THE INTEGERS OF $\mathbf{C}(t)^{\mathrm{alg}}$ INTERPRET Z.

TAYLOR DUPUY AND EHUD HRUSHOVSKI

Abstract. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let $\mathbf{k}(t)^{\mathrm{alg}}$ be the algebraic closure of $\mathbf{k}(t)$. Let $\mathbf{k}[t]^{\mathrm{alg}} = \mathcal{O}_{\mathbf{k}(t)^{\mathrm{alg}}}$, be the integral closure of $\mathbf{k}[t]$ in $\mathbf{k}(t)^{\mathrm{alg}}$. In this paper we prove that $\mathrm{Th}(\mathbf{Z})$ is interpretable in $\mathrm{Th}(\mathbf{k}[t]^{\mathrm{alg}})$ where both theories are viewed with their usual ring structure. This result is to be contrasted with $\mathrm{Th}(\mathbf{Z}^{\mathrm{alg}})$ and $\mathrm{Th}((\mathbf{F}_p^{\mathrm{alg}}[t])^{\mathrm{alg}})$ which are known to have quantifier elimination.

Contents

1. Introduction

In this paper we prove the following theorem.

Theorem 1.1. *Let $\mathbf{k}$ be an infinite algebraically closed field. Let $\mathbf{k}[t]^{\mathrm{alg}} = \mathcal{O}_{\mathbf{k}(t)^{\mathrm{alg}}}$, be the integral closure of $\mathbf{k}[t]$ in $\mathbf{k}(t)^{\mathrm{alg}}$.* $\mathrm{Th}(\mathbf{Z})$*, is interpretable in* $\mathrm{Th}(\mathbf{k}[t]^{\mathrm{alg}})$ *where both are given their usual ring structure.*

This is surprising as both $\mathbf{Z}^{\mathrm{alg}}$ and $\mathbf{F}_p[t]^{\mathrm{alg}}$ have quantifier elimination (see [PS90, vdD88] and [vdDM90] respectively).

1.1. Organization of paper.

In §2, we review how the ideal class group is interpretable in a one dimensional ring and provide and explicit interpretation. In §3, we enrich the ideal class group structure to a structure $\mathcal{S}$, which we will later use to construct useful definable subgroups of the 2-ideal class group. In §4 we construct certain $\mathcal{S}$-definable subgroups of the ideal class group of $\mathbf{k}[t]^{\mathrm{alg}}$ and provide some characterization results. These sections contain the main lemmas feeding into the proof of Theorem 1.1. In §5, we put everything together to prove the main theorem.

The strategy here is to interpret the lattice of subspaces of a three dimensional $\mathbf{Q}$-(vector space), which interprets a projective plane, which, by the fundamental theorem of projective geometry, interprets $\mathbf{Q}$.

Acknowledgements

The authors were supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 291111/ MODAG. The first author also worked on this project while visiting the Institute for Computational and Experimental Research in Mathematics (ICERM) and would like to thank them for their hospitality.

2. Interpretations of ideal class group in the ring structure

Let $X = \operatorname{Spec}(A_X)$ be a variety over $\mathbf{k}$ and assume A_X is an integral domain and that the variety is smooth. We will make the standard abuse of notation and identity the Weil divisor class group $\frac{\operatorname{Div}(X)}{P(X)}$ with the ideal class group of the coordinate ring A_X defined in terms of fractional ideals:

$$\frac{\operatorname{Div}(X)}{P(X)} \cong \operatorname{Cl}(X) \cong \frac{\text{finitely generated fractional ideals}}{\text{principal fractional ideals}}.$$

We will use the formulation of $\operatorname{Cl}(X)$ in terms of divisors for proving "rigidity" theorems and the formulation of $\operatorname{Cl}(X)$ in terms of fractional ideals for constructing interpretations.

Remark 2.1. Let $X/\mathbf{k}$ be a smooth affine algebraic curve. We now recall that much of the time $\operatorname{Cl}(X)$ is uncountable (unlike its arithmetic counterpart). Let $\bar{X}$ denote the compactification of X and let $Z = \bar{X} \setminus X$. Let $s = \#\pi_0(Z)$. We have the exact sequence

$$\mathbf{Z}^s \to \operatorname{Cl}(\bar{X}) \to \operatorname{Cl}(X) \to 0.$$

This tells us that $\operatorname{Cl}(X)$ is uncountable as the quotient of an uncountable group by a countable one is uncountable.

Lemma 2.2. *If $A \subset \mathbf{k}[t]^{\mathrm{alg}}$ is a sub $\mathbf{k}$-algebra, all of its finitely generated ideals are generated by two elements. In particular this is true for $A = \mathbf{k}[t]^{\mathrm{alg}}$.*

This implies for any curve $X = \operatorname{Spec}(A)$, any function $C(\mathbf{k}) \to \mathbf{Z}_{\geq 0}$ with finite support has the form $x \mapsto \min\{\operatorname{ord}_x(f), \operatorname{ord}_x(g)\}$ for some $f, g \in \mathbf{k}[t]^{\mathrm{alg}}$.

