Arithmetic Deformation Classes for Curves

Taylor Dupuy
University of New Mexico

Čech Cohomology

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X scheme

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G sheaf of groups

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$\mathcal{U} = \{U_i\}_{i=1}^n$ open cover

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Cocycles: (g_{ij})

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$$\check{Z}^1(\mathcal{U}, G) \subset \prod_{i,j} G(U_{ij})$$

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$$\Longleftrightarrow \exists\ (h_i) \in \prod_i G(U_i)$$

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$$\boxed{\check{H}^1(\mathcal{U}, G) = \check{Z}^1(\mathcal{U}, G)/\sim}$$

Čech Cohomology

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$\mathcal{U} = \{U_i\}_{i=1}^n$ open cover

$$\boxed{\check{H}^1(X, G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)}$$

Cocycles: (g_{ij})

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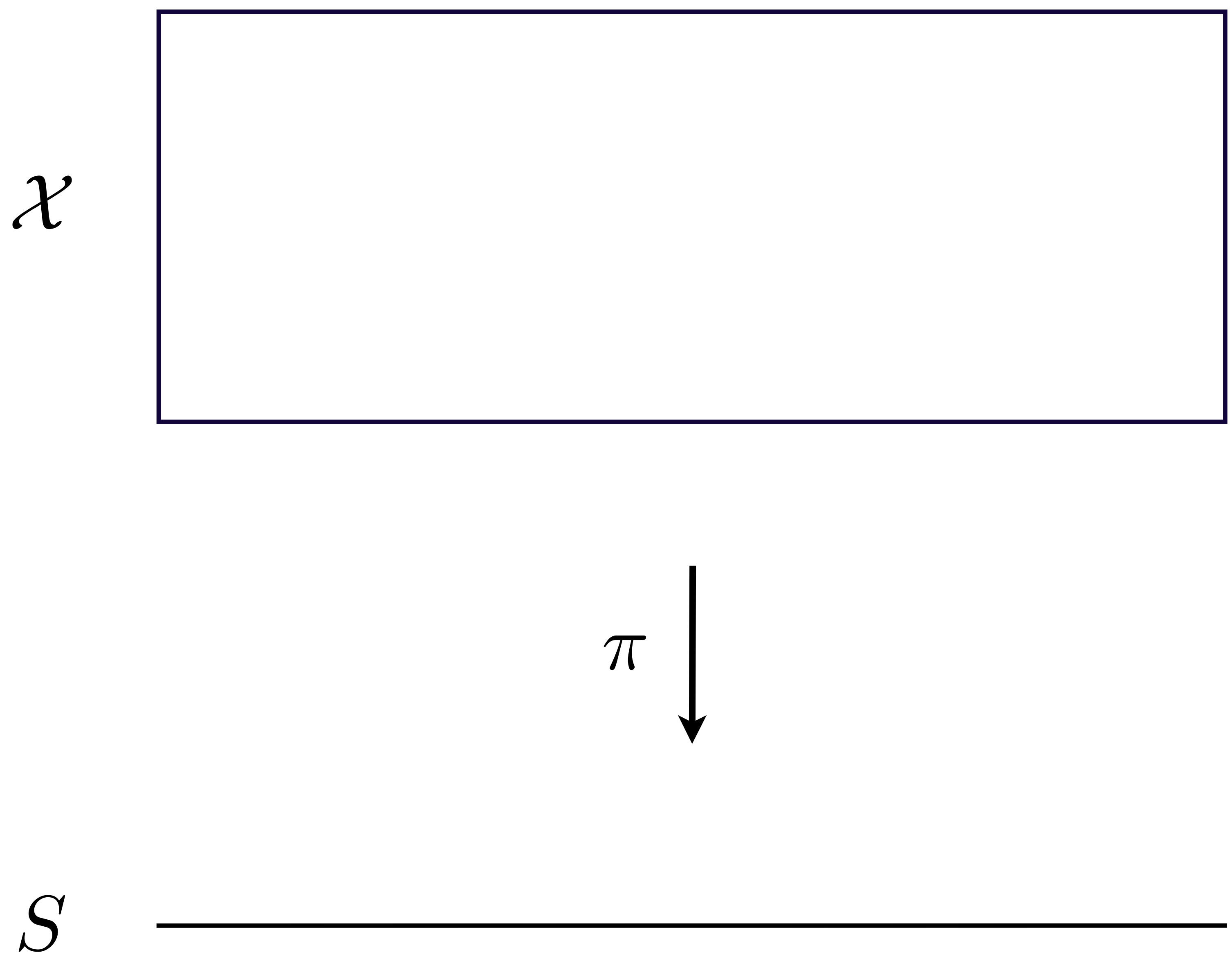
$$\check{H}^1(\mathcal{U}, G) = \check{Z}^1(\mathcal{U}, G) / \sim$$

$$\boxed{\mathrm{KS} : T_P S \to H^1(\mathcal{X}_P, T_{\mathcal{X}_P})}$$



S = Moduli Space
Well-Defined
LES SES

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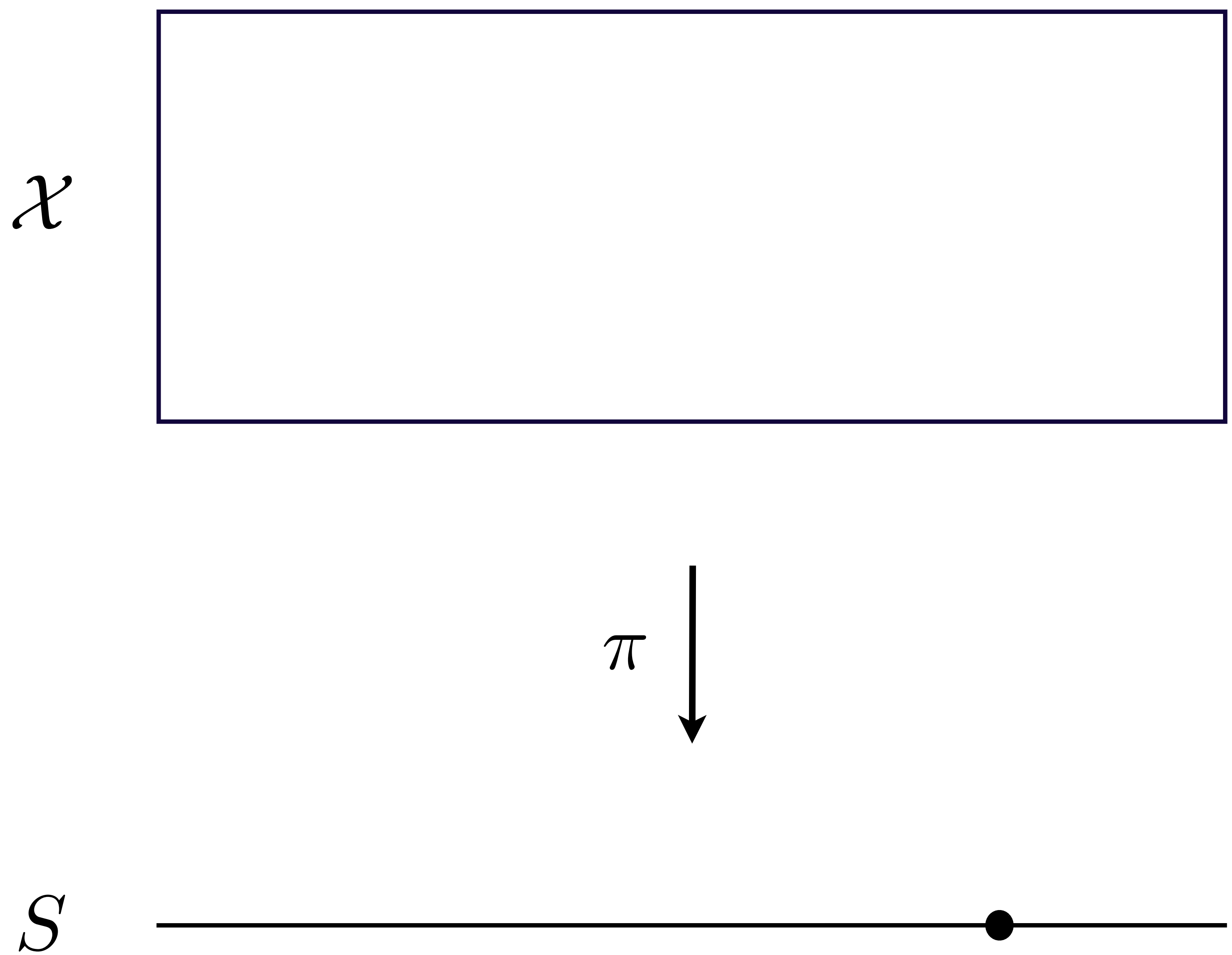

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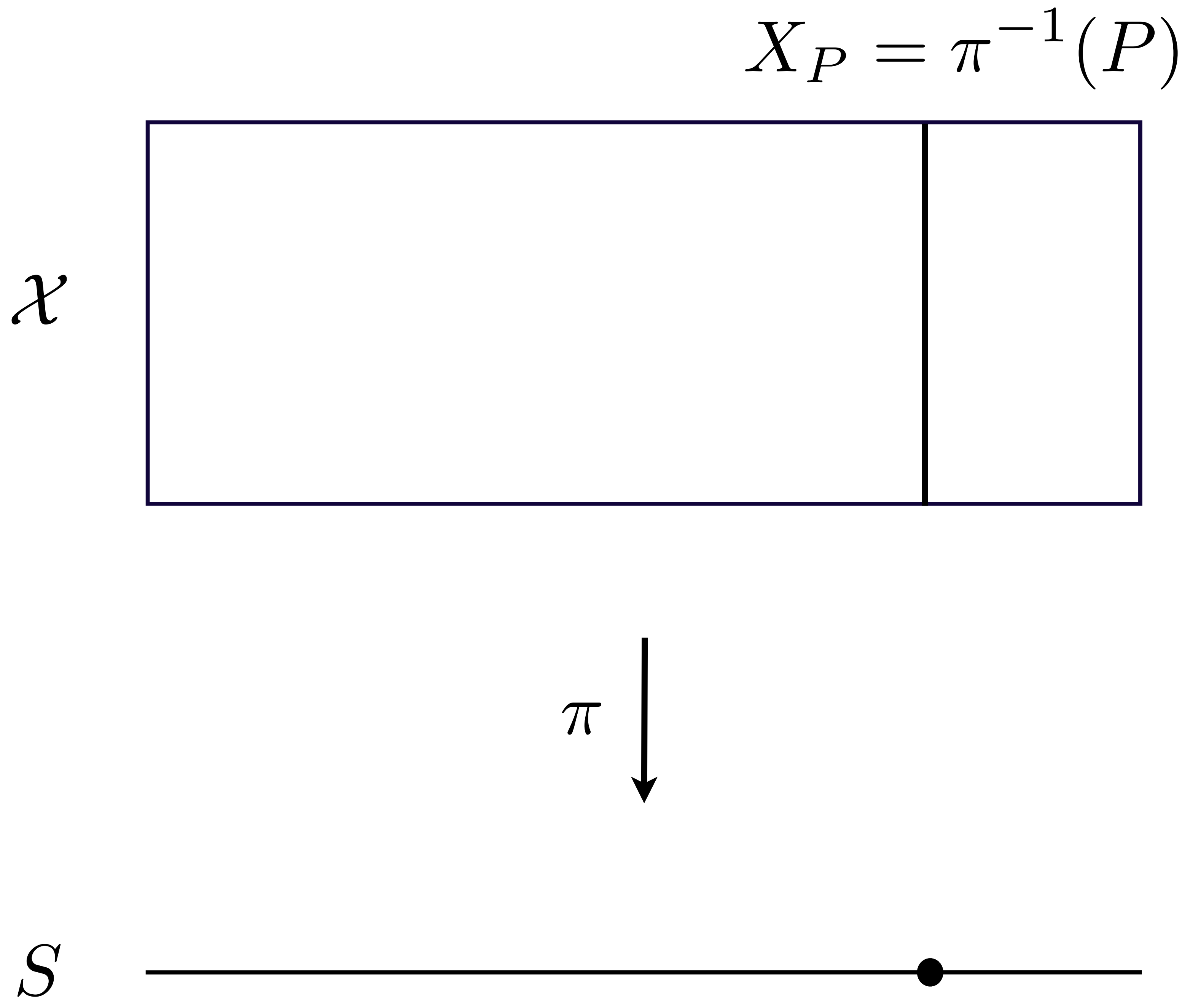

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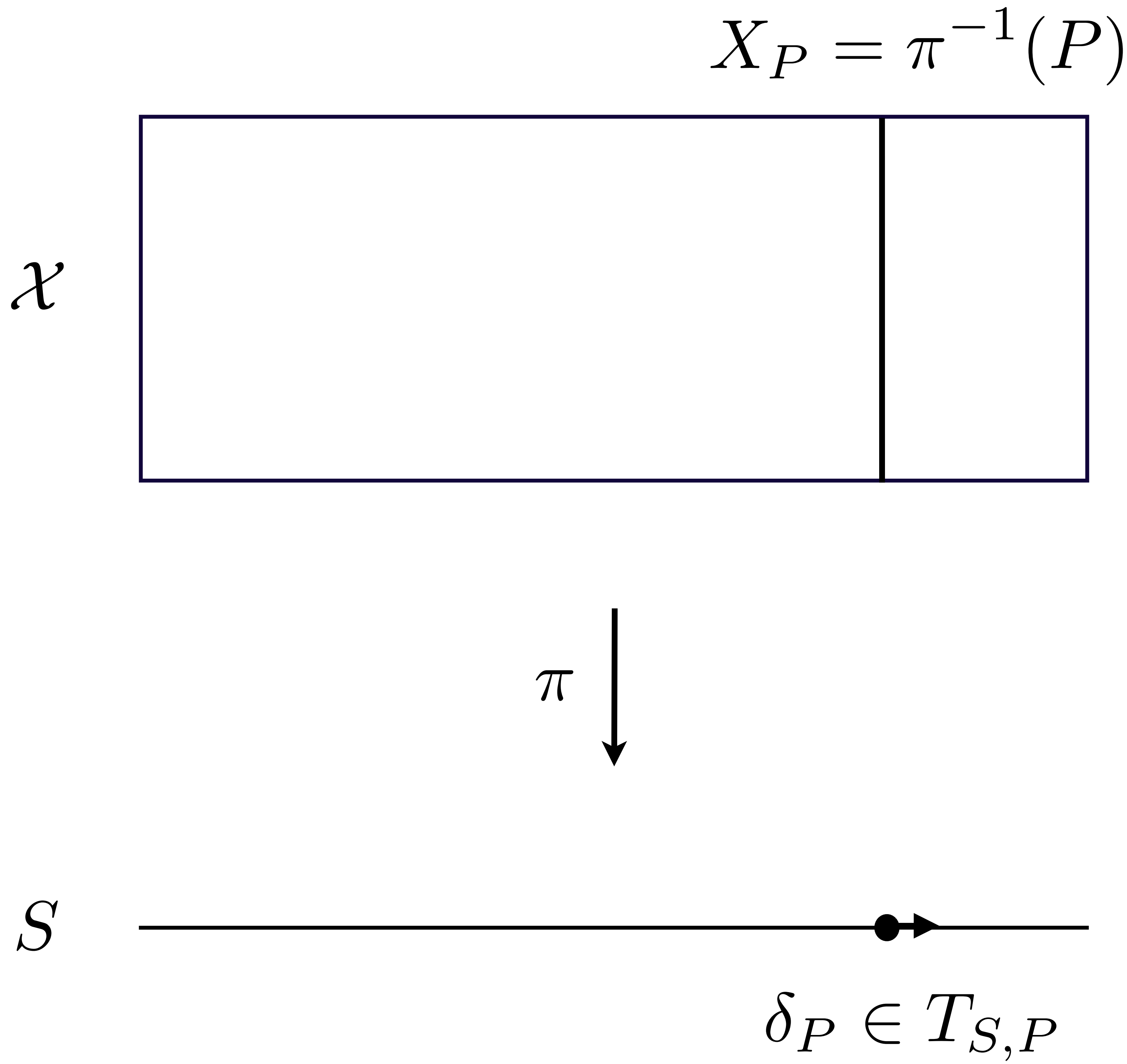

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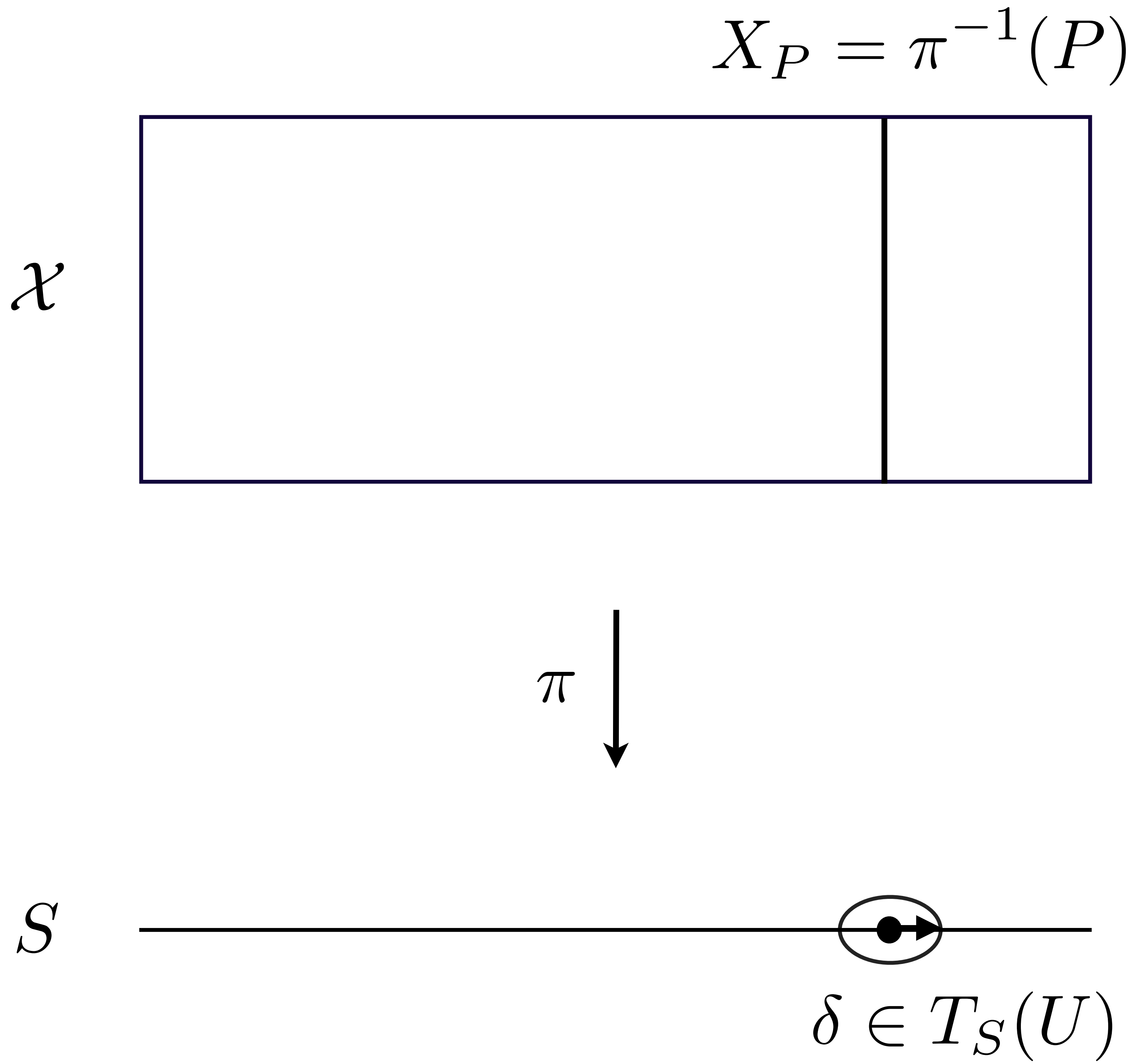

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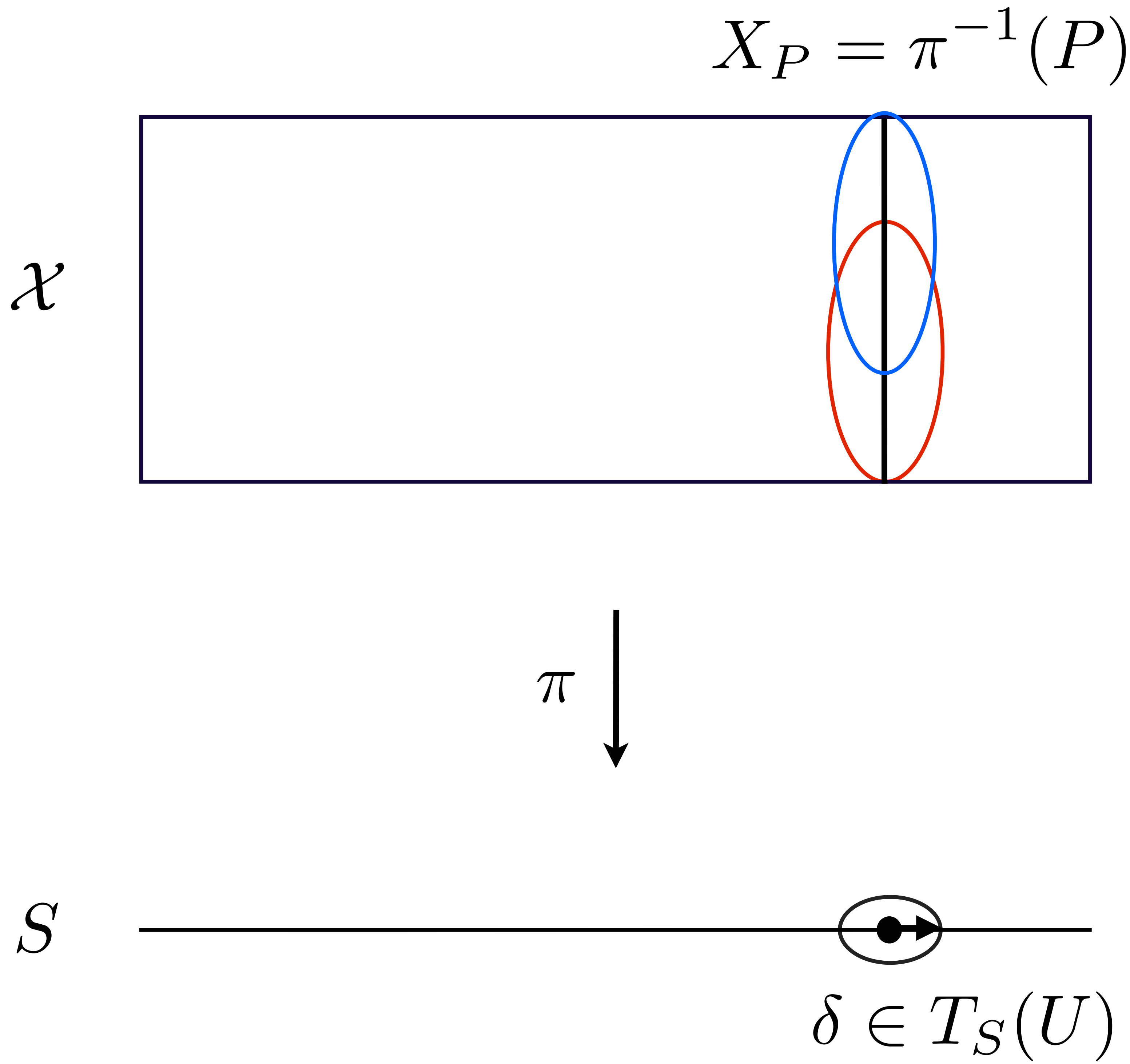
U_i
U_j



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$$\boxed{\mathrm{KS}: T_P S \to H^1(\mathcal{X}_P, T_{\mathcal{X}_P})}$$

