined by the group homomorphism $\mathbb{Z}S' \to \mathbb{Z}T$.

This notion is not as useful.

Example 12.3.2. content...

We can extend the functor Free : $\mathbf{Sets}^p \to \mathrm{Ab}$ to sets with monoid actions

$$\mathrm{Free} : \mathbf{MSets}^p \to \mathbf{MAb},$$

where the output is now the category of Abelian groups with monoid actions **MAb**.

If (M_1, S_1) is an M-set then $\mathrm{Free}(M_1, S_1) = (M_1, \mathbb{Z}S_1) \in \mathbf{MAb}$ is the free abelian group $\mathbb{Z}S_1$ together with the monoid action given by

$$m \cdot \sum_i n_i \underline{s_i} := \sum_i n_i \underline{m \cdot s_i},$$

where $m \in M$ and $\sum_i n_i \underline{s_i} \in \mathbb{Z}S$.

If $(f, g) : (M_1, S_2) \dashrightarrow (M_2, S_2)$ is a partially defined map of M-sets then $\mathrm{Free}(f, g) : (M_1, \mathbb{Z}S_1) \to (M_2, \mathbb{Z}S_2)$ is the map $(f, \mathrm{Free}(g)) : (M_1, \mathbb{Z}S_1) \to (M_2, \mathbb{Z}S_2)$—i.e. the map of free abelian groups becomes naturally equivariant with respect to the monoid actions and monoid maps.

12.4. **Indeterminacy diagrams.** The following definitions allow us to rapidly define collections of morphisms needed to define Mochizuki's ind 3 on regions in tensor products of p-adic fields.

Definition 12.4.1. An **indeterminacy diagram** is a tuple

$$\mathcal{S} = ((f_{ij} : (\Psi_i, R_i) \dashrightarrow (\Psi_j, R_j), (g_i : (\Psi_i, R_i) \to (\Psi_C, C)))$$

consisting of

- Equivariant partially defined maps of M-sets f_{ij}.
- Isomorphisms $g_i : (\Psi_i, R_i) \to (\Psi_C, C)$ in **MSets** (not partially defined).

We draw an indeterminacy diagram as follows:

(12.1)
$$\begin{tikzcd}
\cdots \arrow[r] & R_{i-1} \arrow[r, "f_{i-1,i}"] \arrow[dr, "g_{i-1}"'] & R_i \arrow[r, "f", "{i,i+1}"'] \arrow[d, "g_i"] & R_{i+1} \arrow[r] \arrow[dl, "g_i+1"] & \cdots \\
& & C & &
\end{tikzcd}$$

The special object $C \in \mathbf{MSets}^p$ is called the **core** of the indeterminacy diagram $\mathcal{S}$.

To each indeterminacy diagram we can associate a collection of maps on the core.

Definition 12.4.2. Let $\mathcal{S} = ((f_{ij}, (g_i)))$ be an indeterminacy diagram. The collection of **indeterminacies** of the core C associated to $\mathcal{S}$ is a subset $\mathrm{ind}_{\mathcal{S}} \subset \mathbf{Sets}^p(C, C)$ defined by

$$\mathrm{ind}_{\mathcal{S}} := \bigcup_{j<i} g_i \circ (\mathrm{ind}_{\mathcal{S}})_{ji} \circ g_j^{-1}.$$

Here the collection of maps $\{(\mathrm{ind}_{\mathcal{S}})_{ji} \subset \mathbf{MSets}^p(R_j, R_i) : j < i\}$ is inductively defined by the following properties:

(1) $f_{ji} \in (\mathrm{ind}_{\mathcal{S}})_{ji}$
(2) If $w \in (\mathrm{ind}_{\mathcal{S}})_{ji}$, then for all $\psi_j \in \Psi$, $w \circ \psi_j \in (\mathrm{ind}_{\mathcal{S}})_{ji}$

(3) If $w \in (\mathrm{ind}_{\mathcal{S}})_{ji}$, then for all $\psi_i \in \Psi$, $\psi_i \circ w \in (\mathrm{ind}_{\mathcal{S}})_{ji}$
(4) $(\mathrm{ind}_{\mathcal{S}})_{ki} \circ (\mathrm{ind}_{\mathcal{S}})_{jk} \subseteq (\mathrm{ind}_{\mathcal{S}})_{ji}$ for $j \leq k \leq i$

The most fundamental example of an indeterminacy diagram is a log-Kummer correspondence

Example 12.4.3. ♠♠♠ Taylor: [Give an example of LK without actions.]

A chain of log-links is a sequence

$$\cdots \xrightarrow{\log^{\mathrm{pre}}} K_{\exp^2} \xrightarrow{\log^{\mathrm{pre}}} K_{\exp} \xrightarrow{\log^{\mathrm{pre}}} K \xrightarrow{\log^{\mathrm{pre}}} K_{\log} \xrightarrow{\log^{\mathrm{pre}}} K_{\log^2} \xrightarrow{\log^{\mathrm{pre}}} \cdots$$

where each copy of the field K is log-linked to the next via the partially defined map $\log^{pre} : \mathcal{O}^{\times}_{\log^n,K} \to K_{\log^{n+1}}$.

Let $R_i = K_{\log^i}$, $C = K(\Pi)$, $f_{i,i+1} = \log^{\mathrm{pre}}_{K^{\log^i}}$, and $g_i = \mathrm{kum}_i$. Then our indeterminacy diagram is the log-Kummer correspondence:

$$\begin{tikzcd} \cdots \arrow[r] & K_{\log^{-1}} \arrow[r, "\log^{\mathrm{pre}}_{K^{\log^{-1}}}"] \arrow[dr, "\mathrm{kum}_{-1}"'] & K_{\log^{-1}} \arrow[r, "\log^{\mathrm{pre}}_{K^{\log}}"] \arrow[d, "\mathrm{kum}_0"] & K_{\log} \arrow[r] \arrow[dl, "\mathrm{kum}_1"] & \cdots \\ & & K(\Pi) & & \end{tikzcd} \tag{12.2}$$

Example 12.4.4. ♠♠♠ Taylor: [Give an example of LK with actions.]

Definition 12.4.5. Let $\mathcal{S}$ be an indeterminacy diagram with core C. Let $\Omega \subset C$. We define

$$\Omega^{\mathrm{ind}_{\mathcal{S}}} := \bigcup_{\gamma \in \Gamma_{\mathcal{S}}} \mathrm{ind}(\Omega).$$

Example 12.4.6. Let $I = \{1\}$, $R = (M_1, S_1)$, $C = (M_*, S_1)$, where S_1 and S_* are sets and M_1 and M_* are monoids. Our indeterminacy diagram is as follows:

$$\begin{tikzcd} (S_1, M_1) \arrow[d, "g_1"] \\ (S_*, M_*) \end{tikzcd} \tag{12.3}$$

If M_1 is generated by q, $S_1 = S_* = K$, and $\Omega = \mathcal{O}_K \subseteq K$, then

$$\Omega^{\mathrm{ind}_{\mathcal{S}}} = \bigcup_{n \geq 1} q^n \mathcal{O}_K.$$

12.5. **Log-Kummer correspondence for a single field (without actions).** Let K be a finite extension of $\mathbb{Q}_p$. Consider now the log linked sequence of elements of $(\Pi, \overline{K}_{\log^n}) \in [\Pi_Z, \overline{\mathbb{Q}_p}]$ fitting into the following diagram (with arrows being the partially defined map $\log^{\mathrm{pre}}$)

$$\cdots \to (\Pi, \overline{K}_{\exp}) \xrightarrow{\log^{\mathrm{pre}}} (\Pi, \overline{K}) \xrightarrow{\log^{\mathrm{pre}}} (\Pi, \overline{K}_{\log}) \to \cdots .$$

Note that the arrows Mochizuki uses the anabelian construction given in Lemma **??** and the notion of log-links to construct a so-called log-Kummer correspondence. As a first approximation it a collection of maps

$$\mathrm{LK} = (\log^{\mathrm{pre}} : K_i \to K_{i+1}, \mathrm{kum}_i : K_i \to K_c)$$

defining a diagram whose vertices are Ind-topological fields with Π-actions and whose arrows are Π-equivariant partially defined maps (the maps kum_i are Ind-topological field isomorphisms and the maps $\log^{\mathrm{pre}}$ are pseudo-maps of sets):

$$\begin{tikzcd} \cdots \arrow[r] & \overline{K}_{\exp} \arrow[r, "\log^{\mathrm{pre}}"] \arrow[dr, "\mathrm{kum}_{-1}"'] & \overline{K} \arrow[r, "\log^{\mathrm{pre}}"] \arrow[d, "\mathrm{kum}_0"] & \overline{K}_{\log} \arrow[r] \arrow[dl, "\mathrm{kum}_1"] & \cdots . \\ & & \overline{K}^{\mathrm{et}}(\Pi) & & \end{tikzcd} \tag{12.4}$$

The diagram (12.5) is called the log-Kummer correspondence. We observe that if we fix an isomorphisms of Ind topological fields $(\overline{K}, +, *, 0, 1)$ with $(\overline{K}^{\mathrm{et}}(\Pi), +, *, 0, 1)$ via kum_0 we can witness the non-commutativitiy of the log-kummer correspondence: Let $a \in \mathcal{O}_{\overline{K}}^{\times}$. The image of $a \in \overline{K}^{\mathrm{et}}(\Pi)$ by following the diagram to the right then down gives

$$(\mathrm{kum}_1 \circ \log^{\mathrm{pre}})(a) = \log(a).$$

Alternatively, taking the Kummer map directly down gives

$$\mathrm{kum}_0(a) = a.$$

This show a discrepancy between routes from $\overline{K}$ to. Also note that if we started with $a \in \overline{K} \setminus \mathcal{O}_{\overline{K}}^{\times}$ then a is not in the domain of $\log^{\mathrm{pre}}$ and there is no indeterminacy.

In what follows we will define

$$\mathrm{kum}_{ij} := \mathrm{kum}_i^{-1} \mathrm{kum}_j \,.$$

We think of such an operation as a transition maps between locally defined ring structures.

Lemma 12.5.1. *(1) If the* kum_i *are ring homomorphisms then the induced isomorphism* $\mathrm{kum}_0^{-1} \circ \mathrm{kum}_1$ *must be* $\log^{\mathrm{post}} : K_{\log} \to K$ *up to* $\mathrm{Aut}(K/\log(\mathcal{O}_K^{\times}))$ *(which denotes the set of* $\mathbb{Q}_p$*-automorphisms of* K *which fixes* $\log(\mathcal{O}_K^{\times})$ point-wise*).*

(2) If φ *is a* $\mathbb{Q}_p$*-vector space automorphism if* K *fixing* $\log(\mathcal{O}_K)$ *point-wise then* φ *is the identity.*

(3) If $g : K_{\log} \to K$ *is an isomorphism of topological fields such that* $g(\log^{\mathrm{pre}}(\mathcal{O}_K^{\times})) = \log(\mathcal{O}_K^{\times})$ *then* $g^{-1} \circ \log^{\mathrm{post}}$ *is an automorphism of* $K_{\log}$ *as a* $\mathbb{Q}_p$*-vector space which fixed* $\mathcal{O}_K^{\times\mu}$ *(*$= \log(\mathcal{O}_{K_{\log}})$*).*

(4) If the kum_i *are ring homomorphisms then the induced isomorphism* $\mathrm{kum}_0^{-1} \circ \mathrm{kum}_1$ *must be* $\log^{\mathrm{post}} : K_{\log} \to K$*.*

(5) Suppose $i < j$ *then* $\mathrm{kum}_i^{-1} \circ \mathrm{kum}_j = \log^{\mathrm{post}(j-i)}$*. Where*

$$\log^{\mathrm{post}(j-i)} := \log^{\mathrm{post}}_{K_i} \log^{\mathrm{post}}_{K_{i+1}} \cdots \log^{\mathrm{post}}_{K_{j-1}}$$

Proof. (1) Let's identify the ring structures of $\overline{K}$ with $\overline{K}^{\mathrm{et}}$ via pullback by kum_0. We can then instantiate the log-links afterwards. This means without loss of generality we can suppose $\overline{K}^{\mathrm{et}}(\Pi) = \overline{K}$ as fields and kum_0 is the identity. Let $a \in \mathcal{O}_{\overline{K}}$. Then $\mathrm{kum}_1(\log^{\mathrm{pre}}(a)) = \log(a)$. This means kum_1 is an isomorphism of $\overline{K}_{\log} = \mathcal{O}_K^{\times\,\mathrm{pf}} \to \overline{K}$ such that $\mathrm{kum}_1(\log^{\mathrm{pre}}(a)) = \log(a)$.[17]

(2) Suppose that K is a finite extension. $\log(\mathcal{O}_K) = \bigoplus_i \mathbf{Z}_p g_i$ and $K = \bigoplus_i \mathbb{Q}_p g_i$. If $\log(\mathcal{O}_K)$ is fixed point-wise by φ then $\varphi(g_i) = g_i$ for each i and hence for each $x \in K$ we have

$$\varphi(x) = \varphi(\sum_i a_i g_i) = \sum_i a_i \varphi(g_i) = \sum_i a_i g_i = x.$$

In this series of equalities we used that φ is a $\mathbb{Q}_p$-linear map.

(3) Suppose that $g : K_{\log} \to K$ is an isomorphism and that $g \circ \log^{pre} = \log$ when restricted to $\mathcal{O}_K^{\times}$.

The map $g : \mathcal{O}_K^{\times\mu} \to K$ induces a map via tensorization $g_{\mathbb{Q}} : \mathcal{O}^{\times\mu} \otimes \mathbb{Q} \to \log(\mathcal{O}_K^{\times\mu}) \otimes \mathbb{Q}$ and hence an isomorphism of $\mathbb{Q}$-vectors spaces $(K_{\log}, +) \to (K, +)$. By the axiom of choice this is a $\mathbb{Q}_p$-vector space isomorphism as well. The map $g_{\mathbb{Q}}$ actually must coincide with $\log^{\mathrm{post}}$ (since this was actually the definition of $\log^{\mathrm{post}}$). Hence the map $g^{-1} \circ \log^{\mathrm{post}}$ is an isomorphism of $\mathbb{Q}_p$-vector spaces inducing the identity on $\mathcal{O}_K^{\times\mu}$. By our lemma, this must be the identity.

