EXAMPLES OF GEOMETRIC LANG-BOMBIERI-NOGUCHI OUTSIDE MORDELL-LANG: NON-RIGID VARIETIES WITH AMPLE BUT NOT GLOBALLY GENERATED COTANGENT BUNDLE

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Abstract. In this note we provide examples of nonisotrivial varieties over a function fields with ample cotangent bundle and zero Albanese. This provides explicit examples of varieties which are not embeddable into abelian varieties for which the geometric Lang-Bombieri-Noguchi conjecture is known to be true.

1. Background and Results

The goal of this paper is to provide examples of nonisotrivial varieties over function fields with ample cotangent bundle which can not be embedded in abelian varieties. Our interest in finding such examples is in showing that certain cases of the Geometric Bombieri-Lang-Noguchi conjecture due to Martin-Deschamps [MD84] and Noguchi [Nog81] do not follow from Mordell-Lang-type theorems [Man65] [Col90] [Bui92]. We will now review the relevant definitions.

By a function field we will mean the field of functions on a positive dimensional integral scheme over the complex numbers. Let K be a function field. A scheme X/K is **isotrivial** if there exists some $K_0 \subset K$ with K/K_0 of positive transcendence degree, a variety X_0/K_0, and an isomorphism $X \otimes_K K^{alg} \cong X_0 \otimes_{K_0} K^{alg}$. Here K^{alg} is a choice of algebraic closure of K.

For Y a variety defined over an algebraically closed field F and $S \subset Y(F)$ we will let S^{Zar} denote the Zariski closure of S in $Y(F)$.

Conjecture 1 (Geometric Lang-Bombieri-Noguchi (GLBN))**.** *Let K be a function field. Suppose X is a smooth projective variety of general type defined over K. Then*

$$X(K)^{\mathrm{Zar}} = X(K^{alg}) \implies X \text{ isotrivial}.$$

In order to summerize the main results of [MD84] and [Nog81] we need to introduce some notation. We will make a standard abuse and identify vector bundles with locally free sheaves. For E a vector bundle on a scheme X we let

$$\mathbf{P}(E) = \underline{\mathrm{Proj}}(S(E))$$

where $S(E) = \bigoplus_{r \geq 0} S^r(E)$ is the symmetric algebra of E and $\underline{\mathrm{Proj}}$ denotes the global Proj construction. There is a natural map $\pi : \mathbf{P}(E) \to X$ and a natural isomorphism $\pi_* \mathcal{O}_{\mathbf{P}(E)}(1) \cong E$. We remark that geometrically $\mathbf{P}(E)$ is the space of lines of the physical bundle of $E^\vee$, the dual of E.

Definition 1 ([Har66])**.** Let X be a scheme and let E be a vector bundle on X. We say E **ample** if the line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ is ample.

Remark 1. Papers predating [Ful76] sometimes use "ample" to mean globally generated (see for example [Mat74, pg 312]). The introduction of [Ful76] gives a good review of the various "ampleness" conditions one can impose on a vector bundle and how the notions are related.

Since varieties with ample cotangent bundle are of general type the following theorem is a special case of the GLBN conjecture.

Theorem 1 (Martin-Deschamps + Noguchi (MDN) [MD84] [Nog81])**.** *Let K be a function field. Suppose that X/K be a nonisotrivial projective variety. If X has ample cotangent bundle then $X(K)$ is not dense.*

We wish to compare the MDN theorem to the Falting's theorem/Mordell-Lang Problem. We follow the setup in [Maz00] for the Mordell-Lang problem: Let A be an abelian variety over an algebraically closed field F. We define a **configuration** to be a finite union of translates of abelian subvarieties of A. The set of configurations form the closed sets for a topology on abelian varieties called the configuration topology. For a subset S of $A(K^{alg})$ we will let S^{Zar} denote its Zariski closure and S^{Conf} denote its configuration closure.

Theorem 2 ([Man65] [Col90] [Bui92] GML)**.** *Let K be a function field. Let A/K abelian variety. Suppose no isogeny factor of A is isotrivial (i.e. A has trace zero). If $S \subset A(K^{alg})$ is contained in a finite rank subgroup then $S^{\mathrm{Conf}} = S^{\mathrm{Zar}}$.*

One can show that GML implies GLBN for any subvariety of an abelian variety. One can ask if Theorem 1 actually proves anything new.