$\delta_i \in T_{\mathcal{X}}(U_i)$

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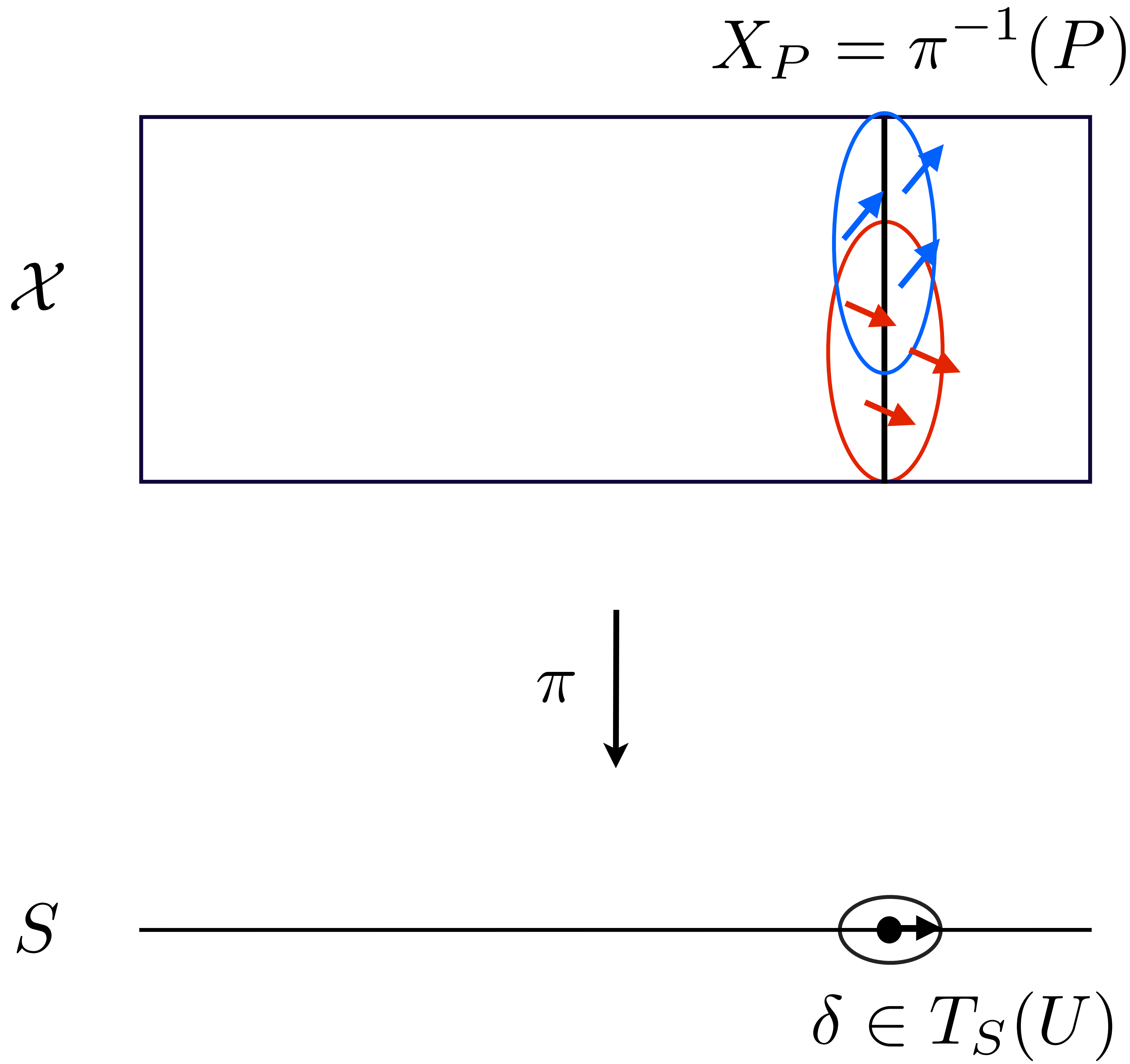

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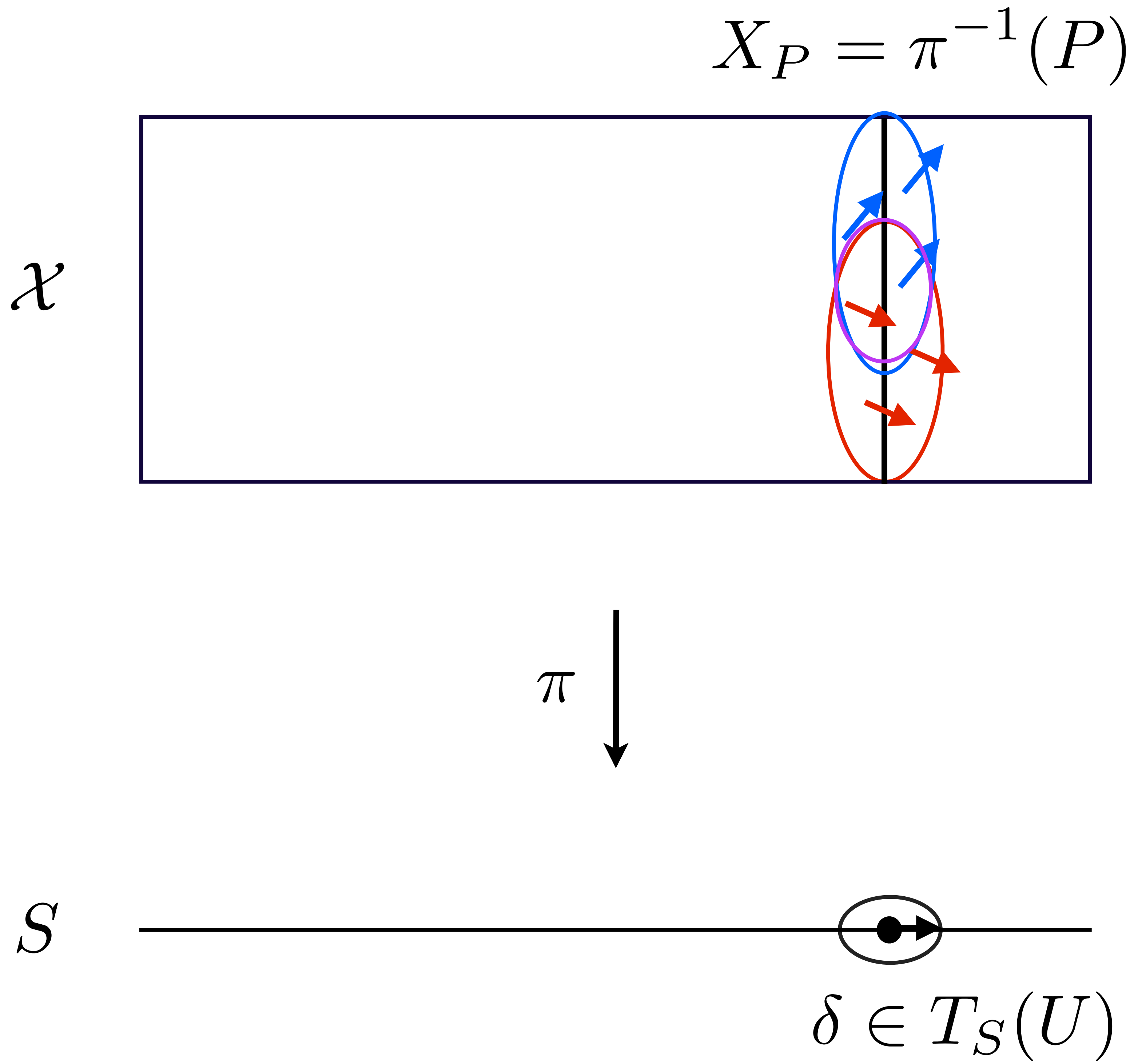

U_{ij}

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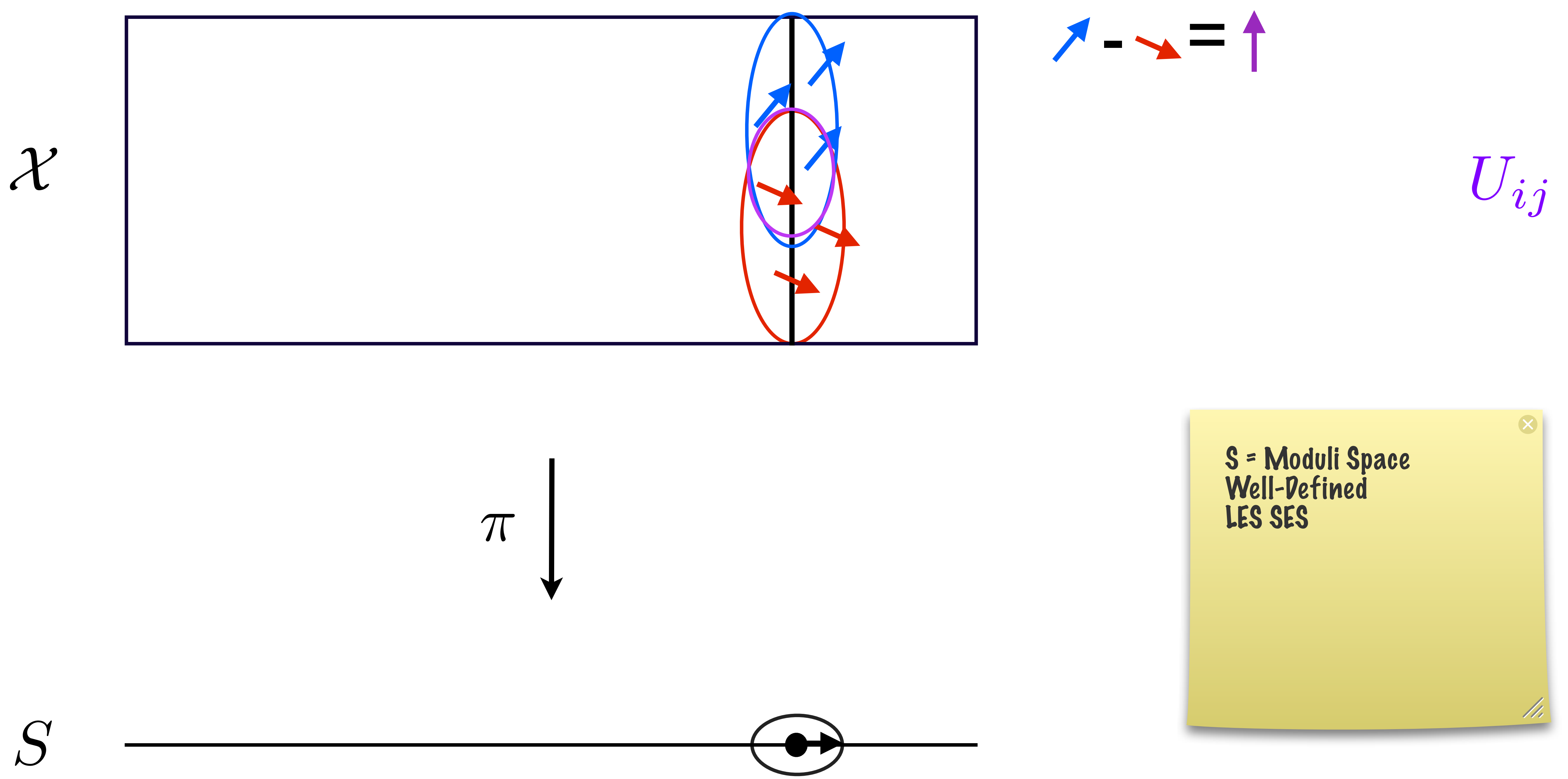

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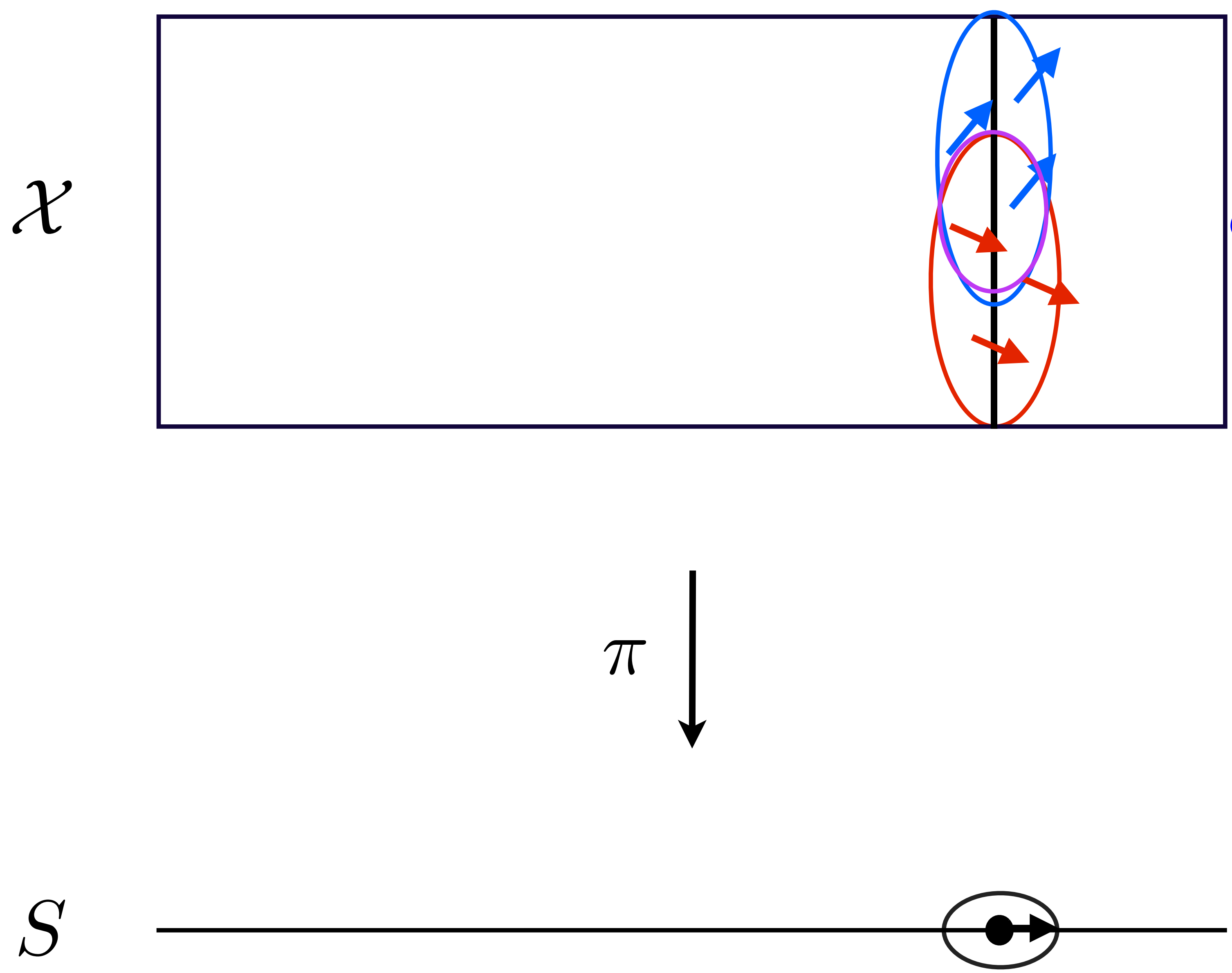

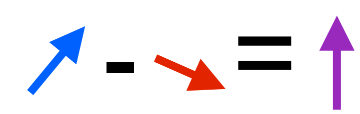

$\delta_i - \delta_j = \delta_{ij}$

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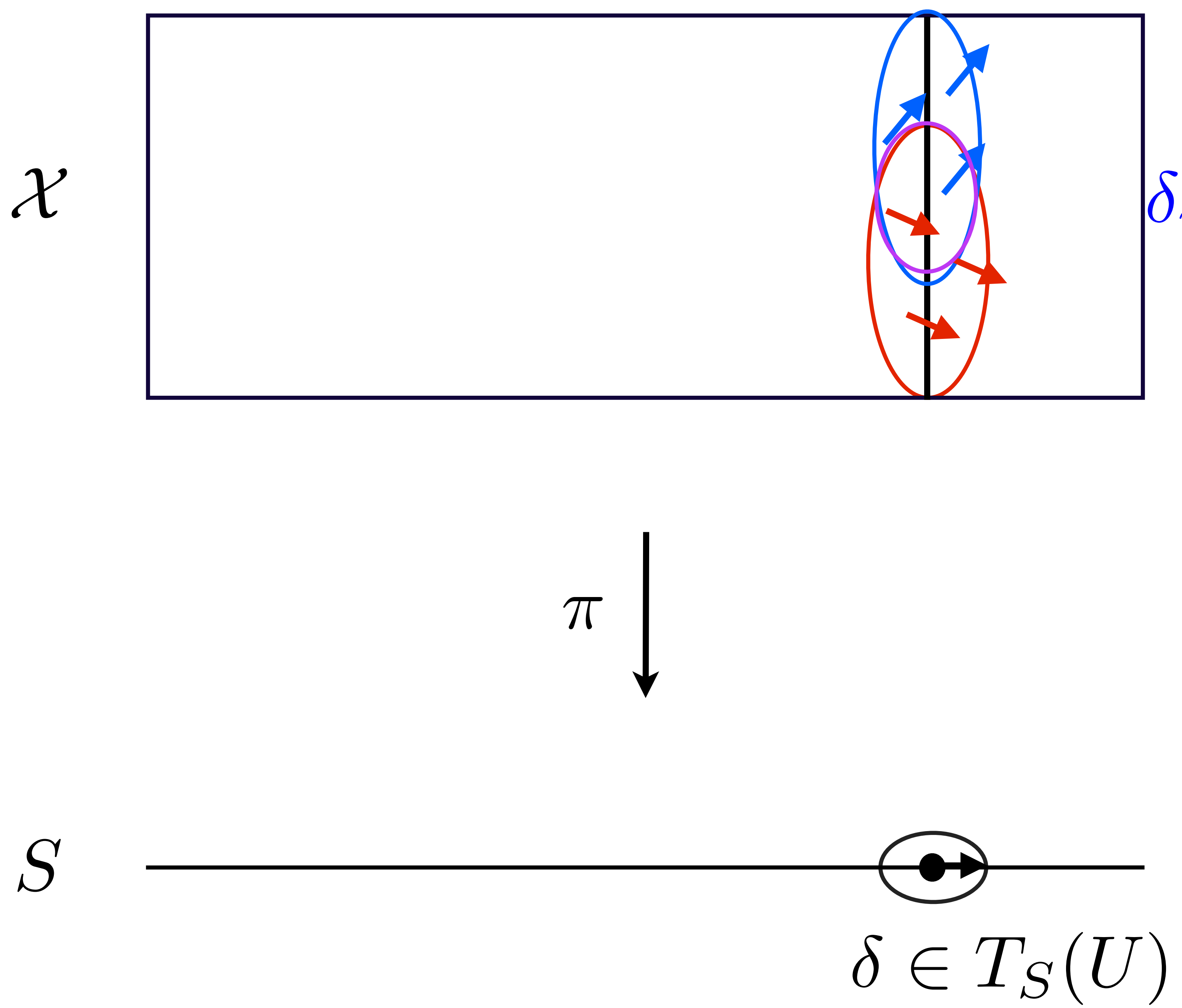

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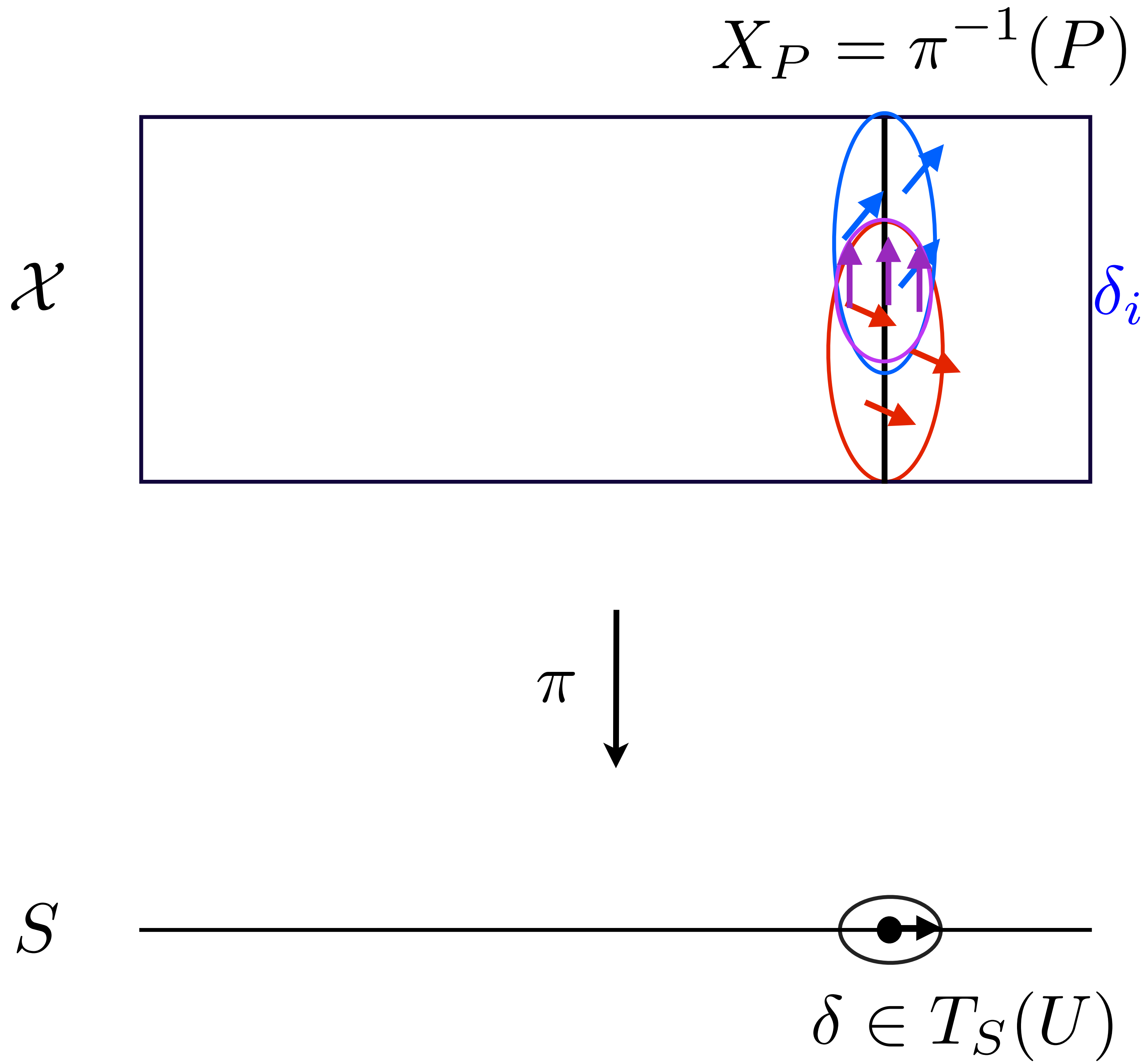

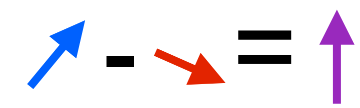

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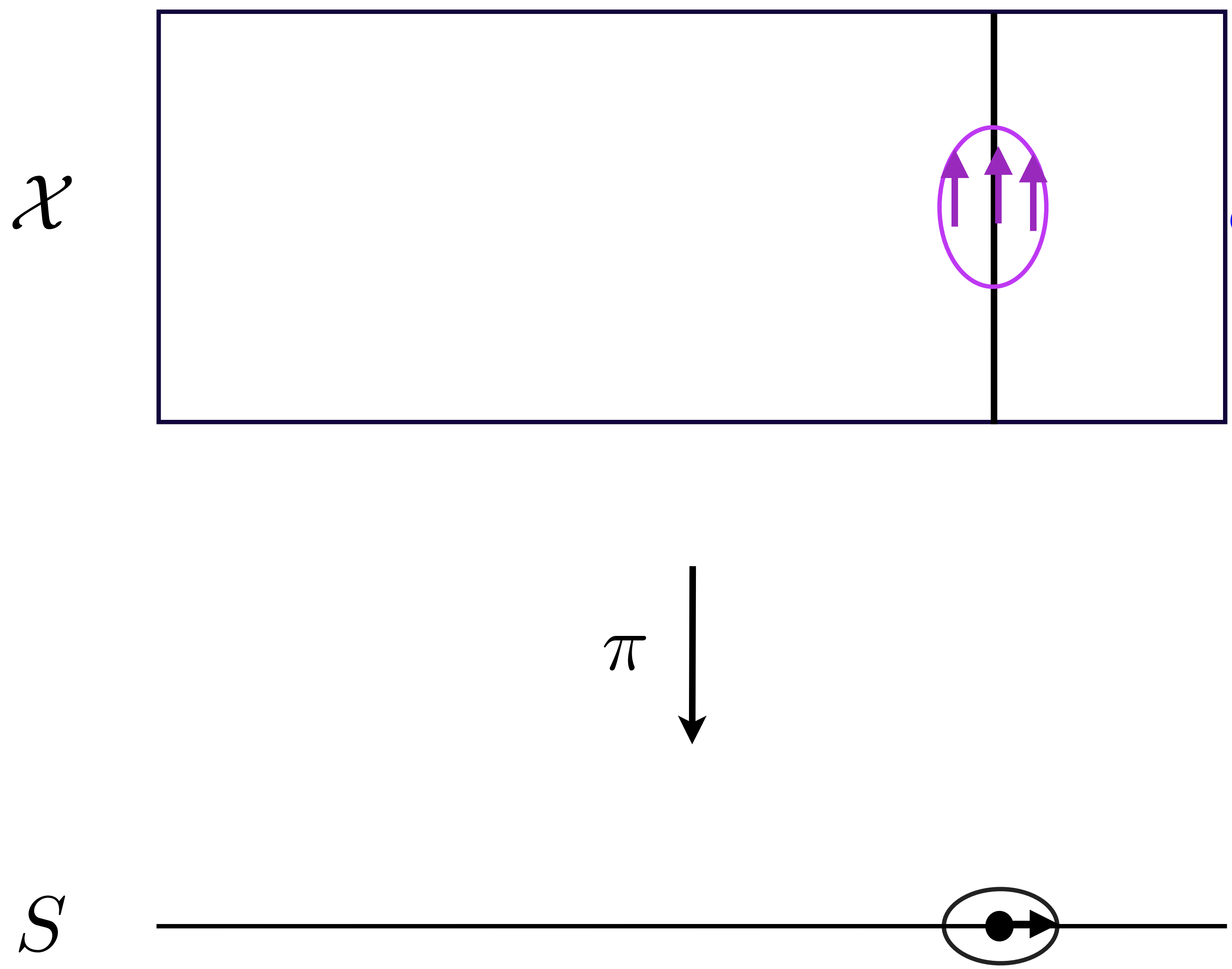

