KODAIRA-SPENCER CLASSES, τ-FORMS AND DERIVED CATEGORIES

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In this note we sketch the construction of the Kodaira-Spencer morphism and the sheaf of τ-forms in extreme generality using the cotangent complex formalism.

Suppose we are given a morphism $f\colon X \to Y$ of schemes over a base S. Then each of the three morphisms f, π_X, π_Y has an associated cotangent complex $\mathbb{L}_{Y/X}, \mathbb{L}_{X/S}, \mathbb{L}_{Y/S}$ [**?**, Tag 08P5]. Moreover, there is a canonical exact triangle

$$Lf^*\mathbb{L}_{X/S} \to \mathbb{L}_{Y/S} \to \mathbb{L}_{Y/X} \to Lf^*\mathbb{L}_{X/S}[1] \tag{0.1}$$

in the derived category $D(Y)$. Here, Lf^* is the left-derived pullback along f. The last arrow is called the Kodaira-Spencer morphism of f.

In general, the cotangent complex is a complex of sheaves with cohomology sheaves concentrated in nonpositive degrees. This implies, for instance, that there is a natural morphism of (complexes of) sheaves $\mathbb{L}_{X/S} \to \Omega^1_{X/S}$, where we view the latter as a complex of sheaves concentrated in degree zero. An S-derivation δ on X determines and is determined by a morphism $\Omega^1_{X/S} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules, and thus also a composite

$$\mathbb{L}_{Y/X} \to Lf^*\mathbb{L}_{X/S}[1] \to f^*\Omega^1_{X/S}[1] \to f^*\mathcal{O}_X[1] \to \mathcal{O}_Y[1]. \tag{0.2}$$

The sheaf of derived τ-forms $\mathbb{L}^\tau_{Y/S}$ is the desuspension of the cone of this composite, i.e. it is a complex of sheaves on Y fitting into an exact triangle

$$\mathbb{L}^\tau_{Y/X} \to \mathbb{L}_{Y/X} \to \mathcal{O}_Y[1]. \tag{0.3}$$

Define $\Omega^\tau_{Y/X}$ to be the zeroth cohomology sheaf of $\mathbb{L}^\tau_{Y/X}$. In the case where f is smooth, $\mathbb{L}_{Y/X} \simeq \Omega^1_{Y/X}$ is locally free and we have a short exact sequence of vector bundles on Y

$$0 \to \mathcal{O}_Y \to \Omega^\tau_{Y/X} \to \Omega^1_{Y/X} \to 0. \tag{0.4}$$

We now suppose we have three morphisms

$$X \to Y \to S \to T. \tag{0.5}$$

There are four nontrivial transitivity triangles associated to this situation. We are interested in the relationship between those associated to $X \to S \to T$ and $Y \to S \to T$. The scheme morphism $f\colon X \to Y$ provides a morphism of triangles

$$\begin{array}{ccccccc} L\pi_X^*\mathbb{L}_{S/T} & \longrightarrow & f^*\mathbb{L}_{Y/T} & \longrightarrow & f^*\mathbb{L}_{Y/S} & \longrightarrow & L\pi_X^*\mathbb{L}_{S/T}[1] \\ \| & & \downarrow & & \downarrow & & \| \\ L\pi_X^*\mathbb{L}_{S/T} & \longrightarrow & \mathbb{L}_{X/T} & \longrightarrow & \mathbb{L}_{X/S} & \longrightarrow & L\pi_X^*\mathbb{L}_{S/T}[1]. \end{array} \tag{0.6}$$

Again, all pullback functors are taken in the derived sense. A T-derivation δ on S now gives a commuting square in $D(X)$ of the form

$$
\begin{tikzcd}
Lf^*\mathbb{L}_{Y/S} \arrow[r] \arrow[d] & f^*\mathcal{O}_Y[1] \arrow[d] \\
\mathbb{L}_{X/S} \arrow[r] & \mathcal{O}_X[1].
\end{tikzcd}
\tag{0.7}
$$

