Covering Spaces, Uniformization and Picard Theorems

Complex Analysis - Spring 2017 - Dupuy

Abstract

The purpose of this note to develop enough covering space theory to allow us to talk about the covering space proof of the Little Picard Theorem.

♠♠♠ Taylor: [add references]

1 Fundamental Groups

Definition 1.1. Let (X, x_0) be a pointed topological space.

Exercise 1.

1. Check that $\pi_1(X, x_0)$ is a group.
2. Assume that X is a path connected topological space. Show that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ as groups.

Given $f : (X, x_0) \to (Y, y_0)$ we get an induced map of fundamental groups

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

$$f_*([\gamma]) = [f \circ \gamma]$$

where $[\gamma]$ denotes the homotopy class of the loop $\gamma : [0, 1] \to X$.

Example 1.2.

1. Let D be the unit disc. We have $\pi_1(D \setminus \{0\}, 1/2) \cong \mathbf{Z}$.
2. Let $S^1 = \{z \in \mathbf{C} : |z| = 1 = \partial D$. We have $\pi_1(S^1, 1) = \mathbf{Z}$.

The above example shows homotopy invariance of π_1. We have that $D \setminus \{0\} \sim S^1$.

Example 1.3. Let X be the figure eight. We have $\pi_1(X, x_0) \cong \mathbf{Z} * \mathbf{Z} = \langle \alpha, \beta \} = F_2$ the free group on two generators. [1]

The figure eight is an example of a "smash product". The figure eight is the smash product of S^1 with itself. A smash product is where you take two topological spaces and identify two points between them.

[1] $G_1 * G_2$ is the free product of two groups. It is just the word build out of elements from G_1 and G_2 subject to the obvious relations.

Exercise 2. Show that the fundamental group of a smash product is the free product of the fundamental groups.

Example 1.4.

1. $\pi_1(\mathbf{P}^1, x_0) = 1$ (it doesn't matter what the base point is).
2. $\pi_1(\mathbf{P}^1 \setminus \{0\}, x_0)$. Note that is doesn't matter which point we remove and we have $\mathbf{P}^1 \setminus \{a\} \cong \mathbf{P}^1 \setminus \{\infty\} \cong \mathbf{C}$.
3. $\pi_1(\mathbf{P}^1 \setminus \{0, 1\}, x_0) = \pi_1(\mathbf{P}^1 \setminus \{\infty, 0\}, x_0) = \pi_1(\mathbf{C} \setminus \{0\}, x_0) \cong \mathbf{Z}$.
4. $\pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\}, x_0) \cong \mathbf{Z} * \mathbf{Z}$.

What about an elliptic curve?

Example 1.5. Let $\Lambda = \omega_1\mathbf{Z} \oplus \omega_2\mathbf{Z}$. Recall the fundamental domain for $\mathbf{C}/\Lambda$. Consider our basepoint $\mathbf{C}/\Lambda \ni z_0 = 0 \equiv \omega_1 \equiv \omega_2 \equiv \omega_1 + \omega_2$. Let α be the path from 0 to ω_1 and β be the path from 0 to ω_2 Any loop can be pushed to the boundary we see that any element is generated by α and β. We can also see from the fundamental domain that

$$\alpha\beta\alpha^{-1}\beta^{-1} = 1$$

and hence that $\alpha\beta = \beta\alpha$. This show that the group is commutative. It remains to show that α and β are not homotopic. We can try to work this out but we won't. Instead we use that $\mathbf{C}/\Lambda \cong S^1 \times S^1$ as topological spaces and then use that $\pi_1(S^1 \times S^1, (x_1, x_2)) \cong \pi_1(S^1, x_1) \times \pi_1(S^1, x_2) \cong \mathbf{Z} \times \mathbf{Z}$. [2]

To completely justify this we need the following exercise.

Exercise 3.