Proof. If suffices to prove this for a cofinal family of finitely generated subrings, namely for affine coordinate rings R of affine curves X. Any ideal in R has for form $\{f : \operatorname{ord}_{x_k}(f) \geq m_k, k = 1, \ldots, n\}$ for some $x_1, \ldots, x_n \in X(\mathbf{k})$ and $m_1, \ldots, m_n \in \mathbf{N}$. Since localizations are regular local rings, the Chinese Remainder Theorem tells us we may find some $g \in R$ with $\operatorname{ord}_{x_k}(g) = m_k$. Now g may have additional zeros at some additional points $y_1, \ldots, y_l$. Buy by independence of valuations, there exists h with $\operatorname{ord}_{x_i}(h) \geq m_i$ and $\operatorname{ord}_{y_i}(h) = 0$. Now it is clear that g and h generate I.

The second statement is proved the same way. □

3. Interpretation of the structure $\mathcal{S}$ in the ring structure.

Let $A \subset \mathbf{k}[t]^{\mathrm{alg}}$ be a sub $\mathbf{k}$-algebra. We now define a structure $\mathcal{S}(A)$ which interprets $\mathrm{Cl}(A)$ in A and allows us to define interesting subgroups of $\mathrm{Cl}(A)$.

Lemma 3.1. (1) *Let A be a smooth noetherian dedekind domain. The ring A interprets the structure*

$$\mathcal{S}(A) := (\mathrm{Div}^+(A), B(A), \mathrm{Div}(A), \mathrm{Cl}(A), \mathrm{supp}_A : \mathrm{Div}^+(A) \to B(A))$$

Here $\mathrm{Div}(A)$ denotes the partially ordered group of fractional ideals, $\mathrm{Div}^+(A)$ is the partially ordered monoid of effective divisors, $B(A)$ is the semi-boolean algebra of radical ideals, supp_A is the support map, and $[-] : \mathrm{Div}(A) \to \mathrm{Cl}(A)$ is the canonical quotient map.

(2) *The ring $\mathbf{k}[t]^{\mathrm{alg}}$ interprets the structure*

$$\mathcal{S}_\infty = (\mathrm{Div}_\infty, \mathrm{Cl}_\infty, B_\infty, \mathrm{supp}_\infty : \mathrm{Div}_\infty \to B_\infty)$$

where if $\mathrm{colim}\, A_i = \bigcup_i A_i = \mathbf{k}[t]^{\mathrm{alg}}$ then the structures are defined as follows.

$$\mathrm{Div}_\infty = \mathrm{colim}\,\mathrm{Div}(A_i)$$

$$\mathrm{Cl}_\infty = \mathrm{colim}\,\mathrm{Cl}(A_i)$$

$$B_\infty = \mathrm{colim}\, B(A_i)$$

$$\mathrm{supp}_\infty = \mathrm{colim}\,\mathrm{supp}_{A_i}$$

Proof. (1) The proof follows from a series of interpretations.

- Let $X = \mathrm{Spec}(A)$. We identify $\mathrm{Div}(X)$ with $D(X) = \{F : X \to \mathbf{Z}$:f has finite support $\}$. Similarly for $D^+(X)$. The interpretation is given by

$$A^2 \to D^+(X)$$

$$(g_1, g_2) \to F_{(g_1,g_2)}$$

  where $F_{(g_1,g_2)}(x) := \max\{\mathrm{ord}_x(g_1), \mathrm{ord}_x(g_2)\}$. One then checks the inverse image of $\Gamma_\leq$, Γ_+, Δ are definable sets.
- We may supplement the interpretation $A^2 \to \mathrm{Div}^+(X)$ of $\mathrm{Div}^+(X)$ with a predicate for "$\in B(X)$".
  First observe that for $I = (g_1, g_2)$ there exists some $N = N_I$ such that

$$\sqrt{I} = \{h : \exists n, h^n \in (g_1, g_2)\} = \{h : h^N \in (g_1, g_2)\}.$$

  This second expression is first order. This means the equality of ideals $\sqrt{(f,g)} = (f,g)$ is first order. We check that $B(X)$ and $\mathrm{supp} : D^+(X) \to B(X)$ are definable. This shows that "$\in B(X)$" is definable. for $\alpha \in \mathrm{Div}^+(X)$.
  Also $\mathrm{supp}(D) = E$ is definable as $\sqrt{(f_1, f_2)} = (g_1, g_2)$ is first order.

