ARITHMETIC DEFORMATION CLASSES ASSOCIATED TO ALGEBRAIC CURVES

TAYLOR DUPUY

ABSTRACT. We prove that for smooth projective curves $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ the first p-jet space $J^1(X)$ admits the structure of a torsor under some line bundle. The cohomology class of the torsor has the property that its reduction mod p is an obstruction to the lift of the Frobenius mod p^2. The class is interesting because its geometric counterpart (a class associated to the usual first jet spaces of a variety over a function field) is the Kodaira-Spencer class.

A consequence of our approach is that the affine bundle structure on p-jet spaces of elliptic curves admit at least two inequivalent reductions of the structure group mod p^2.

CONTENTS

2000 *Mathematics Subject Classification.* Primary 11G99, Secondary 14G40.

1. Main results and overview

1.1. ***p*-Jet spaces.** Arithmetic jet spaces were first introduced by Buium in [Bui96] as a way to extend the differential-algebraic techniques used in a proof of the Geometric/Relative Lang-Conjecture [Bui92]. Roughly speaking, the main motivation of arithmetic differential algebra is to replace derivations used in function field setting with p-derivations in the number field setting in order to extend proof techniques. Philosophically speaking, arithmetic differential algebra introduces a topology finer than the Zariski topology (the *Kolchin topology*) which allows us to apply techniques from algebraic geometry these finer sets.

We should remark that although the Lang conjecture has not been proved in full generality with p-jet space methods, the technique managed to give an effective solution of the Manin-Mumford problem [Bui96].

In the present paper we prove the following theorem:

Theorem 1.1 (Torsor Structure for Curves)**.** *If $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is a smooth projective curve of genus greater than zero and $p > 6g-5$ then first p-jet space of X admits the structure of a torsor under some line bundle.*

This is well-known in the case that X is an abelian variety and is due to Buium (see for example Buium's book [Bui94a]). This is mainly of interest to us because of its connection to Kodaira-Spencer theory. To every torsor one can associate a cohomology class. In the geometric setting (where we consider X a smooth variety over a function field K) the first (geometric) jet space is a torsor and its associated cohomology class is the Kodaira-Spencer class. Hence if p-Jet spaces are arithmetic versions of jet spaces then the cohomology class associated to the torsor structure on the first p-jet space should be viewed as an arithmetic Kodaira-Spencer class.

We should remark that theorem 1.1 is a surprising because apriori there is no obvious action of a line bundle on $J^1(X)$.

Let κ be the cohomology class associated to the torsor structure in Theorem 1.1. What does κ do? The class $\kappa \mod p$ is an obstruction to the lift of the Frobenius modulo p^2 (see section 1.11). In addition κ is known to be related to the Serre-Tate parameters when X is an abelian variety [Bui95]. In her thesis Hurlburt gave a nice presentation of these classes $\mod p$ for E a smooth elliptic curve over $\mathbb{A}^2_R$; if

$$E : y^2 - x^3 - ax - b = 0$$

is viewed as a family over $\mathbb{A}^2$ with parameters a and b after pairing this class we get

$$f^1 := \langle \kappa, \omega \rangle \in \mathbb{Z}_p[a, b, a', b', \Delta^{-1}]\widehat{\ } \tag{1.1}$$

where $\omega = dx/y$, and a' and b' are new variables which stand for the p-derivations of a and b and the hat denotes p-adic completion. This gives an interesting differential modular form f^1. Note that the expression for f^1 depends on the p-derivatives of the coefficients of E—this has two effects; first it allows $\kappa \mod p^n$ to retain information about the $X \mod p^{n+1}$; second, its zero locus defines a set in the parameter space whose zero locus are elliptic curves having a lift of the Frobenius.

The same first effect holds for the classes coming from theorem 1.1 so we justified in calling κ an deformation class.

We can rephrase the above property for f^1 slightly: If we constructed f^1 in equation 1.1 using $E \to Y_1(N)$ where $Y_1(N)$ is a modular family and E the universal elliptic curve then f^1 would be an arithmetic differential modular form which cuts out the canonical lift (CL) curves in the modular family (see [Hur98], [Hur03] [Hul01]):

$$f^1(P) = 0 \iff P = [E_P] \in \mathrm{CL} \subset Y_1(N)(R).$$

Extensions of this idea were used more recently in [BP09] to show that for $\Phi : X_0(N) \to E$ a modular parametrization that

$$\#\mathrm{CL} \cap \Phi^{-1}(E_{finite\ rank}) < \infty,$$

where $E_{finite\ rank}$ is any subgroup of finite rank.

1.2. **Plan of this section.** In order to justify the use of the phrase "arithmetic Kodaira-Spencer class" we will recall some motivating geometric constructions in the first few sections: In section 1.3 we recall the construction of the Kodaira-Spencer class κ in the geometric setting. In section 1.4 we setup our conventions for torsors. In section 1.5 we recall the definitions of $\check{H}^1(X, G)$ for Čech cohomology of a sheaf of (non-abelian) groups G. In section 1.6 we recall how to construct a Čech cohomology class from an algebraic G-torsor T. In section 1.7 we recall the definition of the geometric jet space $J^1(X)$ of a variety X over a field K equipped with a derivation $\delta : K \to K$. In the same section we then explain how $J^1(X)$ is a torsor for the tangent bundle T_X and show that the associated cohomology class is the geometric Kodaira-Spencer class constructed in 1.3. In this section we also state a descent theorem which has strong analogies in the arithmetic setting.

The present paper is about arithmetic versions of the Kodaira-Spencer construction in section 1.3 and the jet space construction in 1.7. To a first approximation the idea is to replace derivations which appear in geometric constructions with *p-derivations* to get arithmetic constructions.

In section 1.8 we recall the definition of p-derivations. In section 1.9 we recall the correspondence between p-derivations and lifts of the Frobenius. In section 1.10 we recall the basic definitions of Frobeniuses on schemes. In section 1.11 we introduce the arithmetic Kodaira-Spencer classes implicit in the work of of Deligne-Illusie.

Having set-up the basic Kodaira-Spencer constructions in both the arithmetic and geometric settings, section 1.12 draws comparisions between the geometric descent theorem (Theorem 1.3) of section 1.7 and the what we interpret as "arithmetic descent". We should two remarks here. First, there exists constructions of which arithmetic jet spaces and geometric jet spaces are both specializations [BW05]. Second, the arithmetic Kodaira-Spencer classes $\mathrm{DI}_0(\delta)$ constructed in section 1.11 are obstructions for a scheme to descend to an object in Borger's category of Λ-schemes (which should be viewed as an $\mathbb{F}_1$-category [Bor09]). These observations make the arithmetic case and the geometric case more than just "analogous".

All of the sections after 1.12 are purely arithmetic and have the purpose of setting up and stating the main results of this paper. In section 1.13 we recall the definition of arithmetic jet spaces [Bui96]. In section 1.14 we recall the definition of an affine bundle and in section 1.15 we show how to associate a cohomology class to an affine bundle. This construction is applied to the arithmetic jet space $J^1(X)$. In section 1.16 we recall the definition of the affine linear group. In section 1.17 we define what it means to give an additional structure to an affine bundle and explain how line bundles are just GL_1-structures on affine bundles. It turns out that affine linear structures are equivalent to torsor structures. In section 1.18 we define the groups A_d of univariate polynomial automorphisms of degree less than d modulo p^d define what it means for an affine bundle to have a degree structure.

In section 1.20 we define the "pairing" between group cohomology and Čech cohomology in a particular case; in this section we also state a non-vanishing result result for elliptic curves which is our second major theorem of this paper. In section 1.21 we show how the non-vanishing result of section 1.20 proves the existence of multiple A_2 stuctures on $J^1(E) \otimes_R R/p^2$ for E and elliptic curve over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$.

1.3. **Geometric Kodaira-Spencer construction.** Let X/K be a smooth projective variety over a field K. Recall that for each derivation $\delta : K \to K$ one can construct a class

$$\kappa = \mathrm{KS}(\delta) \in H^1(X, T_{X/K})$$

called the Kodaira-Spencer class as follows: we cover the variety X by affine open sets $X = \bigcup_{i=1}^N U_i$ and find lifts δ_i of the derivation δ to the rings $\mathcal{O}(U_i)$; i.e. derivations $\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i)$ with the property that $\delta_i|_K = \delta$. The Kodaira-Spencer class associated to the derivation δ on the base K is then the Čech cohomology class

$$\mathrm{KS}(\delta) = \kappa = [\delta_i - \delta_j] \in \check{H}^1(X, T_{X/K}).$$

The Kodaira-Spencer class tells us interesting information about descent. In [Bui] Buium shows that if X is a projective variety defined over an algebraically closed field K of characteristic zero equipped with a derivation δ then $\kappa = 0$ if and only if there exists some $X'/K^\delta = \{f \in K : \delta(f) = 0\}$ such that $X = X' \otimes_{K^\delta} K$. In other words, $\mathrm{KS}(\delta)$ is an obstruction to descent to constants of the derivation.

Vanishing of the Kodaira-Spencer class is also related to triviality of the first jet space of X. We recall the construction of the first jet space following Buium in [Bui94b]. If (R, δ) is an integral domain equipped with a derivation and $A \in \mathsf{CRing}_R$ (CRing_R is the category of commutative R-algebras — all of our rings in this paper will be commutative and have a unit), we can construct the **first jet ring** on A relative to (R, δ) by adjoining to A the symbols a' for every element $a \in A$ and imposing additivity, compatibility with δ and the product rule:

$$A^1 = \frac{A[a' : a \in A]}{((ab)' = a'b + ba', (a+b)' = a' + b', c' = \delta(c) : a, b \in A, c \in R)}. \tag{1.2}$$

There is a universal derivation $\delta_{univ} : A \to A^1$, such that if $B \in \mathsf{CRing}_R$ and $D : A \to B$ is a derivation lifting $\delta : R \to R$ then there exists an A-ring homomorphism $f_D : A^1 \to B$ defined by $f_D(a') = D(a)$ for all $a \in A$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\delta_{univ}"] \arrow[dr, "D"'] & A^1 \arrow[d, "f_D"] \\ & B \end{tikzcd} . \tag{1.3}$$

Remark 1.2. When the derivation $\delta : R \to R$ is trivial, $\delta = 0$, the first jet ring is isomorphic to the symmetric algebra of $\Omega_{A/R}$, where $\Omega_{A/R}$ denotes the module of differentials of A over R. That makes the above construction a relative version (relative to the object (R, δ) in the category of rings with derivations) of jet space constructions commonly found in the literature (c.f. for example [Vak]). Sometimes jet spaces are called arc spaces in the case $\delta = 0$.

The association $A \rightsquigarrow A^1$ is functorial and localizes well meaning $(A_f)^1 = (A^1)_f$ for all $f \in A$. This implies that for X/R a smooth scheme with $\delta : R \to R$ we can define a scheme $J^1(X)/X$ called the first (geometric) jet space of X which has a universal diagram as in equation 1.3 with the arrows reversed.

In what follows, TX/X denotes the physical tangent bundle (the scheme whose sheaf of sections are R-linear derivations on $\mathcal{O}_X$; equivalently $TX = \underline{\mathrm{Spec}}(\mathrm{Sym}(\Omega_{X/R}))$. Recall the following descent result

Theorem 1.3 (Descent to Constants [Bui]))**.** *Let X/K be a smooth projective variety over an algebraically closed field K of characteristic zero, $\delta : K \to K$ be a derivation,* $\mathrm{KS}(\delta) \in H^1(X, T_{X/K})$ *be the Kodaira-Spencer class and* $K^\delta = \{x \in K : \delta(x) = 0\}$. *The following are equivalent*

(1) $\mathrm{KS}(\delta) = 0$
(2) $J^1(X) \cong TX$ *as schemes over* X
(3) X *is defined over* K^δ

We should make some remarks on (2). The scheme $J^1(X)$ is actually an algebraic torsor for the tangent bundle TX whose associated class is the Kodaira spencer class $\mathrm{KS}(\delta)$.

In what follows (subsections 1.4-1.6) we find it useful to recall some basic facts and notation related to torsors.

1.4. **Torsors.** Let G be a group. A (left) G-**torsor** (or **principal homogeneous space**) is a set T with an action $G\times T\to T$ such that the map $G\times T\to T\times T$ given by $(g,x)\to(gx,x)$ is bijective.The condition that $G\times T\to T\times T$ is bijective is equivalent to the action $G\times T\to T$ being free and transitive.

A **sheaf of** G-**torsors** for a sheaf of groups G on a topological space X will then be a sheaf of sets T such that for all open subsets U of X the set $T(U)$ has the structure of a $G(U)$-torsor in the sense of the previous paragraph that behave well with repect to restriction.

Let S be a scheme and G/S be a group scheme. An **algebraic torsor** is a scheme T/S such that $T(U)$ is a torsor for $G(U)$ for all $U\to X$ in a functorial way. Equivalently, an algebraic torsor T/S, is a functor from the category of schemes over S to the category of G-torsors which is representable.

An algebraic torsor T is locally trivial in the Zariski topology if one that has the property that for all $P\in T$ there exists $U\ni P$ an affine open subset of S such that $T(U)$ is non-empty.

We will always assume in what follows that all our torsors are locally trivial in the Zariski topology.

Following [Bae], we find it convenient to view a (left) torsor for a group G as a set T equipped with a division map $\div : T\times T\to G$ with some extra conditions (see section 3.

1.5. **Čech cohomology.** Let G be a sheaf of groups (not necessarily abelian) on a topological space X: If we are given an open cover $\mathcal{U}=\{U_i\}$ we define $\check{H}^1(\mathcal{U},X,G):=\check{Z}^1(\mathcal{U},X,G)/\sim$. Here, $\check{Z}^1(\mathcal{U},X,G)=\{(g_{ij})\in\prod_{i,j}G(U_{ij}):g_{ij}g_{jk}g_{ki}=\mathrm{id}_{ijk}\}$ and $(g_{ij})\sim(g'_{ij})$ if and only if there exists some $(h_i)\in\prod_i G(U_i)$ such that $h_ig_{ij}=g'_{ij}h_j$ for all i and j. Finally for the full definition we take the inductive limit over the refinement of covers and define:

$$\check{H}^1(X,G)=\varinjlim_{\mathcal{U}}\check{H}^1(\mathcal{U},X,G).$$

In this paper $\check{H}^1(X,G)$ and $\check{H}^1(\mathcal{U},X,G)$ will be viewed as objects in the category of pointed sets. These are sets with a distinguished trivial element and morphisms which are maps of sets which map trivial elements to trivial elements. $\check{H}^1(X,G)$ has as its trivial element the class of the trivial cocycle (1_{ij}). The morphism $i_{\mathcal{U}}:\check{H}^1(\mathcal{U},X,G)\to\check{H}^1(X,G)$ is injective in the sense that an element $\eta_{\mathcal{U}}\in\check{H}^1(\mathcal{U},X,G)$ is trivial if and only if $i_{\mathcal{U}}(\eta_{\mathcal{U}})$ is trivial.

1.6. **Čech cohomology classes associated to Zariski locally trivial torsors.** If T is a left algebraic G-torsor on a scheme X, our assumption of local triviality implies that we can cover X by affine open sets $X=\bigcup_i U_i$ and take local sections $s_i\in T(U_i)$ to get a cohomology class $[T]\in[s_i/s_j]\in\check{H}^1(X,G)$. This cohomology class is well-defined and T is completely determined up to isomorphism by $[T]$.

