

# “Linear” Arithmetic Differential Equations

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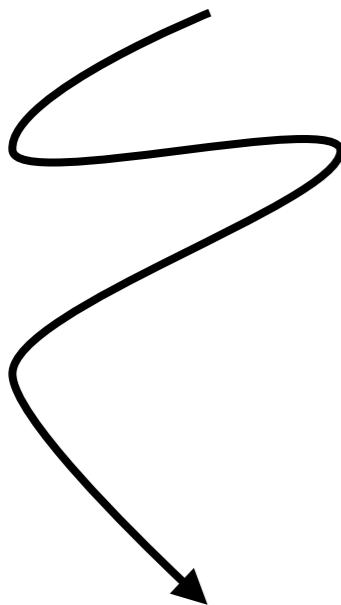
# origins

- Relative Mordell-Lang (characteristic 0)
- $\sim \rightarrow$  p-derivations
- $\sim \rightarrow$  Hrushovski's Relative Manin-Mumford (characteristic p)

Seek a category:

$\text{Sch}_{\mathbf{F}_1}$  = "Category of Schemes over the field with one element"

$\text{Spec}(\mathbf{Z})$



Arakelov Theory (compactification)

$\overline{\text{Spec}(\mathbf{Z})}_{\mathbf{F}_1}$

Finding an appropriate category  
to view as deeper base

# Part I: Introduction to Wittferential Algebra

# Part 2: Linear Arithmetic Differential Equations

# PART I

## Wifferential Algebra

# What is a p-derivation?

Fermat's Little Theorem

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The Frobenius

$$\begin{aligned} F : A/p &\mapsto A/p \\ a &\mapsto a^p \end{aligned}$$

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*non-linear*

Always an  $A$ -algebra



Defn. (Buium, Joyal 90's)

A **p-derivation** is a map of sets  $\delta_p : A \rightarrow B$   
such that

$$\forall a, b \in A$$

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For  $\delta_p : \mathbb{Z} \rightarrow \mathbb{Z}$   
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Example:

$$\begin{aligned}\delta_p(p) &= \frac{p - p^p}{p} \\ &= 1 - p^{p-1}\end{aligned}$$

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**derivations**

$$\delta : A \rightarrow A$$

**ring homomorphisms**

$$f : A \rightarrow A[\varepsilon]/\langle \varepsilon^2 \rangle$$



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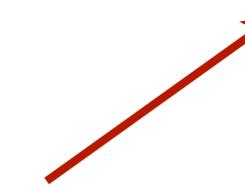
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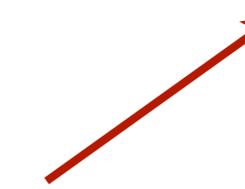
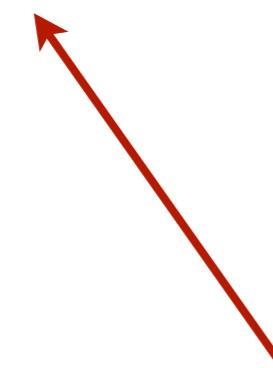
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“Wittferentiation”



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“infinitesimals”

**p-derivations**

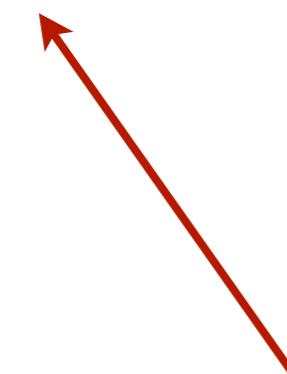
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“Wittdifferentiation”



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## Funky Addition

$$(a_0, a_1) +_W (b_0, b_1) = (a_0 + b_0, a_1 + a_1 + p \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} a_0^{p-j} b_0^j)$$

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exercise:  $W_1(\mathbb{F}_p) = \mathbb{Z}/p^2$