Question 1. Do there exist examples of nonisotrivial varieties with ample cotangent bundle which do not embed into their Albanese?

A negative answer to Question 1 would imply that the MDN theorem is a simple consequence of GML. The following shows that varieties with ample cotangent bundle and zero albanese can be rigid.

Example 1 (rigid varieties with ample cotangent bundle and zero albanese)**.** Fake projective planes provide examples of varieties over $\mathbf{C}$ which have ample cotangent bundle and do not map non-trivially to any abelian variety [Mum79]. Unfortunately, these varieties are rigid [CS10].

In this paper we give a positive answer to Question 1.

Theorem 3. *There exists a function field K and a nonisotrivial projective variety Y/K with ample cotangent bundle and* $\mathrm{Alb}(Y) = 0$.

See Theorem 4 for a more precise statement.

For any variety Y we have $\dim(\mathrm{Alb}(Y)) = h^0(\Omega^1_{\mathrm{Alb}(Y)}) = h^0(\Omega_Y)$.[1] To construct a variety Y/K with $\mathrm{Alb}(Y) = 0$ it suffices to construct one with $H^1_{dR}(Y/K) = 0$. To construct a nonisotrivial Y with $H^1_{dR}(Y/K) = 0$ and Ω^1_Y ample we construct a nontrivial flat family $f : \mathcal{D} \to S$ and base change to a certain open subset of S and take the generic fiber.

[1]Here is a proof: Let $A = \mathrm{Alb}(Y)$. For all abelian varieties a choice of global differential forms gives an isomorphism $\Omega^1_A \cong \mathcal{O}_A^{\oplus \dim(A)}$ of $\mathcal{O}_A$-modules. This proves the first equality. The second equality follows from the fact that the Albanese map $\alpha : X \to A$ induces an isomorphism $\alpha^* : H^0(A, \Omega^1_A) \to H^0(X, \Omega^1_X)$.

Remark 2.
(1) After finishing the paper we realized that Theoerem 3 is stated in [MD84] where the author cites a paper of Bogomolov that never appeared.
(2) Complementary characteristic p results can be found in [GR15].

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2. Proofs

Lemma 1. *There exists a smooth projective variety X such that a general linear section of X of sufficiently high codimension*

(1) *has ample cotangent bundle and*
(2) *is simply connected.*

Proof. This is [Deb05, section 4]. □

Remark 3. In this remark we outline two ways of constructing families of varieties with ample cotangent bundle. The rest of the work of this paper will be in showing that these families are not isotrivial.

(1) Here we outline the construction in [Deb05, section 4]. Let $X_1, \ldots, X_m$ be projective varieties over $\mathbf{C}$ with big cotangent bundle and trivial fundamental group. These exist because of:

Bogomolov's 'big' surface trick: If S is a projective surface,
$$c_1(S)^2 - c_2(S) > 0 \implies \Omega^1_S \text{ big} .$$
See [Rou09, Proposition 1.18] for relevant formulas for $c_1(\mathcal{O}_{\mathbf{P}(\Omega^1_S)}(1))$. Using this fact, together with the Lefschetz hyperplane theorem, one can construct simply connected surfaces S which have Ω^1_S big [Deb05, Proposition 26].

Let $X = X_1 \times \cdots \times X_m$ and embed X into projective space using a very ample line bundle $\mathcal{L}$. Let N denote the dimension of the projective space X sits inside. There exists some $2 \leq c < \dim(X)$ such that a general linear section of codimension c in X will have ample cotangent bundle. This follows from:

Bogomolov's big-to-ample trick : If $X_1, \ldots, X_m$ are projective varieties with big cotangent bundle then a general linear section of $X_1 \times \cdots \times X_m$ of sufficiently small dimension[2] will have ample cotangent bundle. [Deb05, Proposition 23]

[2] If $d = \min_{1 \leq i \leq m} \dim(X_i)$, then the dimension of the general linear section needs to be less than or equal to $(d(m + 1) + 1)/(2(d + 1))$.

Let $G = G(N, N-c)$ denote the Grassmannian of codimension c linear subspaces in $\mathbf{P}^N$. The family the universal family, $f : \mathcal{Y} \subset X \times G \to G$, of general linear sections of X has the property that its general fibers have ample cotangent bundle and are simply connected.

