Configuration Topologies

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Abstract

I played around with the configuation topologies appearing in Mazur's statement of Mordell-Lang in his MSRI notes [**?**].

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1 Configurations

A **c**onfiguration is a subset $C \subset V$ which is the finite union of translates of vector spaces:

$$C = \bigcup_{j=1}^{n} v_j + V_j.$$

Proposition 1 (Configuration Topology)**.** *1. The set of configurations form a topology on V where configurations are closed sets.*

2. The every descending chain of closed sets terminates.

Proof. First it is clear that a finite union of configurations is a configuration. The hard part is showing that an arbitrary intersection of configurations is a configuration. Let I be some indexing set and for every $i \in I$ let

$$C_i = \bigcup_{j=1}^{n_i} v_{ij} + V_{ij}.$$

We need to show that $\bigcup_{i\in I} C_i$ is a configuration. We are going to need some preparations:

- For every configuration there exists a coset contained in it of maximal dimension. If $C = \bigcup_{j=1}^{n} w_j + W_j$ then we will let

$$\operatorname{maxdim}(C) = \max\{\dim(W_j) : 1 \le j \le n\}.$$

- Give a family of configurations $\{C_i\}_{i\in I}$ we consider the set of natural numbers $\{\text{maxdim}(C_i) : i \in I\}$. By the well ordering principle there is a minimal natural number in this set. We will denote that number by minmaxdim:

$$\text{minmaxdim}(\{C_i\}_{i\in I}) = \min\{\text{maxdim}(C_i) : i \in I\}.$$

- As a further preparation we are going to write our family $\{C_i\}_{i\in I}$ in a special way (just re-labeling):

$$\{C_i\} = \{A\} \cup \{B_j\}_{j\in J}$$

  where A satisfies $\text{maxdim}(A) = \text{minmaxdim}(C_i)$ making it one of "minimal configurations". The idea here is that the whole intersection is going to be contained in A.

With everything as above we are going to do induction on $m = \text{minmaxdim}(C_i)$. In this induction we are going to need to do another induction.

Base Case (first induction) Suppose that $m = \text{maxdim}(A) = 0$ (meaning A is just a collection of points). This means that

$$\bigcap_{i\in I} C_i = \left(\bigcap_{j\in J} B_j\right) \cap A \subset A$$

and every subset of A is a configuration so we are done.

Inductive Step Suppose the theorem is true for families $\{C_i\}_{i\in I}$ with $\text{minmaxdim}(C_i) < m$ and show that it is true for families with $\text{minmaxdim}(C_i) = m$. Suppose that $\{C_i\}_{i\in I} = \{A\} \cup \{B_j\}_{j\in J}$ with $\text{maxdim}(A) = m$. We can break up A into piece of dimension exactly m and smaller than m

$$A = A_m \cup A_{<m}$$

Where A_m is a finite union of cosets of dimension exactly equal to m and $A_{<m}$ is a finite union of cosets of dimension strictly less than m. This gives

$$\begin{aligned}
\bigcap_{i\in I} C_i &= \left(\bigcap_{j\in J} B_j\right) \cap A \\
&= \left(\bigcap_{j\in J} B_j\right) \cap (A_m \cup A_{\leq m}) \\
&= \left(\bigcap_{j\in J} B_j \cap A_m\right) \cap \underbrace{\left(\bigcap_{j\in J} B_j \cap A_{<m}\right)}_{\text{done by induct step}}
\end{aligned}$$

So it remains to show that

$$\bigcap_{j\in J} B_j \cap A_m$$

is a configuration. To do this we are going to do another induction, this time on the number of components involved is the high dimensional piece A_m. Let

$$A_m = \bigcup_{l=1}^{L} w_l + W_l$$

Base Case (inside induction) Consier the case when $L = 1$. Here we have

$$A_m = w + W$$

where W is a vector space of dimension m. Consider what happens when you intersect with one configuration:

$$\begin{aligned}
(\bigcup_{j=1} v_j + V_j) \cap (w + W) &= \bigcup_{j=1}^{n} [(v_j + V_j) \cap (w + W)] \\
&= \bigcup_{j=1}^{n} w_j + (V_j \cap W).
\end{aligned}$$

The intersection become one of the following three things:

1. The configuration breaks up into a finite union of cosets of strictly smaller dimension
2. It remains exactly the same.
3. It is empty.

If there is at least one $j \in J$ such that $C_j \cap (w + W)$ into smaller pieces we are done by the inductive step. Alternatively for every $j \in J$ every $C_j \cap (w + W)$ falls into the other two cases. We can throw away the empty set cases. If $C_j \cap (w+W) = w+W$ for every $(j \in J)$ then $\bigcap_{j\in J} C_j \cap (w + W)$ is a configuration and we are done.

Inductive Step (inside induction Now suppose the proposition holds for all $L' < L$ with $L > 1$. We can write

$$A_m = A_{m,1} \bigcup A_{m,1}$$

where the number of components in both $A_{m,1}$ and $A_{m,2}$ is strictly less than L. We can imply the inductive hypothesis to these:

$$\begin{aligned}
\bigcap_{j\in J} B_j \cap A_m &= \bigcap_{j\in J} B_j \cap (A_{m,1} \cup A_{m,2}) \\
&= (\bigcap_{j\in J} B_j \cap A_{m,1}) \cup (\bigcap_{j\in J} B_j \cap A_{m,1})
\end{aligned}$$

where both of those pieces are separate configurations by the inductive hypothesis.

□

Does this theorem hold more generally? If we replace every use of "dimension" with a rank argument the proof carries through with the word "vector space" replaced by "abelian group". We also never used the fact that V was abelian. In fact we only used the fact that the intersection of two cosets is a coset in general. What we did use was the "dimension function" and the property that when two cosets intersect we either get the empty set, a coset of dimension *strictly less* or the exact same coset we started with. This in addition with the fact that that any descending chain of vector spaces terminates allows use to apply induction.

Let V be a vector space. We can replace the dimension of V with the following number: The length of it's longest descending chain of proper subspaces:

$$V_n \supset V_{n-1} \supset \cdots \supset V_1 \supset \{0\}.$$

Here is something we can do: suppose that $\mathcal{R}$ is a family of equivalent relations where

- The family is closed under meet,

$$R, R' \in \mathcal{R} \implies R \wedge R' \in \mathcal{R}$$

- We can order the relations in the following way:

$$R \leq R' \iff [x]_R \supset [x]_{R'} \iff x \sim_R y \implies x \sim_{R'} y$$

  and have a descending chain condition

- The relations "everything is related too everything" and "nothing is related to nothing" are included.

Then we can define a **configuration** to be the finite union of equivalence classes.

Proposition 2. *We can form a topology on a set $(X, \mathcal{R})$ that satisfies the descending chain condition.*

Example: Let T be a G torsor. This means that there exists a map

$$\div : T \times T \to G$$

satisfying

1. $(x/y)(y/z) = x/z$
2. $(x/y)^{-1} = y/x$
3. $x/x = 1_G$

We can now define an equivalence relation on T for every $H \leq G$ in the following way

$$x \sim_H y \iff x/y \in H.$$

The three properties of a torsor correspond exactly to the three properties of an equivalence relation.

Proposition 3. *Let T be a principle homogeneous space of G and H and H' subgroups of G.*

$$\sim_H \wedge \sim_{H'} = \sim_{H \cap H'} .$$

Proof. Suppose that $x \sim_H y$ and $x \sim_{H'} y$. This means that $(x/y) \in H$ and $(x/y) \in H'$ which means that $(x/y) \in H \cap H'$. This shows that $x \sim_{H\cap H'} y$. This arguement is reversible so the converse holds. □

Conjecture 1. *Suppose that T is an algebraic homogeneous space for G an algebraic group. If $S \subset T$ has the property that it's image in G is contained in a finitely generated subgroup then*

$$S^{\mathrm{Conf}_T} = S^{\mathrm{Zar}_T}.$$

Note that when $T = G$ and G is abelian we recover the ordinary Mordell-Lang conjecture.

Given a set $S \subset V$ the **configuration closure** of S is the closure of the set S in the configuration topology, which as usual is the smallest closed set containing S. We will denote it by S^{Conf}.

2 The Zariski Closure of Sets Contained in Finitely Generated Subgroups are Configurations!

The following theorem, which is very awkward the first time you look at it has many important consequences.

Theorem 1. *Let A be an abelian variety defined $K = \overline{K}$ with* $\mathrm{char}(K) = 0$.
If $S \subset A$ is contained in a finitely generated group then

$$S^{\mathrm{Zar}_A} = S^{\mathrm{Conf}_A}. \tag{1}$$

That is the Zariski Closure is the

Here is an equivalent version of the above theorem.

Theorem 2. *Let A be an abelian variety defined over $K = \overline{K}$ with* $\mathrm{char}(K) = 0$.
If $S \subset \Gamma \subset A$ where Γ is a finitely generated subgroup of A then

$$S^{\mathrm{Conf}_V} = V \text{ and } \Gamma^{\mathrm{Zar}_A} = A \implies S^{\mathrm{Zar}_A} = A, \tag{2}$$

where $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

proof that the two versions are equivalent. Suppose that every $S \subset A$ contained in a finitely generated group we have $S^{\mathrm{Zar}_A} = S^{\mathrm{Conf}_A}$ and the hypotheses of the second version:

- That $S \subset \Gamma$ where Γ is finitely generated,
- that Γ is Zariski-dense
- That the configuration closure of S in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is the whole shabam.

We want to show $S^{\mathrm{Zar}} = A$.

Since we are assuming the first theorem

$$S^{\mathrm{Zar}} = S^{\mathrm{Conf}_A} = \bigcup_{j=1}^{n} a_j + A_j.$$

Consider the intersection of this with Γ,

$$\begin{aligned} \Gamma \cap (\bigcup_{j=1}^{n} a_j + A_j) &= \bigcup_{j=1}^{n} \Gamma \cap (a_j + A_j) \\ &= \bigcup_{j=1}^{n} \tilde{\Gamma}_j \\ &= \bigcup_{j} \gamma_j + \Gamma_j \end{aligned}$$

[Why is coset?]

The image of the above in $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ consists of translates of subspaces.

By hypotheses this union is the full vector space which implies one of it's factors is equal to the full vector space. That is there exists a k such that $\Gamma_k + \gamma_k = \Gamma$ which implies that $\Gamma_k = \Gamma$.

This that $A_k \supset \Gamma$. Hence

$$A_k = A_k^{\mathrm{Zar}} \supset \Gamma^{\mathrm{Zar}} = A$$

$S^{\mathrm{Conf}_A} = A$.

□