1. Let $\gamma, \gamma' : J \to X \times Y$ be a morphisms of topological spaces. Write $\gamma(t) = (\alpha(t), \beta(t))$ and $\gamma'(t) = (\alpha'(t), \beta'(t))$ where $\alpha, \alpha' : J \to X$ and $\beta, \beta' : J \to Y$ are the maps obtained by projecting γ, γ' to X and Y respectively. Show that
$$\gamma \sim \gamma' \iff \alpha \sim \alpha' \text{ and } \beta \sim \beta'.$$
2. Show that the map $\pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by
$$[(\alpha, \beta)] \mapsto ([\alpha', \beta'])$$
is a well-defined, bijective group homomorphism.

[2]Warning: Although it is true that for every lattice $\Lambda \subset \mathbf{C}$ we have $\mathbf{C}/\Lambda \cong S^1 \times S^1$ as topological spaces we do not have $\mathbf{C}/\Lambda_1 \cong \mathbf{C}/\Lambda_2$ as Riemann Surfaces for $\Lambda_1 \neq \Lambda_2$ as general topological spaces. The j-invariant, $j = 1728g_2(\Lambda)^3/\Delta(\Lambda)$ where $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_2(\Lambda)^3$ determines whether we have isomorphic Riemann surfaces.

2 Covering spaces

In this section we work with Manifolds.

Definition 2.1. A **covering map** of X is a surjective $p : Y \to X$ such that for every $x \in X$ there exist some $U \ni x$ open with

$$p^{-1}(U) = \coprod_{i \in I} V_i$$

with $V_i \subset Y$ open such that $p|_{V_i} : V_i \to U$ is a homeomorphism.

Morphism of covering spaces: Let $p_1 : Y_1 \to X$ and $p_1 : Y_2 \to X$ be covering maps. A morphisms of coverings of X is a morphisms of topological spaces $f : Y_1 \to Y_2$ such that the following diagram commutes:

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} .$$

The collection of objects and morphisms allow us to define a category $\mathrm{Cov}(X)$ of coverings of X.

We denote the category of coverings of X by

$$\mathrm{Cov}(X).$$

We denote the category of connected coverings by

$$\mathrm{Cov}(X)^0.$$

Definition 2.2. For a pointed topological space we define $\mathrm{Cov}(X, x_0)$ and $\mathrm{Cov}(X, x_0)^0$ in the same way only using morphisms of pointed topological spaces.

Remark 2.3. The space Y in the definition above is a covering space and we will use the term covering space and covering map interchangably understanding that covering spaces come with covering maps.

Definition 2.4. An automorphism of a covering is called a **deck transformation**. For a cover $X \to Y$ the group of deck transformations are denoted by $G(Y/X)$.

Definition 2.5. A **universal covering space** is an initial object of $\mathrm{Cov}(X)^0$ [unique up to isomorphism].

Spelled-out version of universal covering space: suppose that $\widetilde{X}$ is a universal cover of X. This means for every cover $Y \to X$ there exist some map $\widetilde{X} \to Y$ such that

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd} .$$

Definition 2.6. Let $p : Y \to X$ be a covering of topological spaces (or pointed topological spaces). Consider a map $\gamma : Z \to X$ of topological spaces (I can be anything here). A **lift** of γ is a map $\widetilde{\gamma}$ such that $p \circ \widetilde{\gamma} = \gamma$.

Lemma 2.7 (Path Lifting)**.** *Let $p : (Y, y_0) \to (X, x_0)$ be a covering space. For any $\gamma : [0, 1] \to (X, x_0)$ such that $\gamma(0) = x_0$) there exists a unique lifting $\widetilde{\gamma} : [0, 1] \to Y$ such that*

1. *$\widetilde{\gamma}$ is a lift.*
2. *$\widetilde{\gamma}(0) = y_0$*

Proof. Since $[0, 1]$ is compact we can assume that $\gamma([0, 1]) \subset \bigcup_{i=1}^{n} U_i \subset X$ with $p^{-1}(U_i) = \coprod_{j \in J_i} V_{i,j}$ local trivializations. We perform induction on n.