(4) Omitted.

(5) One has

$$\begin{aligned}
\mathrm{kum}_i^{-1}\mathrm{kum}_j &= \mathrm{kum}_i^{-1}\mathrm{kum}_{i+1}\mathrm{kum}_{i+1}^{-1}\cdots\mathrm{kum}_{j-1}\mathrm{kum}_{j-1}^{-1}\mathrm{kum}_j \\
&= \log^{\mathrm{post}} \circ \log^{\mathrm{post}} \cdots \log^{\mathrm{post}} \\
&= \log^{\mathrm{post}(j-i)}.
\end{aligned}$$

□

Consider again the log-kummer correspondence. One can ask what the affect of taking and element of the coric copy of K, and passing it through on of the loops on its spokes will do. More precisely, we can ask what the effect of

$$x \mapsto \mathrm{kum}_{i+1} \log^{\mathrm{pre}} \mathrm{kum}_i^{-1}(x)$$

on $x \in K_{\mathrm{core}}$ is. Is this action via the passage through a log-link the same for every i? We will make use of some notation in the answer to this question: for a map $f_i : K_i \to K_i$, where

[17] Hence the problem is reduced to uniqueness of $\log^{\mathrm{post}}$. Is any map g such that

(a) $g : \overline{K}_{\log} \to \overline{K}$ is an isomorphism.

(b) $g \circ \log^{\mathrm{pre}} = \log$ when restricted to $\mathcal{O}_{\overline{K}}^{\times}$

unique?

It is clear that $g = \log^{\mathrm{post}}$ satisfies these criteria.

Here is a potential counter-example: if φ is an automorphism of the $\mathbb{Q}_p$-vector space $\mathcal{O}_{\overline{K}}^{\times\,\mathrm{pf}}$ fixing $\mathcal{O}_{\overline{K}}^{\times\mu}$ then the map $g = \log^{\mathrm{post}} \circ \varphi : \overline{K}_{\log}$ has the property that 1) it is an isomorphism and 2) $g \circ \log^{\mathrm{pre}} = \log$ when restricted to $\mathcal{O}_{\overline{K}}^{\times}$.

K_i sits inside a log-kummer correspondence then we use the notation

$$f_i^{\mathrm{kum}_i} := \mathrm{kum}_i\, f_i\, \mathrm{kum}_i^{-1}.$$

Lemma 12.5.2. *(1) For every i and j we have*

$$\mathrm{kum}_{i+1} \log^{\mathrm{pre}} \circ\, \mathrm{kum}_i^{-1} = \log_{K_i}^{\mathrm{kum}_i} = \log_{K_j}^{\mathrm{kum}_j}.$$

Proof. We will prove two assertions. First that the effect of passing through a loop just an application of the p-adic logarithm. This is equivalent to the first equality. Second, that the effect of through any loop is the same. In what follows we will let $K_{\mathrm{core}} = K$.

(1) If $\mathrm{kum}_i^{-1}(x) \in \mathcal{O}_{K_i}^{\times}$ then

$$\begin{aligned}\mathrm{kum}_i^{-1}(\mathrm{kum}_{i+1} \log^{\mathrm{pre}} \mathrm{kum}_i^{-1}(x)) &= \log^{\mathrm{post}} \log^{\mathrm{pre}} \mathrm{kum}_i^{-1}(x)\\ &= \log_{K_i} \mathrm{kum}_i^{-1}(x)\\ &= \mathrm{kum}_i^{-1} \log_K(x)\end{aligned}$$

Here we are using that since the field isomorphism kum_i is an isomorphism of topological fields that power series are respected.

(2) This is verified by a Čech-style computation:

$$\begin{aligned}\log_{K_i}^{\mathrm{kum}_i} &= \mathrm{kum}_j\, \mathrm{kum}_j^{-1}\, \mathrm{kum}_i \log_{K_i} \mathrm{kum}_i^{-1}\, \mathrm{kum}_j\, \mathrm{kum}_j^{-1}\\ &= \mathrm{kum}_j\, \mathrm{kum}_{ji} \log_{K_i} \mathrm{kum}_{ij}\, \mathrm{kum}_j^{-1}\\ &= \mathrm{kum}_j\, \mathrm{kum}_{ji}\, \mathrm{kum}_{ij} \log_{K_j} \mathrm{kum}_j^{-1}\\ &= \mathrm{kum}_j \log_{K_j} \mathrm{kum}_j^{-1} = \log_{K_j}^{\mathrm{kum}_j}\end{aligned}$$

□

The results of the above are summarized in the diagram below:

(12.5)
$$\begin{tikzcd}
\cdots \arrow[r] & K_{-1} \arrow[r, "\log^{\mathrm{pre}}_{K_{-1}}"] \arrow[l, bend right] \arrow[dr, "\mathrm{kum}_{-1}"] & \overline{K_0} \arrow[r, "\log^{\mathrm{pre}}_{K_0}"] \arrow[l, bend right, "\log^{\mathrm{post}}_{K_{-1}}"'] \arrow[d, "\mathrm{kum}_0"] & K_1 \arrow[r] \arrow[l, bend right, "\log^{\mathrm{post}}_{K_0}"'] \arrow[dl, "\mathrm{kum}_1"] & \cdots \arrow[l, bend right] \\
& & \overline{K}^{\mathrm{et}}(\Pi) & &
\end{tikzcd}$$

♠♠♠ Taylor: [Do we need to know how automorphism of the pairs $(\Pi, (K_i)_{i\in\mathbf{Z}})_{i\in\mathbf{Z}}$ act on the diagrams? Act on a given diagram?]

♠♠♠ Taylor: [There is an overall conjugacy indeterminacy that we need to contend with.]

12.6. **Log-kummer correspondences.** The aim of this section is to define Mochizuki's third indeterminacy. This will take a set $\Omega \subset K_1 \otimes \cdots \otimes K_m$ where $K_i/\mathbb{Q}_p$ is a finite extension, and some $a \in K_m$ acting on $K_1 \otimes \cdots \otimes K_m$ through the mth tensor factor, via

$$(a, x_1 \otimes x_m) \mapsto x_1 \otimes a x_m.$$

and define a new set

$$\Omega^{\mathrm{ind}\,3(a)} \subset K_1 \otimes \cdots \otimes K_m,$$

which should be thought of as Ω "blurred" by the operation of taking p-adic logarithms on the tensor factors and multiplying the region by a.

The starting point for this construction is the log-kummer correspondence.

Example 12.6.1. Let $\mathrm{LK}_m(a)$ for $a \in K$ be the following monoid indeterminacy diagram

$$(12.6) \qquad \begin{tikzcd} \cdots \arrow[r] & (a^{\mathbb{N}}, \prod_{u=1}^m K_{u,\log^{-1}}) \arrow[r, "f_{-1,0}"] \arrow[dr, "g_{-1}"'] & (a^{\mathbb{N}}, \prod_{u=1}^m K_u) \arrow[r, "f_{0,1}"] \arrow[d, "g_0"] & (a^{\mathbb{N}}, \prod_{u=1}^m K_{u,\log}) \arrow[r] \arrow[dl, "g_1"] & \cdots \\ & & (a^{\mathbb{N}}, \prod_{u=1}^m K_u) & & \end{tikzcd}$$

Here the $f_{i,i+1}$ are pairs $(\alpha_i, \beta_i) : (a^{\mathbb{N}}, \prod_{u=1}^m K_{u,\log^i}) \to (a^{\mathbb{N}}, \prod_{u=1}^m K_{u,\log^{i+1}})$, where $\alpha_i : a^{\mathbb{N}} \to a^{\mathbb{N}}$ is an isomorphism and $\beta_i : \prod_{u=1}^m K_{u,\log^i} \to \prod_{u=1}^m K_{u,\log^{i+1}}$ is given by $\log^{\mathrm{pre}}_{\prod_{u=1}^m K_{u,\log^i}}$. The bijections g_i are given by Mochizuki's Kummer map.

The monoid action

$$a^{\mathbb{N}} \times \prod_{u=1}^m K_{u,\log^i} \to \prod_{u=1}^m K_{u,\log^i}$$

is given by

$$(a, (x_1, \ldots x_m)) \mapsto (x_1, \ldots, ((\log^{\mathrm{post}})^i)^{-1}(a(\log^{\mathrm{post}})^i(x_m))).$$

The above example motivates the following definition.

Definition 12.6.2. For $\Omega \subset K_1 \times \cdots \times K_m$ and $a \in K_m$ define

$$(12.7) \qquad \Omega^{\mathrm{ind}\,3(a)} := \Omega^{\mathrm{ind}_{\mathrm{LK}_m(a)}}.$$

We will use the notation

$$\Omega^{\mathrm{ind}\,3} := \Omega^{\mathrm{ind}\,3(1)}.$$

To pass our definition of the indeterminacy associated to $\mathrm{LK}_m(a)$ to tensor products we need the free group version $\mathrm{Free}(\mathrm{LK}_m(a))$. By functorality this is an indeterminacy diagram and we can define

Definition 12.6.3. For $\widetilde{\Omega} \subset \mathbb{Z}K_1 \times \cdots \times K_m$

$$(12.8) \qquad \widetilde{\Omega}^{\mathrm{ind}\,3(a)} := \widetilde{\Omega}^{\mathrm{ind}_{\mathrm{Free}(\mathrm{LK}_m(a))}}$$

We will use the notation

$$\widetilde{\Omega}^{\mathrm{ind}\,3} := \widetilde{\Omega}^{\mathrm{ind}\,3(1)}.$$

Remark 12.6.4. While kummer maps exists for tensor products of fields, there is no pre-logarithm that can be defined and hence no log-Kummer correspondence. This is why we need to pass to free groups.

Let quot : $\mathbb{Z}K_1 \times \cdots \times K_m \to K_1 \otimes \cdots \otimes K_m$ be the map from the free group on the product of fields p-adic fields to the tensor product defined by

$$\sum_i n_i \underline{(x_{1i}, \cdots, x_{mi})} \mapsto \sum_i n_i x_{1i} \otimes \cdots \otimes x_{mi}.$$

Definition 12.6.5. For $\Omega \subset K_1 \otimes \cdots \otimes K_m$ and $a \in K_m$ we define

$$\Omega^{\mathrm{ind}\,3(a)} := \mathrm{quot}(\mathrm{quot}^{-1}(\Omega)^{\mathrm{ind}\,3(a)}).$$

We will use the notation

$$\Omega^{\mathrm{ind}\,3} := \Omega^{\mathrm{ind}\,3(1)}.$$

We remark that the map quot is equivariant for the action of a^{N} on the mth factor.

12.7. **Explicit description of** $\mathrm{ind}\,3$ **in free groups.** In this subsection we use the notation

$$\overline{\log} = \mathrm{Free}(\log_{K_1 \times \cdots \times K_m}).$$

We also let

$$\psi_a = \psi_a^{(m)}$$

denote the map induced my multiplication by a on the mth component.

Lemma 12.7.1. *Let $\widetilde{\Omega} \subset \mathbb{Z}K_1 \times \cdots \times K_m$ and $a \in K_m$ have $|a|_p < 1$ then*

$$\widetilde{\Omega}^{\overline{\mathrm{ind}\,3}(a)} = \{\overline{\log}, \psi_a\}^* \cdot \widetilde{\Omega}$$

Proof. ♠♠♠ Taylor: [Follows from the description of $\mathrm{ind}_{\mathrm{Free}^\circ(\mathrm{LK}_m(a))}$.] We want to show that for $\mathcal{S} = \mathrm{Free}^p(\mathrm{LK}_m(a))$ that

$$\mathrm{ind}_{\mathcal{S}} = \{\psi_a, \overline{\log}\}^*.$$

Let $w \in \mathrm{ind}_{\mathcal{S}}$. By definition $w \in g_i((\mathrm{ind}_{\mathcal{S}})_{ji})g_j^{-1}$ for some $j < i$. By the definition of $\mathrm{LK}_m(a)$, and the observation that $(\mathrm{ind}_{\mathcal{S}})_{ji}$ is the set of words made from ψ_a and $\overline{\log}$ with exactly $i-j$ instances of $\overline{\log}$, we have

$$w = \mathrm{kum}_i \circ \psi_{a,i}^{n_i} \circ \overline{\log}_{i,i-1} \circ \ldots \circ \overline{\log}_{j+1,j} \circ \psi_{a,j}^{n_j} \, \mathrm{kum}_j^{-1}.$$

Since $\mathrm{kum}_k \circ \mathrm{kum}_k^{-1}$ is just the identity, we can insert it between the ψ_a's and the $\overline{\log}$'s so that we have

$$w = \mathrm{kum}_i \circ \psi_{a,i}^{n_i} \circ \mathrm{kum}_k \circ \mathrm{kum}_k^{-1} \circ \overline{\log}_{i,i-1} \circ \ldots \circ \overline{\log}_{j+1,j} \circ \mathrm{kum}_k \circ \mathrm{kum}_k^{-1} \circ \psi_{a,j}^{n_j} \, \mathrm{kum}_j^{-1}$$

Suppose kum_k is equivariant for every k. Then $\mathrm{kum}_k \circ \psi_{a,k} \circ \mathrm{kum}_k^{-1} = \psi_{a,k}$. Hence

$$w = \psi_{a,i}^{n_i} \circ \overline{\log}_{i,i-1} \circ \ldots \circ \overline{\log}_{j+1,j} \circ \psi_{a,j}^{n_j}.$$

□

12.8. **Final inequality.** ♠♠♠ Taylor: [The point here is that no matter what happens before, $\overline{\log}$ must have an image in $\mathbb{Z} \cdot K_1 \times \cdots \times K_m$.] Let $\Omega \subset K_1 \otimes \cdots \otimes K_m$ and let $a \in K_m$ act on $K_1 \otimes \cdots \otimes K_m$ via the mth tensor factor. Although one would like to prove something like

$$(a \cdot \Omega)^{\mathrm{ind}\,3} = a \cdot \Omega^{\mathrm{ind}\,3}$$

where we have defined $(a \cdot \Omega)^{\mathrm{ind}\,3} := \Omega^{\mathrm{ind}\,3(a)}$, this formula is not true. In this section we prove weaker version of this which is what is used in Mochizuki's inequality.

