SOLUTIONS

Math 264 — Spring 2010 — Test 2

April 15, 2010

Remember to show your work. Take your time and relax.

1. Compute the $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ for

   (a) $f(x,y) = e^{xy}$

   (b) $f(x,y) = x^3 + 4xy + y^2$.

(a) $f_x = ye^{xy}$, $f_y = xe^{xy}$, $f_{xx} = y^2 e^{xy}$, $f_{yy} = x^2 e^{xy}$

$f_{xy} = (xy + 1)e^{xy}$

(b) $f_x = 3x^2 + 4y$, $f_{xx} = 6x$, $f_{xy} = 4$

$f_y = 4x + 2y$, $f_{yy} = 2$

Common mistake: $\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)$

2. Find the plane tangent to the graph of $f(x,y) = x^2 + 2x - y^2 + 6xy$ at the point $(1,1)$.

$z = f(1,1) + \nabla f(1,1) \cdot (x-1, y-1)$ $+5$

$= 8 + (2x+2+6y, -2y+6x)\Big|_{(x,y)=(1,1)} \cdot (x-1, y-1)$ $+3$

$= 8 + (10, 4) \cdot (x-1, y-1)$

$= 8 + 10(x-1) + 4(y-1)$. $+2$

①

3. Compute $D_{\mathbf{u}}f(x_0, y_0)$ in the following cases. If $\mathbf{u}$ is not initially given as a unit vector, please normalize it.

   (a) $f(x, y) = e^{x+y}$, $(x_0, y_0) = (1, 2)$, $\vec{u} = (1, 0)$.

   (b) $f(x, y) = e^{x^2 y}$, $(x_0, y_0) = (1, 3)$, $\vec{u} = (1, 3)$

+5

(a) $(D_{\vec{u}}f)(x_0, y_0) = \frac{\partial f}{\partial x}(1,2) = e^{1+2} = e^3$

(b) $\nabla f = (2xy e^{x^2 y}, x^2 e^{x^2 y})$, +5

$\nabla f(1,3) = (6e^3, 1e^3)$, $\vec{u} = \frac{(1,3)}{\sqrt{1+9}}$

$\nabla f(1,3) \cdot \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{10}}(6e^3 + 3e^3) = \frac{33e^9}{10}$.

4. (a) What unit vector $\mathbf{u}$ maximizes $(D_{\mathbf{u}}f)(x_0, y_0)$?

   (b) Explain/Prove this is true that the statement you gave above is true.

   (c) Find the direction of maximum increase of the function $x^2 + 2xy - 1$ at the point $(-1, 1)$.

(a) +3 $\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$

(b) +4 $D_{\vec{u}}f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$

$= |\nabla f(x_0,y_0)|\,|\vec{u}| \cos\theta$

$\theta = 0$ maximizes the result

$\Rightarrow \vec{u}$ & $\nabla f(x_0,y_0)$ should be $\parallel$.

Since $\vec{u}$ needs to be a unit vector $\vec{u} = \frac{\nabla f}{\|\nabla f\|}$.

(c) $\nabla f = (2x+2y, 2x)$

$\nabla f(-1,1) = (0, -2)$, +3

$\boxed{\vec{u} = (0, -1)}$

5. If $z = y + f(x^2 - y^2)$ where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x.$$

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y\left(\cancel{f'(x^2-y^2)}(2\cancel{x})\right) + x\left(1 + \cancel{f'(x^2-y^2)(-2y)}\right)$

only term left over

$= x$.

problem assuming the conclusion.

6. Evaluate the limit or show it does not exist

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+4y^3}$.

(b) $\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y}$.

+5

(a) Approach along (t,t): $\frac{t^2 t}{t^3+4t^3} = \frac{1}{5} \to \frac{1}{5}$ as $t\to 0$

Approach along $(t,0)$: $\frac{0}{t^3+4(0)} = 0 \to 0$ as $t \to 0$

Since we approached along two different directions & got two diff values the limit doesn't exist. +5

(b) $\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y} = \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x^2+xy+y^2)}{\cancel{(x-y)}} = \lim_{(x,y)\to(1,1)} x^2+xy+y^2$

$= 3.$

7. Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ using the chain rule.

(a) $f(x,y) = e^{xy}$, $x = s\cos t$ $y = s\sin t$.

(b) $f(x,y) = x^2 - y^2$, $x = e^{st}$, $y = e^{-t^2}$.