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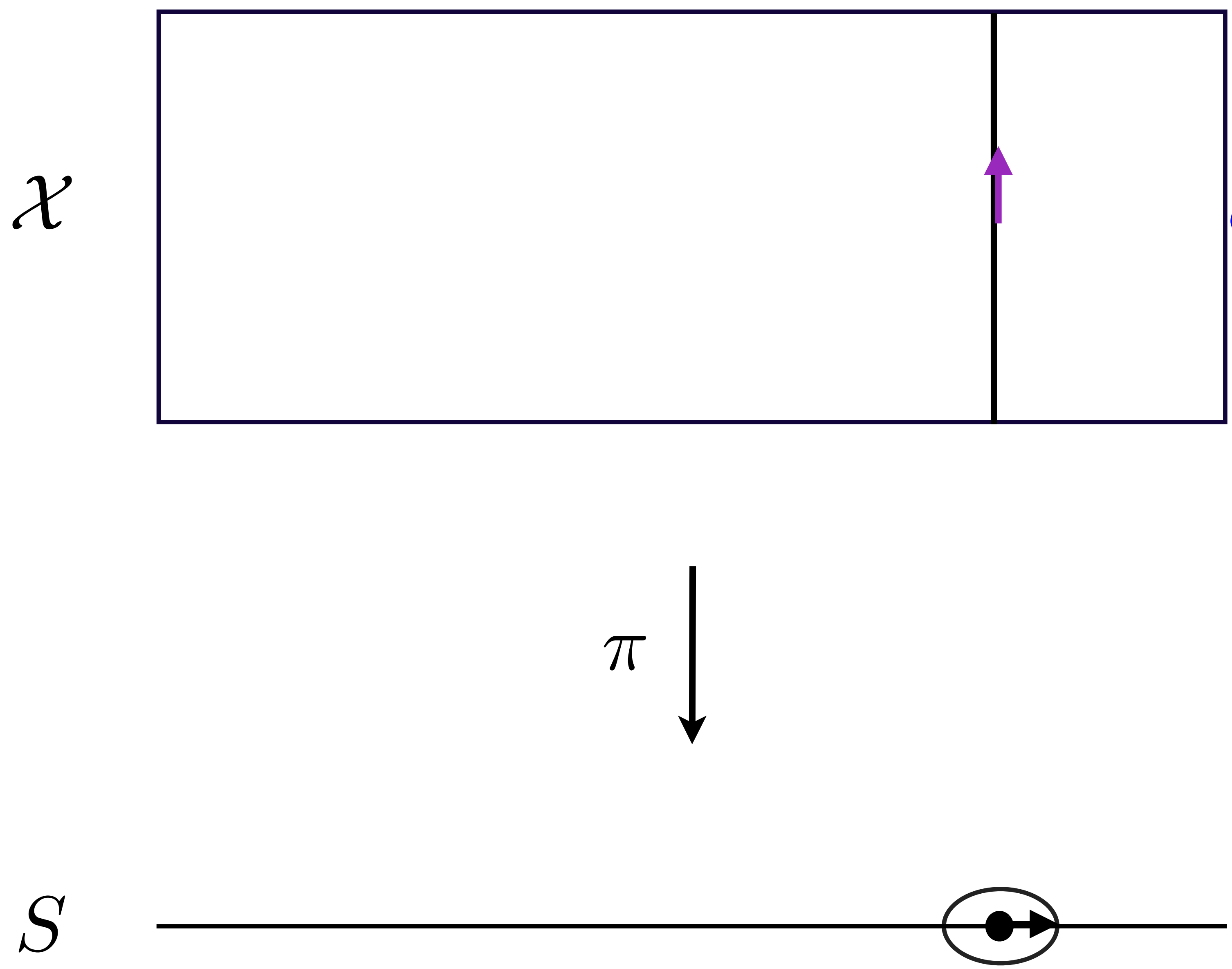

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Kodaira-Spencer Map

$$\boxed{\mathrm{KS} : \{\text{ derivations on } K \ \} \to H^1(X, T_X)}$$

generic fiber

$\delta : K \to K$ $\quad K =$ derivation

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$$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \qquad \delta_i|_K = \delta$$

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Take Differences

$$\delta \mapsto [\delta_i - \delta_j] \in H^1(X, T_{X/K})$$

Deligne-Illusie Map

$$\boxed{\mathrm{DI}_0 : \{\, p\text{-derivations on } R \,\} \to H^1(X_0, F^*T_{X_0})}$$

not surj (missing zero)

Higher Order classes

Does this relate to Serre-Tate for Ab?
What is history for E?
Is this f_1?

Is it injective?
What does it do?
What doesn't it do?
Is it surjective?
How doesn't it do this?

NOTATION!
F, delta, X_0

RETAIN INFORMATION
MOD P^2 not MOD P^3

$$X = \bigcup_i U_i \qquad \delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \qquad \delta_i|_R = \delta$$

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$$\mathrm{mod}\ p$$

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Take Differences

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$$\mathrm{mod}\ p$$

need to explain this doodad

Get Local Lifts

Is this smoothness?

Is this Hensel?

How can I think about this?

Get Local Lifts

Infinitesimal
Lifting Property

Is this smoothness?

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Get Local Lifts

Infinitesimal Lifting Property

$$A \longrightarrow B$$

Is this smoothness?

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Get Local Lifts

Infinitesimal Lifting Property

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C/I \end{tikzcd}$$

Is this smoothness?

Is this Hensel?

How can I think about this?

Get Local Lifts

Infinitesimal Lifting Property

$$I^2 = 0 \qquad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C/I \end{array}$$

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Geometric Setting

$$C = D_1(B) := B[\varepsilon]/\langle \varepsilon^2 \rangle$$

$$I = \langle \varepsilon \rangle$$

How can I think about this?

Get Local Lifts

Infinitesimal Lifting Property

$I^2 = 0$

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$$I = \langle \varepsilon \rangle$$

Arithmetic Setting

$$C =$$

$$I =$$

How can I think about this?

Get Local Lifts

Infinitesimal Lifting Property

$$I^2 = 0 \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & \text{Open} \arrow[d] \arrow[dl] \\ C \arrow[r] & C/I \end{tikzcd}$$

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Geometric Setting

$$C = D_1(B) := B[\varepsilon]/\langle \varepsilon^2 \rangle$$

$$I = \langle \varepsilon \rangle$$

Arithmetic Setting

$$C =$$

$$I =$$

How can I think about this?

The Frobenius Tangent Sheaf

$$D \in F^* T_{X_0}$$

Derivations of the Frobenius

$$\begin{aligned} D(a+b) &= D(a) + D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

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$$\delta_1, \delta_2 : A \to B$$

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Additivity:

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Additivity:

$$\begin{aligned} D(a+b) &= \delta_1(a+b) - \delta_2(a+b) \\ &= \delta_1(a) + \delta_1(b) + \mathrm{CRAP} - (\delta_2(a) + \delta_2(b) + \mathrm{CRAP}) \end{aligned}$$

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$$\delta_1, \delta_2 : A \to B$$
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$$\mathrm{CRAP} = \frac{a^p + b^p - (a+b)^p}{p}$$

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$$\mathrm{CRAP} = \frac{a^p + b^p - (a+b)^p}{p}$$

Additivity:

$$\begin{aligned} D(a+b) &= \delta_1(a+b) - \delta_2(a+b) \\ &= \delta_1(a) + \delta_1(b) + \mathrm{CRAP} - (\delta_2(a) + \delta_2(b) + \mathrm{CRAP}) \\ &= D(a) + D(b) \end{aligned}$$

Product Rule:

$$\delta_i(ab) = \delta_i(a)b^p + a^p\delta_i(b) + p\delta_i(a)\delta_i(b)$$

Derivations of the Frobenius

$$\begin{aligned} D(a+b) &= D(a)+D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

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$$D(a) := \delta_1(a) - \delta_2(a) \quad \text{mod p}$$

$$\mathrm{CRAP} = \frac{a^p + b^p - (a+b)^p}{p}$$

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Arithmetic Jet Spaces

$$R = \widehat{\mathbb{Z}}_p^{ur}$$

EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

Arithmetic Jet Spaces

$R = \widehat{\mathbb{Z}}_p^{ur}$

$$R \xrightarrow[\delta]{} R$$

EXAMPLE USED IN THE FUTURE!!!

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Arithmetic Jet Spaces

$R = \widehat{\mathbb{Z}}_p^{ur}$

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{\delta}} & B \\ \\ R & \xrightarrow[\delta]{} & R \end{array}$$

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$$\begin{tikzcd} A \arrow[r, "\widetilde{\delta}"] & B \\ R \arrow[u] \arrow[r, "\delta"'] & R \arrow[u] \end{tikzcd}$$

EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

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$R = \widehat{\mathbb{Z}}_p^{ur}$

always an A-algebra

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EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

Arithmetic Jet Spaces

$R = \widehat{\mathbb{Z}}_p^{ur}$

$\mathcal{O}(J^1(\mathrm{Spec}(A)))$

always an A-algebra

$$\begin{tikzcd} A \arrow[r, "\widetilde{\delta}"] & B \\ R \arrow[u] \arrow[r, "\delta"'] & R \arrow[u] \end{tikzcd}$$

EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

Arithmetic Jet Spaces

$$R = \widehat{\mathbb{Z}}_p^{ur}$$

$$\begin{tikzcd}
& \mathcal{O}(J^1(\mathrm{Spec}(A))) \arrow[dr] & \\
A \arrow[ur, "\delta_{universal}"] \arrow[rr, "\widetilde{\delta}"] & & B \\
R \arrow[u] \arrow[rr, "\delta"'] & & R \arrow[u]
\end{tikzcd}$$

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EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

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$$
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\end{tikzcd}
$$

always an A-algebra

Example $\mathbb{A}^1_R = \mathrm{Spec}\ R[x]$

$\mathcal{O}(J^1(\mathbb{A}^1_R)) = R[x][\dot{x}]\widehat{\ } = R[x]\{\dot{x}\}$ Rest

EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

${}^1_R \hat{\times} \widehat{\mathbb{A}}^1_R$

r Series

Geometric Descent

What is X/K?

Geometric Descent

X/K smooth projective

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$K = \overline{K} \qquad \mathrm{char}(K) = 0$

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Theorem

What is X/K?

T.F.A.E.

1. $\mathrm{KS}(\delta) = 0$
2. $J^1(X) \cong TX$ as schemes over X
3. $\exists X'/K^\delta$ such that $X' \otimes_{K^\delta} K \cong X$

Geometric Descent

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Descent to the constants

What is X/K?

Arithmetic Descent

$X/\widehat{\mathbb{Z}}_p^{ur}$ smooth

$\delta : \widehat{\mathbb{Z}}_p^{ur} \to \widehat{\mathbb{Z}}_p^{ur}$

How can we get an equation that describes when we have a lift?

Theorem

T.F.A.E.

1. $\mathrm{DI}_0(\delta) = 0$
2. $J^1(X)_0 \cong F^* T_{X_0}$ as schemes over X_0
3. X_1 admits a lift of the p-Frobenius

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Descent to the field with one element

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Descent to the field with one element

X descends to Borger-Buium $\mathbb{F}_1 \implies \mathrm{DI}_0(\delta_p) = 0$

$$\widehat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta : \zeta^n = 1, p \nmid n]\widehat{\ }$$

$$(\widehat{\mathbb{Z}}_p^{ur})^{\delta} = \{r : \delta(r) = 0\} \qquad \delta(r) = \frac{\phi(r) - r^p}{p}$$

$$= \text{ Monoid of roots of unity}$$

$$:= M$$

$$\mathrm{DI}_0(\delta) = 0 \implies \exists X_1'/M_1 \text{ such that } X_1' \otimes_M \widehat{\mathbb{Z}}_p^{ur}/p^2 \cong X$$

Positivity

genus 0 curves

genus 1 curves

genus 2 >= curves

Fano

Monoidal

Al- Geom

General Type

Frobenius Splitting???

DO LIFTS EXIST?

CM/Singular analogy K3 tends to break down?

u

DOES THE RAYNAUD THM WORK IN HIGHER DIMENSIONS

s Lifts

Frobenius Does Not Lift

$\kappa < 0$

$\kappa = 0$

$\kappa > 0$

Simple Problem:

Describe the set of λ such that

What is the principle of computations?
What care about explicit comp?
What makes the computation difficult?

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - \lambda X_0 X_1 X_2 X_3 X_4 X_5 = 0$$

admits a lift of the Frobenius

Relation of Deformation classes to Jet Spaces

PHS ~> torsor

Setup	Torsor	Group Scheme	C C
X/K	$J^1(X)$	$T_{X/K}$	$\mathrm{KS}(\delta)$
$X/\widehat{\mathbb{Z}}_p^{ur}$	$J^1(X)_0$	$F^*T_{X_0}$	$\mathrm{DI}_0(\delta)$

Main Theorem of Talk

	Setup	Torsor	Group Scheme	Cohom Class
	X/K	$J^1(X)$	$T_{X/K}$	$\mathrm{KS}(\delta)$
		$^1(X)_0$	$F^*T_{X_0}$	$\mathrm{DI}_0(\delta$
	$X/\widehat{\mathbb{Z}}_p^{ur}$ curves	$J^1(X)_n$	L_n	$\mathrm{DI}_n(\delta)$
$g \geq 2$ (new)	$\widehat{X}/\widehat{\mathbb{Z}}_p^{ur}$ curves	$J^1(X)$	$\widehat{L}$	$\widehat{\mathrm{DI}}(\delta)$

How do you get this lift?

What happens in elliptic curve case?
Higher Dimensions?
Procedure?
-"Structures" OK
-Cocycle

What does that tensor mean?

Natural To Investigate

- Arithmetic Jet Spaces are affine bundles

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- Affine bundles have associated cohomology classes

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- the cohomology clas[hidden] to the jet space controls the D[hidden]

Natural To Investigate

need p-formal completion

- Arithmetic Jet Spaces are affine bundles
- Affine bundles have associated cohomology classes
- the cohomology class associated to the jet space controls the D

Natural To Investigate

WARNING

- Arithmetic Jet Spaces admit reductions of the structure group.

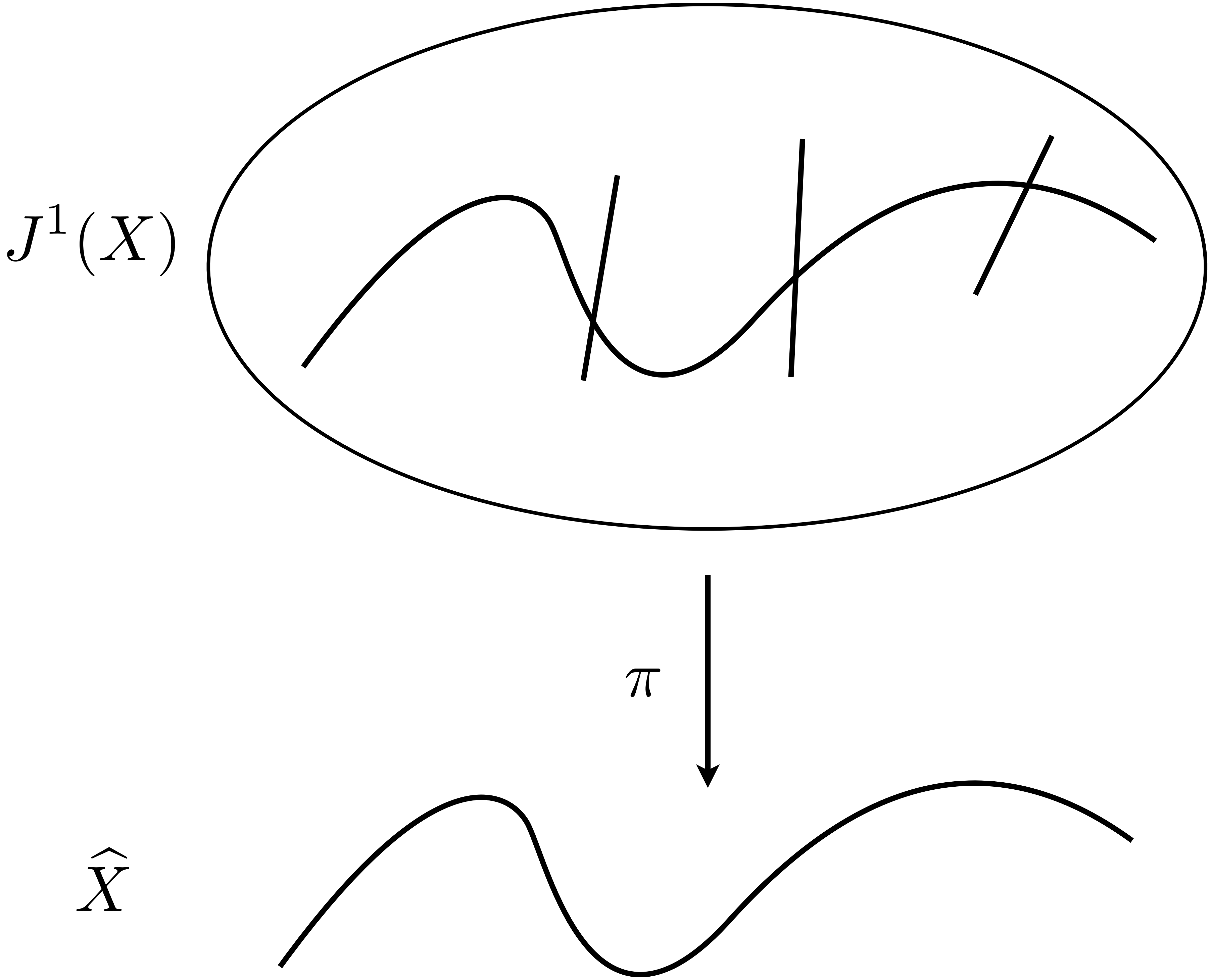
WARNING

- Arithmetic Jet Spaces admit reductions of the structure group.
- Elliptic Curves may admit multiple reductions of the structure group!!!

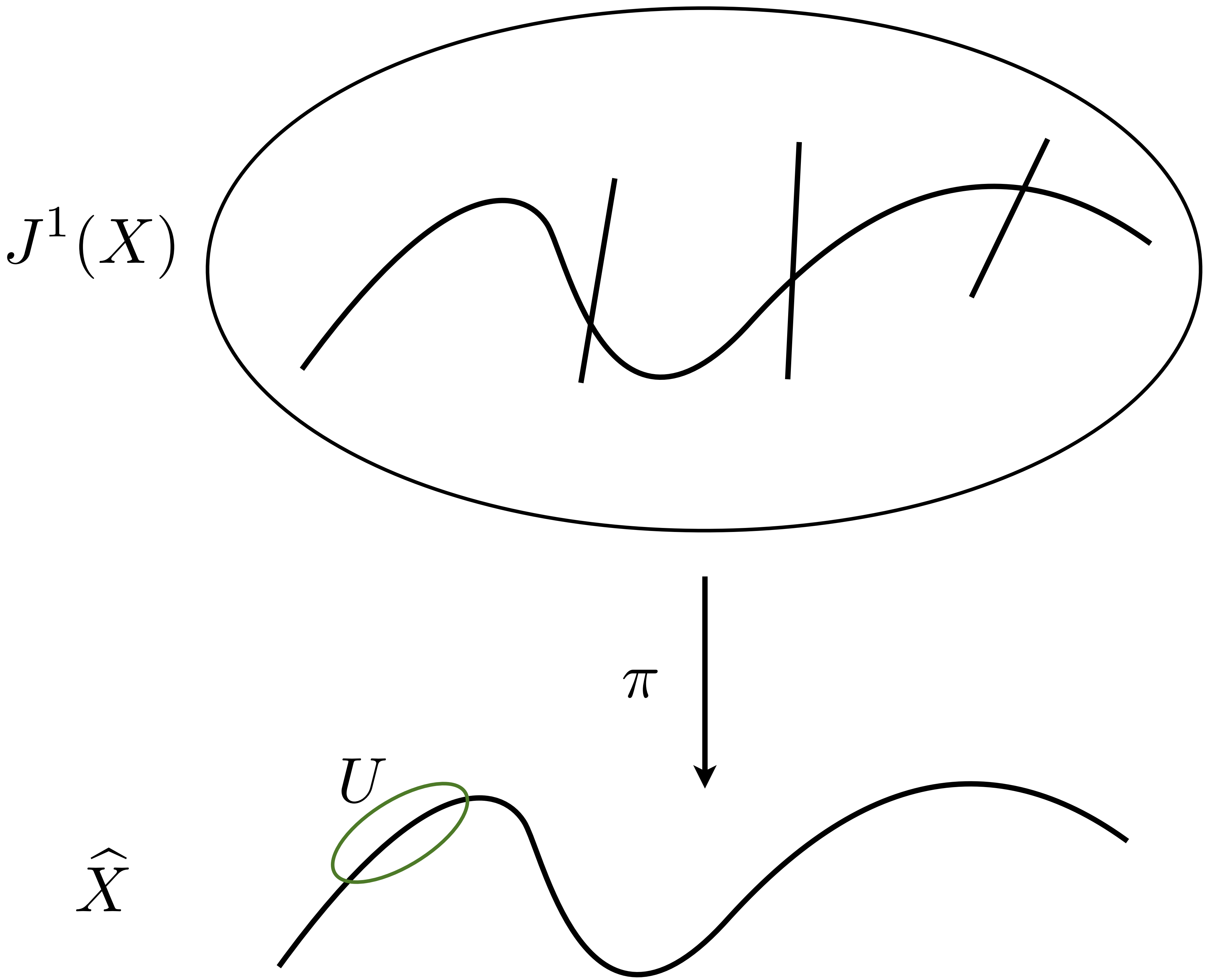
WARNING

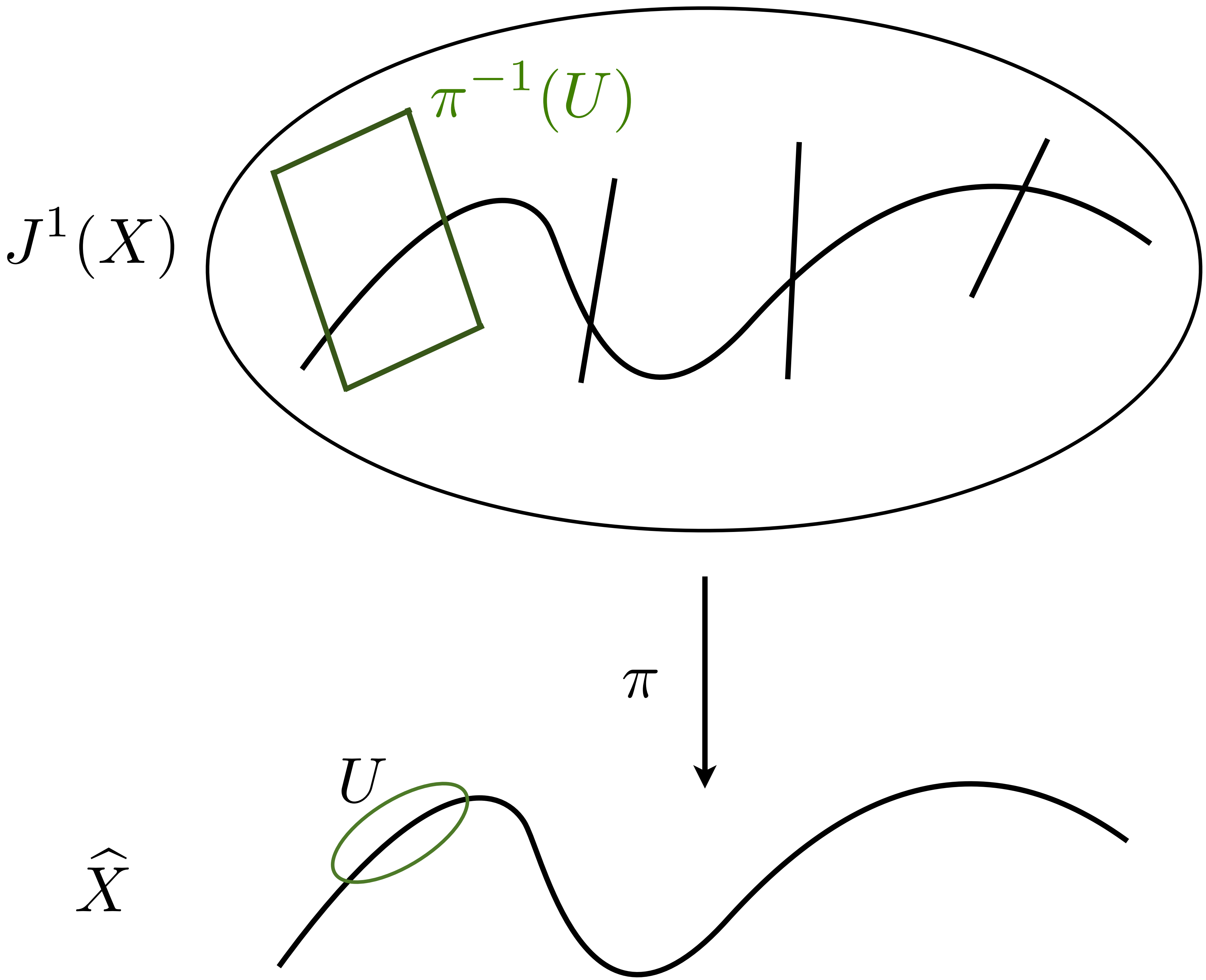
Lemma

$$\boxed{U \to \mathbb{A}^n_R \quad \text{\textbf{étale}} \quad \Longrightarrow \quad J^1(U) \cong \widehat{U} \,\hat{\times}\, \widehat{\mathbb{A}}^n}$$



$J^1(X)$
π
$\widehat{X}$


$J^1(X)$
π
U
$\widehat{X}$


$\pi^{-1}(U)$
$J^1(X)$
π
U
$\widehat{X}$

Local Trivialization of F-bundles

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

What is E and F?

