Dupuy — Complex Analysis Problem Bank

Contents

1 Euler's Formula

Euler's formula states that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for $\theta \in \mathbb{R}$. There are some nice things you can do with this.

1. Compute and draw the 8th roots of unity.
2. Let ζ_n be a primitive nth root of unity. Show that $\sum_{j=0}^{n-1} \zeta_n^j = 0$.
3. (Wallis' Formula) Using the complex representation of cosine, find a formula for

$$\int_0^{2\pi} \cos(\theta)^{2n} d\theta.$$

2 Quaternion Exercise

This exercise show how nice the complex numbers are and how if one tries to develop a notion of holomorphic function in higher for the quaternions. The quaternions are the Division algebra (noncommutative field) over the reals defined by

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k, (\cong \mathbb{R}^4 \text{ as a vector space})$$

where i,j and k satisfy

$$ijk = -1 \text{ and } i^2 = j^2 = k^2 = -1.$$

The norm on the quaternions is defined as

$$|a + bi + cj + dk|^2 = a^2 + b^2 + c^3 + d^2,$$

here $a, b, c, d \in \mathbb{R}$.

1. For $U \subset \mathbb{H}$ open, we say a function $f : U \to \mathbb{H}$ is **holomorphic** if

   $$f(q) = \lim_{h \to 0} \left((f(q + h) - f(q))h^{-1} \right).$$

   Show that the only quaternionic holomorphic functions are of the form

   $$f(q) = \alpha q + \beta.$$

   where $\alpha, \beta \in \mathbb{H}$.

3 Power and Laurent Series

Many of the problems here were taken from old University of New Mexico qualifying exams. They were automatically converted from a scanned file using Google's Gemini so be wary that these were automatically converted.

1. Prove Hadamard's formula for the radius of convergence of a series $\sum_{n=0}^{\infty} a_n z^n$.

$$R = \lim_{n\to\infty} \inf_{m\geq n} |a_m|^{-1/m}.$$

   Also show that the series converges absolutely and uniformly, (the differentiability thing is the next problem).

2. (Analytic implies Holomorphic) Suppose that $f(z) = \sum_{n\geq 0} a_n z^n$ has a radius of convergence R.

   (a) Show $\sum_{n\geq 0} n a_n z^{n-1}$ converges with the same radius of convergence R.

   (b) Show $\frac{d}{dz}\left[\sum_{n=0}^{\infty} a_n z^n\right] = \sum_{n\geq 0} \frac{d}{dz}[a_n z^n]$ on the disc of convergence.

   (Warning: it is not true that for general $u_n(t) \to u(t)$ uniformly that $u_n'(t) \to u_n'(t)$ uniformly! This is a special fact about power series.)

3. Suppose that $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ has a finite radius of convergence. Let $g(z) = a_n + a_{n+1}(z-z_0) + a_{n+2}(z-z_0)^2 + \cdots$. Show that $g(z)$ has the same radius of convergence as $f(z)$ at z_0. (Hint: don't think about this too much)

4. (Extra Credit, see WW page 59) This is a famous example of non-uniform cooonvergence. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

   converges to $\frac{1}{(z-1)^2}$ when $|z| < 1$ and $\frac{1}{z(z-1)^2}$ when $|z| > 1$

5. If the series converges do some analysis to determine the radius of convergence at the boundary.

   (a) Expand $\frac{1}{1+z^2}$ in a power series around $z = 0$, find the radius of convergence.

   (b) Find the radius of convergence of $\sum_{n\geq 0} n! z^n$.

   (c) (New Mexico, Jan 1998) Expand $\frac{z^2+2z-4}{z}$ in a power series around $z = 1$ and find its radius of convergence.

6. (a) Let $B \times A \subset \mathbb{C} \times \mathbb{C}$ be an open region with compact closure. Let $f : B \times A \to \mathbb{C}$ be a function. Let $\gamma \subset A$ be a C^1-curve (so it has finite length). Define $F : B \to \mathbb{C}$ by

$$F(z) = \int_\gamma f(z,s)ds.$$

   Assuming $\frac{\partial f}{\partial z}(z,s)$ exists and is continuous for all $s \in \gamma$ and all $z \in B$ show that

$$\frac{d}{dz}[F(z)] = \int_\gamma \frac{\partial f}{\partial z}(z,s)ds.$$

   (b) Let $\Omega \subset \mathbb{C}$ be an open set. Let $\gamma : [0,1] \to \Omega$ be an C^1 curve. Let $f \in \mathrm{hol}(\Omega)$ and $g \in L^2(\Omega)$. Show that

$$F(z) := \int_\gamma f(\zeta - z) g(\zeta) d\zeta$$

   is holomorphic on Ω.

7. (a) (Whittaker and Watson, page 99) Consider the series

$$\frac{1}{2}\left(z+\frac{1}{z}\right)+\sum_{n=1}^{\infty}\left(z-\frac{1}{z}\right)\left(\frac{1}{1+z^n}-\frac{1}{1+z^{n-1}}\right).$$

Show that this series converges for all values of z with $|z|\neq 1$. Furthermore, show that

$$\frac{1}{2}\left(z+\frac{1}{z}\right)+\sum_{n=1}^{\infty}\left(z-\frac{1}{z}\right)\left(\frac{1}{1+z^n}-\frac{1}{1+z^{n-1}}\right)=\begin{cases} z, & |z|<1 \\ \frac{1}{z}, & |z|>1 \end{cases}$$

(b) (Whittaker and Watson, 2.8, problem 16) By converting the series

$$1+\frac{8q}{1-q}+\frac{16q^2}{1+q^2}+\frac{24q^3}{1-q^3}+\cdots$$

(in which $|q|<1$), into a double series, show that it is equal to

$$1+\frac{8q}{(1-q)^2}+\frac{8q^2}{(1+q^2)^2}+\frac{8q^3}{(1-q^3)^2}+\cdots$$

8. Consider the following meromorphic functions.
   (a) Expand $f(z)=\frac{1}{2z-z^2}$ in a power series about $z=1$.
   (b) Find a Laurent expansion of $f(z)=\frac{1}{z}+\frac{1}{z+2}+\frac{1}{(z-1)^2}$ which is valid in the annulus $1<|z|<2$.

