DENINGER COHOMOLOGY THEORIES

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ABSTRACT. A brief explanation of Denninger's cohomological formalism which gives a conditional proof Riemann Hypothesis. These notes are based on a talk given in the University of New Mexico Geometry Seminar in Spring 2012. The notes are in the same spirit of Osserman and Ile's surveys of the Weil conjectures [Oss08] [Ile04].

Readers who know what the standard conjectures are should skip to section 0.6.

0.1. **Schemes.** We will use the following notation:

$$\begin{aligned} \mathsf{CRing} &= \text{Category of Commutative Rings with Unit}, \\ \mathsf{Sch}_{\mathbb{Z}} &= \text{Category of Schemes over } \mathbb{Z}, \end{aligned}$$

Recall that there is a contravariant functor which assigns to every ring a space (scheme)

$$\begin{array}{ccc} \mathsf{CRing} & \longrightarrow & \mathsf{Sch} \\ \Psi & & \Psi \\ \mathrm{A} & \longmapsto & \mathrm{Spec\ A} \end{array}$$

Where

$$\mathrm{Spec(A)} = \{ \text{ primes ideals of } A \text{ not including } A$$

where the closed sets are generated by the sets of the form

$$V(f) = \{P \in \mathrm{Spec(A)} : \mathrm{f(P)} = 0\}, \quad \mathrm{f} \in \mathrm{A}.$$

By "$f(P) = 0''$ we means $f \equiv 0 \mod P$. If $X = \mathrm{Spec(A)}$ we let

$$|X| := \text{closed points of } X \ = \ \text{maximal ideals of } A$$

i.e. $x \in |X|$ if and only if $\overline{\{x\}} = \{x\}$. The overline here denote the closure of the set in the topology and a singleton in Spec(A) being closed is equivalent to x being a maximal ideal. [1] Another word for a closed point is a geometric point. If a point is not closed it is called generic, and the set of generic points are in one-to-one correspondence with closed subspaces where the associated closed subspace associated to a generic point x is $\overline{\{x\}}$.

Schemes have the additional data of a structure sheaf. To every set $U \subset X$ we can assign a ring $\mathcal{O}(U)$. For $U = D(f) := X \setminus V(f)$, $\mathcal{O}(U) = A_f = A[1/f]$. This is the structure sheaf of the scheme Spec(A). Notice that for $U = D(f)$, $V(f)$ it is just the set of functions which don't have 'poles" on the set $V(f)$.

A scheme is a topological space X with a sheaf of rings $\mathcal{O}$ which is locally isomorphic to an affine scheme.

There is another sense of points which is important. Given an equation,

$$X : x^2 + y^2 - 1 = 0$$

we can consider its solution sets for various rings: for example, $X(\mathbb{C}) = \{(a,b) \in \mathbb{C}^2 : a^2 + b^2 = 1\}$ or $X(\mathbb{R}) = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$. It is an interesting observation that given a points $P = (a,b) \in X(\mathbb{C})$ we define a ring homomorphism $A = \frac{\mathbb{Z}[x,y]}{\langle x^2+y^2-1\rangle} \to \mathbb{C}$ defined by $x \mapsto a$ and $y \mapsto b$. In fact

$$X(\mathbb{C}) \cong \mathsf{CRing}(\mathsf{A}, \mathbb{C}) \cong \mathsf{Sch}_{\mathbb{Z}}(\mathrm{Spec}(\mathbb{C}), \mathrm{X})$$

If $x \in X$ then we get some $x \in X(k(x))$ by $\mathcal{O}_x \to k(x)$ where $k(x)$ is the residue field of $\mathcal{O}_x$.

[1] The Zariski topology was originally invented by Krull and was only taken up later by Grothendieck with the purpose of cooking up spaces where the Lefschetz fixed point theorem would hold [McL03].

