Complex Numbers and Quaternions for Calc III

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September 4, 2009

Abstract

An introduction to complex numbers and quaternions.

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1 Two Ways of Looking at Complex Numbers

A complex number is a number of the from $x + iy$ where x and y are real numbers. We can represent complex numbers in the plane by identifying $x + iy$ with the ordered pair (x, y) in the plane.

Looking at complex numbers as points in the two dimensional real plane allows us to interpret complex multiplication as a rotation in this plane.

For example multiplying some number $x + iy$ by $i = \sqrt{-1}$ is the same thing as rotating that number by $\pi/2$. For example, if I multiply $z = 1$ by i I get $zi = 1 \cdot i = i$. Here the complex number $z = 1$, identified with $(1, 0)$ is rotated to the complex number i which we identify with $(0, 1)$. See the picture below.

You can try this with other numbers and see the same thing. If I multiply $z = 1 + i$ by i, I get $(1 + i)i = -1 + i$.

2 Complex Multiplication

The set of complex numbers is denoted by

$$\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}.$$

Every complex number $z = x + iy$ consists essentially of the data of two real numbers. As stated before we can put the set of complex numbers in correspondence with points in the cartesian plane $\mathbb{R}^2$. As said before, the correspondence is

$$x + iy \leftrightarrow (x, y).$$

Recall that i is the square root of negative one $i = \sqrt{-1}$. It satisfies the identity

$$i^2 = -1.$$

Note that when we write complex numbers in this way, the rule for complex multiplication takes the form

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= x_1(x_2+iy_2)+iy_1(x_2+iy_2)\\ &= x_1x_2+ix_1y_2+iy_1x_2-y_1y_2\\ &= (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)\end{aligned}$$

For example

$$\begin{aligned}(1+i)(2-i) &= (1+i)2+(1+i)(-i)\\ &= (2+2i)+(-i+1)\\ &= 3+i.\end{aligned}$$

The same sort of things is going on with the cross product of two vectors and quaternion multiplication.

3 Another Way of Looking at the Complex Numbers

We are going to construct another set of numbers like the complex numbers called the quaternions. Quaternion multiplication gives rise to vector operations like the cross product. To develop the analogy we'll first look at the complex numbers in a different way.

Abstractly we could define a the complex numbers as being ordered pairs of real numbers (x, y) with the following multiplication operation:

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \tag{1}$$

So if you handed me two vectors say, $(1, 1)$ and $(2, -1)$ and asked me to multiply them I would just use the rule:

$$\begin{aligned}(1,1)*(2,-1) &= (1\cdot 2-1\cdot(-1), 1\cdot(-1)+1\cdot 2)\\ &= (2+1,-1+2)\\ &= (3,1).\end{aligned}$$

under the identification

$$(x, y) \leftrightarrow x + iy$$

our rule crazy looking rule (1) is just the rule for complex multiplication. That we did in the previous section.

4 Norms, Unit Vectors and Complex Conjugation

On $\mathbb{R}^2$ we have **norm/length/magnitude** of vectors defined by

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

So for example the length of $(1, 2)$ is $\sqrt{1 + 4} = \sqrt{5}$. As stated before, the complex numbers are represented by points in $\mathbb{R}^2$. The complex numbers form a vector space over $\mathbb{R}$ and have the additional structure of multiplication. We defined the length of a complex number in the exact same way as we defined the length of a two dimensional vector. If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$.

Recall that **unit vectors** are ones whose magnitude are one.

For example, the complex number $1/\sqrt{2} + i/\sqrt{2}$ is a unit complex number:

$$\left|\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

Also, $1/2 + i\sqrt{3}/2$ is a unit. This is because

$$|\frac{1}{2} + i\frac{\sqrt{3}}{2}| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

These examples may look familiar. Note that $1/\sqrt{2}+i/\sqrt{2} = \cos(\pi/4)+i\sin(\pi/4)$ and $1/2+i\sqrt{3}/2 = \cos(\pi/6) + i\sin(\pi/6)$.

In fact,

Proposition 4.1 (Unit Complex Numbers are on the Unit Circle) *All the unit complex numbers will be of the form $z = \cos(\theta) + i\sin(\theta)$.*

Proof This is because $|z| = 1$, means that $|z|^2 = 1$. But $|z|^2 = x^2 + y^2$ so this means that if $z = x + iy$ is a unit vector the x and y must satisfy $x^2 + y^2 = 1$. ∎

We will show briefly that unit complex numbers are *closed under multiplication*. That is, if z_1 and z_2 are unit complex numbers then $z_1 z_2$ is also a unit complex number. Let first verify this with an example.

We already showed that $z_1 = 1/\sqrt{2} + i/\sqrt{2}$, and $z_2 = 1/2 + i\sqrt{3}/2$ are unit complex numbers. We now can multiply these together then check that $z_1 z_2$ is in fact a unit complex number (it's a pain and I'm not going to do it, but you can check it).

In addition to a multiplication, the set of complex numbers has an operation called **complex conjugation**. Here is the operation:

$$\overline{x + iy} = x - iy.$$

For example, the complex conjugate of the number $1 + i$ is $\overline{1 + i} = 1 - i$.

The operation of complex conjugation splits up over sums and products of complex numbers as well as interacts with

Proposition 4.2

1. *We can write the norm squared of a complex number as a product of a complex number and it's conjugate*

$$|z|^2 = z\bar{z}$$

.

2. *Complex Conjugation breaks up over sums:*

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

3. *Complex Conjugation breaks up over products:*

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

Proof

1. If $z = x + iy$ we can write its norm squared in a special way:

$$\begin{aligned} |z|2 &= x2 + y2 \\ &= (x - iy)(x + iy) \\ &= \overline{z}z. \end{aligned}$$

which shows what we wanted.

2. To see this, first let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{x_1 + x_2 + i(y_1 + y_2)} \\ &= x_1 + x_2 - i(y_1 + y_2) \\ &= x_1 - iy_1 + x_2 - iy_2 \\ &= \bar{z}_1 + \bar{z}_2 \end{aligned}$$

3.

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)} \\
&= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) \\
&= (x_1 - iy_1)(x_2 - iy_2) \\
&= \bar{z}_1\bar{z}_2
\end{aligned}$$

$\blacksquare$

Corollary 4.3 (Norm of the Product is the Product of the Norms.) *For any two complex numbers* $|z_1z_2| = |z_1| \cdot |z_2|$.

Proof To prove this we just apply all the things that we learned before:

$$\begin{aligned}
|z_1z_2|^2 &= \overline{z_1z_2}z_1z_2 \\
&= \bar{z}_1\bar{z}_2z_1z_2 \\
&= \bar{z}_1z_1\bar{z}_2z_2 \\
&= |z_1|^2|z_2|^2.
\end{aligned}$$

$\blacksquare$

In view of this fact we didn't even need to compute the product of our two numbers in the example above: If $|z_1| = 1$ and $|z_2| = 1$ then $|z_1z_2| = |z_1| \cdot |z_2| = 1 \cdot 1 = 1$. Now just take $z_1 = 1/\sqrt{2} + i/\sqrt{2}$, and $z_2 = 1/2 + i\sqrt{3}/2$.

5 Polar Representation of Complex Numbers

Here is an amazing theorem:

Theorem 5.1 (Euler's Formula) *For every real number* t,

$$e^{it} = \cos t + i \sin t.$$

Proof First recall that

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

This means that

$$\begin{aligned}
e^{it} &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \\
&= \cos(t) + i \sin(t).
\end{aligned}$$

$\blacksquare$

This is the formula responsible for the famous

$$e^{i\pi} = -1.$$

By Euler's Formula we have $e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 + i = -1$.

This also shows that $e^{i\pi/2} = i$.

