

ARITHMETIC DIFFERENTIAL EQUATIONS ON GL_n , I: DIFFERENTIAL COCYCLES

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ABSTRACT. The theory of differential equations has an arithmetic analogue [10] in which derivatives are replaced by Fermat quotients. One can then ask what is the arithmetic analogue of a linear differential equation. The study of usual linear differential equations is the same as the study of the differential cocycle from GL_n into its Lie algebra given by the logarithmic derivative [17]. However we prove here that there are no such cocycles in the context of arithmetic differential equations. In sequels of this paper [12, 13] we will remedy the situation by introducing arithmetic analogues of Lie algebras and a skew version of differential cocycles; this will lead to a theory of linear arithmetic differential equations.

1. INTRODUCTION AND MAIN RESULTS

In [7] an arithmetic analogue of differential equations was introduced in which derivations are replaced by Fermat quotient operators; cf. [10] for an overview of the theory which was mostly concerned with Abelian varieties and Shimura varieties. With the exception of [9], however, little attention has been given to the case of arithmetic differential equations attached to linear algebraic groups such as GL_n . The present paper is the first in a series of papers whose purpose is to shed some light into the theory for GL_n . In particular, one of our main motivations is to understand what is the correct notion of “linear arithmetic differential equation”.

The basic clue should come from the Ritt-Kolchin differential algebra [17] whose arithmetic analogue is the theory in [7, 10]. So let us examine Kolchin’s setting first. Since we will later treat Kolchin’s theory and our arithmetic theory simultaneously it is convenient to use the same notation in two different contexts. We call these contexts the δ -algebraic setting (corresponding to Kolchin’s theory) and the δ -arithmetic setting (corresponding to the theory in [7, 10]).

In the δ -algebraic setting we denote by R a field of characteristic zero equipped with a derivation $\delta : R \rightarrow R$ and we assume R is δ -closed (which is the same as *constrainedly closed* in Kolchin’s terminology [17]). We denote by R^δ the field of constants $\{c \in R; \delta c = 0\}$. In this context we will consider smooth schemes of finite type X over R (i.e. nonsingular varieties) and we denote by $X(R)$ the set of R -points of X ; if there is no danger of confusion we often simply write X in place of $X(R)$.

In the δ -arithmetic setting we assume R is the unique complete discrete valuation ring with maximal ideal generated by an odd prime p and residue field $k = R/pR$ equal to the algebraic closure \mathbb{F}_p^a of \mathbb{F}_p ; then we denote by $\delta : R \rightarrow R$ the unique p -derivation on R in the sense of [7]; recall that $\delta x = \frac{\phi(x) - x^p}{p}$ where $\phi : R \rightarrow R$ is the unique ring homomorphism lifting the p -power Frobenius on the residue field

k . We denote by R^δ the monoid of constants $\{\lambda \in R; \delta\lambda = 0\}$; so R^δ consists of 0 and all roots of unity in R . Also we denote by K the fraction field of R . In this context we will consider smooth schemes of finite type X over R or, more generally, smooth p -formal schemes of finite type, by which we mean formal schemes locally isomorphic to p -adic completions of smooth schemes of finite type; we denote by $X(R)$ the set of R -points of X ; if there is no danger of confusion we often simply write X in place of $X(R)$. Groups in the category of smooth p -formal schemes will be called smooth group p -formal schemes.

In the δ -algebraic setting a map $f : R^N \rightarrow R^M$ will be called a δ -map of order n if there exists an M -vector $F = (F_j)$ of polynomials $F_j \in R[x_0, \dots, x_n]$ with coefficients in R in $n+1$ N -tuples of variables x_0, \dots, x_n , such that

$$(1.1) \quad f(a) = F(a, \delta a, \dots, \delta^n a), \quad a \in R^N.$$

In the δ -arithmetic setting a map $f : R^N \rightarrow R^M$ will be called a δ -map of order n if there exists an M -vector $F = (F_j)$ of restricted power series $F_j \in R[x_0, \dots, x_n]^\wedge$ (where $^\wedge$ means p -adic completion) such that 1.1 holds. In both the δ -algebraic and the δ -arithmetic setting one can then consider affine smooth schemes X, Y and define a map $f : X(R) \rightarrow Y(R)$ to be a δ -map of order n if there exist embeddings $X \subset \mathbb{A}^N, Y \subset \mathbb{A}^M$ such that f is induced by a δ -map $R^N \rightarrow R^M$ of order n ; we simply write $f : X \rightarrow Y$. Even more generally if X is any smooth scheme and Y is an affine scheme a set theoretic map $X \rightarrow Y$ is called a δ -map of order n if there exists an affine cover $X = \bigcup X_i$ such that all the induced maps $X_i \rightarrow Y$ are δ -maps of order n . If G, H are smooth group schemes a δ -homomorphism $G \rightarrow H$ is, by definition, a δ -map which is also a homomorphism. We shall review these concepts in section 2 following [7, 10] in the δ -arithmetic setting and [17, 14, 6] in the δ -algebraic setting. We will use a slightly different (but equivalent) approach using jet spaces.

Assume we are in either the δ -algebraic or in the δ -arithmetic setting. Let G be an algebraic group (i.e. smooth group scheme) over R and let $L(G)$ be its Lie algebra which we view as a group scheme over R isomorphic as a scheme with the affine space. Denote by $\star : G \times L(G) \rightarrow L(G)$ the (left) adjoint action.

Definition 1.1. A classical δ -cocycle of G with values in $L(G)$ is a δ -map $f : G \rightarrow L(G)$ which is a cocycle for the adjoint action; i.e. for all $g_1, g_2 \in G(R)$ we have

$$(1.2) \quad f(g_1 g_2) = f(g_1) + g_1 \star f(g_2).$$

Here $G(R)$ is the set of R -points of G . If in the definition above f is a regular map (i.e. a morphism of schemes over R) then we say f is a regular cocycle.

The adjective *classical* in the above definition was used in order to distinguish between the *classical* δ -cocycles introduced above and *skew* δ -cocycles to be introduced in [12]; skew δ -cocycles will then be viewed as “non-classical” objects.

Remark 1.2.

1) For G a smooth closed subgroup scheme of some GL_n , identifying $L(G)$ with a Lie subalgebra of $\mathfrak{gl}_n = L(GL_n)$, we get that condition 1.2 reads

$$f(g_1 g_2) = f(g_1) + g_1 f(g_2) g_1^{-1}.$$

On the other hand if G is a (non-necessarily affine) smooth commutative group then condition 1.2 reads

$$f(g_1 g_2) = f(g_1) + f(g_2)$$

i.e. f is simply a δ -homomorphism. In view of Kolchin's theory [17] it seems reasonable to introduce the following definition (in both the δ -algebraic and the δ -arithmetic contexts): a linear differential equation is an equation of the form $f(u) = \alpha$ where f is a classical δ -cocycle of G with values in $L(G)$, $u \in G(R)$ is the unknown and $\alpha \in L(G)(R)$ is a given R -point of the Lie algebra.