- We may supplement the interpretation of $\mathrm{Div}^+(X)$ with the predicate for membership in principal ideals " $\in P^+(X)$". The inverse image of this predicate under the interpretation $A^2 \to \mathrm{Div}^+(X)$ is given by

$$\exists f : (g_1, g_2) = (f)$$

where one observes that $(g_1, g_2) = (f)$ is first order.
- We may interpret $\mathrm{Div}(X)$ in $\mathrm{Div}^+(X)$ via the map

$$\mathrm{Div}^+(X) \to \mathrm{Div}(X)$$

$$(\alpha, \beta) \mapsto \alpha - \beta.$$

- Using the interpretation of $P^+(X)$ we can also interpret $P(X)$ the collection of principal ideals.
- Since we can interpret the structure $\mathrm{Div}(X)$ with a predicate for membership into the subgroup $P(X)$ the map

$$\mathrm{Div}(X) \to \mathrm{Cl}(X)$$

$$\alpha \mapsto [\alpha]$$

provides an interpretation. Observe that $[\alpha_1] = [\alpha_2]$ if and only if $\alpha_1 - \alpha_2 \in P(X)$ which is first order in $\mathrm{Div}(X)$ with its supplemented structure.

(2) This is a limiting case of item 1 and uses $A = \mathbf{k}[t]^{\mathrm{alg}}$. The key interpretation is given by

$$A^2 \to D^+_\infty$$

$$(g_1, g_2) \mapsto (\tilde{x} \mapsto \max\{\mathrm{ord}_{\tilde{x}}(g_1), \mathrm{ord}_{\tilde{x}}(g_2)\})$$

where $\tilde{x} = (x_i)$ where $x_i \in X_i(\mathbf{k})$ mapping to each other under the system $(X_i \to X_j)_{i>j}$, [1] and $D^+_\infty \cong \mathrm{Div}^+_\infty := \mathrm{colim}\, \mathrm{Div}^+(A_i)$ are the functions from $\lim X_i(\mathbf{k}) \to \mathbf{Z}_{\geq 0}$ which have profinite support.

□

4. $\mathcal{S}$-definable subspaces of the ideal class group and decomposition of the ideal class group

The following lemma will motivate certain subspaces of rational divisor class groups which will be critical in proving the main theorem (c.f. Definition 4.2).

Lemma 4.1. *Let $f : X \to Y$ be a morphism of smooth affine varieties over $\mathbf{k}$. The following direct sum of $\mathbf{Q}$-(vector spaces) holds.*

$$\mathrm{Cl}(X)_{\mathbf{Q}} \cong \ker(f_*) \oplus \mathrm{im}(f^*)$$

[1]Let $\varphi : X \to Y$ be a morphism of varieties. Suppose $\varphi(x) = y$ and let $f \in \kappa(Y) \subset \kappa(X)$ via $\varphi^{\#}$: we have $\mathrm{ord}_{\varphi(x)}(f) = \mathrm{ord}_x(\varphi^{\#} f)$ — we are just extending valuations.

Proof. Let $\deg(f) = n$. It suffices to show the decomposition for a single point. Let $[x] \in \mathrm{Cl}(X)_{\mathbf{Q}}$ and let $y = f(x)$. We have

$$n[x] = n[x] - f^*([y]) + f^*([y]) = f^*([y]) - \left[\sum_{\widetilde{x}\mapsto y} e(\widetilde{x}/y)[\widetilde{x}] - e(\widetilde{x}/y)[x]\right].$$

This implies

$$[x] = \frac{1}{n}\cdot f^*[y] + \frac{1}{n}\cdot\left(\sum_{\widetilde{x}\mapsto y} e(\widetilde{x}/y)([\widetilde{x}] - [x])\right),$$

where the left term is in $\mathrm{im}(f^*)$ and the right term is in $\ker(f_*)$. The fact that these are in direct sum follows from the projection formula $f_*f^* = n$. $\square$

The following definition defines subgroups of $\mathrm{Cl}(X)$.

Definition 4.2. Let $f : X \to Y$ be a morphism of smooth affine curves over $\mathbf{k}$. For a point $y \in Y$ we define the *fiber span*, *inverse image* and *fiber difference span* as

$$\mathrm{Cl}(X)_y^{(0)} = \mathrm{Cl}(X)_y = \{[D] : \mathrm{supp}(D) \leq f^{-1}(y)\},$$

$$\mathrm{Cl}(X)_y^{(1)} = \mathrm{Cl}(X)_y \cap \mathrm{im}(f^*),$$

$$\mathrm{Cl}(X)_y^{(2)} = \mathrm{Cl}(X)_y \cap \ker(f_*),$$

respectively. We make a similar definition for $\beta = \{y_1, \ldots, y_n\} \subset Y$:

$$\mathrm{Cl}(X)^{(i)}_{\{y_1,\ldots,y_n\}} = \sum_{j=1}^{n} \mathrm{Cl}(X)^{(i)}_{y_j}.$$

We will define $\mathbf{Q}$-versions similarly and denote them by $\mathrm{Cl}(X)^{(i)}_{\mathbf{Q},\{y_1,\ldots,y_n\}}$.

