Dupuy — Complex Analysis — Spring 2017 — Homework 01

Contents

1 Euler's Formula

Euler's formula states that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for $\theta \in \mathbb{R}$. There are some nice things you can do with this.

1. Compute and draw the 8th roots of unity.
2. Let ζ_n be a primitive nth root of unity. Show that $\sum_{j=0}^{n-1} \zeta_n^j = 0$.
3. (Wallis' Formula) Using the complex representation of cosine, find a formula for

$$\int_0^{2\pi} \cos(\theta)^{2n} d\theta.$$

2 Quaternion Exercise

This exercise show how nice the complex numbers are and how if one tries to develop a notion of holomorphic function in higher for the quaternions. The quaternions are the Division algebra (noncommutative field) over the reals defined by

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k, (\cong \mathbb{R}^4 \text{ as a vector space})$$

where i,j and k satisfy

$$ijk = -1 \text{ and } i^2 = j^2 = k^2 = -1.$$

The norm on the quaternions is defined as

$$|a + bi + cj + dk|^2 = a^2 + b^2 + c^3 + d^2,$$

here $a, b, c, d \in \mathbb{R}$.

1. For $U \subset \mathbb{H}$ open, we say a function $f : U \to \mathbb{H}$ is **holomorphic** if

$$f(q) = \lim_{h \to 0} \left((f(q + h) - f(q))h^{-1}\right).$$

Show that the only quaternionic holomorphic functions are of the form

$$f(q) = \alpha q + \beta.$$

where $\alpha, \beta \in \mathbb{H}$.

3 Power and Laurent Series

1. Prove Hadamard's formula for the radius of convergence of a series $\sum_{n=0}^{\infty} a_n z^n$.

$$R = \lim_{n\to\infty} \inf_{m\geq n} |a_m|^{-1/m}.$$

Also show that the series converges absolutely and uniformly, (the differentiability thing is the next problem).

2. (Analytic implies Holomorphic) Suppose that $f(z) = \sum_{n\geq 0} a_n z^n$ has a radius of convergence R.

   (a) Show $\sum_{n\geq 0} n a_n z^{n-1}$ converges with the same radius of convergence R.

   (b) Show $\frac{d}{dz}\left[\sum_{n=0}^{\infty} a_n z^n\right] = \sum_{n\geq 0} \frac{d}{dz}[a_n z^n]$ on the disc of convergence.

   (Warning: it is not true that for general $u_n(t) \to u(t)$ uniformly that $u_n'(t) \to u_n'(t)$ uniformly! This is a special fact about power series.)

3. Suppose that $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ has a finite radius of convergence. Let $g(z) = a_n + a_{n+1}(z-z_0) + a_{n+2}(z-z_0)^2 + \cdots$. Show that $g(z)$ has the same radius of convergence as $f(z)$ at z_0. (Hint: don't think about this too much)

4. (Extra Credit, see WW page 59) This is a famous example of non-uniform coonvergence. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

converges to $\frac{1}{(z-1)^2}$ when $|z|<1$ and $\frac{1}{z(z-1)^2}$ when $|z|>1$

5. If the series converges do some analysis to determine the radius of convergence at the boundary.

   (a) Expand $\frac{1}{1+z^2}$ in a power series around $z=0$, find the radius of convergence.

   (b) Find the radius of convergence of $\sum_{n\geq 0} n! z^n$.

   (c) (New Mexico, Jan 1998) Expand $\frac{z^2+2z-4}{z}$ in a power series around $z=1$ and find its radius of convergence.

6. (a) Let $B\times A \subset \mathbb{C}\times\mathbb{C}$ be an open region with compact closure. Let $f : B\times A \to \mathbb{C}$ be a function. Let $\gamma \subset A$ be a C^1-curve (so it has finite length). Define $F : B \to \mathbb{C}$ by

$$F(z) = \int_\gamma f(z,s)ds.$$

   Assuming $\frac{\partial f}{\partial z}(z,s)$ exists and is continuous for all $s\in\gamma$ and all $z \in B$ show that

$$\frac{d}{dz}[F(z)] = \int_\gamma \frac{\partial f}{\partial z}(z,s)ds.$$

   (b) Let $\Omega \subset \mathbb{C}$ be an open set. Let $\gamma : [0,1] \to \Omega$ be an C^1 curve. Let $f \in \mathrm{hol}(\Omega)$ and $g \in L^2(\Omega)$. Show that

$$F(z) := \int_\gamma f(\zeta - z)g(\zeta)d\zeta$$

   is holomorphic on Ω.