# Analogies

Dual Numbers

$$D_1(A) = A[t]/\langle t^2 \rangle$$

Truncated Witt Vectors

$$W_1(A)$$

Power Series

$$D(A) = A[[t]]$$

Witt Vectors

$$W(A)$$

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$$(a_0, a_1, a_2, \dots) +_W (b_0, b_1, b_2, \dots) = (s_0(a, b), s_1(a, b), s_2(a, b), \dots)$$

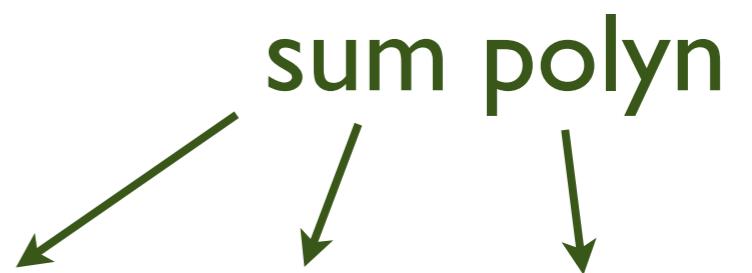
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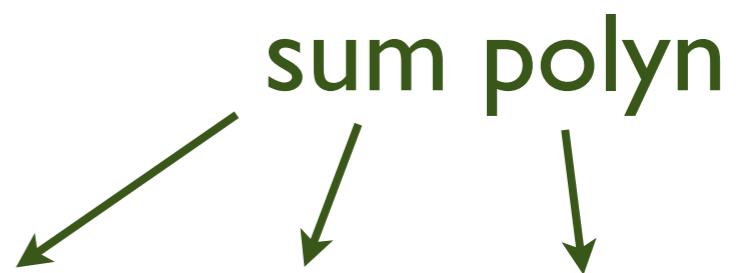
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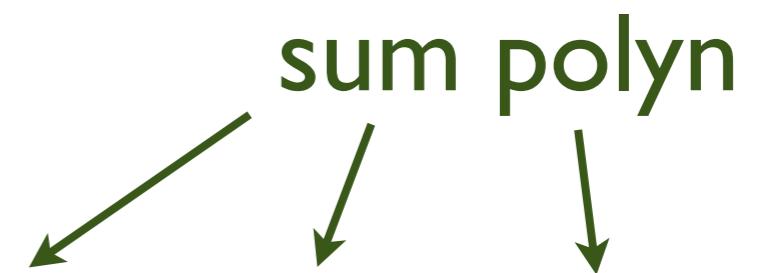
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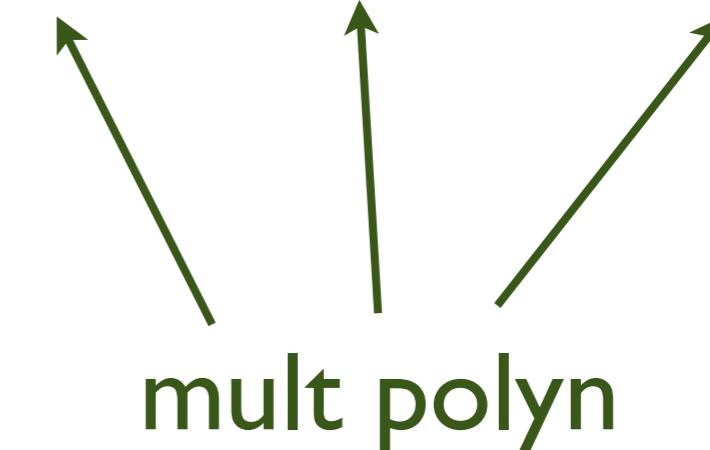
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$$W(A) \cong_{sets} A^{\mathbb{N}}$$

## Witt polynomials

$$w_0(a) = a_0$$

$$w_1(a) = a_0^p + pa_1$$

$$w_2(a) = a_0^{p^2} + pa_1^p + p^2 a_2$$

$$w_3(a) = a_0^{p^3} + pa_1^{p^2} + p^2 a_2^p + p^3 a_3$$

⋮  
⋮

## Defining property of Witt vectors:

The map  $W(A) \rightarrow A^{\mathbb{N}}$  defined by

$$(a_0, a_1, a_2, \dots) \mapsto (w_0(a), w_1(a), w_2(a), \dots)$$

is a ring homomorphism.