9. Expand $e^{1/z}$ in a Laurent series about $z=0$, determining the Laurent coefficients.

10. Show that

$$\frac{1}{n!}=\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)d\theta.$$

11. Expand $f(z)=\frac{z+6}{z^2-2z-3}$ in
    (a) Taylor series around $z=0$.
    (b) Laurent series in the annulus $1<|z|<3$.
    (c) Laurent series in the region $3<|z|<\infty$.

12. Expand in Laurent series in region indicated: $e^{\frac{1}{z-1}}, |z|>1$.

13. Let $f(z):=\frac{1}{(z-1)(z-2)}$. Write $f(z)$ as a Laurent series centered at $z=0$ which converges on the annulus $1<|z|<2$.

14. Let $f(z)=\frac{1}{z^2(e^z-e^{-z})}$, $0<|z|<\pi$. Compute the first three non-zero terms of the Laurent expansion of $f(z)$ in $0<|z|<\pi$.

15. Prove that if f is analytic on the region U (open and simply connected), $z_0\in U$, and $f'(z_0)=0$, then f is not one-to-one in any neighborhood of z_0.

16. Let $\{a_n\}$ be a sequence of complex numbers. Assume that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ converges for all z satisfying $|z|\leq r$. Prove that if $|a_1|>\sum_{n=2}^{\infty}n|a_n|r^{n-1}$, then f is an injective function on the disc $|z|\leq r$.

17. Prove that if $w=f(z)$ is holomorphic in the disc $D(0,2)$, then $w=f(z)$ has a Taylor expansion centered at $z_0=0$ which converges for $|z|\leq 1$.

18. Assume that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of analytic functions defined on the region Ω, such that $\lim_{n\to\infty}f_n(z)=f(z)$ uniformly on compact subsets of Ω. Show that $f(z)$ is analytic in Ω and that $\lim_{n\to\infty}f_n'(z)=f'(z)$ uniformly on compact subsets of Ω.

19. Classify the singularities at $z = 0$ of the following functions $f(z)$ and find their residue.

   (a) $f(z) = \frac{1}{z}$
   (b) $f(z) = z \cos\left(\frac{1}{z}\right)$
   (c) $f(z) = z^{-3}\csc(z^2)$

20. Let $f(z) := \frac{e^z}{(z-1)^4}$.

   (a) Classify all of the singularities and find the associated residues.
   (b) Determine the Laurent expansion of f centered at $z = 1$.
   (c) If C denotes the positively oriented circle of radius 2 centered at $z = 0$, evaluate $\oint_C f(z)dz$.

21. Classify the singularities (including the point at ∞) and find the residues for

   (a) $f(z) = \sin\left(\frac{1}{z}\right)$
   (b) $f(z) = \frac{\sin(z^2)}{z^7}$
   (c) $f(z) = \frac{1}{z^2}\cot z$

22. Classify all of the singularities and find the associated residues for each of the following functions:

   (a) $\frac{(z+3)^2}{z}$
   (b) $\frac{e^{-z}}{(z-1)(z+2)^2}$

23. Classify all of the singularities and find the associated residues for each of the following functions:

   (a) $z\cos(2z)$
   (b) $\frac{z^3-z^2+2}{z-1}$

24. Classify the singularities of the following functions in the extended complex plane $\mathbb{C} \cup \{\infty\}$, and find the singular part at each of the isolated singular points:

   (a) $\frac{1-\cos z}{z^4}$
   (b) $\sqrt{1-\sin z}$
   (c) $\frac{1-z^3}{1-z^2}$

25. Expand (if possible) in Laurent series in the indicated region:

   (a) $e^{1/(x-1)}$ for $|z| > 1$
   (b) $\frac{1}{(z-a)(z-b)}$ for (i) $0 < |a| < |z| < |b|$ and (ii) $|a| < |b| < |z|$
   (c) $\log\left(\frac{1}{1-z}\right)$ for $|z| > 1$

26. Let $f(z) = \frac{1}{z-1} + \frac{1}{(z-2)^2}$. Expand $f(z)$ in:

   (a) Taylor series in $|z| < 1$
   (b) Laurent series in $1 < |z| < 2$

27. Find the Laurent expansion of $f(z) = (1-z^2)e^{1/z}$ around $z = 0$. Determine its annulus of convergence and the residue of $f(z)$ at $z = 0$.

28. (a) Find the first 3 non-vanishing terms of the Taylor series expansion of $\tan z$ around the origin.
   (b) Also, find its radius of convergence.

29. Expand the function $f(z) = \frac{z^3+2z-4}{z}$ in a power series around $z = 1$ and give its radius of convergence.

30. Compute the radius of convergence of the following:

    (a) $\sum_{n=1}^{\infty} \frac{(3n)!}{(3n)^{3n}} z^n$

    (b) $\sum_{n=0}^{\infty} [3 + (-1)^n]^n z^n$

    (c) The Taylor series around zero for the function $\frac{z}{e^z-1}$

31. Compute the radius of convergence of the following:

    (a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} z^n$

    (b) $\sum_{n=0}^{\infty} (n + a^n) z^n$ where $a \in \mathbb{C}$

    (c) The Taylor series around zero for the function $z \cot z$

32. Determine the three Laurent series around 0 of the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the three regions $|z| < 1$, $1 < |z| < 2$, and $|z| > 2$, respectively.

4 Sequences of Analytic Functions

1. (CUNY, Fall 2005) Let D be the closed unit disc. Let g_n be a sequence of analytic functions converging uniformly to f on D.

