Math 180 — Test 2.5 — Fall 2011

November 14, 2011

Instructions You are to work on all the problems BY YOURSELF and return the test at the beginning of the next class period. You may use your book and the internet. To get full credit you must show all of your work. Each problem is worth 10 points.

1. Consider the function
$$f(x) = \frac{x^5}{5} + \frac{x^4}{4} - 2x^3 + 1.$$

(a) Find all the critical points of $f(x)$.

solution

$$\begin{aligned} f'(x) &= x^4 + x^3 - 6x^3 \\ &= x^2(x^2 + x - 6) \\ &= x^2(x+3)(x-2) \end{aligned}$$

so the critical points are

$$x = 0, \quad x = -3, \quad x = 2.$$

$\square$

(b) Find all of the relative extrema of $f(x)$ by applying the second derivative test to the critical points.

solution

$$f''(x) = 4x^3 + 3x^2 - 12x.$$

$$\begin{aligned} f''(0) &= 0, && \text{so the second derivative test is inconclusive} && (1) \\ f''(-3) &= -4(-3)^3 + 3(-3)^2 - 12(-3) > 0, && \text{concave down, } x = -3 \text{ maximizer} && (2) \\ f''(2) &= -4(2)^4 + 2(2)^3 - 12(-3) < 0, && \text{concave up, } x = 2 \text{ minimizer} && (3) \end{aligned}$$

$\square$

2. Sketch that graph of

$$f(x) = \frac{x^2 - 9}{x^2 - 4}.$$

You graph should include

(a) x and y intercepts.

x-intercepts $f(x) = 0$ when $x^2 - 9 = 0$ so

$$x = 3, \quad x = -3,$$

are the x-intercepts.

y-intercepts Since $f(0) = -9/(-4) = 9/4$ the y-intercept is

$$(0, 9/4).$$

(b) Relative extrema.

solution Using the quotient rule we find that

$$f'(x) = \frac{2x(x^2-4) - 2x(x^2-9)}{(x^2-4)^2} = \frac{2x(5)}{(x^2-4)^2} = \frac{10x}{(x^2-4)^2}$$

so the only critical point is when $x = 0$. Now since $f(0) = 9/4$ the critical point on the graph will be

$$(0, 9/4)$$

we apply the first derivative test to show that $(0, 9/4)$ is a maximum:

$$\begin{aligned} f(.001) &> 0 \\ f(-.001) &< 0 \end{aligned}$$

This tells us that slope of $f(x)$ goes negative to positive at $x = 0$ and hence

$$(0, 9/4) \text{ is a local minimum.}$$

It is the only extrema.

$\square$

(c) Points of inflection.

solution

$$f''(x) = \frac{10(x^2-4)^2 - 10x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} = \frac{10x^2 - 40 - 40x^2}{(x^2-4)^3} = \frac{-30x^2 - 40}{(x^2-4)^3}$$

for $f''(x) = 0$ we must have

$$-30x^2 - 40 = 0,$$

which implies that

$$x^2 = -4/3$$

since this means $x = \pm\sqrt{-4/3}$ which is imaginary there are no inflection points.

$\square$

(d) Vertical and horizontal asymptotes.

Vertical: These are where the denominator of $f(x)$ is equal to zero. The vertical asymptotes are the lines

$$x = -2, \text{ and } x = 2.$$

Horizontal: To compute these we see if the limits as $x \to \pm\infty$ exist:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1 - 9/x^2}{1 - 4/x^2} = 1,$$

similarly we see that

$$\lim_{x\to-\infty} f(x) = 1,$$

so the only horizontal asymptotes is $y = 1$ and $f(x)$ approaches this asymptote as $x \to \pm\infty$.

3. The velocity of blood (in centimeters/second) in a cylindrical artery can be given as a function of the distance r (in centimeters) to the central axis of an artery by

$$v(r) = k(R^2 - r^2)$$

where k is constant and R is the radius of the artery (also constant). Show that the velocity of blood is greatest along the central axis of the artery. What is the maximum velocity?

solution Since $v'(r) = -2kr$, So the only critical point is when $r = 0$. Since the domain of v is a closed interval $r \in [0, R]$ we can look for absolute extrema by plugging in the critical points and the end points:

$$v(0) = kR^2$$
$$v(R) = 0$$

There are two cases:

- If k is positive then $r = 0$ is the maximizer and the maximum velocity is

$$v(0) = kR^2$$

- If k is negative that $r = R$ is the maximizer and the maximum velocity is

$$v(R) = 0,$$

(In this case we would actually have the velocity being negative which would make the blood vessil actually a vein)

$\square$

4. The quantity demanded each month of Sicard wristwatches is related to the unit price by the equation

$$p = \frac{50}{x^2/100 + 1}$$

where $0 \leq x \leq 20$, p is measured in the thousands of dollars and x in measures in units of a thousand. To yield a maximum revenue how many watches must be sold?

solution The revenue function is

$$R(x) = p(x) \cdot x = \frac{50x}{x^2/100 + 1}.$$

Since the domain is a closed interval we can just find the maximum value at the critical points and the end-points of the interval. We first solve for the critical points:

$$R'(x) = \frac{50(x^2/100 + 1) - (x/50) \cdot 50x}{(x^2/100 + 1)^2} = \frac{x^2/2 + 50 - x^2}{(x^2/100 + 1)^2} = \frac{-x^2/2 + 50}{(x^2/100 + 1)^2}$$

so $R'(x) = 0$ means that

$$-x^2/2 + 50 = 0$$

which tells us that $x = \pm 10$. Since $x = -10$ is not it the domain of the function we only need to consider $x = 10$. We now evaluate our function at this critical point and the endpoints:

$$\begin{aligned} R(0) &= 0, \\ R(20) &= \frac{50 \cdot 20}{(20^2)/100 + 1} = 200, \\ R(10) &= \frac{50 \cdot 10}{10^2/100 + 1} = 500/2 = 250. \end{aligned}$$

which tells us that $x = 10$ is a maximizer for revenue, or that 10 thousand watches must be sold to maximize revenue.

$\square$

5. A Norman windox has the shape of a rectangle surmounted by a semicircle. If a Norman window is to have a perimeter of $4+\pi$ ft what should the dimensions be in order to maximize the amount of light passing through the window. In your answer be sure to include the radius of the semicircle and the height of the rectangular portion of the window. (see page 320 problem 16 for a picture).

solution The formula for perimeter:

$$\begin{aligned} P &= \text{(semicircle perimeter)} + \text{(lower rectangle perimeter)} \\ &= \pi r + (2r + 2y). \end{aligned}$$

The formular for area:

$$\begin{aligned} A &= \text{(area of semi-circle)} + \text{(area of lower rectangle)} \\ &= \frac{\pi r^2}{2} + 2r \cdot y. \end{aligned}$$

So, we have a constrained optimization problem

$$\begin{cases} 4+\pi = (2+\pi)r + 2y, & \text{constrait} \\ A = \frac{\pi r^2}{2} + 2ry, & \text{what you want to optimize} \end{cases}.$$

Solving for y in terms of r gives

$$y = \frac{4+\pi-(2+\pi)r}{2}.$$

We plug this back into our formula for area to get

$$\begin{aligned} A &= \frac{\pi r^2}{2} + r[(4+\pi) - (2+\pi)r] \\ &= \frac{\pi r^2}{2} + (4+\pi)r - (2+\pi)r^2 \end{aligned}$$

$$\begin{aligned} \frac{dA}{dr} &= \pi r + (4+\pi) - (4+2\pi)r \\ &= -(4+\pi)r + (4+\pi). \end{aligned}$$

So the solution of $\frac{dA}{dr} = 0$ is

$$r = 1$$

Note that $\dfrac{d^2A}{dr^2} = -(4+\pi) < 0$ which tells us that $r = 1$ is a maximizer.
The dimension are then

$$r = 1, \qquad y = \frac{4+\pi-2-\pi}{2} = 1.$$

$\square$