We then pass to cones and rotate back to obtain a morphism of triangles

$$
\begin{tikzcd}
Lf^*\mathcal{O}_Y \arrow[r] \arrow[d] & Lf^*\mathbb{L}^{\tau}_{Y/S} \arrow[r] \arrow[d, "f^{\tau}"'] & Lf^*\mathbb{L}_{Y/S} \arrow[r] \arrow[d] & lf^*\mathcal{O}_Y[1] \arrow[d] \\
\mathcal{O}_X \arrow[r] & \mathbb{L}^{\tau}_{X/S} \arrow[r] & \mathbb{L}_{X/S} \arrow[r] & \mathcal{O}_X[1].
\end{tikzcd}
\tag{0.8}
$$

Here we use exactness of the left-derived functor Lf^* to obtain the equivalence

$$Lf^*\mathbb{L}^{\tau}_{Y/S} \simeq Cone\left(Lf^*\mathbb{L}_{Y/S} \to Lf^*\mathcal{O}_Y[1]\right)[-1].$$

We will now specialize to our case of interest

$$C \to J \to \operatorname{Spec} K \to \operatorname{Spec} \mathbb{Z}. \tag{0.9}$$

Here C is a smooth projective curve over K of genus ≥ 2, J its Jacobian, $j\colon C \to J$ an Abel-Jacobi map. This situation allows for various simplifications on the above setup. Most importantly, π_C and π_J are smooth morphisms, which yields two short exact sequences of vector bundles

$$0 \to \mathcal{O}_C \to \Omega^{\tau}_{C/K} \to \Omega^1_{C/K} \to 0 \tag{0.10}$$

and

$$0 \to \mathcal{O}_J \to \Omega^{\tau}_{J/K} \to \Omega^1_{J/K} \to 0. \tag{0.11}$$

The local freeness of these sheaves on J implies that the pullback j^*preserves the exactness of (**??**), and we obtain a morphism of short exact sequences of $\mathcal{O}_C$-modules

$$
\begin{tikzcd}
0 \arrow[r] & j^*\mathcal{O}_J \arrow[r] \arrow[d] & j^*\Omega^{\tau}_{J/K} \arrow[r] \arrow[d] & j^*\Omega^1_{J/K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & \Omega^{\tau}_{C/K} \arrow[r] & \Omega^1_{C/K} \arrow[r] & 0.
\end{tikzcd}
\tag{0.12}
$$

We now take sheaf cohomology and obtain morphisms of triangles in $D(K)$

$$
\begin{tikzcd}
R\Gamma(J,\mathcal{O}_J) \arrow[r] \arrow[d] & R\Gamma(J,\Omega^\tau_{J/K}) \arrow[r] \arrow[d] & R\Gamma(J,\Omega^1_{J/K}) \arrow[r] \arrow[d] & R\Gamma(J,\mathcal{O}_J)[1] \arrow[d] \\
R\Gamma(C,j^*\mathcal{O}_J) \arrow[r] \arrow[d] & R\Gamma(C,j^*\Omega^\tau_{J/K}) \arrow[r] \arrow[d] & R\Gamma(C,j^*\Omega^1_{J/K}) \arrow[r] \arrow[d] & R\Gamma(C,j^*\mathcal{O}_J)[1] \arrow[d] \\
R\Gamma(C,\mathcal{O}_C) \arrow[r] & R\Gamma(C,\Omega^\tau_{C/K}) \arrow[r] & R\Gamma(C,\Omega^1_{C/K}) \arrow[r] & R\Gamma(C,\mathcal{O}_C)[1].
\end{tikzcd}
\tag{0.13}
$$

Passing to long exact sequences in cohomology and composing vertically we obtain a diagram

$$
\begin{tikzcd}
0 \arrow[r] & H^0(J,\mathcal{O}_J) \arrow[r] \arrow[d] & H^0(J,\Omega^\tau_{J/K}) \arrow[r] \arrow[d] & H^0(J,\Omega^1_{J/K}) \arrow[r] \arrow[d] & H^1(J,\mathcal{O}_J) \arrow[d] \\
0 \arrow[r] & H^0(C,\mathcal{O}_C) \arrow[r] & H^0(C,\Omega^\tau_{C/K}) \arrow[r] & H^0(C,\Omega^1_{C/K}) \arrow[r] & H^1(C,\mathcal{O}_C)
\end{tikzcd}
\tag{0.14}
$$

Usual properties of the Abel-Jacobi map imply that each map besides the middle is an isomorphism, and so the five-lemma implies

$$H^0(J,\Omega^\tau_{J/K}) \cong H^0(C,\Omega^\tau_{C/K}). \tag{0.15}$$

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