0.2. **Zeta Functions.** In 1948 Weil in proposed to study defined Zeta functions for varieties over finite fields [Wei49]. If $X/\mathbb{Z}$ scheme we define

$$\zeta(X,s) := \prod_{y\in|X|}(1-Ny^{-s})^{-1}, \qquad Ny := \#k(y)$$

where $Ny = \#\kappa(y)$. This function converges for Re(s) > dim(X) and is analytic on this domain. Observe that

$$\zeta(\mathrm{Spec}(\mathbb{Z}),\mathrm{s}) = \prod_{(\mathrm{p})\in\mathrm{Spec}(\mathbb{Z})}(1-\mathrm{p}^{-\mathrm{s}})^{-1} = \sum_{\mathrm{n}\geq 1}\mathrm{n}^{-\mathrm{s}} = \zeta(\mathrm{s})$$

so we recover the Riemann-Zeta function from this definition.

For $X/\mathbb{F}_q$ we can define $q = p^m$ and p Weil defined

$$Z(X,t) = \exp\left(\sum_{r\geq}\#X(\mathbb{F}_{p^r})\frac{t^r}{r}\right).$$

Notice that this is some sort of generating function for $\mathbb{F}_{q^r}$-points of the variety X. This function is a Zeta function as we will now show. Let $B_d = \#\{y\in|X| : \deg y = d\}$ where $\deg(y) = [\kappa(y):\mathbb{F}_q]$. Also let $G = G(\kappa(y))/\mathbb{F}_q)$. Let $P\in X(\mathbb{F}_{q^d})$ comes from the residue map of $y\in|X|$. Observe that for all $g\in G$, we have $g(y)\in X(\mathbb{F}_{q^d})$ corresponding to the same closed point (we just are just taking an automorphism of the residue field). This means that the number of $\kappa(y)$ points come in multiples of $G(\kappa(y)/\mathbb{F}_q)$ or that

$$\#X(\mathbb{F}_{q^r}) = \sum_{d|r} dB_d.$$

This observation now shows

$$\begin{aligned}
Z(X,t) &= \exp\left(\sum_{r\geq 1}\#X(\mathbb{F}_{q^r})\frac{t^r}{r}\right)\\
&= \exp\left(\sum_{r\geq 1}\sum_{r|d} dB_d\frac{t^r}{r}\right)\\
&= \exp\sum_{d\geq 1, i\geq 1} dB_d\frac{t^{id}}{id}\\
&= \exp\sum_{d\geq 1} B_d\sum_{i\geq 1}\frac{t^{id}}{i}\\
&= \exp\sum_{d\geq 1} -B_d\log(1-t^d)\\
&= \prod_{d\geq 1}(1-t^d)^{-B_d}, \qquad t = q^{-s}\\
&= \prod_{d\geq 1}(1-q^{-ds})^{-B_d}\\
&= \prod_{y\in|X|}(1-Ny^{-s})^{-1}\\
&= \zeta(X,s),
\end{aligned}$$

so we see that the funky generating function $Z(X,t)$ for a curve over a finite field is really one of the zeta functions which generalize Riemann's.

0.3. **Weil's Problem and the Riemann Hypothesis for Varieties over Finite Fields.** Weil proved the following for curves over finite fields and conjectured the following for varieties over finite fields [Wei49]. Let $X/\mathbb{F}_p$ be a variety over a finite field with $\dim(X) = n$. Then its zeta function $Z(X,t)$ should satisfy:1

Rationality:

$$Z(X,t) = \prod_{w=0}^{n} P_w(t)^{(-1)^{j+1}}, \qquad P_w(t) \in \mathbb{Q}[t]$$

Connection to Topology: If $\tilde{X}/\mathbb{Z}$ has the property that $\tilde{X} \mod p = X$ then

$$\deg(P_w) = \dim H^w_{sing}(\tilde{X}(\mathbb{C}), \mathbb{C})$$

where "sing" denotes the plain old singular cohomology from Topology.

Functional Equation: $P_w(\rho) = 0 \implies P_{2n-w}(1/q^w\rho) = 0$ via

$$Z(X, \frac{1}{q^w t}) = \pm q^{w\chi/2} t^{\chi} Z(X, \frac{1}{q^w t}),$$

where $\chi = \sum_{w=0}^{2\dim(X)} (-1)^n \dim H^w_{sing}(\tilde{X})$

Riemann Hypothesis: $P_w(\rho) = 0 \implies |\rho| = p^{-j/2}$

We should remark that the rationality puts severe restrictions on the roots and the functional equation expresses an extra symmetry for the $\mathbb{F}_{q^r}$ points of X.