Remark 12.8.1. It would be sufficient to show that $\overline{\log} \circ \psi_a = 0$ on $\mathbb{Z}K_1 \otimes \cdots K_m$ but this is not true. Consider $m = 1$, $\Omega = \mathcal{O}_L = \mathcal{O}_K$, and $\Psi = a^{\mathbb{N}}$, with $a \in K$ and $|a|_p = \frac{1}{p}$. Then $p\underline{\frac{1}{p}} \in \mathbf{Z} \cdot K$ is an element of $\mathrm{quot}^{-1}(\mathcal{O}_K)$. Then the action of a on the element $p\underline{\frac{1}{p}}$ is given by $\psi_a(p\underline{\frac{1}{p}}) = p\underline{\frac{1}{p}}$. Since $|a|_p = \frac{1}{p}$ and $|\frac{1}{p}|_p = p$, we have $\frac{a}{p} \in \mathcal{O}_K^\times$ and therefore $\frac{a}{p}$ is in the domain of the logarithm. Then

$$\overline{\log}\psi_a(p\frac{1}{p}) = p\underline{\log\frac{a}{p}}$$

which is nonzero.

Lemma 12.8.2 (ind3 inequalities)**.** *Let $\Omega \subset K_1 \otimes \cdots \otimes K_m$, and $a \in K_m$ with $|a|_p < 1$. Then*

$$\Omega^{\mathrm{ind}\,3(a)} \subset a^{\mathrm{N}} \cdot \left(\Omega \cup \bigotimes_{i=1}^{m} \log(\mathcal{O}_{K_i}^\times) \right). \tag{12.9}$$

$$a^{\mathrm{N}} \cdot \Omega^{\mathrm{ind}\,3} \subset a^{\mathrm{N}} \cdot \left(\Omega \cup \bigotimes_{i=1}^{m} \log(\mathcal{O}_{K_i}^\times) \right) \tag{12.10}$$

Proof. Let $\widetilde{\Omega} \subset \mathbb{Z}K_1 \times \cdots \times \mathbb{K}_m$. We will first compute $\widetilde{\Omega}^{\mathrm{ind}\,3(a)}$ and then use this result to prove our Lemma. First recall that by Lemma 12.7.1 we have

$$\widetilde{\Omega}^{\mathrm{ind}\,3} = \{\psi_a, \overline{\log}\}^*(\widetilde{\Omega})$$

First recall that $a^{\mathrm{N}} \subset K_m$ acts through the mth factor of $K_1 \times \cdots \times K_m \to K_1 \times \cdots \times K_m$ via $\psi_a := \psi_a^{(m)} : K_1 \times \cdots \times K_m$ via $\psi_a(x_1, \ldots, x_m) = (x_1, \ldots, a x_m)$. By abuse of notation we let ψ_a also denote induced maps on $\mathbb{Z}K_1 \times \ldots \times K_m$ given by $\psi_a(\underline{(x_1, \ldots, x_m)}) = \underline{(x_1, \ldots, a x_m)}$. Again by another abuse of notation we let ψ_a denote the induced map on $K_1 \otimes \cdots \otimes K_m$ given by $\psi_a(x_1 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes a x_m$.

Also recall that $\overline{\log} = \mathrm{Free}(\log_{K_1 \times \cdots \times K_m})$. Note that every element of $\{\psi, \overline{\log}\}^*$ takes the form

$$\psi_a^{n_1} \circ \overline{\log} \circ \psi_a^{n_2} \circ \overline{\log} \circ \ldots \psi_a^{n_r} \circ \overline{\log} \circ \psi_a^{n_{r+1}}$$

for some non-negative integer r and some non-negative integers $n_1, \ldots, n_{r+1}$. When $x \in \widetilde{\Omega}$ is acted on by such an element we have

$$\psi_a^{n_1} \circ \overline{\log} \circ \psi_a^{n_2} \circ \overline{\log} \circ \ldots \psi_a^{n_r} \circ \overline{\log} \circ \psi_a^{n_{r+1}}(x) = \psi_a^{n_1}(y)$$

for either $y \in \overline{\log}(\mathbf{Z}K_1 \times \cdot \times K_m)$ or $y \in \tilde{\Omega}$. Since $\overline{\log}(\mathbf{Z}K_1 \times \cdot \times K_m) \subset \mathbb{Z}\log(\mathcal{O}_{K_1}^\times) \times \cdots \times \log(\mathcal{O}_{K_m}^\times)$ this proves

$$\{\psi_a, \overline{\log}\}^* \cdot \tilde{\Omega} \subseteq \{\psi_a\}^* \cdot \left(\tilde{\Omega} \cup \mathrm{Free}(\prod_{i=1}^m \log(\mathcal{O}_{K_i}^\times))\right).$$

This proves

$$\widetilde{\Omega}^{\mathrm{ind}\,3} \subset a^{\mathrm{N}} \cdot \left(\widetilde{\Omega} \cup \mathrm{Free}(\prod_{i=1}^m \log(\mathcal{O}_{K_i}^\times)\right). \tag{12.11}$$

We will now prove the statement of the Theorem. Let $\Omega \subset K_1 \otimes \cdots \otimes K_m$ and take $\widetilde{\Omega} = \mathrm{quot}^{-1}(\Omega)$. Since $\mathrm{quot} : \mathrm{Free}(\prod_{i=1}^m K_i) \to \bigotimes_{i=1}^m K_i$ is a^{N}-equivariant, (12.11) implies that

$$\Omega^{\mathrm{ind}\,3(a)} \subseteq a^{\mathbb{N}} \cdot \left(\Omega \cup \bigotimes_{i=1}^m \log(\mathcal{O}_{K_i}^\times)\right).$$

Note that when $a = 1$ that this proves

$$\Omega^{\mathrm{ind}\,3} \subset \Omega \cup \bigotimes_{i=1}^m \log(\mathcal{O}_{K_i}^\times).$$

□

Remark 12.8.3. One may be able to refine this computation with the following technique: if K is a discretely valued field then

$$K = \coprod_{n\in\mathbb{Z}} \{x \in K : \mathrm{ord}_K(x) = n\}.$$

If we let $S_n = \{x \in K : \mathrm{ord}_K(x) = n\}$ then one has $\mathbb{Z} \cdot K = \bigoplus \mathbb{Z}S_n$. We know how multiplication by a acts on each of these components.

Remark 12.8.4. We have $\mathrm{quot}(\mathbb{Z}\mathcal{O}_{K_1}^\times \times \cdots \times \mathcal{O}_{K_m}^\times) = K_1 \otimes \cdots \otimes K_m$. One can define

$$\Omega^{\mathrm{ind}\,3'(a)} = \mathrm{quot}([\mathbb{Z}\mathcal{O}_{K_1}^\times \times \cdots \times \mathcal{O}_{K_m}^\times \cap \mathrm{quot}^{-1}(\Omega)]^{\mathrm{ind}\,3(a)}).$$

Here we do have $\overline{\log} \circ \psi_a = 0$ and one gets $\Omega^{\mathrm{ind}\,3'(a)} = a^{\mathrm{N}} \cdot \Omega^{\mathrm{ind}\,3}$. ♠♠♠ Taylor: [Check again]

12.9. **Application.** Let $L = K_1 \otimes \cdots \otimes K_m$ and let q be a number with $|a| < 1$. We have

$$\mathcal{O}_L^{\mathrm{ind}\,3(a)} \subset a^{\mathrm{N}} \cdot \left(\mathcal{O}_L \cup \bigotimes_{i=1}^m \log(\mathcal{O}_{K_i})\right)$$
$$\subset a^{\mathrm{N}} \cdot \bigotimes_{i=1}^m \mathcal{I}_{K_i}.$$

The first line follows from ♠♠♠ Taylor: [I think this is False now since it isn't in our section on log shells]

12.10. **Equivalence relation approach to the third indeterminacy.** The following is another approach to the third indeterminacy suggested to us by Emmanuel Lepage. ♠♠♠ Taylor: [FILL IN]

- Explain alternative approach.

Let $\Omega \subseteq K_1 \otimes \ldots \otimes K_m$. In this approach, we define

$$\Omega^{\mathrm{Ind}\,3} = \bigcup_{x \in \Omega} [x]_{\mathrm{Ind}\,3}$$

where $[x]_{\mathrm{Ind}\,3}$ stands for the equivalence class of x under an equivalence relation which we will now define,.

In the case of a single field, $[x]$ is the equivalence class of x under the equivalence relation

$$x \sim \log(x).$$

Next we will define the equivalence relation for direct products of fields, free groups over direct products of fields, and tensor products of fields.

Having defined the equivalence relation for each of the fields $K_1, \ldots, K_m$, we define the equivalence relation on $K_1 \times \ldots \times K_m$ via

$$[(x_1, \ldots, x_m)] = [x_1] \times \ldots \times [x_m].$$

Then we can define the equivalence relation on the free group $\mathbf{Z} \cdot [K_1 \times \ldots \times K_m]$ by defining

$$\sum_{i=0}^{r} n_i s_i \sim \sum_{i=0}^{r} n_i s_i'$$

whenever $s_i \sim s_i'$ for all i.

Now we can define $\Omega^{\mathrm{Ind}\,3}$ for $\Omega \subseteq K_1 \otimes \ldots \otimes K_m$ as

$$\mathrm{quot}((\mathrm{quot}^{-1}(\Omega))^{\mathrm{Ind}\,3}).$$

Note that $\mathrm{quot}^{-1}(\Omega) \subseteq \mathbf{Z} \cdot [K_1 \times \ldots \times K_m]$.

13. Ind1 and Ind 2

♠♠♠ Taylor: [The goal of this section is to obtain a representation

$$\mathrm{Aut}(G) \to \mathrm{Aut}(\overline{K} : \log(\mathcal{O}_{\overline{K}}^{\times})).$$

]

♠♠♠ Taylor: [This is explained in [Moc15b, Proposition 3.4.ii]. Here Mochizuki states that there are three intrepretations of $(\Pi_{\underline{\underline{X}}_{\underline{v}}}, \mathcal{O}^{*}_{\overline{K}_{\underline{v}}})$, a tempered fundamental group of a particular cover coming from initial theta data acting on on the multiplicative monoid of a ring of integers. In each of these interpretations the fundamental group sort, $\Pi_{\underline{v}}$ is exactly the same and he constructs natural isomormophism starting from the tempered Frobenioid level, passing to an interpretation in a mono-theta environment interpreted in the tempered frobenioid, passing to an interpretation in a mono-theta environment interpreted from the topological group alone:

$$\Psi_{\mathcal{C}_{\underline{v}}} \to \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \to \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$$

The composition is Mochizuki's Kummer map and is induced by an isomorphism of monotheta environements $\mathrm{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{M}^{\Theta}_{*}(\Pi_{\underline{v}})$, a structure cooked-up as an intermediary between interpretations in tempered frobenioids and interpretations in tempered fundamental groups which can be interpreted in both $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ and $\Pi_{\underline{v}}$.

The isomorphism

$$(\Pi_{\underline{v}}, \Psi_{\mathcal{C}_{\underline{v}}}) \to (\Pi_{\underline{v}}, \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))) \to (\Pi_{\underline{v}}, \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\Pi_{\underline{v}})))$$

pass to isomorphisms

$$(G_{\underline{v}}(\Pi_{\underline{v}}), \Psi_{\mathcal{C}_{\underline{v}}}) \to (G_{\underline{v}}(\Pi_{\underline{v}}), \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))) \to (G_{\underline{v}}(\Pi_{\underline{v}}), \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\Pi_{\underline{v}}))).$$

and the natural transformation

$$(G_{\underline{v}}(\Pi_{\underline{v}}), \Psi^{\times\mu}_{\mathcal{C}_{\underline{v}}}) \to (G_{\underline{v}}(\Pi_{\underline{v}}), \Psi_{\mathrm{cns}}(\mathrm{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times\mu})$$

is not full. This (I think) is a consequence of the simple fact that

$$(G, M) \to (G, \overline{\mathcal{O}}^{\times\mu}(G))$$

is not full.]

The following lemma shows that the log of Galois invariants are not the same as invariants of logarithms. This is the source of Ind2 as finite extensions of number fields (or monoids therein) are typically constructed as Galois invariants of a structure which is a reduct of an algebraically closed field.

Lemma 13.0.1. *Let K be a galois extension of $\mathbb{Q}_p$ with galois group G. In general, the map*

$$(\mathcal{O}_K^{\times})^G \to \log(\mathcal{O}_K^{\times})^G$$

is not surjective.

Proof. This is a galois cohomology, short exact sequence to long exact sequence calculation. Consider the exact sequence

$$1 \to (\mathcal{O}_{\overline{K}}^{\times})_{\mathrm{tors}} \to \mathcal{O}_{\overline{K}}^{\times} \to \mathcal{O}_{\overline{K}}^{\times\mu} \to 1,$$

of Galois modules. Taking invariants gives the sequence

$$\begin{array}{c} 1 \longrightarrow (\mathcal{O}_K^{\times})_{\mathrm{tors}} \longrightarrow \mathcal{O}_K^{\times} \longrightarrow \log(\mathcal{O}_{\overline{K}})^G \\ \longrightarrow H^1(G, (\mathcal{O}_{\overline{K}}^{\times})_{\mathrm{tors}}) \longrightarrow H^1(G, \mathcal{O}_K^{\times}) \longrightarrow H^1(K, \log(\mathcal{O}_{\overline{K}})) \end{array}$$

♠♠♠ Taylor: [I think we need to use that H^2 vanishes together with Hilbert's theorem 90 to finish. This might be in the other notes.] ∎

A "Mochizuki isometry" of the G_K-module $\overline{K}$ is defined to be the $\mathbb{Q}_p$-vector space automorphism that preserves $\log(\mathcal{O}_{K_1})$ for every finite extension K_1 of K.