Local Trivialization of F-bundles

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Local Trivialization
of F-bundles

$$\begin{tikzcd}
E \supset \pi^{-1}(U) \arrow[r, "\sim", "\psi"'] \arrow[d, "\pi"'] & U \times \\
X \supset U
\end{tikzcd}$$

What is E and F?

$$
\begin{tikzcd}
E \supset \pi^{-1}(U_i) \arrow[r, "\sim", "\psi_i"'] \arrow[d, "\pi"'] & U_i \times F \\
X \supset U_i &
\end{tikzcd}
$$

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Trivializing Cover

$$X = \bigcup_i U_i$$

$$
\begin{array}{ccc}
E \supset \pi^{-1}(U_i) & \xrightarrow[\psi_i]{\sim} & U_i \times F \\
\downarrow{\scriptstyle \pi} & & \\
X \supset U_i & &
\end{array}
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Trivializing Cover

$$X = \bigcup_i U_i$$

$$
\begin{array}{c}
\pi^{-1}(U_{ij}) \\
\downarrow \\
U_{ij} = U_i \cap U_j
\end{array}
$$

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$$
\begin{tikzcd}
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Trivializing Cover

$$X = \bigcup_i U_i$$

Transition Map

$$\psi_{ij} := \psi_i \circ \psi_j^{-1}$$

$$
\begin{tikzcd}
U_{ij} \times F & \pi^{-1}(U_{ij}) \arrow[l, "\sim"', "\psi_i"] \arrow[r, "\sim", "\psi_j"'] \arrow[d] & U_{ji} \times F \arrow[ll, bend right] \\
& U_{ij} = U_i \cap U_j &
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$\rightsquigarrow$

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& U_{ij} &
\end{tikzcd}$$

Class associated to Bundle

$$\boxed{\rightsquigarrow [\psi_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(F))}$$

$$\begin{tikzcd}
J^1(X) \supset \pi^{-1}(\widehat{U}_i) \arrow[r, "\sim", "\psi_i"'] \arrow[d, "\pi"'] & \widehat{U}_i \hat{\times} \widehat{\mathbb{A}}^m \\
\widehat{X} \supset \widehat{U}_i &
\end{tikzcd}$$

$$\rightsquigarrow \quad \beta := [\psi_{ij}] \in H^1(\widehat{X}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^m))$$

Who is the Big Class?

What is the data actually given by?

What is the data given by mod p and mod p^2?

$$
\begin{tikzcd}
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\widehat{X} \supset \widehat{U}_i &
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$$

$$m = \dim(X)$$

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$$m = \dim(X)$$

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Controls Deligne-Illusie

Who is the Big Class?

What is the data actually given by?

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Consider the whole situation mod p

How to prove B? (General Nonsense)

Consider the whole situation mod p

Theorem

$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

$\mathrm{DI}_0(\delta)$ $\qquad\qquad$ $[F^*T_{X_0}]$

How to prove B? (General Nonsense)

Consider the whole situation mod p

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$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

$$\mathrm{DI}_0(\delta) \xrightarrow{B} [a_{ij}] \qquad [F^* T_{X_0}] \xrightarrow{A} [b_{ij}T]$$

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B A

$\mathrm{DI}_0(\delta)$ $[F^*T_{X_0}]$

How to prove B? (General Nonsense)

idea used in A

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How to prove B? (General Nonsense)

idea used in A

$$\mathrm{AL}_1 = \underline{\mathrm{Aut}}(\mathbb{A}^1_{\mathbb{F}_p}) \quad \cdots \longrightarrow \quad \mathcal{O}_{X_0} \rtimes \mathcal{O}^\times_{X_0}$$

$$a + bT \circ c + dT = a + bc \cdots \quad \cdots b) \cdot (c, d) = (a + bc, bd)$$

Consider the whole situation mod p

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$\mathrm{DI}_0(\delta)$ B

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$$\mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^\times \xrightarrow{\pi} \mathcal{O}_{X_0}^\times$$

Consider the whole situation mod p

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$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

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idea used in A

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$$\mathcal{O}_{X_0} \rtimes \mathcal{O}^\times_{X_0} \xrightarrow{\pi} \mathcal{O}^\times_{X_0}$$

$$H^1(X, \mathcal{O}_X \rtimes \mathcal{O}^\times_X) \xrightarrow{\pi} H^1(X, \mathcal{O}^\times_X)$$

Consider the whole situation mod p

Theorem

$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

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idea used in A

$$\mathrm{AL}_1 = \underline{\mathrm{Aut}}(\mathbb{A}^1_{\mathbb{F}_p}) \quad \ldots \longrightarrow \quad \mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^{\times}$$

$$a + bT \circ c + dT = a + bc \ \ldots \ \ldots, b) \cdot (c, d) = (a + bc, bd)$$

$$\mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^{\times} \xrightarrow{\ \pi\ } \mathcal{O}_{X_0}^{\times}$$

$$H^1(X, \mathcal{O}_X \rtimes \mathcal{O}_X^{\times}) \xrightarrow{\ \pi\ } H^1(X, \mathcal{O}_X^{\times})$$

$$\pi(\beta_0) = [b_{ij}] = [F^*T_{X_0}]$$

Consider the whole situation mod p

Theorem

$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

$\mathrm{DI}_0(\delta)$ —B→ a_{ij} A: $[F^* T_{X_0}]$ → $b_{ij}T$

How to prove B? (General Nonsense)

idea used in A

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Conventions

$$\varphi_i : \mathcal{O}(U_i) \to L(U_i)$$

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idea of B

$$\pi : \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$$

Lemma

$$\boxed{\begin{array}{l} \pi^{-1}([L]) \xrightarrow[\iota_{left}]{\sim} H^1(X, L) \\ \subset H^1(X, \mathcal{O} \rtimes \mathcal{O}^\times) \end{array}}$$

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Proof

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$$(0,1) = (a_{ij}, b_{ij}) \cdot (a_{jk}, b_{jk}) \cdot (a_{ki}, b_{ki})$$

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\begin{aligned}
(0,1) &= (a_{ij}, b_{ij}) \cdot (a_{jk}, b_{jk}) \cdot (a_{ki}, b_{ki}) \\
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Strategy: Find "images" of β in semi-direct products

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Strategy: Find "images" of β in semi-direct products

use GROUP COCYCLES

Defn. $$\Phi : G \to A$$

$$\Phi(g_1 g_2) = \Phi(g_1) + g_1 \cdot \Phi(g_2)$$

$$\Phi \quad \rightsquigarrow \quad \begin{gathered} G \to A \rtimes \mathrm{Aut}(A) \\ g \mapsto (\Phi(g), \rho(g)) \end{gathered}$$

GROUP COCYCLES

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GROUP COCYCLES

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What happens when F is Affine space and H is GL_n?

What is weird about the A_2 structures?

How does this compare to smooth structures?

Fiber Bundle

$$
\begin{tikzcd}
E \supset \pi^{-1}(U_i) \arrow[r, "\sim", "\psi_i"'] & U_i \times F \\
X \supset U_i \arrow[from=1-1, d, "\pi"'] &
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$$\psi_{ij} \in H(U_{ij})$$

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Definition

$\{(U_i, \psi_i)\}$ = $\boxed{H\text{-atlas}}$ for E

Definition

Σ = $\boxed{H\text{-structure}}$ = Maximal H-atlas

$$A_n \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_{n-1}})$$

$$\psi(T) = c_0 + c_1 T + p c_2 T^2 + \cdots + p^{n-1} c_{n-1} T^n \mod p^n$$

What is a Degree Structure?

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Transition maps for Jet Spaces lie in wacky subgroups!

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Prop.
$$p > 6g - 5 \implies \exists \text{ tri-canonical } A_{n+1}\text{-structure on } J^1(X)_n$$

$\boxed{A_2}$

What is the point of the cocycle you are going to cook up?

$\boxed{A_2}$ (transition maps for $J^1(X) \mod p^2$)

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$$\psi(T) = T + pT^2$$

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$$\boxed{A_2}$$

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What does the 2 in tau_2 stand for?

What was iota_{right}?

cook up right cocycle + map:

$$f_{\tau_2} : H^1(X_1, A_2) \to H^1(X_0, \mathcal{O}^\times \rtimes \mathcal{O})$$

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$$\beta_1 = [\psi_i \circ \psi_j^{-1}] \qquad f_{\tau_2}(\psi_{ij}) = f_{\tau_2}(\psi_{\alpha_i}) f_{\tau_2}(\psi_{\alpha_j}^{-1})$$

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$$\psi_{\alpha_i}^{-1} \psi_{ij} \psi_{\alpha_j} \in \mathrm{Ker}(f_{\tau_2}) = \mathrm{AL}_1(\mathcal{O}_{X_1})$$

What is going on in the elliptic curves case?

Find a Group Cocycle

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Find a Group Cocycle

$\downarrow$

map to a semi-direct product

$$H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})$$

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Find a Group Cocycle

$\downarrow$

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$$H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})$$

$\downarrow$

line bundle

$$H^1(X, L)$$

$\downarrow$

What is going on in the elliptic curves case?

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↓

map to a semi-direct product
$H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})$

↓

line bundle
$H^1(X, L)$

↓

Does the class vanish?

What is going on in the elliptic curves case?

Find a Group Cocycle

↓

map to a semi-direct product

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↓

line bundle

$$H^1(X, L)$$

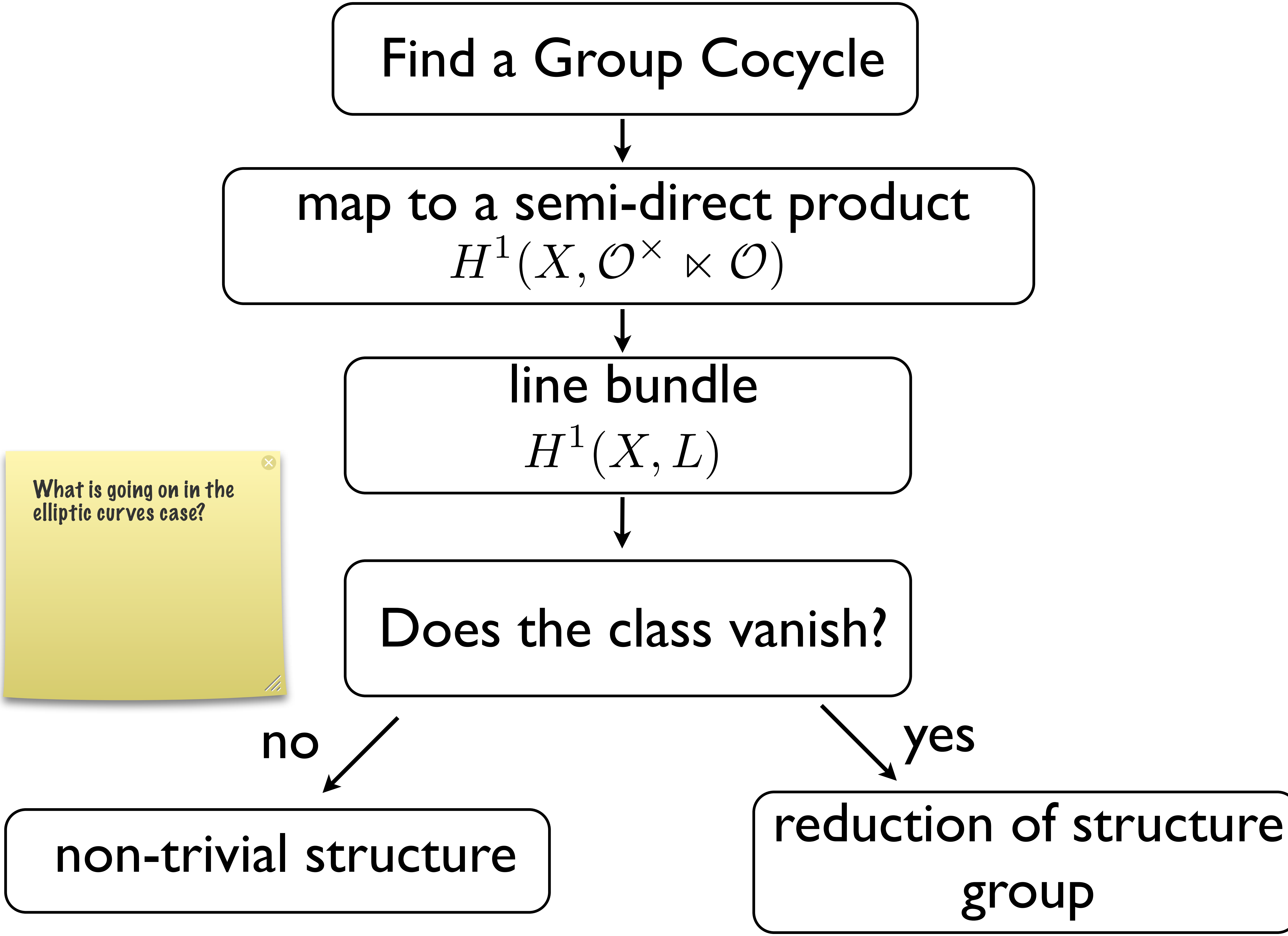
↓

Does the class vanish?

↘ yes

reduction of structure group

What is going on in the elliptic curves case?



What is going on in the elliptic curves case?

EXAMPLE

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{aligned} U_1 &= \text{Spec } R[x] \\ U_2 &= \text{Spec } R[y] \end{aligned}$$

Do we need to know about the A^1 example?

$$U_1 \cap U_2 = \text{Spec } R[x, y]/\langle xy - 1 \rangle$$

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$$U_1 \cap U_2 = \text{Spec } R[x, y]/\langle xy - 1\rangle$$

$$J^1(\mathbb{P}^1_R) = ???$$

$$\boxed{\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{array}{l} U_1 = \mathrm{Spec}\ R[x] \\ U_2 = \mathrm{Spec}\ R[y] \end{array}}$$

$$\mathcal{O}(J^1(U_1)) = \mathcal{O}(U_1)[\dot{x}]\widehat{\ } \xrightarrow{\sim} \mathcal{O}(U_1)[T]\widehat{\ } \qquad \dot{x} \mapsto T$$

$$\mathcal{O}(J^1(U_2)) \xrightarrow{\sim} \mathcal{O}(U_2)[T]\widehat{\ } \qquad \dot{y} \mapsto T$$

$$\boxed{\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{matrix} U_1 = \mathrm{Spec}\ R[x] \\ U_2 = \mathrm{Spec}\ R[y] \end{matrix}}$$

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$$\mathcal{O}(J^1(U_2)) \xrightarrow{\sim} \mathcal{O}(U_2)[T]\widehat{\ } \qquad \dot{y} \mapsto T$$

$$\mathcal{O}(J^1(U_{12}))$$

$$\boxed{\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{aligned} U_1 &= \mathrm{Spec}\ R[x] \\ U_2 &= \mathrm{Spec}\ R[y] \end{aligned}}$$

$$\mathcal{O}(J^1(U_1)) = \mathcal{O}(U_1)[\dot{x}]\widehat{\ } \xrightarrow{\ \sim\ } \mathcal{O}(U_1)[T]\widehat{\ } \qquad \dot{x} \mapsto T$$

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$$\begin{tikzcd} & \mathcal{O}(U_{12})[\dot{x}]\widehat{\ } = \mathcal{O}(J^1(U_{12})) \arrow[dl, "\sim"', "\dot{x} \mapsto T"] \\ \mathcal{O}(U_{12})[T]\widehat{\ } & \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=small]
& \mathcal{O}(U_{12})[\dot{x}]\widehat{\ } = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]\widehat{\ } \arrow[dl, "\sim"', "\dot{x} \mapsto T"] \arrow[dr, "\sim", "\dot{y} \mapsto T"'] & \\
\mathcal{O}(U_{12})[T]\widehat{\ } & & \mathcal{O}(U_{12})[T]\widehat{\ }
\end{tikzcd}$$

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$$T \mapsto \dot{x}$$

$$\boxed{\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{array}{l} U_1 = \mathrm{Spec}\ R[x] \\ U_2 = \mathrm{Spec}\ R[y] \end{array}}$$

$$\begin{tikzcd}
& \mathcal{O}(U_{12})[\dot{x}]^{\widehat{}} = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]^{\widehat{}} \arrow[dl, "\dot{x} \mapsto T\ \sim"'] \arrow[dr, "\sim\ \dot{y} \mapsto T"] & \\
\mathcal{O}(U_{12})[T]^{\widehat{}} & & \mathcal{O}(U_{12})[T]^{\widehat{}}
\end{tikzcd}$$

$$T \mapsto \dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})}$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{aligned} U_1 &= \operatorname{Spec} R[x] \\ U_2 &= \operatorname{Spec} R[y] \end{aligned}$$

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 & \mathcal{O}(U_{12})[\dot{x}]^\wedge = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]^\wedge \arrow[dl, "\dot{x} \mapsto T\ \sim"'] \arrow[dr, "\sim\ \dot{y} \mapsto T"] & \\
\mathcal{O}(U_{12})[T]^\wedge & & \mathcal{O}(U_{12})[T]^\wedge
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$$T \mapsto \dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})}$$

$$x = 1/y$$

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$$\begin{tikzcd} & J^1(U_{12}) \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ \widehat{U}_{12} \hat{\times} \widehat{\mathbb{A}}^1 & & \widehat{U}_{12} \hat{\times} \widehat{\mathbb{A}}^1 \end{tikzcd}$$

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$$\mapsto \frac{-T}{y^p(y^p + pT)} = \psi_2^{-1*}\psi_1^*(T)$$

what is the point?

can you tell that PP^1 has a lift of the frobenius mod p^2 from this class already?

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$$

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$$
\psi_{12}(T) = -\frac{T}{y^p(y^p+pT)}
$$

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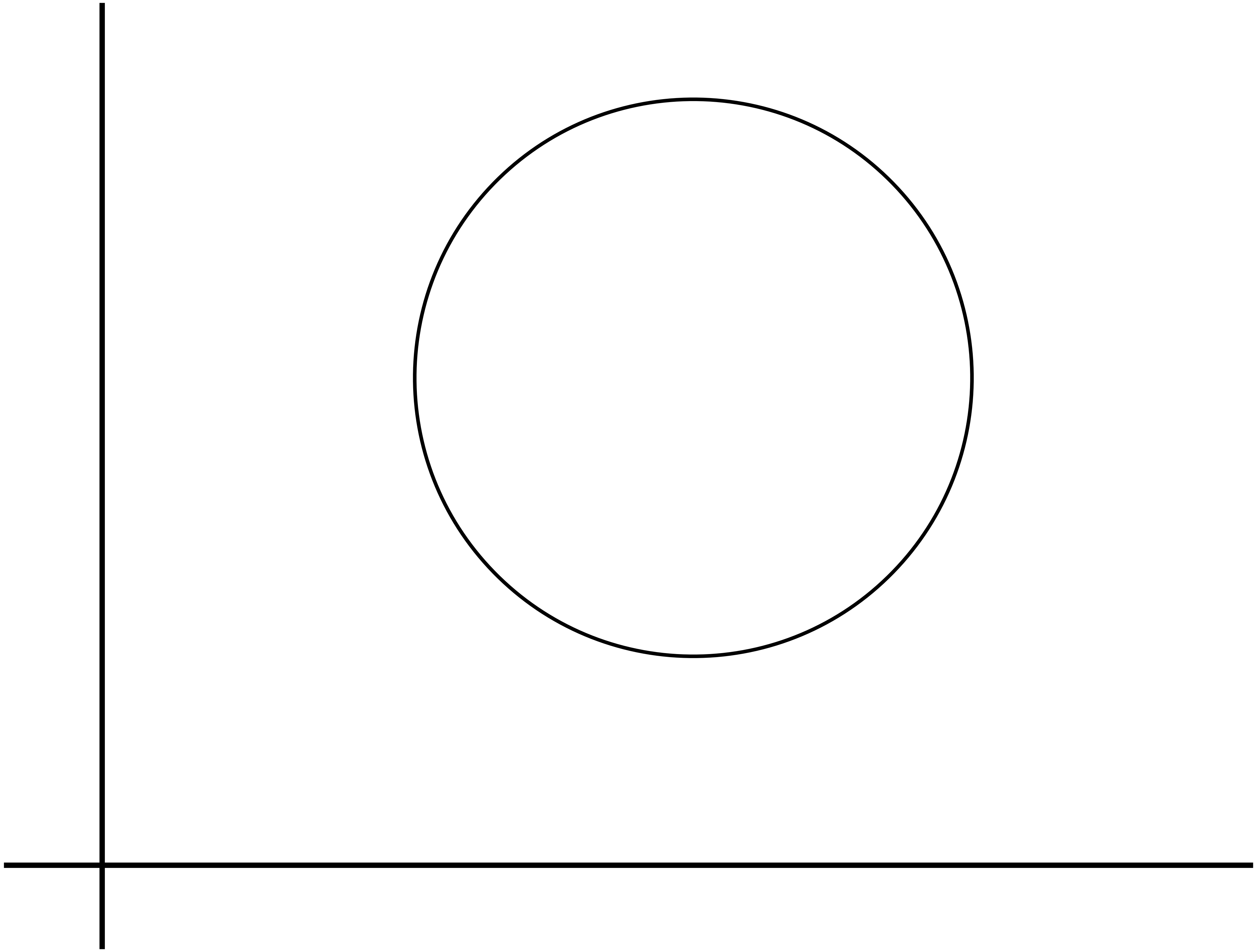
what is the point?

can you tell that PP^1 has a lift of the frobenius mod p^2 from this class already?

$$\psi_{12}(T) = -\frac{T}{y^p(y^p + pT)}$$

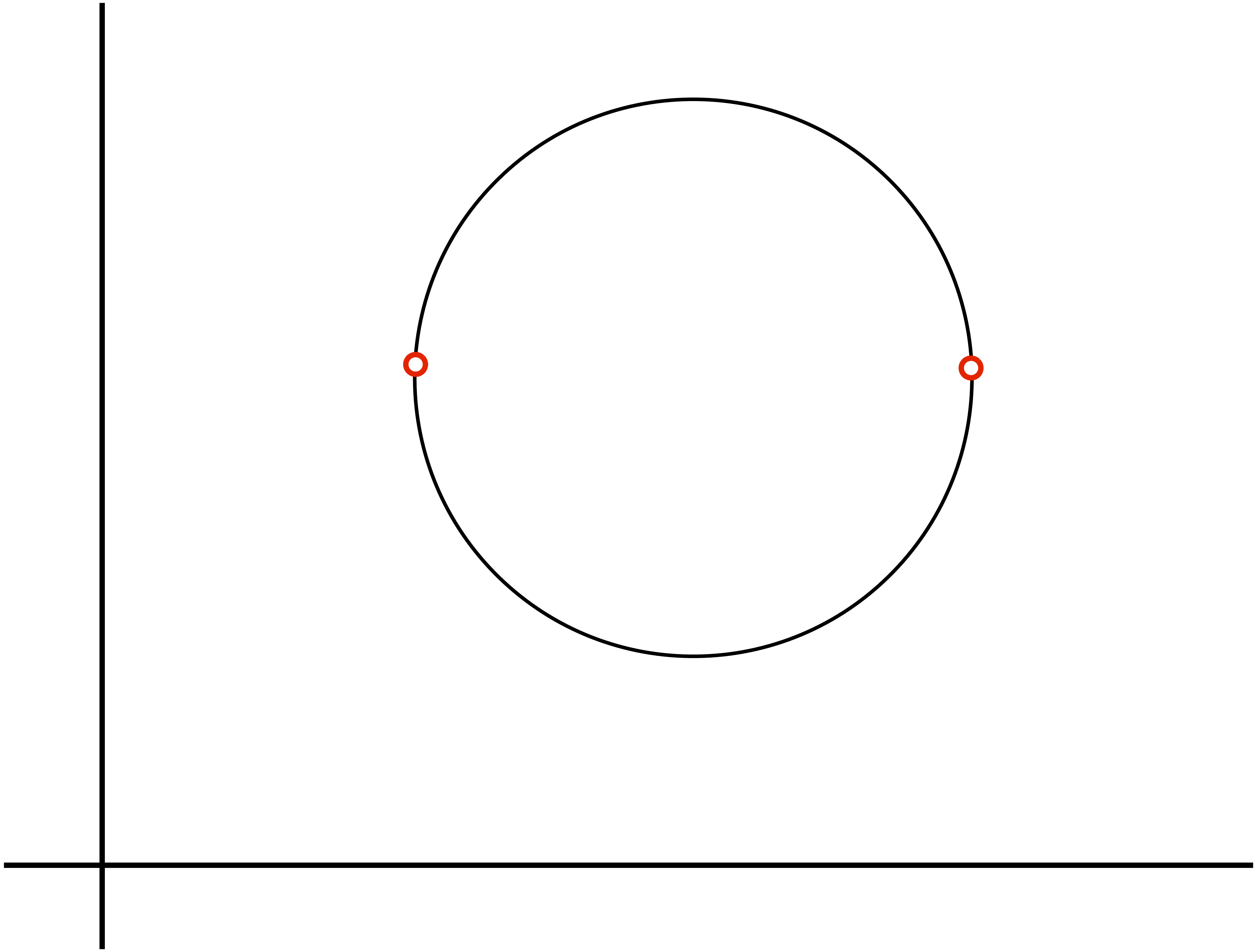
$$[\psi_{12}] \in H^1(\mathbb{P}^1, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1))$$

EXAMPLE $X : f(x, y) = 0$



$$X = \mathrm{Spec}\ R[x, y]/\langle f(x, y)\rangle$$
$$= U_1 \cup U_2$$

EXAMPLE $X : f(x, y) = 0$



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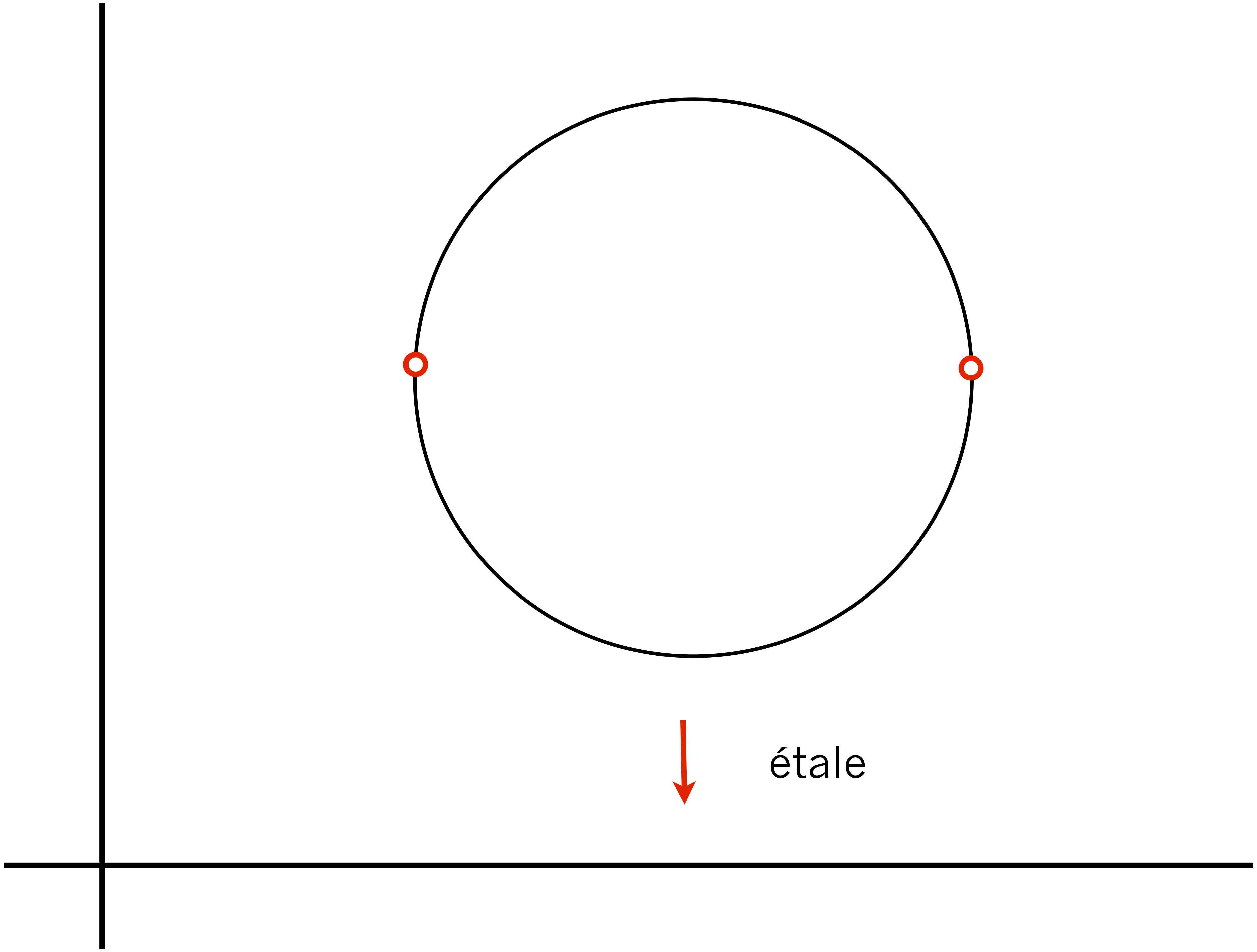

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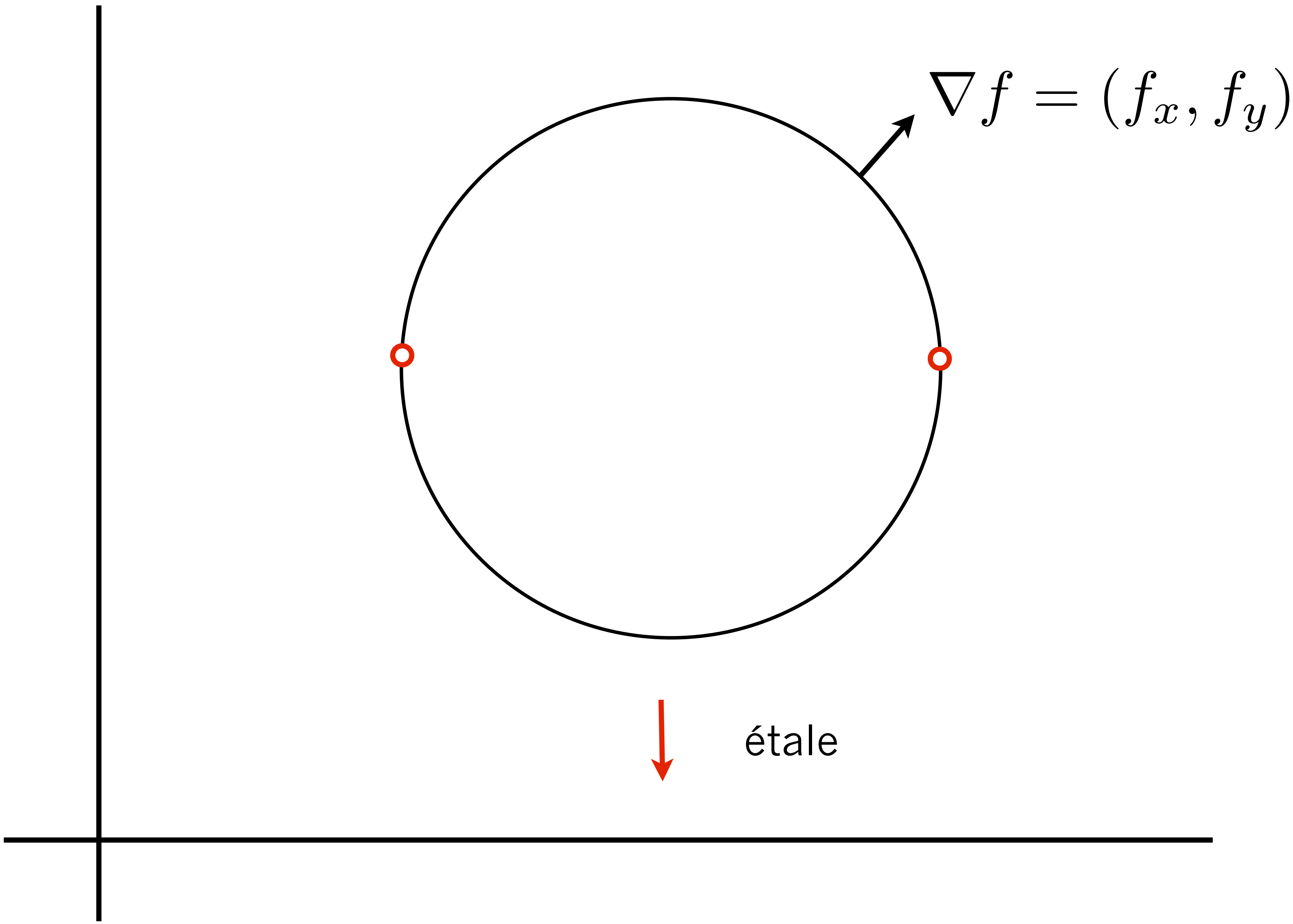

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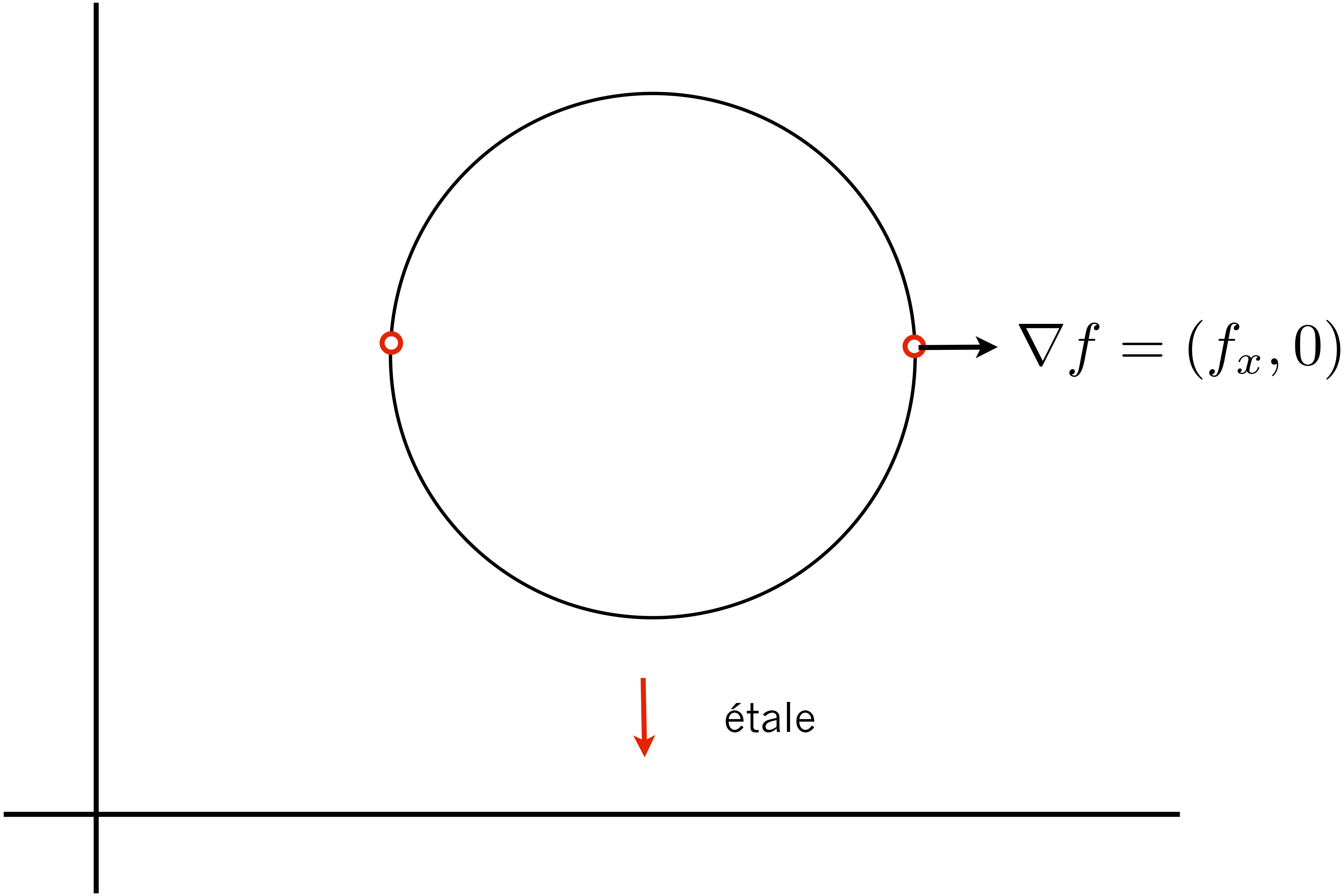

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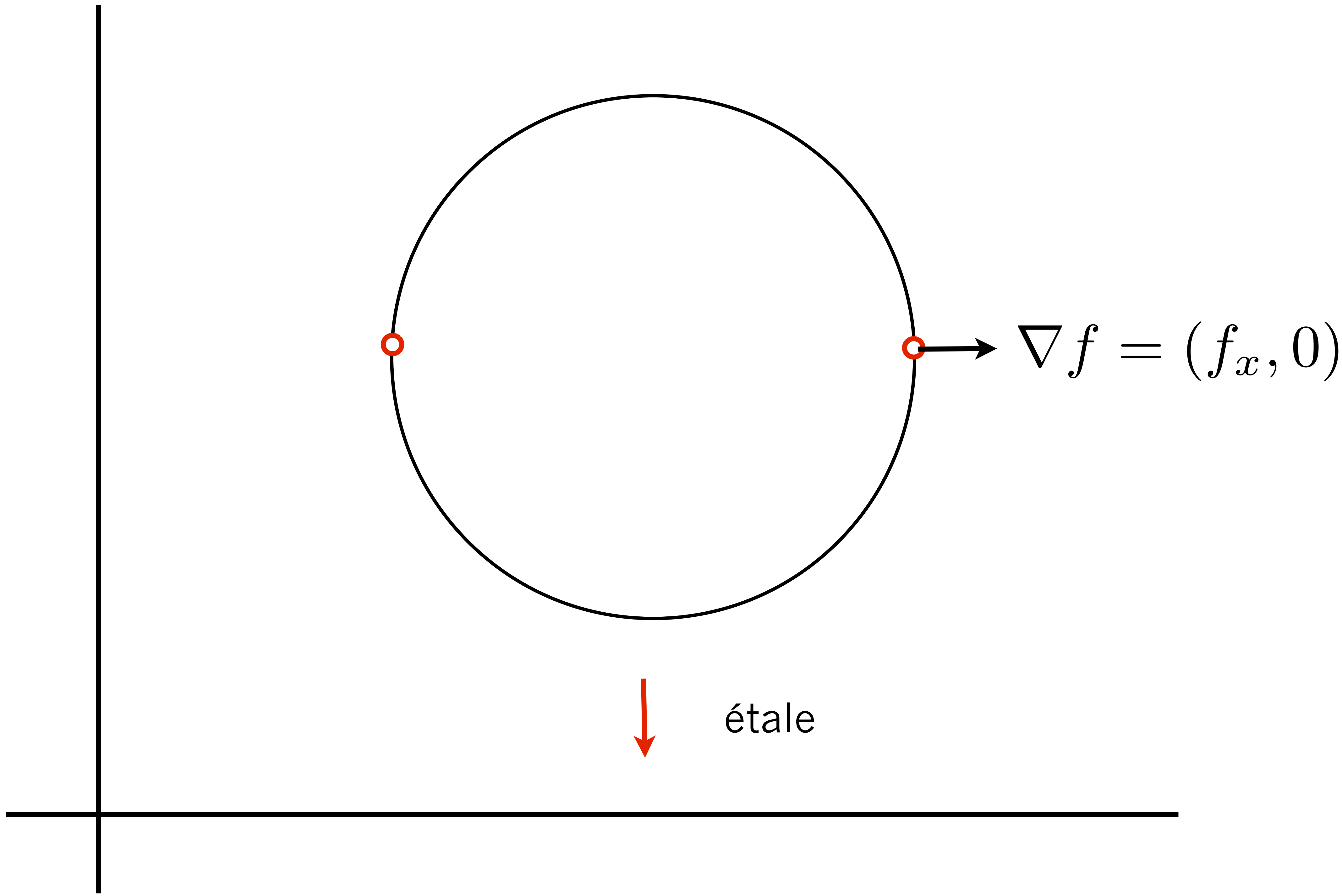

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$U_2 = X \setminus V(f_y)$

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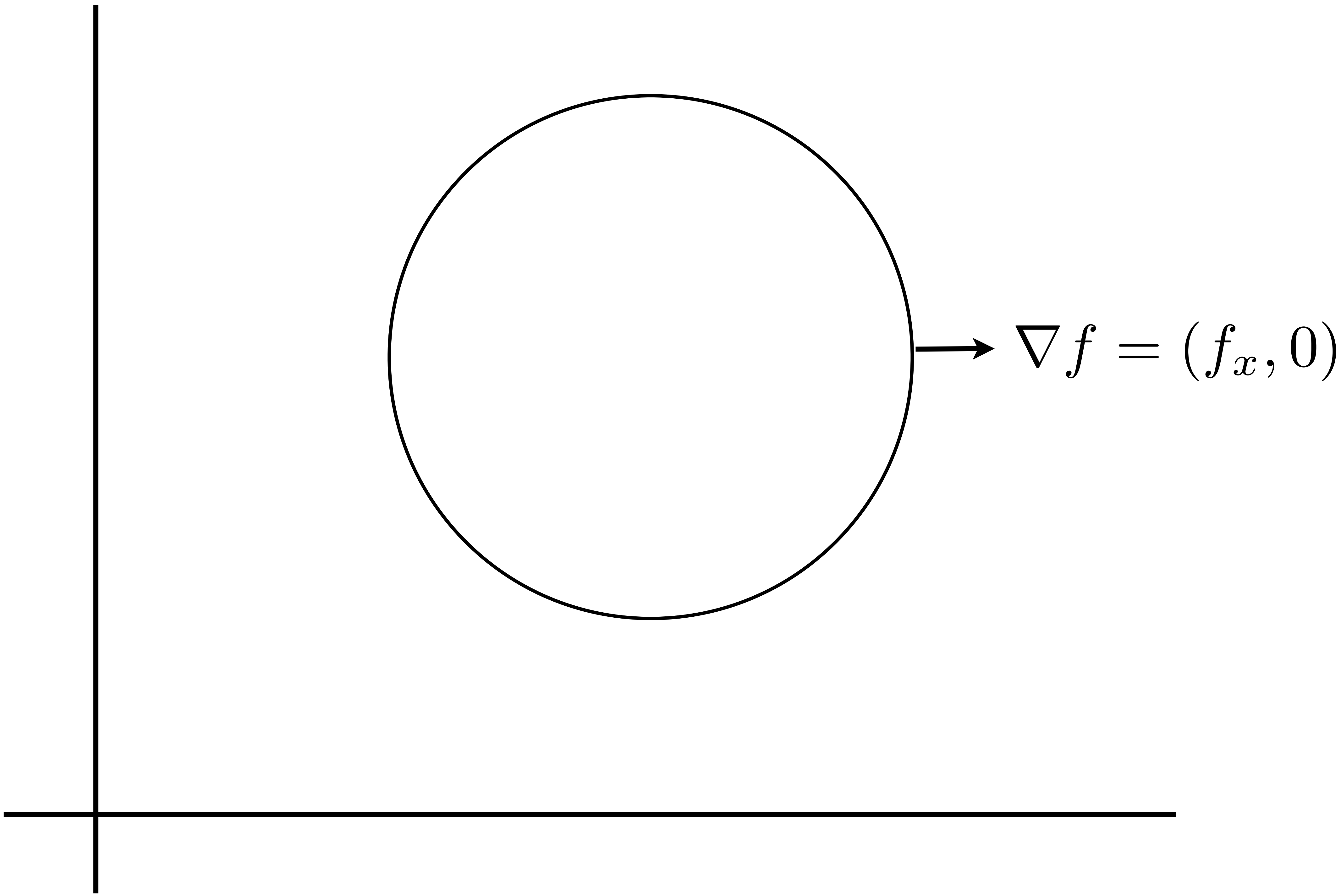

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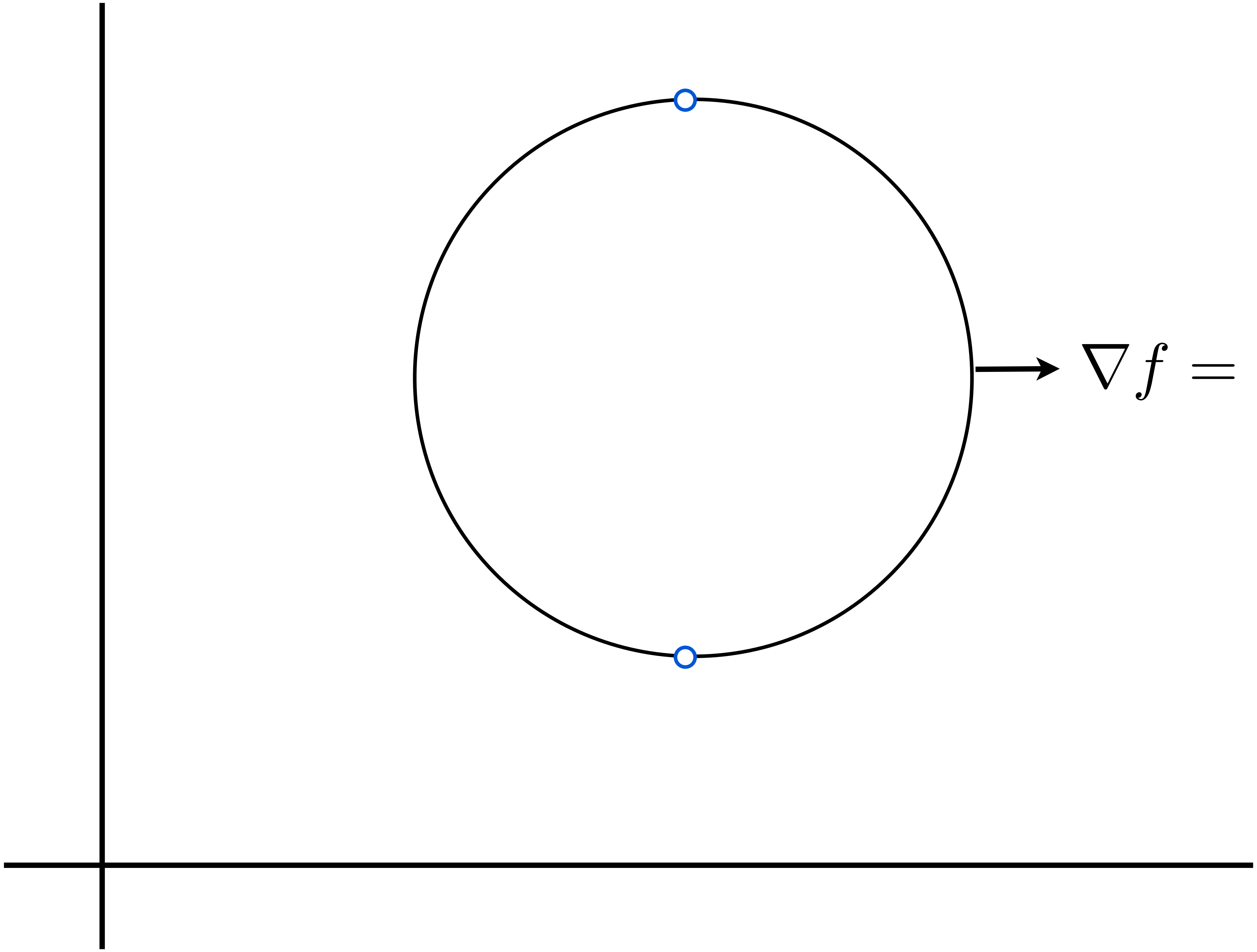

$$\nabla f = (f_x, 0)$$

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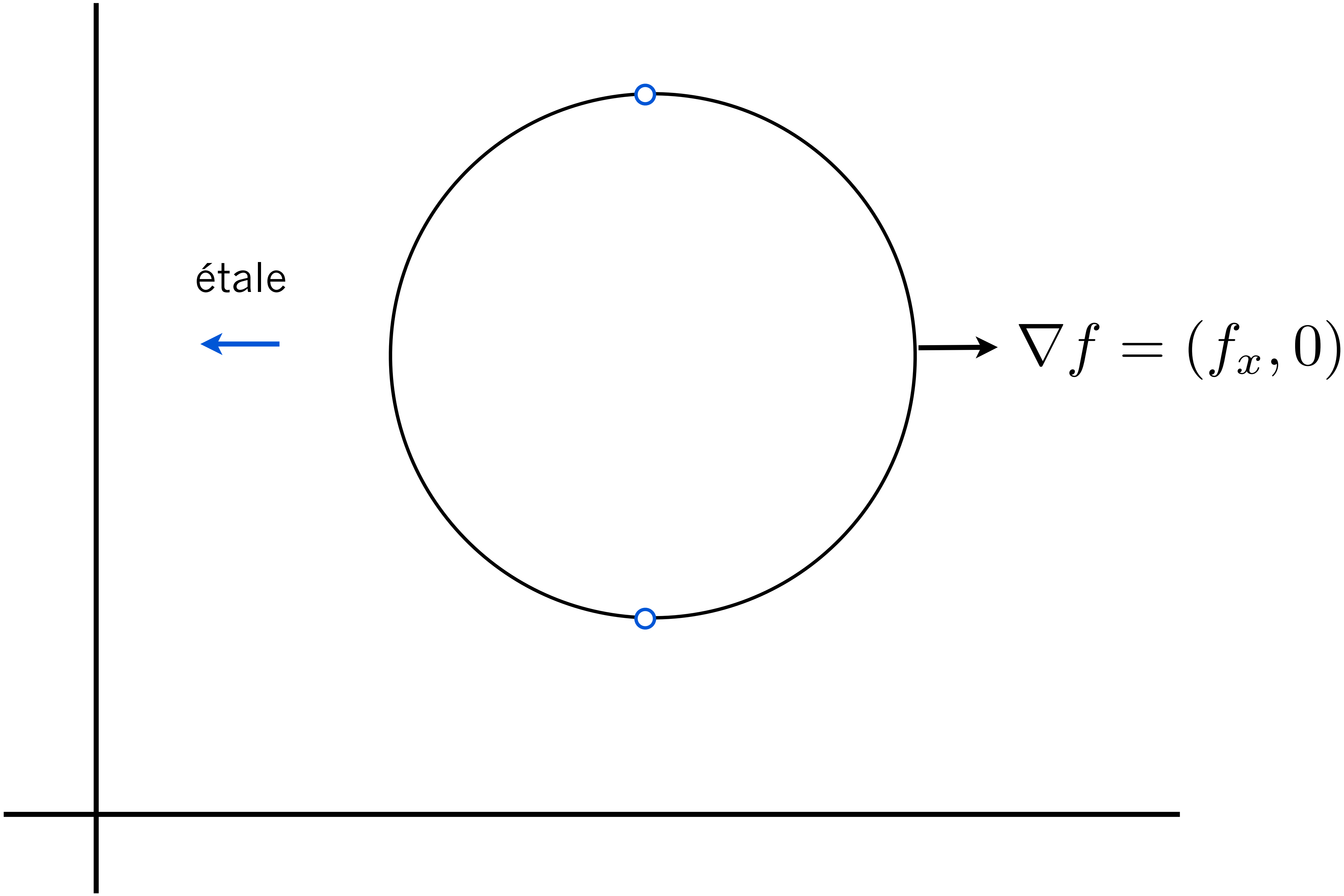