7. (Whittaker and Watson, page 99) Consider the series

$$\frac{1}{2}\left(z+\frac{1}{z}\right) + \sum_{n=1}^{\infty}\left(z-\frac{1}{z}\right)\left(\frac{1}{1+z^n} - \frac{1}{1+z^{n-1}}\right).$$

Show that this series converges for all values of z with $|z|\neq 1$. Furthermore, show that

$$\frac{1}{2}\left(z+\frac{1}{z}\right) + \sum_{n=1}^{\infty}\left(z-\frac{1}{z}\right)\left(\frac{1}{1+z^n} - \frac{1}{1+z^{n-1}}\right) = \begin{cases} z, & |z|<1 \\ \frac{1}{z}, & |z|>1 \end{cases}$$

8. (Whittaker and Watson, 2.8, problem 16) By converting the series

$$1 + \frac{8q}{1-q} + \frac{16q^2}{1+q^2} + \frac{24q^3}{1-q^3} + \cdots$$

(in which $|q| < 1$), into a double series, show that it is equal to

$$1 + \frac{8q}{(1-q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1-q^3)^2} + \cdots$$

4 Sequences of Analytic Functions

1. (CUNY, Fall 2005) Let D be the closed unit disc. Let g_n be a sequence of analytic functions converging uniformly to f on D.

   (a) Show that g_n' converges.

   (b) Conclude that f is analytic.

   (Hint/Discussion: Normally, taking derivatives makes things numerically behave worse and integration makes things nicer. What is nice about complex analysis is that integration and differentiation are the same thing. Here is the hint now: use the integral formula for derivatives to get this done (I think). A basic philosophical point here is that differentiation of holomorphic functions is actually easy because it is secretely integration.)

2. Here is a first example of an analytic continuation "from the wild".

   (a) Show that the Riemann Zeta function
   $$\zeta(z) := \sum_{n\geq 1} \frac{1}{n^z}$$
   converges for $\operatorname{Re} z > 1$ and is analytic on this domain. (You need to use the "analytic convergence theorem", which states that a uniform limit of analytic functions is analytic. This is just a slight generalization of the previous problem.)

   (b) (Whittaker and Watson, 2.8, problem 10)

      i. Show that when $\operatorname{Re} s > 1$,
      $$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty} \left[\frac{1}{n^s} + \frac{1}{s-1}\left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}} \right) \right]$$

      ii. Show that the series on the right converges when $0 < \operatorname{Re} s < 1$. (This means the series above gives us access to the interesting part of the Riemann-Zeta function. Hint: $\int_n^{n+1} x^{-s} dx = \frac{(n+1^{-s+1}}{1-s} - \frac{n^{-s+1}}{1-s}$)

5 Liouville's Theorem

The proof of Liouville's Theorem is basic of taking limits in Cauchy's formula. There are variants of this proof which are featured in this problem. The proof of the estimate of the partial sum for a power series expansion is based of expanding Cauchy's Integral Formula in a geometric series and then truncating the series.

1. Prove Liouville's Theorem: any bounded entire function is constant.

2. (New Mexico, not sure which year) Let f be analytic on $\mathbb{C}$. Assume that $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of valued r going to infinity. Show that f is a polynomial of degree less than or equal to n.

3. (New Mexico, not sure which year) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume that g is not identically zero. Show that f/g is rational.

4. Prove Goursat's theorem. Let γ be a simple contour. If $f : \overline{\gamma^+} \to \mathbb{C}$ is holomorphic (but whose derivative is not necessarily continuous) then
   $$\int_\gamma f(\zeta) d\zeta = 0.$$

5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let R be the radius of convergence (which is possibly infinite). Let $S_N(f)(z) = \sum_{n=0}^{N} a_n z^n$. Show that for all $r < R$ and all $z \in \mathbb{C}$ with $|z| < r$ we have

$$|f(z) - S_N(f)(z)| \leq \frac{M(f,r)}{r - |z|} \frac{|z|^{N+1}}{r^N}$$

where $M(f,r) = \max_{|z|=r} |f(z)|$.

