

DUPUY - HW08 - FALL 2016

Problem 1a

$$\vec{F}'(r, \theta) = \begin{bmatrix} \cosh(\theta) & r \sinh(\theta) \\ \sinh(\theta) & r \cosh(\theta) \end{bmatrix}$$

$$\begin{aligned} \frac{J(xy)}{J(r, \theta)} &= r \cosh(\theta)^2 - r \sinh(\theta)^2 \\ &= r. \end{aligned}$$

# Problem 1b

$$\vec{F}'(p, \theta, \phi)$$

$$= \begin{bmatrix} a \sin(\phi) \cos(\theta) & -a p \sin(\phi) \sin(\theta) & a p \cos(\phi) \cos(\theta) \\ b \sin(\phi) \sin(\theta) & b p \sin(\phi) \cos(\theta) & b p \cos(\phi) \sin(\theta) \\ c \cos(\phi) & 0 & -c \sin(\phi) \end{bmatrix}$$

Jacobian Matrix

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \det(\text{Jacobian Matrix})$$

$$= a \sin(\phi) \cos(\theta) (bp \sin(\phi) \cos(\theta) (-c \sin(\phi)))$$

$$+ a p \sin(\phi) \sin(\theta) (b \sin(\phi) \sin(\theta) (-c \sin(\phi)) - bp \cos(\phi) \sin(\theta) (c \cos(\phi)))$$

$$+ ap \cos(\phi) \cos(\theta) (0 - bp \sin(\phi) \cos(\theta) (c \cos(\phi)))$$

$$= abc p^2 \sin(\phi) [-\sin(\phi)^2 \cos(\theta)^2 - \sin(\phi)^2 \sin(\theta)^2 - \cos(\phi)^2 \sin(\theta)^2 - \cos(\phi)^2 \cos(\theta)^2]$$

$$= -abc p^2 \sin(\phi) [\sin(\phi)^2 (\cos(\theta)^2 + \sin(\theta)^2) + \cos(\phi)^2 (\sin(\theta)^2 + \cos(\theta)^2)]$$

$$= -abc\rho^2 \sin(\phi) [\sin(\phi)^2 + \cos(\phi)^2]$$

$$= -abc\rho^2 \sin(\phi) . //$$

## Problem 2

$$\int \int \int_E dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \left| \frac{\partial(x,y,z)}{\partial(\rho, \phi, \theta)} \right| d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 abc \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= abc \left( \text{vol}(\text{Unit Ball}) \right)$$

$$= abc \left( \frac{4}{3} \pi (1)^3 \right) = \frac{4}{3} \pi abc. //$$

### Problem 3

$$A = \iiint_E x^2 \rho(x,y,z) dV$$

$$B = \iiint_E y^2 \rho(x,y,z) dV$$

$$C = \iiint_E z^2 \rho(x,y,z) dV$$

compute  
these separately

Note that

$$I_x = B + C, \quad I_y = A + C, \quad I_z = A + B.$$

we will show

$$\begin{cases} A = \frac{\pi R^4 h k}{20} \\ B = \frac{\pi R^4 h k}{20} \\ C = \frac{\pi R^2 h^2 k}{5} \end{cases}$$

Computation for A:

$$\iiint_E x^2 \rho(x,y,z) dV$$

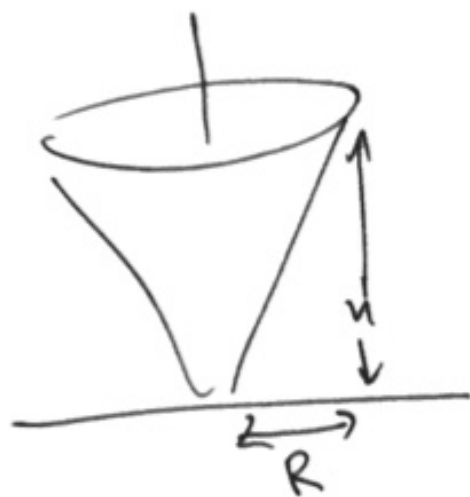
$$= k \iiint_E x^2 dV$$

$$= k \int_0^{2\pi} \int_0^R \int_{\frac{h}{R}r}^h r^2 \cos^2(\theta)^2 r dz dr d\theta$$

$$= k \int_0^{2\pi} \int_0^R \left(h - \frac{h}{R}r\right) r^3 \cos^2(\theta)^2 dr d\theta$$

$$= kh \left[ \int_0^{2\pi} \int_0^R r^3 \cos^2(\theta)^2 dr d\theta \right] + \frac{-kh}{R} \left[ \int_0^{2\pi} \int_0^R r^4 \cos^2(\theta)^2 dr d\theta \right]$$