Remark 4.3. Note that $\mathrm{Cl}(X)^{(i)}_{\mathbf{Q},\{y_1,\ldots,y_n\}} = (\mathrm{Cl}(X)^{(i)}_{\{y_1,\ldots,y_n\}})_{\mathbf{Q}}$. That is, the groups defined by taking all the maps by first taking $\mathbf{Q}$-divisors and then forming subgroups and quotients is the same as taking the full groups and tensoring with $\mathbf{Q}$. This is because $\mathbf{Q}$ is flat over $\mathbf{Z}$.

This is convenient because the integral versions of these groups are definable:

$$\mathrm{Cl}(X)_y = \{\alpha \in \mathrm{Cl}(X) : (\exists D_1, D_2 \in \mathrm{Div}^+(X))(\mathrm{supp}(D_1) \leq f^*y \text{ and } \mathrm{supp}(D_2) \leq f^*y \text{ and } [D_1]-[D_2] = \alpha)\}$$

and the groups $\mathrm{Cl}(X)_{\mathbf{Q}}$ are subgroups of $\mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}})$.

The following lemma summerizes the situation regarding torsion.

Lemma 4.4 (Torsion Freeness in the Limit)**.** *Let A_i be an exhaustive system of rings of integers for $\mathbf{k}[t]^{\mathrm{alg}}$ i.e. $\mathbf{k}[t]^{\mathrm{alg}} = \mathrm{colim}\, A_i = \bigcup_i A_i$. The following equalities hold:* $\mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}}) \cong \mathrm{colim}\,\mathrm{Cl}(A_i) \cong \mathrm{colim}[\mathrm{Cl}(A_i)_{\mathbf{Q}}]$.

Proof. We will check each equality separately. Each $A_i \to \mathbf{k}[t]^{\mathrm{alg}}$ gives

$$\mathrm{Cl}(A_i) \to \mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}}),$$

given by $[(f_i, g_i)]_{\mathrm{Cl}(A_i)} \mapsto [(f_i, g_i)]_{\mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}})}$. Here the square brackets are used to denote equivalence classes of ideals.

We will now check that $\mathrm{colim}\,\mathrm{Cl}(A_i) \cong \mathrm{colim}[\mathrm{Cl}(A_i)_{\mathbf{Q}}]$. This is just saying that in the colimit, the divisor class groups are torsion free. It suffices to show that for all curves $X/\mathbf{k}$ and all $[D] \in \mathrm{Cl}(X)_{\mathrm{tors}}$ there exists some cover $p : X' \to X$ such that $p^*[D] = 0$.

Let $m \in \mathbf{N}$ be such that $mD = \mathrm{div}(f)$ for some $f \in A$. There exists a cover $p : X' \to X$ and some $g \in \mathcal{O}(X')$ such that $p^{\#}(f) = g^m$. This implies the following string of equalities in $\mathrm{Div}(X')$:

$$mp^*D = p^*(mD) = p^*(\mathrm{div}(f)) = \mathrm{div}(g^m) = m\,\mathrm{div}(g).$$

Hence $p^*D = \mathrm{div}(g)$ and $[D] \in \ker(p^* : \mathrm{Cl}(X) \to \mathrm{Cl}(X))$ as desired. □

Remark 4.5. Lemma 4.4 is exactly the reason why we assume $\mathbf{k} \neq \mathbf{F}_p^{\mathrm{alg}}$. If we let $\mathbf{k} = \mathcal{F}_p^{\mathrm{alg}}$, then for every X, every element of $\mathrm{Cl}(X)$ would be map to zero in the limit since all elements are torsion.

Lemma 4.4 is nice because it allows us to use the nice interpretability properties of $\mathrm{Cl}(X)_y^{(i)}$ at the finite level and extend them to the infinite level.

Lemma 4.6 (First Decomposition Lemma). *Let $f : X \to Y$ be a morphism of smooth affine curves over $\mathbf{k}$. For all $\beta = \{y_1, \dots, y_n\} \subset Y(\mathbf{k})$, we have the following direct sum decomposition of $\mathbf{Q}$-vector spaces:*

$$\mathrm{Cl}(X)_{\mathbf{Q},\beta} \cong \mathrm{Cl}(X)^{(1)}_{\mathbf{Q},\beta} \oplus \mathrm{Cl}(X)^{(2)}_{\mathbf{Q},\beta}.$$

Proof. This follows directly from Lemma 4.1. □

The next decomposition lemma (Lemma 4.8) requires some representability of the divisor class group (in order to talk about transcendence degrees and Zariski closures).

Lemma 4.7. *Let X be a smooth affine curve over $\mathbf{k} = \mathbf{k}^{\mathrm{alg}}$. Let $\overline{X}$ be its compactifcation. We have*

$$\mathrm{Cl}(X) \cong \mathrm{Cl}^0(\overline{X}) / \mathrm{Cl}^0(\overline{X})_{\overline{X}\setminus X}.$$

In other words, the ideal class group is just the Jacobian modded out by the subgroup generated by points at infinity.

Proof. We define $\rho : \mathrm{Div}(X) \to \mathrm{Cl}^0(X)/\mathrm{Cl}^0(X)_{\overline{X}\setminus X}$ given by $\rho(D) = [D - D_\infty]$ where D_∞ is any effective divisor with $\mathrm{supp}(D_\infty) \subset \overline{X} \setminus X$ and $\deg(D_\infty) = \deg(D)$. We claim that this map is independent of the choice of D_∞. To see this observe that for any other choice D'_∞ we have $D_\infty - D'_\infty \in \mathrm{Cl}^0(X)_{\overline{X}/X}$. This map is clearly surjective. It hence suffices to show $\ker(\rho) = P(X)$.

We will now show $\ker(\rho) = P(X)$. Observe that $\rho(D) = 0$ if and only if

$$D - \deg(D) \cdot x = \sum (x_i - x_i') + \mathrm{div}_{\overline{X}}(g)$$

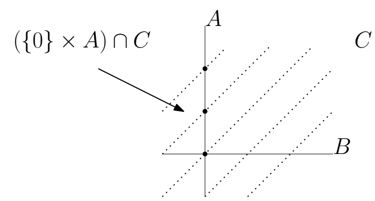

FIGURE 1. A graph of the situation in Lemma 4.8.

where x, x_i, x_i' are points of $f^{-1}(\overline{Y} \setminus Y)$ and $g \in \kappa(\overline{X})$. By manipulating this equality we find

$$\operatorname{div}_X(g) - D = -\operatorname{div}_{\overline{X} \setminus X}(g) + \sum_i (x_i - x_i') - \deg(D)x.$$

The left hand side of this equality has support in X and the right hand side has support in $\overline{X} \setminus X$. This means both sides must be zero which implies that $D = \operatorname{div}_X(g)$. ☐

In what follows we refer to [Lan58] for Weil style notions of generic points.

Lemma 4.8 (Second Decomposition Lemma)**.** *Let X and Y be affine curves with compactifications $\overline{X}$ and $\overline{Y}$. Suppose in addition $\overline{Y}$ is an elliptic curve whose origin is $\overline{Y} \setminus Y$. Let $f : X \to Y$ be a morphism. Let $\bar{\beta}_1, \bar{\beta}_2 \in \mathrm{Cl}(Y)_{\mathbf{Q}}$ are linearly independent then $\mathrm{Cl}(X)^{(2)}_{\mathbf{Q},\bar{\beta}_1}, \mathrm{Cl}(X)^{(2)}_{\mathbf{Q},\bar{\beta}_2}$ and $\mathrm{Cl}(X)^{(2)}_{\mathbf{Q},\bar{\beta}_1+\bar{\beta}_2}$ are in direct sum as subspaces of $\mathrm{Cl}(X)^{(2)}_{\mathbf{Q}}$.*

Proof. Let $\bar{\alpha}_2 \in \mathrm{Cl}(X)^{(2)}_{\bar{\beta}_2}$ and $\bar{\alpha}_3 \in \mathrm{Cl}(X)^{(2)}_{\bar{\beta}_3}$ and set $\bar{\alpha}_1 = \bar{\alpha}_2 + \bar{\alpha}_3$. Let $\beta_i \in \overline{Y} = \mathrm{Pic}^0(\overline{Y})$ and $\alpha_i \in \mathrm{Pic}^0(\overline{X})$ be points lifting β_i. Suppose that $\bar{\beta}_i$ and $\bar{\alpha}_i$ are the images of β_i and α_i in the quotients $\mathrm{Cl}(Y)$ and $\mathrm{Cl}(X)$ respectively.

We will show that $\alpha_i \in \mathrm{Pic}^0(\overline{X})(\mathbf{k})$ are torsion (and hence zero in the limit). To do this we show the α_i are in $\mathrm{Pic}^0(\overline{X})(k_0)$ where k_0 is the field of definition of $\overline{X}$ and that they simultaneously have zero trace (from being in the difference of fibers).

Let $A = \mathrm{Pic}^0(\overline{X})$ and $B = \mathrm{Pic}^0(\overline{Y}) \cong \overline{Y}$. Let $z_i = (\beta_i, \alpha_i) \in \overline{Y} \times \mathrm{Pic}^0(\overline{X})$ and observe that

$$\operatorname{trdeg}(\beta_i, \alpha_i) = \operatorname{trdeg}(\beta_i) = 1 \tag{4.9}$$

for $i = 1, 2, 3$.[2] Define $Z_i := \langle z_i \rangle^{\mathrm{Zar}} \subset \overline{Y} \times \mathrm{Pic}^0(\overline{X})$ where $\langle z_i \rangle$ denotes the group generated by z_i and the superscript Zar denotes Zariski closure. All of these subvarieties have dimension one by 4.9. Also, by Mordell-Lang, all of the Z_i are translates of an abelian subvariety. Observe that $Z_2 + z_3 = Z_1$ and $z_2 + Z_3 = Z_1$ which means all of the Z_i are cosets of the same abelian variety. $Z_i = C + \gamma_i$ where $C \subset \overline{Y} \times \mathrm{Pic}^0(\overline{X})$ is a fixed abelian subvariety and $\gamma_i = (0, \alpha_i')$ is a point.