- If $n = 1$ then let V_{1,j_1} be the unique open set containing y_0. The lifting of the path follows from the isomorphism $V_{1,j_1} \cong U_1$.
- Suppose now the theorem holds for covers containing $m < n$ open sets. Break γ into parts $\gamma = \gamma_1 \cdot \gamma_2$ such that $\gamma_1([0, 1])$ and $\gamma_2([0, 1])$ are contained in a union of fewer than n elements of $\mathcal{U} = \{U_i : 1 \leq i \leq n\}$.[3] By inductive hypothesis, γ_1 has a unique lift $\widetilde{\gamma}_1$ with $\widetilde{\gamma}_1(0) = y_0$. By inductive hypothesis, γ_2 has a unique lift $\widetilde{\gamma}_2$ with $\widetilde{\gamma}_2(0) = \widetilde{\gamma}_1(1)$. The conjuction $\widetilde{\gamma} := \widetilde{\gamma}_1 \cdot \widetilde{\gamma}_2$ works.

□

Lemma 2.8. *Let $Z = [0, 1]$. Let $p : Y \to X$ be a covering. Let $\gamma : Z \to X$ be a map with a lift $\widetilde{\gamma} : Z \to Y$. Any homotopy $H : [0, 1] \times Z \to X$ lifts to a unique homotopy $\widetilde{H} : [0, 1] \times Z \to Y$.*

Proof.

- First we prove the uniqueness statement in the theorem. Suppose $\widetilde{H}_1, \widetilde{H}_2 : [0, 1] \times Z \to Y$ are two lifts of $H : [0, 1] \times Z \to X$ with $\widetilde{H}_1(0, z) = \widetilde{H}_2(0, z) = \widetilde{\gamma}(z)$. Then for each $z_0 \in Z$, and each $i \in \{1, 2\}$, the function $\widetilde{H}_i(t, z_0)$ is a path lifting of $H(t, z_0)$ with $\widetilde{H}_i(0, z_0) = \widetilde{\gamma}(z_0)$ the lifted point. By uniqueness of path lifting we have $\widetilde{H}_1(t, z_0) = \widetilde{H}_2(t, z_0)$.
- Since $I \times Z$ is compact there exists some n and some $\mathcal{U} = \{U_i : 1 \leq i \leq n\}$ a collection of trivializing open sets such that $H(I \times Z) \subset \bigcup_{i=1}^{n} U_i \subset X$, where the U_i are trivializing: $p^{-1}(U_i) \cong \coprod_{j \in J_i} V_{i,j}$ where J_i is some index set. Let's suppose in addition that such a collection is minimal. We perform induction on n.
- Base case of induction: Suppose $n = 1$. Then $U_1 \cong V_{1,j_1}$ for some unique $V_{1,j_1} \ni y_0$ (we are just using something here to pin down the sheet in the cover that we want to lift to). We use this isomorphism to transport the homotopy.

[3] If no such break point exists then for every subinterval $[a, b]$ we have $\gamma([a, b])$ touching every one of these U_i's. Letting $a \to b$ we get a discontinuity which is a contradiction.

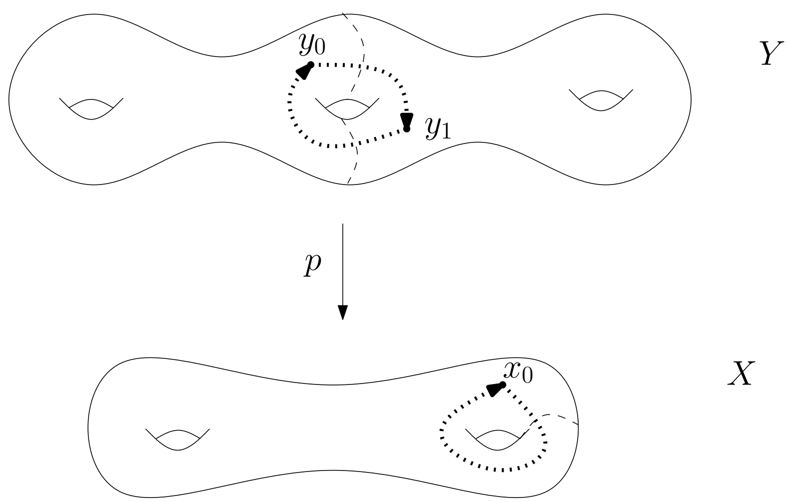

Figure 1: Here is a cover of a surface with three holes to a surface with holes. The map here is degree two. Note that loops on the bottom do not necessarily lift to loops on the top.