$$= kh \left[ \left( \int_0^{2\pi} \cos^2(\theta)^2 d\theta \right) \left( \int_0^R r^3 dr \right) \right]$$



$$-\frac{1}{R} \left( \int_0^R \cancel{5}^4 dr \right) \left( \int_0^{2\pi} \cos(\theta)^2 d\theta \right)$$

$$= kh \left[ \left( \frac{\pi}{4} \cdot 4 \right) \left( \frac{R^4}{4} \right) - \frac{1}{R} \left( \frac{\pi}{4} \cdot 4 \right) \left( \frac{R^5}{5} \right) \right]$$

$$= kh \pi R^4 \left( \frac{1}{4} - \frac{1}{5} \right) = kh \pi R^4 \left( \frac{5}{20} - \frac{1}{20} \right)$$

$$= \frac{kh \pi R^4}{20}$$

$$\boxed{A = \frac{kh \pi R^4}{20}}$$



Computation for B:

$$\iiint_E y^2 \rho(x, y, z) dV = \text{same}$$

$$A = B = \frac{4\pi R^4}{20}$$

QED

Computation for C:

$$C = \iiint z^2 \rho(x,y,z) dV$$

$$= k \int_0^{2\pi} \int_0^R \int_{\frac{h}{R}r}^h z^2 r dz dr d\theta$$

$$= k \int_0^{2\pi} \int_0^R \left( \frac{h^3}{3} - \frac{1}{3} \left( \frac{h}{R} r \right)^3 \right) r dr d\theta$$

$$= k \cdot 2\pi \int_0^R \left( \frac{h^3}{3} r - \frac{h^3}{3} \frac{r^4}{R^3} \right) dr$$

$$= k \cdot 2\pi \left( \frac{h^3}{3} \frac{R^2}{2} - \frac{h^3}{3} \frac{1}{R^3} \frac{R^5}{5} \right) \quad \frac{\pi R^2 h^2 k}{5}$$

$$= k (2\pi) \left( \frac{h^3}{3} \right) \left( \frac{R^2}{2} - \frac{R^2}{5} \right) \quad //$$

$$= \frac{2\pi k h^3}{3} R^2 \left( \frac{5}{10} - \frac{2}{10} \right) = \boxed{\frac{2\pi k h^3 R^2}{10} = C}$$

Final Computations:

$$I_x = B + C$$

$$= \frac{\pi R^4 h k}{20} + \frac{\pi R^2 h^2 k}{5},$$

$$I_y = A + C$$

$$= (\text{Same as above}) = \frac{\pi R^4 h k}{20} + \frac{\pi R^2 h^2 k}{5},$$

$$I_z = A + B$$

$$= 2 \left( \frac{\pi R^4 h k}{20} \right) = \frac{\pi R^4 h k}{10}.$$

Wikipedia:

$$\begin{cases} I_x = I_y = \frac{3h}{20} (R^2 + 4h^2) m \\ I_z = \frac{3h R^2}{10} m \end{cases}$$

Our Answer:

$$\begin{cases} I_x = I_y = \frac{\pi R^4 h k}{20} + \pi \frac{R^2 h^3}{5} k \\ I_z = \cancel{\frac{\pi R^4 h k}{10}} \end{cases}$$

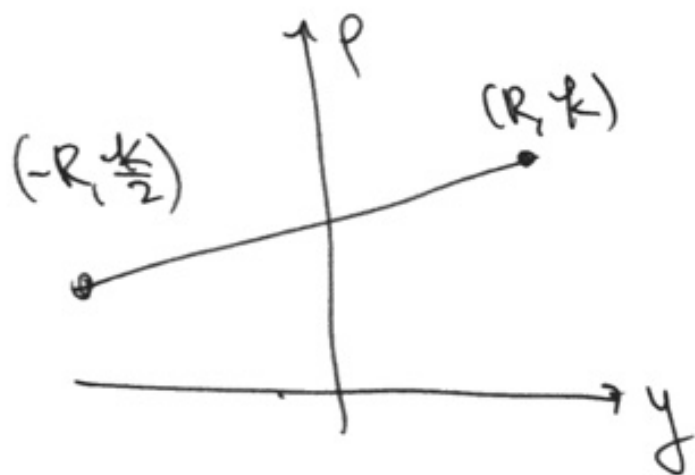
Relationship:

$$\text{Density} = \frac{\text{mass}}{\text{volume}} = \frac{m}{\pi R^2 \frac{h}{3}} = \frac{3m}{\pi R^2 h} = k.$$

$$\begin{aligned} (\text{our } I_x) &= \frac{\pi R^2 h}{20} k (R^2 + 4h^2) \\ &= \frac{\pi R^2 h}{20} \frac{3m}{\pi R^2 h} (R^2 + 4h^2) \\ &= \frac{3m}{20} (R^2 + 4h^2) = (\text{wikipedia's } I_x) \end{aligned}$$