Using C we will define a function $h : \overline{Y} \to \mathrm{Pic}^0(\overline{X})$ defined by

$$h(\beta) = \pi_{\mathrm{Pic}^0(\overline{X})}([m]\pi_{\overline{Y}}^{-1}(\beta)),$$

where $\pi_{\mathrm{Pic}^0(\overline{X})} : C \to \mathrm{Pic}^0(\overline{X})$ and $\pi_{\overline{Y}} : C \to \overline{Y}$ are induced from the natural projections of $\overline{Y} \times \mathrm{Pic}^0(\overline{X})$ and

$$m := \#[C \cap (\{0\} \times \mathrm{Pic}^0(\overline{X}))].$$

[2]This is where we needed the representability

This is well-defined: if $(\beta, a_1), (\beta, a_2) \in \pi^{-1}_{\mathrm{Pic}^0(\overline{X})}(\beta) \subset C$ observe that $(\beta, a_1) - (\beta, a_2) = (0, a_1 - a_2) \in C \cap (\{0\} \times \mathrm{Pic}^0(\overline{X}))$ which is a finite group, so multiplication by the order, m, returns zero.

We have shown:

- Let $\mathbf{K}$ be a saturated algebraically closed field containing $\mathbf{k}$. For generic $\beta_2 \in \overline{Y}(\mathbf{K})$ we have $h(\beta_2) = m(\alpha_2 - \alpha_2')$ ♠♠♠ Taylor: [FIXME: I think this is because we can do the computation for $y = \beta_i$ where we can actually compute it.]

We claim now that $h = 0$ identically. Since the value of h at points is determined by specialization of the value of h at generic points it suffices to show $h = 0$ at a generic point. Let $K = \mathbf{k}(y)$. Let $L = K(\overline{f}^{-1}(y))$.

There is a transitive action of $\mathrm{Aut}(L/K)$ on $\overline{f}^{-1}(y)$. This means that for all $x, \sigma(x) \in f^{-1}(y)$

$$\mathrm{Tr}(\sigma(x) - x) = \sum_{\tau \in \mathrm{Aut}(L/K)} \tau(\sigma(x) - x) = 0.$$

On the other hand, since the morphism h is defined over K this means $h^\sigma = h$ and hence that $\mathrm{Tr}_{L/K}(h(y)) = [L : K]h(y)$. [3] Putting these two computations together gives $[L : K]h(y) = 0$ which implies $h(y) = 0$.

♠♠♠ Taylor: [From here we conclude that α_1 is in the field of definition and that it is torsion.]

□

The following Lemma tells us how to define one dimensional vector spaces over $\mathbf{Q}$ using parameters.

Lemma 4.10 (Definable $\mathbf{Q}$-subspaces)**.** *For each $[y_0] \in \mathrm{Cl}_\infty$ the subgroups*

$$L(y_0) := \bigcap \{W : y_1, y_2 \in B_\infty \text{ and } y_0 \in W = \mathrm{Cl}_{\infty, y_1} + \mathrm{Cl}_{\infty, y_2}\}$$

are $\mathcal{S}_\infty$-definable and $L(y_0) = \mathbf{Q}[y_0] \subset \mathrm{Cl}_\infty$.

Proof. Since $\mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}}) \cong \operatorname{colim} \mathrm{Cl}(X_i)_{\mathbf{Q}}$ there exists some affine smooth curve Y in the inverse system $(X_i)_{i \in I}$ such that $[y_0] \in \mathrm{Cl}(Y)_{\mathbf{Q}}$. Let $\epsilon = [y_1]$ be linearly independent from $[y_0]$ in $\mathrm{Cl}(Y)_{\mathbf{Q}}$ and take $[y_2] := [x] + \epsilon \in \mathrm{Cl}(Y)_{\mathbf{Q}}$. Observe that with these choices of y_1 and y_2 we have $[y_0] \in \mathrm{Cl}_{\infty, y_1} + \mathrm{Cl}_{\infty, y_2}$. We will show that any $\alpha_0 \in \mathrm{Cl}_{\infty, y_1} + \mathrm{Cl}_{\infty, y_2}$ is a rational multiple of $[y_0]$. To see this, write

$$\alpha_0 = \alpha_1 + \alpha_2$$

where $\alpha_i \in \mathrm{Cl}_{\infty, y_i}$. Using the decomposition $\mathrm{Cl}(Y)_{\mathbf{Q}} = \mathrm{Cl}(Y)^{(1)}_{\mathbf{Q}} \oplus \mathrm{Cl}(Y)^{(2)}_{\mathbf{Q}}$ we may write

$$\alpha_j = \alpha_j^{(1)} + \alpha_j^{(2)} = n[y_j] + (\alpha_j - n[x])$$

for $j = 0, 1, 2$ at some finite level $X \to Y$ of degree n and we have

$$\alpha_0^{(1)} = \alpha_1^{(1)} + \alpha_2^{(1)},$$

[3] Again see [Lan58].