Corollary 5.2 (Unit Complex Numbers) *Every unit complex number takes the form $e^{i\theta}$ for some θ.*

Proof We already show that unit complex number looks like $z = \cos\theta + i\sin\theta$ by looking at the unit circle. Now from Euler's formula we know that these numbers are really of the form $e^{i\theta}$. ∎

Now recall that every complex number can be written as

$$z = |z|\left(\frac{z}{|z|}\right) = (\text{ nonnegative real number }) * (\text{ unit }).$$

This gives us the following

Proposition 5.3 (Polar Representation of Complex Numbers) *Every complex number can be written in the form $re^{i\theta}$ for some θ.*

Proof Let $z = |z|\left(\frac{z}{|z|}\right)$, here $|z| = r$ and $\frac{z}{|z|}$ is a unit complex number so it have the form $e^{i\theta}$ for some θ. ∎

To actually convert from the polar expression to the cartesian (and conversely) some relations. If $z = x + iy$ then

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

Sometimes people use the notation $\arg(z) = \theta$ where arg is called the **argument** of the complex number (just a fancy word for angle).

Here is an example: the complex number $z = 1 + i$ has a radius of $r = \sqrt{2}$ and an angle of $\theta = \pi/4$. That is

$$1 + i = \sqrt{2}e^{i\pi/4}$$

Here is a conversion that goes backwards:

$$2e^{\pi/3} = 2(\cos(\pi/3) + i\sin(\pi/3)) = \sqrt{3} + i.$$

Now we are at a point where we can justify our claim that every complex number is really an expansion an a rotation: take $z = re^{\theta}$ and $w = \rho e^{i\phi}$. Now note that ANY complex number can be put into this form. Let's look at what happens when we multiply z by w

$$\begin{aligned} z \mapsto zw &= (re^{i\theta})(\rho e^{i\phi}) \\ &= (r\rho)e^{i(\theta+\phi)} \\ &= (\text{ } r \text{ expanded by } \rho \text{ })(\text{ unit in direction } \theta \text{ rotated by } \phi \text{ }). \end{aligned}$$

6 Quaternions

Definition A **quaternion** is a number of the form

$$q = a + bi + cj + dk$$

where a, b, c and d are real numbers and i, j, k[1] The quaternionic numbers i, j and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

[1] You can think of the i in the quaternions as the same or different from the i found in the complex numbers it doesn't really matter.

The relations given in equation (2) between i, j and j are just an extremely concise way of specifying special relations in the same way that the relation $i^2 = -1$ did for complex numbers. I don't ever use these rules to work them out. I use an expanded version which I will show you.

When working with quaternions we have to keep track of the ORDER in which we multiply i, j and k. QUATERNION MULTIPLICATION IS NOT COMMUTATIVE!!

Don't stress too much though, because any real number commutes with any quaternion. So it doesn't matter if I write $5k$ or $k5$... that is $5k = k5$.

Here is an example. The multiplicative inverse of i is $-i$. This is because $i(-i) = -i^2 = (-1)(-1) = 1$. Similarly, the multiplicative inverses of j and k are $-j$ and $-k$ respectively.

Proposition 6.1 (Uncompressed Version of Quaternion Relations) *1. The following are how i, j and k relate to each other*

$$\begin{aligned} ij &= k, \\ jk &= i, \\ ki &= j. \end{aligned}$$

2. Unlike the case of complex multiplication, quaternion multiplication is NONCOMMUTATIVE*:*

$$\begin{aligned} ij &= -ji \\ ik &= -ki \\ jk &= -jk \end{aligned}$$

Proof 1.

$$\begin{aligned} ijk = -1 \quad &\implies \quad i^2jk = -i \\ &\implies \quad -jk = -i \\ &\implies \quad jk = i. \end{aligned}$$

Next we show that $ij = k$

$$\begin{aligned} ijk = -1 \quad &\implies \quad ijk2 = -k \\ &\implies \quad -ij = -k \\ &\implies \quad ij = k. \end{aligned}$$

Next we show that $ki = j$. Using the

$$\begin{aligned} ki &= -ki(-1) \\ &= -ki(j2) \\ &= -k(ij)j \\ &= -k^2j \\ &= j. \end{aligned}$$

On the first line we introduced two negatives, and the first to second we used that fact that $j2 = -1$, on the second to third we rearranged, on the third to fourth we used the fact that $ij = k$ and the fourth to fifth we used that $ij = k$.