When $\dim(X) = 1$ we have

$$Z(X, p^{-s}) = \frac{P_1(t)}{P_0(t)P_2(t)}$$

so the zeros of Zeta come from $P_1(t) = 0$. Making the substitution $t = p^{-s}$ the Riemann hypothesis above says that the zeros of $Z(X, p^{-s})$ has real part equal to 1/2.

0.4. **Weil's Big Idea!** From plain-old algebra we know that $\mathbb{F}_{p^m}$ are fixed points of $\overline{\mathbb{F}}_p$ under F^m the mth power of the Frobenius map $F : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ defined by $a \mapsto a^p$. We have something similar for points on varieties:

$$X(\mathbb{F}_{p^m}) = \text{ fixed points of } X(\overline{\mathbb{F}}_p) \text{ under } F^m \ .$$

There is a classical theorem for compact manifolds which allows you to count fixed points of a continuous function on a compact manifold, given $F : X \to X$ a continuous endo-map of a complex manifold we have

$$\#\text{ fixed points of } F \ = \sum_{w=0}^{2\dim(X)} (-1)^w \mathrm{Tr}(\mathrm{F}^*|\mathrm{H}^w(\mathrm{X}))$$

where the H^j on the right is singular cohomology. Weil's idea was then to pretend we had such a cohomology theory and see what would happen if this was the case; let's suppose

$$\#X(\mathbb{F}_{p^r}) = \sum_{w=0}^{2n} (-1)^w \mathrm{Tr}(\mathrm{F}^{r*}|\mathrm{H}^w)$$

for some suitable cohomology theory. This implies

$$
\begin{aligned}
Z(X,t) &= \sum_{r\geq 1} \#X(\mathbb{F}_{p^n})\frac{t^r}{r} \\
&= \sum_{r\geq 1}\sum_{w=0}^{2n}(-1)^j \mathrm{Tr}(\mathrm{F^r}|\mathrm{H^w})\frac{\mathrm{t^r}}{\mathrm{r}} \\
&= \sum_{w=0}^{2n}(-1)^w \sum_{r\geq 1}(\lambda_{w,1}^r + \cdots + \lambda_{w,\beta_w}^r)\frac{t^r}{r} \\
&= \sum_{w=0}^{2n}(-1)^w(\log(1-\lambda_{w,1}t) + \cdots + \log(1-\lambda_{w,\beta_w}t)) \\
&= \prod_{w=0}^{2n}\det(1-tF^*|H^w)^{(-1)^{w+1}} \\
&= \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}.
\end{aligned}
$$

This is the main observation that lead Weil to his conjectures. We should note that in 1950 there was no such thing as a scheme and finding the right "space" (if one even existed) was considered very very far afield. At the time much of this work was viewed as just numerology. [McL03]

0.5. **The Standard Conjectures/Weil Cohomology Theories.** Much of Grothendieck and Serre's work in the 50's and 60's on schemes was motivated by the Weil's conjectures. Grothendieck wrote down the following Standard Conjectures for regarding the appropriate cohomology theory in [Gro68]. It is essentially a copy of the properties of singular cohomology that you need in order to prove the Lefschetz fixed point theorem plus some additional properties would would imply the Riemann Hypothesis

A **Weil Cohomology** is a functor

$$H^\bullet : \{\text{ Smooth Varieties}/\ k\ \} \to \{\text{ Graded } K\text{-algebras}\}$$

$$H^\bullet(X) = \bigoplus_{w=0}^{2\dim(X)} H^w(X)$$

where k is a perfect field of characteristic p and K is a field of characteristic zero.