$$\alpha_0^{(2)} = \alpha_1^{(2)} + \alpha_2^{(2)}.$$

By the second part of Lemma 4.6, $\alpha_i^{(2)}$ are in direct sum and hence must all be zero. This implies that

$$\alpha_0 = \alpha_0^{(1)} + \alpha_0^{(2)} = \alpha_0^{(1)} = n[x].$$

□

5. Proof of the main theorem

We give the following definition of projective incidence geometry informally. We will omit the first order formulas for the axioms as they are cumbersome. More information can be found in [EH91], [Art, Chapter II, sections 3 and 10], [Hil02, section 15, section 24].

Definition 5.1. A **projective incidence geometry** (PIG) is a two-sorted structure

$$(S, L, * \in *, l)$$

with "$\in$" being a relation on $S \times L$ and $l : S \times S \to L$. The elements of S are called points and the elements of L are called lines. The axioms of the theory are as follows:

(1) There exists a point and a line not intersecting.
(2) Every line has three points.
(3) For every two points there is a unique line passing through it.
(4) Planes are spanned by two intersecting lines: For every distinct $s, s', t, t' \in S$, if $l(s, s') \cap l(t, t') \neq \emptyset$ then $l(s, t) = l(s', t') \neq \emptyset$.

The fundamental example of such structures come from considering $\mathbf{P}_k(V)$ where V is a vector space over a field k. We remark that the notion of dimension is definable within such a structure. To see this one just needs to define what it means for $n + 1$ points to exist in general position. If such a configuration exists then the incidence geometry has dimension at least n.

For two dimensional projective geometries we may impose an additional axiom called Pappus' axiom. This is pictured in figure 5.

Lemma 5.2 (Fundamental Theorem of Projective Geometry)**.** *Suppose*

$$(S, L) \models PIG + PAP + (\dim(S, L) \geq 2)$$

(1) *There exists some field k, some vector space V and an isomorphism of incidence geometries*

$$\sigma : \mathbf{P}_k(V) \to S.$$

(2) *If (σ, V, k) and (σ', V', k') are two such triples there exists isomorphisms*

$$k \to k'$$

$$V \to V'$$

which are compatible with the isomorphisms σ and σ'.

(3) *If in addition we admit to the structure*
   - *two intersecting lines lines l_1, l_2*

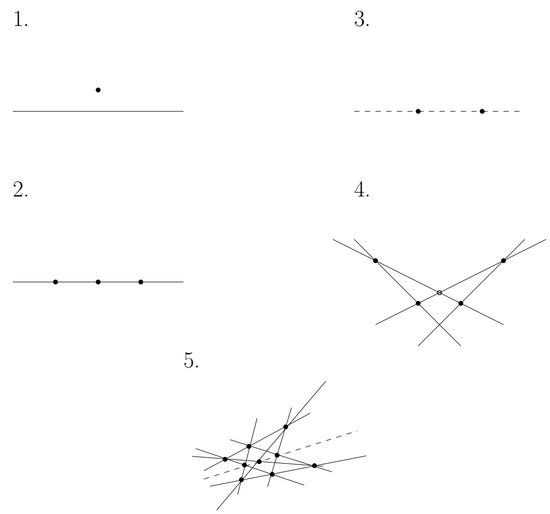

FIGURE 2. Axioms of PIGs.

- *four distinct points $P_1, P_2 \in l_1$ and $Q_1, Q_2 \in l_2$ not equal to the unique point of intersection.*

Then $(S, L) \cup \{l_1, l_2, P_1, P_2, Q_1, Q_2\}$ interprets the field k.

Definition 5.3. Let V be a finite dimensional vector space over a field F. We defined the **vector subspaces structure** $\mathrm{VSS}(V)$ to be the structure

$$\mathrm{VSS}(V) = (\mathcal{F}, \cap, +)$$

consisting of $\mathcal{F} = \{W \leq V, F\text{-vector subspaces}\}$ together with the join operation, $+$, and the intersection operation $\cap$. We will just denote this by V.

Note that the collection of a vector subspaces of a fixed dimension are definable as $\subsetneq$ is definable.

Lemma 5.4. *The let $\mathcal{S}'$ be the structure $\mathcal{S}_\infty$ (section 3) together with three $\mathbf{Q}$-(linearly independent) points $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Cl}_2(\mathbf{k}[t]^{\mathrm{alg}})$ as parameters. The structure $\mathcal{S}'$ interprets the lattice of vectors subspaces for $V = \mathbf{Q}\alpha_1 + \mathbf{Q}\alpha_2 + \mathbf{Q}\alpha_3 \subset \mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}})$.*

We can now prove the main theorem of the paper (Theorem 1.1).