6. (UIC, Spring 2016) Describe all entire functions such that $f(1/n) = f(-1/n) = 1/n^2$ for all $n \in \mathbb{Z}$.

6 Riemann Extension Theorem

For functions which are analytic in some punctured neighborhood and which are bounded there is a natural way to extend the function to the point. This again uses the Cauchy integral formula and is another nice part of complex analysis.

1. (a) Prove the Riemann Extension Theorem: Let $U \subset \mathbb{C}$ be a region containing a point z_0. Let $f \in \mathrm{hol}(U \setminus \{z_0\})$. If f is bounded on U show that there exists a unique $\widetilde{f} \in \mathrm{hol}(U)$ such that $\widetilde{f}|_{U\setminus\{z_0\}} = f \in \mathrm{hol}(U \setminus \{z_0\})$.

   (b) Recall that a morphism of topological spaces $f : X \to Y$ is "proper" if and only if the inverse image of every compact set is compact. Show that an analytic map $f : \mathbb{C} \to \mathbb{C}$ is proper if and only if for all $z_j \to \infty$ we have $f(z_j) \to \infty$.

   (c) Show that the only proper maps $f : \mathbb{C} \to \mathbb{C}$ are polynomials. (see page 27 of McMullen, you need to consider the function $g(z) = 1/f(1/z)$ and show that $g(z) = z^n g_0(z)$ where $g_0(z)$ is analytic and non-zero. This will allows you to conclude $|g(z)| > c|z|^n$ for some n which will allows you to conclude behavious about the growth of $f(z)$ as $z \to \infty$.)

7 Topological Things

I collected a bunch of topological exercises here.

Background:

- Let X and Y be topological spaces. We define the topology on $X \times Y$ to be the smallest topology such that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous (this means the open sets are generated by sets of the form $U \times Y$ or $X \times V$ for $U \subset X$ open or $X \times V$ for $V \subset Y$ open.
- A topological space X is **compact** if every open cover has a finite subcover. An open cover is just a union of open sets that equal X.
- A **proper map** is a morphism of topological spaces such that the inverse image of compact sets is compact.

Side Remark: The third condition is interesting because Grothendieck realized we can use it to extend this definition to categories other than topological spaces. In particular to the category of "schemes".

1. Let $U \subset \mathbb{C}$ be a connected open set. Consider $U \subset \mathbb{C}$ with the subspace topology (open subset of U are the intersection of open subsets of $\mathbb{C}$ with U and closed subset are closed subset of $\mathbb{C}$ intersected with U). Show that the only subset of U which are open, closed and nonempty is U itself.

2. (Green and Krantz, Ch 11) A subset $S \subset \mathbb{R}^n$ is **path connected** if for all $a, b \in S$ there exists a continuous $\gamma : [0,1] \to S$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

   Let U be an open subset of $\mathbb{R}^n$. Show that U is path connected if and only if U is connected. (Hint: show that the collection of path connected elements is open and closed. Also, you can use that the only nonempty open and closed subset of a connected open set is the entire set itself.)

3. Show that the following conditions are equivalent for a topological space X:

   (a) For all $a, b \in X$ there exists open sets $U \ni a$ and $V \ni b$ with $U \cap V = \emptyset$.

   (b) For all $a, b \in X$, if every neighborhood of a intersects every neighborhood of b then $a = b$.

   (c) The diagonal map $X \to X \times X$ given by $x \mapsto (x, x)$ is proper.

   (d) The diagonal subset is closed.

   If any of these conditions hold we call the topological space **separated** or **hausdorff**. (Hint: You should use the fact that a morphism f is proper if and only if f is closed and the inverse image of every point is compact.)

8 Harmonic Functions

Let $u(x + iy) = u(x, y)$ be a real valued harmonic function on some region $U \subset \mathbb{C}$. A **harmonic conjugate** is a function $v(x, y)$ such that $f(x + iy) := u(x, y) + iv(x, y)$ is holomorphic.