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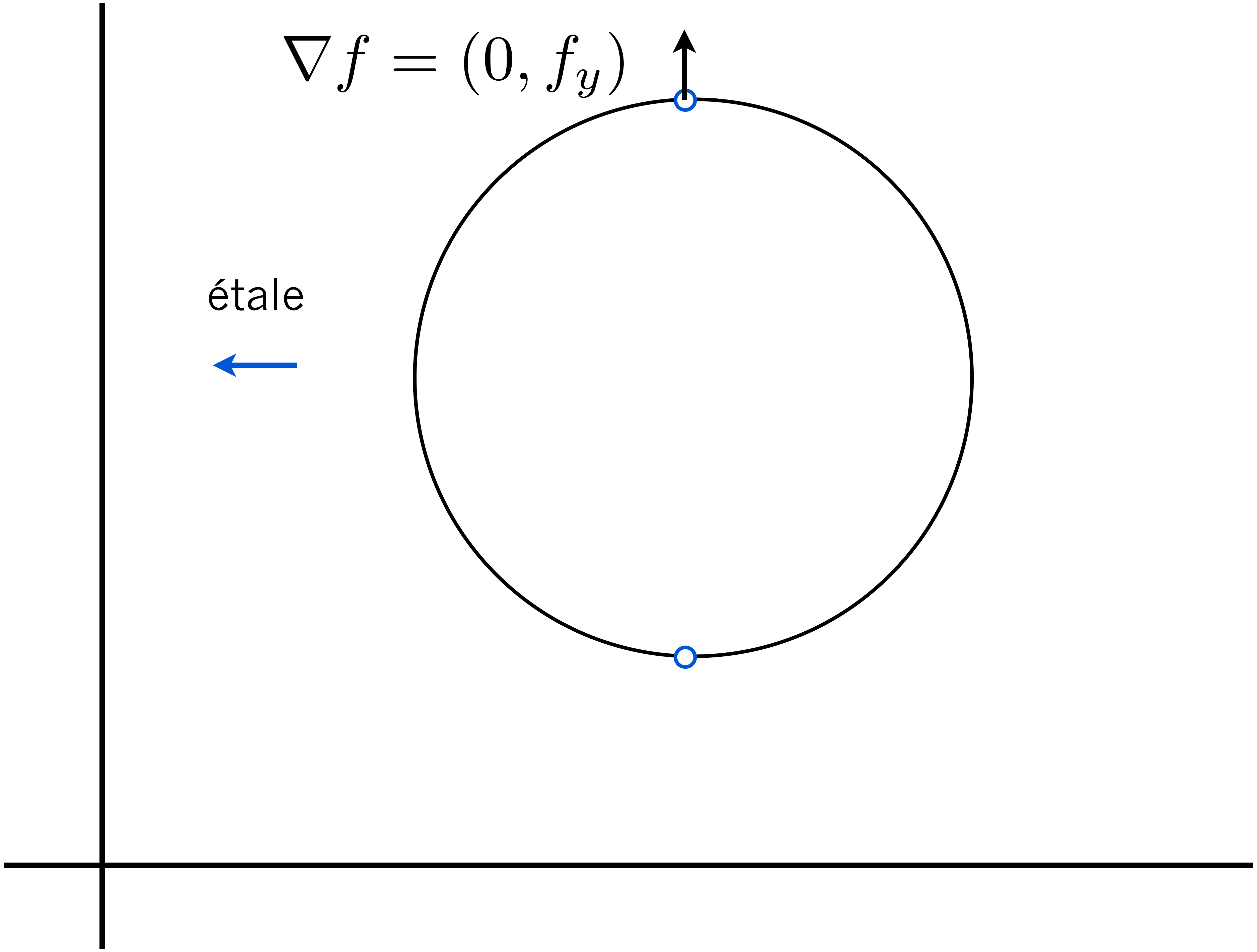

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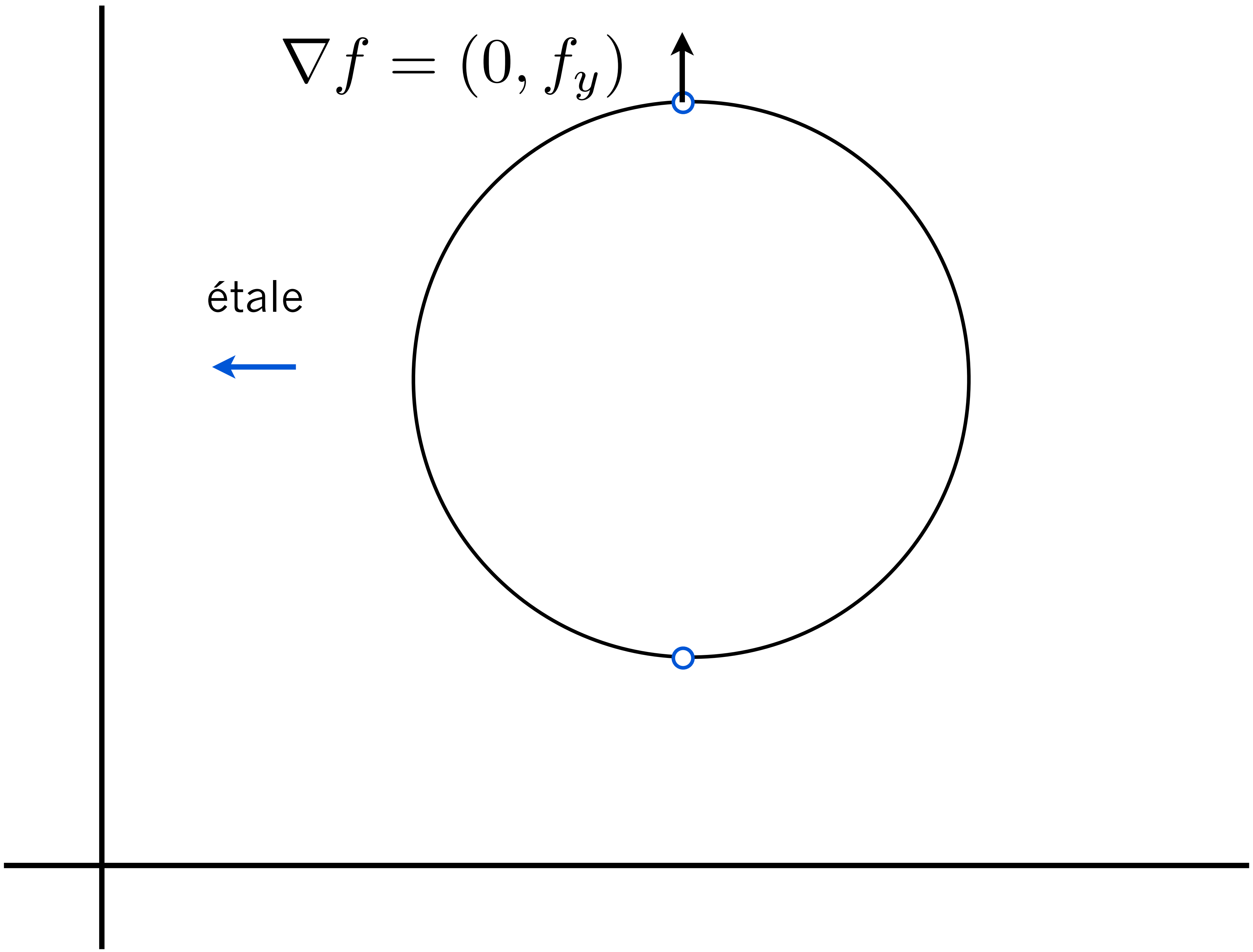

$U_1 = X \setminus V(f_x)$

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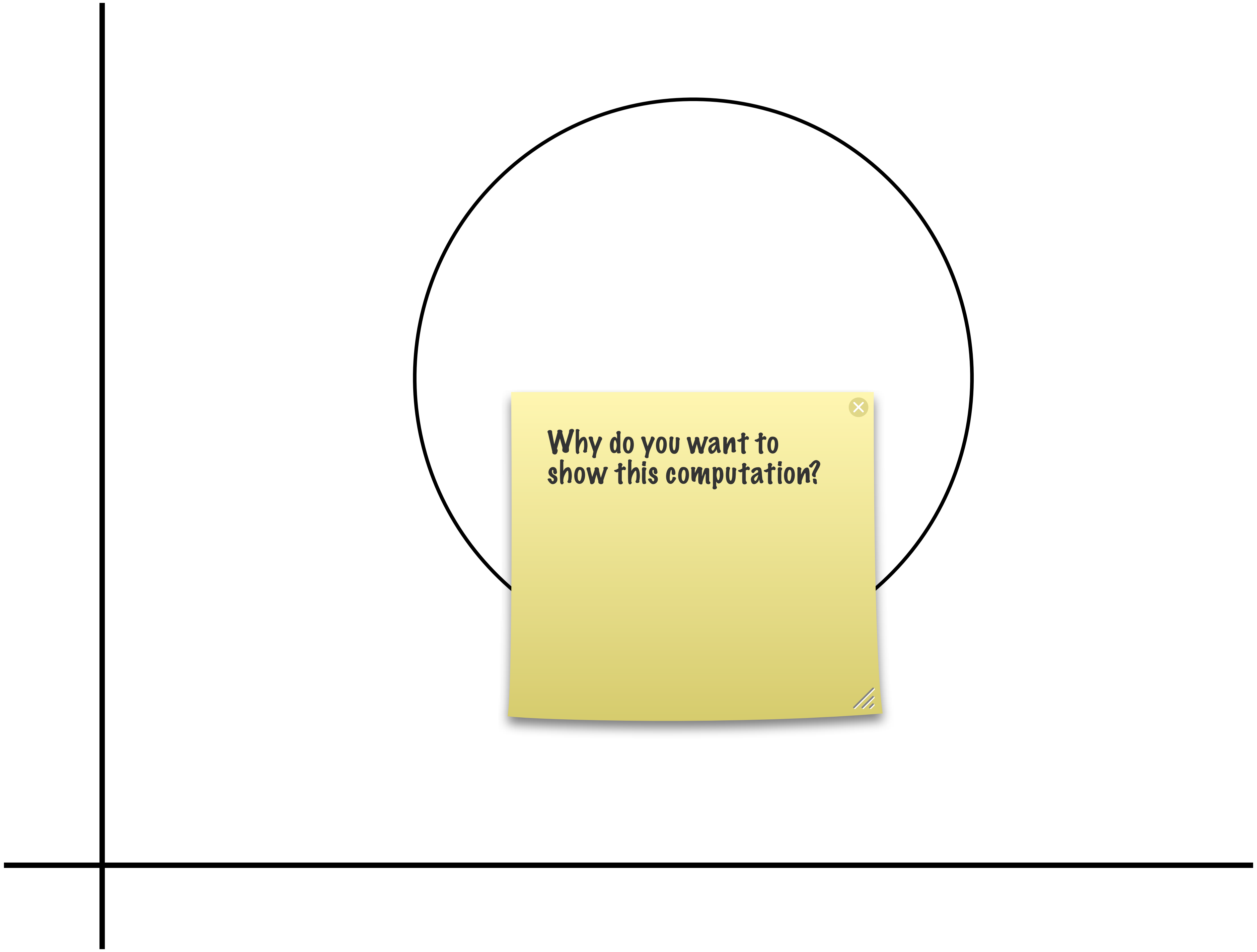

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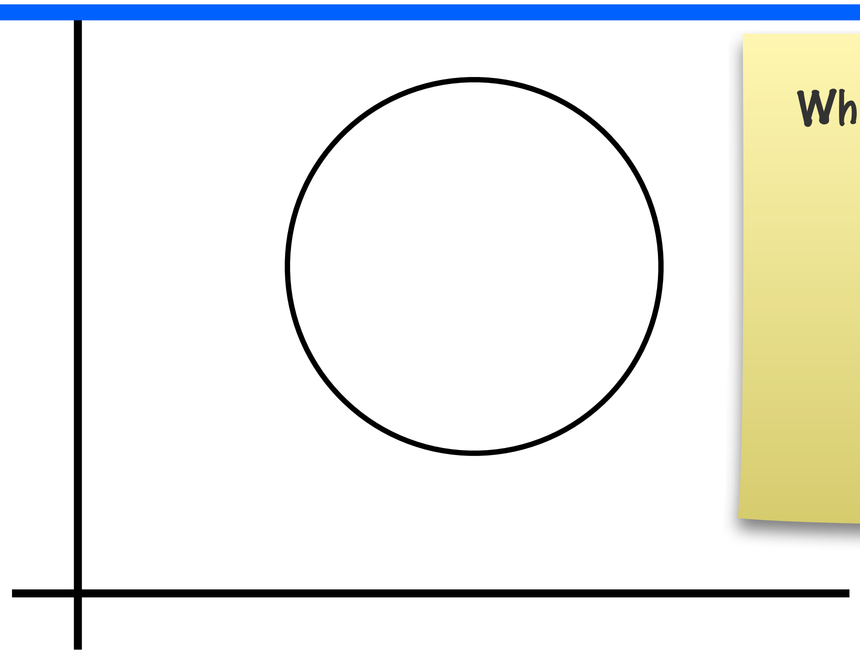

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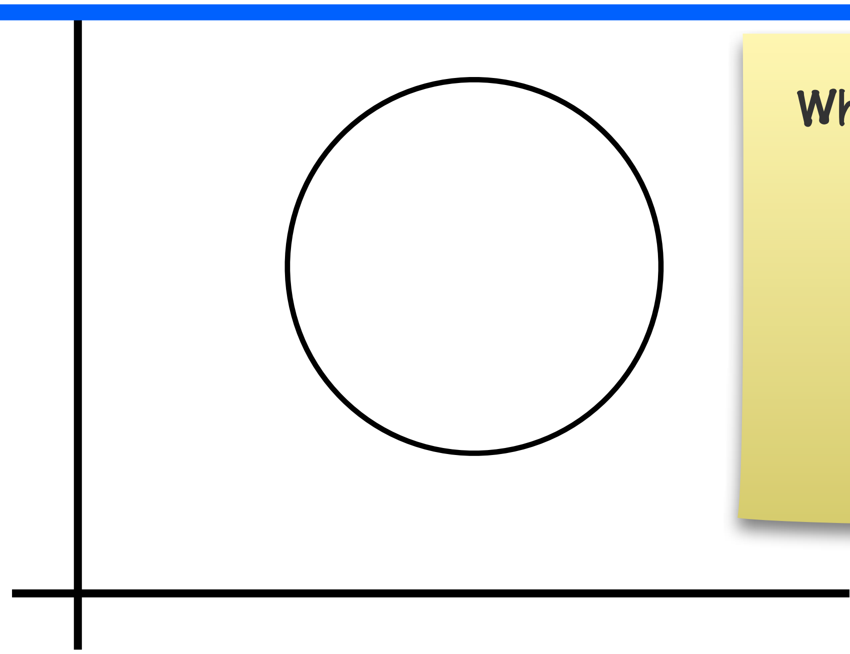

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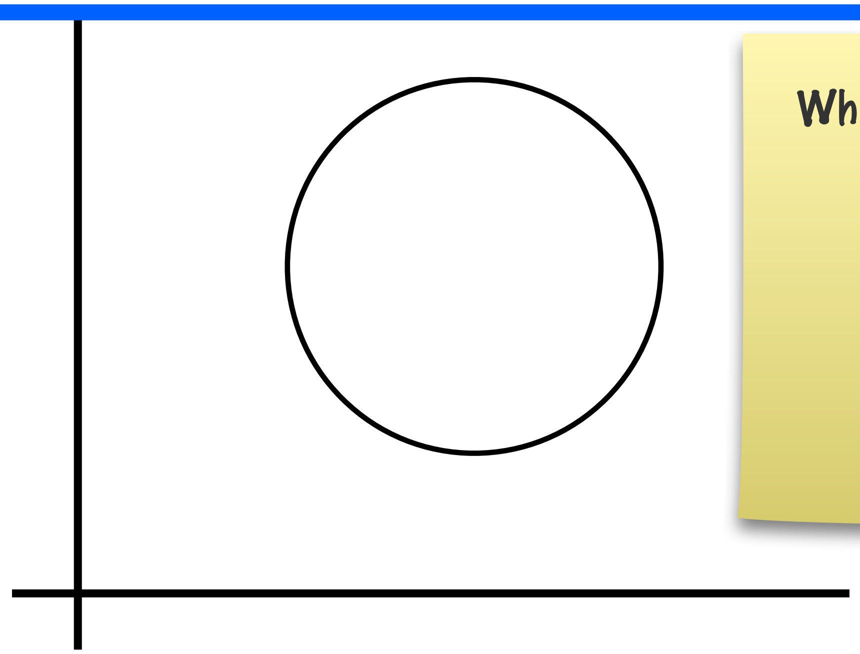

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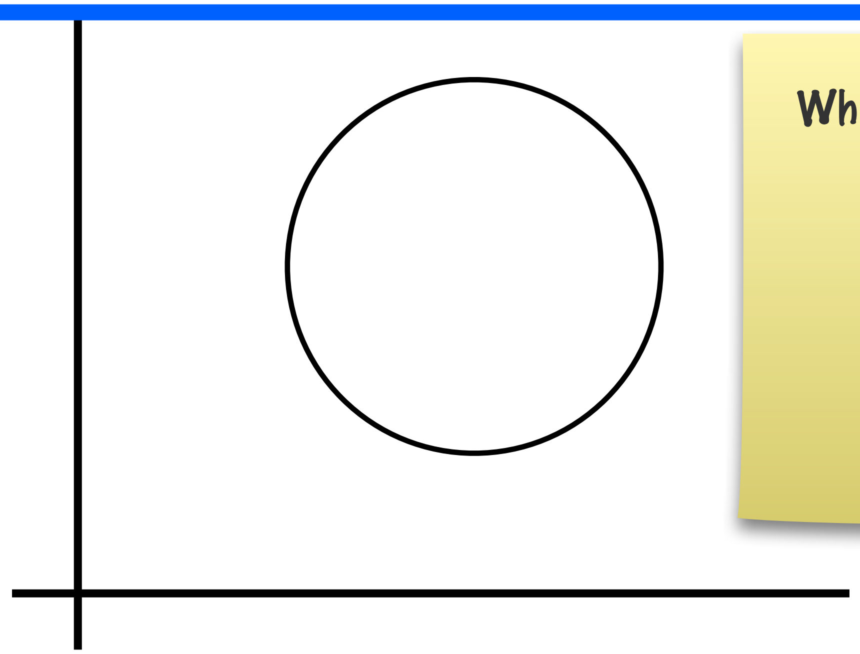

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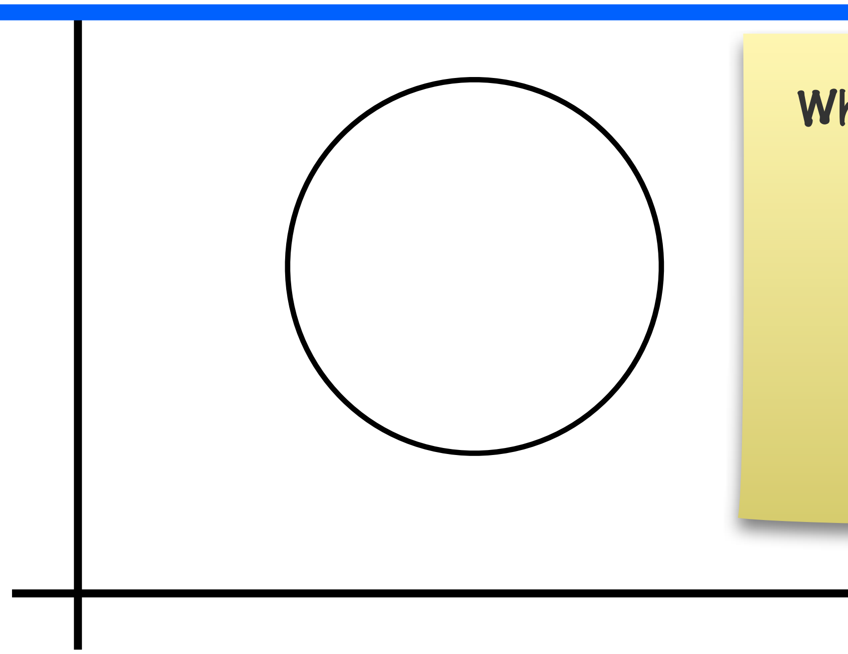

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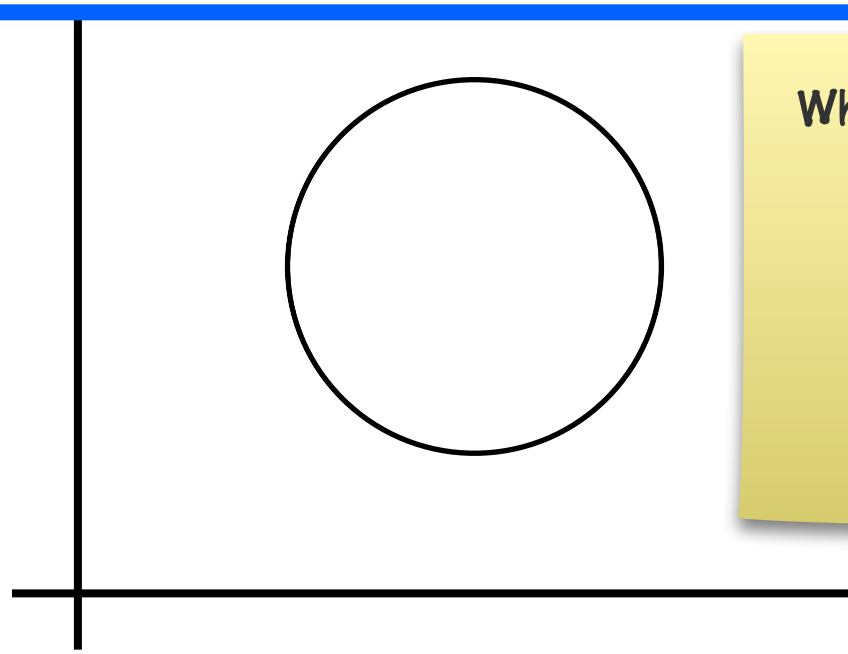