- To perform the inductive step we must suppose there exists a partition

$$[0,1] \times Z = [0,1] \times [0,1] = \bigcup_{1 \le i \le r, 1 \le j \le s} [t_i, t_{i+1}] \times [z_j, z_{j+1}] \tag{2.9}$$

  such that $H([t_i, t_{i+1}] \times [z_j, z_{j+1}])$intersects a proper subset of the $\{U_i : 1 \le i \le n\}$. We will show that such a partition exists in a subsequent bullet.

- Supposing the existence of the partition (2.9) exists, we perform the inductive step: By inductive hypothesis there exist a unique $\widetilde{F}_{1,j} : [0, t_1] \times Z_j \to X$ for $j = 1, \dots, s$ that has $\widetilde{F}_{1,j}(0, z) = \widetilde{\gamma}(z)$ lifting $\gamma(z)$. By the uniqueness lemma,

$$\widetilde{F}_{1,j}(t_1, z) = \widetilde{F}_{1,j+1}(t_1, z),$$

  and hence we get $\widetilde{H}_1 : [0, t_1] \to Y$ lifting $H|_{[0,t_1]\times Z}$: defined by

$$\widetilde{H}_1(t,z) = \begin{cases} \widetilde{F}_{1,1}(t,z), & z \in [z_1, z_2] \\ \widetilde{F}_{1,2}(t,z), & z \in [z_2, z_3] \\ & \vdots \\ \widetilde{F}_{1,s-1}(t,z), & z \in [z_{s-1}, z_s]. \end{cases}$$

This fits into a diagram

$$\begin{tikzcd} & Y \arrow[d] \\ {[0,t_1] \times Z} \arrow[ur, "\widetilde{H}_1"] \arrow[r, "H"] & X \end{tikzcd} .$$

Now we repeat this process applying the inductive hypothesis the next intervals $[t_1, z] \times Z_j$ where we apply our lifting to "initial data" $\widetilde{H}_1(t_1, z)$ (the lift we just constructed). This gives a lifts $\widetilde{F}_{2,j} : [t_1, t_2] \times Z_j \to Y$ lifting $H|_{[t_1,t_2]\times Z_j}$. As before these glue together to give some $\widetilde{H}_2 : [t_1, t_2] \times Z \to Y$ lifting $H|_{[t_1,t_2]\times Z} : [t_1, t_2] \times Z \to X$.

Repeating this process we get a sequence $\widetilde{H}_j : [t_j, t_{j+1}] \times Z \to Y$ and we define

$$\widetilde{H}(t,z) := \begin{cases} \widetilde{H}_1(t,z), & t \in [t_1, t_2] \\ \widetilde{H}_2(t,z), & t \in [t_2, t_3] \\ & \vdots \\ \widetilde{H}_{r-1}(t,z), & t \in [t_{r-1}, t_r]. \end{cases} .$$

By uniqueness of lifting, all of the components agree on their intersection and we have a lifting.

- Proof of existence of the partition (2.9): Suppose no such partition exists. We derive a contradiction in the style of Goursat's Theorem or Bolzano-Weierstrass: By subdividing $I \times Z = [0,1] \times [0,1]$ into fourths there exists some B_1 in that partition which intersects all of the U_i in our cover. By subdividing B_1 into quarters again, there exists some $B_2 \subset B_1$ one quarter of the size whose image intersects all of the U_i again. Repeating this process gives us a decreasing sequence

$$B_1 \supset B_2 \supset B_3 \supset \dots$$

with $\bigcap B_i = \{b_\infty\} \subset I \times Z$ (as the sets are converging we get a single point) such that for all j,

$$H(B_j) \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq n \ .$$

If we pick $b_j \in B_j$ such that if $j \equiv i \mod n$ we have

$$H(b_j) \in U_i \setminus \bigcup_{k:k\neq i} U_k$$

then $\lim_{j\to\infty} H(b_j)$ doesn't exist while $(b_j) \to b_\infty \in I \times Z$. Since H is continuous this is a contradiction.

$\square$

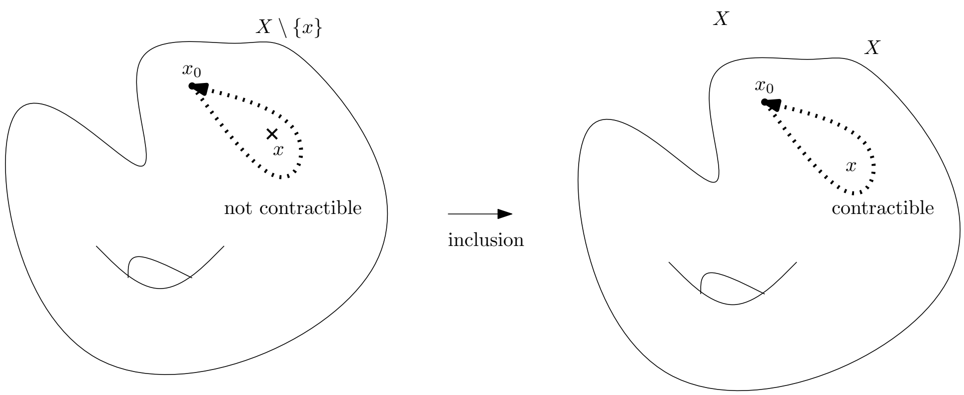

Figure 2: A loop around x in $X \setminus \{x\}$ is not contractible, but after adding the point x back it, it becomes contractible.

Lemma 2.10 (Coverings give Subgroups of Fundamental Groups)**.** *Let* $p : (Y, y_0) \to (X, x_0)$ *be a covering map. The map* $p_* : (Y, y_0) \to (X, x_0)$ *is injective.*

$$p_*\pi_1(Y, y_0) = \{ \text{ loops based at } x_0 \text{ with lifts that are loops based at } y_0 \}/\sim$$

Remark 2.11. Can we draw a picture of Lemma 2.10? Not really. Lemma 2.10 states essentially that is impossible to draw two loops in Figure 1 which are not homotopic in the bottom but homotopic in the top.

Note that for non-covering maps this is a common thing. Consider for example $(Z, z_0) = (X \setminus \{x\}, x_0) \subset (X, x_0)$ together with its inclusion map. Here loops around x in $X \setminus \{x\}$ can be not homotopic to zero but become homotopic to zero one we consider their pushforward in X. See figure 2.[4]

Proof. By the Homotopy Lifting Lemma, the map is injective. □

Remark 2.12. For covering map $(Y, y_0) \to (X, x_0)$ it is useful to make the notational convention

$$\pi_1(Y, y_0) = p_*\pi_1(Y, y_0) \subset \pi_1(X, x_0)$$

viewing fundamental groups of coverer as subgroups of the fundamental group of the coveree.

[4] The kernel of the map $\pi_1(X \setminus \{x\}, x_0) \to \pi_1(X, x_0)$ is called the **inertia group** and generated by loops around x. These actually correspond to stabilizers for actions of fundamental groups on universal covers. For ramified coverings of Riemann surface, if one deletes the branch points and branch values one gets a covering space. The inertia group of a ramification value is then a subgroup of the group of deck transformations corresponding to the kernel of this inclusion. ♠♠♠ Taylor: [write down exact sequence in later version]

Lemma 2.13 (Lifting Criterion)**.** *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map. Let $f : (Z, z_0) \to (X, x_0)$ be any morphism. There exists some $f' : (Z, z_0) \to (Y, y_0)$ such that the following diagram commutes*

$$\begin{tikzcd} (Y, y_0) \arrow[d, "p"] & (Z, z_0) \arrow[l, "f'"] \arrow[dl, "f"] \\ (X, x_0) & \end{tikzcd}$$

if and only if

$$f_* \pi_1(Z, z_0) \subset \pi_1(Y, y_0) \subset \pi_1(X, x_0).$$

Proof. We define the map ☐

Lemma 2.14 (Universal Covers = Simply Connected Covers)**.** *Consider the diagram*

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "\textit{covering map}"] \\ (Z, z_0) \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

If Z is simply connected then there exists a unique $\widetilde{f} : (Z, z_0) \to (Y, y_0)$ such that

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "\textit{covering map}"] \\ (Z, z_0) \arrow[r, "f"] \arrow[ur, "\widetilde{f}"] & (X, x_0) \end{tikzcd} .$$

In particular any covering map $(Z, z_0) \to (X, x_0)$ with Z simply connected is a universal covering map.

Proof. ♠♠♠ Taylor: [Add the proof from class. This is essentially from Munkres.] ☐

Theorem 2.15 (Galois Correspondence)**.** *Let $\mathrm{Cov}(X, x_0)^0$ be the category of connected pointed covers.*

$$\pi_1 : \mathrm{Cov}(X, x_0)^0 \to \{ \textit{ subgroups of } \pi_1(X, x_0) \ \}.$$

♠♠♠ Taylor: [Finish Adding proofs] Covers of covers are covers so higher covers corrspond to smaller subgroups.

The universal cover corresponds to the trivial subgroup.

Note that $N_{\pi_1(X, x_0)}(\pi_1(Y, y_0))$ is a subcover. This is the Galois hull.

3 Aside: Seifert-Van Kampen

We can now have enough machinery to prove some things about fundamental groups.

Theorem 3.1 (Seifert-Van Kampen)**.** *Let X be a manifold and assume $X = U_1 \cup U_2$ with U_1 an U_2 non-empty open sets. Assume that $U_1 \cap U_2$ is also non-empty. Then for $x_0 \in U_1 \cap U_2$*

$$\begin{tikzcd} \pi_1(U_1 \cap U_2, x_0) \arrow[r] \arrow[d] & \pi_1(U_1, x_0) \arrow[d] \\ \pi_2(U_2, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

is a pushout.

Proof. The idea of the proof is to use the Galois correspondence to construct a cover. ♠♠♠ Taylor: [Finish me] □

♠♠♠ Taylor: [Applications]
♠♠♠ Taylor: [In the category of topological spaces the universal cover exists because it is the inverse limit of topological spaces.]

4 Galois Coverings

Definition 4.1. A covering $p : Y \to X$ is called **normal** or **Galois** if and only if $\forall x \in X, \forall y, y' \in p^{-1}(x), \exists g \in G(Y/X)$ such that

$$g(y) = y'.$$

Lemma 4.2. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map.*

1. *$G(Y/X) \cong N_{\pi_1(X,x_0)}(\pi_1(Y, y_0))/\pi_1(Y, y_0)$.*[5]
2. *The covering p is Galois if and only if $\pi_1(Y, y_0) \subset \pi_1(X, x_0)$ is normal. In this case we have*

$$G(Y/X) = \pi_1(X, x_0)/\pi_1(Y, y_0).$$

Lemma 4.3. *If $p : Y \to X$ is Galois then $X \cong Y/G$ where $G = G(Y/X)$.*

Remark 4.4. In the theory of Riemann surfaces (and more generally algebraic geometry) it is useful to consider so called "branched coverings". There are maps $p : Y \to X$ where outside some small set $R \subset X$ we have a genuine covering:

$$Y \setminus p^{-1}(R) \to X \setminus R.$$

We will not consider these here, but you should keep this in mind when reading things elsewhere.