# Problem 4

The density  $\rho$  is a line so we compute it...



$$\text{slope} = \frac{k - k/2}{R - (-R)} = \frac{k/2}{2R} = \frac{k}{4R}.$$

Point-slope:

$$(p - k) = \frac{k}{4R}(y - R)$$

$$\Rightarrow p = \frac{k}{4R}y - \frac{k}{4} + k = \frac{k}{4R}y + \frac{3k}{4}.$$

$$\rho(x, y, z) = \frac{k}{4R}y + \frac{3k}{4} \quad \int \text{Density function.}$$

We now compute the mass,

$$m = \iiint_E \rho(x, y, z) dV = \star$$

$$\star = \iiint \left( \frac{k}{4R} y + \frac{3k}{4} \right) dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left( \frac{k}{4R} \rho \sin(\phi) \sin(\theta) + \frac{3k}{4} \right) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{k}{4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left[ \frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R} + 3 \rho^2 \sin(\phi) \right] d\rho d\phi d\theta$$

$$= \frac{k}{4} [A + B], \quad \text{where}$$

$$A = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R} d\rho d\phi d\theta$$

$$= \frac{1}{R} \left( \int_0^R \rho^3 d\rho \right) \left( \int_0^{\pi/2} \sin(\phi)^2 d\phi \right) \left( \int_0^{2\pi} \sin(\theta) d\theta \right)$$

$$= \frac{1}{R} \left( \frac{R^4}{3} \right) \left( \frac{\pi}{4} \right) \cancel{(2\pi)} (0)$$

$$= 0$$

$$\star = \iiint \left( \frac{k}{4R} y + \frac{3k}{4} \right) dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left( \frac{k}{4R} \rho \sin(\phi) \sin(\theta) + \frac{3k}{4} \right) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{k}{4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left[ \frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R} + 3 \rho^2 \sin(\phi) \right] d\rho d\phi d\theta$$

$$= \frac{k}{4} [A + B], \quad \text{where}$$

$$A = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R} d\rho d\phi d\theta$$

$$= \frac{1}{R} \left( \int_0^R \rho^3 d\rho \right) \left( \int_0^{\pi/2} \sin(\phi)^2 d\phi \right) \left( \int_0^{2\pi} \sin(\theta) d\theta \right)$$

$$= \frac{1}{R} \left( \frac{R^4}{3} \right) \left( \frac{\pi}{4} \right) \cancel{(2\pi)} (0)$$

$$= 0$$

$$B = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 3\rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= 3 \frac{4\pi R^3}{6}$$

$$= 2\pi R^3$$

(because this is just  $\frac{1}{2}$  the volume of the sphere)

$$\Rightarrow m = \frac{k}{4} (A+B) = \frac{k}{4} (0 + 2\pi R^3) \\ = \frac{k\pi R^3}{2}$$

$$\boxed{m = \frac{k\pi R^3}{2}}$$

Note:

$$\frac{\frac{k}{2} \left( \frac{\pi R^3}{3} \right)}{2} \leq m \leq \frac{k \left( \frac{\pi R^3}{3} \right)}{2}$$

half density.

full density



Moments:

$$m_x = \iiint_E x \rho(x, y, z) dV$$

$$= \boxed{0} \text{ by symmetry.}$$

$$m_y = \iiint_E y \rho(x, y, z) dV$$

$$= \iiint_E y \left( \frac{k}{4R} y + \frac{3k}{4} \right) dV$$

$$= \frac{k}{4} \left( \iiint_E \frac{y^2}{R} dV + \iiint_E 3y dV \right)$$

$$= \frac{k}{4} (A + B)$$

where

$$A = \iiint_E \frac{y^2}{R} dV, \quad B = \iiint_E 3y dV.$$

$$A = \iiint \frac{y^2}{R} dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left( \frac{\rho \sin(\phi) \sin(\theta)}{R} \right)^2 \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{\rho^4 \sin(\phi)^3 \sin(\theta)^2}{R} d\rho d\phi d\theta$$

$$= \frac{1}{R} \left[ \int_0^R \rho^4 d\rho \right] \left[ \int_0^{\pi/2} \sin(\phi)^3 d\phi \right] \left[ \int_0^{\pi/2} \sin(\theta)^2 d\theta \right]$$