1.7. **Kodaira-Spencer class and the class of the geometric jet space.** Now let X/K be a smooth projective scheme over an algebraically closed field equipped with a derivation δ. It turns out that $J^1(X)$ is a TX-torsor [Bui09]. Indeed, by Yoneda's lemma it is enough to show that this is true on the level of points. Let $\pi : J^1(X) \to X$ be the canonical projection. Local sections $s \in \Gamma(U, J^1(X)) = \{s : U \to J^1(X) : \pi \circ s = \mathrm{id}\}$ correspond to lifts of the derivation $\delta : K \to K$ to $\delta_U : \mathcal{O}(U) \to \mathcal{O}(U)$; since s gives $s^* : \mathcal{O}(J^1(U)) = \mathcal{O}(U)^1 \to \mathcal{O}(U)$ which by the universal property corresponds to a derivation. Given two section $s, t : U \to J^1(X)$ we can consider the corresponding derivations δ_s and δ_t and consider the difference $\delta_s - \delta_t$. Since they are both equal to δ when restricted to K we have $\delta_t - \delta_s \in T_{X/K}(U) = TX(U)$ which gives $J^1(X)$ the structure of a TX-torsor. This construction is exactly the same as the construction of the Kodaira- Spencer class which tells us that

$$\mathrm{KS}(\delta) = [J^1(X)] \in \check{H}^1(X, TX).$$

This observation explains the relation between parts 1 and 2 of Theorem 1.3. [1]

1.8. ***p*-derivations.** The following definition of a p-derivation was given by Buium [Bui96] and Joyal [Joy85] independently. A ***p*-derivation** is a map of sets $\delta_p = \delta : R \to A$ where R is a ring and A is an R-algebra which satisfy a sum and product rule:

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) + C_p(x,y) \\ \delta(xy) &= \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y), \end{aligned}$$

where $C_p(X,Y) = \frac{X^p+Y^p-(X+Y)^p}{p} = \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} X^j Y^{p-j} \in \mathbb{Z}[X,Y]$ (observe that all of the binomial coefficients in the sum are divisible by p).

The category of rings with p-derivations are a good arithmetic analog of the category of rings with derivations. For example, a derivation $\delta : R \to A$ is equivalent to a ring homomorphism $R \to D_1(A) = A[\varepsilon]/(\varepsilon^2)$ given by $r \mapsto r + \varepsilon\delta(r)$, a p-derivation is equivalent to a ring homomorphism $R \to W_1(A)$ given by $r \mapsto (r, \delta(r))$ where $W_1(R)$ is the ring of truncated p-typical Witt vectors. Recall that $W_1(R) = R \times R$ as sets with addition $+_W$ and multiplication $\cdot_W$ defined by

$$\begin{aligned} (r_0, r_1) +_W (s_0, s_1) &= (r_0 + s_0, r_1 + s_1 + C_p(r_0, s_0)), \\ (r_0, r_1) \cdot_W (s_0, s_1) &= (r_0 s_0, r_1 s_0^p + r_0^p s_1 + p r_1 s_1). \end{aligned}$$

where $C_p(X,Y) = \frac{X^p+Y^p-(X+Y)^p}{p} \in \mathbb{Z}[X,Y]$ as above. For example $W_1(\mathbb{F}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$ via the map $\mathbb{Z}/p^2 \to W_1(\mathbb{F}_p)$ is given by $a \mapsto (a \mod p, \delta(a) \mod p)$ and $\delta(a) = (a - a^p)/p$.

1.9. ***p*-derivations, lifts of the Frobenius, derivations of the Frobenius and a torsor structure.** The set p-derivations on a ring R are intimately tied to **lifts of the Frobenius** which are $\mathsf{CRing}_{\mathbb{Z}}$ homomorphism $\phi : R \to A$ such that for all $x \in R$ we have

$$\phi(x) \equiv x^p \mod pA.$$

If R and A are p-torsion free rings (for all x, $px = 0$ if and only if $x = 0$) then p-derivations and lifts of the Frobenius are in one-to-one correspondence via the

[1] Buium's original proof is in a more general setting uses "prologation sequences" which might be useful for generalizing these constructions to other plethories. See [BW05].

equation

$$\delta(x) = \frac{\phi(x) - x^p}{p}.$$

We should remark that if $\delta_1, \delta_2 : R \to R/p$ are p-derivations then the map $D(x) := (\delta_1 - \delta_2)(x) := \delta_1(x) - \delta_2(x)$ is a derivation of the Frobenius on R/p i.e. $D : R \to R/p$ such that $D(x+y) = D(x) + D(y)$ and $D(xy) = D(x)y^p + x^p D(y)$. More precisely, the collection of p-derivations are a torsor for the derivations of the Frobenius (Proposition 2.2).

Remark 1.4. The collection of p-derivations are not a torsor under derivations of the Frobenius when the target does not have characteristic p.

1.10. **The Absolute Frobenius.** Let X/R be a smooth scheme where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Here $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension of the p-adic integers. Equivalently, $\widehat{\mathbb{Z}}_p^{\mathrm{ur}} = W(\overline{\mathbb{F}}_p)$, the ring of p-typical Witt vectors of $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. If X_0 is its reduction mod p (using the notation $X_n = X \otimes_R R/p^{n+1}$), then p-**Frobenius** F_{X_0} (or simply "the Frobenius", when the prime p we are working with is clear) is the inseperable morphism of schemes in characteristic p, $F_{X_0} : X_0 \to X_0$ defined by $F_{X_0}^*(f) = f^p$ for every local section f of $\mathcal{O}_{X_0}$ (which implies that F_{X_0} is the identity on the topological space X_0). Lifts of the Frobenius are morphisms of schemes $\phi : X \to X$ such that for all local sections f of $\mathcal{O}_X$ we have $\phi(f) \equiv f^p \mod p$. See section 2.3 for more on this. We will sometimes denote the absolute Frobenius by F when the context is clear.

1.11. **Arithmetic Kodaira-Spencer classes.** Let X/R be a smooth projective scheme where R is a discrete valuation ring whose maximal ideal is generated by p. The Deligne-Illusie class $\mathrm{DI}_0(\delta) \in H^1(X_0, F_{X_0}^* T_{X_0/\overline{\mathbb{F}}_p})$ associated to a p-derivation δ on the base scheme first appeared implicitly in the paper of Deligne-Illusie ([PD], e.g. page 252)[2] who were concerned with lifts of the Frobenius on a scheme. It was introduced by Buium in [Bui95] as a general object of interest. Let X/R be a smooth projective scheme where R is a discrete valuation ring whose maximal ideal is generated by p (e.g. the p-adics), and admits a lift of the Frobenius mod p^2. We define the arithmetic Kodaira-Spencer class as

$$\mathrm{DI}_0(\delta) = [\frac{\phi_i - \phi_j}{p} \mod p] \in H^1(X_0, F_{X_0}^* T_{X_0})$$

where the ϕ_i are local lifts of the p-Frobenius on affine open sets U_i which cover X_1. The construction is independent of the lifts ϕ_i and cover $\{U_i\}$ and vanishes if and only if $X_1 = X \mod p^2$ admits a lift of the Frobenius.

To make $\mathrm{DI}_0(\delta)$ look more differential-algebraic we can write it as $\mathrm{DI}_0(\delta) = [\delta_i - \delta_j]$ where δ_i and δ_j are the p-derivations associated to the local lifts of the Frobenius ϕ_i. This first appeared in [Bui95].

1.12. **Arithmetic descent and the field with one element.** An *analogy* can be adopted: since vanishing of $\mathrm{KS}(\delta)$ in the geometric setting is equivalent to descent to the constants of the derivation (Theorem 1.3) then *vanishing of* $\mathrm{DI}_0(\delta)$ *in the arithmetic setting (equivalently, the existence of lifts of the Frobenius) should be viewed as descent data to some more primitive base* .

[2] In this paper the u's are the p-derivations

We would like to mention that the constants of p-derivations

$$R^\delta = \{x \in R : \delta(x) = 0\} \tag{1.4}$$

are not a ring but are a monoid (of roots of unity together with zero when $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}} = \mathbb{Z}_p[\zeta : \zeta^n = 1, p \nmid n]^\frown$ and δ its unique p-derivation) so there is no hope of a scheme X' defined over R^δ such that

$$X' \otimes_{R^\delta} R = X. \tag{1.5}$$

The replacement candidate could be an object X' in an algebro-geometric category C with a base change functor $\mathsf{C} \to \mathrm{Sch}_R$ such that

$$\mathrm{BaseChange}(X') = X.$$

One can call such a category C an $\mathbb{F}_1$-**category** and intentionally leave what we mean by "algebro-geometric" vague. Some authors like to write the base-change functor suggestively as $X' \otimes_{\mathbb{F}_1} R = X$ where the notation $\mathbb{F}_1$ suggests the field with one element.

We should warn the reader of the proliferation of definitions of field with one element. For a large commutative diagram (13 nodes not including $\mathrm{Sch}_{\mathbb{Z}}$) between various $\mathbb{F}_1$-categories we refer the reader to ([PL09], page 19).

One major branch of $\mathbb{F}_1$ theories (related by the diagram in [PL09]) includes the monoidal geometry categories of Töen-Vasquié [Toe] (often specialized to the usual set-monoids) and the category of generalized rings of Durov [Dur07] (see [Fre09] for an excellent survey). Both of these approaches are inspired by the original category of $\mathbb{F}_1$-Gadgets introduced by Soulé [Sou04]. Another notable approach is that of Lorscheid [Lor12], [Lor].

The second approach we would like to mentioned, and excluded form the diagram in Peña-Lorscheid (only mentioned in [PL09] and omitted from the diagram) views the existence of lifts of the Frobenius for every prime p as descent data. This descent viewpoint was first introduced explicitly by Borger in [Bor09]. A smooth scheme $X/\mathbb{Z}$ to descends to $\mathbb{F}_1$ in this theory if and only if X admits a lifts δ_p for every prime p such that the corresponding lifts of the Frobenius commute. Here $\delta_p : \mathbb{Z} \to \mathbb{Z}$ is the p-derivation corresponding to the identity map on $\mathbb{Z}$. In particular if $X/\mathbb{Z}$ descends to $\mathbb{F}_1$ in this theory then we have $\mathrm{DI}_0(X, \delta) = 0$ for every prime p (but not conversely).

We should point out that both approaches have been able to justify some of the numerology of Tits [Tit]— Both Borger([Bor09], Corollary 4.3.1) and Töen-Vasquies ([Toe], Proposition 4.4) show[3] in their respective categories that

$$\mathrm{GL}_n(\mathbb{F}_1) = S_n,$$

Remark 1.5.
- Tits observed that for certain varieties over finite fields (projective space, flag varieties) on can count the $\mathbb{F}_q$ points and plug in $q = 1$ to get meaningful values (finite sets, collections of finite sets) while at the same time preserving much of the representation theory. In particular Tits posits that one should view $G(\mathbb{F}_1)$ as $\mathrm{Weyl}(G)$ where Weyl denotes the Weyl group of the algebraic group G. Tits then conjectured that there should exist an "$\mathbb{F}_1$-category" where the $\mathbb{F}_1$-points of algebraic groups should be equal to the Weyl groups of the corresponding algebraic group. Whether

[3]Technically, $\mathrm{GL}_n/\mathbb{Z}$ does not descent to $\mathbb{F}_1$ in Borger's theory but rather $\mathrm{GL}_n(\mathbb{F}_1)$ is the automorphism group of some object which does descend to $\mathbb{F}_1$.

Tits' conjecture holds in greater generality in the categories of Borger and Töen-Vasquies we believe is still an open problem. The ideas in [Lor12] and [Lor] are developed precisely for attacking this issue.

- Whether the fact that the constants of p-derivations are a monoid (equation 1.4) of roots of unity represents some connection between the Töen-Vasquies-Durov-Soulé categories and the Borger-Buium category is not understood. A survey of some connections between cyclotomy and the field with one elements can be found in [Mana]. We know that set-monoidal descent implies Borger-Buium descent but not conversely and it is an open problem whether all Λ-schemes actually come from monoids.

In a fashion similar to the way that existence of a Weil-Cohomology theory gives a conditional proof of the Weil-conjectures for varieties over finite fields one of the hopes is that a modification or combination of these $\mathbb{F}_1$-constructions will yield a candidate for a "Weil-Deninger" cohomology theory that gives spectral/cohomological intepretation of the Riemann zeta function which would prove the Riemann Hypothesis [Den94] [Den92] [Den91] [Kur92]. See [Manb] for a review of the Denninger-Kurakowa approach to zeta functions. A recent revival of these ideas can be found in connection with p-adic Hodge theory [Ked09] and non-commutative geometry (See for example [CC11] for an introduction and [CC12] for real-Fontaine rings) has yielded interesting results. To our knowledge there is currently no candidate for an "Weil-Deninger" cohomology theory in an $\mathbb{F}_1$-category although we believe Deninger's $H^1(\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathbb{F}_1}, j_*\mathcal{R})$ to be related to the Bost-Connes system. For a connection between the BC-system and Λ-schemes see Yalkinoglu [Yal].

1.13. **Arithmetic Jet Spaces** [Bui96]**.** Let R be a p-torsion free ring, $S \in \mathsf{CRing}_R$, $\delta : R \to S$ be a p-derivation and $f \in R$ not a unit or zero-divisor. We may try to extend δ to a map $\delta : R[1/f] \to S[1/f]$ by using the rules for p-derivations. Attempting this we find that

$$\delta(1/f) = -\frac{\delta(f)}{f^p(f^p + p\delta(f))} = -\frac{\delta(f)}{f^p\phi(f)} \notin S[1/f]. \tag{1.6}$$

This problem with localization is fixed by p-adic completions since $1/(f^p + p\delta(f))$ makes sense as an element of $S[1/f]\hat{}$. This leads us to the following definition: The **first p-jet ring** of A/R relative to $\delta : R \to S$ a p-derivation is

$$A^1 := \left(\frac{S[a' : a \in A]}{I}\right)^{\widehat{}} \tag{1.7}$$

where the hat denotes a p-adic completion and

$$I = ((a+b)' = a'+b'+C_p(a,b), (ab)' = a'b^p + a^p b' + pa'b', r' = \delta(r) : a, b \in A, r \in R).$$

In later sections we will find it convenient to use the $\dot{x}$ notation rather than x' notation.

For example if $R = \mathbb{Z}[x]$ then $R^1 = \mathbb{Z}[x][\dot{x}]\hat{}$. These are elements of the form $\sum_{j=0}^\infty a_j(x)\dot{x}^j$ where $\mathrm{ord}_p(a_j(x)) \to \infty$ as $j \to \infty$ where $\mathrm{ord}_p(f) = \max\{n : f \equiv 0 \mod p^n\}$. For non affine schemes X, $J^1(X)$ is obtained by gluing together its affine pieces to get a formal scheme. We refer to [Con07],starting on page 31) for a treatment of formal schemes.

In order to state the univeral property we need a definition. Suppose that $\delta : R \to S$ is a p-derivation. A p-derivation $\tilde{\delta}$ is a **prolongation** of δ if $\tilde{\delta} : A \to B$ where $A \in \mathsf{CRing}_R$ and $B \in \mathsf{CRing}_S$ such that

$$\tilde{\delta}|_R = \delta.$$

In the statement that $\tilde{\delta}$ is a prolongation of δ implies that the R algebra structure on B coming from A is the same as the R algebra structure on B coming from S.

Proposition 1.6 (Universal Property of p-derivations)**.** *Suppose that $\delta : R \to S$ is a p-derivation with $S = \widehat{S}$. If $\tilde{\delta} : A \to B$ is a p-derivation prolonging δ where $B = \widehat{B}$ then there exists a unique ring homomorphism $f_{\tilde{\delta}} : A^1 \to B$ such that*

$$\tilde{\delta} = f_{\tilde{\delta}} \circ \delta_{univ}$$

Remark 1.7.

- The map $f_{\tilde{\delta}}$ sends the symbols $a' \in A^1$ to $\delta(a)$.
- We may form an incomplete version of the first jet ring satisfying an analogous univeral property by considering Λ-rings [Bor11].