2) Assume we are in the δ -algebraic context and assume G is defined over the field of constants R^δ of δ . In that case there is a remarkable order 1 classical δ -cocycle $l\delta : G \rightarrow L(G)$ called the *Kolchin logarithmic derivative*. For G closed in GL_n and δ -horizontal (i.e. defined by equations with coefficients in R^δ) $l\delta$ is defined by $l\delta g = \delta g \cdot g^{-1}$, $g \in G(R)$. This essentially models, in the context at hand, concepts that go back to Lie and Cartan. Also the equation $l\delta(u) = \alpha$ for $G = GL_n$ reduces to an equation $\delta u = \alpha u$ which is the familiar form of a linear differential equation. The above construction can be generalized to the case when G has a structure of D -group [5, 19] i.e. G comes equipped with a derivation $\mathcal{O}_G \rightarrow \mathcal{O}_G$ extending that of R and compatible with the group structure.

3) Assume again that we are in the δ -algebraic context. Then for any Abelian variety A over R (not necessarily defined over the field of constants R^δ !) there is a surjective, order 2, δ -homomorphism $f : A \rightarrow L(A)$ (hence a classical δ -cocycle) into the Lie algebra $L(A)$ of A ; this follows from Manin's work [18] (cf. [4, 6] for a different construction of Manin's map f). It turns out that there is an analogue of Manin maps in the δ -arithmetic setting [7]: for any abelian scheme A over R there is an order 2 surjective δ -homomorphism $f : A \rightarrow L(A)$ with remarkable properties (e.g one has a precise description of its kernel which is analogous to the Manin theorem of the kernel [18]). Similar maps exist for linear tori in place of abelian schemes; in particular there is a complete description of the δ -homomorphisms $\mathbb{G}_m \rightarrow \mathbb{G}_a = L(\mathbb{G}_m)$; cf. Lemma 3.4 (which follows immediately from [7]).

To summarize, one has a good definition of linear differential equations in the δ -algebraic setting. Moreover there is a nice δ -arithmetic analogue of linear differential equations in the case of Abelian varieties. However our Theorem 1.3 below shows that there is no "naive" δ -arithmetic analogue of the linear differential equations in the case of GL_n : what we show is that there is no "naive" δ -arithmetic analogue of the Kolchin logarithmic derivative for GL_n , $n \geq 2$.

Theorem 1.3. *Assume we are in the δ -arithmetic setting. Let $f : GL_n \rightarrow \mathfrak{gl}_n$ be a classical δ -cocycle. Then there exists a δ -homomorphism $\omega : \mathbb{G}_m \rightarrow \mathbb{G}_a$ and there exists $v \in \mathfrak{gl}_n(R)$ such that for all $g \in GL_n(R)$ we have:*

$$f(g) = \omega(\det(g))1_n + gv g^{-1} - v.$$

The corresponding statement in algebraic geometry, saying that if a map of varieties $GL_n \rightarrow \mathfrak{gl}_n$ over an algebraically closed field of characteristic zero is a cocycle then the map must be a coboundary, is "well known" and follows easily from Whitehead's lemma; cf. Lemma 3.1.

Note also that our computations have a δ -algebraic variant leading to a characterization of Kolchin's logarithmic derivative that seems to be new. Indeed assume we are in the δ -algebraic setting and let $f : GL_n \rightarrow \mathfrak{gl}_n$ be a classical δ -cocycle. Let us say that f is δ -coherent if for any algebraic subgroup $G \subset GL_n$ defined over the field of constants R^δ with Lie algebra $L(G)$ we have that $f(G) \subset L(G)$. It is well known (and trivial to check) that Kolchin's logarithmic derivative $l\delta : GL_n \rightarrow \mathfrak{gl}_n$, $l\delta g = \delta g \cdot g^{-1}$, is a δ -coherent classical δ -cocycle. Conversely we will prove:

Theorem 1.4. *Assume we are in the δ -algebraic setting. Let $f : GL_n \rightarrow \mathfrak{gl}_n$ be a δ -coherent classical δ -cocycle and assume $n \geq 2$. Then there exists $\nu \in R$ such that for all $g \in GL_n(R)$,*

$$f(g) = \nu \cdot \delta g \cdot g^{-1}.$$

For $n = 1$ the above fails; instead one has a complete description of the situation in this case due to Cassidy [14]; cf. Lemma 3.4 for a review of this.

As already mentioned Theorem 1.3 shows that there is no “naive” analogue of Kolchin’s logarithmic derivative for GL_n when $n \geq 2$. In order to find, then, an arithmetic analogue of linear differential equations we will be led in [12, 13] to replace Lie algebras of algebraic groups by objects that are less linear and, arguably, are better adapted to the arithmetic jet theory. This will lead to a theory of linear arithmetic differential equations.

The present paper is organized as follows. Section 2 reviews some of the basic concepts in [7, 10]. In section 3 we present the proofs of Theorems 1.3 and 1.4 about classical δ -cocycles.

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2. REVIEW OF p -JETS

The aim of this section is to review the relevant material on the δ -arithmetic setting in [7, 10]. We will also include the relevant corresponding comments on differential algebra [17, 14, 6].

Assume first we are in the δ -arithmetic setting. Recall from the introduction that we denote by R the complete discrete valuation ring with maximal ideal generated by an odd prime p and algebraically closed residue field $k = \mathbb{F}_p^a$. Then R comes equipped with a unique lift of Frobenius $\phi : R \rightarrow R$ i.e. with a ring homomorphism whose reduction mod p is the p -th power map. For x a tuple of indeterminates over R and tuples of indeterminates $x', \dots, x^{(n)}, \dots$ we let $R\{x\} = R[x, x', x'', \dots]$ and we still denote by $\phi : R\{x\} \rightarrow R\{x\}$ the unique lift of Frobenius extending ϕ on R such that $\phi(x) = x^p + px'$, $\phi(x') = (x')^p + px''$, etc. Then we let $\delta : R\{x\} \rightarrow R\{x\}$ be defined as $\delta f = p^{-1}(\phi(f) - f^p)$; so $\delta x = x'$, $\delta x' = x''$, etc. We view δ as an analogue of the total derivative operator in differential algebra. Now for any affine scheme of finite type $X = \text{Spec } R[x]/(f)$ over R , where f is a tuple of polynomials, we define the p -jet spaces of X as being the p -formal schemes

$$(2.1) \quad J^n(X) = \text{Spf } \frac{R[x, x', \dots, x^{(n)}]^\wedge}{(f, \delta f, \dots, \delta^n f)}$$

For X of finite type but not necessarily affine we define $J^n(X) = \bigcup J^n(X_i)$ where $X = \bigcup X_i$ is an affine cover and the gluing is an obvious one. The spaces $J^n(X)$ have an obvious universality property for which we refer to [7, 10] and can be defined for X a p -formal scheme of finite type as well. If X/R is smooth then $J^n(X)$ is locally the p -adic completion of a smooth scheme. The universality property yields natural maps on sets of R -points $\nabla^n : X(R) \rightarrow J^n(X)(R)$; for $X = \mathbb{A}^m$ (the affine space), $J^n(X) = \mathbb{A}^{m(n+1)}$ and $\nabla^n(a) = (a, \delta a, \dots, \delta^n a)$. Let X, Y be schemes of finite type over R ; by a δ -map (of order n) $f : X \rightarrow Y$ we understand a map of

p -formal schemes $J^n(X) \rightarrow J^0(Y) = \widehat{Y}$. Two δ -maps $X \rightarrow Y$ and $Y \rightarrow Z$ of orders n and m respectively can be composed (using the universality property) to yield a δ -map of order $n + m$. Any δ -map $f : X \rightarrow Y$ induces a set theoretic map $f_* : X(R) \rightarrow Y(R)$ defined by $f_*(P) = f(\nabla^n(P))$; if X, Y are smooth the map f_* uniquely determines the map f and, in this case, we simply write f instead of f_* (and X, Y instead of $X(R), Y(R)$). A δ -map $X \rightarrow Y$ of order zero is nothing but a map of p -formal schemes $\widehat{X} \rightarrow \widehat{Y}$. The functors J^n commute with products and send groups into groups. By a δ -homomorphism $f : G \rightarrow H$ between two group schemes (or group p -formal schemes) we understand a group homomorphism $J^n(G) \rightarrow J^0(H) = \widehat{H}$.