2. Now we use these along with the fact that we know the multiplicative inverses of i,j and k:

$$\begin{aligned} ij = k \quad &\implies \quad iji = ki \\ &\implies \quad iji = j \\ &\implies \quad -iiji = -ij \\ &\implies \quad ji = -ij \end{aligned}$$

the first implication we multiplied by i on the right, the second implication we used the fact that $ki = j$, the third we multiplied both sides by $-i$ and on the fourth we use the fact that $-ii = 1$.

Similarly,

$$\begin{aligned} jk = i \quad &\implies \quad kjk = ik \\ &\implies \quad kjk = j \\ &\implies \quad -kkjk = -kj \\ &\implies \quad jk = -kj. \end{aligned}$$

and,

$$\begin{aligned} ki = j \quad &\implies \quad iki = ij \\ &\implies \quad iki = k \\ &\implies \quad -iiki = -ik \\ &\implies \quad ki = -ik. \end{aligned}$$

Let $q = (1 + 2i + k)$ and $p = (1 + j)$

$$\begin{aligned} qp &= (1 + 2i + k)(1 + j) \\ &= (1 + 2i + k) + (1 + 2i + k)j \\ &= 1 + 2i + k + (j + 2ij + kj) \\ &= 1 + 2i + k + j + 2k - i \\ &= 1 + i + j + 3k. \end{aligned}$$

while

$$\begin{aligned} pq &= (1 + j)(1 + 2i + k) \\ &= (1 + j) + (1 + j)(2i) + (1 + j)k \\ &= (1 + j) + (2i + 2ji) + k + jk \\ &= 1 + j + 2i - 2k + k + i \\ &= 1 + 3i + j - k. \end{aligned}$$

Every quaternion can be written as $q = q_0 + \vec{v}$ where $\vec{v} = xi + yj + zk$ and sometimes these are called the **real part** and the **vector part** of a quaternion number in the same sense that a complex number has a real and imaginary part. It was through studying how these vector parts transform the cross product of two vectors arose.

6.1 Connection to the Usual Vector Operations

We can define the cross product is defined as a function form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (b_2c_2 - c_1b_1, -a_1c_2 + a_2c_1, a_1b_2 - a_2b_1)$$

If we use the identification

$$ai + bj + ck \leftrightarrow (a, b, c)$$

then we get the following relation

Proposition 6.2 (Quaternion Multiplication Relates to Dot Products and Cross Products) *Let q*

$$(a_1i + b_1j + c_1k)(a_2i + b_2j + c_2k) = -\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \times \vec{v}_2$$

Proof Expanding the multiplication out on the left hand side above is a little bit of a pain, but it gets things done.

$$\begin{aligned}
(a_1 i + b_1 j + c_1 k)(a_2 i + b_2 j + c_2 k) &= a_1 i(a_2 i + b_2 j + c_2 k) + b_1 j(a_2 i + b_2 j + c_2 k) + c_1 k(a_2 i + b_2 j + c_2 k) \\
&= a_1 a_2 i^2 + a_1 b_2 ij + a_1 c_1 ik + b_1 a_2 ji + b_1 b_2 j^2 + b_1 c_1 jk + c_1 a_2 ki + c_1 b_2 kj + c_1 c_2 k^2 \\
&= a_1 a_2(-1) + a_1 b_2 k + a_1 c_1(-j) + b_1 a_2(-k) + b_1 b_2(-1) \\
&\quad + b_1 c_1(i) + c_1 a_2(j) + c_1 b_2(-i) + c_1 c_2(-1) \\
&= -(a_1 a_2 + b_1 b_2 + c_1 c_2) + (b_2 c_2 - c_1 b_1)i - (a_1 c_2 - a_2 c_1)j + (a_1 b_2 - a_2 b_1)k \\
&= \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \times \vec{v}_2
\end{aligned}$$