Compute some of the addition polynomials

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$$w_0(a) + w_0(b) = w_0(s)$$

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$$a_0 + b_0 = s_0$$

$$(a_0^p + pa_1) + (b_0^p + pb_1) = s_0^p + ps_1$$

$$(a_0^{p^2} + pa_1^p + p^2a_2) + (b_0^{p^2} + pb_1^p + p^2b_2) = s_0^{p^2} + ps_1 + p^2s_2$$

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**Exercise.**  $W(\mathbb{F}_p) = \mathbb{Z}_p$

lifts of the Frobenius     $\approx$     p-derivations

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$\delta_p : A \rightarrow B$  + rules

$$\phi : A \rightarrow B$$

$$\phi(a) \equiv a^p \pmod{p}$$

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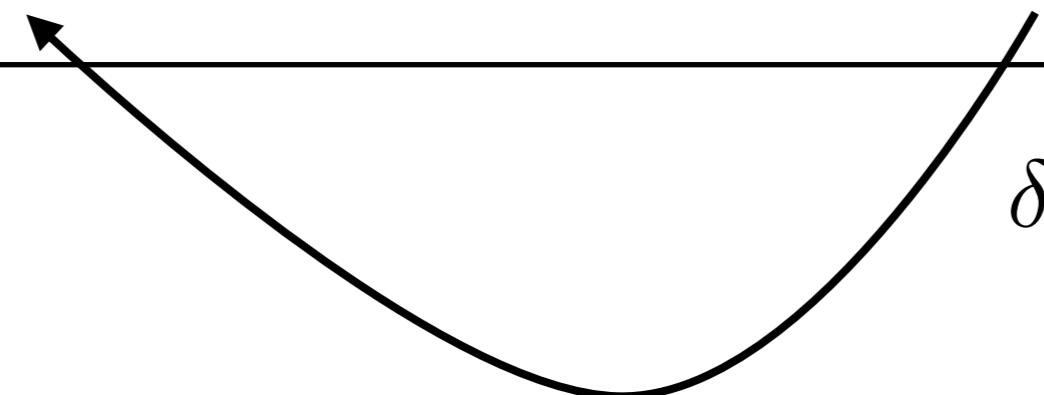
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$$\delta_p : A \rightarrow B + \text{rules}$$

$$\phi(a) := a^p + p\delta_p(a)$$



$$\delta_p(a) := \frac{\phi(a) - a^p}{p}$$

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*p-torsion free*

**lifts of the Frobenius**  $\approx$  **p-derivations**

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# Summary

Always an  $A$ -algebra



Defn. (Buium, Joyal 90's)

A **p-derivation** is a map of sets  $\delta_p : A \rightarrow B$   
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$$\forall a, b \in A$$

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## The poor man's model companion:

$$R = \widehat{\mathbf{Z}}_p^{\text{ur}}$$

### Properties:

- 1) unramified CDVR, residue field  $\overline{\mathbf{F}}_p$
- 2)  $\widehat{\mathbf{Z}}_p^{\text{ur}} = \mathbf{Z}_p[\zeta; \zeta^n = 1, p \nmid n]^\wedge$
- 3) unique lift of the Frobenius:  $\exists! \phi : R \rightarrow R$ 
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R WILL ALWAYS MEAN  
THIS RING IN THIS  
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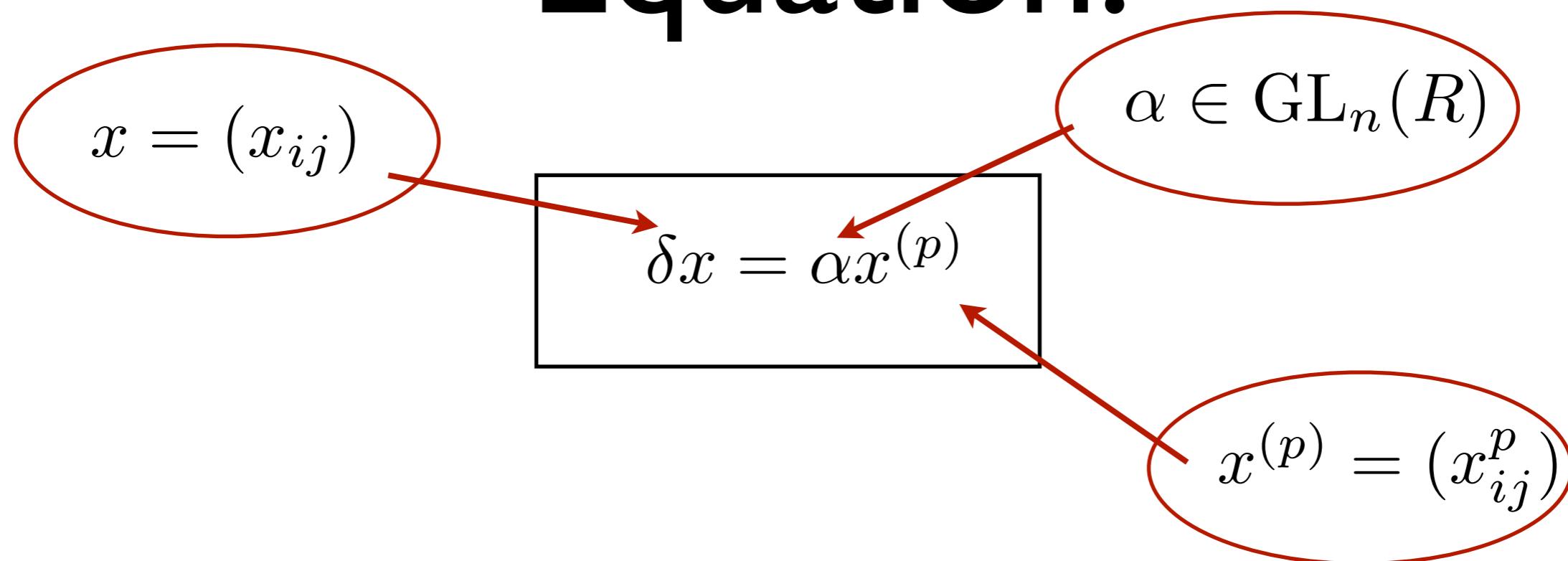
# PART 2