The Kolchin δ -algebraic setting can be presented in a similar way. In this setting one defines jet spaces $J^n(X)$ of schemes of finite type X over R by the same formula 2.1 in which one drops the symbol $\widehat{}$, one replaces Spf by $Spec$, and one takes $\delta : R\{x\} \rightarrow R\{x\}$ to be the unique derivation extending δ on R and sending $\delta x = x'$, $\delta x' = x''$, etc. If X is smooth then $J^n(X)$ are smooth over R . Again one has natural set theoretic maps $\nabla^n : X(R) \rightarrow J^n(X)(R)$. A δ -map $f : X \rightarrow Y$ is, as before, a morphism of schemes $J^n(X) \rightarrow Y$; one defines similarly the composition of δ -maps. Any δ -map $f : X \rightarrow Y$ induces a set theoretic map $f_* : X(R) \rightarrow Y(R)$ which determines f uniquely if X and Y are smooth; in this case we write f in place of f_* . A δ -map of order zero is nothing but a map of schemes. One defines δ -homomorphisms in the expected way. The formalism of flows and prime integrals is similar and boils down, of course, to the classical picture.

Remark 2.1. The p -jet spaces $J^n(X)$ have natural algebraizations which, in their turn, have multi-prime versions. The resulting jet functors are adjoint to the Witt functors. Cf. Borger's work [1, 2].

3. PROOFS OF THEOREMS 1.3 AND 1.4

We start with the “non-differential” case which is “well known”; for convenience we include a proof (for which we are indebted to A. Minchenko):

Lemma 3.1. *Let $f : GL_n(F) \rightarrow \mathfrak{gl}_n(F)$ be a regular map of algebraic varieties over an algebraically closed field F of characteristic zero and assume f is a cocycle i.e. for all $g_1, g_2 \in GL_n(F)$ equation 1.2 holds. Then f is a coboundary, i.e. there exists $v \in \mathfrak{gl}_n(F)$ such that for all $g \in GL_n(F)$, $f(g) = gvg^{-1} - v$.*

Proof. If in 1.2 we set $g_1 = x \in GL_n(F)$, $g_2 = 1 + \epsilon\xi \in GL_n(F[\epsilon])$ where 1 is the identity, $\epsilon^2 = 0$, and $\xi \in \mathfrak{gl}(F)$ and if we take the coefficient of ϵ we get

$$d_x f(x\xi) = xd_1 f(\xi)x^{-1},$$

equivalently

$$(3.1) \quad d_x f(\xi) = xd_1 f(x^{-1}\xi)x^{-1}.$$

Similarly setting $g_1 = 1 + \epsilon\xi$, $g_2 = x$ in 1.2 we get

$$(3.2) \quad d_x f(\xi) = d_1 f(\xi x^{-1}) + [\xi x^{-1}, f(x)].$$

From 3.1 and 3.2 we get

$$(3.3) \quad xd_1 f(x^{-1}\xi)x^{-1} = d_1 f(\xi x^{-1}) + [\xi x^{-1}, f(x)]$$

for all $x \in GL_n(F)$ and $\xi \in \mathfrak{gl}_n(F)$ and hence for all $x \in GL_n(A)$, $\xi \in \mathfrak{gl}_n(A)$, A an F -algebra. Setting in 3.3 $x = 1 + \epsilon\eta \in GL_n(F[\epsilon])$, with $\eta \in \mathfrak{gl}_n(F)$ and taking the coefficient of ϵ we get

$$d_1f([\xi, \eta]) = [d_1f(\xi), \eta] + [\xi, d_1f(\eta)]$$

so d_1f is a derivation on $\mathfrak{gl}_n(F)$. Note that d_1f preserves $\mathfrak{sl}_n(F)$. Indeed by 1.2 we have

$$\mathrm{tr} f(g_1g_2) = \mathrm{tr} f(g_1) + \mathrm{tr} f(g_2).$$

Since GL_n has no regular non-zero homomorphism into \mathbb{G}_a we get $\mathrm{tr} f = 0$ hence f takes values in $\mathfrak{sl}_n(F)$. Now by the first Whitehead lemma [16], p. 77, all derivations of $\mathfrak{sl}_n(F)$ are inner hence there exists $v \in \mathfrak{sl}_n(F)$ such that

$$(3.4) \quad d_1f(\xi) = [\xi, v]$$

for all $\xi \in \mathfrak{sl}_n(F)$. By 3.3, for all $\lambda \in F$, $f(\lambda 1)$ centralizes all of $\mathfrak{gl}_n(F)$ and, since $f(\lambda 1) \in \mathfrak{sl}_n(F)$, we must have $f(\lambda 1) = 0$. We conclude that 3.4 holds for all $\xi \in \mathfrak{gl}_n(F)$. Let $f^v : GL_n(F) \rightarrow \mathfrak{gl}_n(F)$ be the coboundary $f^v(g) = gvg^{-1} - v$ and note that $d_1f = d_1f^v$. Since, by 3.3, d_xf is determined by d_1f it follows that $d_xf = d_xf^v$ for all $x \in GL_n(F)$. Since $f(1) = f^v(1) = 0$ it follows that $f = f^v$. \square

As a consequence we get:

Lemma 3.2. *Assume we are in the δ -arithmetic setting. Let $f : GL_n \rightarrow \mathfrak{gl}_n$ be a map of schemes over R and assume f is a cocycle i.e. for all $g_1, g_2 \in GL_n(R)$ equation 1.2 holds. Then f is a coboundary, i.e. there exists $v \in \mathfrak{gl}_n(R)$ such that for all $g \in GL_n(R)$, $f(g) = gvg^{-1} - v$.*

Proof. Let K be the fraction field of R and F its algebraic closure. By Lemma 3.1 there exists $v_F \in \mathfrak{gl}_n(F)$ such that $f(g) = gv_Fg^{-1} - v_F$ for all $g \in GL_n(F)$. Denoting by v_K the sum of the conjugates of v_F over K we have $f(g) = gv_Kg^{-1} - v_K$ for all $g \in GL_2(K)$. Let $m \geq 1$ be an integer such that $p^mv_K \in \mathfrak{gl}_2(R)$. If $w = p^mv_K - \alpha 1 = (w_{ij})$ then we have $gwg^{-1} - w = p^mf(g) \in p^m\mathfrak{gl}_n(R)$ for all $g \in GL_n(R)$. Taking g to be an arbitrary diagonal matrix we get that $w_{ij} \equiv 0 \pmod{p^m}$ for $i \neq j$. Taking $g = 1 + e_{ij}$, $i \neq j$, with e_{ij} having all its entries 0 except the (i, j) entry which is 1 we get $w_{ii} \equiv w_{jj} \pmod{p^m}$; since $w_{11} \equiv 0 \pmod{p^m}$ we get that all the entries of w are divisible by p^m in R . So $w = p^mv$, $v \in \mathfrak{gl}_n(R)$, and hence $f(g) = gvg^{-1} - v$ for all $g \in GL_n(R)$, which proves our claim. \square

Lemma 3.3.