Definition 13.0.2. We have

$$\mathrm{Ism}(\overline{K}) = \{\varphi \in \mathrm{Aut}(\overline{K}/\mathbb{Q}_p) : \forall K_1/K \text{ finite }, \varphi(\log(\mathcal{O}_{K_1}^\times)) \subset \log(\mathcal{O}_{K_1}^\times)\}$$

For (M, G) an isomorph of $\overline{K}$ as a $\mathbb{Q}_p$-vector space with action, we define $\mathrm{Ism}(M)$ as the pull-back under an isomorphism. ♠♠♠ Taylor: [Is this well-defined?] ♠♠♠ Taylor: [Mochizuki also defined this for a suitable interpretation for $G \in [G_K]$, I don't know if we need this]

Lemma 13.0.3. *[[?, 6.12]] Let K be a finite extension of $\mathbb{Q}_p$.*

(1) One may produce a section of the forgetful functor

$$[G_K, (\overline{K}, +)] \to [G_K]$$

up to $\mathrm{Ism}(\overline{K})$.

(2) Let Π be a hyperbolic curve defined over K. Let α and β be the forgetful functors in the diagram below. There exists an interpretation Ψ fitting into the diagram

$$\begin{tikzcd} {[\Pi_X, (\mathcal{O}_{\overline{K}}^*, *)]} \arrow[r, "\alpha"] \arrow[d, "\beta"] & {[\Pi_X]} \arrow[dl, "\Psi"] \\ {[\Pi_X, \log(\mathcal{O}_{\overline{K}}^\times)]} & \end{tikzcd}$$

such that $\Psi \circ \alpha$ and β are naturally isomorphic.

Proof. (1) ♠♠♠ Taylor: [Fix our interpretation $\Psi : [G_K] \to [G_K, \log(\mathcal{O}_{\overline{K}}^\times)$ and write $\Psi(G)$ for the interpreted copy of $\log(\mathcal{O}_{\overline{K}}^\times)$.

Fix an isomorphism $f : (G, M) \to (G, \Psi(G))$ and write it as $f = (f_G, f_M)$ where f_G is the identity on G and $f_M : M \to \Psi(G)$ is an isomorphism of Galois modules. An automorphism γ of (G, M) takes the form $\gamma = (\gamma_G, \gamma_M)$ where $\gamma_G \in \mathrm{Aut}(G)$ and $\gamma_M \in \mathrm{Aut}(M)$ and they are subject to

$$\gamma_G(\sigma) \cdot \gamma_M(m) = \gamma_M(\sigma \cdot m)$$

for every $\sigma \in G$ and $m \in M$. This is just saying that they are equivariant. We now need to understand how our isomophism f changes under application of γ. We have a diagram

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[d, "\gamma_M"] & \Psi(G) \arrow[d, "\Psi(\gamma_G)"'] \\ M & \Psi(G) \end{tikzcd},$$

to get two automorphisms of M we need to transport $\Psi(\gamma_G)$ to M via f_M:

$$f_M^{-1} \circ \Psi(\gamma_G) \circ f_M \in \mathrm{Aut}(M).$$

The claim is that

$$f_M^{-1} \Psi(\gamma_G) f_M \gamma_M^{-1} \in \mathrm{Ism}(M).$$

]

(2) ♠♠♠ Taylor: [We need to know how $\Psi^{\times}_{\text{cns}}$ works.]

□

14. Ind 1 and Ind 2 from the perspective of explicit bases

Let M be $\mathbf{Z}_p$-lattices inside a $\mathbb{Q}_p$-vectors space V (meaning the $\mathbb{Q}_p$-span of M is all of V and M injects into its $\mathbb{Q}_p$-span.) Let $\mathrm{Aut}_{\mathbb{Q}_p}(A : M)$ denote the group of $\mathbb{Q}_p$-vector space automorphisms of V inducing a $\mathbb{Z}_p$-module automorphism of M. Let N be another sublattice of V. Can we compute $\bigcup_{\varphi \in \mathrm{Aut}(V:M)} \varphi(N)$?

Example 14.0.1. $V = L = \bigotimes_{i=1}^{n} K_i$, $N = \mathcal{O}_L$, $M = \bigotimes \mathcal{I}_{K_i}$

We can pick a $\mathbb{Z}_p$-basis $m_1, \dots, m_d$ for M and a $\mathbb{Z}_p$-basis for $n_1, \dots, n_d$ for N. They are both going to be a $\mathbb{Q}_p$-basis for V and hence there exists some $(a_{ij}) \in \mathrm{GL}_d(\mathbb{Q}_p)$ such that

$$n_i = \sum_{j=1}^{d} a_{ij} m_j.$$

Also, any automorphism φ of M will give a matrix $(\varphi_{ij}) \in \mathrm{GL}_d(\mathbb{Z}_p)$ via

$$\varphi(m_i) = \sum_{j=1}^{d} \varphi_{ij} m_j.$$

How does φ act on elements $x \in N$? Well, this is completely determined by how it acts on the n_i and we can now compute this:

$$\begin{aligned}
\varphi(n_i) &= \varphi(\sum_{j=1}^{d} a_{ij} m_j) \\
&= \sum_{j=1}^{d} a_{ij} \sum_{k=1}^{d} \varphi_{jk} m_k \\
&= \sum_{k=1}^{d} \left(\sum_{j=1}^{d} a_{ij} \varphi_{jk} \right) m_k \\
&= \sum_{k=1}^{d} \left(\sum_{j=1}^{d} a_{ij} \varphi_{jk} \right) \sum_{l=1} b_{kl} n_l \\
&= \sum_{l=1}^{d} \left(\sum_{k=1}^{d} \sum_{j=1}^{d} a_{ij} \varphi_{jk} b_{kl} \right) n_l
\end{aligned}$$

this shows that if we choose an isomorphism $f : \mathbb{Z}_p^d \to N$ given by $f(e_i) = n_i$ then $f^{-1}\varphi f = A\Phi A^{-1}$ or in matrices

$$\begin{bmatrix} \varphi(n_1) \\ \varphi(n_2) \\ \vdots \\ \varphi(n_d) \end{bmatrix} = A\Phi A^{-1} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_d \end{bmatrix}.$$

This proves the following:

Lemma 14.0.2. *If $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Q}_p)$ is the change of basis matrix given by $n_i = \sum_{j=1}^d a_{ij}m_j$ and $f : \mathbf{Z}_p^d \to N$ is the isomorphism given by $f((x_1, \dots, x_d)^t) = x_1n_1 + \cdots + x_dn_d$ then between M and N then after fixing an isomorphism*

$$\bigcup_{\varphi \in \mathrm{Aut}(V:M)} \varphi(N) = fA\,\mathrm{GL}_2(\mathbb{Z}_p)A^{-1}f^{-1}N.$$

Example 14.0.3. Let K be totally ramified extension of $\mathbb{Q}_p$ of degree d with uniformizer π. Let $V = K$, $M = \log(\mathcal{O}_K^\times)$ and $N = \mathcal{O}_K$. Then $1, \pi, \pi^2, \dots, \pi^{e-1}$ is a basis for $\mathcal{O}_K$.

♠♠♠ Taylor: [We need to figure out a way to write down a basis of $\log(\mathcal{O}_K^\times)$ to finish this, then compute the change of basis matrix. This is the same as a basis for $(1 + \pi\mathcal{O}_K)/(tors)$.]

Once we know the change of basis matrix between M ($M = \bigotimes \log(\mathcal{O}_K^\times)$ or $M = \log(\mathcal{O}_K^\times)$) and ($N = \mathcal{O}_K$ or $N = \mathcal{O}_L$) we will be able to write down things like here $n_1, \dots, n_d$ is a basis for N

$$\varphi(n_i) = \sum_{j=1}^d c_{ij}n_j$$

where $a_{ij} \in \mathbb{Q}_p$. Then we can define $C_\varphi = \lceil \max_{1 \le i,j \le d}\{|a_{ij}|_p\} \rceil$ and we will have

$$\varphi(N) \subset \frac{1}{C_\varphi}N.$$

Note that $\max_{1 \le i,j \le d} = \|A\Phi A^{-1}\|_\infty$ which is the p-adic l^∞ norm, (Φ is the matrix for φ with respect to the basic $m_1, \dots, m_n$) We also have that for all $\Phi \in \mathrm{GL}_2(\mathbb{Z}_p)$, $\|\Phi\|_\infty$.

While the matrix ∞-norm is not submultiplicative for Archimedean norms, it is submultiplicative for non-archimedean norms.

Lemma 14.0.4. *(1) Let $A, B \in M_d(\mathbb{Q}_p)$.*

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$$

(2) Let $A \in \mathrm{GL}_d(\mathbb{Q}_p)$. Then

$$\|A^{-1}\|_\infty = \frac{1}{|\det(A)|_p}\|A\|_\infty$$

(3)

Proof. (1) We examine the absolute value of each entry and get:

$$|(AB)_{ik}|_p = |\sum_{j=1}^{d} a_{ij}b_{jk}|_p \leq \max_{1\leq j\leq k} |a_{ij}b_{jk}|_p \leq \|A\|_\infty \|B\|_\infty.$$

Since this is true over all entries of AB we get

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty.$$

(2) We can write $A^{-1} = \frac{1}{\det(A)}\widetilde{A}$ where $\widetilde{A}$ is the adjugate matrix. Since the adjugate matrix is just a permutation of the entries of A (with a possible sign change) we have $\|\widetilde{A}\|_\infty = \|A\|_\infty$. Hence the result follows.

□

This means that for every $A\Phi A^{-1}$ we have

$$\|A\Phi A^{-1}\|_\infty \leq \|A\|_\infty \|\Phi\|_\infty \|A^{-1}\|_\infty \leq \frac{\|A\|^2}{|\det(A)|_p}.$$

Note that $\|\Phi\|_\infty = 1$ since $\Phi \in \mathrm{GL}_2(\mathbb{Z}_p)$. Hence for every φ (provided we show basis independence) we have

$$\max_{1\leq i,j\leq d} |c_{ij}|_p \leq \frac{\|A\|_\infty^2}{|\det(A)|_p}$$

This implies

$$\log_p(C_\varphi) = \lceil \log_p(\frac{\|A\|_\infty^2}{|\det(A)|_p})\rceil$$

Note that this is independent of φ. We actually claim it is independent of the choice of basis matrix. Such a change will have the effect of multiplying by some change of basis matrix $Q \in \mathrm{GL}_d(\mathbb{Z}_p)$ which have non-archimedean norm one. ♠♠♠ Taylor: [this needs to be checked and proved rigorously] hence we have a quantity which is independent the basis and only depends on relating a basis of M to a basis of N. We define the constant $C(N : M) \in p^{\mathrm{N}}$ by the formula

$$\log_p C(M : N) := \lceil \log_p(\frac{\|A\|_\infty^2}{|\det(A)|_p})\rceil.$$

Lemma 14.0.5. $\bigcup_{\varphi\in \mathrm{Aut}(\mathbb{Q}_p:M)} \varphi(N) \subset \frac{1}{C(N:M)}N$

Lemma 14.0.6. *Let $K_1, \dots, K_n$ be p-adic fields and consider $L = K_1 \otimes \cdots \otimes K_n$. Letting $N = \bigotimes \log(\mathcal{O}_{K_i})$ and $M = \mathcal{O}_L$ be $\mathbb{Z}_p$-modules of L. We have*

$$\bigcup_{\varphi\in \mathrm{Aut}_{\mathbb{Q}_p}(L:\bigotimes_{i=1}^n \log(\mathcal{O}_{K_i}^\times))} \varphi(\mathcal{O}_L) \subset \frac{1}{C(\mathcal{O}_L : \bigotimes_{i=1}^n \log(\mathcal{O}_{K_i}^\times))}\mathcal{O}_L.$$

♠♠♠ Taylor: [$C(\mathcal{O}_K : p^{-1}\mathcal{O}_K) = p$]
♠♠♠ Taylor: [What is

$$\bigcup_{\varphi} \varphi(\mathrm{peel}^n(q)\mathcal{O}_L)?$$

I think we have]

Example 14.0.7. In the very special case of $K = \mathbb{Q}_p(p^{1/p^2})$ we can consider the case $M = \mathcal{O}_K$ and $N = \log(\mathcal{O}_K^\times)$. We need to $\pi = p^{1/p^2}$ and we showed that $\log(1+\pi^i)$ form a basis where $1 \leq i \leq p^2\frac{p}{p-1}$ and $(i,p)=1$. Let's write

$$\{i : 1 \leq i \leq p^3/(p-1), (i,p) = 1\} = \{ap+b : 0 \leq a < p^2/(p-1) \text{ and } 1 \leq b \leq p-1\}$$

which index the p^2 basis elements ♠♠♠ Taylor: [we needed to check this]. We need to find constants $c_{ap+b,j} \in \mathbb{Q}_p$ such that

$$\pi^j = \sum_a \sum_b c_{ap+b,j} \log(1+\pi^{ap+b})$$

which will give us our change of basis matrix. Here the matrix A in our formulas above is $A = (c_{ap+b,j})$. ♠♠♠ Taylor: [It is enough to find an upper and lower bounds on $|c_{ap+b,j}|_p$ to estimate $\|A\|_\infty^2/|\det(A)|_p$.]