Finiteness: $h^i(X) < \infty$
Vanishing: $h^i(X) = 0$ for $i > 2n$ or $i < 0$
Trace Map: $\mathrm{Tr} : \mathrm{H^{2n}} \cong \mathrm{K}$
Poincare Duality: $H^i \cong H^{2n-i*}$ via cup product
Kunneth: $H^\bullet(X\times Y) = H^\bullet(X)\otimes H^\bullet(Y)$
Lefschetz Axioms:
- **Cycle Map:** $Z^i(X) \to H^{2i}$ has some nice properties
- **Weak Lefschetz:** The pullback map on cohomology to hyperplane sections behaves nicely.
- **Hard Lefschetz:** $H^i \to H^{i+2}$ via $\xi \mapsto \xi \smile \omega$ where $\omega = [H]$, H a hyperplane section.

The last three are essentially cooked up to give you the Riemann Hypothesis in Weil's Setting. We should remark that Grothendieck, Serre and their collaborators were able to prove everything but the Riemann Hypothesis using étale cohomology [2]. and that these conjectures were formulated for the expressed purpose of terminating the Weil Conjectures. It is still an open problem whether the following cohomology theories satisfy the axioms:

- Algebraic De Rham, $H^\bullet$,
- Crystalline, $H^\bullet_{crys}$,
- l-adic étale cohomology $H^\bullet(\mathbb{Q}_l)$.

[2]Serre invented flat cohomology and Grothendieck jazzed it up to get étale cohomology.

It should also be mentioned that understanding these properties are related to the Grothendieck group of Chow Motives which is essentially a free group on varieties over $\overline{\mathbb{F}}_p$.

The Weil conjectures were finially resolved in the early 70's by Deligne in [Del74] but not by means of a Weil Cohomology theory. I believe that Deligne showed the Lefschetz hard property. A concise overview of Deligne's proof is [Kow08].

0.6. **Deninger Cohomology Theories.** The basic motto is

" Singular cohomology is to a Weil Cohomology theory as Foliation cohomology is to a Denninger Cohomology"

There should exist a category $\mathsf{Sch}_{\mathbb{F}_1}$ equipt with a base change functor

$$- \otimes_{\mathbb{F}_1} \mathbb{Z} : \mathsf{Sch}_{\mathbb{F}_1} \to \mathsf{Sch}_{\mathbb{Z}}.$$

There should also exists a completion procedure so that we can compactify $\mathrm{Spec}(\mathbb{Z})_{\mathbb{F}_1}$ in the category $\mathsf{Sch}_{\mathbb{F}_1}$ to get $\overline{\mathrm{Spec}(\mathbb{Z})_{\mathbb{F}_1}}$ a complete object in this category. We also should have $\dim_{\mathbb{F}_1}(\mathrm{Spec}(\mathbb{Z})_{\mathbb{F}_1}) = 1$.

$$H_D^\bullet : \mathsf{Sch}_{\mathbb{F}_1} \to \text{ Graded } \mathbb{R}\text{-algebras}$$

which will be some sheaf cohomology

- **Frechetness:** The spaces $H^w(X_D, j_*\mathcal{C})$
- **Vanishing:** $H^w(X) = 0$ for $w > 2n$ or $w < 0$
- **Trace Map:** $\mathrm{Tr} : \mathrm{H}^{2n} \cong \mathrm{K}$
- **Poincare Duality:** $H^i \cong H^{2n-i*}$ via cup product
- **Kunneth:** $H^\bullet(X \times Y) = H^\bullet(X) \otimes H^\bullet(Y)$
- **Lefschetz Axioms:** ???, I'm not sure exactly how these should look but there should be something here.
- **Hodge $*$:** $* : H^w \cong H^{2n-w}$
- **Real Action:** An real action $\phi : \mathbb{R} \times X_D \to X_D$, which should be thought of as a replacement of the Frobenius.

See [Den92], [Den94], [Man95]. For video lectures and accompanied lectures see [**?**].

Remark 0.1. (1) Note that the Hodge Star operator together with Poincare Duality gives us an inner product on middle cohomology: If $f, g \in H^i$ then

$$\langle f, g \rangle := \mathrm{Tr}(\mathrm{f} \smile *\mathrm{g}).$$

Allows us to complete the Frechet spaces to Hilbert Spaces. These spaces would be those conjectured by Hilbert and Polya for the Riemann Hypothesis.