1) [7] *Assume we are in the δ -arithmetic setting. Then any δ -homomorphism $f : \mathbb{G}_a \rightarrow \mathbb{G}_a$, viewed as a map $f : R \rightarrow R$, has the form*

$$f(a) = \sum_{i=0}^r \lambda_i \phi^i(a), \quad a \in R$$

for some $\lambda_0, \dots, \lambda_r \in R$.

2) [14] *Assume we are in the δ -algebraic setting. Then any δ -homomorphism $f : \mathbb{G}_a \rightarrow \mathbb{G}_a$, viewed as a map $f : R \rightarrow R$, has the form*

$$f(a) = \sum_{i=0}^r \lambda_i \delta^i(a), \quad a \in R$$

for some $\lambda_0, \dots, \lambda_r \in R$.

Proof. 2) is due to Cassidy [14], p. 936. To check 1), note that by [7] there exists $\nu \geq 1$ and $\lambda_i \in R$ such that $p^\nu f(a) = \sum \lambda_i \phi^i(a)$, $a \in R$. We are left to check that if for $c_i \in R$ we have that for any $a \in R$, $\sum c_i \phi^i(a) \in pR$ then we must have $c_i \in pR$ for all i . But this follows from the fact that $\sum c_i a^{p^i} \equiv 0 \pmod{p}$ for all $a \in R$ and from the fact that the residue field of R is algebraically closed. \square

Next recall from [7], p. 313, that there exists a remarkable δ -homomorphism $\psi : \mathbb{G}_m \rightarrow \mathbb{G}_a$ given on R -points by

$$\psi(a) = \sum_{n \geq 1} (-1)^{n-1} \frac{p^{n-1}}{n} \left(\frac{\delta a}{a^p} \right)^n.$$

With this notation we have:

Lemma 3.4.

1) [7] Assume we are in the δ -arithmetic setting. Then any δ -homomorphism $f : \mathbb{G}_m \rightarrow \mathbb{G}_a$, viewed as a map $f : R^\times \rightarrow R$, has the form

$$f(a) = \sum_{i=0}^r \lambda_i \phi^i(\psi(a)), \quad a \in R^\times$$

for some $\lambda_0, \dots, \lambda_r \in R$.

2) [14] Assume we are in the δ -algebraic setting. Then any δ -homomorphism $f : \mathbb{G}_m \rightarrow \mathbb{G}_a$, viewed as a map $f : R^\times \rightarrow R$, has the form

$$f(a) = \sum_{i=0}^r \lambda_i \delta^i(\delta a \cdot a^{-1}), \quad a \in R^\times$$

for some $\lambda_0, \dots, \lambda_r \in R$.

Proof. 2) is due to Cassidy [14]. To check 1) note that by [7] there exist $\nu \geq 1$ such that $p^\nu f$ has the above form; we conclude exactly as in Lemma 3.3. \square

If instead of looking at δ -homomorphisms $\mathbb{G}_m \rightarrow \mathbb{G}_a$ we are looking at cocycles of non-trivial actions the situation is very different:

Lemma 3.5. Assume we are either in the δ -algebraic setting or in the δ -arithmetic setting. Assume $f : \mathbb{G}_m \rightarrow \mathbb{G}_a$ is a δ -morphism and $0 \neq s \in \mathbb{Z}$ is an integer such that for any $a_1, a_2 \in R^\times$ we have

$$f(a_1 a_2) = f(a_1) + a_1^s f(a_2).$$

Then there exists $\mu \in R$ such that

$$f(a) = \mu(1 - a^s), \quad a \in R^\times;$$

in particular f is a regular cocycle.

Proof. Let us assume first we are in the δ -arithmetic setting. By induction on $r \geq 1$ one gets, for all $a \in R^\times$:

$$f(a^r) = (1 + a^s + a^{2s} + \dots + a^{(r-1)s})f(a).$$

In particular for any $\nu \geq 0$ one gets

$$(3.5) \quad (1 - a^s)f(a^{p^\nu}) = (1 - a^{sp^\nu})f(a)$$

Now recall that

$$\mathcal{O}(J^n(\mathbb{G}_m)) = R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge.$$

Identify the point set map f with a restricted power series $f \in R[x, x^{-1}, x', \dots, x^{(n)}]^\wedge$ and write

$$f = \sum_{i_1, \dots, i_n \geq 0} f_{i_1 \dots i_n}(x) (x')^{i_1} \dots (x^{(n)})^{i_n}$$

where $f_{i_1 \dots i_n} \in R[x, x^{-1}]^\wedge$ tend to 0 p -adically for $|i_1| + \dots + |i_n| \rightarrow \infty$. By 3.5 we get

$$\begin{aligned} (3.6) \quad (1 - x^s) \sum_{i_1, \dots, i_n \geq 0} f_{i_1 \dots i_n}(x^{p^\nu}) \delta(x^{p^\nu})^{i_1} \dots \delta^n(x^{p^\nu})^{i_n} \\ = (1 - x^{sp^\nu}) \sum_{i_1, \dots, i_n \geq 0} f_{i_1 \dots i_n}(x) (x')^{i_1} \dots (x^{(n)})^{i_n} \end{aligned}$$

By induction on ν it is easy to see that for any $\nu \geq r \geq 1$ we have

$$(3.7) \quad \delta^r(x^{p^\nu}) \in p^{\nu-r+1} R[x, x', \dots, x^{(r)}];$$

cf. also [11]. Fix $(i_1, \dots, i_n) \neq (0, \dots, 0)$ and let $\nu \geq n$ be arbitrary. Picking out the coefficient of $(x')^{i_1} \dots (x^{(n)})^{i_n}$ in the equation 3.6 we get

$$(1 - x^{sp^\nu}) f_{i_1 \dots i_n} \in p^{\nu-n+1} R[x, x^{-1}]^\wedge.$$

Since ν is arbitrary we get

$$f_{i_1 \dots i_n} = 0 \quad \text{for } (i_1, \dots, i_n) \neq (0, \dots, 0),$$

hence $f = f_{0 \dots 0}$. For $\nu = 1$ we may rewrite equation 3.6 as

$$(1 - x^s) f_{0 \dots 0}(x^p) = (1 - x^{sp}) f_{0 \dots 0}(x)$$

The latter is an equality in $R[x, x^{-1}]^\wedge$ so we can also view it as an equality in the ring

$$A = \left\{ \sum_{i=-\infty}^{\infty} c_i x^i; c_i \rightarrow 0 \text{ for } i \rightarrow -\infty \right\}$$

If $s \geq 1$ we get the following equality in A :

$$g(x^p) = g(x), \quad g(x) = \frac{f_{0 \dots 0}(x)}{1 - x^s} = f_{0 \dots 0}(x)(1 + x^s + x^{2s} + \dots);$$

if $s \leq -1$ we get the following equality in A :

$$g(x^p) = g(x), \quad g(x) = \frac{-x^{-s} f_{0 \dots 0}(x)}{1 - x^{-s}} = -x^{-s} f_{0 \dots 0}(x)(1 + x^{-s} + x^{-2s} + \dots);$$

In either case we get $g = \mu \in R$ and we are done.