We claim that if the line bundle $\mathcal{L}$ is chosen to be sufficiently ample, the family obtained has non-zero Kodaira-spencer map.

(2) Given the existence of simply connected X with ample cotangent bundle and $\dim(X) > 2$, there is an easy way to obtain families of simply-connected varieties with ample cotangent bundle. Namely, take a family of ample divisors in X; these are simply connected by the Lefschetz hyperplane theorem. Furthermore, these divisors have ample cotangent bundle as any smooth subvariety Z of a smooth variety X with ample cotangent bundle has ample cotangent bundle, because Ω^1_Z is a quotient of $\Omega^1_X|_Z$. We claim that if the divisors chosen are sufficiently ample, the family obtained has non-zero Kodaira-spencer map.

We now show that a sufficiently positive complete intersection in any smooth projective variety admits embedded deformations with non-trivial Kodaira-Spencer class; that is, sufficiently positive complete intersections of positive dimension always deform non-trivially as abstract varieties.

Lemma 2. *Let X be a smooth projective variety and $\mathcal{L}$ an ample line bundle on X. Let $n \gg 0$ and $D_1, \cdots, D_r \in |\mathcal{L}^{\otimes n}|$ be general divisors, with $r < \dim(X)$. Let*

$$Z = D_1 \cap \cdots \cap D_r.$$

Then the natural map

$$\partial : H^0(Z, \mathcal{N}_{Z/X}) \to H^1(Z, T_Z)$$

is non-zero.

Baby Proof. The prototype for this situation is when D is an ample divisor so that $I_D = \mathcal{O}_X(-D)$ and we can use Serre-Duality and Serre Vanishing to conclude prove these statements:

The normal sequence is

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_{D/X} \to 0.$$

This implies

$$H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{N}_{D/X}) \to H^1(\mathcal{O}_X).$$

Since $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{N}_{D/X})$ is trapped between $H^0(\mathcal{O}_X)$ and $H^1(\mathcal{O}_X)$ which are independent of D, this implies the rank of $H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{N}_{D/X})$ is very large for sufficiently ample D, since the rank of $H^0(\mathcal{O}_X(D))$ tends to infinity as D is replaced by a multiple. □

We now prove Lemma 2.

Proof. Recall that for general $D_1, \cdots, D_r$ as in the statement, Z is smooth, by Bertini's theorem. Hence there is a short exact sequence of locally free sheaves

$$0 \to T_Z \to T_X|_Z \to \mathcal{N}_{Z/X} \to 0.$$

The map

$$\partial : H^0(Z, \mathcal{N}_{Z/X}) \to H^1(Z, T_Z)$$

arises as the boundary map in the cohomology long exact sequence obtained from this short exact sequence. Thus it suffices to show that for $n \gg 0$ and $D_1, \cdots, D_r$ general, $h^0(Z, T_X|_Z)$ is bounded, while $h^0(Z, \mathcal{N}_{Z/X})$ tends to infinity.

We first show that $h^0(Z, T_X|_Z)$ is bounded as n increases, with $D_1, \cdots, D_r$ general. Choose N such that for all $n \geq N$,

$$H^0(X, T_X \otimes (\mathcal{L}^\vee)^{\otimes n}) = H^1(X, T_X \otimes (\mathcal{L}^\vee)^{\otimes n}) = \cdots = H^{\dim(X)-1}(X, T_X \otimes (\mathcal{L}^\vee)^{\otimes n}) = 0,$$

which is possible by Serre vanishing and Serre duality, and let $D_1, \cdots, D_r$ be general elements of $|\mathcal{L}^{\otimes N}|$.