1. Show that $u(x, y) = u(z)$ has a harmonic conjugate locally. (Hint: Use the fundamental theorem of line integrals $v(\vec{P}) - v(\vec{Q}) = \int_C \nabla v \cdot d\vec{r}$ if C is a path starting a $\vec{Q}$ and ending at $\vec{P}$)

2. Find all of the harmonic conjugates of $u(x, y) = x^3 - 3xy^2 + 2x$.

3. Let $f(z) = u(z) + iv(z)$ be analytic. Show that the level sets of $u(z)$ and $v(z)$ are orthogonal.

4. (New Mexico, Summer 2000) Show that the pullback of a harmonic function by an holomorphic map is harmonic (what these words means is explained below). Assume that $w = f(z) = u(z) + iv(z)$ is holomorphic map $f : D \to D' \subset \mathbb{C}$. We consider D in the z-plane to a domain D' in the w-plane. If ϕ is harmonic on D', show that

$$\Phi(x, y) := \phi(u(x, y), v(x, y))$$

is harmonic in D.

(The function Φ is called the pullback of ϕ by f. Sometimes in the literature these you will see the notation $f^*\phi$ for Φ.)

5. (New Mexico, not sure which year) Let $f(z)$ and $g(z)$ be entire functions. Show that if $f(g(z))$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials. (Hint: this relates to the problem on properness from the previous homework).
6. Find all entire functions $f(z)$ which satisfy $\operatorname{Re} f(z) \leq 2/|z|$ when $|z| > 1$. (Hint: Consider $e^{-f(z)}$ or $e^{f(z)}$. You will need the maximum modulus principle and Liouville's theorem.)
7. Let $u(z)$ be a real valued harmonic function on a domain $D \subset \mathbb{C}$
8. Show that for all $D_r(z_0) \subset D$ we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

(Hint: use a harmonic conjugate)

9. If $z_0 \in D$ has the property that there exists some $r > 0$ with $D_r(z_0) \subset D$ and

$$u(z_0) \geq u(z)$$

for all $z \in D_r(z_0)$ then $u(z)$ is constant. (Hint: Consider a function such that $f(z) = u(z) + iv(z)$ then consider the maximum of $e^{f(z)}$.)

10. Let $u_0(\theta)$ be a continuous 2π-periodic function. Let D be a disc of radius r. The Dirichlet boundary value problem asks to find a function $u(x, y)$ such that:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in D \\ u(e^{i\theta}) = u_0(\theta), & \end{cases}$$

Show that convolution with the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

gives a solution to this problem.

9 Residue Integrals

1. (Whittaker and Watson, 6.24,3) If $-1 < z < 3$ then

$$\int_0^\infty \frac{x^z}{(1 + x^2)^2} dx = \frac{\pi(1 - z)}{4\cos(\pi z/2)}$$

2. (Whittaker and Watson, 6.21, Example 4) Let $a > b > 0$ be real numbers. Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

3. (Whittaker and Watson, 6.23, 2) If $a > 0$ and $b > 0$ show that

$$\int_{-\infty}^\infty \frac{x^4 dx}{(a + bx^2)^4} = \frac{\pi}{16a^{3/2}b^{5/2}}$$

4. (Whittaker and Watson, 6.22, 1) Show that if $a > 0$ then

$$\int_0^\infty \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}.$$

5. (Whittaker and Watson, 6.22) If the $\operatorname{Re} z > 0$ then

$$\int_0^\infty (e^{-t} - e^{-tz}) \frac{dt}{t} = \log z$$

6. (Whittaker and Watson, 6.24,2) If $0 \leq z \leq 1$ and $-\pi < a \leq \pi$ then

$$\int_0^\infty \frac{t^{z-1}}{t + e^{ia}} dt = \frac{\pi e^{i(z-1)a}}{\sin(\pi z)}$$

7. (Whittaker and Watson 6.24, 1, pg118) If $0 < a < 1$ show that

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi \csc a\pi$$

8. (Whittaker and Watson, 6.24, 4)Show that if $-1 < p < 1$ and $-\pi < \lambda < \pi$ we have

$$\int_0^\infty \frac{x^{-p} dx}{1 + 2x\cos(\lambda) + x^2} = \frac{\pi}{\sin(p\pi)} \frac{\sin(p\lambda)}{\sin(\lambda)}$$

9. (Whittaker and Watson, 6.21, Example 3) Let n be a positive integer. Show that

$$\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$$

10 Rouche's Theorem and Argument Principal

1. (New Mexico, Jan 1997) How many roots does $p(z) = z^4 + z + 1$ have in the first quadrant?
2. (New Mexico, Aug 1993) How many roots does $e^z - 4z^n + 1 = 0$ have inside the unit disc $|z| < 1$?

11 Conformal Maps

1. (New Mexico, Summer 1999) Let $\mathcal{H}$ be the upper half complex plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let D be the unit disc $D = \{w : |w| < 1\}$. Show that the map $f(z) = w = (z-i)/(z+i)$ defines a bijection $\mathcal{H} \to D$.
2. Find the points where $w = f(z)$ is conformal if
   (a) $w = \cos(z)$
   (b) $w = z^5 - 5z$
   (c) $w = 1/(z^2 + 1)$
   (d) $w = \sqrt{z^2 + 1}$.
3. Find a conformal map of the strip $0 < \operatorname{Re} z < 1$ onto the unit disc $|w| < 1$ in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.
4. Find the Möbius transformation that maps the left have plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ to the unit disc $\{w \in \mathbb{C} : |w| < 1$ and has $z = 0$ and $z = 1$ as fixed points.
5. Find a conformal map from the following regions onto the unit disc $D = \{z : |z| < 1\}$
   (a) $A = \{z : |z| < 2, \operatorname{Arg}(z) \in (0, \pi/4)\}$
   (b) $B = \{z : \operatorname{Re}(z) > 2$
   (c) $C = \{z : -1 < \operatorname{Re}(z) < 1\}$
   (d) $D' = \{z : |z| < 1 \text{ and } \operatorname{Re} z < 0\}$

6. Let D be the unit disc. Let $f : D \to D$ be a conformal map.

   (a) If $f(0) = 0$ show that $f(z) = \omega z$ for some $\omega \in \partial D$.

   (b) If $f(0) \neq 0$ show that there exists some $a \in D$ and $\omega \in \partial D$ such that

   $$f(z) = \omega \frac{z-a}{1-\bar{a}z}.$$

7. (a) Show that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $S(z) = -1/z$ and $T(z) = z+1$ and hence has the presentation

   $$\langle S, T : S^2 = 1, (ST)^3 = 1 \rangle.$$

   (b) Show that a fundamental domain[1] for this action is the complement of the unit disc in a vertical strip of length 1 centered around zero in the upper half plane. In other words

   $$\Omega = \{z : |z| \geq 1 \text{ and } -1/2 \leq \mathrm{Re}(z) \leq 1/2\}$$

   is a fundamental domain for this action.

   (c) Show that the following points are fixed points of $\overline{\Omega}$ with the following stabilizers:

      i. $\mathrm{Stab}(i) = \{1, S\}$
      ii. $\mathrm{Stab}(e^{2\pi i/2}) = \{1, ST, (ST)^2\}$
      iii. $\mathrm{Stab}(e^{\pi i/3}) = \{1, TS, (TS)^2\}$

   (Note: this exercise gives you an example of an action that is not free.)

12 Elliptic and Modular Functions

1. Show that

   $$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

   is elliptic with period lattice Λ.

2. For a lattice $\Lambda \subset \mathbb{C}$ and $m \geq 3$ define $G_m = G_m(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-m}$.

   (a) Show that $\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k$.[2]

   (b) Conclude that

   $$\wp'(z)^2 - 4\wp(z)^3 + g_2 \wp(z) + g_3 = O(z^2),$$

   as $z \to 0$, which shows that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is analytic at the origin of $\mathbb{C}$. Here $g_2 = 60G_4$ and $g_3 = 140G_6$.

   (c) Conclude that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is constant. (Hint: use that elliptic functions without poles are constant.)

   (d) Show the constant in the previous number is zero.

3. The zeros of $\wp(z) - c$ are simple with precisely double zeros at the points congruent to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$. (Hint: what are the zeros of $\wp'(z)$ and what does this mean?)

[1] A fundamental domain for an action $\Gamma \times X \to X$ is a closed subset $\Omega \subset X$ such that

i. $X = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$
ii. For all $\gamma \neq 1$ the set $\gamma(\Omega) \cap \Omega$ has empty interior.

Note that this definition is different from what I had originally said in class. We had our fundamental domains have the property that $\gamma(\Omega) \cap \Omega = \emptyset$. Unfortunately, as this example shows, we can't always arrange for this.

