Dupuy - Homework 06 solns - 121 Fall 2016

Problem 1:

$$\begin{cases} P = 2pq + 2pr + 2rq \\ p + q + r = 1 \\ p, q, r \geq 1 \end{cases}$$

$$\nabla P = \left(\frac{\partial P}{\partial p}, \frac{\partial P}{\partial q}, \frac{\partial P}{\partial r} \right)$$

$$= (2q + 2r, 2p + 2r, 2p + 2q)$$

$$\nabla g = (1, 1, 1)$$

$$\begin{cases} \nabla P = \lambda \nabla g \\ p + q + r = 1 \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} (2q + 2r, 2p + 2r, 2p + 2q) = \lambda (1, 1, 1) \\ p + q + r = 1 \end{cases}$$

$$\begin{cases} 2q+2r = \lambda & (1) \\ 2p+2r = \lambda & (2) \\ 2p+2q = \lambda & (3) \\ p+q+r = 1 & (4) \end{cases}$$

$$(1)-(2) \Rightarrow \begin{cases} 2p-2q = 0 \\ 2p+2q = \lambda \\ p+q+r = 1 \end{cases}$$

$$\Rightarrow 4p = \lambda \Rightarrow p = \lambda/4.$$

<u>back substitute</u>:

- $2q = \lambda - 2p \Rightarrow 2q = \lambda - \lambda/2 = \lambda/2$
  $\Rightarrow q = \lambda/4.$

- similarly $r = \lambda/4.$

$$\lambda/4 + \lambda/4 + \lambda/4 = 1 \Rightarrow \lambda/4 = 1/3$$

$$\Rightarrow p = q = r = 1/3.$$

to check that this is a local max lets compare against another point:

$$P\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = 2\left(\frac{1}{9}\right)+2\left(\frac{1}{9}\right)+2\left(\frac{1}{9}\right) = \frac{6}{9} = \frac{2}{3}$$

$$P(1,0,0) = 0$$

Since the constraint is a compact set this is a maximum.

Problem 2:

$$\begin{cases} f(x,y,z,t) = y+2z+t \\ g(x,y,z,t) = x^2+y^2+z^2+t^2 = 1. \end{cases}$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (0,1,2,1) = \lambda(2x,2y,2z,2t) \\ x^2+y^2+z^2+t^2 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 0 = \lambda 2x & (1) \\ 1 = \lambda 2y & (2) \\ 2 = \lambda 2z & (3) \\ 1 = \lambda 2t & (4) \\ x^2+y^2+z^2+t^2 = 1. & (5) \end{cases}$$

From the first equation we have

$$0 = \lambda(2x)$$

$\Rightarrow 2x = 0$ $\underline{\text{or}}$ $\lambda = 0$.

If $\lambda = 0$ then (2) $\Rightarrow$ $1 = 0$. which is a contradiction.

We can suppose now $\lambda \neq 0$ & $x = 0$. This gives

$$\begin{cases} 1 = 2\lambda y & (1') \\ 2 = 2\lambda z & (2') \\ 1 = 2\lambda t & (3') \\ y^2 + z^2 + t^2 = 1. & (4') \end{cases}$$

Eqn (2') $\Rightarrow$ $2 - 2\lambda z = 0$ $\cancel{\Leftrightarrow \lambda(1-z) = 0}$

~~since $\lambda \neq 0 \Rightarrow z = 1$.~~ $z = 1/\lambda$

Eqn (1') $\Rightarrow$ $y = 1/2\lambda$. Similarly, $t = 1/2\lambda$.