$\blacksquare$

Corollary 6.3 (Anti-Symmetric Property of Cross Product) $\vec{a} \times b = -\vec{b} \times \vec{a}$

Proof Everything in the computation above will stay the same when flipping $\vec{b}$ and $\vec{a}$ with the exception of the cross terms. These are the terms with any of the following combinations

$$ij, ji, ik, ki, kj, jk.$$

Look at the second line in the computation above (I know a pain but still...) — if we multiplied out $\vec{b}\vec{a}$ rather than $\vec{a}\vec{b}$ the terms connected to ij would be replaced with ji and the terms with ik with be replace with ki etc.

Since these are the only terms that contribute to the cross product, and since $ji = -ij$, $ik = -ki$ etc, we will just pick up an overall minus sign for all of these terms. $\blacksquare$

Proposition 6.4 (Quaternion Formula For the Cross Product)

$$\vec{a} \times \vec{b} = \frac{\vec{a}\vec{b} - \vec{b}\vec{a}}{2}$$

Proof The proof is a straightforward computation,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \frac{1}{2}(-\vec{a} \cdot \vec{b} + \vec{a} \times \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \times \vec{a}) \\
&= \frac{1}{2}(\vec{a} \times \vec{b} - \vec{b} \times \vec{a}) \\
&= \frac{1}{2}(\vec{a} \times \vec{b} + \vec{a} \times \vec{a}).
\end{aligned}$$

In the computations we have just used the fact that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

From this we can easily see now that $\vec{a} \times \vec{a} = 0$. This is because

$$\vec{a} \times \vec{a} = \frac{\vec{a}\vec{a} - \vec{a}\vec{a}}{2}.$$

We can also prove that the cross product distributes over sums and that you can pull constants out:

Proposition 6.5 (Distributive Properties) *For and vector $\vec{v}$, $\vec{w}$ and $\vec{u}$, and any constant c we have*

1. $(c\vec{w} + \vec{u}) \times \vec{v} = c\vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.
2. $\vec{v} \times (c\vec{w} + \vec{u}) = c\vec{v} \times \vec{w} + \vec{v} \times \vec{u}$.

Proof 1. The proof is a straight forward computation:

$$\begin{aligned}
\vec{v} \times (c\vec{w} + \vec{u}) &= \frac{1}{2}(\vec{v}(c\vec{w} + \vec{u}) - (c\vec{w} + \vec{u})\vec{v}) \\
&= \frac{1}{2}(c\vec{v}\vec{w} + \vec{v}\vec{u} - c\vec{w}\vec{v} - \vec{u}\vec{v}) \\
&= c\frac{1}{2}(\vec{v}\vec{w} - \vec{w}\vec{v}) + \frac{1}{2}(\vec{v}\vec{u} - \vec{u}\vec{v}) \\
&= c\vec{v} \times \vec{w} + \vec{v} \times \vec{u}.
\end{aligned}$$

2. Here we just used the antisymmetric property to flip the terms and introduce a minus sign then use part 1

$$\begin{aligned}(c\vec{w}+\vec{u})\times\vec{v} &= -\vec{v}\times(c\vec{w}+\vec{u})\\ &= -c\vec{v}\times\vec{w}-\vec{v}\times\vec{u}\\ &= c\vec{w}\times\vec{v}+\vec{u}\times\vec{v}.\end{aligned}$$