Assume now we are in the δ -algebraic setting. Identify f with a polynomial in $A[x, x^{-1}, x', \dots, x^{(n)}]$ where $A \subset R$ is a subring finitely generated over \mathbb{Z} . Now select a prime $p \in \mathbb{Z}$ which is not invertible in A ; hence by Krull's intersection theorem $\bigcap_{n=1}^{\infty} p^n A = 0$. Then one can conclude by the same argument as above in which R is replaced by A (and all completions are dropped); by the way 3.7 is, in this case, obvious. \square

Lemma 3.6. *Assume we are either in the δ -algebraic setting or in the δ -arithmetic setting. Assume we are given a δ -map $\epsilon : \mathbb{G}_m \times \mathbb{G}_a^{n-1} \rightarrow \mathfrak{gl}_{n-1}$ and a row vector $\lambda \in R^{n-1}$ such that for any $(a_1, b_1), (a_2, b_2) \in R^\times \times R^{n-1}$:*

$$(3.8) \quad \epsilon(a_1 a_2, b_1 + a_1 b_2) = \epsilon(a_1, b_1) + \epsilon(a_2, b_2) + a_1^{-1} (a_2^{-1} - 1) \lambda^t b_1.$$

Then, for any $a_1, a_2, a \in R^\times$, $b \in R^{n-1}$:

$$(3.9) \quad \epsilon(a_1 a_2, 0) = \epsilon(a_1, 0) + \epsilon(a_2, 0)$$

$$(3.10) \quad \epsilon(a, b) = \epsilon(a, 0) + a^{-1} \lambda^t b.$$

Proof. Assume first we are in the δ -arithmetic setting. Setting $b_1 = b_2 = 0$ in equation 3.8 we get 3.9. Setting $a_1 = a_2 = 1$ in equation 3.8 we get

$$\epsilon(1, b_1 + b_2) = \epsilon(1, b_1) + \epsilon(1, b_2).$$

So if $\epsilon = (\epsilon_{kl})$ then by Lemma 3.3 there exist $\lambda_{ikl} \in R^{n-1}$, $i \geq 0$, such that

$$(3.11) \quad \epsilon_{kl}(1, b) = \sum \phi^i(b) \lambda_{ikl}^t, \quad b \in R^{n-1}.$$

Now setting $a_1 = a \in R^\times$, $a_2 = 1$, $b_1 = 0$, $b_2 = a^{-1}b \in R^{n-1}$ in equation 3.8 we get

$$(3.12) \quad \epsilon_{kl}(a, b) = \epsilon_{kl}(a, 0) + \epsilon_{kl}(1, a^{-1}b) = \epsilon_{kl}(a, 0) + \sum \phi^i(a)^{-1} \phi^i(b) \lambda_{ikl}^t.$$

Setting $a_1 = 1$, $a_2 = a \in R^\times$, $b_1 = b$, $b_2 = 0$, in equation 3.8 we get

$$(3.13) \quad \epsilon_{kl}(a, b) = \epsilon_{kl}(1, b) + \epsilon_{kl}(a, 0) + (a^{-1} - 1)(\lambda^t b)_{kl}$$

hence, using equation 3.12 we get

$$(3.14) \quad \sum (\phi^i(a)^{-1} - 1) \phi^i(b) \lambda_{ikl}^t = (a^{-1} - 1)(\lambda^t b)_{kl}.$$

This can be viewed as an identity in $\phi^i(a), \phi^i(b)$ so we get $\lambda_{ilk} = 0$ for all $i \geq 1$ and all k, l and also $b \lambda_{0kl}^t = (\lambda^t b)_{kl}$. Then 3.10 follows from 3.12.

Assume now we are in the δ -algebraic setting. The above argument can be modified as follows. Instead of equation 3.11 one gets

$$(3.15) \quad \epsilon_{kl}(1, b) = \sum (\delta^i b) \lambda_{ikl}^t, \quad b \in R^{n-1}.$$

Instead of equation 3.12 one gets

$$(3.16) \quad \epsilon_{kl}(a, b) = \epsilon_{kl}(a, 0) + \sum \delta^i(a^{-1}b) \lambda_{ikl}^t.$$

Equation 3.13 remains then valid. Then instead of 3.14 we get

$$(3.17) \quad \sum (\delta^i(a^{-1}b) - \delta^i b) \lambda_{ikl}^t = (a^{-1} - 1)(\lambda^t b)_{kl}.$$

Setting $b = e_j = (0, \dots, 1, \dots, 0)$ (with 1 on the j th place) we get

$$(a^{-1} - 1) e_j \lambda_{0kl}^t + \sum_{i \geq 1} \delta^i(a^{-1}) e_j \lambda_{ikl}^t = (a^{-1} - 1) \lambda_k \delta_{lj}$$

(δ_{lj} the Kronecker symbol). Hence $e_j \lambda_{ikl}^t = 0$ for $i \geq 1$ and all j, k, l . Hence $\lambda_{ilk} = 0$ for $i \geq 1$ and all k, l and also $e_j \lambda_{0kl}^t = \lambda_k \delta_{lj}$. So $a^{-1} b \lambda_{0kl}^t = (a^{-1} \lambda^t b)_{kl}$. This ends the proof. \square

Lemma 3.7. Assume we are either in the δ -algebraic setting or in the δ -arithmetic setting. Assume we are given a δ -map $\alpha : \mathbb{G}_m \times \mathbb{G}_a^{n-1} \rightarrow \mathbb{A}^1$ and a row vector $\lambda \in R^{n-1}$ such that for any $(a_1, b_1), (a_2, b_2) \in R^\times \times R^{n-1}$:

$$(3.18) \quad \alpha(a_1 a_2, b_1 + a_1 b_2) = \alpha(a_1, b_1) + \alpha(a_2, b_2) + a_1^{-1} (1 - a_2^{-1}) b_1 \lambda^t.$$

Then for any $a_1, a_2, a \in R^\times$, $b \in R^{n-1}$:

$$(3.19) \quad \alpha(a_1 a_2, 0) = \alpha(a_1, 0) + \alpha(a_2, 0)$$

$$(3.20) \quad \alpha(a, b) = \alpha(a, 0) - a^{-1} b \lambda^t.$$

Proof. Entirely similar to the proof of Lemma 3.6. \square

Lemma 3.8. *Assume we are either in the δ -algebraic setting or in the δ -arithmetic setting. Assume we are given a δ -map $\beta : \mathbb{G}_m \times \mathbb{G}_a^{n-1} \rightarrow \mathbb{A}^{n-1}$, a δ -homomorphism $\psi : \mathbb{G}_m \rightarrow \mathfrak{gl}_{n-1}$, and a row vector $\lambda \in R^{n-1}$ such that for any $(a_1, b_1), (a_2, b_2) \in R^\times \times R^{n-1}$:*

$$\begin{aligned} \beta(a_1 a_2, b_1 + a_1 b_2) &= \beta(a_1, b_1) + a_1 \beta(a_2, b_2) \\ (3.21) \quad &+ b_1 \psi(a_2) \\ &+ a_2^{-1} b_2 \lambda^t b_1 + a_2^{-1} b_1 \lambda^t b_2 + a_1^{-1} (a_2^{-1} - 1) b_1 \lambda^t b_1 \end{aligned}$$

1) *Assume we are in the δ -arithmetic setting. Then $\psi = 0$ and there exist $\mu \in R^{n-1}$ and $\nu \in \mathfrak{gl}_{n-1}(R)$ such that for any $a \in R^\times$, $b \in R^{n-1}$ we have*

$$(3.22) \quad \beta(a, b) = (1 - a)\mu + a^{-1} b \lambda^t b + b \nu.$$

2) *Assume we are in the δ -algebraic setting. Then there exist $\mu \in R^{n-1}$ and $\nu, \eta \in \mathfrak{gl}_{n-1}(R)$ such that for any $a \in R^\times$, $b \in R^{n-1}$ we have*

$$(3.23) \quad \psi(a) = -(a^{-1} \delta a) \nu;$$

$$(3.24) \quad \beta(a, b) = (1 - a)\mu + a^{-1} b \lambda^t b + b \eta + a \delta(a^{-1} b) \nu.$$

Proof. Assume first we are in the δ -arithmetic setting. Setting $b_1 = b_2 = 0$ in equation 3.21 we get

$$\beta(a_1 a_2, 0) = \beta(a_1, 0) + a_1 \beta(a_2, 0).$$

So by Lemma 3.5 there exists $\mu \in R^{n-1}$ such that

$$\beta(a, 0) = (1 - a)\mu, \quad a \in R^\times.$$

Setting $a_1 = a_2 = 1$ in equation 3.21 we get

$$\beta(1, b_1 + b_2) = \beta(1, b_1) + \beta(1, b_2) + b_2 \lambda^t b_1 + b_1 \lambda^t b_2,$$

because $\psi(1) = 0$. Let $\beta^*(b) = \beta(1, b) - b \lambda^t b$. Then

$$\beta^*(b_1 + b_2) = \beta^*(b_1) + \beta^*(b_2).$$

So by Lemma 3.3 there exist $\lambda_i \in \mathfrak{gl}_{n-1}(R)$, $i \geq 0$, such that

$$\beta^*(b) = \sum \phi^i(b) \lambda_i, \quad b \in R^{n-1}$$

hence

$$(3.25) \quad \beta(1, b) = b \lambda^t b + \sum \phi^i(b) \lambda_i.$$

Now setting $a_1 = a \in R^\times$, $a_2 = 1$, $b_1 = 0$, $b_2 = a^{-1} b \in R^{n-1}$ in equation 3.21 we get

$$\begin{aligned} \beta(a, b) &= \beta(a, 0) + a \beta(1, a^{-1} b) \\ (3.26) \quad &= (1 - a)\mu + a (a^{-2} b \lambda^t b + \sum \phi^i(a)^{-1} \phi^i(b) \lambda_i). \end{aligned}$$

Setting $a_1 = 1$, $a_2 = a \in R^\times$, $b_1 = b$, $b_2 = 0$ in equation 3.21 we get

$$(3.27) \quad \beta(a, b) = \beta(1, b) + \beta(a, 0) + b \psi(a) + (a^{-1} - 1) b \lambda^t b,$$

hence, using equation 3.26 we get, after cancellation:

$$(3.28) \quad \sum (a\phi^i(a)^{-1} - 1)\phi^i(b)\lambda_i = b\psi(a).$$

Let $a = \zeta$ be an arbitrary root of unity. Since $\psi(\zeta) = 0$ and $\phi(\zeta) = \zeta^p$ we get:

$$\sum (\zeta^{1-p^i} - 1)\phi^i(b)\lambda_i = 0.$$

Since, for each ζ this can be seen as an identity in $\phi^i(b)$ we get that for all ζ and all $i \geq 0$,

$$(\zeta^{1-p^i} - 1)\lambda_i = 0.$$

We get that $\lambda_i = 0$ for all $i \geq 1$. By 3.28 we get $\psi = 0$. By 3.26, if $\nu := \lambda_0$, we get 3.24. This ends to the proof of assertion 1) in the Lemma.

Assume now we are in the δ -algebraic setting. The above argument can be modified as follows. Instead of 3.25 we get

$$(3.29) \quad \beta(1, b) = b\lambda^t b + \sum (\delta^i b)\lambda_i.$$

Instead of equation 3.26 we get

$$(3.30) \quad \beta(a, b) = (1 - a)\mu + a(a^{-2}b\lambda^t b + \sum \delta^i(a^{-1}b)\lambda_i).$$

Equation 3.27 is valid. Instead of equation 3.28 we then get

$$(3.31) \quad \sum (a\delta^i(a^{-1}b) - \delta^i b)\lambda_i = b\psi(a).$$

Let $\psi = (\psi_{kl})$ and $\lambda_i = (\lambda_{ikl})$. Setting $b = e_j = (0, \dots, 1, \dots, 0)$ we get

$$\sum_{i \geq 1} a\delta^i(a^{-1})\lambda_{ijl} = \psi_{jl}(a)$$

Replacing a by a^{-1} and using $\psi(a^{-1}) = -\psi(a)$ we get

$$\sum_{i \geq 1} (\delta^i a)\lambda_{ijl} = -a\psi_{jl}(a)$$

Using the fact that ψ_{jl} is a δ -homomorphism $\mathbb{G}_m \rightarrow \mathbb{G}_a$ and δ is a δ -homomorphism $\mathbb{G}_a \rightarrow \mathbb{G}_a$ and setting $a = a_1 a_2$ one immediately concludes that $\lambda_{ijl} = 0$ for $i \geq 1$ and all j, l . Hence $\psi(a) = -(a^{-1}\delta a)\lambda_1$. Setting $\nu = \lambda_1$ and $\eta = \lambda_0$ we conclude by 3.30. \square

Lemma 3.9. *Let x be an $n \times n$ matrix of indeterminates over \mathbb{Z} and let $\Delta_i(x)$ be the determinant of the matrix obtained from x by removing the first i rows and the first i columns. Let $\Delta(x) = \prod_{i=1}^{n-1} \Delta_i(x)$. Let W be the group of all $n \times n$ matrices obtained from the identity matrix 1_n by permuting its columns. Then there exist $w_0, \dots, w_N \in W$ and there exist elements*

$$a_1(x), \dots, a_N(x) \in \mathbb{Z}[x, \Delta(x)^{-1}],$$

and $1 \times (n-1)$ vectors

$$b_1(x), \dots, b_N(x) \in \mathbb{Z}[x, \Delta(x)^{-1}]^{n-1},$$

such that if

$$s_i(x) = \begin{pmatrix} a_i(x) & b_i(x) \\ 0 & 1_{n-1} \end{pmatrix}$$

then

$$x = w_0 \cdot s_1(x) \cdot w_1 \cdot s_2(x) \cdot w_2 \cdot \dots \cdot s_N(x) \cdot w_N$$