[5]Here $N_{\pi_1(X,x_0)}(\pi_1(Y, y_0))$ denotes the normalizer.

5 Coverings of Riemann Surfaces

Lemma 5.1. *Let $p : Y \to X$ be a covering. If X is a Riemann surface then so is Y.*

Proof Idea. Y is locally isomorphic to X. $\square$

Theorem 5.2 (Uniformization Theorem)**.** *Every simply connected Riemann surface is isomorphic to one of the following:*

- $\mathbf{P}^1$
- $\mathbf{C}$
- $H \cong D$ *the upper-half plane or unit disc.*

Lemma 5.3 (Classification of Riemann Surfaces)**.** *Any Riemann surface is isomorphic to one of the following:*

1. $X = \mathbf{P}^1$
2. $X = \mathbf{C}/\Gamma$, $\Gamma \subset \mathrm{Aut}(\mathbf{C}) = \mathrm{AL}_1(\mathbf{C})$
3. $X = H/\Gamma$, $\Gamma \subset \mathrm{Aut}(H) = \mathrm{PSL}_2(\mathbf{R})$.

Proof. Let X be a Riemann surface. Let $\widetilde{X}$ be its universal cover.

- By the characterization of universal covers (Lemma 2.14) we know that $\widetilde{X}$ is simply connected.
- By the characterization of simply connected Riemann surfaces we know that $\widetilde{X} \cong \mathbf{P}^1, \mathbf{C}$ or H.
- Since $\pi_1(\widetilde{X}) = 1$ we have that $\widetilde{X} \to X$ is Galois and hence that

$$X \cong \widetilde{X}/G(\widetilde{X}/X).$$

$\square$

Lemma 5.4. *Consider $X = \mathbf{P}^1 \setminus \{p_1, p_2, \dots, r\}$ if $r \geq 3$ then the universal cover $\widetilde{X}$ is D.*

6 Little Picard Theorem

Theorem 6.1 (Little Picard)**.** *Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function. If the image of f omits more than two points then f is constant.*

Proof.
- If suppose f omits more than two points. We can view this as a map $f : \mathbf{C} \to X = \mathbf{P}^1 \setminus \{p_1, p_2, \dots, p_r\}$ with $r \geq 3$.

- There is a cover of $\mathbf{P}^1 \setminus \{p_1, p_2, \ldots, p_r\}$ by $D \cong H = \{z : \operatorname{Im} z > 0\}$. [6]
- By the lifting lemma, since $\mathbf{C}$ is simply connected there exists some $\mathbf{C} \to D$ such that

$$\begin{tikzcd} X = \mathbf{C} \arrow[r, "F"] \arrow[dr, "f"] & Y = D \arrow[d, "p"] \\ & X = \mathbf{P}^1 \setminus \{p_1, p_2, \ldots, p_r\} \end{tikzcd}$$

.

- By Liouville's Theorem F is constant and hence so is f.

$\square$

♠♠♠ Taylor: [It suffices to prove this for two points missing since having more points missing still defines a map $f : \mathbf{C} \to \mathbf{C} \setminus \{0, 1\}$.]

7 Galois theory of functions

Theorem 7.1. *The contravariant functor*

$$\{\textit{Compact Riemann surfaces}\} \to \{\textit{Algebraic extensions of } \mathbf{C}(z)\ \}$$

$$X \mapsto \operatorname{Mer}(X)$$

which assigns a compact Riemann surface to its field of meromorphic functions is an equivalence of categories.

First things first, it is not clear that for a compact Riemann surface X that $\operatorname{Mer}(X)$

[6] In the case of three points it is enough to show that there is a map $H \to \mathbf{C} \setminus \{0, 1\}$ and the map

$$j(z) = 1728 g_2^3(z)/\Delta(z)$$

where $\Delta(z) = g_2^3(z) - 27 g_3^2(z)$ for $z \in H$ does the trick. If we let $q = \exp(2\pi i z)$ then

$$j = 1/q + 744 + 196844q + 21493760q^2 + 86429970q^3 + \cdots.$$