For a scheme or formal scheme X/R where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ we will write

$$X_n = (X \mod p^{n+1}) = X \times_S \operatorname{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}/p^{n+1}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}) = X \otimes_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}}} \widehat{\mathbb{Z}}_p^{\mathrm{ur}}/p^{n+1}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}.$$

Note that the construction of Jet Spaces is performed schemes over R (including non-reduced ones). This implies that $J^1(X_1) = J^1(X)_0$ since $\delta(p^2) = p(1-p^{2p-2})$. See [Bui96] for more details.

1.14. **Affine bundles.** This paper is mainly a study of a particular "big" cohomology class

$$[J^1(X)] \in \check{H}^1(\widehat{X}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1))$$

attached to a smooth projective curve $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. In what follows we define the sheaf $\underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1)$

Definition 1.8. Let $f : J \to X$ be a morphism of schemes over another scheme S,

- An $\mathbb{A}^1$**-trivialization** (or simply a trivialization) of f will be a pair (U, ψ) consisting of an open subset U of X together with an isomorphism (sometimes also called the trivialization) $\psi : f^{-1}(U) \to U \times_S \mathbb{A}^1_S$ which satisfies $p_1 \circ \psi = f$.
- A **trivializing cover** will be a collection of trivializations $\{(U_i, \psi_i)\}$ such that $\{U_i\}$ is an open cover of X.
- A morphism $f : J \to X$ of schemes over S which admits a trivializing cover is called an $\mathbb{A}^1_S$**-bundle**.

Remark 1.9. We can make a similar definition in the category of p-formal schemes (see [Con07] for a review of formal schemes).

The collection of $\mathbb{A}^1_S$-bundles modulo isomorphism is in one-to-one correspondence with $\check{H}^1(X, \underline{\mathrm{Aut}}(\mathbb{A}^1))$. We will sometimes use the notation

$$[J] \in H^1(X, \underline{\mathrm{Aut}}(\mathbb{A}^1_S)).$$

Here $\underline{\mathrm{Aut}}(\mathbb{A}^1_S)$ is the sheaf of groups defined by

$$\underline{\mathrm{Aut}}(\mathbb{A}^1_S)(U) = \{\psi : \mathbb{A}^1_S \times_S U \cong \mathbb{A}^1_S \times_S U : p_2 \circ \psi = p_2\},$$

for each U an open subset of X where p_2 is the second projection. In the rest of the paper we may omit certain subscripts S. Note that $\psi \in \underline{\mathrm{Aut}}(\mathbb{A}^1_S)(U)$ can be identified with a polynomial $a_0 + a_1 T + \ldots + a_d T^d$ with $a_i \in \mathcal{O}(U)$ so we can talk about the degree of ψ as being the degree of this polynomial.

In the case of p-formal schemes we will be considering $\widehat{\mathbb{A}}^1_S$-bundles, the transition maps correspond to restricted power series, and for such a bundle J we can associate a cohomology classes $[J] \in H^1(X, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1_S))$.

1.15. **Cohomology classes associated to affine bundles.** Let X be a smooth curve over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Then by [Bui95] the map $J^1(X) \to \widehat{X}$ is an $\widehat{\mathbb{A}}^1$-bundle. Given a trivializing cover $\mathcal{U} = \{(U_i, \psi_i)\}$ of $J^1(X) \to \widehat{X}$ where $\psi_i : \pi^{-1}(U_i) \to U_i \hat{\times} \widehat{\mathbb{A}}^1$ we define $\psi_{ij} = \psi_i \circ \psi_j^{-1} \in \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1_{\widehat{\mathbb{Z}}_p^{\mathrm{ur}}})(U_{ij})$. The collection (ψ_{ij}) satisfies the cocycle condition $\psi_{ij}\psi_{jk}\psi_{ki} = \mathrm{id}_{ijk}$) and hence defines a cohomology class

$$[J^1(X)] := [\psi_{ij}] \in \check{H}^1(\widehat{X}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1)).$$

This construction is well-defined and independent of the choice of trivializing cover.

1.16. **Affine linear cocycles.** Let AL_1 be the group scheme of affine linear automorphisms. This is just a functor from rings to groups and when we restrict it to open subsets of a scheme we can view it as a subgroup of $\underline{\mathrm{Aut}}(\mathbb{A}^1)$ where elements (local sections) of AL_1 are identified with polynomials $a+bT$ and group law is given by composition of polynomials and $a \in \mathcal{O}$ and $b \in \mathcal{O}^\times$.

When we reduce $\beta = [J^1(X)]$ (the cocycle constructed in section 1.15) mod p we get some class $\beta_0 \in \check{H}^1(X_0, \mathrm{AL}_1)$, and the natural map $\mathrm{AL}_1 \to \mathcal{O}^\times$ induces a map on cohomology $\alpha : \check{H}^1(X_0, \mathrm{AL}_1) \to \check{H}^1(X_0, \mathcal{O}_{X_0}^\times) = \mathrm{Pic}(X_0)$. It turns out that $\alpha(\beta_0) = [F_{X_0}^* T_{X_0/\overline{\mathbb{F}}_p}]$ where $F_{X_0} : X_0 \to X_0$ is the absolute Frobenius (see Proposition 5.15). This in conjunction with an "embedding lemma" (Proposition 5.10) allows us to convert the "constant part" of $\beta_0 = [a_{ij} + b_{ij}T]$ into the Deligne-Illuse class $\mathrm{DI}_0(\delta) \in \check{H}^1(X_0, F_{X_0}^* T_{X_0/\overline{\mathbb{F}}_p})$ (see section 5.8 on page 46).

1.17. **Structures on affine bundles.** We should mention that $\mathbb{A}^1_S$-bundles are more general than line bundles in that line bundles require $\psi_{ij} := \psi_i \circ \psi_j^{-1}$ to be in GL_1 while $\mathbb{A}^1$-bundles only have ψ_{ij} in $\underline{\mathrm{Aut}}(\mathbb{A}^1_S)$ in general. So line bundles $L \to X$ are $\mathbb{A}^1$-bundles which admit a special trivialzing cover $\mathcal{U} = \{(U_i, \psi_i)\}$ that satisfies $\psi_i \circ \psi_j^{-1} \in \mathrm{GL}_1(U_{ij})$ where $U_{ij} := U_i \cap U_j$. In other words, line bundles are just $\mathbb{A}^1_S$-bundles with an additional GL_1-structure. The GL_1 transition maps allow us to give the sheaf of local sections of an $\mathbb{A}^1$-bundle L, which we denote by $\Gamma(-, L)$, the structure of an invertible sheaf.

To state our result we need to generalize the notion of a GL_1-structure by replacing $\mathrm{GL}_1 \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_S)$ with more general subsheaves of groups $A \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_S)$ (we use the symbol "$\leq$" to denote subgroups or sheaves of subgroups).

Definition 1.10. Fix a subsheaf of groups $A \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_S)$, an $\mathbb{A}^1_S$-bundle $f : J \to S$ and a trivializing cover $\mathcal{U} = \{(U_i, \psi_i)\}$,

- trivializations $\psi_i : f^{-1}(U_i) \to U \times_S \mathbb{A}^1_S$ and $\psi_j : U_j \to U_j \times_S \mathbb{A}^1_S$ are A-**compatible** if $\psi_i \circ \psi_j^{-1} \in A(U_{ij})$
- An A-**Atlas** on an $\mathbb{A}^1_S$-bundle will be a trivializing cover $\mathcal{U}$ such that all trivializations in the cover are A-compatible.
- An A-**structure** Σ will be a maximal A-atlas.

Remark 1.11. Two trivializations don't need to be in a cover in order to speak of their A-compatibility. Also the same definitions can be made for formal schemes.

Given an A-atlas $\{(U_i, \psi_i)\}$ which gives an A-structure Σ for an affine bundle $f : J \to X$ we will use the notation

$$[\Sigma] = [\psi_i \circ \psi_j^{-1}] \in \check{H}^1(X, A).$$

1.18. **Structures of bounded degree on affine bundles.** The proof of Theorem 1.1 uses succesive reductions of the structure group. For X a scheme of finite type over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ we can define the subgroup $A_{n+1} \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_n})$ consisting of automorphisms of the form

$$\psi(T) = a_0 + a_1 T + p a_2 T^2 + \cdots + p^n a_{n+1} T^{n+1} \mod p^{n+1}$$

where $a_i \in \mathcal{O}_{X_n}$ (in section 4.1 we prove that it is a group)

Theorem 1.12 (Degree Structures on Projective Curves)**.** *Let $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ be a smooth projective curve of genus g. If $p > 6g-5$ then for each $n \geq 1$ the first p-jet space mod p^{n+1}, $J^1(X)_n$, admits an A_n-structure.*

The degree structures appearing in Theorem 1.12 arise naturally from cocycle computations and are quite canonical. It is obvious that they should exist after doing several examples by hand but proving their existence is non-trivial.

After having shown that the jet space of every sufficiently general curve admit such a structure we then use a "pairing" with group cocycles to succesively reduce the structure group to eventually prove Theorem 1.1.

1.19. **Group cocycles.** Let G be a group (or sheaf of groups) and A be an abelian group (or sheaf of $\mathcal{O}_X$-modules). A (left) group cocycle $\tau : G \to A$ with respect to a left group action $\rho : G \to \mathrm{Aut}(A)$ is a morphism of set (resp. sheaves of sets) which satisfies

$$\tau(g_1 g_2) = \tau(g_1) + g_1 \tau(g_2).$$

The collection of group cocycles will be denoted by $Z^1_{group}(G, A)$.

1.20. **A group cohomology/Čech cohomology "pairing" and the main non-triviality result.** Let X be a smooth projective curve over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$.

In the present paper given the cohomology class of some A_{n+1}-structure $[\Sigma_n] \in H^1(X_n, A_{n+1})$ and some $\tau \in Z^1_{group}(A_n, \mathcal{O}_{X_0})$ we construct a cohomology class

$$\kappa(\Sigma_n, \tau) \in H^1(X_0, L_\Sigma).$$

which aids the proof of Theorem 1.1.

- The subscript "group" denotes group cohomology with respect to a "pull-back right-action" $\rho_n : \mathcal{O}_{X_0} \times A_{d+1} \to \mathcal{O}_{X_0}$ defined by
$$\rho_n(f, g(T)) = f \cdot g'(T)^n.$$
where $f \in \mathcal{O}_{X_0}$ and $g(T) = a + bT + pcT^2 + O(p^2) \in A_{d+1}$. From the form of $g(T)$ we can see that $g'(T) \in \mathcal{O}_{X_0}^\times$ so the action is well-defined.
- $[L_\Sigma] \in \mathrm{Pic}(X_0) = H^1(X, \mathcal{O}_{X_0}^\times)$ is the image of Σ in the Picard group via the map induced by
$$A_{d+1} \to A_{d+1} \otimes \overline{\mathbb{F}}_p = \mathrm{AL}_1(\mathcal{O}_{X_0}) \to \mathcal{O}_{X_0}^\times.$$

The first map is reduction mod p and the second map the quotient of the semi-direct product

$$\mathrm{AL}_1(\mathcal{O}_{X_0}) \cong \mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^\times \to \mathcal{O}_{X_0}^\times.$$

Example 1.1.
(1) If we let $\tau : A_2 \to \mathcal{O}_{X_0}$ be defined by $\tau(a + bT + pcT^2) = a$ mod p on local sections then τ is a group cocycle with respect to the action ρ_1 and $\kappa(X, \Sigma, \tau) \in H^1(X_0, F_{X_0}^* T_{X_0/\overline{\mathbb{F}}_p})$ is equal to the Deligne-Illusie class of defined section 1.11 in terms of differences local lifts of p-derivations.
(2) The map $\tau_2 : A_2 \to \mathcal{O}_{X_0}$ defined by $\tau_2(a + bT + pcT^2) = c$ on local sections is a group cocycle with respect to ρ_2.

1.21. **Multiple structures result.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and let X/R be a smooth elliptic curve. In this setting its first arithmetic jet space $J^1(X)_1$ is a group (scheme) and admits a trivializing cover $\mathcal{U}_{elliptic}$ induced by the group structure with A_2-compatible trivializations—In fact, the transition maps coming from $\mathcal{U}_{elliptic}$ have degree one (see section 7.1). This implies that in addition to the structure coming from Theorem 1.12, which we will call Σ_{plane} (emb is for embedding as it comes from the embedding in projective space), there exists a distinct A_2-structure $\Sigma_{elliptic}$ and $(\Sigma_{elliptic}, \tau_2) = 0$. Now note that

$$\kappa(\Sigma_{plane}, \tau_2) \neq 0 \text{ and } \kappa(\Sigma_{elliptic}, \tau_2) = 0 \implies \Sigma_{plane} \neq \Sigma_{elliptic}.$$

We get the following:

Corollary 1.13 (Multiple A_2-Structures)**.** *Fix the following:*

$$\begin{array}{rcl} p & = & \textit{any prime} \neq 2 \\ S & = & \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{ur}}/p^2\mathbb{Z}_p^{\mathrm{ur}}) \\ X & = & \textit{Elliptic Curve over } S \\ J & = & \textit{First Arithmetic Jet Space of } X \\ A & = & A_2 = \textit{ Subgroup of } \underline{\mathrm{Aut}}(\mathbb{A}^1_S) \textit{ of automorphism of degree less than or equal to two} \end{array}$$

The scheme J is an $\mathbb{A}^1_S$-bundle which can be given two distinct A-structures, $\Sigma_{elliptic}$ and Σ_{plane} which are incompatible. In other words, $\Sigma_{elliptic} \neq \Sigma_{plane}$.

Remark 1.14. The existence of multiple A_2-structure on $J^1(E)$ is quite surprising and in stark contrast to the case of line bundles (when $A = \mathrm{GL}_1$) which are classified by the Picard group $\mathrm{Pic}(X) = \check{H}^1(X, \mathrm{GL}_1)$ where the same physical $\mathbb{A}^1$-bundle will never admit multiple line bundle structures. We will now prove the latter. Suppose that $\eta_1 = (a_{ij}), \eta_2 = (b_{ij}) \in H^1(X, \mathcal{O}_X^\times)$ define $\mathcal{O}^\times$-structures of the same affine bundle. This means we can suppose that

$$\frac{\coprod_i U_i \times \mathbb{A}^1}{\eta_1} \cong \frac{\coprod U_i \times \mathbb{A}^1}{\eta_2}$$

where the quotients denotes gluing of the open sets $U_{ij} \times \mathbb{A}^1 \subset U_i \times \mathbb{A}^1$ and $U_{ji} \times \mathbb{A}^1 \subset U_j \times \mathbb{A}^1$ using the cocycle η_k, $k = 1, 2$ and the isomorphism is an isomorphism of schemes. Such an isomorphism of schemes over X is defined by a collection of maps $f_i : U_i \times \mathbb{A}^1 \to U_i \times \mathbb{A}^1$ which are compatible with the gluing maps. This means that we have

$$b_{ij} T = f_i \circ a_{ij} T \circ f_j^{-1}$$

$$\begin{tikzcd}
 & \beta = [J^1(E)] \arrow[d, mapsto] & \check{H}^1(\widehat{E}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1)) \arrow[d, "\text{reduction mod } p^2"] \\
 & \beta_1 = [J^1(E)_1] & \check{H}^1(E_1, \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})) \\
{[\Sigma_{elliptic}]} \arrow[ur, mapsto] \arrow[dr, mapsto] & {[\Sigma_{plane}]} \arrow[u, mapsto] \arrow[d, mapsto] & \check{H}^1(E_1, A_2) \arrow[u, "\text{include into larger group}"'] \arrow[d, "\text{reduction mod } p"] \\
 & \beta_0 = [J^1(E)_0] & \check{H}^1(E_0, \mathrm{AL}_1)
\end{tikzcd}$$

FIGURE 1. Diagram of various arithmetic cohomology classes associated to elliptic curves

in terms of polynomials we have $b_{ij}T = f_i(a_{ij}f_j^{-1}(T))$. Taking derivatives implies

$$b_{ij} = f_i'(a_{ij}f_j^{-1}(T))a_{ij}f_j^{-1\prime}(T) = f_i'(a_{ij}f_j^{-1}(T))a_{ij}f_j'(f_j^{-1}(T))^{-1}.$$

Making the substitution gives $T = f_j(S)$ gives $b_{ij} = f_i'(a_{ij}S)a_{ij}f_j'(S)^{-1}$ and letting $S = 0$ shows that

$$b_{ij} = f_i'(0)a_{ij}f_j'(0)^{-1},$$

which shows that $\eta_1 = \eta_2$.