(2) The Hodge star operation would make the graded ring algebra into a $\mathbb{C}^*$-algebra. This explains the interest of the non-commutative geometry community in these problems. See for example [CC11]

(3) I do not claim that Deninger or anyone I have cited would state the conjectures as I have. In fact, my formulation of the $\mathbb{F}_1$-category above is the most imprecise part. One could for example ask that it have the six-operation formalism of Grothendieck or not.

0.7. **Necessary Conditions.** The conjectures above come from considering the (false) formula

$$\zeta(s) = \frac{P_1(t)}{P_0(t)P_2(t)}, \qquad P_w = \det(\theta - s|H^w) \tag{0.1}$$

where θ is some operator that acts on some cohomology theory[3]. If 0.1 is true then the zeros of the Riemann Zeta function must be in the spectrum of $\theta|H^1$ (θ restricted to H^1). This gives two contradictions. First, unlike the case of the zeta function for varieties over finite fields the Riemann zeta function has "trivial zeros" off the line Re(s) = 1/2. Second, the product of the eigenvalues is not a finite one.

There is a fix though by considering the completed Riemann Zeta function:

$$\begin{aligned} \widehat{\zeta}(s) &:= \zeta(s)\zeta_1(s) \\ &= \zeta(s)2^{-1/2}\pi^{-s/2}\Gamma(s/2). \end{aligned}$$

[3]This operator comes from the ϕ that is supposed to replace the Frobenius in the characteristic p setting

$\widehat{\zeta}(s)$ has a meromorphic continuation to the entire plane, has simple poles at $s = 0$ and $s = 1$, and has zeros only at the non-trivial zeros of $\zeta(s)$.

In general $\zeta_1(s)$ is a product of Γ-factors determined by the Hodge structure at 1.[4] The fact that extra factors are needed to correct the product can be taken as (perhaps weak) evidence that some compactification procedure of $\mathrm{Spec}(\mathbb{Z})$ is needed.

The next section deals with the fact

0.8. **Regularized Products and Regularized Determinants.** Let H be a $\mathbb{C}$ vector space and $\theta : H \to H$. Suppose further that

$$H = \bigoplus_{i=1}^{\infty} H_i$$

where for each i, H_i is a finite dimensional θ-invariant subspace.

We define the regularized determinant of θ on H as

$$\det_\infty(\theta|\mathrm{H}) = \begin{cases} 0, & 0 \in \sigma_p(\theta) \\ \prod_{\lambda\in\sigma_p(\theta)} \lambda, & \lambda \notin \sigma_p(\theta) \end{cases}$$

where

$$\begin{aligned} \prod_{j=1}^{\infty} \lambda_j &:= \exp(-\zeta_\theta'(0)), \\ \zeta_\theta(s) &:= \sum_{\lambda\in\sigma_p(\theta)} \lambda^{-s}, \quad \mathrm{Re(s)} >> 0, \quad -\pi < \arg(\alpha) < \pi \end{aligned}$$

where the arguments are all taken so that $-\pi < \arg(\alpha) \leq \pi$, and the derivative at zero is obtained by analytic continuation (which exists by considerations involving the Hurwitz zeta function).

Example 0.1. This is how Ramanujan determined that the product of all the the natural numbers is $\sqrt{\pi}$

$$\prod_{\nu=1}^{\infty} = \exp(-\zeta'(0)) = \sqrt{2\pi}$$

We should also remark that this procedure is used in Quantum Field Theory to determinants of infinite matrices which appear in Feynmann integrals. Peruse for example [**?**].

0.9. **Recovering Euler Products.** We want to recognize all of the Euler factors of $\overline{\mathrm{Spec}(\mathcal{O}_\mathrm{K})}$ as regularized determinants. Here is the recipe: for each y let us define

$$\mathcal{R}_y = \begin{cases} \mathbb{R}/\log Ny\mathbb{Z}, & y \nmid 1 \\ \mathbb{R}[\exp(-2y)], & y \mid 1, y \text{ real} \\ \mathbb{R}[\exp(-y)], & y \mid 1, y \text{ complex} \end{cases}$$

These spaces have real actions

$$\phi^t f(y) := f(y+t)$$

which gives

$$\theta = \frac{d}{dy}$$

as the infinitesimal generatator of the $\mathbb{R}$-action.