Proof of Theorem 1.1. Let α_1, α_2 and α_3 be three $\mathbf{Q}$-linearly independent points in $\mathrm{Cl}(\mathbf{k}[t]^{\mathrm{alg}})$ and let V be the $\mathbf{Q}$-vector space

$$V = \mathbf{Q}\alpha_1 \oplus \mathbf{Q}\alpha_2 \oplus \mathbf{Q}\alpha_3.$$

Theorem 1.1 follows from the following series of interpretations:

- The $\mathbf{k}[t]^{\mathrm{alg}}$ with its standard ring structure interprets $\mathcal{S}(\mathbf{k}[t]^{\mathrm{alg}})$ (Lemma 3.1).
- The structure $\mathcal{S}(\mathbf{k}[t]^{\mathrm{alg}}) \cup \{\alpha_1, \alpha_2, \alpha_3\}$ interprets the lattice of subspaces of V.
- The lattice of subspaces of V interpret $\mathbf{P}_{\mathbf{Q}}(V)$ as a projective plane structure.
- By the fundamental theorem of projective geometry the structure $\mathbf{P}_{\mathbf{Q}}(V)$ interprets $\mathbf{Q}$ with its standard field structure.

□

As we give it, the interpretations requires parameters for three linearly independent points of a certain interpretable group. The following can make such an interpretation uniform.

Lemma 5.5 (Definable Parameters implies No Parameters)**.** *Let P be a definable set and suppose that $\{Z_c\}_{c\in P}$ is a family of interpretable structures in some finite language given uniformly in c. Suppose that*

(1) *For each $c, d \in P$ there exists an definable isomorphism $f_{c,d} : Z_c \to Z_d$ possibly with additional parameters.*
(2) *For each $c \in P$, $\mathrm{Aut}(Z_c) = 1$.*

Then there exists a structure Z interpretable without parameters an isomorphisms $g_c : Z \to Z_c$.

Proof. Let $Z = \{(a, z) : a \in P, z \in Z_a\}$ modulo the equivalence relations $(c, y)E(d, z)$ if and only if there exists a definable isomorphism $f_{c,d} : Z_c \to Z_d$ with $f_{c,d}(y) = z$. The equivalence relation is Ind-definable but since for any c, d, y there exists a unique z with $(c, y)E(d, z)$ we have $(c, y)!E(d, z)$ if and only if $\exists(z' \neq z) : (c, y)E(d, z')$ so it is definable. etc. □

Lemma 5.6 (Definability of our Parameters)**.** *The situation of Lemma 5.4 satisfies the hypotheses of Lemma 5.5 (This implies that $\mathcal{S}_\infty$ interprets $\mathbf{Q}$ without parameters and hence that $\mathbf{k}[t]^{\mathrm{alg}}$ interprets $\mathbf{Q}$).*

Proof. In our case Z_c is a copy of $\mathbf{Q}$ with its usual field structure. It is clear that $\mathrm{Aut}(\mathbf{Q}) = 1$. It remains to check the definable isomorphism condition (item 1). It is easy to see that since two projective geometries P_c, P_d used to define Z_c, Z_d can be embedded into a single projective geometry of higher dimension. By the Fundamental Theorem of Projective Geometry, the division ring of P_c is isomorphic to that of P and thus to that of P_d. Also, these isomorphisms are given in a definable way with parameters. In other words, all the relevant copies of $\mathbf{Q}$ are isomorphic. □

References

[Art] Emile Artin, *Geometric algebra. 1957*, Interscience, New York.

[EH91] David M Evans and Ehud Hrushovski, *Projective planes in algebraically closed fields*, Proceedings of the London Mathematical Society **3** (1991), no. 1, 1–24.

[Hil02] David Hilbert, *The foundations of geometry*, Open court publishing Company, 1902.

[Lan58] Serge Lang, *Introduction to algebraic geometry*, Interscience Publishers, Inc., Chapman & Hall, 1958.

[PS90] Alexander Prestel and Joachim Schmid, *Existentially closed domains with radical relations*, J. reine angew. Math **407** (1990), 178–201.

[vdD88] Lou van den Dries, *Elimination theory for the ring of algebraic integers*, J. reine angew. Math **388** (1988), 189–205.

[vdDM90] Lou van den Dries and Angus Macintyre, *The logic of Rumely's local-global principle*, J. reine angew. Math **407** (1990), 33–56.

Einstein Institute of Mathematics Edmond J. Safra Campus The Hebrew University of Jerusalem Givat Ram. Jerusalem, 9190401, Israel
E-mail address: `dupuy@math.huji.ac.il`

Einstein Institute of Mathematics Edmond J. Safra Campus The Hebrew University of Jerusalem Givat Ram. Jerusalem, 9190401, Israel
E-mail address: `ehud@math.huji.ac.il`