   (a) Show that g_n' converges.

   (b) Conclude that f is analytic.

   (Hint/Discussion: Normally, taking derivatives makes things numerically behave worse and integration makes things nicer. What is nice about complex analysis is that integration and differentiation are the same thing. Here is the hint now: use the integral formula for derivatives to get this done (I think). A basic philosophical point here is that differentiation of holomorphic functions is actually easy because it is secretely integration.)

2. Here is a first example of an analytic continuation "from the wild".

   (a) Show that the Riemann Zeta function

   $$\zeta(z) := \sum_{n \geq 1} \frac{1}{n^z}$$

   converges for $\operatorname{Re} z > 1$ and is analytic on this domain. (You need to use the "analytic convergence theorem", which states that a uniform limit of analytic functions is analytic. This is just a slight generalization of the previous problem.)

   (b) (Whittaker and Watson, 2.8, problem 10)

      i. Show that when $\operatorname{Re} s > 1$,

      $$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty} \left[\frac{1}{n^s} + \frac{1}{s-1} \left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}} \right) \right]$$

      ii. Show that the series on the right converges when $0 < \operatorname{Re} s < 1$. (This means the series above gives us access to the interesting part of the Riemann-Zeta function. Hint: $\int_n^{n+1} x^{-s} dx = \frac{(n+1^{-s+1}}{1-s} - \frac{n^{-s+1}}{1-s}$)

5 Liouville's Theorem

The proof of Liouville's Theorem is basically taking a limit in Cauchy's Integral Formula for the first derivative (if I don't understand what I mean here, this is your first exercise). There are variants of this proof which are featured in this problem. The proof of the estimate of the partial sum for a power series expansion is based of expanding Cauchy's Integral Formula in a geometric series and then truncating the series. Many of the problems here were taken from University of New Mexico qualifying exams.

1. Prove Liouville's Theorem: any bounded entire function is constant.
2. (New Mexico, not sure which year) Let f be analytic on $\mathbb{C}$. Assume that $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of valued r going to infinity. Show that f is a polynomial of degree less than or equal to n.
3. (New Mexico, not sure which year) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume that g is not identically zero. Show that f/g is rational.
4. Prove Goursat's theorem. Let γ be a simple contour. If $f : \overline{\gamma^+} \to \mathbb{C}$ is holomorphic (but whose derivative is not necessarily continuous) then
$$\int_\gamma f(\zeta)d\zeta = 0.$$
5. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and let R be the radius of convergence (which is possibly infinite). Let $S_N(f)(z) = \sum_{n=0}^N a_n z^n$. Show that for all $r < R$ and all $z \in \mathbb{C}$ with $|z| < r$ we have
$$|f(z) - S_N(f)(z)| \leq \frac{M(f,r)}{r - |z|} \frac{|z|^{N+1}}{r^N}$$
where $M(f,r) = \max_{|z|=r} |f(z)|$.
6. (UIC, Spring 2016) Describe all entire functions such that $f(1/n) = f(-1/n) = 1/n^2$ for all $n \in \mathbb{Z}$.
7. (a) State and prove a form of the maximum principle.
   (b) State Schwarz's lemma and give a proof.

   (a) Deduce the fundamental theorem of algebra from Liouville's Theorem.
8. Let f be analytic in $\mathbb{C}$. Assume $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of values of r going to infinity. Show that f is a polynomial of degree less than or equal to n.
9. Assume that $f(z)$ is an entire function with
$$\lim_{|z|\to\infty} \frac{f(z)}{z^2} = 0.$$
Prove that $f(z)$ must be linear, that is $f(z) = a + bz$, with $a, b \in \mathbb{C}$. Please provide all the details.
10. Let $f(z)$ and $g(z)$ be entire functions. Show that if $f \circ g(z)$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials.
11. Let $f(z)$ and $g(z)$ be entire functions satisfying $|f(z)| \leq 10|g(z)|$ for all $z \in \mathbb{C}$.

    Does it follow that there exists $\lambda \in \mathbb{C}$ with $f(z) = \lambda g(z)$ for all $z \in \mathbb{C}$? Give a proof or a counterexample.
12. Assume that $w = f(z)$ is an entire function, and that a and b are two positive constants so that $f(z)$ satisfies $|f(z)| \leq a + b|z|^2$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial of degree no larger than two.

13. (a) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume g is not identically zero. Show f/g is rational.

    (b) Let u be harmonic in $\mathbb{C}$ and $u(x, y) \geq -2$ for all $x + iy \in \mathbb{C}$. Show u is constant in $\mathbb{C}$.

14. $\varphi(z) = |f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2$.

    (a) Show that $\varphi(z)$ is harmonic on the domain only if all the functions $f_k(z)$ $(k = 1, 2, \ldots, n)$ reduce to constant functions.

    (b) Show that $\varphi(z)$ has no local maximum unless all the functions $f_k(z)$ $(k = 1, 2, \ldots, n)$ reduce to constant functions.

15. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

16. (a) Find a bound for the modulus of the integral shown below:

    $$\int_\gamma \sin^2(z)\, dz,$$

    where γ is the simple contour $\gamma(t) = (1 - t) + i\pi t$ and $0 \leq t \leq 1$.

    (b) Evaluate exactly the modulus of the integral in (a).

17. Let $f(z)$ be an entire function for which the real part $\operatorname{Re} f(x + iy) = u(x, y)$ is a bounded function. Does it follow that $f(z)$ is a constant function? Give a proof or a counterexample.

18. Give two distinct harmonic functions on $\mathbb{C}$ that vanish on the entire real axis. Why is this not possible for analytic functions?

19. Let $P_1, P_2, \ldots, P_n$ be arbitrary points of a plane and let $\overline{PP_k}$ denote the distance between P_k and a variable point P. If P is confined to the closure of a bounded domain D, show that the product $\prod_{k=1}^{n} \overline{PP_k}$ attains its maximum on the boundary of D.

20. Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and set $g(z) := f(1/z)$. Prove that f is a polynomial if and only if $g(z)$ has a pole at $z = 0$.

21. Find all entire functions that satisfies the Lipschitz condition on $\mathbb{C}$. A function $f(z)$ is said to satisfy the Lipschitz condition on $\mathbb{C}$ if there exists a positive constant M such that $|f(z_1) - f(z_2)| \leq M \cdot |z_1 - z_2|$ for all $z_1,\ z_2 \in \mathbb{C}$.

6 Riemann Extension Theorem

For functions which are analytic in some punctured neighborhood and which are bounded there is a natural way to extend the function to the point. This again uses the Cauchy integral formula and is another nice part of complex analysis.

1. (a) Prove the Riemann Extension Theorem: Let $U \subset \mathbb{C}$ be a region containing a point z_0. Let $f \in \mathrm{hol}(U \setminus \{z_0\})$. If f is bounded on U show that there exists a unique $\widetilde{f} \in \mathrm{hol}(U)$ such that $\widetilde{f}|_{U\setminus\{z_0\}} = f \in \mathrm{hol}(U \setminus \{z_0\})$.

   (b) Recall that a morphism of topological spaces $f : X \to Y$ is "proper" if and only if the inverse image of every compact set is compact. Show that an analytic map $f : \mathbb{C} \to \mathbb{C}$ is proper if and only if for all $z_j \to \infty$ we have $f(z_j) \to \infty$.

   (c) Show that the only proper maps $f : \mathbb{C} \to \mathbb{C}$ are polynomials. (see page 27 of McMullen, you need to consider the function $g(z) = 1/f(1/z)$ and show that $g(z) = z^n g_0(z)$ where $g_0(z)$ is analytic and non-zero. This will allows you to conclude $|g(z)| > c|z|^n$ for some n which will allows you to conclude behavious about the growth of $f(z)$ as $z \to \infty$.)

7 Topological Things

I collected a bunch of topological exercises here.

Background:

- Let X and Y be topological spaces. We define the topology on $X \times Y$ to be the smallest topology such that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous (this means the open sets are generated by sets of the form $U \times Y$ or $X \times V$ for $U \subset X$ open or $X \times V$ for $V \subset Y$ open.
- A topological space X is **compact** if every open cover has a finite subcover. An open cover is just a union of open sets that equal X.
- A **proper map** is a morphism of topological spaces such that the inverse image of compact sets is compact.

Side Remark: The third condition is interesting because Grothendieck realized we can use it to extend this definition to categories other than topological spaces. In particular to the category of "schemes".

1. Let $U \subset \mathbb{C}$ be a connected open set. Consider $U \subset \mathbb{C}$ with the subspace topology (open subset of U are the intersection of open subsets of $\mathbb{C}$ with U and closed subset are closed subset of $\mathbb{C}$ intersected with U). Show that the only subset of U which are open, closed and nonempty is U itself.
2. (Green and Krantz, Ch 11) A subset $S \subset \mathbb{R}^n$ is **path connected** if for all $a, b \in S$ there exists a continuous $\gamma : [0, 1] \to S$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

   Let U be an open subset of $\mathbb{R}^n$. Show that U is path connected if and only if U is connected. (Hint: show that the collection of path connected elements is open and closed. Also, you can use that the only nonempty open and closed subset of a connected open set is the entire set itself.)
3. Show that the following conditions are equivalent for a topological space X:

   (a) For all $a, b \in X$ there exists open sets $U \ni a$ and $V \ni b$ with $U \cap V = \emptyset$.

   (b) For all $a, b \in X$, if every neighborhood of a intersects every neighborhood of b then $a = b$.

   (c) The diagonal map $X \to X \times X$ given by $x \mapsto (x, x)$ is proper.

   (d) The diagonal subset is closed.

   If any of these conditions hold we call the topological space **separated** or **hausdorff**. (Hint: You should use the fact that a morphism f is proper if and only if f is closed and the inverse image of every point is compact.)

8 Harmonic Functions

Let $u(x+iy) = u(x,y)$ be a real valued harmonic function on some region $U \subset \mathbb{C}$. A **harmonic conjugate** is a function $v(x,y)$ such that $f(x+iy) := u(x,y) + iv(x,y)$ is holomorphic.

1. Assume that $w = f(z) = u(z) + iv(z)$ is an analytic function mapping a domain D in the z-plane onto a domain D' in the w-plane. If $\phi(u,v)$ is a harmonic function in D', show that the function $\Phi(x,y) = \phi(u(x,y), v(x,y))$ is harmonic in D.

2. Answer only one of the following questions:

   (a) Prove that if $u(z)$ is a non-constant harmonic function in the domain D, then $u(z)$ does not have a local maximum in D.

   (b) Let $u(x,y)$ be a harmonic function on the entire plane, and let $v(x,y)$ be a harmonic conjugate of $u(x,y)$. Assume that $u(x,y) \le v^2(x,y)$ for all $(x,y) \in \mathbb{C}$. Prove that both $u(x,y)$ and $v(x,y)$ must be constant.

3. (a) Show that $u(x,y) = (x+1)y$ is harmonic in the entire plane.

   (b) Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

   (c) Give explicitly an analytic function $w = f(z)$ with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

4. (a) Show that $u(x,y) = x^3 - 3xy^2 + y^2 - x^2$ is harmonic in the entire plane.

   (b) Find a harmonic conjugate $v(x,y)$.

   (c) Give explicitly an analytic function $w = f(z)$ with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

5. A complex-valued function $f = U + iV$ is said to be harmonic on a domain $D \subset \mathbb{C}$ if U and V are harmonic on D. Show that f is holomorphic on D if and only if both f and zf are harmonic on D.

6. Let $u(x,y)$ be the bounded harmonic function in the upper half-plane $\{z = x+iy \in \mathbb{C} | y > 0\}$ that has the boundary value

$$u(x,0) = \operatorname{sgn} x = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

   Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

7. Show that $u(x,y) = u(z)$ has a harmonic conjugate locally. (Hint: Use the fundamental theorem of line integrals $v(\vec{P}) - v(\vec{Q}) = \int_C \nabla v \cdot d\vec{r}$ if C is a path starting a $\vec{Q}$ and ending at $\vec{P}$)

8. Find all of the harmonic conjugates of $u(x,y) = x^3 - 3xy^2 + 2x$.

9. Let $f(z) = u(z) + iv(z)$ be analytic. Show that the level sets of $u(z)$ and $v(z)$ are orthogonal.

10. (New Mexico, Summer 2000) Show that the pullback of a harmonic function by an holomorphic map is harmonic (what these words means is explained below). Assume that $w = f(z) = u(z) + iv(z)$ is holomorphic map $f : D \to D' \subset \mathbb{C}$. We consider D in the z-plane to a domain D' in the w-plane. If ϕ is harmonic on D', show that

$$\Phi(x,y) := \phi(u(x,y), v(x,y))$$

    is harmonic in D.

    (The function Φ is called the pullback of ϕ by f. Sometimes in the literature these you will see the notation $f^*\phi$ for Φ.)

11. (New Mexico, not sure which year) Let $f(z)$ and $g(z)$ be entire functions. Show that if $f(g(z))$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials. (Hint: this relates to the problem on properness from the previous homework).

12. Find all entire functions $f(z)$ which satisfy $\mathrm{Re}\, f(z) \leq 2/|z|$ when $|z| > 1$. (Hint: Consider $e^{-f(z)}$ or $e^{f(z)}$. You will need the maximum modulus principle and Liouville's theorem.)

13. Let $u(z)$ be a real valued harmonic function on a domain $D \subset \mathbb{C}$

14. Show that for all $D_r(z_0) \subset D$ we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

(Hint: use a harmonic conjugate)

15. If $z_0 \in D$ has the property that there exists some $r > 0$ with $D_r(z_0) \subset D$ and

$$u(z_0) \geq u(z)$$

for all $z \in D_r(z_0)$ then $u(z)$ is constant. (Hint: Consider a function such that $f(z) = u(z) + iv(z)$ then consider the maximum of $e^{f(z)}$.)

16. Let $u_0(\theta)$ be a continuous 2π-periodic function. Let D be a disc of radius r. The Dirichlet boundary value problem asks to find a function $u(x, y)$ such that:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in D \\ u(e^{i\theta}) = u_0(\theta), & \end{cases}$$

Show that convolution with the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

gives a solution to this problem.

9 Residue Integrals

This is what I call the "Residue Integral Gaunlet". You should do all of these before your qualifying exam. Many of the integrals from from a course given by Vladimir Zakharov at University of Arizona. Some were taken from Whittaker and Watson. Some were taken from University of New Mexico qualifying exams.

1. (Whittaker and Watson, 6.24,3) If $-1 < z < 3$ then
$$\int_0^\infty \frac{x^z}{(1+x^2)^2} dx = \frac{\pi(1-z)}{4\cos(\pi z/2)}$$
2. (Whittaker and Watson, 6.21, Example 4) Let $a > b > 0$ be real numbers. Show that
$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos(\theta))^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$
3. (Whittaker and Watson, 6.23, 2) If $a > 0$ and $b > 0$ show that
$$\int_{-\infty}^\infty \frac{x^4 dx}{(a+bx^2)^4} = \frac{\pi}{16a^{3/2}b^{5/2}}$$
4. (Whittaker and Watson, 6.22, 1) Show that if $a > 0$ then
$$\int_0^\infty \frac{\cos(x)}{x^2+a^2} dx = \frac{\pi}{2a} e^{-a}.$$
5. (Whittaker and Watson, 6.22) If the $\operatorname{Re} z > 0$ then
$$\int_0^\infty (e^{-t} - e^{-tz}) \frac{dt}{t} = \log z$$
6. (Whittaker and Watson, 6.24,2) If $0 \leq z \leq 1$ and $-\pi < a \leq \pi$ then
$$\int_0^\infty \frac{t^{z-1}}{t+e^{ia}} dt = \frac{\pi e^{i(z-1)a}}{\sin(\pi z)}$$
7. (Whittaker and Watson 6.24, 1, pg118) If $0 < a < 1$ show that
$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi \csc a\pi$$
8. (Whittaker and Watson, 6.24, 4)Show that if $-1 < p < 1$ and $-\pi < \lambda < \pi$ we have
$$\int_0^\infty \frac{x^{-p} dx}{1+2x\cos(\lambda)+x^2} = \frac{\pi}{\sin(p\pi)} \frac{\sin(p\lambda)}{\sin(\lambda)}$$
9. (Whittaker and Watson, 6.21, Example 3) Let n be a positive integer. Show that
$$\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$$
10. Answer only one of the following questions:
    (a) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$.
    (b) Evaluate the integral $\int_0^\infty \frac{x\sin x}{1+x^2} dx$.

11. Consider $I = \int_\gamma \frac{z^2 dz}{1+z^4}$, where γ is the contour shown below:

    [Insert image of contour here]

12. Evaluate I when $R < 1$.

13. Evaluate I when $R > 1$.

14. Discuss the results obtained when $R \to \infty$.

15. Choose a branch of $\sqrt{z^2-1}$ that is analytic on $\mathbb{C}\backslash\{z+0i \mid -1 \le z \le 1\}$ and has the value $\sqrt{3}$ at $z=2$. Evaluate $\int_\gamma \sqrt{z^2-1}dz$, where γ is a contour (not specified).

16. Evaluate the following integrals

    (a)

    (b) $\int_0^\infty \frac{\ln x}{4+x^2}dx$.

    (c) $\oint_\gamma \frac{e^z}{(z+1)(z-2i+1)}dz$, where γ is the ellipse given by $\frac{x^2}{4}+y^2=1$, with positive orientation (counterclockwise). Evaluate the following integrals:

    (d) $\int_0^\infty \frac{\ln x}{4+x^2}dx$.

17. Define the function $F(z) = \int_{|\zeta|=z} \frac{d\zeta}{\zeta(\zeta-z)(\zeta-z+1)}$. Determine the limit of $F(z)$ as $z \to 2$ from:

    (a) Inside the circle $|z|=2$.

    (b) Outside the circle $|z|=2$.

    (c) Is $F(z)$ continuous at $z=2$?

18. Evaluate:

    (a) $\int_{|z|=2} \frac{z+6}{z^2-2z-3}dz$.

    (b) $\int_0^\pi \frac{d\theta}{6-3\cos\theta}$.

19. Let $\gamma \subset \mathbb{C}$ be the square centered at $z=0$ with vertices at $z=\pm 2 \pm 2i$. Compute $\oint_\gamma \frac{z}{z^3+1}dz$, where γ is traversed once in the counterclockwise direction.

20. Evaluate $\int_{|z|=1} \frac{1-\cos z}{(e^z-1)\sin z}dz$.

21. Given $f(z) = \frac{1}{z} - \frac{2}{z^2}$, find $\int_C z^2 \exp(\frac{1}{z}) f(z)dz$, where C is the unit circle traversed counterclockwise.

22. Evaluate the integral $\int_0^\pi \frac{d\theta}{a+\sin^2(\theta)}$, where $a > 0$.

23. Evaluate the integral $\int_0^\pi \frac{dt}{5+4\cos t}$.

24. Assume that a, b, c are real numbers satisfying $ac-b^2 > 0$. Prove using residues that $\int_{-\infty}^\infty \frac{dx}{ax^2+2bx+c} = \frac{\pi}{\sqrt{ac-b^2}}$.

25. Show that $\int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\cos(2x)dx = \frac{2\pi}{\sinh(\pi)}$.

26. Compute the integral $I = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\sin x}{x(x^2+1)}dx$. Carefully justify all your steps.

27. Evaluate the real integral $\int_0^\infty \frac{x^2}{(x^2+1)^2}dx$ and justify all steps.

28. Use the theory of residues to evaluate the integral $\int_0^\infty \frac{\sqrt{x}dx}{x^2+4}$.

29. Use the method of contour integration and the calculus of residues to evaluate the integral $\int_0^\infty \frac{x^p}{1+x^2}dx$, where $-1 < p < 1$. Draw the relevant contour and justify all steps.

30. Compute $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{e^{ikx}}{\pi^2+1}dx$, where $k \in \mathbb{R}$.

31. Find the Fourier transform of the function $f(x) = e^{-x/4}$; i.e., find the function $\hat{f}(t)$ defined for all $t \in \mathbb{R}$ by $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{-itz} dz$.

32. Evaluate $\int_{-\infty}^{\infty} \frac{\exp(ikx)}{(1+x^2)} dx$ by contour integration (assume $k > 0$ real).

33. (a) $\int_0^{\infty} e^{-x^2} \cos(\lambda x) dx$.

    (b) $\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}$.

    It may be helpful to know that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.

34. Show by the method of complex contour integration that the following identities hold:

    (a) $\int_0^{\infty} \frac{\sin ax}{x(x^2+1)} dx = \frac{1-e^{-a}}{2}$.

    (b) $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \pi$.

35. Answer only one of the following questions:

    (a) Evaluate the integral $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx$, where $0 < \alpha < 1$.

    (b) Evaluate the integral $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx$, where $0 < \alpha < 1$.

36. Evaluate the integral $\int_0^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} dx$.

37. Evaluate the improper integral $\int_{-\infty}^{\infty} \frac{\cos(x) dx}{1+x^4}$. Include justifications for all steps in your calculation.

38. Use the theory of residues to evaluate the integral $\int_0^{\infty} \frac{\sqrt{x} dx}{x^2+4}$.

39. Evaluate $\int_0^{\infty} \frac{\sqrt{x}}{(1+x)^3} dx$ by contour integration.

40. Evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin^3 x}{\pi^3} dx$.

41. Show that $\int_{-\infty}^{+\infty} \frac{\cos x - \cos a}{x^2-a^2} dx = -\pi \frac{\sin a}{a}$, where $a \in \mathbb{R}^+$.

10 Rouche's Theorem and Argument Principal

Many of the problems were taken from University of New Mexico's Complex Analysis qualifying exams. The year they appeared has been lost with time (with legwork you can find them on the University of New Mexico Department of Mathematics and Statistics webpage).