♠♠♠ Taylor: [If you convert this into a matrix then]

15. A first intermediate inequality: log volume computations on tensor powers of fake adeles for the field of moduli

The goal of this section is to compute our first explicit upper bound for $\overline{\log \mu_{\mathbb{L}}}(\mathrm{hull}(U_\Theta))$. We recall that

$$\mathrm{hull}(U_\Theta) = \bigoplus_{p,j} \mathrm{hull}((U_\Theta)_p^{(j)}) = \bigoplus_{p,j} \bigoplus_{\underline{\vec{v}} \in \underline{V}_p^{j+1}} \mathrm{hull}(U_\Theta)_{\underline{\vec{v}}},$$

where $U_\Theta \subset \mathbb{L}$ is the orbit of the region associated to the Theta pilot object under indeterminacies.

In what follows we will let $\mathbf{E}_p^2 = \mathbf{E}_{\{1,\ldots,(l-1)/2\}} \circ \mathbf{E}_{V(F_0)^{j+1}}$ denote taking averages over tuples $(v_0,\ldots,v_j)$ followed by taking uniform averages over the set $\{1,\ldots,(l-1)/2\}$

Theorem 15.0.1. *Fix initial theta data*

$$(\overline{F}/F, l, E_F, \underline{C}_K, \underline{V}, \underline{V}_{mod}^{\mathrm{bad}}, \underline{\epsilon}),$$

and let U_Θ be the multiradial region associated to this data. The following inequality holds:
(15.1)
$$\overline{\log \mu_{\mathbb{L}}}(\mathrm{hull}(U_\Theta)) \leq \sum_p \mathbf{E}_p^2(-\mathrm{ord}_p(\underline{\underline{q}}_{\underline{v}}^{j^2}))\ln(p) + \sum_p \mathbf{E}_p^2(1 + \|\,\mathrm{diff}_{\underline{\vec{v}}}\,\|_1 - \|\,\mathrm{diff}_{\underline{\vec{v}}}\,\|_\infty + C_{\vec{v}}\ln(p))$$

where

♠♠♠ Taylor: [

- It looks like there is a conductor from removing a floor in $\mathrm{ord}_p(q_{\underline{v}_j}^{j^2})$ but it doesn't come from here! It comes from the different term!
- The term

$$\sum_p \mathbf{E}_p^2(1 + \|\,\mathrm{diff}_{\underline{\vec{v}}}\,\|_1 - \|\,\mathrm{diff}_{\underline{\vec{v}}}\,\|_\infty + C_{\underline{\vec{v}}})\ln(p)$$

  is really what will give rise to Szpiro.

]

- *The expression* $-\operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}}^{j^2})$ *is understood to mean the map*

$$V(F_0)_p \ni v \mapsto \begin{cases} -\operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}}^{j^2}), & v \text{ a bad place of } E/F_0 \\ 0, & \text{otherwise} \end{cases}$$

- *The expression* $1 + \| \operatorname{diff}_{\vec{\underline{v}}} \|_1 - \| \operatorname{diff}_{\vec{\underline{v}}} \|_\infty - C_{\vec{v}}$ *is understood to mean the map*

$$V(F_0)_p^{j+1} \ni \vec{v} \mapsto \begin{cases} 1 + \| \operatorname{diff}_{\vec{\underline{v}}} \|_1 - \| \operatorname{diff}_{\vec{\underline{v}}} \|_\infty + C_{\vec{v}}, & \vec{v} \text{ ramified} \\ 0, & \text{otherwise} \end{cases}$$

- *For a vector* $\vec{\underline{v}} = (\underline{v}_0, \ldots, \underline{v}_j) \in \underline{V}_p^{j+1}$ *the vector* $\operatorname{diff}_{\vec{\underline{v}}}$ *is the vector of different exponents*

$$\operatorname{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)} = (\operatorname{diff}(K_{\underline{v}_0}/\mathbb{Q}_p), \ldots, \operatorname{diff}(K_{\underline{v}_j}/Q_p)),$$

and $\| \operatorname{diff}_{\vec{\underline{v}}} \|_1$ *and* $\| \operatorname{diff}_{\vec{\underline{v}}} \|_\infty$ *denote the* l^1*-norm and* l^∞*-norm of this vector respectively.*

- *For* $(v_0, \ldots, v_j) \in V(F)_p^{j+1}$ *the constant* $C_{(v_0,\ldots,v_j)}$ *is given by*

$$C_{(v_0,\ldots,v_j)} = \log_p \left(b_p^{j+1} e(\underline{v}_0/p) \cdots e(\underline{v}_j/p) \right),$$

and

$$b_p = \frac{1}{\log(p)\exp(1)|2p|_p}.$$

Here $\exp(1) = 2.71828182\ldots$ *is Euler's constant.*

We will now discuss how we will arrive at this upper bound, but first it is important that we understand the regions

$$\operatorname{hull}(U_\Theta)_p^{(j)}$$

are themselves are products of regions $\operatorname{hull}(U_\Theta)_{\vec{v}} \subset L_{\vec{\underline{v}}}$ for $\vec{v} \in V(F)_p^{j+1}$ with corresponding theta-data list $\vec{\underline{v}} \in \underline{V}_p^{j+1} \subset V(K)_p^{j+1}$. First, let's describe the regions $(U_\theta)_{\vec{v}}$. ♠♠♠ Taylor: [I started relabeling things with $v \in V(F)$ instead of $\underline{v}$ wherever possible]

Here $\operatorname{hull}(U_\Theta)_{\vec{\underline{v}}}$ is by definition equal to $\operatorname{hull}((U_\Theta)_{\vec{\underline{v}}})$.

We remind the reader that in all of the cases, $(U_\Theta)_{\vec{\underline{v}}}$ is a component of jth component of the p-part of the region determined by the jth component of the theta pilot divisor $(P_{\Theta,j})$ up to indeterminacies $\mathrm{Ind1}, \mathrm{Ind2}, \mathrm{Ind3}$ (this is not a typo).

$$(U_\Theta)_p^{(j)} = \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2 \rangle \cdot (((\underline{\Omega}_{P_{\Theta,j}}^{(j)})_p)^{\mathrm{ind}\,3}).$$

This means the indeterminacies act component-wise at the level of $\mathbb{Q}_p$-vector spaces indexed by $\vec{\underline{v}} \in \underline{V}_p^{j+1}$. Explicitly, we have

$$\underline{\Omega}_{P_{\Theta,j}}^{(j)} = \operatorname{peel}^j((a_v)_{\underline{v}|p}) \cdot \mathcal{O}_{\mathbb{A}_{K,\underline{v},p}^{\otimes j+1}}.$$

Where $P_{\Theta,j} \in \widehat{\operatorname{Div}}(F_0)$ is defined as

$$P_{\Theta,j} = \sum_{v \text{ bad}} \operatorname{ord}_v(\underline{\underline{q}}_{\underline{v}}^{j^2})[v].$$

As a remark, $E_{F_{0,v}}$ may not have split multiplicative reduction (although it has multiplicative reduction at its bad places). We take F to be a field extension of the field of moduli $F_0 = \mathbb{Q}(j_E)$ so Tate parameters only make sense for bad places of F. In particular $q_{\underline{v}}$ is a Tate parameter for $E_{F_{\underline{v}_F}}$ and lives in the field $F_{\underline{v}_F}$. Here $\underline{v}_F$ denote the image of $\underline{v} \in V(K)$ under the map $V(K) \to V(F)$.

In the fake-adelic fractional ideal encoding we use that

$$P_{\Theta,j} = \sum_{v \in V(F_0)} \mathrm{ord}_v(a_v)[v]$$

where

$$a_v = \begin{cases} \underset{=}{q}_{\underline{v}}, & v \text{ bad place of } E \\ 1, & v \text{ good place} \end{cases}.$$

Note that we could have chosen $u_{\underline{v}} a_v$ in place of a_v for any unit $u_v \in \mathcal{O}_{K_{\underline{v}}}$.

For any $\underline{\vec{v}} \in \underline{V}_p^{j+1}$ we have

$$(U_\Theta)_{\underline{\vec{v}}} = \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2 \rangle \cdot ((\mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{ind}\,3}). \tag{15.2}$$

We will now describe these regions futher using the bad/good but ramified/unramified cases of $\underline{\vec{v}} \in \underline{V}_p^{j+1}$. For $\underline{\vec{v}} = (\underline{v}_0, \ldots, \underline{v}_j) \in \underline{V}_p^{j+1}$ the description of $(U_\Theta)_{\underline{\vec{v}}}$ falls into two cases, the case where $\underline{v}_j$ is bad and we have a Tate parameter, and the case where $\underline{v}_j$ is good and here $a_{\underline{v}} = 1$ (in reality we only need $a_{\underline{v}} \in K_{\underline{v}}^\times / \mathcal{O}_{K_{\underline{v}}}^\times$ which is what Mochizuki considers).

We will now find upper bounds for $\mathrm{hull}(U_{\underline{\vec{v}}})$ by breaking studying $\underline{v}$ in various cases. There are three cases to consider:

Bad Case: The place $\underline{v}_j$ in the tuple $(\underline{v}_0, \ldots, \underline{v}_j)$ is a place of bad reduction for the elliptic curve $E_{K_{\underline{v}_j}}$ (by the way, this automatically implies that $e(\underline{v}_j/p) > 0$).

Unramified case: Here, the ramification indices satisfy $e(\underline{v}_i/p) = 0$ for all i.

Good, but ramified case: The place $\underline{v}_j$ in the tuple $(\underline{v}_0, \ldots, \underline{v}_j)$ is a place of good reduction for the elliptic curve E_K, but there exists some i such $\underline{v}_i$ in the tuple $(\underline{v}_0, \ldots, \underline{v}_j)$ has $e(\underline{v}_i/p) > 0$.

We will do the bad case last since it is the most interesting.

15.1. **Unramified case.** The following lemma shows why the log-volume computation is not infinite.

Lemma 15.1.1. *Suppose that* $\underline{\vec{v}} = (\underline{v}_0, \ldots, \underline{v}_j) \in \underline{V}_p^{j+1}$ *with* $e(\underline{v}_i/p) = 1$ *for* $i = 0, \ldots, j$. *Then* $\mathrm{hull}((U_\Theta)_{\underline{\vec{v}}}) = \mathcal{O}_{L_{\underline{v}}}$.

Proof. Let $\underline{\vec{v}} = (\underline{v}_0, \ldots, \underline{v}_j)$. Since for each i where $1 \le i \le j$ we have $e(\underline{v}_i/p) = 1$ we have

$$\mathcal{I}_{\underline{\vec{v}}} = \bigotimes_{i=1}^{j} \mathcal{O}_{K_{\underline{v}_i}} = \mathcal{O}_{L_{\underline{\vec{v}}}}.$$

The first equality follows from the first assertion in Lemma 6.2.4 and the second equality follows from Lemma 3.0.1 3. We claim that $\mathcal{O}_{L_{\underline{\vec{v}}}}^{\mathrm{ind}\,3} = \mathcal{O}_{L_{\underline{v}}}$. This follows from the fact that

$$\begin{aligned}
\mathcal{O}_{L_{\underline{\vec{v}}}}^{\mathrm{ind}\,3} &= \mathrm{quot}(\mathrm{quot}^{-1}(\mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{ind}\,3}) \\
&= \mathrm{quot}(\mathrm{quot}^{-1}(\mathcal{O}_{L_{\underline{\vec{v}}}}) \cup \log(\mathbb{Q}_p[\prod_{i=1}^n \mathcal{O}_{K_{\underline{v}_i}}^\times] \cap \mathrm{quot}^{-1}(\mathcal{O}_{L_{\underline{\vec{v}}}})) \\
&= \mathcal{O}_{L_{\underline{\vec{v}}}} \cup \mathrm{quot}(\log(\mathbb{Q}_p[\prod_{i=1}^n \mathcal{O}_{K_{\underline{v}_i}}^\times] \cap \mathrm{quot}^{-1}(\mathcal{O}_{L_{\underline{\vec{v}}}})))
\end{aligned}$$

♠♠♠ Taylor: [FIXME]

Hence

$$\begin{aligned}
(U_\Theta)_{\underline{\vec{v}}} &= \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle \cdot ((\mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{ind}\,3}) \\
&= \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle \cdot (\mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{ind}\,3} \\
&\subset \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle \cdot \mathcal{I}_{\underline{\vec{v}}} \\
&= \mathcal{I}_{\underline{\vec{v}}} \\
&= \mathcal{O}_{L_{\underline{v}}}.
\end{aligned}$$

Since $\mathcal{O}_{L_{\underline{\vec{v}}}} \subset \mathrm{hull}((U_\Theta)_{\underline{\vec{v}}}) \subset \mathrm{hull}(\mathcal{O}_{L_{\underline{\vec{v}}}}) = \mathcal{O}_{L_{\underline{\vec{v}}}}$ we have equality. □

Remark 15.1.2. In general $\mathcal{O}_{L_{\underline{\vec{v}}}} \not\subseteq \mathcal{I}_{\underline{\vec{v}}}$. Indeed, there exists finite extensions $K_1, \ldots, K_n$ of $\mathbb{Q}_p$ such that $\bigotimes_{i=1}^n \mathcal{I}_{K_i}$ does not contain $\mathcal{O}_L$ where $L = \bigotimes_{i=1}^n K_i$. We have shown that

$$\bigotimes_{i=1}^n \mathcal{I}_{K_i} \subset D_L(0, (pc_p)^{j+1} \prod_{i=1}^n e(K_i/\mathbb{Q}_p)).$$

The ramification is small so that

$$(pc_p)^{j+1} \prod_{i=1}^n e(K_i/\mathbb{Q}_p) < 1$$

and hence we can't have $\mathcal{O}_L = D_L(0,1) \subset D_L(0,r)$ for $r < 1$.

Note that this implies that we do not have $(\mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{Ind}\,3*} := \mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{I}_{\underline{\vec{v}}}$ or $(\mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{O}_{L_{\underline{\vec{v}}}})^{\mathrm{Ind}\,3*} \subset \mathrm{peel}^j(a_{\underline{v}_j}) \cdot \mathcal{I}_{\underline{\vec{v}}}$ in general.

15.2. **Good, but ramified case.**

Lemma 15.2.1. *Let $\underline{\vec{v}} = (\underline{v}_0, \ldots, \underline{v}_j) \in \underline{V}_p^{j+1}$ and suppose that $\underline{v}_j$ is a place of good reduction of E_K (so that we don't have a Tate parameter). Then*

(1) $(U_\Theta)_{\underline{\vec{v}}} \subset p^{\|\mathrm{diff}_{\underline{\vec{v}}}\|_\infty - \|\mathrm{diff}_{\underline{\vec{v}}}\|_\infty} \mathcal{I}_{\underline{\vec{v}}}$.

(2) $\mathrm{hull}^*(U_{\Theta,\underline{\vec{v}}}) \subset D_{L_{\underline{v}}}(0, R)$ *where*

$$\log_p(R) = \|\mathrm{diff}_{\underline{\vec{v}}}\|_1 - \|\mathrm{diff}_{\underline{\vec{v}}}\|_\infty - C_{\underline{\vec{v}}}$$

and hence

$$\overline{\log \mu}_{L_{\underline{\vec{v}}}}(\mathrm{hull}^*(U_{\Theta,\underline{\vec{v}}})) \leq (\|\mathrm{diff}_{\underline{\vec{v}}}\|_1 - \|\mathrm{diff}_{\underline{\vec{v}}}\|_\infty - C_{\underline{\vec{v}}}) \log(p)$$

where

$$-C_{\vec{\underline{v}}} = (j+1) + \sum_{i=0}^{j} \log(c_p e(\underline{v}_i/p))$$

♠♠♠ Anton: [Shouldn't there be a 1 (coming from the floor) here just as in the bad case?]

Proof. (1) By Lemma **??** we have

$$(\mathcal{O}_L)^{\operatorname{Ind} 3*} \subset \mathcal{O}_L \cup \mathcal{I}_L.$$

The action of $\operatorname{Aut}_{\mathbb{Q}_p}(L:\mathcal{I}_L)$ on $\mathcal{I}_L$ is trivial so it remains to find an upper bound for $\operatorname{Aut}_{\mathbb{Q}_p}(L:\mathcal{I}_L)\cdot\mathcal{O}_L$. We have

$$\mathcal{O}_L \subset p^{-\|\operatorname{diff}\|_1 + \|\operatorname{diff}\|_\infty} \bigotimes_{i=0}^{j} \mathcal{O}_{K_i} \subset p^{-\|\operatorname{diff}\|_1 + \|\operatorname{diff}\|_\infty} \bigotimes_{i=0}^{j} \mathcal{I}_{K_i}.$$

This proves

$$(\mathcal{O}_L)^{\operatorname{Ind} 3*} \subset p^{-\|\operatorname{diff}\|_1 + \|\operatorname{diff}\|_\infty} \mathcal{I}_L.$$

(2) We have $\log(\mathcal{O}_{K_i}^\times) \subset D_{K_i}(0, B_i)$ with $B_i = c_p e_i$. To compute an upper bound for $\operatorname{hull}^*(U_{\Theta,\vec{\underline{v}}})$ we need to convert these bounding sets into principal fractional ideals.

$$p^{-\|\operatorname{diff}\|_1 + \|\operatorname{diff}\|_\infty} \mathcal{I}_L \subset p^{\|\operatorname{diff}\|_1 + \|\operatorname{diff}\|_\infty} D_L(0, B),$$

where $B = c_p^{j+1} p^{j+1} (e_0 \cdots e_j)$.

□

15.3. Bad case. ♠♠♠ Taylor: [This term needs to be fixed]

Let $\vec{\underline{v}} = (\underline{v}_0, \dots, \underline{v}_j) \in \underline{V}_p^{j+1}$ be such that $\underline{v}_j$ is a place of bad reduction of E_K.

Lemma 15.3.1. *(1)* $\langle \operatorname{Ind} 1, \operatorname{Ind} 2\rangle \cdot (\underline{\Omega}^{(j)}_{P_{\Theta,j},\vec{\underline{v}}})^{\operatorname{Ind} 3*} \subset p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}_j})\rfloor + \|\operatorname{diff}_{\vec{\underline{v}}}\|_\infty - \|\operatorname{diff}_{\vec{\underline{v}}}\|_1} \mathcal{I}_{\vec{\underline{v}}}$

(2) We have the containment $U_{\Theta,\vec{\underline{v}}} = \langle \operatorname{Ind} 1, \operatorname{Ind} 2\rangle \cdot (\underline{\Omega}^{(j)}_{P_{\Theta,j},\vec{\underline{v}}})^{\operatorname{Ind} 3*} \subset D_{L_{\vec{\underline{v}}}}(0, R)$ *where*

$$\log_p(R) < 1 - j^2 \operatorname{ord}_p(\underline{\underline{q}}) + \|\operatorname{diff}_{\vec{\underline{v}}}\|_1 - \|\operatorname{diff}_{\vec{\underline{v}}}\|_\infty - C_{\vec{\underline{v}}}.$$

This proves

$$\overline{\log\mu}_{L_{\vec{\underline{v}}}}(\operatorname{hull}(U_{\Theta,\vec{\underline{v}}})) \le (1 - j^2 \operatorname{ord}_p(\underline{\underline{q}}) + \|\operatorname{diff}\|_1 - \|\operatorname{diff}\|_\infty - C_{\vec{\underline{v}}}) \log(p).$$

where

$$-C_{\vec{\underline{v}}} = j + 1 + \sum_{i=0}^{j} \log_p(c_p e_i)$$

Proof. For simplicity we will use the notation $K_i = K_{\underline{v}_i}$, $L = L_{\vec{\underline{v}}}$, $\mathcal{I}_L = \mathcal{I}_{\vec{\underline{v}}}$, $\underline{\underline{q}} = \underline{\underline{q}}_{\underline{v}_j}$ and $e_i = e(K_i/\mathbb{Q}_p)$. We will also use $\operatorname{diff} = \operatorname{diff}_{\vec{\underline{v}}}$.

(1) We have

$$(\underline{\underline{q}}^{j^2} \cdot \mathcal{O}_L)^{\mathrm{Ind}\,3*} = \underline{\underline{q}}^{j^2} \cdot (\mathcal{O}_L)^{\mathrm{Ind}\,3*}$$
$$= \underline{\underline{q}}^{j^2} \cdot (\mathcal{O}_L \cup \mathcal{I}_L)$$

First, equality follows from the non-interference property and the second equality follows from a computation of $(\mathcal{O}_L)^{\mathrm{Ind}\,3}$. We will now compute upper bounds for $\langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle \cdot \underline{\underline{q}}^{j^2}\mathcal{O}_L$ and $\langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle \cdot \underline{\underline{q}}^{j^2}\mathcal{I}_L$.

- We recall that by Lemma 3.0.1 we have

$$\mathcal{O}_L \subseteq p^{\|\operatorname{diff}_{\underline{\vec{v}}}\|_\infty - \|\operatorname{diff}_{\underline{\vec{v}}}\|_1} \bigotimes_{i=1}^{j} \mathcal{O}_{K_i}.$$

Since $\mathcal{O}_{K_i} \subset \frac{1}{2p}\log(\mathcal{O}_{K_i}^\times)$ we have $\bigotimes_{i=0}^{j} \mathcal{O}_{K_i} \subseteq \bigotimes_{i=1}^{j} \frac{1}{2p}\log(\mathcal{O}_{K_i}^\times)$, and hence

$$\underline{\underline{q}}^{j^2}\mathcal{O}_L \subseteq p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor + \|\operatorname{diff}_{\underline{\vec{v}}}\|_\infty - \|\operatorname{diff}_{\underline{\vec{v}}}\|_1}\mathcal{I}_L \tag{15.3}$$

We can now consider the effect of Ind 1 and Ind 2, as encoded in the $\mathbf{Q}_p$-linear automorphism φ of L acting as an automorphism on the lattice $\mathcal{I}_L$. The action is trivial on the left hand side of (15.3). This proves containment.

- We now consider the action of $\langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle$ on $\underline{\underline{q}}^{j^2} \cdot \mathcal{I}_L$. We have $\underline{\underline{q}}^{j^2}\mathcal{I}_L \subset p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor}\mathcal{I}_L$. This gives

$$\langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle(\underline{\underline{q}}^{j^2}\mathcal{I}_L) \subset \langle \mathrm{Ind}\,1, \mathrm{Ind}\,2\rangle(p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor}\mathcal{I}_L)$$
$$= p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor}\mathcal{I}_L$$

(2) By Lemma 1, we have $\log(\mathcal{O}_{K_i}) \subset D_K(0, B_i)$ where $B_i = c_p e_i$ where $c_p = (\log(p)\exp(1))^{-1}$. Taking the tensor product over all of these inclusions gives

$$\mathcal{I}_L = \bigotimes_{i=0}^{j} \mathcal{I}_{K_i} \subset \bigotimes_{i=0}^{j} D_L(0, B)$$

where

$$\log_p(B) < j + 1 - \sum_{i=0}^{j} \log(c_p e_i).$$

In addition we have

$$\bigotimes_{i=1}^{j} \mathcal{O}_{K_i} \subseteq \mathcal{O}_L.$$

This gives

$$p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor + \|\operatorname{diff}_{\underline{\vec{v}}}\|_\infty - \|\operatorname{diff}_{\underline{\vec{v}}}\|_1}\mathcal{I}_{\underline{\vec{v}}}$$
$$\subseteq p^{\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}})\rfloor + \|\operatorname{diff}_{\underline{\vec{v}}}\|_\infty - \|\operatorname{diff}_{\underline{\vec{v}}}\|_1} D_L(0, B) \subset D_L(0, R)$$

where B

$$\log_p(R) < -\lfloor j^2 \operatorname{ord}_p(\underline{\underline{q}}) \rfloor - \| \operatorname{diff}_{\underline{\vec{v}}} \|_\infty + \| \operatorname{diff}_{\underline{\vec{v}}} \|_1 + j + 1 + \sum_{i=0}^{j} \log_p(c_p e_i)$$

which proves

$$\log_p(R) < 1 - j^2 \operatorname{ord}_p(\underline{\underline{q}}) + \| \operatorname{diff} \|_1 + \| \operatorname{diff} \|_\infty + j + 1 + \sum_{i=0}^{j} \log_p(c_p e_i).$$

□

15.4.

16. A second intermediate inequality: computing probabilities

Theorem 16.0.1.

Remark 16.0.2 (Fesenko's Inequality c.f. [Fes15, §2.12]). From this one can already see the inequality as stated in [Fes15, §2.12]. Combining the inequalities $-\widehat{\underline{\deg}}(P_q) \leq \overline{\log \nu_{\mathbf{L}}}(\operatorname{hull}(U_\Theta))$ and $\overline{\log \nu_{\mathbf{L}}}(\operatorname{hull}(U_\Theta)) \leq a(l) - b(l)\widehat{\underline{\deg}}(P_q)$ one has obtains $(b(l)-1)\widehat{\underline{\deg}}(P_q) \leq a(l)$ and hence

$$\widehat{\underline{\deg}}(P_q) \leq \frac{a(l)}{b(l)-1}.$$

Playing this this a little bit more is what gives the Szpiro inequality for Elliptic curves in initial theta data.

It remains to find upper bounds of $U_{\underline{\vec{v}}}$, and compute the hulls of those regions and compute the normalized log-volumes of these regions.

16.1. **Multiplying discrete random variables.** We are going to treat the spaces $V(F_0)_p$ as a discrete probability spaces where $v \in V(F_0)_p$ occur with probability $\mathbb{P}(v) = [F_{0,v} : \mathbb{Q}_p]/[F_0 : \mathbb{Q}_p]$.

If we want to compute

$$\mathbf{E}(g(v_1)g(v_2) \cdot g(v_n) : \vec{v} = (v_1, \ldots, v_n) \in V(F_0)_p)$$

for $g : V(F_0) \to \mathbb{R}$ and we are treating each v_i like an independent random variable we can do the following:

$$\begin{aligned}
\mathbf{E}(g(v_1)g(v_2) \cdot g(v_n)) &= \frac{1}{[F_0 : \mathbb{Q}]^n} \sum_{(v_1,\ldots,v_n)\in V(F_0)_p^n} g(v_1)g(v_2) \cdot g(v_n)) \prod_{i=1}^{n} [F_{0,v_i} : \mathbb{Q}_p] \\
&= \frac{1}{[F_0, \mathbb{Q}]^n} \sum_{(v_1,\ldots,v_n)\in V(F_0)_p^n} \prod_{i=1}^{n} g(v_i)[F_{0,v_i} : \mathbb{Q}_p] \\
&= \frac{1}{[F_0, \mathbb{Q}]^n} (\sum_{v\in V(F_0)_p} g(v)[F_{0,v} : \mathbb{Q}_p])^n \\
&= \mathbf{E}(g(v))^n.
\end{aligned}$$

16.2. **Jensen's inequality.** Jensen's inequality states that for a convex function $g(x)$ and a random variable X that

$$g(\mathbf{E}(X)) \leq \mathbf{E}(g(X)).$$

The inequality goes the other way for concave functions and one can test for convexity using the second derivative test: a function of a real variable $g(x)$ is convex if and only $g''(x) \geq 0$. In particular $g(x) = \exp(x)$ is a convex function and $g(x) = \log(x)$ in concave. This allows us to say that

$$\exp(\mathbf{E}(\log(X))) \leq \mathbf{E}(X) \leq \log(\mathbf{E}(\exp(X))). \tag{16.1}$$

16.3. **The different term.** In our local computations it turns out that we need to compute (at a single prime p) the value of

$$\mathbf{E}^2(\| \operatorname{diff}_{\underline{\vec{v}}} \|_1 - \| \operatorname{diff}_{\underline{\vec{v}}} \|_\infty).$$

Let's explain this notation. The first $\mathbf{E}$ is averaging over $V(F_0)_p^{j+1}$ (where $v \in V(F_0)_p$ is weighted by $[F_{0,v} : \mathbb{Q}_p]$) for a fixed $j \in \{1, \ldots, (l-1)/2\}$. The second $\mathbf{E}$ is averaging that result over $\{1, \ldots, (l-1)/2\}$ with the uniform probability distribution.