## Linear Arithmetic Differential Equations

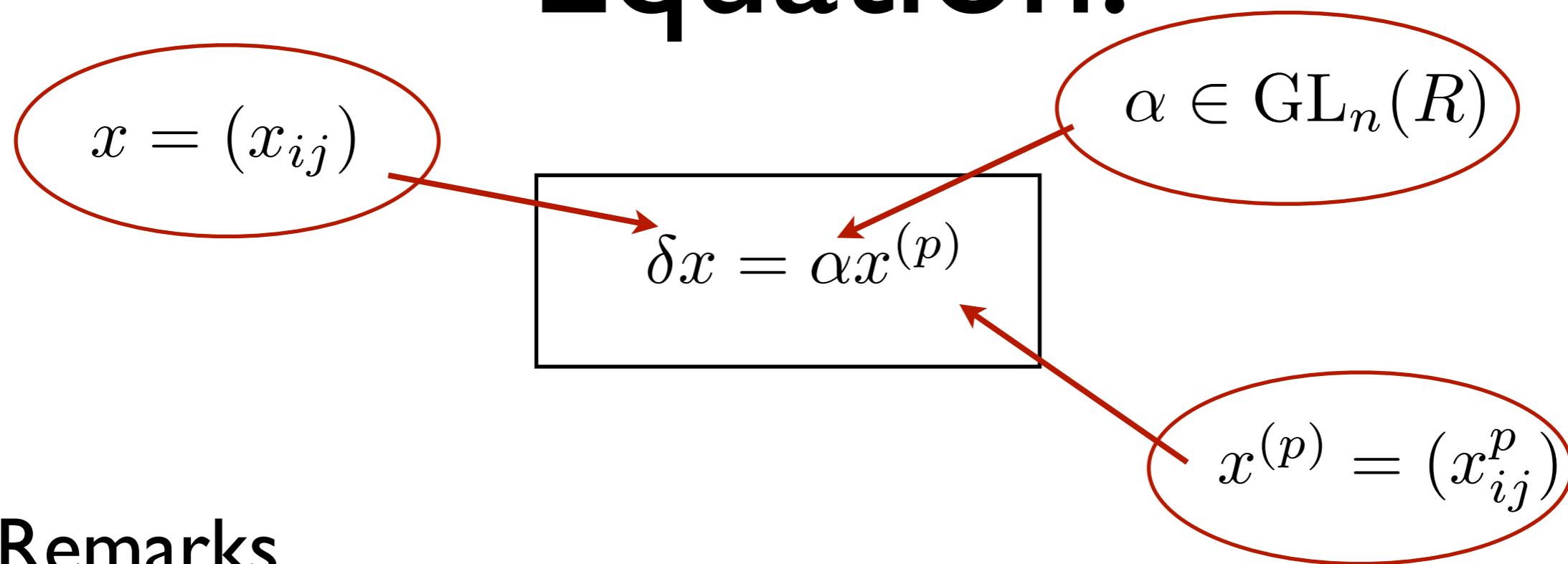
# Simplest Possible Equation:

$$\delta x = \alpha x^{(p)}$$

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Remarks.

$x \mapsto \delta(x)(x^{(p)})^{-1}$  almost a cocycle

## Theorem (existence and uniqueness)

$\alpha \in \mathfrak{gl}_n(R)$ ,  $u_0 \in \mathrm{GL}_n(R)$

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \pmod{p} \end{cases}$$

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$$u = \lim_{n \rightarrow \infty} f^n(u_0)$$

## Theorem (coeffs in CDVR)

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \pmod{p} \end{cases}$$

$\mathcal{O}$  = Complete Discrete Valuation Subring

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O})$$

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Proof.  $\mathcal{O} = R^{\phi^\nu}$

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Proof.  $\mathcal{O} = R^{\phi^\nu}$  (**characterization**)

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Proof.  $\mathcal{O} = R^{\phi^\nu}$  (**characterization**)

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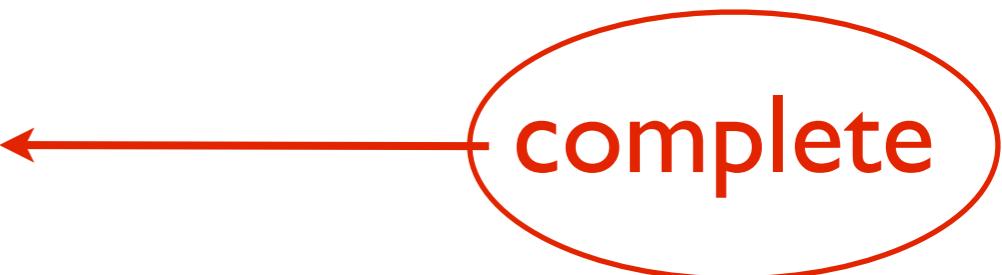
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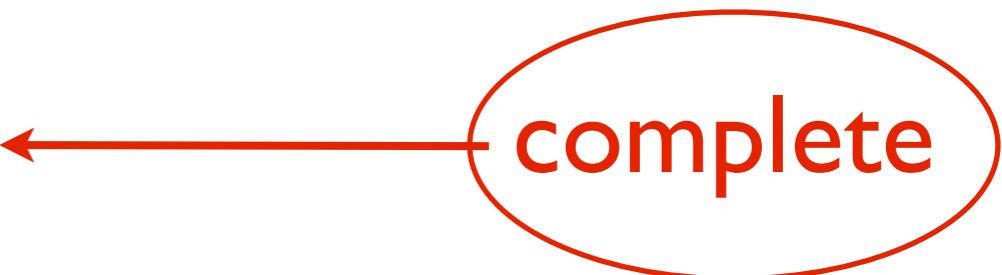
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general principle:

$$\begin{aligned} \varphi/F &\implies \text{fixed points of } \varphi : \mathbf{A}^n(K^a) \rightarrow \mathbf{A}^n(K^a) \\ &\quad \subset \mathbf{A}^n(F^a) \end{aligned}$$