We should remark that this proof technique does not extend to proving the uniqueness of A_2-structures.

1.22. **Acknowledgements.** I am indebted to Alexandru Buium for his excellent guidance and encouragement.

1.23. **Structure of the paper.** Section 2 recalls the basic definition and facts about p-derivations and Witt vectors. Section 3 we gather some general facts about fiber bundles and their cohomology classes that we need to develop the theory for pairings (developed in 5). In section 4 we construct the A_n-structures which appear in the statement theorem 1.12. Section 5 is about cohomology classes coming from the affine bundle on the first arithmetic jet space. In particular section 5.8 studies the Deligne-Illusie class introduced in section 1.11 and explains how $[J^1(X)]$ is a lift of it. In section 6 we apply the methods developed in section 5 to prove the main theorem (Theorem 1.1). Section 7 proves the result on the existence of multiple A_2-structures for $J^1(E)_1$ and section 8 provides some examples for better understanding this result. Section 9 contains some extra computations with superelliptic curves which may be useful for future work.

2. WITT VECTORS, p-DERIVATIONS AND ARITHMETIC JET SPACES

This brief section is about p-derivations, Jet spaces and the Frobenius. Section 2.1 recalls the definition of p-derivations, section 2.2 states the adjointness of the witt and jet functors functors W_1 and J^1 and section 2.3 gives a slightly more in-depth look at Frobeniuses on schemes.

2.1. **p-derivations.** Let p be a fixed prime in $\mathbb{Z}$, R be a p-torsion free ring and S an R-algebra. A **p-derivation** is a map of sets $\delta : R \to S$ such that for all $a, b \in R$,

$$\delta(a+b) = \delta(a) + \delta(b) + C_p(a,b), \tag{2.1}$$

$$\delta(ab) = \delta(a)b^p + a^p\delta(b) + p\delta(a)\delta(b) \tag{2.2}$$

where $C_p(X,Y) = \frac{X^p+Y^p-(X+Y)^p}{p} \in \mathbb{Z}[X,Y]$ since all the relevant binomial coeffiecients in the numerator are divisible by p. Also the sum rule shows that this operation is not linear. One should observe that such a map is not even linear mod p.

For any ring of characteristic p there exists a Frobenius endomorphism denoted by F which takes x to x^p. A **lift of the Frobenius** is a ring endomorphism $\phi : R \to S$ such that

$$\phi(x) \equiv x^p \mod pS. \tag{2.3}$$

There is a connection between p-derivations and lifts of the Frobenius:

Proposition 2.1 (p-derivations and lifts of the Frobenius)**.** *Suppose that R is a p-torsion free ring and $S \in \mathsf{CRing}_R$. There is a one-to-one correspondence between lifts of Frobenius $\phi : R \to S$ and p-derivations $\delta : R \to S$:*

$$\delta(x) = \frac{\phi(x) - x^p}{p} \leftrightarrow \phi(x) = x^p + p\delta(x). \tag{2.4}$$

When $\mathrm{char}(S) = p$, the set of p-derivations have the structure of a principal homogeneous spaces for abelian group of **derivations of the Frobenius**. These are maps $D : R \to S$ such that for all $r_1, r_2 \in R$ we have $D(r_1+r_2) = D(r_1)+D(r_2)$ and $D(r_1r_2) = D(r_1)r_2^p + r_1^pD(r_2)$.

Proposition 2.2 (p-derivations into rings of characteristic p form a torsor for derivations of the Frobenius)**.** *If $\delta_1, \delta_2 : R \to S$ are p-derivations on R then the map D defined by*

$$D(x) := (\delta_1(x) - \delta_2(x)) \mod pS$$

is a derivation of the Frobenius. Conversely, every derivation of the Frobenius $D : R \to S/p$ is obtained in this way.

A more geometric description of this relationship can be found in [Bui96].

Remark 2.3.
- Note that this is not true in characteristic zero and is precisely what makes Theorem 1.1 (on page 3) interesting as we have a lift of this torsor structure in characteristic zero.
- There do not exists p-derivations whose domain has characteristic p.
- Differences of p-derivations in characteristic zero are $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ linear maps.

2.2. **p-derivations and Witt vectors.** Recall that if $B \in \mathsf{CRing}_R$ that the set of derivations $\delta : R \to B$ is in bijection with ring homomorphisms $f : R \to D_1(B) := B[\varepsilon]/(\varepsilon^2)$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\text{algebra map}"'] & D_1(B) \arrow[d, "p_1"] \\ & B \end{tikzcd}$$

commutes, where $p_1(b_0+\varepsilon b_1)=b_0$. Explicity, given a derivation δ the map $r\mapsto r+\varepsilon\delta(r)$ defines a ring homomorphism.

There is a similar characterization of p-derivations where we replace the dual ring of B with the truncated Witt vectors of B, $W_1(B)$. Here the functor $W_1 : \mathsf{CRing}_{\mathbb{Z}} \to \mathsf{CRing}_{\mathbb{Z}}$ takes a ring A to $W_1(A)$. As sets $W_1(A)=A\times A$, and to give it a ring structure one defines

$$\begin{aligned}(a_0,a_1)+(b_0,b_1) &= (a_0+b_0, a_1+b_1+C_p(a_0,b_0)),\\ (a_0,a_1)\cdot(b_0,b_1) &= (a_0b_0, a_1b_0^p+a_0^pb_1+pa_1b_1),\end{aligned}$$

where $C_p(X,Y)=\frac{X^p+Y^p-(X+Y)^p}{p}\in\mathbb{Z}[X,Y]$.

We can now state the characterization of p-derivations in terms of ring maps: Fix and R-algebra B. The set of p-derivations $\delta: R\to B$ is in bijection with ring homomorphisms $f: R\to W_1(B)$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\text{algebra map}"'] & W_1(B) \arrow[d, "p_1"] \\ & B \end{tikzcd}$$

commutes where p_1 denotes the projection onto the first factor.

2.3. **Absolute and relative Frobenius on schemes.** Following [PD], let S be a scheme of characterstic p and let F_S denote the the **Frobenius endomorphism** which is the identity on the topological space and acts as $a\mapsto a^p$ on $\mathcal{O}_S$. If $u: X\to S$ is a morphism of schemes we have the following commutative diagram which defines the scheme $X^{(p)}$ and the maps $F_{X/S}$

$$\begin{tikzcd} X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "u"'] & & \\ & X^{(p)} = X\times_{S,F_S} S \arrow[r] \arrow[d] & X \arrow[d, "u"] \\ & S \arrow[r, "F_S"] & S \end{tikzcd} \quad . \tag{2.5}$$

The morphism $F_{X/S}$ is called a **relative Frobenius**. If x is a local section of $\mathcal{O}_X$ then the image of $x\otimes 1\in\mathcal{O}_{X^{(p)}}$ under $F_{X/S}^*$ is

$$F_{X/S}^*(x\otimes 1)=F_X^*(x)=x^p.$$

For example if X is defined by the equations $f_\alpha=\sum a_{\alpha m}T^m$ in affine space $S[T_1,\ldots,T_n]=\mathbb{A}_S^n$ then $X^{(p)}$ is defined by the equations $f^{(p)}=\sum a_{\alpha,m}^pT^m$ in $\mathbb{A}_S^n$ and $F_{X/S}$ is defined by $T_i\mapsto T_i^p$.

If $\widetilde{X}$ is a scheme whose reduction mod p is X then a **lift of the absolute Frobenius** is a map $\phi_{\widetilde{X}}:\widetilde{X}\to\widetilde{X}$ whose reduction mod p if F_X.

Now in addition to supposing that $\tilde{X}$ is a scheme whose reduction mod p is X, suppose that S has a lift $\widetilde{S}$ and a morphism $\phi_{\widetilde{S}}:\widetilde{S}\to\widetilde{S}$ lifting the absolute Frobenius on it. One can construct the pullback $\widetilde{X}^\phi$ of $\widetilde{X}$ by ϕ as giving the similar commutative diagram

$$
\begin{tikzcd}
X \arrow[dr, "\phi_{\widetilde{X}/\widetilde{S}}"] \arrow[drr, bend left, "\phi_X"] \arrow[ddr, bend right, "\tilde{u}"'] & & \\
& \widetilde{X}^{(\phi)} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\tilde{u}"] \\
& \widetilde{S} \arrow[r, "\phi"] & \widetilde{S}
\end{tikzcd} \tag{2.6}
$$

The diagram 2.6 shows that a lift of the absolute Frobenius $\phi_{\widetilde{X}}$ on $\widetilde{X}$ exists if and only if a **lift of the relative Frobenius** $\phi_{\widetilde{X}/\widetilde{S}}$ exists. In this paper we will be mostly concerned with the case $\widetilde{S} = \mathrm{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$ where ϕ is the unique lift of the Frobenius on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$.

3. Fiber bundles, atlases and structures

This brief section sets up the Language of bundles and torsors to be used in section 4. In section 3.1 we recall what a torsor is. In section 3.2 we define the notion of a "structure" on a bundle. In section 3.3 we recall the construction of a cohomology class which classifies bundles. In section 3.4 we perform a similar construction for sheaves which are locally isomorphism to a more simple sheaf. In section 4 we make similar definitions for sheaves of $\mathcal{O}$-modules.

3.1. **Torsors and bundles.** A sheaf of sets T on scheme X (or formal scheme) is called a **sheaf of G-torsors** (or sheaf of **principal homogeneous spaces**) for a sheaf of groups G if for all open subsets U of X the set $T(U)$ has the structure of a $G(U)$-torsor in the sense that there exists some $\div : T(U) \times T(U) \to G(U)$ such that for all $x, y, z \in T(U)$ we have

(1) (Transitive) $(x/y)(y/z) = x/z$.
(2) (Symmetric) $(x/y)^{-1} = y/x$.
(3) (Reflexive) $(x/x) = 1_G$.
(4) (Cancellation) $x/y = 1 \implies x = y$.
(5) (Surjectivity) For all $x \in T(U)$ and all $g \in G(U)$ there exists some $y \in T(U)$ such that $x/y = g$.

and the map $\div$ behaves well with respect to restriction.

An F-**bundle** (**fiber bundle** with fiber F) is some morphism $\pi : E \to B$ such that for all P in B there exists some open U containing P and some isomorphism $\psi_U : \pi^{-1}(U) \to U \times F$ such that $p_1 \circ \psi_U = \pi$ (One usually refers to the above as locally trivial F-bundles; we will omit the words "locally trivial".)

Given two trivializations $\psi, \varphi : \pi^{-1}(U) \to U \times F$ we can divide them to get an automorphism $\psi \circ \varphi^{-1} : U \times F \to U \times F$. If we let

$$T(U) = \{ \text{ trivializations } \varphi : \pi^{-1}(U) \to U \times F\},$$

then $T(U)$ is a $\underline{\mathrm{Aut}}(F)(U)$-torsor where $\underline{\mathrm{Aut}}(F)(U) = \{\psi : \pi^{-1}(U) \cong U \times F : p_1 \circ \psi = \pi\}$.

Proposition 3.1. *(1) T the collection of trivializations is a sheaf.*
(2) $G = \underline{\mathrm{Aut}}(F)$ is a sheaf.
(3) T is a G-torsor.

3.2. **Atlases and structures.**

Definition 3.2. Let $\pi : E \to B$ be an F-bundle and $H \leq \underline{\mathrm{Aut}}(F)$ as sheaves of groups.

- Let U be an open set of B and ψ and φ trivializations. We say that ψ and φ are H-**compatible** if $\psi \circ \varphi^{-1} \in H(U)$.
- An H-**atlas** consist of a collection of open sets and trivializations $\{(U_i, \varphi_i)\}$ such that
  (1) The open sets cover the base: $B = \bigcup_{i \in I} U_i$.
  (2) The trivializations are H-compatible on φ_i.
- An H-**structure** is is a maximal H-atlas.

Remark 3.3. Note that every H-atlas is contained in a unique H-structure.

If Σ is an H-structure we will let $[\Sigma] \in H^1(X, H)$ denote the associated cohomology class.

In the terminology above vector bundles are just $\mathbb{A}^n$-bundles with an additional GL_n-structure. Let X be a projective variety over a field K. Let $V \to X$ be a n-dimensional vector bundle in the category of varieties over K. A vector bundle V has the property that for every P in X there exists some U open in X containing P such that $\pi^{-1}(U) \cong U \times \mathbb{A}^n$. This gives V the structure of an affine bundle. Since V is a vector bundle there exists some $\mathcal{U} = \{(U, \varphi_i) : i \in I\}$, a trivializing cover such that $\varphi_i \circ \varphi_j^{-1} \in \mathrm{GL}_n(U_{ij})$. This property gives a natural $\mathcal{O}_X$-module structure to $V_X(-)$, the functor of X-points of the scheme V. GL_n-structures on affine bundles are unique if they exists.

3.3. **Classification of F-bundles.** Let E,F and X be objects in Sch_S or ForSch_S. Suppose that $\pi : E \to X$ is a surjection such that there exists an open cover $\{(U_i, \varphi_i)\}$ of X giving E the structure of an F-bundle. The transition maps $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} \in \underline{\mathrm{Aut}}(F)(U_{ij})$ define a class

$$\beta := [E] := [\varphi_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(F)).$$

Proposition 3.4. *Let $\pi : E \to X$ and $\pi' : E' \to X$ be F-bundle with associated cohomology classes $\beta, \beta' \in \check{H}^1(X, \underline{\mathrm{Aut}}(F))$*

(1) $\beta = \beta'$ if and only if $E \cong E'$ as F-bundles.
(2) $\beta = 1$ if and only if $E \cong X \times_S F$.

In the proposition, 1 denotes the trivial element in cohomology. The proof of 3.4 is straight forward.

3.4. **Locally trivial sheaves of modules.** Let F and B be a sheaves of quasi-coherent $\mathcal{O}_X$-modules on a scheme (or formal scheme) X. Let $\mathcal{U} = \{U_i\}$ be a cover of X and $\varphi_i : F(U_i) \to B(U_i)$ be isomorphisms of $\mathcal{O}(U_i)$-modules. We will call such a collection an B-atlas for F and the collection $\{(U_i, \varphi_i)\}$, B-trivializations. We call such a sheaf F locally B-trivial.

To each locally B-trivial sheaf F we can associate a cohomology class:

$$(F, \{(U_i, \varphi_i)\}) \mapsto [\varphi_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(B)).$$

Here $\underline{\mathrm{Aut}}(B)$ is the sheaf of automorphisms of quasi-coherent $\mathcal{O}$-modules. We will use the notation $[F] = [\varphi_{ij}]$. Notice that when $B = \mathcal{O}^{\oplus n}$ that a locally B-trivial sheaf with the same thing as a GL_n-atlas.

Proposition 3.5. *Let F, F' and L be quasi coherent $\mathcal{O}$-module on scheme or formal scheme X. Suppose in addition that F and F' be are locally L-trivial.*

(1) $[F] = [F']$ in $\check{H}^1(X, \underline{\mathrm{Aut}}(L))$ if and only if $F \cong F'$ as sheaves of $\mathcal{O}$-modules.
(2) Suppose that $\varphi_i : F(U_i) \to L(U_i)$ are trivializations for F and $\varphi_i' : F(U_i) \to L(U_i)$ are trivializations for F'. If

$$\varphi_{ij} = g_i \varphi_{ij}' g_j^{-1}.$$

Then the map $\gamma : F' \to F$ defined by

$$\gamma(s) = (\varphi_i^{-1} \circ g_i \circ \varphi_i')(s), s \in F'$$

is well-defined and induces an isomorphism.

Proof. The first item just says when two sheaves are isomorphic and the second item gives an explicit description this isomorphism. It suffices to explain the second item. The map γ is what one would construct if you were asked to construct such a map. Examining the relation $\varphi_{ij} = g_i \varphi_{ij}' g_j^{-1}$ immediately gives

$$\varphi_j^{-1} g_j \varphi_j' = \varphi_i^{-1} g_i \varphi_i'$$

as morphisms of sheaves. This shows that the map is well-defined: if $s \in F'(U)$ let $s_i = s|U_i$ and define $\gamma(s)$ to be the unique section which lifts $\gamma(s_i)$ (which agree on the intersections of their domains). Since every factor involved in the definition of γ is an isomorphism it follows that γ itself is an isomorphism. ☐

4. A_n-Structures on p-Jet Spaces

Throughout this section $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ for p is a fixed prime. The goal of this section is to prove the following:

Theorem 4.1. *If $X \subset \mathbb{P}_R^N$ a smooth irreducible curve of genus g with $p > 6g - 6$ then for every $n \geq 1$, $J^1(X)_n$ admits a canonical A_{n+1}-structure.*

Recall that any smooth scheme X/R of relative dimension d admits a cover by affine opens $\{U_i\}$ which admit étale maps $f_i : U_i \to \mathbb{A}_R^d$. Recall the following from [Bui95],[Bui94a]

Lemma 4.2. *Let X and Y be finite dimensional smooth schemes over R,*

(1) If $f : X \to Y$ is étale then $J^1(X) \cong \widehat{X} \hat{\times}_Y J^1(Y)$.
(2) If $f : X \to \mathbb{A}_R^d$ is étale then $J^1(X) \cong \widehat{X} \hat{\times} \widehat{\mathbb{A}}_R^d$

Remark 4.3. If $X = \mathrm{Spec} A$ and $f^* : R[T_1, \ldots, T_n] \to A$ induces the étale map then $\mathcal{O}(J^1(X)) = \mathcal{O}(X)[\dot{T}_1, \ldots, \dot{T}_n]\widehat{\ }$. Here we have identified the étale parameters T_i with their image under f^*. We will make use of this later.

We will prove theorem 4.1 by considering certain étale maps $\varepsilon_i : U_i \to \mathbb{A}^1$ on an open cover $\{U_i\}$ of X then proving that the associated trivialization maps $\psi_i : J^1(U_i) \to \widehat{U}_i \hat{\times} \widehat{\mathbb{A}}^1$ induce an A_{n+1}-structure on $J^1(X)_n$ for every $n \geq 1$. Here, the groups $A_n \leq \mathrm{Aut}(\mathbb{A}_{R_n}^1)$ consist of univariate polynomial automorphisms of the form

$$a_0 + a_1 T + p a_2 T^2 + \cdots + p^{n-1} a_n T^n \mod p^{n+1}$$

where the a_i are local sections of the structure sheaf (section 4.1).

The proof is a reduction to the case of affine plane curves which is proved in section 4.2. For $V(f) \subset \mathbb{A}_R^2$, the computation in section 4.2 relies on the fact that the partial derivatives of f are not both identically zero modulo p. Section 4.3 says that if $\deg(f) > p$ then the partial derivatives of f cannot both be identically zero modulo p. Section 4.5 examines the transition maps coming from two arbitrary étale maps $\varepsilon_1 : U \to \mathbb{A}^1$ and $\varepsilon_2 : U \to \mathbb{A}^1$ and shows how to reduce their A_n-compatibility to the local computations of section 4.5. In particular section 4.5 says that if the degree of the closure of the image curve $\varepsilon_1 \times \varepsilon_2(U) \subset \mathbb{A}^1_R$ is less than p then the computations of section **??** imply the associated trivializations mod p^n are A_n-compatible.

The remainder of this section is then devoted to constructing the cover $\{U_i\}$ of X and étale maps $\varepsilon_i : U_i \to \mathbb{A}^1$ with the property that $\overline{\varepsilon_i \times \varepsilon_j(U_{ij})} \subset \mathbb{A}^2_R$ are irreducible curves of degree less than p (for p sufficiently large) and that every point of X is contained in the domain of one of these étale projections. Section 4.5 proposes to construct the collection $\{\varepsilon_i : U_i \to \mathbb{A}^1\}$ which cover $X \subset \mathbb{P}^n$ via projections to lines in $\mathbb{P}^n_R$. The difficult part is then to confirm that such a system of lines exist that give projections with the right properties. Section 4.7 introduces a notion of a "decomposition of projective space" and section 4.15 spells out the conditions for such a decomposition to induce the family of étale projections that we seek. The necessary theory for these admissible decompositions is developed/recalled as needed in sections 4.4,4.6, 4.8, 4.9, 4.10, 4.13, 4.14, 4.16, 4.17.

Remark 4.4. Finally, we have included an appendix at the end of the paper that contains computations with superelliptic curves. In section 9.1 we recall the definition of a superelliptic curves and then in section 9.2 provide a canonical A_2-structure for them. In section 9.3 we give an A_2-structure for plane curves. In section 9.4 we show that when our superelliptic curve is a plane curve that the two structures Σ_{plane} and Σ_{super} coincide. These computations motivated many of the results of this section. We would also like to remark that section 7.1 gives an A_n-structure structure on elliptic curves which has nothing to do with this section.

4.1. A_n **is a group.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. We define a subset of automorphisms of degree n mod p^n

$$A_n := \{a_0 + a_1 T + pa_2 T^2 + \cdots + p^{n-1} a_n T^n : a_1 \in \mathcal{O}_{X_n}^\times, a_i \in \mathcal{O}_{X_n}\} \subset \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_{n-1}}).$$

Proposition 4.5. *$A_n \subset \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_{n-1}})$ is a subgroup.*

Proof. We will first show that A_n is closed under composition and then show that A_n is closed under taking inverses. Let

$$\begin{aligned} f(T) &= a_0 + a_1 T + pa_2 T^2 + \cdots + p^{n-1} a_n T^n, \\ g(T) &= b_0 + b_1 T + pb_2 T^2 + \cdots + p^{n-1} b_n T^n \end{aligned}$$

be elements of A_n. We claim that $g(f(T)) \in A_n$.

If is sufficient to show that every term in

$$p^{j-1} b_j (f(T))^j, 1 < j \leq n-1$$

of degree d is divisible by p^{d-1}.

A typical term in the expansion above takes the form $A = p^{j-1} \cdot (p^{i_1 - 1} a_{i_1} T^{i_1}) \cdots (p^{i_j - 1} a_{i_j} T^{i_j})$ has degree greater than d. This means that $i_1 + i_2 + ... + i_j = d$ and that $p^{d-j} = p^{i_1 + i_2 + ... + i_j - j}$ which means that A is of the form $A = p^{d-1} a_{i_1} ... a_{i_j} T^d$ and that every coefficient T^d in the expansion of $g(f(T))$ is divisible by p^{d-1}. In particular note that $g(f(T))$ has degree n mod p^n which shows that A_n is closed under composition.

We will now show that if $f \in A_n$ then $f^{-1} \in A_n$. Fix $f(T) = a_0 + a_1 T + pa_2 T^2 + \cdots + p^{n-1} a_n T^n$. We proceed by induction on n. The base case is $n = 2$ we have proved everything. Now suppose that

$$f(g(T)) = g(f(T)) = T \mod p^n$$

we need to show that $g \in A_n$. By induction we know that we can write (by rearranging terms if necessary)

$$g(T) = g_{n-1}(T) + p^{n-1} G(T)$$

where $G(T)$ has order greater than n and

$$g_{n-1}(T) = b_0 + b_1 T + pb_2 T^2 + \cdots + p^{n-2} b_{n-1} T^{n-1}.$$

We will assume that $G(T)$ has degree greater strictly greater that n and derive a contradiction. Examining

$$g(f(T)) = g_{n-1}(f(T)) + p^n G(f(T)) \mod p^n$$

we know from the previous proposition that

$$\deg(g_{n-1}(f(T))) \leq n.$$

We also know that

$$p^{n-1} G(f(T)) = p^{n-1} G(a_0 + a_1 T)$$

and that the degree of $G(f(T))$ is exactly the degree of $G(T)$ since a_1 is a unit. This means that $g(f(T)) = T \mod p^n$ has degree strictly greater than n which is a contradiction. This shows that $g(T)$ actually has degree n and hence $g(T) \in A_n$ which completes the proof. $\square$

It is not obvious that the groups A_n are the correct ones to study in the context of transition maps on Jet spaces. To illustrate this point section 4.1.1 gives an interesting subgroup of $\mathrm{Aut}(\mathbb{A}^1_{R_1})$ which was a first of many interesting subgroups that we found.

4.1.1. *Automorphisms affine line.* Let R be a p-torsion free ring with $pR \in \mathrm{Spec}(R)$. Since $R/pR = R_0$ is an integral domain, polynomials $g \in R_1[t]$ which induce automorphisms of $R_0[t]$ are affine linear. This means that polynomials which induce automorphisms of $R_1[t]$, where $R_1 = R/p^2$, can be represented as

$$g(t) = a_0(g) + a_1(g)t + pF_g(t)$$

with $F_g(t) \in R_0[t]$, $a_1(g) \in R_1^\times$, and $a_0(g) \in R_1$. We can further arrange that $\mathrm{ord}_t(F_g) \geq 2$ in which case the representation is unique.

Let $\tilde{\mathrm{A}}_d(R_1) \leq \mathrm{Aut}_{R_1}(R_1[t])^{\mathrm{op}}$ denote the collection of automorphisms of $\tilde{\mathrm{A}}_1[t]$ of degree less than or equal to d.

Proposition 4.6. *Let R be a p-torsion free ring with $pR \in \mathrm{Spec}(R)$. For each $d \geq 1$, $\tilde{\mathrm{A}}_d(R_1)$ is a subgroup of $\mathrm{Aut}_{R_1}(\mathcal{O}_1[t])^{\mathrm{op}}$.*

Proof. If we let $f(t) = a_0(f) + a_1(f)t + pF_f(t)$ and $g(t) = a_0(g) + a_1(g)t + pF_g(t)$ with $\mathrm{ord}_t(F_f), \mathrm{ord}_t(F_f) \geq 2$ we get

$$\begin{aligned} f(g(t)) &= a_0(f) + a_1(f)[a_0(g) + a_1(g)t + pF_g] + pF_f(a_0(g) + a_1(g)t) \\ &= a_0(f) + a_1(f)a_0(g) + (a_1(f)a_1(g))t \\ &\quad +p(a_1(f)F_g(t) + F_f(a_0(g) + a_1(g)t) \end{aligned} \tag{4.1}$$

Note that

$$F_{f\circ g} = a_1(f)F_g(t) + F_f(a_0(g) + a_1(g)t)$$

has degree no larger than $\max\{\deg F_g, \deg F_f\}$. Also since $a_1(fere)$ and $a_1(g)$ are units the degree of $f \circ g$ is exactly $\max\{\deg f, \deg g\}$ in the case that $\deg f \neq \deg g$. This means that if f and g are inverse to each other then $\deg f = \deg g$. This is enough information to show that $f \circ g^{-1} \in \tilde{\mathrm{A}}_d$ which shows $\tilde{\mathrm{A}}_d$ is a subgroup. $\square$

Remark 4.7. We do not know of a natural generalization of these group mod p^3. For example the polynomials in $\underline{\mathrm{Aut}}(\mathbb{A}^1_{R_2})$ of degree less than or equal to (say) $d = 3$ do not form a subgroup mod p^3: if we take $p = 101$, $f(T) = T + pT^3$ we find its inverse is $g(T) = T + 10200pT^3 + 3p^2T^5$ which does not have degree less than or equal to three. (In case you are wondering $10200 = 2^3 \cdot 3 \cdot 5^2 \cdot 17$).

4.2. **Local computations for transition maps.** In this section $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. For $f \in R[X,Y]$ we let $V(f) = \mathrm{Spec}(R[X,Y]/(f))$ and $D(f) = \mathbb{A}^2_R \setminus V(f)$.

Lemma 4.8 (Local Computations)**.** *Let $C = V(f)$ be a plane curve over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with $f \in \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[x,y]$. Let $U = D(f_x)$ and $V = D(f_y)$ and ε_U and ε_V be the étale projections to the y and x axes of $\mathbb{A}^2$ respectively.*

Let $\psi_U : J^1(U) \to \widehat{U} \hat{\times} \widehat{\mathbb{A}}^1$ and $\psi_V : J^1(V) \to \widehat{V} \hat{\times} \widehat{\mathbb{A}}^1$ be the trivializations associated to the projections as in Lemma 4.2. Then the transition map $\psi_{VU} := \psi_V \circ \psi_U^{-1}$ has the property that

$$\psi_{UV} \otimes_R R_n \in A_n$$

for each $n \geq 2$.

In the statement of the theorem f_x and f_y are just the usual partial derivatives of f with respect to x and y respectively.

Proof. Assume without loss of generality that $f_y \not\equiv 0 \mod p$. The maps $\varepsilon_U : U \to \mathbb{A}^1$ given by $(x,y) \mapsto y$ and $\varepsilon_V : V \to \mathbb{A}^1$ given by $(x,y) \mapsto x$ are étale. On these open sets we have $\mathcal{O}^1(U) = O(U)[\dot{y}]\hat{}$ and $\mathcal{O}^1(V) = \mathcal{O}(V)[\dot{x}]^{\widehat{p}}$. This means we have

$$\mathcal{O}(J^1(U \cap V)) = \mathcal{O}(U \cap V)^1 = \mathcal{O}(U \cap V)[\dot{x}]^{\widehat{p}} = \mathcal{O}(U \cap V)[\dot{y}]^{\widehat{p}}.$$

Let $\psi_U : J^1(U) \to \widehat{U} \hat{\times} \widehat{\mathbb{A}}^1$ be given by $t \mapsto \dot{y}$ and $\psi_V : J^1(V) \to \widehat{V} \hat{\times} \widehat{\mathbb{A}}^1$ be given by $t \mapsto \dot{x}$. We can compute the transition map $\psi_V \circ \psi_U^{-1} \in \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1)(U \cap V)$ by first computing what $\dot{y}$ is in terms of $\dot{x}$. We first have

$$\begin{aligned} \delta f &\equiv \frac{1}{p}[f^\phi(x^p, y^p) - f(x,y)^p] + \nabla f^\phi(x^p, y^p) \cdot (\dot{x}, \dot{y}) \\ &\quad + \frac{p}{2}[f^\phi_{xx}(x^p, y^p)\dot{x}^2 + 2f^\phi{}_{xy}(x^p, y^p)\dot{x}\dot{y} + f^\phi_{yy}(x^p, y^p)\dot{y}^2] \\ &\equiv 0 \mod p^2 \text{ in } \mathcal{O}(U \cap V)[\dot{y}]\hat{} \end{aligned}$$

where for a polynoimal $g(x) = a_0 + a_1 x + \cdots + a_n x^n$ the polynomial $g^\phi(x) := \phi(a_0) + \phi(a_1)x + \cdots + \phi(a_n)x^n$ as usual and $\nabla f = (f_x, f_y)$ is the usual gradient from calculus.