[see side work]

$$= \frac{1}{R} \left( \frac{R^5}{5} \right) \left( \frac{2}{3} \right) \left( \frac{\pi}{4} \right)$$

$$= \frac{2\pi R^4}{60} = \frac{\pi R^4}{30}$$

Side work (for my)

$$\int_0^{\pi/2} \sin(\phi)^3 d\phi = \int_0^{\pi/2} (1 - \cos(\phi)^2) \sin(\phi) d\phi$$

$$> \int_0^{\pi/2} \sin(\phi) d\phi - \int_0^{\pi/2} \cos(\phi)^2 \sin(\phi) d\phi$$

$$= \int_0^{\pi/2} \frac{d}{d\phi} [-\cos(\phi)] d\phi$$

$$+ \int_0^{\pi/2} \frac{d}{d\phi} \left[ \frac{\cos(\phi)^3}{3} \right] d\phi$$

$$= \left[ -\cos(\phi) \right]_0^{\pi/2} + \frac{\cos(\phi)^3}{3} \Big|_0^{\pi/2}$$

$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

$$B = \iiint_E z_y dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 3 \rho \sin(\phi) \sin(\theta) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 3 \rho^3 \sin(\phi)^2 \sin(\theta) d\rho d\phi d\theta$$

$$= 3 \left( \int_0^{2\pi} \sin(\theta) d\theta \right) \left( \int_0^{\pi/2} \sin(\phi)^2 d\phi \right) \left( \int_0^R \rho^3 d\rho \right)$$

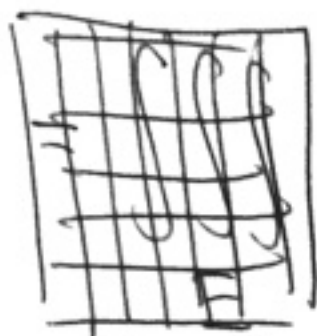
$$= 0$$

$$\Rightarrow \boxed{m_y = \frac{k}{4} \left( \frac{\pi R^4}{30} + 0 \right)}$$

$$= \frac{\pi R^4 k}{120}$$

$$m_z = \iiint_E z \rho(x, y, z) dV$$

$$= \iiint_E z \left( \frac{k}{4} \left( \frac{y^2}{R} + 3 \right) \right) dV$$



$$= \frac{k}{4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho \cos(\phi) \left( \frac{\rho \sin(\phi) \sin(\theta)}{R} + 3 \right) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{k}{4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \left[ \rho^4 \frac{\cos(\phi) \sin(\phi)^2 \sin(\theta)}{R} + 3 \rho^3 \cos(\phi) \sin(\phi) \right] d\rho d\phi d\theta$$

$$= \frac{k}{4} (A+B)$$

$$\begin{aligned}
 A &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{\rho^4 \cos(\phi) \sin(\phi)^2 \sin(\theta)}{R} d\rho d\phi d\theta \\
 &= \frac{1}{R} \left( \int_0^R \rho^4 d\rho \right) \left( \int_0^{\pi/2} \cos(\phi) \sin(\phi)^2 d\phi \right) \left( \int_0^{2\pi} \sin(\theta) d\theta \right) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 B &= 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^3 \cos(\phi) \sin(\phi) d\rho d\phi d\theta \\
 &= 3 \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\pi/2} \cos(\phi) \sin(\phi) d\phi \right) \left( \int_0^R \rho^3 d\rho \right) \\
 &= 3(2\pi) \left( \int_0^{\pi/2} \frac{\sin(2\phi)}{2} d\phi \right) \left( \frac{R^4}{4} \right) \\
 &= \frac{3\pi R^4}{2} \left( \int_0^{\pi/2} \frac{d}{d\phi} [-\cos(2\phi)] d\phi \right) \\
 &= \frac{3\pi}{2} R^4 (-\cos(\pi) + \cos(0)) = \frac{3\pi R^4}{2}.
 \end{aligned}$$

this means

$$m_2 = \frac{k}{4} \left( 0 + \frac{3\pi R^4}{2} \right)$$

$$= \boxed{\frac{3\pi R^4 k}{8} = m_2}$$

We now put everything together to get the center of mass,

$$\text{COM} = \frac{1}{m} (m_1, m_2, m_3)$$

$$= \frac{1}{\left(\frac{k\pi R^3}{2}\right)} \left( 0, \frac{\pi R^4 k}{120}, \frac{3\pi R^4 k}{8} \right)$$

$$= \frac{2}{k\pi R^3} \left( 0, \frac{\pi R^4 k}{120}, \frac{3\pi R^4 k}{8} \right)$$

$$= \boxed{\left( 0, \frac{R}{60}, \frac{3R}{4} \right)}$$

### Problem 4: (Errata)

There is a missing factor of 4 on the computation of  $m_y$ .

Correct Answer:

$$m_y = \frac{R}{15}$$

$$\Rightarrow \text{COM} = \left( 0, \frac{R}{15}, \frac{3R}{4} \right).$$