+5

(a) $\frac{\partial f}{\partial s} = \boxed{\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}} = (ye^{xy})(\cos t) + (xe^{xy})\sin t$

$\frac{\partial f}{\partial t} = (ye^{xy})(-s\sin t) + (xe^{xy})(s\cos t)$

$\boxed{\text{replace } x \,\&\, y \text{ w/ fns of } s \,\&\, t}$

(b) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = (2x)(te^{st}) + (-2y)(0)$

$= 2x(te^{st})$

$\frac{\partial f}{\partial t} = (2x)(se^{st}) + (-2y)(-2te^{-t^2})$

8. Find the critical points of $x^2 - 4y + 2$ and determine if they are maxima, minima or saddles.

(a) There are no critical points! +5

(b) $x^2 - y^2 - 2x - 2y$. $(1,-1)$ saddle pt. +5

9. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin. (Hint: that surface is a constraint and you want to optimize the distance to the origin. Tip: Optimize the distance squared rather than the straight up distance.)

$$\begin{cases} f(x,y,z) = x^2+y^2+z^2 \\ g(x,y,z) = xy^2z^3 - 2, \end{cases} \quad \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$
$$\nabla g = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k},$$

$$\begin{cases} 2x = \lambda y^2 z^3 \\ 2y = \lambda 2xyz^3 \\ 2z = \lambda 3xy^2z^2 \\ xy^2z^3 = 2 \end{cases}$$

+10

SOLVE THIS SYSTEM (on attached)

10. Suppose f is differentiable. Prove that vectors tangent to the level sets $L_C = \{(x,y) : f(x,y) = C\}$ are perpendicular to ∇f. (Hint: suppose that $\vec{r}(t) = (x(t), y(t))$ parametrizes L_C).

+5

$$f(x(t), y(t)) = C \implies \frac{d}{dt}\left[f(x(t), y(t))\right] = 0$$

By the chain rule

$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$

+5

$$\implies \nabla f(\vec{r}(t)) \ \& \ \vec{r}\,'(t) \text{ are perpendicular.}$$

Extra Credit: Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are both continuous at the point $\mathbf{r}_0 \in \mathbb{R}^2$ then function $(f+g) : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $\mathbf{r}_0$. (Using the ε-δ definition of continuity.)

$$|f(r) + g(r) - f(r_0) - g(r_0)| = |(f(r) - f(r_0)) + g(r) - g(r_0)|$$
$$\le |f(r) - f(r_0)| + |g(r) - g(r_0)|$$

Since f & g are continuous there exists some δ_1 & δ_2 such that

$$\|r - r_0\| < \delta_1 \implies |f(r) - f(r_0)| < \varepsilon/2$$

&

$$\|r - r_0\| < \delta_2 \implies |g(r) - g(r_0)| < \varepsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ so that

$$\|r - r_0\| < \delta \implies \|f(r) - f(r_0)\| + \|g(r) - g(r_0)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

$$\begin{cases} 2x = \lambda y^2 z^3 \\ 2y = 2\lambda x y z^3 \\ 2z = 3\lambda x y^2 z^2 \\ x y^2 z^3 = 2 \end{cases} \implies \begin{cases} 2x^2 = \lambda x y^2 z^3 = 2\lambda \\ 2y^2 = 2\lambda x y^2 z^3 = 4\lambda \\ 2z^2 = 3\lambda x y^2 z^3 = 6\lambda \end{cases}$$

$$\implies \boxed{\begin{aligned} x^2 &= \frac{\cancel{3}}{2}\lambda \\ y^2 &= \lambda \\ z^2 &= \frac{3\lambda}{2} \end{aligned}}$$

$$x = \pm\sqrt{\lambda} = \pm\mu$$

$$y = \pm\sqrt{\lambda/2} = \pm\mu/\sqrt{2}$$

$$z = \pm\sqrt{\tfrac{3}{2}\lambda} = \pm\sqrt{3}\,\mu$$

Let $\sqrt{\lambda} = \mu$.

$$\implies x y^2 z^3 = 2$$

$$(\pm\mu)\,\lambda/2\,(\pm\mu^3(\sqrt{3})^3) = 2$$

$$\implies \pm\frac{3}{2}\sqrt{3}\,\lambda^3 = 2$$

$$\implies \lambda^3 = \pm\frac{4}{3\sqrt{3}} \implies \boxed{\lambda = \pm\left(\frac{4}{3\sqrt{3}}\right)^{1/3}}.$$

~~If $\lambda = \pm\left(\frac{4}{3\sqrt{3}}\right)^{1/3}$ then~~

λ must be positive since y needs to be a real number.

$$\boxed{\begin{aligned} &x = \pm\left(\frac{4}{3\sqrt{3}}\right)^{1/6},\quad y = \pm\left(\left(\frac{4}{3\sqrt{3}}\right)^{1/3}\Big/2\right)^{1/2} \\ &z = \pm\sqrt{3\left(\frac{4}{3\sqrt{3}}\right)^{1/2}} \end{aligned}}$$

6 possible points.