Let

$$\begin{aligned} A &= R + f^\phi{}_x(x^p, y^p)\dot{x} + pf^\phi{}_{xx}(x^p, y^p)\dot{x}^2/2, \\ B &= f^\phi{}_y(x^p, y^p) + pf^\phi{}_{xy}(x^p, y^p)\dot{x}, \\ C &= f^\phi{}_{yy}(x^p, y^p)/2, \\ R &= (f^\phi(x^p, y^p) - f(x,y)^p)/p \end{aligned}$$

then, solving the equation $A + B\dot{y} + C\dot{y}^2 = 0$ gives

$$\dot{y} = -\frac{A}{B} + p\frac{A^2 C}{B^3}.$$

Since

$$\begin{aligned} pB^{-3}A^2C &= p\frac{(R + f^\phi{}_x(x^p, y^p)\dot{x})^2 f^\phi{}_{yy}(x^p, y^p)}{2f^\phi{}_y(x^p, y^p)^3} \\ AB^{-1} &= \frac{1}{f^\phi{}_y(x^p, y^p)}[R + f^\phi{}_x(x^p, y^p)\dot{x} + pf^\phi{}_{xx}(x^p.y^p)\dot{x}^2/2 \\ &\quad -p\frac{f^\phi{}_{xy}(x^p, y^p)\dot{x}}{f^\phi{}_y(x^p, y^p)}(R + f^\phi{}_x(x^p, y^p)\dot{x})] \end{aligned}$$

we get

$$\dot{y} = \alpha + \beta\dot{x} + p\gamma\dot{x}^2 \tag{4.2}$$

where

$$\begin{aligned} \alpha &= -\frac{R}{f^\phi{}_y(x^p, y^p)} + p\frac{R^2 f^\phi{}_y y(x^p, y^p)}{2f^\phi{}_y(x^p, y^p)^3} \\ \beta &= -\frac{-f^\phi{}_x(x^p, y^p)}{f^\phi{}_y(x^p, y^p)} + p\frac{f^\phi{}_{xy}(x^p, y^p)R}{f^\phi{}_y(x^p, y^p)^2} + \frac{pRf^\phi{}_x(x^p, y^p)f^\phi{}_y y(x^p, y^p)}{f^\phi{}_y(x^p, y^p)^3}, \\ \gamma &= -\frac{f^\phi{}_{xx}(x^p, y^p)}{2f^\phi{}_y(x^p, y^p)} + \frac{f^\phi{}_{xy}(x^p, y^p)f^\phi{}_x(x^p, y^p)}{f^\phi{}_y(x^p, y^p)^2} + \frac{f^\phi{}_x(x^p, y^p)^2 f^\phi{}_{yy}(x^p, y^p)}{2f^\phi{}_y(x^p, y^p)^3}. \end{aligned}$$

We will now show that $\dot{y} \equiv a_0 + a_1\dot{x} + pa_2\dot{x}^2 + \cdots + p^n a_{n+1}\dot{x}^{n+1} \mod p^{n+1}$ by induction. We have proven the base case and proceed to solve for $\dot{y}$ in terms of $\dot{x}$ as we did before inductively. As before we have $\delta(f(x,y)) = \frac{1}{p}\left(f^\phi(x^p + p\dot{x}, y^p + p\dot{y}) - f(x,y)^p\right) = 0$. We use the expansion

$$f^\phi(x^p + p\dot{x}, y^p + p\dot{y}) = \sum_{d \geq 0} p^{d-1} h_d(\dot{x}, \dot{y})$$

where h_d are homogeneous polynomials of degree d in $\dot{x}$ and $\dot{y}$ with coefficients in $R[x,y]/(f)$; this gives

$$\frac{f^\phi(x^p, y^p) - f(x,y)^p}{p} + \sum_{d=1}^{n} p^{d-1} h_d(\dot{x}, \dot{y}) \equiv 0 \mod p^{n+1}. \tag{4.3}$$

By inductive hypothesis we may assume $\dot{y} = A + p^n B$ where $A = a_0 + \sum_{j=1}^n p^{j-1} a_j \dot{x}^j$. Expanding the homogeneous polynomials gives

$$h_d(\dot{x}, \dot{y}) = h_d(\dot{x}, A + p^n B) = h_d(\dot{x}, A) + \frac{\partial h_d}{\partial \dot{y}}(\dot{x}, A)p^n B \mod p^{n+1}$$

and substituting into equation 4.3 we get

$$r+\sum_{d=1}^{n} p^{d-1}\left(h_d(\dot{x},A)+\frac{\partial h_d}{\partial \dot{y}}(x,A)p^nB\right) = r+\sum_{d=1}^{n} p^{d-1}h_d(\dot{x},A)+\sum_{d=1}^{n} p^{d-1}\frac{\partial h_d}{\partial \dot{y}}(\dot{x},A)p^nB \tag{4.4}$$

where $r = \frac{f^{\phi}(x^p,y^p)-f(x,y)^p}{p}$. Note that the left terms on the right side of equation 4.4 can be written as

$$r+\sum_{d=1}^{n} p^{d-1}h_d(\dot{x},A) = p^nC$$

and the term on the right can be written as

$$\sum_{d=1}^{n} p^{d-1}\frac{\partial h_d}{\partial \dot{y}}(\dot{x},A)p^nB \equiv \frac{\partial h_1}{\partial \dot{y}}(\dot{x},A)p^nB \mod p^{n+1}.$$

Using the fact that $h_1 = f^{\phi}{}_x(x^p,y^p)\dot{x}+f^{\phi}{}_y(x^p,y^p)\dot{y}$ we have $\frac{\partial h_1}{\partial \dot{y}}(x,A) = f^{\phi}{}_y(x^p,y^p)$ which tells us that $p^nC + f^{\phi}{}_y(x^p,y^p)p^nB \equiv 0 \mod p^{n+1}$ and hence that $C + f^{\phi}{}_y(x^p,y^p)B \equiv 0 \mod p$ and finally that

$$B = -C/f^{\phi}{}_y \mod p.$$

It remains to show that B has degree less than or equal to n in $\dot{x}$.

We note that $p^nC = r + \sum_{d=1}^{n+1} p^{j-1}h_d(\dot{x},A) \mod p^{n+1}$ where we can write $h_d(S,T) = \sum_{j+k=d} a^d_{j,k}S^jT^k$, where $a^d_{j.k} \in R[S,T]/(f)$. We can expand the expression

$$p^{d-1}h_d(\dot{x},A) = p^{d-1}h_d(\dot{x},a_0+a_1\dot{x}+\cdots+p^{n-2}a_{n-1}\dot{x}^{n-1}) \tag{4.5}$$

so that its general term takes the form

$$p^{d-1}a^d_{i,j}\dot{x}^i(a_0+a_1\dot{x}+pa_2\dot{x}^2+\cdots+p^{n-2}a_{n-1}\dot{x}^{n-1})^j.$$

We expand this general term further to get

$$\begin{aligned}
& (a_0+a_1\dot{x}+pa_2\dot{x}^2+\cdots+p^{n-2}a_{n-1}\dot{x}^{n-1})^j \\
= & \sum_{j_0+j_1+\cdots+j_{n-1}=j} a_0^{j_0}(a_1\dot{x})^{j_1}(pa_2\dot{x}^2)^{j_2}\cdots(p^{n-2}a_{n-1}\dot{x}^{n-1})^{j_{n-1}} \\
= & \sum_{j_0+j_1+\cdots+j_{n-1}=j} a_0^{j_0}a_1^{j_1}a_2^{j_2}\cdots a_{n-1}^{j_{n-1}}p^{j_2+2j_3+3j_4+\cdots+(n-2)j_{n-1}}\dot{x}^{j_1+2j_2+3j_3+\cdots+(n-1)j_{n-1}}
\end{aligned}$$

So that a general term of equation 4.5 takes the form

$$\alpha p^a\dot{x}^b$$

where $\alpha \in \mathcal{O}(U)$ and

$$\begin{aligned}
i+j &= d \\
a &= d-1+j_2+2j_3+\cdots+(n-2)j_{n-1} \\
b &= i+j_1+2j_2+\cdots+(n-1)j_{n-1} \\
j &= j_0+j_1+\cdots+j_{n-1}
\end{aligned}$$

Using these relations we show

$$\begin{aligned} a &= d-1+j_2+2j_3+\cdots+(n-2)j_{n-1} \\ &= i+j-1+j_2+2j_3+\cdots+(n-2)j_{n-1} \\ &= i-1+j_0+j_1+2j_2+3j_3+\cdots+(n-1)j_{n-1} \\ &= i-1+j_0+(b-i) \\ &= b-1+j_0 \end{aligned}$$

Which tells us the $a = b-1+j_0 \geq b-1$. Notice that the degree of the general term is b and we want to show that $b \leq n+1$. Suppose this is not the case and that $b > n+1$. This implies that $a > n$ which implies $\alpha p^a \dot{x}^b \equiv 0 \mod p^{n+1}$; so such a term doesn't contribute to $\dot{y} \mod p^{n+1}$. This concludes the proof. □

4.3. **Degree bounds give non-vanishing of partial derivatives.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. $f \in R[S,T]$, $f(S,T) = \sum_{k=0}^d f_k(S,T)$ f_k homogeneous of degree d i.e. $f_0 = a_{00}$, $f_1 = a_{10}S + a_{01}T$, $f_2 = a_{20}S^2 + a_{11}ST + a_{02}T^2$ and so on. We have $f_d \neq 0$ since f is of degree d

Using this decomposition we can compute the partial derivatives term-wise to get

$$\frac{\partial f}{\partial S} = \sum_{k=1}^d \frac{\partial f_k}{\partial S}, \qquad \frac{\partial f}{\partial T} = \sum_{k=1}^d \frac{\partial f_k}{\partial T}.$$

If $\frac{\partial f}{\partial S} \equiv \frac{\partial f}{\partial T} \equiv 0 \mod p$ identically then

$$S\frac{\partial f}{\partial S} + T\frac{\partial f}{\partial T} = \sum_{k=1}^d \left(S\frac{\partial f_k}{\partial S} + T\frac{\partial f_k}{\partial T} \right) = \sum_{k=1}^d k f_k \equiv 0 \mod p$$

and since $R_0[S,T] \equiv \bigoplus_{k\geq 0}(R_0[S,T])_k$ we must have that $kf_k(S,T) \equiv 0 \mod p$ for $k = 1, \ldots, d$. If $p \nmid k$ this means that $f_k(S,T) = 0$ which tells us that

$$f(S,T) = h(S^p, T^p) + pg(S,T).$$

Note in particular that

$$\frac{\partial f}{\partial S} \equiv \frac{\partial f}{\partial T} \equiv 0 \mod p \implies \deg(f) \geq p.$$

We have just proved the following (stated in the contrapositive)

Lemma 4.9. *Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. For all $f \in R[S,T]$ if $\deg(f) < p$ then it is impossible for both partial derivatives of f to vanish identically mod p.*

Remark 4.10. Under the hypotheses of this lemma the computations in the "local computations" Lemma (Lemma 4.8) are valid.

4.4. **Tricks for studying transition maps.** Let X be a scheme and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. And let $(\psi_{ij}) \in \check{Z}^1(\mathcal{U}, X, G)$ for some sheaf of groups G. In order to compute the cocycle we actually compute $\psi_{ij} \in G(U_{ij})$ for a spanning set of indices. To be precise $S \subset I \times I$ will be called **spanning** if for all $(i,j) \in I \times I$ there exists some $(i_1,i_2),(i_2,i_3),\ldots,(i_{m-1},i_m) \in S$ with $i_1 = 1$ and $i_m = j$ such that

$$\psi_{ij} = \psi_{i_1 i_2}\psi_{i_2 i_2}\ldots\psi_{i_{m-1}i_m} \text{ in } G(U_{i_1 i_2 \ldots i_m}).$$

Remark 4.11. The combinatorial problem is equivalent to finding a spanning tree in the graph $K^I = (V, E)$ where $V = I$ and $E = \{\{i, j\} : i, j \in I \text{ and } i \neq j\}$.

For the sheaves will be dealing with, a spanning set of ψ_{ij} will be enough to compute all of the components of our Čech cocycle.

Lemma 4.12. *Let A and B be abelian groups and suppose that B is p-torsion free. Let $\varphi : A \to B$ homomorphism. We claim that $\varphi_1 : A/p^2A \hookrightarrow B/p^2B$ provided $\varphi_0 : A/pA \hookrightarrow B/pA$.*

The proof of this is straight forward using the definitions.

Proof. First observe that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "can"] & B \arrow[d, "can"] \\
A/p^2A \arrow[r, "\varphi_1"] \arrow[d, "can"] & B/p^2B \arrow[d, "can"] \\
A/pA \arrow[r, "\varphi_0"] & B/pB
\end{tikzcd}$$

commutes and $\varphi_i(a) := \varphi(a) \mod p^{i+1}B$ is well defined. Now let $a \in A$,

$$\begin{aligned}
\varphi(a) \equiv 0 \mod p^2B \quad &\Longrightarrow \quad \varphi(a) \equiv 0 \mod pB \\
&\Longrightarrow \quad a \in pA, \text{ so } \exists a' \in A \text{ such that } a = pa'. \\
&\qquad\;\;\; \varphi(a) = p\varphi(a') \equiv 0 \mod p^2B \\
&\Longrightarrow \quad \varphi(a') \equiv 0 \mod pB \\
&\Longrightarrow \quad a' \in pA \Longrightarrow \exists a'' \in A \text{ such that } a' = pa'' \\
&\Longrightarrow \quad \varphi(a) = p^2\varphi(a'') \equiv 0 \mod p^2B
\end{aligned}$$

So φ_1 is injective. □

Remark 4.13. We can modify this proof to get a number of results:

- $\widehat{A} \hookrightarrow \widehat{B}$ provided B is p-torsion free and φ_0 is injective.
- If B is p-torsion free then φ_n is injective provided there exists some φ_r and φ_s which are injective and $r + s = n$.
- If B is p-torsion free, p-separable and φ_0 is injective then φ is injective. p-separability means $\bigcap_{n \geq 1} p^n B = 0$.

Recall that for an ideal I in a ring B we have

$$(I : f) := \{g \in B : gf \in I\}.$$

Corollary 4.14. *Suppose B is a p-torsion free ring and $g \notin pB$. Then $(pB : g) = pB$ implies $B/p^n \to B_g/p^n = (B/p^n)_{\overline{g}}$ is injective.*

Proof. Note that $B \to B_g$ B_g is also is p-torsion free. This puts us in the hypotheses of Lemma 4.12. We will show that $B/p \to B_g/p$ is injective by showing that g is a not a zero divisor mod p. If g is a zero divisor the $\overline{g}\hbar = 0 \mod p$, which means $gh \in pB$ which means $h \in (pB : g) = pB$ which means $\hbar = 0$. This means $\overline{g}$ is a not a zero divisor, and we are done. □

Remark 4.15. Note that if pB is prime in B then $(pB : g) = pB$ for all $g \notin pB$. We claim that if X is an irreducible projective scheme over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, then its special fiber will be a projective variety over $\overline{\mathbb{F}}_p$ is also smooth and irreducible. We argue by contradiciton. Suppose the special fiber X_0 has multiple components then X/R could not be smooth to begin with as the components of the special fiber would intersect which gives a contradiction. In particular the sheaf $p\mathcal{O}$ is prime will be prime if $\mathcal{O}$ is the structure sheaf of X.

Lemma 4.16. *Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and let $X \subset \mathbb{P}^n_R$ be smooth and irreducible. If $U = \mathrm{Spec}(B)$ is an affine open subset of X and $U_b = \mathrm{Spec}(B_b)$ for some $b \in B$ then the restiction maps*

$$\mathrm{res}^U_{U_b} : \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U) \to \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U_b)$$

are injective.

Proof. Since

$$\begin{aligned} \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U) &= \{\psi : U \times \mathbb{A}^d_{R_n} \to U \times \mathbb{A}^d_{R_n}, p_1 \circ \psi = p_1\} \\ &= \mathrm{Aut}_{B_n}(B_n[T])^{\mathrm{op}} \\ &\subset (B_n[T], \circ) = \mathcal{O}(U_{R_n})[T] \end{aligned}$$

it is sufficient to show that $B_n[T] \to (B_b)_n[T]$ is injective as sets (here T denotes a tuple of d variables). Now because $\mathcal{O}_{X_n}[T] \cong \varinjlim \mathcal{O}_{X_n}^{\oplus n}$ as a sheaf of modules it is sufficient to show $B_n \hookrightarrow (B_b)_n$ —which follows from Lemma 4.12. $\square$

In view of the above result (where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$) we can view $\underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U)$ as a subset of $\underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U_b)$ for U an affine open and $b \in \mathcal{O}(U)$ not divisible by p and a non-zero divisor. This is crucial for computations.

Corollary 4.17. *Let $X \subset \mathbb{P}^n_R$ be a smooth irreducible scheme with R a p-torsion free ring. Let $U = \mathrm{Spec}(A), V = \mathrm{Spec}(B), W = \mathrm{Spec}(C)$ be affine open subsets of X with $U \cap V \cap W = U_a = V_b = W_c$ for some $a \in A, b \in B$ and $c \in C$. Let*

$$\psi_{UV} \in \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U \cap V), \psi_{VW} \in \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U \cap W), \psi_{WV} \in \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(W \cap U)$$

such that

$$\psi_{UV}|_{U\cap V\cap W} = (\psi_{UV}|_{U\cap V\cap W}\psi_{VW}|_{U\cap V\cap W}) \in \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U \cap V \cap W)$$

Then

$$\psi_{UV} = (\psi_{UV}|_{U\cap V\cap W} \circ \psi_{VW}|_{U\cap V\cap W}) \in \underline{\mathrm{Aut}}(\mathbb{A}^d_{R_n})(U \cap V)$$

Let G be a sheaf on a scheme X, let H be subsheaf. We say that H **acts like a subsheaf of a constant sheaf** with respect to G if for all affine open subsets $U = \mathrm{Spec}(A) \supset U_f = \mathrm{Spec}(A_f)$ where $f \in A$ we have

(1) $G(U) \subset G(U_f)$
(2) $H(U) \subset H(U_f)$
(3) $H(U_f) \cap G(U) = H(U)$

Note that if $H \leq G$ is almost a subsheaf of a constant sheaf then

$$\psi \in G(U) \text{ and } \psi|_{U'} \in H(U') \implies \psi \in H(U). \tag{4.6}$$

Corollary 4.18. *If $X \subset \mathbb{P}^n_R$ is smooth and irreducible over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, then the group of polynomial automorphisms A_{n+1} acts like subsheaf of a constant sheaf with respect to $\underline{\mathrm{Aut}}(\mathbb{A}^1_{R_n})$ on X_n.*

The following will allows us to prove A_n-structures exists on smooth curves

Corollary 4.19. *Let $X \subset \mathbb{P}^m_R$ be a smooth irreducible scheme where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. let U, V and W be affine open subsets of X with $U \cap V \cap W = U_a = V_b = W_c$ for some $a \in \mathcal{O}(U), b \in \mathcal{O}(V)$ and $c \in \mathcal{O}(W)$. Let*

$$\psi_{UV} \in \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_n})(U \cap V), \psi_{VW} \in A_{n+1}(U \cap W), \psi_{WV} \in A_{n+1}(W \cap U)$$

such that

$$\psi_{UV} = \psi_{UV}\psi_{VW} \in A_{n+1}(U \cap V \cap W)$$

then

$$\psi_{UV} \in A_{n+1}(U \cap V).$$

Proof. This is just a special case of equation 4.6. □

4.5. **Degree bounds coming from projections.** Let $X \subset \mathbb{P}^N_R$ be a smooth and irreducible curve of degree d. Let $\pi : \mathbb{P}^N_R \dashrightarrow \mathbb{P}^2_R$ be a projection from a hyperplane to $\mathbb{P}^2_R$. Furthermore suppose that the center of projection does not intersect X. Let π_1 and π_2 be projections to distinct lines in $\mathbb{P}^2$. If $U \subset X$ is an affine open subset of X where both π_1 and π_2 are étale onto their image then the etale maps $\varepsilon_1 := \pi_1|_U, \varepsilon_2 := \pi_1|_U : U \to \mathbb{A}^1_R$ will be called **compatible**.

If $\{U_i\}$ is an open cover of X and $\varepsilon_i : U_i \to \mathbb{A}^1_R$ are a family of étale morphisms which are pairwise compatible the family will be called **admissible**.

Proposition 4.20. *If $X \subset \mathbb{P}^N_R$ is a smooth and irreducible curve of degree less than p and we can find an affine open cover $\{U_i\}$ of X and an admissible family of étale morphisms $\varepsilon_i : U_i \to \mathbb{A}^1$ then $J^1(X)_n$ admits an A_{n+1}-structure for each $n \geq 1$.*

Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, $U \subset X$ an affine open subset of a smooth irreducible projective curve X/R. Notice that U_0 is also smooth and irreducible. Note that this puts us in the hypotheses of the sections 4.4 (where $B = \mathcal{O}(U)$).

Lemma 4.21. *If $X \subset \mathbb{P}^N_R$ be smooth and irreducible of degree $d < p$ and U an affine open subset of X. If étale morphisms $\varepsilon_i : U \to \mathbb{A}^1_R$ for $i = 1, 2$ are compatible then*

$$\sigma_0 : R_0[S,T]/(\overline{f}) \to \mathcal{O}(U)/p$$

is injective, where $\sigma = (\varepsilon_1 \times \varepsilon_2)^* : R[S,T] \to \mathcal{O}(U)$ and σ_0 is the reduction modulo p of σ.

Proof. Let $X \subset \mathbb{P}^n_R$ be flat over R so that $\deg(X) = \deg(X_K) = \deg(X_0) = d$. Let $\pi : X \to \mathbb{P}^n_R$ and let $U = \mathrm{Spec}(B)$ an open subset of X where $\pi|_U$ is étale onto its image contained in $\mathbb{A}^2_R = \mathrm{Spec}(A)$. Let $V(F) = \pi(X) \subset \mathbb{P}^2_R$ be the image of X and let $(\pi|_U)^* = \sigma : A \to B$. Let $\ker(\sigma) = (f)$. We know that f is irreducible by topological considerations. Similarly f_0 is not identically zero due to degree considerations. Note that $V(f) = \overline{\pi(U)}$ is the Zariski closure of $\pi(U)$ in $\mathbb{A}^2_R$.

We claim that $\overline{\pi_0(U_0)} \subset (\overline{\pi(U)})_0$ as closed subschemes of $\mathbb{A}^2_R$. First note that $(\overline{\pi(U)})_0 = V(f,p) \subset \mathbb{A}^2_R$. On the other hand if we let $\pi_0^* = \sigma_0 : A/p \to B/p$ and $\overline{\pi_0(U_0)} = V(J) \subset \mathbb{A}^2_R$ where $\ker(\alpha \circ \sigma_0) = J$ where

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ A/p & \xrightarrow{\sigma_0} & B/p. \end{array}$$

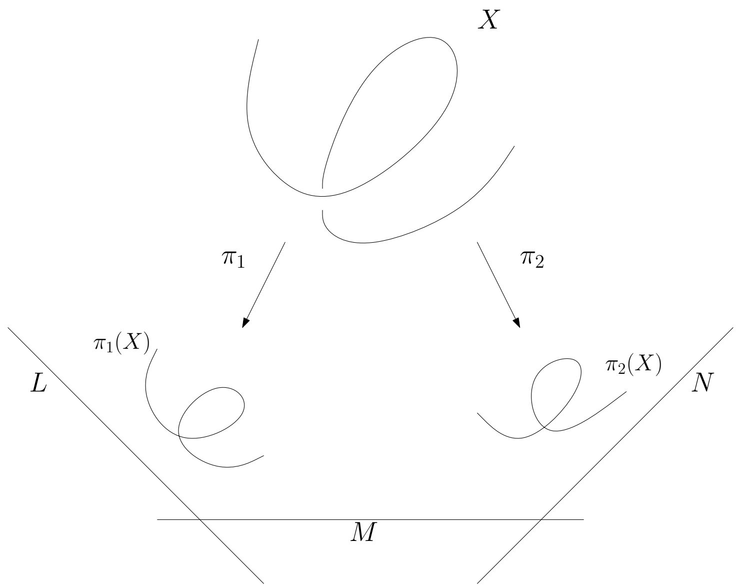

FIGURE 2. A curve $X \subset \mathbb{P}^N$ with two projections onto $\Lambda_1 = \overline{L, M}$ and $\Lambda_2 = \overline{M, N}$ both isomorphic to $\mathbb{P}^2$. The étale projections $\varepsilon_L, \varepsilon_M$ and ε_N to the lines L, M and N which induce the trivializations on the $J^1(X)$ factor through the projections π_1 and π_2

commutes then we clearly have $f \in J$ and $p \in J$. This implies that $(f, p) \subset J$ which proves $\overline{\pi_0(U_0)} \subset (\overline{\pi(U)})_0$ as closed subschemes of $\mathbb{A}^2_R$.

We will now show that $\overline{\pi_0(U_0)} = \overline{\pi(U)}_0$ as closed subschemes by degree considerations. First, note that $\deg(\overline{\pi_0(U_0)}) = \deg(\pi_0(X_0)) = \deg(X_0) = d$. One the other hand note that $\deg(\overline{\pi(U)}_0) = \deg(\pi(X_0)) = \deg(F \mod p) \leq d$ since some terms of the homogeneous polynomial F may be killed off when we reduce modulo p.

We next claim that if $X = X_1 \cup \ldots X_r$ is a decomposition into irreducible components then $\deg(X) \geq \deg(X_i)$. Furthermore, if $\deg(X) = \deg(X_i)$ then $X = X_i$. Since each X_i is irreducible we can write $X_i = V(f_i)$ where f_i is an irreducible polynomial. This means we can write X as $X = V(f)$ where $f = \prod_{i=1}^r f_i$ which proves that $\deg(X) \geq \deg(X_i)$. Now if $f_i | f$ and $\deg(f_i) = \deg(f)$ then $\deg(f/f_i) = 0$ which implies that $(f) = (f_i)$ and that $X = X_i$. This proves the second claim.

We can now conclude that $(f, p) = J$. This implies that $\ker(\sigma_0) = J/(p) = (\overline{f})$. Since $A_0/(\overline{f}) = A_0/\ker(\sigma_0) \hookrightarrow \mathcal{O}(U_0)$ the proof is complete. $\square$

Proposition 4.22. *Let $X \subset \mathbb{P}^n_R$ be a projective curve of degree less than p and let $\varepsilon_1, \varepsilon_2 : U \to \mathbb{A}^1_R$ be compatible étale morphisms. If*

$$\psi_1, \psi_2 : J^1(U) \cong \widehat{\mathbb{A}}^1 \hat{\times} \widehat{U}$$

are the trivializations associated to the étale morphisms ε_1 and ε_2 as in Lemma 4.2 then for every $n \geq 1$ we have

$$\psi_{21} \otimes_R R_n \in A_{n+1}.$$

Proof. Let $\varepsilon_1^*(T) = x$ and $\varepsilon_2^*(T) = y$ where T is the étale parameter on $\mathbb{A}^1$. Define $\sigma : R[S,T] \to \mathcal{O}(U)$ by $S \mapsto x$ and $T \mapsto y$. Since the image of σ is an integral domain we know that $\ker(\sigma)$ is a prime ideal. Since $R[S,T]$ is a UFD and the $\ker(\sigma)$ has height 1 we know that there exists some irreducible $f \in R[S,T]$ such that $\ker(\sigma) = (f)$. This f is the minimal relation among x and y and we have the equation $f(x,y) = 0$. Geometrically we have

$$\overline{\varepsilon_1 \times \varepsilon_2(U)} = V(f) \subset \mathbb{A}^2,$$

and f is a dehomogenization of F where F defines $\pi(X) = V(F) \subset \mathbb{P}^2$ and π is the projection in the definition of compatibility. Note that the image is not necessarily non-singular or even flat.

The equation $f(x,y) = 0$ implies that

$$\begin{aligned} &r + f^{\phi}{}_x(x^p, y^p)\dot{x} + f^{\phi}{}_y(x^p, y^p)\dot{y} \\ &+\frac{p}{2}(f^{\phi}{}_{xx}(x^p, y^p)\dot{x}^2 + 2f^{\phi}{}_{xy}(x^p, y^p)\dot{x}\dot{y} + f^{\phi}{}_{yy}(x^p, y^p)\dot{y}^2 \equiv 0 \mod p^2 \end{aligned}$$

where $r = \frac{f^{\phi}(x^p,y^p) - f(x,y)^p}{p} \in \mathcal{O}(U)$.

Hence ψ_{21} can be computed by solving for either $\dot{x}$ in terms of $\dot{y}$ or $\dot{y}$ in terms of $\dot{x}$. This is possible mod p^n for every $n \geq 2$ if either $f_x(x^p, y^p)$ or $f_y(x^p, y^p)$ is invertible in $\mathcal{O}(U)_0$. By localizing if necessary this is equivalent to having f_x or f_y being not identically zero mod p. This is true since the morphisms $\sigma_0 : R_0[S,T]/(f) \to \mathcal{O}(U)/p$ is injective (proved in Lemma 4.21) and by Lemma 4.9 one of the partial derivative $\partial f/\partial S$ or $\partial f/\partial T$ is non-zero in $R_0[S,T]/(f)$. We now apply the local computations of section 4.2 and the results of section 4.3 to establish that the transition map ψ_{21} mod p^n lies in the subgroup A_n for each $n \geq 2$. ☐

This concludes the proof of proposition 4.20.

We have just shown that if we are given an admissible family of projections $\varepsilon_i : U_i \to \mathbb{A}^1_R$ where U_i cover X then $J^1(X)_n$ admits A_{n+1}-structures for every $n \geq 1$. The remainer of this section devoted to proving that for $p > 6g - 5$ an admissible cover exists.

4.6. **Projections to hyperplanes: varieties.** If V is a K vector space then we will let $\mathbb{P}V = (V \setminus \{0\})/\sim$ where for all $v \in V$ we have $v \sim \lambda v$ for all $\lambda \in K^\times$. Note that $\mathbb{P}$ defines a functor

$$\mathbb{P} : \{ \text{ finite dim'l } K \text{ vector spaces } \} \to \{ \text{ Varieties}/K + \text{rational maps } \}.$$

Example 4.1. $V = Ke_0 \oplus Ke_1 \oplus Ke_2$ and $W = Kv_0 \oplus Kv_1$ where $v_0 = e_1 + e_2$ and $v_1 = e_0 + e_2$. A projection onto the subspace W is a map $P : V \to V$, $P^2 = P$ and $Pv_i = v_i$ for $i = 0, 1$. If we let e_i correspond to column vectors then

$$P = \begin{bmatrix} 1-\beta & -\beta & \beta \\ -\alpha & -\alpha & \alpha \\ 1-\alpha-\beta & 1-\alpha-\beta & \alpha+\beta \end{bmatrix}$$

where $Pe_3 = \alpha v_0 + \beta v_1$. One can fix a projection P onto the subspace W by choosing some $v \in V \setminus W$ and demanding that $Pv = 0$.