[2] You may need to use that you can interchange some series. If $f_n(z) = \sum a_j^{(n)} z^j$ and $A_j = \sum_{n=0}^{\infty} a_j^{(n)}$ converges then $\sum_{n=0}^{\infty} f_n(z) = \sum_{j=0}^{\infty} A_j z^j$.

13 Riemann Surfaces

This uses some basic properties of Riemann Surfaces.

1. (a) Show that every automorphism of $\mathbb{C}$ extends to an automorphism of $\mathbb{P}^1$.
   (b) Show that $\mathrm{Aut}(\mathbb{C}) := \{az+b : a \in \mathbb{C}^\times \text{ and } b \in \mathbb{C}\}$ (This sometimes called the one dimensional affine linear group and is denoted $\mathrm{AL}_1(\mathbb{C})$.).
2. Show that $\mathbb{C}$ is not conformally equivalent to $D = \{z \in \mathbb{C} : |z| < 1\}$.
3. Show that $\mathrm{Aut}(H) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\}$ (This is sometimes called the two dimensional projective special linear groups with coefficients in $\mathbb{R}$, and is denoted $\mathrm{PSL}_2(\mathbb{R})$).

14 Infinite Products

1. Show the Gauss formula for the Gamma function:
$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)}.$$
(Take the definition of the Gamma function to be from its product formula).
2. Verify that $F(z) = \int_0^\infty t^{z-1}e^{-t}dt$ and $\Gamma(z)$ (via $1/\Gamma(z)$ being defined by the product formula) satify the hypotheses of Weilandt's Theorem. In particular that $F(z)$ and $\Gamma(z)$ are bounded when $1 < \mathrm{Re}\, z < 2$.
3. Show that $\int_0^{2\pi} \log|1 - e^{i\theta}|d\theta = 0$.
4. (New Mexico, Jan 2006) Consider $f(z) = \prod_{n=1}^\infty (1 - z/n^3)$. What is the order of $f(z)$?
5. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of finite order ρ. Show that
$$\rho = \liminf_{n\to\infty} \frac{\log(n)}{\log|a_n|^{-1/n}}.$$

15 The Big Picard Theorem

1. (a) Prove the Casorati-Weiestrass Theorem: Let $f(z)$ is analytic in a punctured disc of radius R at the origin. If $f(z)$ has an essential singularity at $z = 0$ show that for every r with $0 < r < R$ the set $f(D_r(0) \setminus \{0\})$ is dense in $\mathbb{C}$. (This is a corollary of Big Picard).
   (b) Let p be a polynomial. Show that there exists infinitely many z_j such that $p(z_j) = e^{z_j}$.
2. The following exercise is intended to introduce you to the j function which plays a role in the proof of the Big Picard Theorem from class.

   Let H be the upper-half plane. A **modular form** of weight k and level $N = 1$ is a function $f : H \to \mathbb{C}$ such that
$$f(\frac{az+b}{cz+d}) = (cz+d)^{-2k}f(z). \tag{1}$$
   for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

   (a) Let M_k denote the collection of modular forms of weight k and level 1. Show that $M = \bigoplus_{k\geq 0} M_k$ is a graded ring (i.e. that $M_{k_1}M_{k_2} \subset M_{k_1+k_2}$.
   (b) Show that $G_{2k}(\frac{az+b}{cz+d}) = (cz+d)^{2k}G_{2k}(z)$ has weight $2k$ (Hint: check this on the generators of $\mathrm{SL}_2(\mathbb{Z})$.)

   Using the first part conclude that the we have the following modular forms of the indicated weights:

i. $g_2(\tau) = 60G_4(\tau)$, $k = 4$

ii. $g_3(\tau) = 140G_6(\tau)$, $k = 6$

iii. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, $k = 12$

iv. $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$, $k = 0$

3. Explain in words the ideas that go into the proof of Montel's Theorem in Green and Krantz (page 193). How is Arzela-Ascoli used?

4. Let $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. What is the universal cover of X? What is group of deck transformations for this cover?

5. Use Van Kampen's theorem to rigorously compute $\pi_1(\mathbf{P}^1 \setminus \{p_1, \ldots, p_r\}, z_0)$ for arbitrary r. (Hint: apply Van Kampen to open sets U, V where $U \cap V$ is simply connected).