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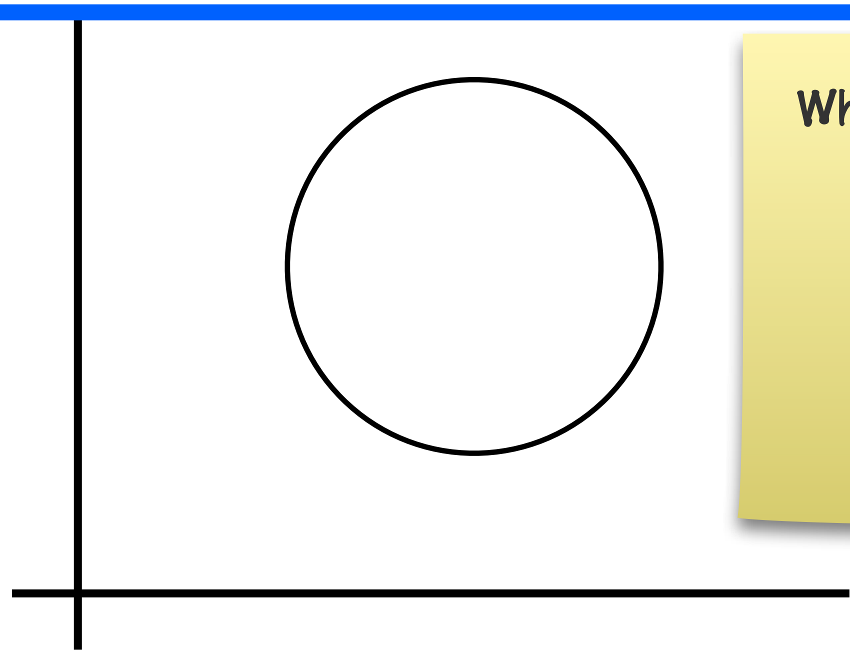

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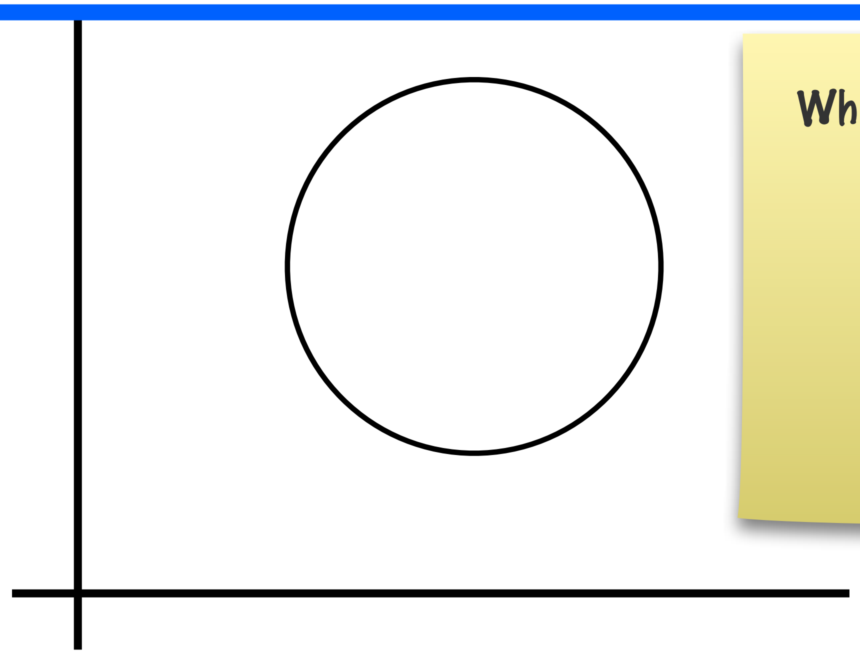

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$$g(T) = a_0 + a_1 T + \cdots \text{ then } g^{\phi}(T) = \phi(a_0) + \phi(a_1)T + \cdots$$

$$\boxed{X = \mathrm{Spec}\ R[x,y]/\langle f(x,y)\rangle}$$

$$f^{\phi}(x^p + p\dot{x}, y^p + p\dot{y}) \ = \ f^{\phi}(x^p, y^p) \qquad \text{0th order}$$

$$+ \quad p\left[\frac{\partial f^{\phi}}{\partial x}(x^p, y^p)\dot{x} + \frac{\partial f^{\phi}}{\partial y}(x^p, y^p)\dot{y}\right] \qquad \text{1st order}$$

$$+ \ \frac{p^2}{2}\left[\frac{\partial^2 f^{\phi}}{\partial x^2}(x^p, y^p)\dot{x}^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x^p, y^p)\dot{x}\dot{y} + \frac{\partial^2 f^{\phi}}{\partial y^2}(x^p, y^p)\dot{y}^2\right]$$

2nd order

$$+\cdots$$

$$= \ \sum_{d \geq 0} \frac{p^d}{d!} h_d$$

various orders

$$\boxed{X = \mathrm{Spec}\ R[x,y]/\langle f(x,y)\rangle}$$

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Transition maps take on a very particular form!

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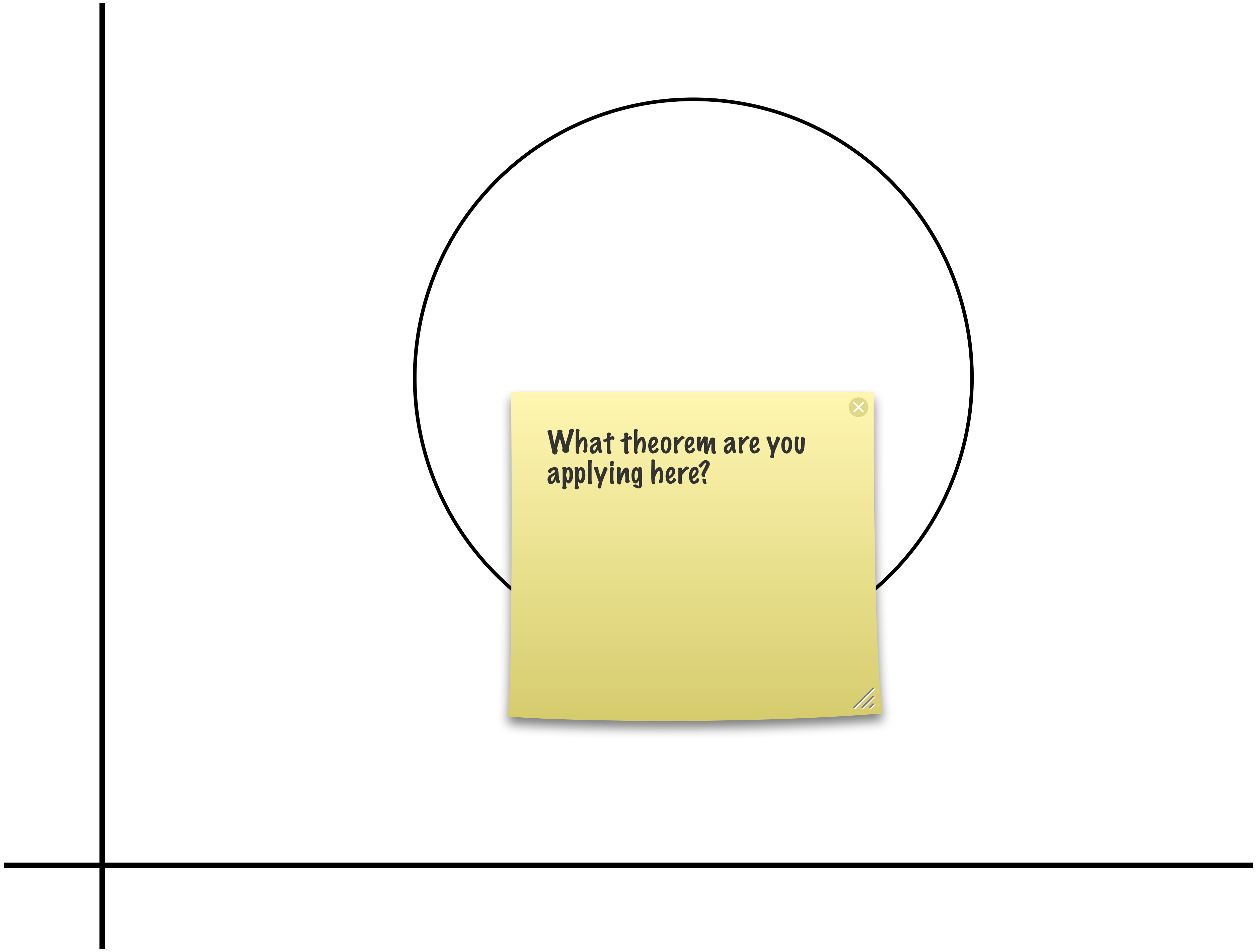

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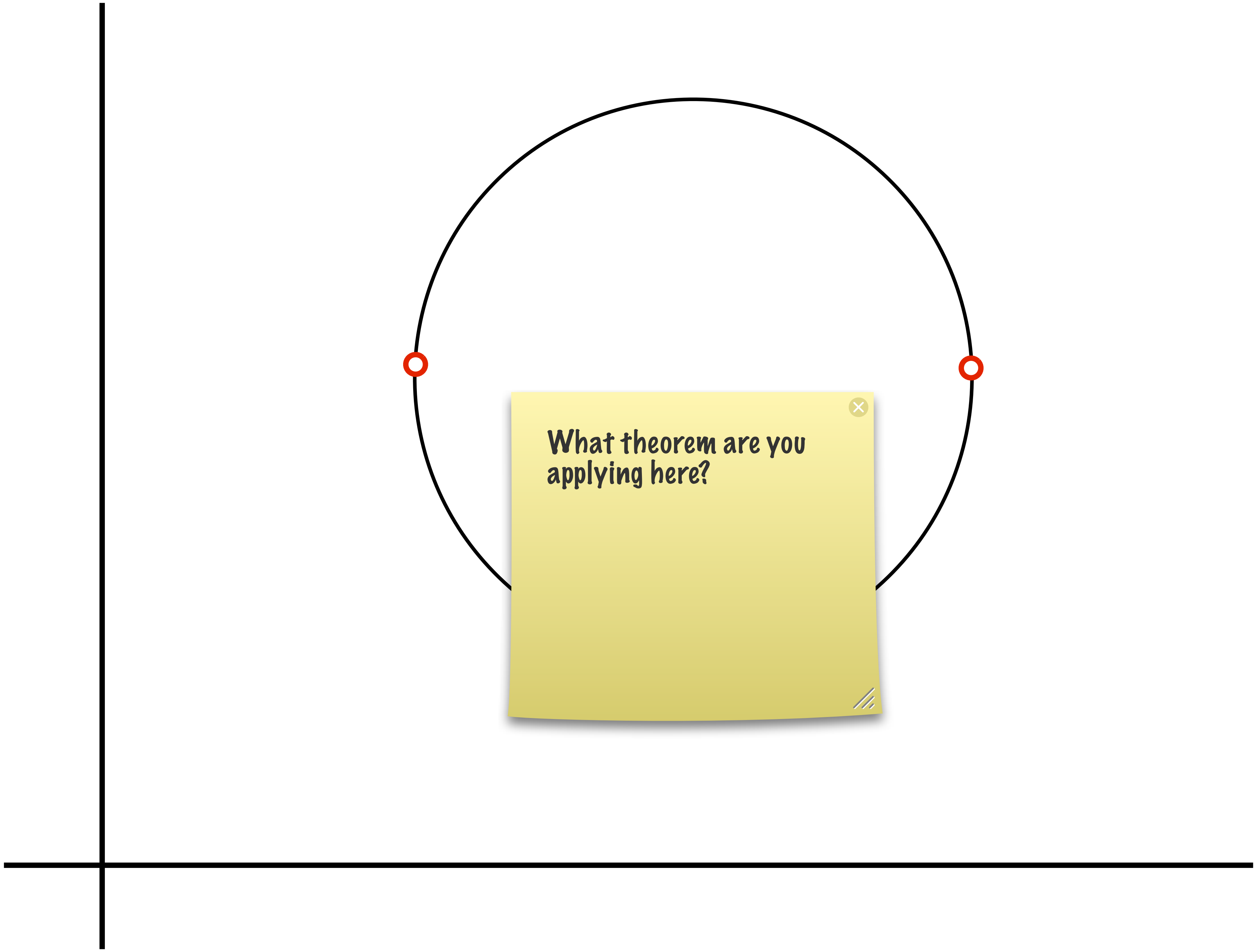

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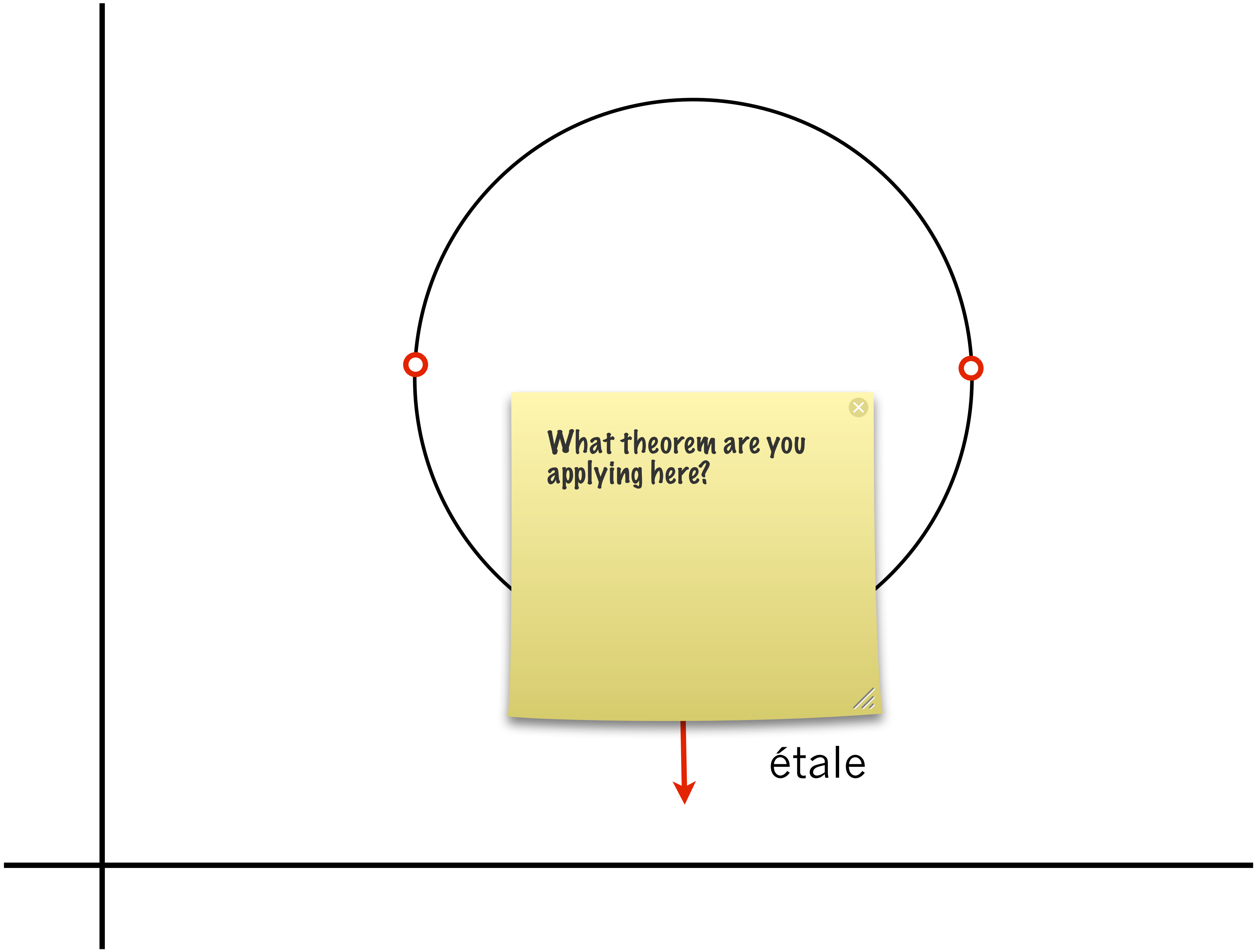

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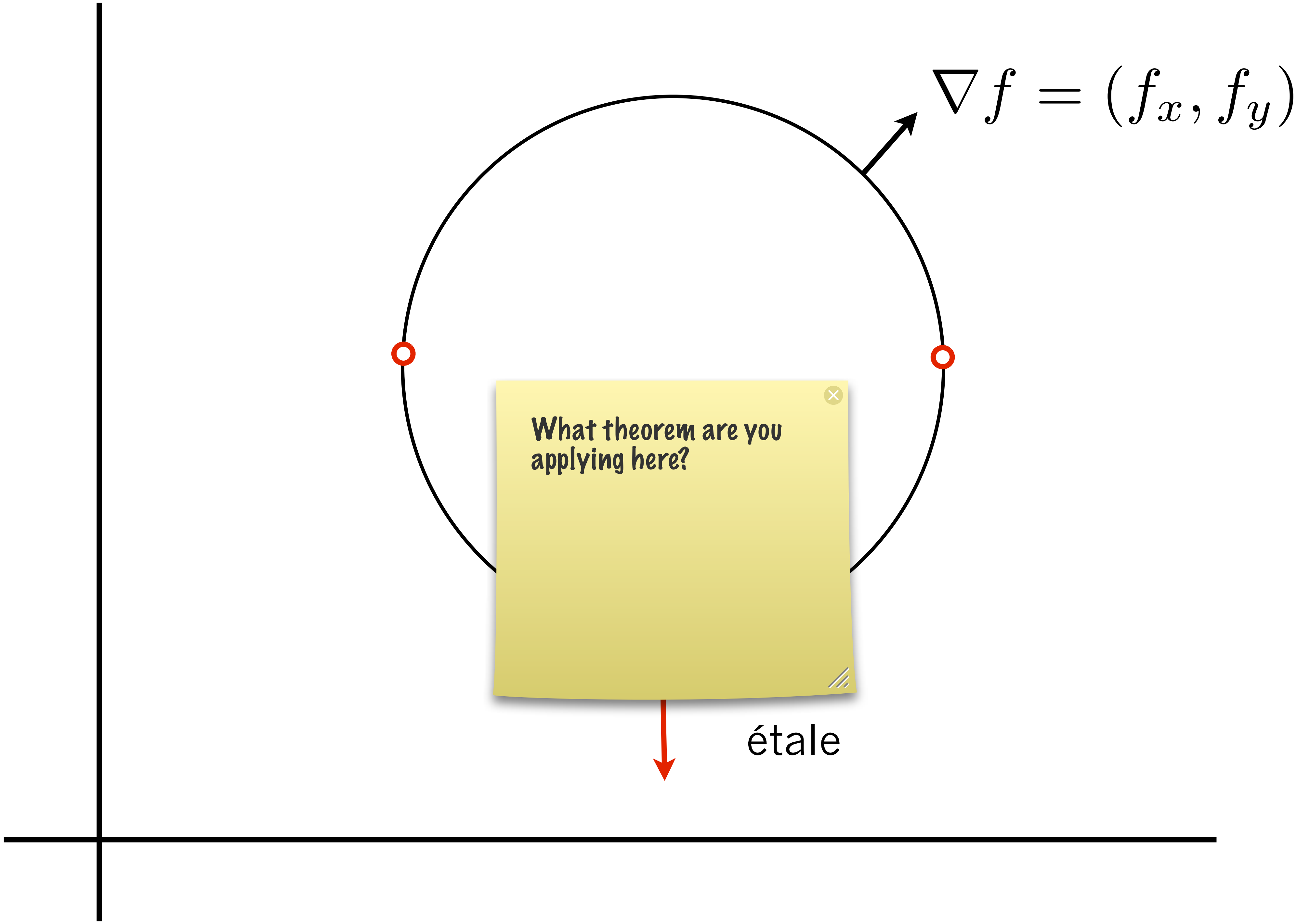

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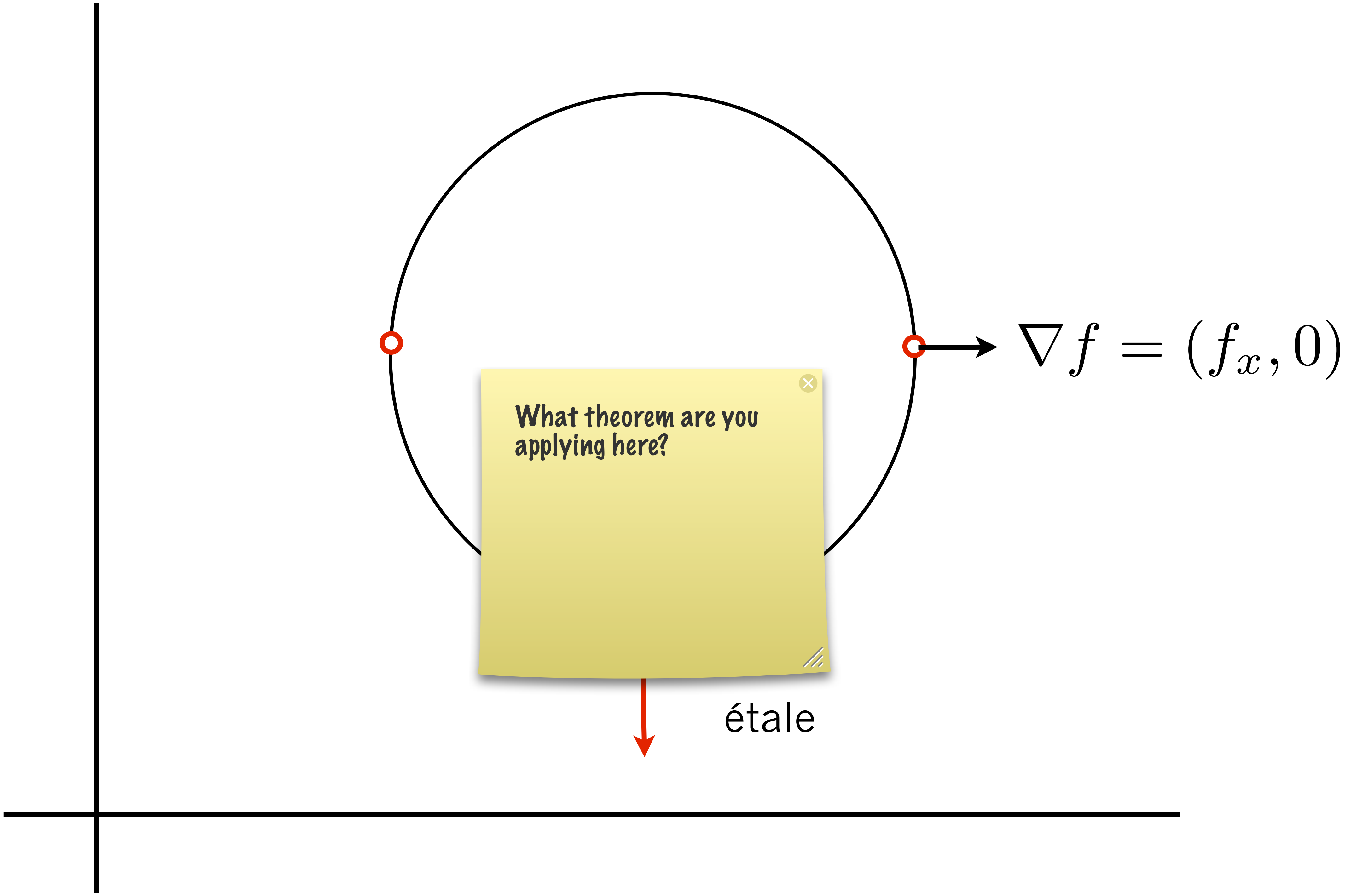