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$$

$$\Rightarrow \quad \lambda^2 = \frac{1}{4} + 1 + \frac{1}{4} = 3/2.$$

$$\lambda = \pm \frac{\sqrt{3}}{\sqrt{2}}.$$

$\underline{\lambda = +\frac{\sqrt{3}}{\sqrt{2}}}$: $\quad (x,y,z,t) = \left(0, \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\underline{\lambda = -\frac{\sqrt{3}}{\sqrt{2}}}$: $\quad (x,y,z,t) = \left(0, \frac{-1}{\sqrt{3}}, \frac{-\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right).$

$$f\left(0, \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + 2\left(\frac{\sqrt{2}}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} = \underbrace{\frac{2+2\sqrt{2}}{\sqrt{3}}}_{\text{max}}$$

$$f\left(0, \frac{-1}{\sqrt{3}}, \frac{-\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \underbrace{\frac{-2-2\sqrt{2}}{\sqrt{3}}}_{\text{min.}}$$

Problem 3:

$$\begin{cases} f(x,y,z) = x^2+y^2+z^2 \\ x-y=1 \\ y^2-z^2=1 \end{cases}$$

$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g = (1, -1, 0)$$
$$\nabla h = (0, 2y, -2z)$$

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ x-y=1 \\ y^2-z^2=1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2x = \lambda & (1) \\ 2y = -\lambda + \mu 2y & (2) \\ 2z = -2\mu z & (3) \\ x-y=1 & (4) \\ y^2-z^2=1 & (5) \end{cases}$$

From (3) we have

$$\lambda z = -2\mu z$$

$$\Rightarrow 2z + 2\mu z = 0$$

$$\parallel$$

$$2z(\mu+1) = 0$$

so $\mu = -1$ or $z = 0$.

<u>Suppose $z = 0$</u>:

$z = 0 \underset{(5)}{\Rightarrow} y^2 = 1 \Rightarrow y = \pm 1.$

<u>if $y = +1$</u>: (4) $\Rightarrow x = 1 + y$

$\Rightarrow x = 2$

$(2, 1, 0)$

<u>if $y = -1$</u>: (4) $\Rightarrow x = 0$ (similar)

$(0, -1, 0)$.

<u>Suppose $\mu = -1$</u>:

(1) $\Rightarrow x = \lambda/2$

(2) $\Rightarrow 4y = -\lambda \Rightarrow y = -\lambda/4$

$$(4) \Rightarrow \lambda/2 - \left(\frac{-\lambda}{4}\right) = 1$$

$$\Leftrightarrow 2\lambda/4 + \lambda/4 = 1$$

$$\Leftrightarrow \frac{3\lambda}{4} = 1 \Rightarrow \lambda = \frac{4}{3}.$$

$$\therefore \begin{cases} x = \lambda/2 = \frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3} \\ y = -\lambda/4 = -1/3. \end{cases}$$

$$(5) \Rightarrow y^2 - z^2 = 1$$

$$\Leftrightarrow z^2 = \left(\frac{-1}{3}\right)^2 - 1 = \frac{1}{9} - 1 = -\frac{8}{9}.$$

$$\Rightarrow \underbrace{z = \pm\sqrt{-8/9}}$$

these are imaginary so we throw ~~them~~ them out.

$\therefore \mu = -1$ gives no solutions.

This means the only critical points are

$(2,1,0)$

$(0,-1,0)$.

$f(2,1,0) = 2^2+1^2+0^2 = 5$, "max"

$f(0,-1,0) = 0^2+(-1)^2+0^2 = 1$, min.

Technical Note: $(2,1,0)$ only gives a local min too. We can't apply the extreme value theorem because the set

$$\{(x,y,z) : x-y=1 \ \&\ y^2-z^2=1\} = S$$

is <u>not compact</u>!!! To see this is not a max lets parametrize this set:

$$x-y=1 \implies x=y+1$$
$$y^2-z^2=1 \implies z=\pm\sqrt{y^2-1}, \ |y|\geq 1.$$

so we have

$$\alpha(y) = \underbrace{\left(y+1, y, +\sqrt{y^2-1}\right)}_{\text{just the positive part of } z.}, \quad y \geq 1$$

plugging this in gives

$$f(\vec{x}(y)) = x(y)^2 + y^2 + z(y)^2$$
$$= (y+1)^2 + y^2 + y^2 - 1$$
$$= 3y^2 + 2y, \quad |y| \geq 1.$$

You can see this function is unbounded as $|y| \to \infty$.

Here is the graph of what is going on: