Problem 1:

$$\begin{cases} L = 2pq + 2pr + 2rq \\ p + q + r = 1 \\ p, q, r \geq 1 \end{cases}$$

$$\nabla L = \left(\frac{\partial L}{\partial p}, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial r} \right)$$

$$= \left(2q + 2r, \; 2p + 2r, \; 2p + 2q \right)$$

$$\nabla g = (1, 1, 1)$$

$$\iff \begin{cases} \nabla L = \lambda \nabla g \\ p + q + r = 1 \end{cases}$$

$$\begin{cases} (2q + 2r, \; 2p + 2r, \; 2p + 2q) = \lambda(1, 1, 1) \\ p + q + r = 1 \end{cases}$$

$$\begin{cases} 2q+2r = \lambda & (1) \\ 2p+2r = \lambda & (2) \\ 2p+2q = \lambda & (3) \\ p+q+r = 1 & (4) \end{cases}$$

$$(1)-(2) \implies \begin{cases} 2p-2q = 0 \\ 2p+2q = \lambda \\ p+q+r = 1 \end{cases}$$

$$\implies 4p = \lambda \implies p = \lambda/4.$$

back substitute:

- $2q = \lambda - 2p \implies 2q = \lambda - \lambda/2 = \lambda/2$
$$\implies q = \lambda/4.$$

- similarly $r = \lambda/4$.

$$\lambda/4 + \lambda/4 + \lambda/4 = 1 \implies \lambda/4 = 1/3$$

$$\implies p = q = r = 1/3.$$

to check that this is a local max lets compare against another point:

$$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2\left(\frac{1}{9}\right) + 2\left(\frac{1}{9}\right) + 2\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}$$

$$P(1, 0, 0) = 0$$

Since the constraint is a compact set this is a maximum.

Problem 2:

$$\begin{cases} f(x,y,z,t) = y + 2z + t \\ g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1. \end{cases}$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ \\ g = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} (0,1,2,1) = \lambda(2x, 2y, 2z, 2t) \\ \\ x^2 + y^2 + z^2 + t^2 = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} 0 = \lambda 2x & (1) \\ 1 = \lambda 2y & (2) \\ 2 = \lambda 2z & (3) \\ 1 = \lambda 2t & (4) \\ x^2 + y^2 + z^2 + t^2 = 1. & (5) \end{cases}$$

From the first equation we have

$$0 = \lambda(2x)$$

$$\Rightarrow 2x = 0 \quad \overset{\text{or}}{\otimes} \quad \lambda = 0.$$

If $\lambda = 0$ then $(2) \Rightarrow 1 = 0$, which is a contradiction.

We can suppose now $\lambda \neq 0$ & $\boxed{x = 0}$. This gives

$$
\begin{cases}
1 = 2\lambda y & (1') \\
2 = 2\lambda z & (2') \\
1 = 2\lambda t & (3') \\
\bullet \ y^2 + z^2 + t^2 = 1, & (4')
\end{cases}
$$

Eqn $(2') \Rightarrow 2 - 2\lambda z = 0 \ \xLeftarrow{\ \ \ \ \ \ \ \ \ \ } $ ~~since $x \neq 0 \Rightarrow z = t$.~~ & $z = \tfrac{1}{\lambda}$

Eqn $(1') \Rightarrow y = \tfrac{1}{2}\lambda$, Similarly, $t = \tfrac{1}{2}\lambda$.

$$\left(\tfrac{1}{2\lambda}\right)^2 + \left(\tfrac{1}{\lambda}\right)^2 + \left(\tfrac{1}{2\lambda}\right)^2 = 1$$

$$\implies \lambda^2 = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$$

$$\implies \lambda = \pm\sqrt{3/2}$$

$\underline{\lambda = +\sqrt{3/2}}$: $(x, y, z, t) = \left(0, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right)$

$\underline{\lambda = -\sqrt{3/2}}$: $(x, y, z, t) = \left(0, \frac{-1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{6}}\right)$

$$f\left(0, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}} + 2\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

max.

$$f\left(0, \frac{-1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{6}}\right) = -\sqrt{6}.$$

min

Problem 3:

$$\begin{cases} f(x,y,z) = x^2 + y^2 + z^2 \\ \quad x - y = 1 \\ \quad y^2 - z^2 = 1 \end{cases}$$

$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g = (1, -1, 0)$$
$$\nabla h = (0, 2y, -2z)$$

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ \quad x - y = 1 \\ \quad y^2 - z^2 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2x = \lambda & (1) \\ 2y = -\lambda + \mu 2y & (2) \\ 2z = -2\mu z & (3) \\ x - y = 1 & (4) \\ y^2 - z^2 = 1 & (5) \end{cases}$$

from (3) we have

$$2z = -2\mu z$$

$$\Rightarrow 2z + 2\mu z = 0$$

$$\parallel$$

$$2z(\mu+1) = 0$$

So $\mu = -1$ or $z = 0$.

Suppose $z = 0$:

$$z = 0 \underset{(5)}{\Rightarrow} y^2 = 1 \Rightarrow y = \pm 1.$$

if $y = +1$: (4) $\Rightarrow$ $x = 1+y$

$$\Rightarrow x = 2$$

$$\boxed{(2,1,0)}$$

if $y = -1$: (4) $\Rightarrow$ $x = 0$ (similar)

$$\boxed{(0,-1,0)}.$$

Suppose $\mu = -1$:

(1) $\Rightarrow$ $x = \frac{1}{2}$

(2) $\Rightarrow$ $4y = -x \Rightarrow y = -\frac{1}{4}$

$(4) \Rightarrow \quad \lambda/2 - \left(\frac{-\lambda}{4}\right) = 1$

$\Longleftrightarrow \quad 2\lambda/4 + \lambda/4 = 1$

$\Longleftrightarrow \quad \frac{3\lambda}{4} = 1 \Rightarrow \lambda = \frac{4}{3},$

$\therefore \begin{cases} x = \lambda/2 = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \\ y = -\lambda/4 = -1/3, \end{cases}$

$(5) \Rightarrow \quad y^2 - z^2 = 1$

$\Longleftrightarrow \quad z^2 = \left(\frac{-1}{3}\right)^2 - 1 = \frac{1}{9} - 1 = \frac{-8}{9},$

$\Rightarrow \quad z = \pm\sqrt{-8/9}$

these are imaginary, so we
throw ~~them~~ them out.

$\therefore \mu = -1$ gives no solutions.

This means the only critical points are

~~●~~ $(\frac{2}{3}, 1, 0)$

$(0, -1, 0).$

$$f(2,1,0) = 2^2 + 1^2 + 0^2 = 5,$$
$$f(0,-1,0) = 0^2 + (-1)^2 + 0^2 = 1,$$

"max"

min.

local

Technical Note $(2,1,0)$ only gives a local extreme value

min too.

We can't apply the extreme value

theorem because the set

$$\{(x,y,z): x-y=1 \ \& \ y^2-z^2=1\} = S$$

is **not compact**!!! To see this is not a max

lets parametrize this set:

$$x-y = 1 \implies x = y+1$$
$$y^2-z^2 = 1 \implies z = \pm\sqrt{y^2-1} \ , \ |y| \geq 1.$$

so we have

$$\alpha(y) = (y+1, y, +\sqrt{y^2-1}) \ , \ y \geq 1$$

$\underbrace{\qquad\qquad}$
just the positive
part of z.

plugging this in gives

$$f(\vec{z}(y)) = x(y)^2 + y^2 + z(y)^2$$

$$= (y+1)^2 + y^2 + y^2 - 1$$

$$= 3y^2 + 2y, \qquad |y| \geq 1.$$

You can see this function is unbounded as $|y| \to \infty$.

Here is the graph of what is going on:

this region is not in the domain of $f(\vec{z}(y))$

$w = f(\alpha(y))$
$= 3y^2 + 2y$

$(1,5)$

w

$(-1,1)$

$y = +1$

$y = -1$

$(0,0)$

$(-\frac{1}{3}, 0)$

$3y^2 + 2y = 3(y+\frac{1}{3})^2 - \frac{1}{3}$

y