Let $Z_0 = X, Z_1 = D_1, Z_2 = D_1 \cap D_2, \cdots, Z = Z_r = D_1 \cap \cdots \cap D_r$. Then there are exact sequences

$$0 \to T_X|_{Z_i}(-D_{i+1}) \to T_X|_{Z_i} \to T_X|_{Z_{i+1}} \to 0.$$

Thus it is enough to show that $h^1(Z_i, T_X|_{Z_i}(-D_{i+1})) = 0$ for $i = 1, \cdots, r-1$, $n \gg 0$, and $D_1, \cdots, D_r$ general. But considering the exact sequence

$$0 \to T_X|_{Z_{i-1}} \otimes (\mathcal{L}^\vee)^{\otimes Nk} \to T_X|_{Z_{i-1}} \otimes (\mathcal{L}^\vee)^{\otimes N(k-1)} \to T_X|_{Z_i} \otimes (\mathcal{L}^\vee)^{\otimes N(k-1)} \to 0$$

and induction on i, k, we see that

$$H^j(Z_i, T_X|Z_i \otimes (\mathcal{L}^\vee)^{\otimes Nk}) = 0$$

for $k > 0$ and $j < \dim Z_i = \dim(X) - i$. Taking $j = 1$ and $k = 0$ proves the desired claim.

Now we show that $h^0(Z, \mathcal{N}_{Z/X})$ is unbounded as n grows, with $D_1, \cdots, D_r$ general; the argument is quite similar. Observe that

$$\mathcal{N}_{Z/X} = \mathcal{O}(D_1)|_Z \oplus \cdots \oplus \mathcal{O}(D_r)|_Z.$$

As before, let $D_1, \cdots, D_r$ be general elements of $|\mathcal{L}^{\otimes N}|$. Let $Z_0 = X, Z_1 = D_1, Z_2 = D_1 \cap D_2, \cdots, Z = Z_r = D_1 \cap \cdots \cap D_r$. Then we have short exact sequences

$$0 \to \mathcal{O}_X^{\oplus r} \to \mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r) \to \mathcal{O}(D_1)|_{Z_1} \oplus \cdots \oplus \mathcal{O}(D_r)|_{Z_1} \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}_{Z_{r-2}}^{\oplus r} \to \mathcal{O}(D_1)|_{Z_{r-2}} \oplus \cdots \oplus \mathcal{O}(D_r)|_{Z_{r-2}} \to \mathcal{O}(D_1)|_{Z_{r-1}} \oplus \cdots \oplus \mathcal{O}(D_r)|_{Z_{r-1}} \to 0$$

$$0 \to \mathcal{O}_{Z_{r-1}}^{\oplus r} \to \mathcal{O}(D_1)|_{Z_{r-1}} \oplus \cdots \oplus \mathcal{O}(D_r)|_{Z_{r-1}} \to \mathcal{N}_{Z/X} \to 0$$

But $h^0(Z_i, \mathcal{O}_{Z_i}^{\oplus r}) = r$, so

$$h^0(Z, \mathcal{N}_{Z/X}) > h^0(X, \mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)) - r^2.$$

By asymptotic Riemann-Roch, this tends to infinity with n. □

Theorem 4. *There exists a family $f : \mathcal{Y} \to S$ with the property that*

1. $\mathrm{Alb}(\mathcal{Y}/S) = 0$
2. *Has generically ample cotangent bundle.*
3. *f is nonisotrivial.*

Consequently, by taking the generic fiber of f, we get a variety as in Theorem 3

Proof. By Lemma 1 there exists a family $f : \mathcal{Y} \to S$ of linear sections of a fixed projective variety X with such that each section has ample cotangent bundle and trivial Albanese. We will show there exists an open subset of S where

$$\mathrm{KS}_f : T_S \to R^1 f_* T_{\mathcal{Y}/S}$$

is nonzero.

To do this we will show that infinitesimal embedded deformations give rise to abstract infinitesimal deformations. Let D be a general linear section of X in the family f. Recall that $H^1(D, T_D)$ parametrizes abstract first order deformations of D [Ols07, Proposition 2.6]. Also recall that $H^0(\mathcal{N}_{D/X})$ parametrizes first order embedded deformations of D [Ols07, Summary 4.4]. These two cohomology groups are related by the exact sequence

$$0 \to T_D \to T_X|D \to \mathcal{N}_{D/X} \to 0,$$

and upon taking the associated long exact sequence

$$\cdots \longrightarrow H^0(T_X|D) \xrightarrow{\psi} H^0(\mathcal{N}_{D/X}) \xrightarrow{\rho} H^1(T_D) \longrightarrow \cdots$$

we observe that embedded deformations map abstract deformations under the map ρ. To show nontriviality of Kodaira-Spencer at a point, it suffices to show that embedded deformations give rise to nontrivial abstract deformations. Hence it suffices to show that ρ has positive rank. This is implied by Lemma 2.

□

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