Proof. We proceed by induction on n . Write

$$x = \begin{pmatrix} u & y \\ z^t & w \end{pmatrix}$$

where u is one variable, y, z are $1 \times (n-1)$ matrices of variables and w is an $(n-1) \times (n-1)$ matrix of variables. We may write

$$(3.32) \quad x = \begin{pmatrix} u - yw^{-1}z^t & yw^{-1} \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w^{-1}z^t & 1_{n-1} \end{pmatrix}.$$

Now for $n \times n$ matrices m, m' we say m and m' are W -equivalent if $WmW = Wm'W$. We then have that $\begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}$ is W -equivalent to $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$. Also, if $w^{-1}z^t = (c_1(x), \dots, c_{n-1}(x))^t$ then

$$\begin{pmatrix} 1 & 0 \\ w^{-1}z^t & 1_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_1(x) & 1 & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c_2(x) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-3} \end{pmatrix} \dots$$

and each of the matrices in the right hand side of the above equation is W -equivalent to its transpose. We may conclude from 3.32 by the induction hypothesis applied to w . \square

Proof of Theorem 1.3. Write matrices in $GL_n(R)$ in block form $\begin{pmatrix} a & b \\ c^t & d \end{pmatrix}$ where $a \in R$, $b, c \in R^{n-1}$ are viewed as a $1 \times (n-1)$ matrices, c^t is the transpose of c , and d is an $(n-1) \times (n-1)$ matrix. Also let 1_{n-1} be the identity $(n-1) \times (n-1)$ matrix. We may write

$$f \begin{pmatrix} a & b \\ 0 & 1_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha(a, b) & \beta(a, b) \\ \gamma(a, b)^t & \epsilon(a, b) \end{pmatrix}$$

where $\alpha, \beta, \gamma, \epsilon$ are δ -morphisms from $\mathbb{G}_m \times \mathbb{G}_a^{n-1}$ to $\mathbb{A}^1, \mathbb{A}^{n-1}, \mathbb{A}^{n-1}, \mathfrak{gl}_{n-1}$ respectively. If we let $(a_i, b_i) \in R^\times \times R^{n-1}$, $i = 1, 2$, and if we set

$$\alpha_i = \alpha(a_i, b_i), \quad \beta_i = \beta(a_i, b_i), \quad \gamma_i = \gamma(a_i, b_i), \quad \epsilon_i = \epsilon(a_i, b_i), \quad i = 1, 2,$$

then the cocycle relation for f yields:

$$(3.33) \quad \begin{aligned} (1) \quad & \alpha(a_1 a_2, b_1 + a_1 b_2) = \alpha_1 + \alpha_2 + a_1^{-1} b_1 \gamma_2^t \\ (2) \quad & \beta(a_1 a_2, b_1 + a_1 b_2) = \beta_1 + a_1 \beta_2 - \alpha_2 b_1 + b_1 \epsilon_2 - a_1^{-1} b_1 \gamma_2^t b_1 \\ (3) \quad & \gamma(a_1 a_2, b_1 + a_1 b_2) = \gamma_1 + a_1^{-1} \gamma_2 \\ (4) \quad & \epsilon(a_1 a_2, b_1 + a_1 b_2) = \epsilon_1 + \epsilon_2 - a_1^{-1} \gamma_2^t b_1 \end{aligned}$$

By equation 3.33 (3) and by Lemma 3.5 there exists $\lambda \in R^{n-1}$ such that

$$(3.34) \quad \gamma(a, b) = (1 - a^{-1})\lambda, \quad a \in R^\times, b \in R^{n-1}.$$

So equation 3.33 (4) reads:

$$(3.35) \quad \epsilon(a_1 a_2, b_1 + a_1 b_2) = \epsilon(a_1, b_1) + \epsilon(a_2, b_2) + a_1^{-1} (a_2^{-1} - 1) \lambda^t b_1.$$

By Lemma 3.6 we get

$$(3.36) \quad \epsilon(a_1 a_2, 0) = \epsilon(a_1, 0) + \epsilon(a_2, 0)$$

$$(3.37) \quad \epsilon(a, b) = \epsilon(a, 0) + a^{-1} \lambda^t b.$$

Similarly, equation 3.33 (1) reads

$$(3.38) \quad \alpha(a_1 a_2, b_1 + a_1 b_2) = \alpha(a_1, b_1) + \alpha(a_2, b_2) + a_1^{-1}(1 - a_2^{-1})b_1 \lambda^t.$$

By Lemma 3.7 we get

$$(3.39) \quad \alpha(a_1 a_2, 0) = \alpha(a_1, 0) + \alpha(a_2, 0)$$

$$(3.40) \quad \alpha(a, b) = \alpha(a, 0) - a^{-1}b\lambda^t.$$

Finally by equations 3.34, 3.37, 3.40, equation 3.33 (2) reads

$$(3.41) \quad \begin{aligned} \beta(a_1 a_2, b_1 + a_1 b_2) &= \beta(a_1, b_1) + a_1 \beta(a_2, b_2) \\ &+ b_1(\epsilon(a_2, 0) - \alpha(a_2, 0)1_{n-1}) \\ &+ a_2^{-1}b_2\lambda^t b_1 + a_2^{-1}b_1\lambda^t b_2 + a_1^{-1}(a_2^{-1} - 1)b_1\lambda^t b_1. \end{aligned}$$

By Lemma 3.8

$$(3.42) \quad \epsilon(a, 0) = \alpha(a, 0)1_{n-1}, \quad a \in R^\times$$

and there exist $\mu \in R^{n-1}$, $\nu \in \mathfrak{gl}_{n-1}(R)$ such that

$$(3.43) \quad \beta(a, b) = (1 - a)\mu + a^{-1}b\lambda^t b + b\nu.$$

Let $\omega : \mathbb{G}_m \rightarrow \mathbb{G}_a$ be the δ -homomorphism defined by $\omega(a) = \alpha(a, 0)$. Consider the classical δ -cocycle $f_0 : GL_n \rightarrow \mathfrak{gl}_n$ defined by

$$f_0(g) := f(g) - \omega(\det g)1_n.$$

By Lemma 3.2 it is sufficient to show that f_0 is induced on points by a regular map. Let $H \subset GL_n$ the natural closed subgroup scheme such that

$$H(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1_{n-1} \end{pmatrix}; a \in R^\times, b \in R^{n-1} \right\}.$$

By equations 3.40, 3.37, 3.34, the restriction of f_0 to H is a regular map. On the other hand by Lemma 3.9 there exist finitely many matrices $w_0, w_1, \dots, w_N \in W$ such that the map

$$H \times \dots \times H \rightarrow GL_n$$

defined on points by

$$(h_1, \dots, h_N) \mapsto w_0 h_1 w_1 \cdots h_N w_N$$

has a rational section

$$s : U \rightarrow H \times \dots \times H, \quad s(g) = (s_1(g), \dots, s_N(g))$$

defined on a Zariski open set U of GL_n which meets the special fiber at p . So for $g \in U(R)$ we have

$$f_0(g) = f_0(w_0 \cdot s_1(g) \cdot w_1 \cdots s_N(g) \cdot w_N).$$

By the cocycle condition we get that the restriction of f_0 to U is regular and hence, using translations and the cocycle condition again we get that f_0 is regular on the whole of GL_n . \square