$\implies$  columns are in finite algebraic extensions

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homogeneity gives rest of finiteness.

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Galois Group

=

$$\{c \in \mathrm{GL}_n(\mathcal{O}) : \exists \sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]), \sigma \circ \delta = \delta \circ \sigma, \sigma(u) = uc\}$$

$G_{u/\mathcal{O}}$

$$\begin{aligned}
 & \text{Galois Group of } \delta u = \alpha u^{(p)} \\
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 & G_{u/\mathcal{O}}
 \end{aligned}$$

## Alternative Descriptions:

1)  $\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$

$$\Gamma_{u/\mathcal{O}} \cong G_{u/\mathcal{O}}$$

2)  $0 \rightarrow I_{u/\mathcal{O}} \rightarrow \mathcal{O}[x, 1/\det(x)] \rightarrow \mathcal{O}[u] \rightarrow 0$

$$\mathrm{Stab}_{\mathrm{GL}_n(R)}(I_{u/\mathcal{O}}) \cong G_{u/\mathcal{O}}$$

$$\delta u = \alpha u^{(p)}$$

**Question: For what  $c \in \mathrm{GL}_n(R)$  do we get**

$$\delta(uc) = \alpha(uc)^{(p)} ?$$

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**Claim.**

- 1)  $\{u, c\}^* = 0 \implies \delta(uc) = \alpha(uc)^{(p)}$
- 2)  $\delta c = 0$

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**Main Example:**

$T$  = maximal torus of diagonals

$W$  = permutation matrices

$N = WT = TW$

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subgroup of  
 $GL_n(\mathbf{F}_1^a)$

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**Theorem.**

$\mathcal{O} \subset R$  delta subring

$\mathcal{O} \subset R^{\phi^\nu}$   $u \in \mathcal{O} \implies G_{u/\mathcal{O}}$  finite

$$\Gamma_{u/\mathcal{O}} = \{\sigma \in \text{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \text{GL}_n(\mathcal{O})\}$$

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finite to begin with

## Theorem A.

$\exists \Omega \subset \mathbf{Q}^2$  “thin set”

$\forall \alpha \in \mathbf{Z}^2 \setminus \Omega , \exists u \in \mathrm{GL}_n(R) \quad \delta u = \alpha u^{(p)}$

$G_{u/\mathcal{O}}$  = finite group containing  $W$

## Theorem B.

$$X = \{u \in \mathrm{GL}_n(R); u \equiv 1 \pmod{p}\}$$

= ball around identity

$\exists \Omega \subset X$  of the second category

$\forall u \in X \setminus \Omega, \forall \mathcal{O} \subset R$   $\delta$ -closed subring

$$\begin{array}{lcl} 1) \ \delta(u)(u^{(p)})^{-1} \in \mathfrak{gl}_n(\mathcal{O}) & \implies & G_{u/\mathcal{O}} = N^\delta \\ 2) \ R^\delta \subset \mathcal{O} & & \end{array}$$

## Theorem C.

$\exists \Omega \subset \mathrm{GL}_n(K^a)$  **Zariski closed**

$\forall u \in \mathrm{GL}_n(R) \setminus \Omega$

$$u' = \alpha u^{(p)}$$

$\mathcal{O} \ni \alpha$   $\delta$ -closed subring of  $R$

$$\dim((Z \cdot G_{u/\mathcal{O}})^{\mathrm{Zar}}) \leq n$$

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- Talk about proof of theorem B if time allows.

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$$Z$$

$$N$$