The main result of this subsection is that

$$\mathbf{E}^2(\| \operatorname{diff}_{(\underline{v}_0, \ldots, \underline{v}_j)} \|_1 - \| \operatorname{diff}_{(\underline{v}_0, \ldots, \underline{v}_j)} \|) \leq \frac{(l+1)}{4} \overline{\operatorname{diff}}_p \tag{16.2}$$

which is proved in Lemma 16.3.3.

Let look at this first average as $\vec{v} = (\underline{v}_1, \ldots, \underline{v}_n)$ varies over $V(F_0)_p^n$ for some n. First, we will obtain a stupid inequality. We can probably do better, but we will make do with this for now.

Lemma 16.3.1. *For $\vec{v} \in V(F_0)_p^n$ we have*

$$\| \operatorname{diff}_{\underline{\vec{v}}} \|_1 - \| \operatorname{diff}_{\underline{\vec{v}}} \|_\infty \leq \frac{n-1}{n} \| \operatorname{diff}_{\underline{\vec{v}}} \|_1$$

Proof. This has nothing to do with the particular vector. We can just consider positive numbers $a_1, \ldots, a_n$ we have

$$\begin{aligned} n(\sum_{i=1}^n a_i - \max_{1 \leq i \leq n} a_i) =& n(\sum_{i=1}^n a_i) - n \max_{1 \leq i \leq n} a_i) \\ \leq& n(\sum_{i=1}^n a_i) - \sum_{i=1}^n a_i \\ =& (n-1) \sum_{i=1}^n a_i, \end{aligned}$$

so this proves

$$\|\vec{a}\|_1 - \|\vec{a}\|_\infty \leq \frac{n-1}{n} \|\vec{a}\|_1,$$

if we let $\vec{a} = (a_1, \ldots, a_n)$. □

For this we will consider

$$\mathbf{E}(\mathrm{diff}_{\underline{\vec{v}}})$$

as $\vec{v}$ varies in $V(F_0)_p^n$.

Lemma 16.3.2. *We have*

$$\mathbf{E}(\mathrm{diff}_{(\underline{v}_1,\ldots,\underline{v}_n)}) \leq n\overline{\mathrm{diff}}_p$$

where $\overline{\mathrm{diff}}_p := \log_p(\mathbf{E}(p^{\mathrm{diff}_{\underline{v}}}))$.

Proof. We will apply Jensen's inequality, to turn an expectation of a sum

$$\mathbf{E}(\mathrm{diff}_{\underline{v}_1} + \cdots + \mathrm{diff}_{\underline{v}_n})$$

into (the log of) an expectation of a product

$$\mathbf{E}(p^{\mathrm{diff}_{\underline{v}_1}} \cdots p^{\mathrm{diff}\,\underline{v}_n}).$$

We are able to compute via the trick we explained in §16.1. Namely, that $\mathbf{E}(p^{\mathrm{diff}_{\underline{v}_1}} \cdots p^{\mathrm{diff}\,\underline{v}_n}) = \mathbf{E}(p^{\mathrm{diff}_{\underline{v}}})^n$. This shows $\mathbf{E}(\mathrm{diff}_{(\underline{v}_1,\ldots,\underline{v}_n)}) \leq n\log_p \mathbf{E}(p^{\mathrm{diff}_{\underline{v}}})$ which is our desired result. $\square$

We can put all of this information together to get the following

Lemma 16.3.3. *We have*

$$\mathbf{E}^2(\|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_1 - \|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|) \leq \frac{(l+1)}{4}\overline{\mathrm{diff}}_p.$$

Proof. We have

$$\begin{aligned}
\mathbf{E}(\|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_1 - \|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_\infty) &\leq \frac{j}{j+1}\mathbf{E}(\|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_1) \\
&\leq \frac{j}{j+1}((j+1)\overline{\mathrm{diff}}_p) \\
&= j\overline{\mathrm{diff}}_p,
\end{aligned}$$

where the first line follows from Lemma **??** and the second line follows from Lemma 16.3.2 (which as an application of Jensen's inequality together with the way expectations of products of random variables behave).

It remains to compute the expectation of these over $\{1,\ldots,j\}$. We have

$$\mathbf{E}^2(\|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_1 - \|\,\mathrm{diff}_{(\underline{v}_0,\ldots,\underline{v}_j)}\,\|_\infty) \leq \mathbf{E}(j\overline{\mathrm{diff}_p}) = \left(\frac{2}{l-1}\sum_{j=1}^{(l-1)/2} j\right)\overline{\mathrm{diff}}_p = \frac{l+1}{4}\overline{\mathrm{diff}}_p,$$

which gives our result. $\square$

16.4. **Minimal discriminant term.** In this section we will compute

$$\mathbf{E}^2(j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}_j}))),$$

and relate this quantity to $\widehat{\deg}_{\mathrm{lgp},F_0}(P_\Theta)$, the log-gaussian procession degree of the theta pilot divisor. As usual, we first compute the average over $(v_0,\ldots,v_j) \in V(F_0)^{j+1}$ (where probabilities vary according to degrees of completions) and then average again over $\{1,\ldots,(l-1)/2\}$. Also, when computing this average, for tuples $(v_0,\ldots,v_j)$ where v_j is not a bad place of E/F_0 we will understand the $j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}_j})$ to be zero.

In what follows below for $(v_0,\ldots,v_n) \in V(F_0)^{n+1}$ we will let

$$F_{0,(v_0,\ldots,v_n)} = F_{0,v_0} \otimes_{\mathbb{Q}_p} \cdots \otimes_{\mathbb{Q}_p} F_{0,v_n}$$

so that $[F_{0,(v_0,\ldots,v_n)} : \mathbb{Q}_p]$ is the dimension of this algebra as a $\mathbb{Q}_p$-vector space (these divided by $[F_0 : \mathbb{Q}]^{n+1}$ are used as probability weights). We remind the reader that

$$[F_0,\mathbb{Q}]^{n+1} = \left(\sum_{v|p}[F_{0,v} : \mathbb{Q}_p]\right)^{n+1} = \sum_{(v_0,\ldots,v_n)} \prod_{i=0}^{n}[F_{0,v_i} : \mathbb{Q}_p] = \sum_{(v_0,\ldots,v_n)} [F_{0,(v_0,\ldots,v_n)} : \mathbb{Q}_p].$$

Again, the last equality follows from the fact that the dimension of a tensor product is product of the dimensions. Keeping these things in mind we can show that the expected value over $V(F_0)^{j+1}$ is the same as expected value over $V(F_0)_p$. As we will need this fact again in a subsequent section we introduce lemma:

Exercise 16.4.1. Let S be a discrete probability space. Let $(X_1,\ldots,X_n)$ be a random variable on S^n. If $f(X_1,\ldots,X_n)$ only depends on X_n (i.e. $f(X_1,\ldots,X_n) = g(X_n)$ for some function of a single variable g) then the expected value of $f(X_1,\ldots,X_n)$ can be computed by just varying over what the function depends on. In symbols:

$$\mathbf{E}_{S^n}(f(X_1,\ldots,X_n)) = \mathbf{E}_S(g(X)).$$

Exercise 16.4.1 shows that

$$\mathbf{E}(j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}_j}) \text{ over } V(F_0)_p^{j+1}) = \mathbf{E}(j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}}) \text{ over } V(F_0)_p)$$

This expectation is related to the p-component $P_{\Theta,j}$ of the theta pilot divisor via the computation below.

$$\begin{aligned}
\mathbf{E}(j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}})) &= \frac{1}{[F_0 : \mathbb{Q}]} \sum_{v|p \ \text{bad}} j^2 \operatorname{ord}_p(\underline{\underline{q}}_{\underline{v}})[F_{0,v} : \mathbb{Q}_p] \\
&= \frac{1}{[F_0 : \mathbb{Q}]} \sum_{v|p,\ \text{bad}} \operatorname{ord}_v(\underline{\underline{q}}_{\underline{v}}^{j^2}) f(v/p) \\
&= \underline{\widehat{\deg}}_{F_0}\Big(\sum_{v|p,\ \text{bad}} \operatorname{ord}_v(\underline{\underline{q}}_{\underline{v}}^{j^2})[v]\Big).
\end{aligned}$$

The Arakelov degree in the formula above is the degree of the p-part of $P_{\Theta}^{(j)}$ which is the jth-component of the Θ-pilot divisor $P_{\Theta} \in \widehat{\mathrm{Div}}_{\mathrm{lgp}}(F_0)$. In formulas we have

$$P_{\Theta}^{(j)} = \sum_{p} \left(\sum_{v|p \ \text{bad}} \mathrm{ord}_v(\underline{\underline{q}}^{j^2})[v] \right) \in \widehat{\mathrm{Div}}(F_0).$$

This means that once we average again over $j \in \{1, \ldots, (l-1)/2\}$ and sum over p we will get the lgp-Arakelov degree of the theta pilot lgp-divisor. In formulas, we have

$$\sum_{p} \mathbf{E}^2(j^2 \, \mathrm{ord}_p(\underline{\underline{q}}_{\underline{v}_j}))) = \widehat{\underline{\deg}}_{\mathrm{lgp},F_0}(P_{\Theta}). \tag{16.3}$$

We remind the reader that via the formulas given in Lemma 4.2.1 the degree of the theta pilot divisor, the q-pilot divisor, and the absolute minimal discriminant of E/F_0 are practically equal.

16.5. ♠♠♠ Taylor: [Stuff from the old branch may need to be moved here]

The estimate

$$1 - \frac{2}{l-1} \sum_{j=1}^{(l-1)/2} \mathbb{P}_{\mathrm{unr},p}^{j+1} \leq 1 - \mathbb{P}_{\mathrm{unr},p}^{\frac{l+1}{2}}$$

is derived by replacing $\mathbb{P}_{\mathrm{unr},p}^{j+1}$ with $j = \frac{l-1}{2}$.

In the second term we have to consider

$$\frac{2}{l-1} \sum_{j=1}^{\frac{l-1}{2}} (j+1) \log_p(b_p)$$

$$\sum_{j=1}^{\frac{l-1}{2}} (j+1) = -1 + \sum_{j=1}^{\frac{l+1}{2}} j$$

16.6. The following describes the appearence of the $1/6$ in Szpiro's inequality.

$$\left(\frac{4}{l+1} + \frac{l}{3} \right) \widehat{\deg}_{F_0}(P_q) \leq \sum_{p} \left[\overline{\mathrm{diff}}_p + \left(1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}} \right) \left(\log_p(b_p) + \log_p(\overline{e}_p) + \left(\frac{4}{l+1} \right)^2 \right) \right] \ln(p). \tag{16.4}$$

We then plug in $\widehat{\deg}_{F_0}(P_q) = \frac{1}{2l} \ln |\Delta_{E/F}^{\min}|$ to see that $1/6$ appearing in Szpiro's inequality.

17. Mochizuki's third inequality: IUT4 – Theorem 1.10

Fix initial theta data

$$(\overline{F}/F, l, E_F, \underline{C}_K, \underline{V}, \underline{V}_{mod}^{\mathrm{bad}}, \underline{\epsilon}).$$

We proved the following:

$$\left(\frac{1}{6} + O(\frac{1}{l}) \right) \ln |\Delta_{E/F}^{\min}| \leq \sum_{p} \left[\overline{\mathrm{diff}}_p + \left(1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}} \right) \left(\log_p(b_p) + \log_p(\overline{e}_p) + O(\frac{1}{l^2}) \right) \right] \ln(p). \tag{17.1}$$

We will now use information about how primes ramify in the tower of fields

$$F_0 \subset F_{\rm tpd} \subset F \subset K$$

in order to relate the above inequality to a uniform Szpiro-like inequality. Here $F_0 = \mathbf{Q}(j_E)$ and

$$F_{\rm tpd} = F_0(E_{F_0}[2]).$$

We recall that from hypotheses of initial theta data that F/F_0 is required to be Galois. The following Lemma describes the ramification information we need in order to relate the different term in (17.1) to the conductor.

♠♠♠ Taylor: [We should really replace the Lemma below with an elementary Lemma about ramification in l-torsion.]

Lemma 17.0.1 ([Moc15d, Proposition 1.8],[Yam17, Lemma 1.8])**.** *If $w \in V(K)$ doesn't divide $30lC_{E/F}$ then $e(w/w_{\rm tpd}) = 1$.*

Proof.

- After extending a field by torsion of an elliptic curve over that field, the primes that ramify are exactly the ones that divide the order of the torsion.
-

□

♠♠♠ Taylor: [The statement above and the following statement are mutually exclusive: If F is in initial theta data for E then $F(\sqrt{l'})$
]

In particular the conductor terms comes from examining the different and *not* from the floor term of the discriminant terms — which is something one might expect naively.

Recall that for an extension K_1/K_0 an extension of p-adic field we defined $\mathrm{diff}(K_1/K_0) = \mathrm{diff}(K_1/\mathbb{Q}_p) - \mathrm{diff}(K_0/\mathbb{Q}_p)$ where $\mathrm{diff}(K/\mathbb{Q}_p) := \mathrm{ord}_p(\mathrm{Diff}(K/\mathbb{Q}_p)$, where $\mathrm{Diff}(K/\mathbb{Q}_p)$ is the different ideal. In particular $\mathrm{diff}(K_1/K_0)$ is not the different exponent. If K_1 has uniformizer π_1 then we can write $\mathrm{Diff}(K_1/K_0) = (\pi_1^d)$. The number d_1 is the different exponent.