Proof of Theorem 1.4. One can redo the argument in the proof of Theorem 1.3 as follows. First we assume f is any classical δ -cocycle (not necessarily δ -coherent). The formulae giving γ, α, ϵ (3.34, 3.36, 3.37, 3.39, 3.40) are unchanged. Hence

equation 3.41 is valid. By Lemma 3.8 there exist $\mu, \lambda \in R^{n-1}$ and $\nu, \eta \in \mathfrak{gl}_{n-1}$ such that

$$(3.44) \quad \epsilon(a, 0) = \alpha(a, 0)1_{n-1} - (a^{-1}\delta a)\nu, \quad a \in R^\times$$

$$(3.45) \quad \beta(a, b) = (1 - a)\mu + a^{-1}b\lambda^t b + b\eta + a\delta(a^{-1}b)\nu.$$

Set $g = \begin{pmatrix} a & b \\ 0 & 1_{n-1} \end{pmatrix} \in H(R)$. By the above we conclude that

$$(3.46) \quad f(g) = \begin{pmatrix} \alpha(a, 0)1_n - a^{-1}b\lambda^t & (1 - a)\mu + a^{-1}b\lambda^t b + b\eta + a\delta(a^{-1}b)\nu \\ (1 - a^{-1})\lambda^t & \alpha(a, 0)1_{n-1} - (a^{-1}\delta a)\nu + a^{-1}\lambda^t b \end{pmatrix}$$

Now let us assume f is δ -coherent. Since f must send H into its Lie algebra, consisting of the matrices $\begin{pmatrix} \star & \star \\ 0 & 0_{n-1} \end{pmatrix}$, we get that $\lambda = 0$ and

$$\alpha(a, 0)1_{n-1} = (a^{-1}\delta a)\nu.$$

This forces ν to be a scalar matrix, say $\nu = \epsilon 1_{n-1}$ and $\alpha(a, 0) = \epsilon \cdot a^{-1}\delta a$. If for $v \in \mathfrak{gl}_n(R)$ we write $f^v(g) = gv g^{-1} - v$, $g \in GL_n(R)$, then note that

$$f^v \begin{pmatrix} a & b \\ 0 & 1_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & (1 - a)\mu + b\eta \\ 0 & 0 \end{pmatrix} \quad \text{for } v = \begin{pmatrix} 0 & -\mu \\ 0 & \eta \end{pmatrix}$$

We conclude that

$$f(g) = f^v(g) + \epsilon \cdot \delta g \cdot g^{-1}$$

for all $g \in H(R)$ and hence (as in the proof of Theorem 1.3) for all $g \in GL_n(R)$. Now since f and $g \mapsto \delta g \cdot g^{-1}$ are δ -coherent it follows that f^v is δ -coherent. It is enough to prove that v is a scalar matrix. Assume v is not scalar. Then we claim there exists $u \in GL_n(R^\delta)$ such that uvu^{-1} is not in the Lie algebra $\mathfrak{t}(R) \subset \mathfrak{gl}_n(R)$ of diagonal matrices. Indeed if one assumes

$$\{uvu^{-1}; u \in GL_n(R^\delta)\} \subset \mathfrak{t}(R),$$

since $\{uvu^{-1}; u \in GL_n(R^\delta)\}$ is Zariski dense in $\{uvu^{-1}; u \in GL_n(R)\}$ we get that $\{uvu^{-1}; u \in GL_n(R)\} \subset \mathfrak{t}(R)$. Note that v itself is in $\mathfrak{t}(R)$. So the stabilizer of v in $GL_n(R)$ under the adjoint action has dimension $n_1^2 + \dots + n_s^2$ where n_1, \dots, n_s are the multiplicities of the eigenvalues of v ; note that $s \geq 2$ and $n_1 + \dots + n_s = n$. Hence the dimension of $\{uvu^{-1}; u \in GL_n(R)\}$ is $n^2 - (n_1^2 + \dots + n_s^2) > n = \dim \mathfrak{t}$ (because $s \geq 2$), a contradiction. Let now $G = u^{-1}Tu$ where $T \subset GL_n$ is the torus of diagonal matrices; so G is defined over R^δ and has Lie algebra $L(G) = u^{-1}L(T)u$. Since f^v is δ -regular we have $u^{-1}duvu^{-1}d^{-1}u - v \in u^{-1}L(T)(R)u$ for all $d \in T(R)$. Hence $duvu^{-1}d^{-1} - uvu^{-1} \in \mathfrak{t}$ for all $d \in T(R)$. But uvu^{-1} is not diagonal so there are indices $i \neq j$ such that then (i, j) -entry of uvu^{-1} is non zero. Taking d with diagonal entries $d_i \neq d_j$ we get that the (i, j) -entry of $duvu^{-1}d^{-1} - uvu^{-1}$ is non-zero, a contradiction. This ends the proof. \square

REFERENCES

- [1] J. Borger, *The basic geometry of Witt vectors, I: the affine case*, Algebra and Number Theory 5 (2011), no. 2, pp 231-285.
- [2] Borger, J., *The basic geometry of Witt vectors, II: Spaces*, arXiv:1006.0092v1.
- [3] J. Borger, *Λ -rings and the field with one element*, arXiv:0906.3146 [math.NT]

- [4] A. Buium, *Intersections in jet spaces and a conjecture of S.Lang*, Ann. of Math. 136 (1992), pp. 583-593.
- [5] A. Buium, *Differential algebraic groups of finite dimension*, LNM 1506, Springer, 1992.
- [6] A. Buium, *Differential Algebra and Diophantine Geometry*, Hermann, Paris, 1994.
- [7] A. Buium, *Differential characters of Abelian varieties over p -adic fields*, Invent. Math., 122 (1995), pp. 309-340.
- [8] A. Buium, *Geometry of p -jets*, Duke J. Math. 82, (1996), 2, pp. 349-367.
- [9] A. Buium, *Differential subgroups of simple algebraic groups over p -adic fields*, Amer. J. Math. 120 (1998), 1277-1287.
- [10] A. Buium, *Arithmetic Differential Equations*, Math. Surveys and Monographs, 118, American Mathematical Society, Providence, RI, 2005. xxxii+310 pp.
- [11] A. Buium, *p -jets of finite algebras, I: p -divisible groups*, Documenta Mathematica, to appear.
- [12] A. Buium, T. Dupuy, *Arithmetic differential equations on GL_n , II: arithmetic Lie theory*, preprint.
- [13] A. Buium, T. Dupuy, *Arithmetic differential equations on GL_n , III: Galois groups*, preprint.
- [14] P. J. Cassidy, *Differential algebraic groups*, Amer. J. Math. 94 (1972), 891-954.
- [15] P. J. Cassidy, *The classification of semisimple differential algebraic groups and the linear semisimple differential algebraic Lie algebras*, J. Algebra (1989), 169-238.
- [16] N. Jacobson, *Lie Algebras*, Dover, New York, 1979.
- [17] E.R. Kolchin, *Differential algebra and algebraic groups*. Pure and Applied Mathematics, Vol. 54. Academic Press, New York-London, 1973. xviii+446 pp.
- [18] Yu.I. Manin, *Algebraic curves over fields with differentiation*, Izv. Akad. Nauk SSSR, Ser. Mat. 22 (1958), pp. 737-756 = AMS translations Series 2, 37 (1964), pp. 59-78.
- [19] A. Pillay, *Algebraic D -groups and differential Galois theory*, Pacific J. Math 216, 2004, 343-360.

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