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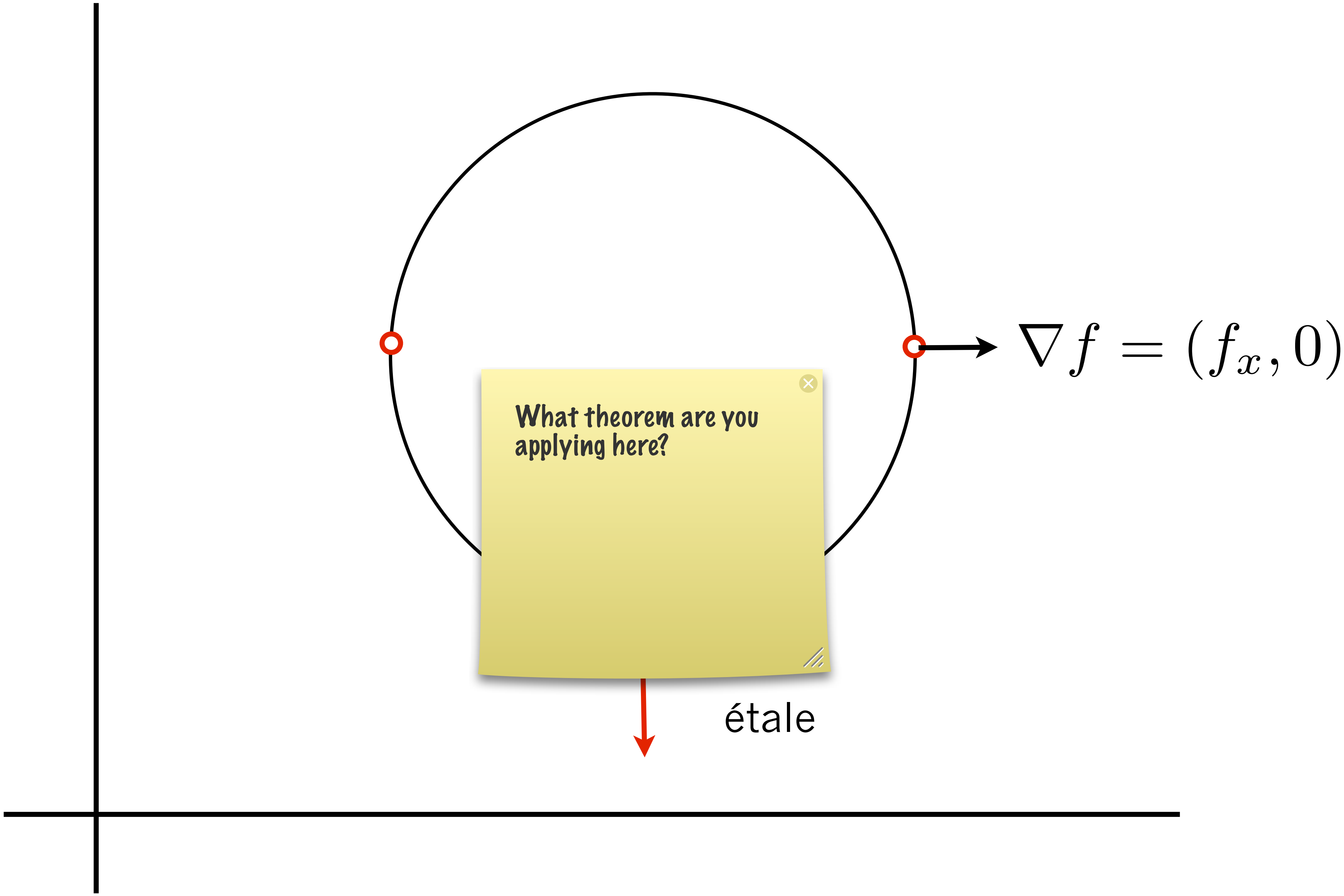

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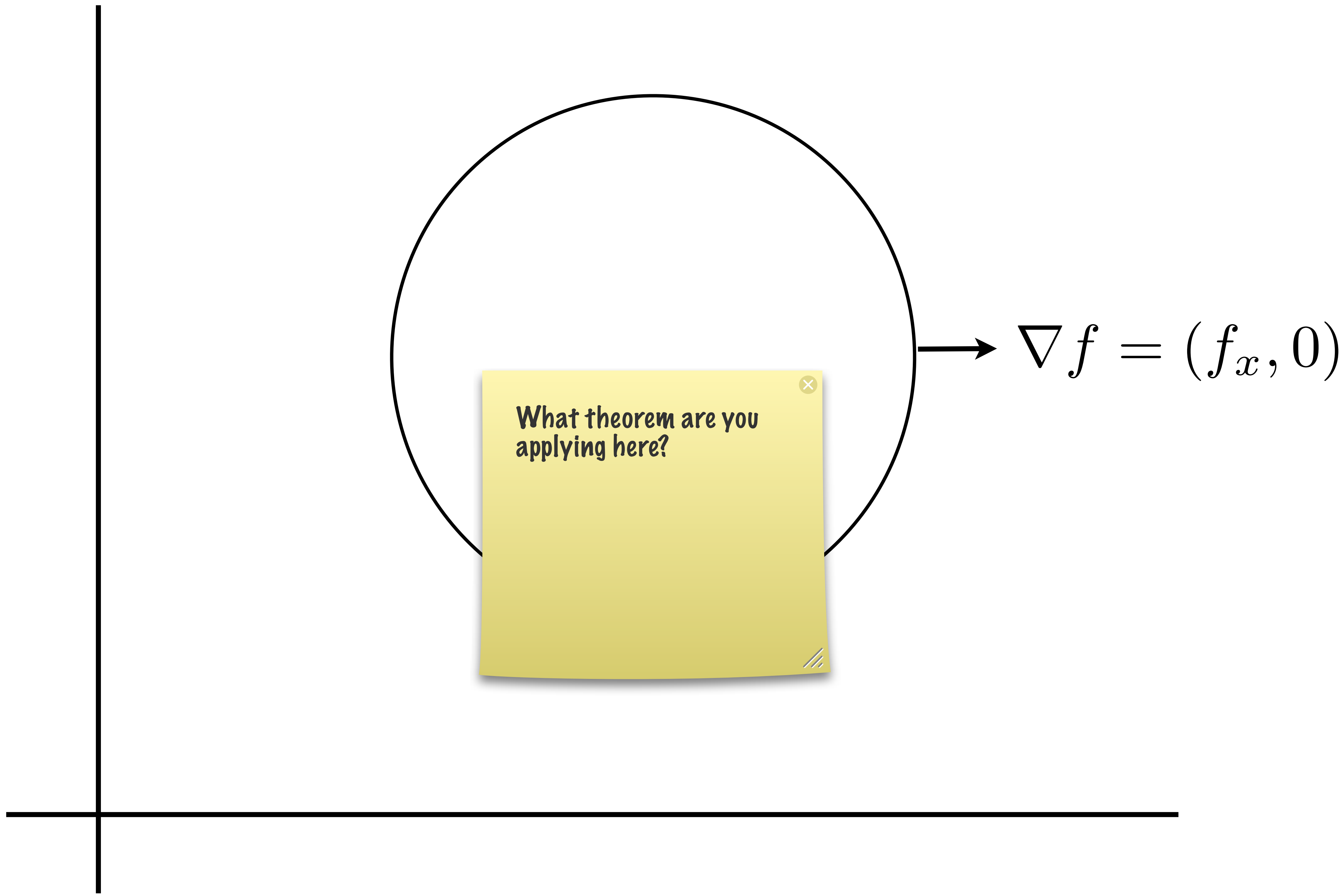

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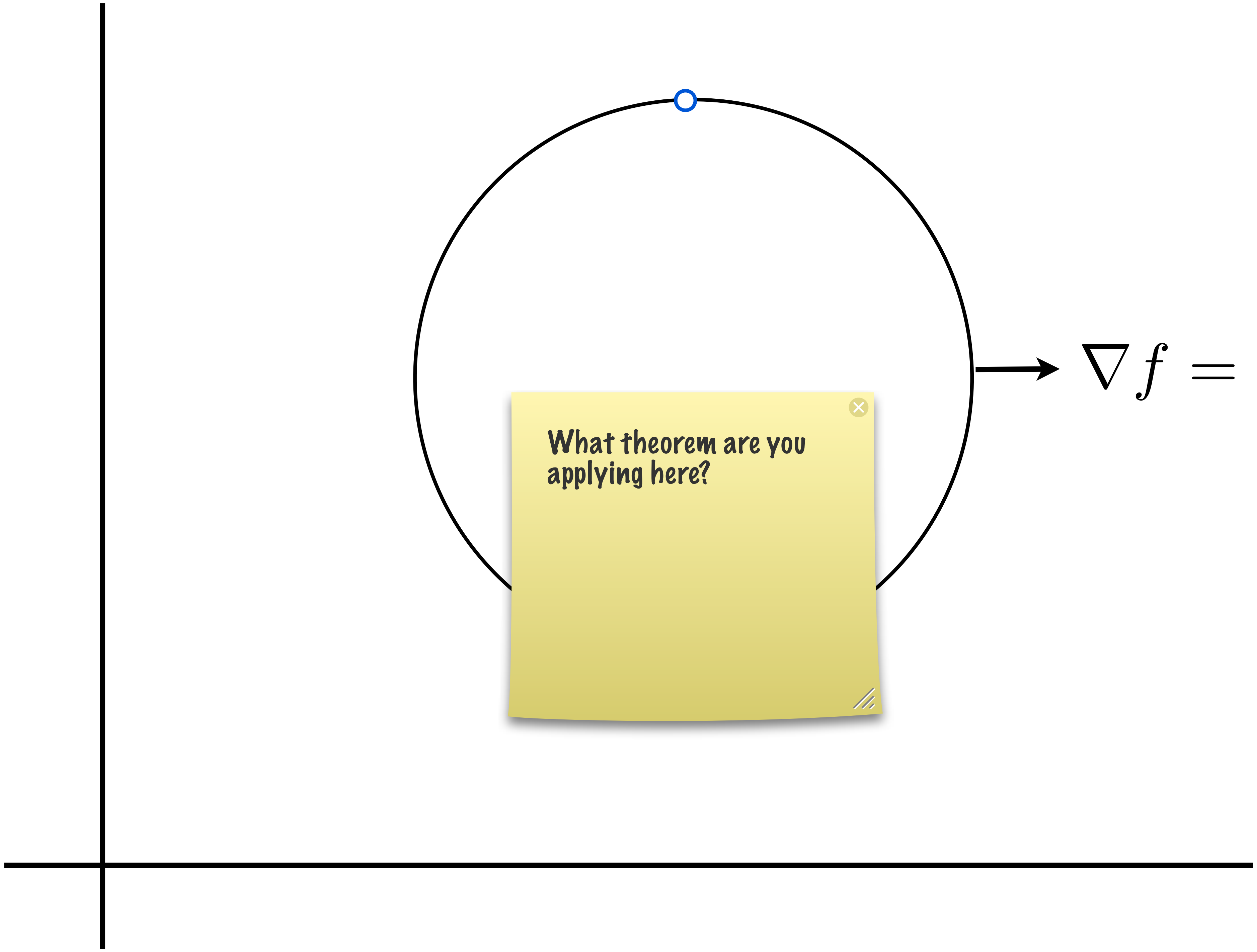

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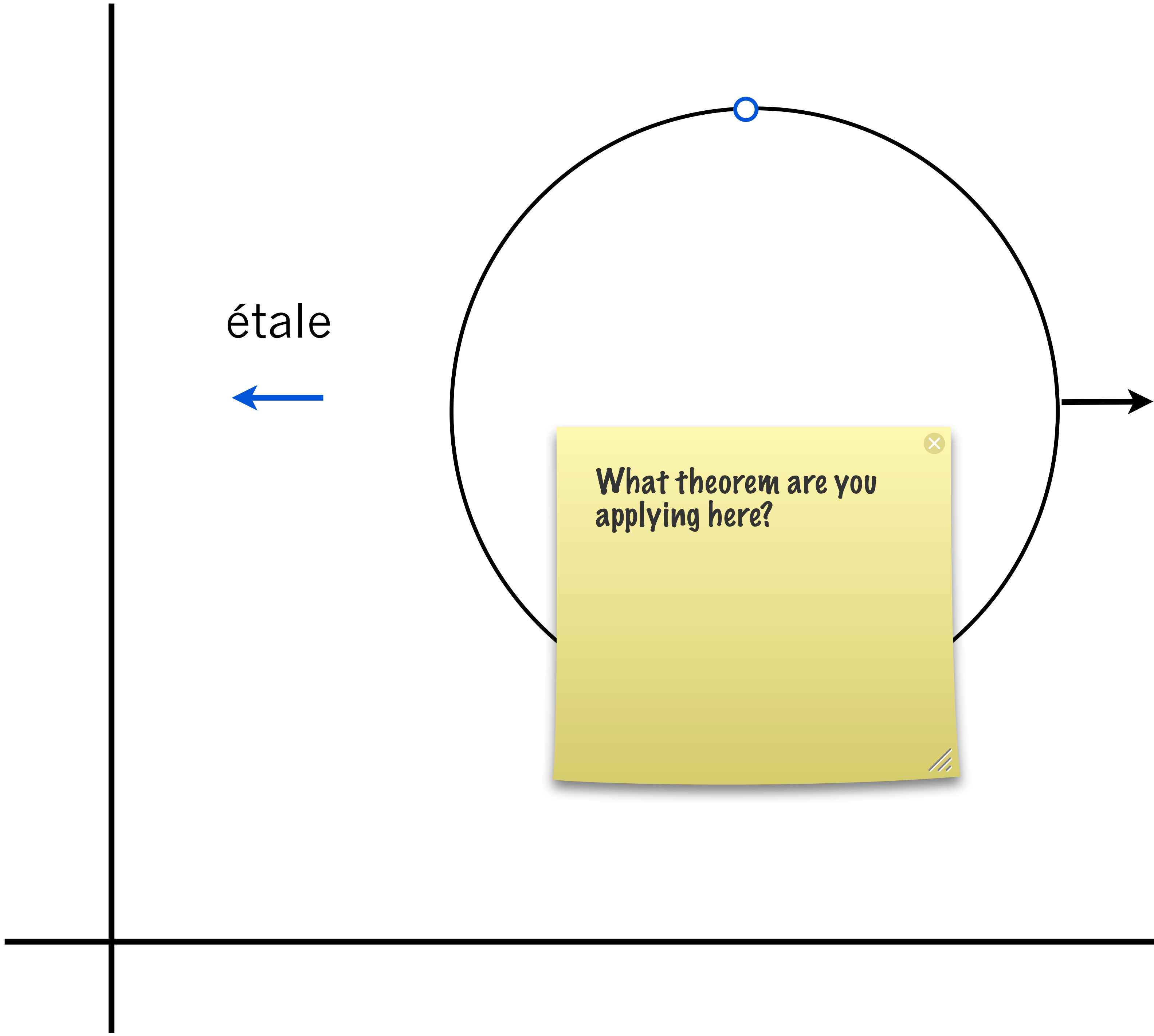

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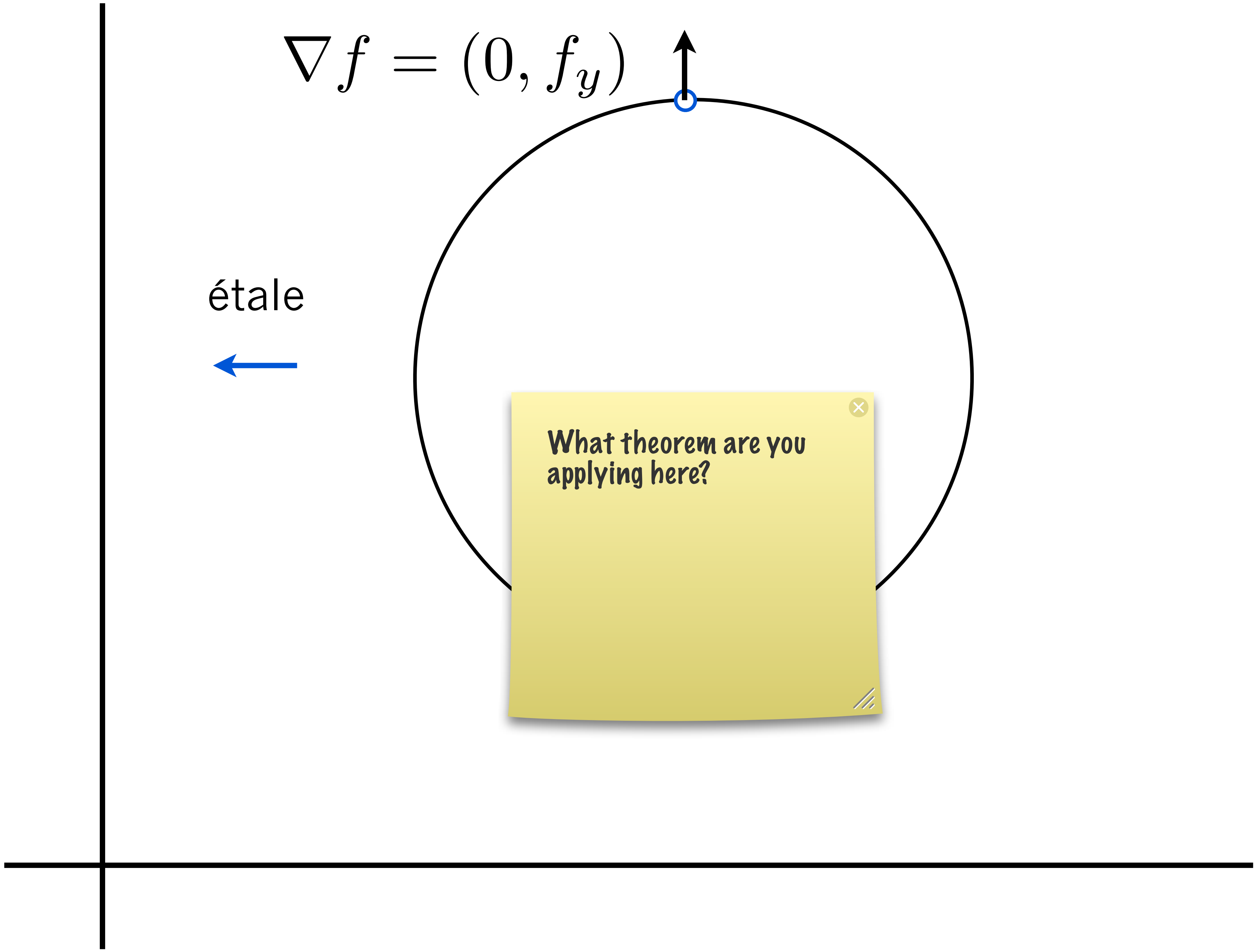

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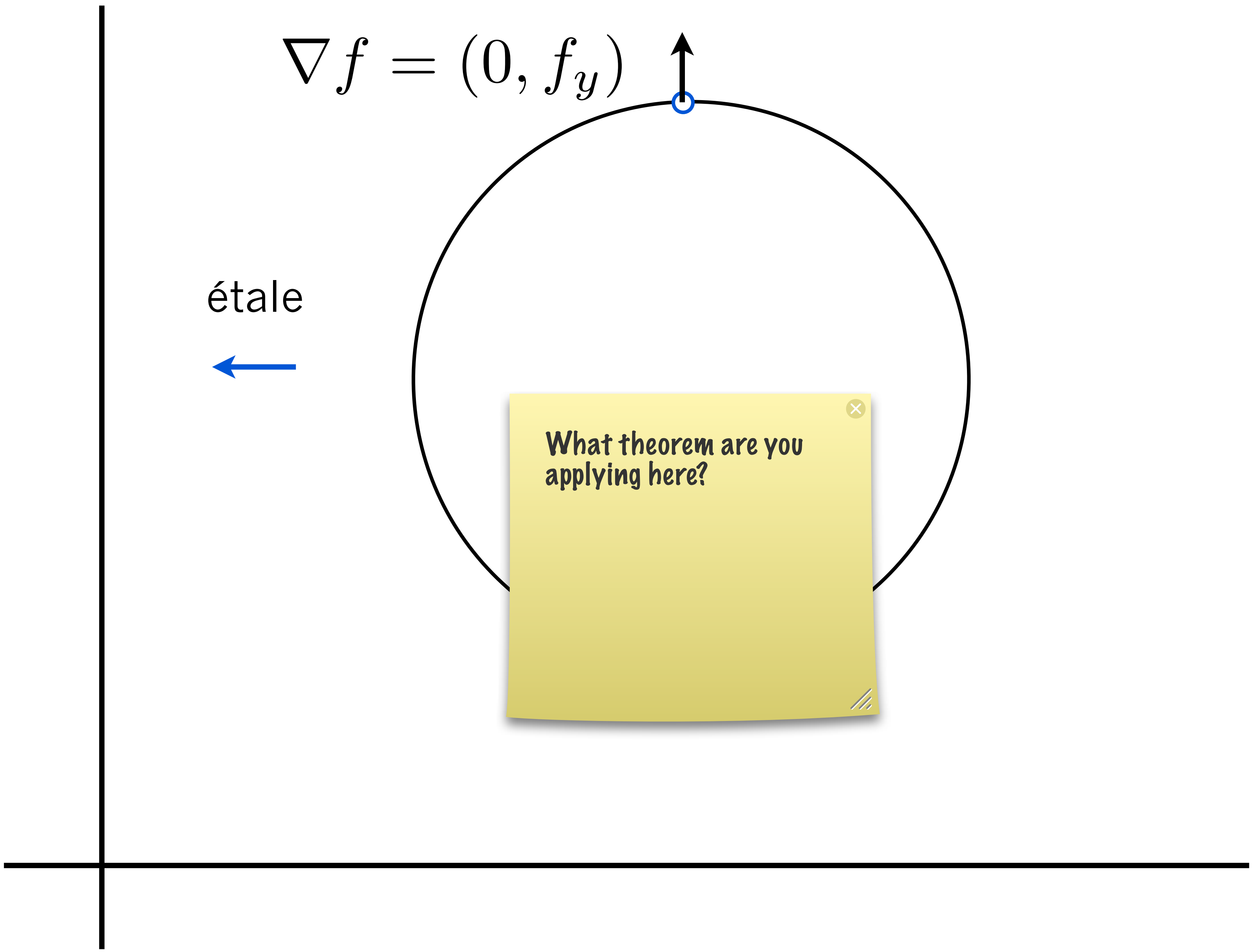

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What was the goal?

Does this say that transition maps lie in wacky subgroups?

Convinced you that theorem is plausible.

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What is the point of this slide?

What does the coeycle that determines reducitons look like?

A_2

(transition maps for $J^1(X) \mod p^2$)

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Example 1

$$\psi(T) = T + pT^2$$

$$\psi^{-1}(T \quad T \quad T^2$$

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Example 2

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$$= (a_0 + a_1 b_0 + pa_2 b_0^2) + (a_1 b_1 + 2pa_2 b_0 b_1)T + p(a_1 b_2 + a_2 b_1^2)T^2$$

Prop/Def

$$\tau_2(c_0 + c_1 T + pc_2 T^2) := \frac{c_2}{c_1}$$

$$\tau_2(f \circ g) = \tau_2(f) \cdot m(g) + \tau_2(g)$$

$$\frac{a_1 b_2 + a_2 b_1^2}{a_1 b_1} = \frac{b_2}{b_1} + \frac{a_2}{a_1} b_1$$

Convince you about elliptic curve phenomena being weird.

Prop.

You can NOT have the same physical affine bundle
th two different GL_1 structures

Multiple Structures?

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h ndles are built

$$\begin{tikzcd} L = \dfrac{\cdots \times \mathbb{A}^1}{\sim} \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

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\begin{array}{c}
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\Big\downarrow \pi \\
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$$
\begin{array}{ccc}
U_{ij} \times \mathbb{A}^1 & \xrightarrow{f_i} & U_{ij} \times \mathbb{A}^1 \\
\psi_{ij} \Big\downarrow & & \Big\downarrow \psi'_{ij} \\
U_{ji} \times \mathbb{A}^1 & & U_{ji} \times \mathbb{A}^1
\end{array}
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$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

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$$b_{ij} = f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} (f_i^{-1})'(T)$$

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$$b_{ij} = f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} (f_i^{-1})'(T)$$

$$= f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} / f'_i(f_i^{-1}(T))$$

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$$T = f_i(S)$$

$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

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$$T = f_i(S)$$

$$= f'_j(a_{ij} S)) \cdot a_{ij} / f'_i(S)$$

$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

$$b_{ij} T = f_j(a_{ij} f_i^{-1}(T))$$

$$\begin{aligned} b_{ij} &= f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} (f_i^{-1})'(T) \\ &= f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} / f'_i(f_i^{-1}(T)) \\ &= f'_j(a_{ij} S)) \cdot a_{ij} / f'_i(S) \end{aligned}$$

$$T = f_i(S)$$

$$S = 0$$

$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

$$b_{ij} T = f_j(a_{ij} f_i^{-1}(T))$$

$$\begin{aligned} b_{ij} &= f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} (f_i^{-1})'(T) \\ &= f'_j(a_{ij} f_i^{-1}(T))) \cdot a_{ij} / f'_i(f_i^{-1}(T)) \\ &= f'_j(a_{ij} S)) \cdot a_{ij} / f'_i(S) \\ &= f'_j(0) \cdot a_{ij} f'_i(0)^{-1} \end{aligned}$$

$$T = f_i(S)$$

$$S = 0$$

THE END