Lemma 17.0.2. *Let K/K_0 be an extension of p-adic fields.*

$$\mathrm{diff}(K/K_0) \leq \mathrm{ord}_p([K_1 : K_0]) + 1.$$

Proof. By [Neu99, III.2], $\mathrm{Diff}(K/\mathbb{Q}_p)$ □

Lemma 17.0.3. $\overline{\mathrm{diff}_p} \leq d_{F_{\rm mod}}(30lC_{E/F}, p)(\mathrm{ord}_p([K : F]) + 1)) + \sum_{v|p} \mathrm{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)$

Proof. We have

$$p^{\overline{\mathrm{diff}_p}} = \sum_{v|p} \frac{[F_{\rm mod} : \mathbf{Q}_p]}{[F : \mathbf{Q}]} p^{\mathrm{diff}(\underline{v}|p)} \leq \sum_{v|p} \frac{[F_{\rm mod} : \mathbf{Q}_p]}{[F : \mathbf{Q}]} p^{\sum_{v|p} \mathrm{diff}(\underline{v}|p)} \leq p^{\sum_{v|p} \mathrm{diff}(\underline{v}|p)}$$

which implies

$$
\begin{aligned}
\overline{\mathrm{diff}}_p &\le \sum_{v|p} \mathrm{diff}(K_{\underline{v}}/\mathbf{Q}_p) \\
&\le \sum_{\substack{v|p \\ \underline{v}|30lC_{E/F}}} \mathrm{diff}(K_{\underline{v}} : F_{\underline{v}_F}) + \sum_{v|p} \mathrm{diff}(F_{\underline{v}_F}/\mathbf{Q}_p) \\
&\le \sum_{\substack{v|p \\ \underline{v}|30lC_{E/F}}} (\mathrm{ord}_p([K_{\underline{v}} : F_{\underline{v}_F}]) + 1) + \sum_{v|p} \mathrm{diff}(F_{\underline{v}_F}/\mathbf{Q}_p) \\
&\le d_{F_{\mathrm{mod}}}(30lC_{E/F}, p)(\mathrm{ord}_p([K : F]) + 1)) + \sum_{v|p} \mathrm{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)
\end{aligned}
$$

where we have used the different inequality in going from the second to the third line, and where $d_{F_{\mathrm{mod}}}(30lC_{E/F}, p)$ refers to the number of places $\underline{v}_F$ of F_{mod} that divide the quantity $30l$ times the conductor $C_{E/F}$, as well as p. □

Lemma 17.0.4.

$$\sum_p d_{F_{\mathrm{mod}}}(30lC_{E/F}, p)\ln(p) \le [F_{\mathrm{mod}} : \mathbf{Q}]\ln(30l) + \ln|C_{E/F}|.$$

♠♠♠ Anton: [I have to include $\mathrm{ord}_p[K : F]+1$ in the computation. See "splitting the different term" in the notes.]

♠♠♠ Taylor: [

- The d in 17.0.3 and the d in 17.0.4 are counting divisors over different fields.
- 17.0.3 on the second line, we are using that a 1) $\mathrm{diff}(w/v)$ non-zero implies $e(w/v)$ non-zero. 2) $e(w/v)$ non-zero implies that $v|30lC_{E/F}$. The second statement here has a shell in 17.0.1.
- In 17.0.3 we are using 17.0.2 for relative extensions but don't have it here (it is still true, but is in our old notes).

]

Proof. We have

$$
\begin{aligned}
\sum_p d_{F_{\mathrm{mod}}}(30lC_{E/F}, p)\ln(p) &= \sum_{p|30l} d_{F_{\mathrm{mod}}}(30l, p) + \sum_{p|30l} d_{F_{\mathrm{mod}}}(C_{E/F}, p))\ln(p) \\
&= \sum_{p|30l} d_{F_{\mathrm{mod}}}(p)\ln(p) + \sum_{p|30l} d_{F_{\mathrm{mod}}}(C_{E/F}, p)\ln(p) \\
&= \sum_{p|30l} \ln p^{d_{F_{\mathrm{mod}}}(p)} + \sum_{p|30l} d_{F_{\mathrm{mod}}}(C_{E/F}, p)\ln(p) \\
&\ln(2^{d_{F_{\mathrm{mod}}}(2)}3^{d_{F_{\mathrm{mod}}}(3)}5^{d_{F_{\mathrm{mod}}}(5)}l^{d_{F_{\mathrm{mod}}}(l)}) + \sum_{p|30l} d_{F_{\mathrm{mod}}}(C_{E/F}, p)\ln(p)
\end{aligned}
$$

Now we just have to show that $\sum_{p|30l} d_{F_{\text{mod}}}(C_{E/F},p)\ln(p) \le \ln|C_{E/F}|$. We have

$$|C_{E/F}| = \prod_p \prod_{Q|(C_{E/F},p)} p^{f(Q/p)} = \prod_p p^{\sum_{Q|(C_{E/F},p)} f(Q/p)} \ge \prod_p p^{d_p(C_{E/F})}$$

Taking natural logarithms gives us

$$\ln|C_{E/F}| \ge \ln(\prod_p p^{d_{F_{\text{mod}}}(C_{E/F},p)}) = \sum_p d_{F_{\text{mod}}}(C_{E/F},p)\ln(p) \ge d_{F_{\text{mod}}}(C_{E/F})$$

Where we used that if $p \neq 2$ then $\ln(p) > 1$. $\square$

Lemma 17.0.5.

$$\sum_p \sum_{v|p} \operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)\ln(p) \le \ln|\operatorname{Disc}(F/\mathbf{Q})|.$$

Proof. Since $|\operatorname{Disc}(F/\mathbf{Q})| = |\operatorname{Diff}(F/\mathbf{Q})|$, we just need to show that

$$\ln|\operatorname{Diff}(F/\mathbf{Q})| \ge \sum_p \sum_{\underline{v}_F|p} \ln p^{\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)}$$

By the definition of the norm of an ideal we have

$$\ln|\operatorname{Diff}(F/\mathbf{Q})| = \sum_p \sum_{\underline{v}_F|p} \ln(p_{\underline{v}_F}^{d(F_{\underline{v}_F}/\mathbf{Q}_p)})^{f(K_{\underline{v}_F}/\mathbf{Q}_p)}$$

where $d(F_{\underline{v}_F}/\mathbf{Q}_p)$ is the exponent of $p_{\underline{v}_F}$ in the different ideal. We have defined

$$\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p) = \frac{d(F_{\underline{v}_F}/\mathbf{Q}_p)}{e(F_{\underline{v}_F}/\mathbf{Q}_p)}$$

Therefore we have

$$\begin{aligned}\ln|\operatorname{Diff}(F/\mathbf{Q})| &= \sum_p \sum_{\underline{v}_F|p} \ln(p^{\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)})^{f(K_{\underline{v}_F}/\mathbf{Q}_p)e(K_{\underline{v}_F}/\mathbf{Q}_p)} \\ &= \sum_p \sum_{\underline{v}_F|p} \ln(p^{\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)})^{[K_{\underline{v}_F}:\mathbf{Q}_p]}\end{aligned}$$

and

$$\sum_p \sum_{\underline{v}_F|p} \ln(p^{\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)})^{[K_{\underline{v}_F}:\mathbf{Q}_p]} \ge \sum_p \sum_{\underline{v}_F|p} \ln p^{\operatorname{diff}(F_{\underline{v}_F}/\mathbf{Q}_p)}$$

which gives us our result. $\square$

♠♠♠ Anton: [The following is the result that we want.]

Corollary 17.0.6.

$$\begin{aligned}(\frac{1}{6} + O(\frac{1}{l}))\ln|\Delta_{E/F}^{\min}| \le \sum_p [K:F]([F_{\text{mod}}:\mathbf{Q}]\ln(30l) + \ln|C_{E/F}|) + \ln|\operatorname{Disc}(F/\mathbf{Q})| \\ + (1 - (\mathbb{P}_{\text{unr},p})^{\frac{l+1}{4}})(\log_p(b_p) + \log_p(\overline{e}_p) + O(\frac{1}{l^2}))]\end{aligned}$$

♠♠♠ Anton: [Still need to finish the proof.]

Lemma 17.0.7.

$$\sum_p ((1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}})(\log_p(\overline{e}_p)) \ln(p) \leq \ln|\operatorname{Disc}(F/\mathbf{Q})|$$

Proof. We have

$$\sum_{p \text{ ramified}} (\ln(\overline{e}_p)) = \sum_{p \text{ ramified}} (\log_p(\overline{e}_p)) \ln(p) \geq \sum_p ((1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}})(\log_p(\overline{e}_p)) \ln(p)$$

Since we have

$$\overline{e}_p = \frac{1}{[F : \mathbf{Q}]} \sum_{v|p} [F_v : \mathbf{Q}_p] e(K_{\underline{v}}/\mathbf{Q}_p)$$

and

$$\sum_{v|p} e(K_{\underline{v}}/\mathbf{Q}_p) \geq \frac{1}{[F : \mathbf{Q}]} \sum_{v|p} [F_v : \mathbf{Q}_p] e(K_{\underline{v}}/\mathbf{Q}_p)$$

therefore

$$\sum_{p \text{ ramified}} (\ln(\overline{e}_p)) \geq \sum_p ((1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}})(\log_p(\overline{e}_p)) \ln(p).$$

Then we have the bound

$$\#(p \text{ ramified})(\ln(e_p)) \geq \sum_{p \text{ ramified}} (\ln(\overline{e}_p)).$$

But the number of ramified primes is just the number of primes dividing the discriminant, so we have

$$d(\operatorname{Disc}(F/\mathbf{Q}))(\ln(e_p)) \geq \sum_{p \text{ ramified}} (\ln(\overline{e}_p)).$$

Also, the ramification index is always bounded by the index of the field, so we have

$$d(\operatorname{Disc}(F/\mathbf{Q}))(\ln([K : \mathbf{Q}])) \geq \sum_{p \text{ ramified}} (\ln(\overline{e}_p)).$$

□

Lemma 17.0.8.

$$\sum_p ((1 - (\mathbb{P}_{\mathrm{unr},p})^{\frac{l+1}{4}})(\log_p(b_p)) \ln(p) \leq \ln|\operatorname{Disc}(F/\mathbf{Q})|$$

References

[Con] Keith Conrad, *Differents*, Notes of course, available on-line. 2.7

[Fes15] Ivan Fesenko, *Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki*, Eur. J. Math. **1** (2015), no. 3, 405–440. MR 3401899 1, 16.0.2

[FV02] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, second ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002, With a foreword by I. R. Shafarevich. MR 1915966 2.6.1, 2.6.1, 2.6.1

[Hos15] Yuichiro Hoshi, *IUT Hodge-Arakelov-theoretic evalutation*, 2015. 1

[Kob12] Neal Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, vol. 58, Springer Science & Business Media, 2012. 6.1

[Lan94] Serge Lang, *Algebraic number theory*, second ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994. MR 1282723 2.4.1

[Mat13] Nils Matthes, *On the equivalence of the Vojta height inequality and the uniform abc conjecture*, Master's thesis, Fachbereich Mathematik der Universitat Hamburg, 2013. 1

[Min16] Minemide, *Mono-anabelian geometry ii: Mono-anabelian geometry over mixed characteristic local fields*, 2016. 6.8

[Moc04] Shinichi Mochizuki, *The absolute anabelian geometry of hyperbolic curves*, Galois theory and modular forms, Dev. Math., vol. 11, Kluwer Acad. Publ., Boston, MA, 2004, pp. 77–122. MR 2059759 6.8

[Moc05] ______, *Galois sections in absolute anabelian geometry*, Nagoya Math. J. **179** (2005), 17–45. MR 2164400 6.8

[Moc10] ______, *Arithmetic elliptic curves in general position*, Math. J. Okayama Univ. **52** (2010), 1–28. MR 2589843 1

[Moc15a] ______, *Inter-universal Teichmüller theory I: Construction of Hodge theaters*, RIMS Preprint (2015). 4.3, 7, 8, 3

[Moc15b] ______, *Inter-universal Teichmüller theory II: Hodge-arakelov-theoretic evaluation*, RIMS preprint (2015). 1, 6.8.4, 13

[Moc15c] ______, *Inter-universal Teichmüller theory III: Canonical splittings of the log-theta-lattice*, RIMS preprint (2015). 1, 1.0.2, 1, 3, 4.2, 4.4, 5, 2, 6.7, 6.7.1, 6.8, 6.8, 6.8.4, 6.9, 7, 9, 11.2

[Moc15d] ______, *Inter-universal Teichmüller theory IV: log-volume computations and set-theoretic foundations*, RIMS preprint **1** (2015). 1, 1.0.2, 1, 2.9, 3.0.1, 3.1, 3.1, 8, 6.2.5, 7, 11, 17.0.1

[Moc15e] ______, *Topics in absolute anabelian geometry III: global reconstruction algorithms*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 4, 939–1156. MR 3445958 1, 1, 2, 1, 6.5, 6.7, 6.7.5, 6.8, 6.8, 6.8, 6.9, 7, 8.1, 8.2, 8.4.1, 8.5, 9, 10.3.1, 10.3

[Neu99] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859 2.2, 2.6, 2.7, 17

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 2.4.1, 2.7.2

[Sil09] Joseph H Silverman, *The arithmetic of elliptic curves*, vol. 106, Springer Science & Business Media, 2009. 4.1, 1

[Sil13] ______, *Advanced topics in the arithmetic of elliptic curves*, vol. 151, Springer Science & Business Media, 2013. 4.1, 2, 4.1, 7, 4.3

[Sut15] Drew Sutherland, *Notes for 18.785 - number theory i*, MIT course notes (2015). 2.7

[Tan18] Fucheng Tan, *Note on IUT*, 2018. 1, 1

[Yam17] Go Yamashita, *A proof of the ABC conjecture after Mochizuki*, RIMS preprint (2017). 1, 17.0.1