Here are some exercises:

(1) The eigenvalues of θ on $\mathcal{R}_y$ are the poles of $\zeta_y(s) = (1 - Ny^{-1})^{-1}$
(2) $\zeta_y(s) = \det_\infty(\frac{\mathrm{s}-\theta}{2\pi}|\mathcal{R}_\mathrm{y} \otimes \mathbb{C})^{-1}$

[4] I am using the non-standard notation 1 in place of the ∞ that Arakelov geometers prefer.

To show this one needs to use the Hurwitz zeta functions.

For $y \nmid 1$

$$\det_\infty((\mathrm{s}-\theta)/2\pi|\mathcal{R}_{\mathrm{y}})^{-1} = \prod_{\nu\in\mathbb{Z}} \frac{1}{2\pi}\left(\mathrm{s} - \frac{2\pi\mathrm{i}}{\log \mathrm{Ny}}\right)$$

For $y \mid 1$

$$\begin{aligned}
\det_\infty(\frac{\mathrm{s}-\theta}{2\pi}|\mathrm{R_y}) &= \prod_{\nu=0}^{\infty}\frac{s+2\nu}{2\pi} \\
&= \pi^{s/2}\sqrt{2}\Gamma(s)^{-1}, \quad y \text{ real} \\
&= \pi^{s/2}\sqrt{2}\Gamma(s/2)^{-1}, \quad y \text{ complex}
\end{aligned}$$

0.10. **Deninger's Conditional Proof.** Denninger proved that

$$\widehat{\zeta}(s) = \frac{\prod_{\widehat{\zeta}(\rho)=0}\frac{s-\rho}{2\pi}}{\frac{s}{\pi}\cdot\frac{s-1}{\pi}}.$$

Let's assume that this coms from a cohomology theory and that we have

$$\widehat{\zeta}(s) = \frac{\det_\infty(\frac{\mathrm{s}-\theta}{2\pi}|\mathrm{H}^1)}{\det_\infty(\frac{\mathrm{s}-\theta}{2\pi}|\mathrm{H}^0)\det_\infty(\frac{\mathrm{s}-\theta}{2\pi}|\mathrm{H}^2)}.$$

Let's also assume that

- θ acts as zero on H^0
- θ acts as identity on H^2

Which is consistent with Denninger's formula. Since $\theta = \lim_{t\to 0}\frac{\phi^{t*}-\mathrm{id}}{t}$ where ϕ^{*t} is the induced action on cohomology, we have that θ is a derivation:

$$\theta(f \smile g) = \theta(f)\smile g + f \smile \theta(f).$$

We will assume the pairing

$$\langle f, g\rangle := \mathrm{Tr}(\mathrm{f_1} \smile *\mathrm{f_2})$$

is non-degenerate. This allows us to proceed with a Hilbert-Polya argument:

For $f_1, f_2 \in H^1$ we have

$$\begin{aligned}
f_1 \smile *f_2 &= \theta(f_1 \smile *f_2) \\
&= \theta(f_1)\smile *f_2 + f_1 \smile \theta(*f_2) \\
&= \theta(f_1)\smile *f_2 + f_1 \smile *\theta(f_2)
\end{aligned}$$

there the first equality is because $f_1 \cup *f_2 \in H^2$ and θ acts trivially on H^2. Taking traces of the above equation gives

$$\begin{aligned}
& b\langle f_1, f_2\rangle = \langle\theta(f_1), f_2\rangle + \langle f_1, \theta(f_2)\rangle \\
\implies\ & \theta - 1/2 \text{ is antisymmetric on } H^1 \\
\implies\ & \sigma(\theta - 1/2|H^1) \subset i\mathbb{R} \\
\implies\ & \sigma(\theta|H^1) \subset 1/2 + i\mathbb{R}.
\end{aligned}$$

We should remark that the above is consistent with the Random Matrix theory conjectures (that the distribution of zeros of the zeta function is like a the distribution of eigenvalues for a large random unitary matrix after rescaling). See [KS99] for an overview of Random Matrix theory connections to L-functions.

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