1. Answer only one of the following questions:
   (a) State and prove Rouche's theorem.
   (b) State and prove the argument principle.
2. (New Mexico, Jan 1997) How many roots does $p(z) = z^4 + z + 1$ have in the first quadrant?
3. (New Mexico, Aug 1993) How many roots does $e^z - 4z^n + 1 = 0$ have inside the unit disc $|z| < 1$?
4. All of the problems below are essentially the same but appeared in different years in the forms they are given.
   (a) For each $n \in \mathbb{N}$ set $f_n(z) := \sum_{j=1}^{n} \frac{z^{-j}}{j!}$. For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $f_n(z)$ lie within $D(0, \rho)$.
   (b) Let $p_n(z) = 1 + \frac{z}{1!} + \frac{x^2}{2!} + \cdots + \frac{z^n}{n!}$ Prove that for every $R > 0$ there exists a positive integer $n(\hat{R})$ such that all roots of $p_n(z) = 0$ for $n \geq n(R)$ belong to the set $\{z \in \mathbb{C} || z| > R\}$.
   (c) For each $n \in \mathbb{N}$ set $p_n(z) := \sum_{j=0}^{n} (-1)^j \frac{z^{-2j}}{j!}$.
      i. For each fixed $n \in \mathbb{N}$, show that $p_n(z) = 0$ has precisely $2n$ solutions.
      ii. For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $p_n(z)$ lie within $D(0, \rho)$.
5. State Rouche's theorem and use it to determine how many roots of the polynomial $z^4 + 5z + 3$ lie inside:
   (a) the unit disc.
   (b) the annulus $1 < |z| < 2$.
6. State Rouche's theorem and find the number of zeros of the function $f(z) = 2z^5 + 7z^3 + z^2 - 3$ in the annulus $1 \leq |z| < 2$.
7. Show that the equation $(z - 2)^2 = e^{-z}$ has two distinct roots in the disc $|z - 2| \leq 1$.
8. State and prove Rouche's theorem. Use Rouche's theorem to show that there is $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$ the equation $z^3 - \epsilon z - 1 = 0$ has three distinct roots.
9. State some version of Rouche's theorem, and then use it to show that $f(z) := ze^{3-z} - 1$ has only one real zero in $D(0, 1)$.
10. State some version of Rouche's theorem, and then use it to show that all of the zeros for $f(z) := z^8 - 4z^3 + 10$ lie in the annulus $D(0, 2) \backslash \overline{D}(0, 1)$.
11. Prove the argument principle. How many roots does the polynomial $p(z) = z^4 + z + 1$ have in the first quadrant?
12. State Rouche's theorem and use it to show that all zeros of the polynomial $p(z) = z^4 + 6z + 3$ lie in the circle $|z| < 2$. How many zeros of $p(z)$ lie in the annulus $1 < |z| < 2$?
13. Let $f(z) = z^{10} + \frac{1}{2}z^6 + \frac{1}{100}\exp(z^5)$.
    (a) State Rouche's theorem.
    (b) Show that $f(z)$ has no zeros on $|z| = 1$.
    (c) How many zeros does $f(z)$ have inside $\{z : |z| = 1\}$? Justify your answer.

14. (a) Show that $z^5 - 15z + 1 = 0$ has one root in the disc $|z| < \frac{1}{8}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

    (b) Show that $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$, four roots in the annulus $\frac{3}{2} < |z| < 2$.

15. Let $f(z) = z^4 - 5z + 1$.

    (a) How many zeros does $f(z)$ have in the disc $\{z \in \mathbb{C} : |z| < 1\}$?

    (b) How many zeros does $f(z)$ have in the annulus $1 < |z| < 2$?

16. Find the number of zeroes of the equation $e^z - 4z^n + 1 = 0$ in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

11 Conformal Maps

Marsden-Hoffman, Basic Complex Analysis, 3rd Edition, Chapter 5 is helpful here. Particularly, the table on page 341.

1. (New Mexico, Summer 1999) Let $\mathcal{H}$ be the upper half complex plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let D be the unit disc $D = \{w : |w| < 1\}$. Show that the map $f(z) = w = (z - i)/(z + i)$ defines a bijection $\mathcal{H} \to D$.

2. Find the points where $w = f(z)$ is conformal if

   (a) $w = \cos(z)$

   (b) $w = z^5 - 5z$

   (c) $w = 1/(z^2 + 1)$

   (d) $w = \sqrt{z^2 + 1}$.

3. Find a conformal map of the strip $0 < \operatorname{Re} z < 1$ onto the unit disc $|w| < 1$ in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

4. Find the Möbius transformation that maps the left have plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ to the unit disc $\{w \in \mathbb{C} : |w| < 1\}$ and has $z = 0$ and $z = 1$ as fixed points.

5. Find a conformal map from the following regions onto the unit disc $D = \{z : |z| < 1\}$

   (a) $A = \{z : |z| < 2, \operatorname{Arg}(z) \in (0, \pi/4)\}$

   (b) $B = \{z : \operatorname{Re}(z) > 2\}$

   (c) $C = \{z : -1 < \operatorname{Re}(z) < 1\}$

   (d) $D' = \{z : |z| < 1 \text{ and } \operatorname{Re} z < 0\}$

6. Let D be the unit disc. Let $f : D \to D$ be a conformal map.

   (a) If $f(0) = 0$ show that $f(z) = \omega z$ for some $\omega \in \partial D$.

   (b) If $f(0) \neq 0$ show that there exists some $a \in D$ and $\omega \in \partial D$ such that
   $$f(z) = \omega \frac{z - a}{1 - \bar{a}z}.$$

7. This is a harder exercise, which is central to the theory of elliptic curves and modular forms. I promise you that this will appear over and over again in seminar talks.

   (a) Show that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $S(z) = -1/z$ and $T(z) = z + 1$ and hence has the presentation $\langle S, T : S^2 = 1, (ST)^3 = 1\rangle$.[1]

   (b) Compute the following stabilizers for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{C} \cup \{\infty\}$.

      i. $\mathrm{Stab}(i) = \{1, S\}$

      ii. $\mathrm{Stab}(e^{2\pi i/2}) = \{1, ST, (ST)^2\}$

      iii. $\mathrm{Stab}(e^{\pi i/3}) = \{1, TS, (TS)^2\}$

   (c) Show that a fundamental domain[2] for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on
   $$\mathcal{H} = \{x + iy \in \mathbb{C} \colon y > 0\}$$
   is the complement of the unit disc in a vertical strip of length 1 centered around zero in the upper half plane. In other words
   $$\Omega = \{z : |z| \geq 1 \text{ and } -1/2 \leq \operatorname{Re}(z) \leq 1/2\}.$$

[1] For every element in $\mathrm{PSL}_2(\mathbb{Z})$ you want to "reduce" it to the identity by succesively applying S or T. For $g \in \mathrm{PSL}_2(\mathbb{Z})$ it is helpful to keep track of $g(\infty) \in \mathbb{Q} \cup \{\infty\}$. If $g(\infty) = \infty$ you can do something, if $g(\infty) \in \mathbb{Q} \setminus \{0\}$ you can do something and if $g(\infty) = 0$ you can do something. Start with the easiest case. What are the elements where $g(\infty) = \infty$?