∎

Like the complex numbers we have an operation of similar to complex conjugation called **quaternion conjugation**: It is defined as follows

$$(a+bi+cj+dk)^* = a-bi-cj-dk.$$

Proposition 6.6 (Norm conjugation formula) $qq^* = |q|^2$

Proof Let $q = a+\vec{q}$, where $\vec{q} = ai+bj+ck$,

$$\begin{aligned}qq^* &= (a+\vec{q})(a+\vec{q})*\\ &= a^2+a\vec{q}-a\vec{q}-\vec{q}\vec{q}\\ &= a^2-(-\vec{q}\cdot\vec{q}+\vec{q}\times\vec{q})\\ &= a^2+\vec{q}\cdot\vec{q}\\ &= a^2+b^2+c^2+d^2\\ &= |q|^2.\end{aligned}$$

Unlike complex conjugation, quaternion conjugation is not linear:

Proposition 6.7 *For any quaternion q and p*

$$(qp)^* = p^*q^*.$$

Proof Let $p = p_0+\vec{p}$ and $q = q_0+\vec{q}$, where p_0 and q_0 are the real parts of the quaternion, and $\vec{p}$ and $\vec{q}$ are the vector parts,

$$\begin{aligned}pq &= (p_0+\vec{p})(q_0+\vec{q})\\ &= p_0q_0+\vec{p}q_0+p_0\vec{q}+\vec{p}\vec{q}\\ &= p_0q_0+q_0\vec{p}+p_0\vec{q}+(-\vec{p}\cdot\vec{q}+\vec{p}\times\vec{q})\\ &= (p_0q_0-\vec{p}\cdot\vec{q})+(q_0\vec{p}+p_0\vec{q}+\vec{p}\times\vec{q}),\end{aligned}$$

on the last line we have separated the product into real and vector parts. This allows us to say

$$(pq)^* = (p_0q_0-\vec{p}\cdot\vec{q})-(q_0\vec{p}+p_0\vec{q}+\vec{p}\times\vec{q}).$$

Now a similar computation shows,

$$\begin{aligned}q^*p^* &= (q_0-\vec{q})(p_0-\vec{p})\\ &= p_0q_0-p_0\vec{q}-q_0\vec{p}+\vec{q}\vec{p}\\ &= p_0q_0-p_0\vec{q}-q_0\vec{p}+(-\vec{q}\cdot\vec{p}+\vec{q}\times\vec{p})\\ &= (p_0q_0-\vec{q}\cdot\vec{p})-p_0\vec{q}-q_0\vec{p}-\vec{p}\times\vec{q}\\ &= (pq)^*\end{aligned}$$

∎

Proposition 6.8

$$|qp|^2 = |q|^2|p|^2$$

Another computation,

Proof

$$\begin{aligned}
|qp|^2 &= (qp)^*(qp) \\
&= p^*q^*qp \\
&= p^*|q|^2p \\
&= |q|^2p^*p \\
&= |q|^2|p|^2
\end{aligned}$$

$\blacksquare$

Proposition 6.9 $|\vec{w} \times vecv| = |\vec{v}||\vec{w}| \sin\theta$

Proof

$$|\vec{v}\vec{w}|^2 = |-\vec{v}\cdot\vec{w} + \vec{v}\times\vec{w}| \tag{3}$$

$$= |\vec{v}\cdot\vec{w}|^2 + |\vec{v}+\vec{w}|^2 \tag{4}$$

$$= |\vec{v}|^2|\vec{w}|^2(\cos\theta)^2 + |\vec{v}\times\vec{w}|^2 \tag{5}$$

Since $|\vec{v}\vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2$ we get

$$|\vec{v}|^2|\vec{w}|^2 - |\vec{v}|^2|\vec{w}|^2(\cos\theta)^2 = |\vec{v}\times\vec{w}|^2$$

which means

$$|\vec{v}\times\vec{w}| = |\vec{v}|^2|\vec{w}|^2(1-(\cos\theta)^2) = |\vec{v}|^2|\vec{w}|^2(\sin\theta)^2$$

which proves the result. $\blacksquare$