[2] A *fundamental domain* for an action $\Gamma \times X \to X$ is a closed subset $\Omega \subset X$ such that

i. $X = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$

ii. For all $\gamma \neq 1$ the set $\gamma(\Omega) \cap \Omega$ has empty interior.

12 Elliptic and Modular Functions

1. Show that
$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$$
is elliptic with period lattice Λ.

2. For a lattice $\Lambda \subset \mathbb{C}$ and $m \geq 3$ define $G_m = G_m(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-m}$.
   (a) Show that $\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty}(k+1)G_{k+2}z^k$.[3]
   (b) Conclude that
   $$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = O(z^2),$$
   as $z \to 0$, which shows that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is analytic at the origin of $\mathbb{C}$. Here $g_2 = 60G_4$ and $g_3 = 140G_6$.
   (c) Conclude that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is constant. (Hint: use that elliptic functions without poles are constant.)
   (d) Show the constant in the previous number is zero.

3. The zeros of $\wp(z) - c$ are simple with precisely double zeros at the points congruent to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$. (Hint: what are the zeros of $\wp'(z)$ and what does this mean?)

[3] You may need to use that you can interchange some series. If $f_n(z) = \sum a_j^{(n)} z^j$ and $A_j = \sum_{n=0}^{\infty} a_j^{(n)}$ converges then $\sum_{n=0}^{\infty} f_n(z) = \sum_{j=0}^{\infty} A_j z^j$.

13 Riemann Surfaces

This uses some basic properties of Riemann Surfaces.

1. (a) Show that every automorphism of $\mathbb{C}$ extends to an automorphism of $\mathbb{P}^1$.
   (b) Show that $\mathrm{Aut}(\mathbb{C}) := \{az+b : a \in \mathbb{C}^\times \text{ and } b \in \mathbb{C}\}$ (This sometimes called the one dimensional affine linear group and is denoted $\mathrm{AL}_1(\mathbb{C})$.).
2. Show that $\mathbb{C}$ is not conformally equivalent to $D = \{z \in \mathbb{C} : |z| < 1\}$.
3. Show that $\mathrm{Aut}(H) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\}$ (This is sometimes called the two dimensional projective special linear groups with coefficients in $\mathbb{R}$, and is denoted $\mathrm{PSL}_2(\mathbb{R})$).

14 Infinite Products

1. Show the Gauss formula for the Gamma function:
$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)}.$$
(Take the definition of the Gamma function to be from its product formula).

2. Verify that $F(z) = \int_0^\infty t^{z-1}e^{-t}dt$ and $\Gamma(z)$ (via $1/\Gamma(z)$ being defined by the product formula) satify the hypotheses of Weilandt's Theorem. In particular that $F(z)$ and $\Gamma(z)$ are bounded when $1 < \operatorname{Re} z < 2$.

3. Show that $\int_0^{2\pi} \log|1-e^{i\theta}|d\theta = 0$.

4. (New Mexico, Jan 2006) Consider $f(z) = \prod_{n=1}^\infty (1 - z/n^3)$. What is the order of $f(z)$?

5. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of finite order ρ. Show that
$$\rho = \liminf_{n\to\infty} \frac{\log(n)}{\log|a_n|^{-1/n}}.$$

15 The Big Picard Theorem

1. (a) Prove the Casorati-Weiestrass Theorem: Let $f(z)$ is analytic in a punctured disc of radius R at the origin. If $f(z)$ has an essential singularity at $z = 0$ show that for every r with $0 < r < R$ the set $f(D_r(0) \setminus \{0\})$ is dense in $\mathbb{C}$. (This is a corollary of Big Picard).
   (b) Let p be a polynomial. Show that there exists infinitely many z_j such that $p(z_j) = e^{z_j}$.

2. The following exercise is intended to introduce you to the j function which plays a role in the proof of the Big Picard Theorem from class.
   Let H be the upper-half plane. A **modular form** of weight k and level $N = 1$ is a function $f : H \to \mathbb{C}$ such that
$$f(\frac{az+b}{cz+d}) = (cz+d)^{-2k} f(z). \tag{1}$$
   for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
   (a) Let M_k denote the collection of modular forms of weight k and level 1. Show that $M = \bigoplus_{k\geq 0} M_k$ is a graded ring (i.e. that $M_{k_1}M_{k_2} \subset M_{k_1+k_2}$.
   (b) Show that $G_{2k}(\frac{az+b}{cz+d}) = (cz+d)^{2k}G_{2k}(z)$ has weight $2k$ (Hint: check this on the generators of $\mathrm{SL}_2(\mathbb{Z})$.)
   Using the first part conclude that the we have the following modular forms of the indicated weights:
      i. $g_2(\tau) = 60G_4(\tau)$, $k = 4$
      ii. $g_3(\tau) = 140G_6(\tau)$, $k = 6$
      iii. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, $k = 12$
      iv. $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$, $k = 0$

3. Explain in words the ideas that go into the proof of Montel's Theorem in Green and Krantz (page 193). How is Arzela-Ascoli used?

4. Let $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. What is the universal cover of X? What is group of deck transformations for this cover?

5. Use Van Kampen's theorem to rigorously compute $\pi_1(\mathbf{P}^1 \setminus \{p_1, \ldots, p_r\}, z_0)$ for arbitrary r. (Hint: apply Van Kampen to open sets U, V where $U \cap V$ is simply connected).