We will have $\mathbb{P}v = [v] = p$, $\mathbb{P}V = \mathbb{P}^2$, $\mathbb{P}W = L \cong \mathbb{P}^1$ and the map $\pi_L^p = \mathbb{P}P$ is the map

$$\pi_L^p : \mathbb{P}^2 \setminus \{p\} \to L$$

is from the point p to the line L given by $q \mapsto \overline{qp} \cap L$. If $q = [v'], p = [v]$ then $\overline{qp}$ corresponds to the vector space spanned by v and v' and the intersection $\overline{qp} \cap L$ corresponds to the intersection of plane spanned by v and v' and W which is a one dimensional subspace.

Let $V = K^{n+1}$ and W is a subspace of dimension $l+1$ and $P : V \to V$ is a projection to the subspace W with $U = \ker(P)$ so that U and W are complementary subspaces (i.e. $V = U \oplus W$). We have

$$\pi_{\Lambda}^{\Lambda'} : \mathbb{P}^n \setminus \Lambda \to \Lambda',$$

$\mathbb{P}P = \pi_{\Lambda}^{\Lambda'}$, $\mathbb{P}^n = \mathbb{P}V$, $\Lambda = \mathbb{P}U$ and $\Lambda' = \mathbb{P}W \cong \mathbb{P}^l$. Geometrically the map is given by

$$\pi_{\Lambda}^{\Lambda'} : q \mapsto \overline{\Lambda, q} \cap \Lambda'.$$

$q \in \mathbb{P}^n \setminus \Lambda$. Again, note that $\overline{\Lambda, q}$ corresponds to $U + Kv$ and $(U + Kv) \cap W$ is a one dimensional subspace.

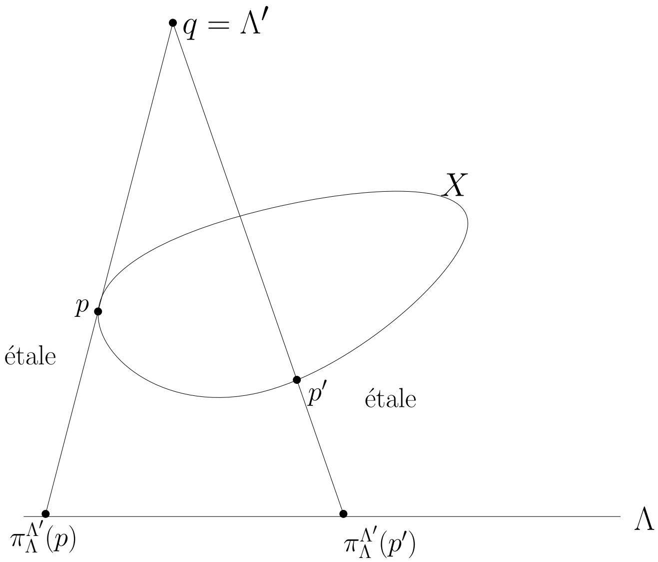

FIGURE 3. A projection in to Λ with center Λ'.

Figure 3 shows the projection from a line to another line.

4.7. **Decompositions.** Let V be an $n+1$ dimensional K vector space. We will call a collection of one dimensional vectorspaces $V_0, \dots, V_n$ such $K^n \cong V_0 \oplus \cdots \oplus V_n$ a **decomposition** of V. The set of k-dimensional vector spaces in V **associated** to the decomposition λ will be the set of $\binom{n}{k}$ k-dimensional subspaces spanned by k element subset of λ. We will denote the set of k dimensional subspaces by λ_k. For example the set of associated 1-dimensional spaces, λ_1 is $V_0, \dots, V_n$. If $W \in \lambda_k$ we let $W' \in \lambda_{n+1-k}$ denote the unique complementary subspace such that $V = W \oplus W'$. If $W \in \lambda_k$ then we let $P_W = P : V \to W$ be the projection such that $P^2 = P$, $P|_W = \mathrm{id}_W$ and $P|_{W'} = 0$.

A **decomposition** of projective space $\mathbb{P}^n = \mathbb{P}V$ will be the projectified version of the above notation. A decomposition then consists of a collection of $n+1$ points in general position. For example λ_2 is a set of $\binom{n+1}{2}$ $\mathbb{P}^1$'s. If $\Lambda \cong \mathbb{P}^{k-1}$ is a $(k-1)$-plane associated to the decomposition λ then the associated complementary hyperplane Λ' has dimension $n-k$. If x is a point not contained in complementary hyperplane Λ and Λ' then there is a unique line passing through x and meeting both Λ and Λ'. The line is the intersection is given by

$$L(\Lambda, \Lambda', x) = \overline{x, \Lambda} \cap \overline{x, \Lambda'}. \tag{4.7}$$

In figure 3 the line $L(\Lambda, \Lambda', Q)$ is the line $\overline{PQ}$

4.8. **Projections to hyperplanes: schemes.** Let $\varphi : B \to C$ be a morphism of graded rings then we get a morphism of schemes

$$\mathrm{Proj}(C) \setminus V_+(M) \to \mathrm{Proj}(B)$$

where $M = \varphi(B_+)C$ where $B_+ = \bigoplus_{d>0} B_d$ which is compatible with the associated maps between affine rings.

Let φ be the inclusion of $B = R[X_0, \ldots, X_l]$ into $C = R[X_0, \ldots, X_N]$ where $N > l$. Then $M = B_+C = (X_0, \ldots, X_l) \subset C$ so $V_+(X_0, \ldots, X_l) = \Lambda \cong \mathbb{P}_R^{N-l-1}$ is a hyperplane of dimension $N-l-1$ and $\mathrm{Proj}(B) = \Lambda' \cong \mathbb{P}^l_R$ and the induced map is

$$\pi_\Lambda : \mathbb{P}_R^N \setminus \Lambda \to \Lambda' \tag{4.8}$$

given at points by $\pi_\Lambda([a_0, a_1, \ldots, a_N]) = [a_0, \ldots, a_l]$. Notice the map is not defined when $a_{l+1} = \cdots = a_N = 0$.

4.9. **Projections to hyperplanes; compatibility between versions for schemes and varieties.** Let K be a field. The functors $V \mapsto \mathbb{P}V$ and $V \mapsto (\mathrm{Proj}\ \mathrm{Sym}V^*)(K)$ from finite dimensional vector spaces over K to varieties over K together with rational maps are naturally isomorphic.

W and V vector spaces over a field K, and $\sigma : W \to V$ we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}W \arrow[r, "\mathbb{P}\sigma"] \arrow[d, "\Phi_W"] & \mathbb{P}V \arrow[d, "\Phi_V"] \\
(\mathrm{Proj}\ \mathrm{Sym}W^*)(K) \arrow[r, "\mathrm{Proj\ Sym}\ \sigma^*"] & (\mathrm{Proj}\ \mathrm{Sym}V^*)(K)
\end{tikzcd}.$$

Let $V = K^n$ then $[v] \in \mathbb{P}V$ corresponds to $[a_0, \ldots, a_n]$ where the map $\mathrm{Spec}(K) \to \mathrm{Proj}\ \mathrm{Sym}V^* \cong \mathbb{P}^n_K$ defined by the standard affine open sets $D_+(X_i)$ is given by $X_j/X_i \mapsto a_j/a_i$ on the level of rings. It therefore makes sense to use

$$\mathbb{P}^n_R = \mathrm{Proj}\ \mathrm{Sym}M^*$$

where M is a free R module of rank $n+1$. We will use this viewpoint later.

4.10. **Hilbert polynomial and the degree of a subscheme of projective space.** Since $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is a local ring we can defined the Hilbert function of a projective scheme $X \subset \mathbb{P}^N$ which is flat over R defined by the $I(X)$ by $h(X, n) := \mathrm{rk}(S_X)_n$ where $S_X = R[X_0, \ldots, X_n]/I(X)$ and $(S_X)_n$ is the degree n part of S_X. It is a theorem that $(S_X)_n$ is in fact a free R-module. It is also a well-known theorem that there exists a unique polynomial $p(X, t)$ and M such that for $n \geq M$ we have $p(X, t) = h(X, n)$. This polynomial is called the Hilbert Polynomial. If we write

$$p(X, t) = \sum_{n=0}^{m} \frac{d_n(X)}{n!} t^n,$$

where $m = \dim(X)$ then we define we define the degree of a subscheme X of $\mathbb{P}^n_R$ by

$$\deg(X \subset \mathbb{P}^N) := d_m(X).$$

If $\mathcal{X} \to R$ is flat then then $P(X_\eta, t) = P(X_p, t)$ where η is the generic point of R and p is the closed point of r. In fact, flatness is equivalent to all of the fibers having the same Hilbert polynomial.

4.11. **Degree of the equations of a hypersurface in $\mathbb{P}^n_R$ and the degree of its associated Hilbert Polynomial.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and $K = \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ where X/K be an irreducible variety in $\mathbb{P}^N_K$ of codimension 1. Then $X = V(f)$ where f is a homogeneous polynomial. It is well known that the degree of X in the sense of the Hilbert polynomial is equal to the degree of the defining homogeneous polynomial:

$$\deg(X_K \subset \mathbb{P}^N_K) = \deg(f).$$

This in particular implies that if X/R in $\mathbb{P}^N_R$ a flat model of X_K then $X = V(f) \subset \mathbb{P}^N_R$, and $\deg(X \subset \mathbb{P}^N_R) = \deg(f)$ as we will now show.

Suppose $X = V(I) \subset \mathbb{P}^N_R$ is flat over R and have codimension 1 and generic fiber X_K. Suppose that X has degree d. By intersection theory we know that $X_K = V(f)$ where f is homogeneous of degree d. One can define another flat projective variety $\tilde{X}$ over R whose generic fiber is X_K by taking the equations that defines X_K and multiplying by an appropriate power of p to clear the denominators; that is, $\tilde{X} = V_+(p^\nu f)$. The problem now is to show that our two models are the same:

$$X = \tilde{X},$$

i.e. that X is defined by a homogeneous polynomial of degree d with coefficients in R. The problem is local and a consequence of the following lemma:

Lemma 4.23. *A a ring over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and let I and J be prime ideals in A. Assume that A/I and A/J are flat over R. If $I \otimes_R K = I \otimes_R K$ then $I = J$.*

Proof. To see this let $f \in I$. Since $I \otimes_R K = J \otimes_R K$ there exists some $\nu \geq 0$ such that $p^\nu f \in J$. We know that $p \notin J$ since A/J is flat over R which implies that multiplication by p is injective (in fact, if multiplication by an element is injective on a ring it will be injective on the module). We also know that J prime. This implies that $f \in J$ which shows $I = J$. □

4.12. **Preservation of degree under projections.** Let X/K be a variety in $\mathbb{P}^N_K$ of degree d and let $\Lambda \cong \mathbb{P}^{N-l-1}$ and $\Lambda' \cong \mathbb{P}^l$ be hyperplanes in $\mathbb{P}^N$ be complementary. If $\Lambda \cap X = \emptyset$ then

$$\deg(\pi_\Lambda(X) \subset \mathbb{P}^l) = \deg(X \subset \mathbb{P}^N).$$

provided $\pi_\Lambda : X \to \mathbb{P}^l$ is birational onto its image.

Proof. This is proved by taking π_Λ to be given by a series of projections from points and applying Bezout's Theorem. Alternatively, this can be viewed as one of the characterizing properties of the interesection product. See Fulton [Ful94] section 11.4 (on moving lemmas); the projection formula axiom and the multiplicity one axiom are cooked-up to reduce intersections by projecting. □

4.13. **Admissible families of projections.** It remains to show that a situation as described in section 4.5 exists. That is for every smooth curve X/R we have the following

(1) An embedding $\varphi : X \to \mathbb{P}^N_R$ of degree less than p
(2) An *admissible* family of projections (or equivalently an admissible decomposition)

For us an admissible family of projections consists of a decomposition λ of $\mathbb{P}^N$ such that for every $x \in X$ there exists some projective line $\Lambda \in \lambda_2$ where $\pi_\Lambda^{\Lambda'} : X \to \Lambda$ is étale at x.

This gives us an open covering by the smooth locus of $\pi_\Lambda^{\Lambda'}$,

$$X = \bigcup_{\Lambda \in \lambda_2} X_\Lambda$$

where $X_\Lambda = \{x \in X : \pi_\Lambda^{\Lambda'} \text{ étale at } x\}$.

Recall that a map being étale at a point is an open property so it is enough to show that for all closed points x there exists some $\Lambda \in \lambda_2$ such that $\pi_\Lambda^{\Lambda'}$ is étale at x.

4.14. **Criterion for étaleness of a projection.** In this subsection everything is done over the special fiber.

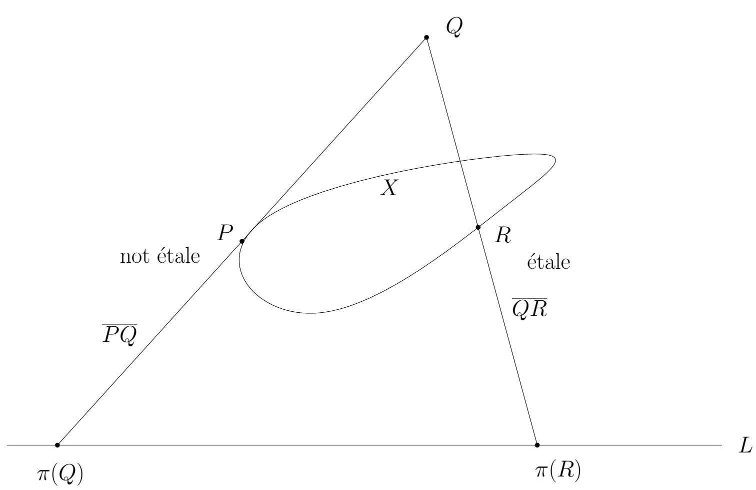

Figure 4. The point Q and the line L are complementary spaces in two dimensions. Since the unique line through P and Q is the tangent space of X at P it is not étale. On the other hand the projection is étale at the point R.

Proposition 4.24 (failure for a projection to be étale at a point)**.** *Let $\pi : \mathbb{P}^n \dashrightarrow \Lambda \cong \mathbb{P}^1$ be a projection with center Λ'. Let $X \subset \mathbb{P}^n$ be a curve. The projection is etale at $x \in X$ if and only if $\overline{x, \pi(x)} \neq T_x X$ where we interpret $T_x X \cong \mathbb{P}^1$ as the physical line tangent to the curve X at the point x.*

See figure 4.14 for a picture of proposition 4.24.

4.15. **There always exists an étale projection.** In this subsection everything is done for varieties over $\overline{\mathbb{F}}_p$. Suppose in addition that $X \subset \mathbb{P}^N$ is a curve and that for all $\Lambda' \in \lambda_{N+1-2} \cup \lambda_{N+1-3}$ we have $X \cap \Lambda' = \emptyset$ so that all of the projections

$$\pi_{\Lambda}^{\Lambda'} : X \to \Lambda \cong \mathbb{P}^1 \text{ or } \mathbb{P}^2$$

are well-defined. Without loss of generality we can assume that the decomposition λ comes from the coordinates $X_0, \ldots, X_N$ on $\mathbb{P}^N$.

Suppose that there exists some $x \in X$ such that for all $\Lambda \in \lambda_2$ that $\pi_{\Lambda}^{\Lambda'}(x)$ is not étale at x. Using the notation introduced in equation 4.7 we would have

$$L(\Lambda', \Lambda, x) = T_x X$$

for all $x \in X$. Here $T_x X$ is interpreted as the physical tangent line for the embedded curve X. This leads a silly situation which we will show cannot be possible by means of synthetic argument. See figures 4.15 and 4.15 for a picture of this situation.

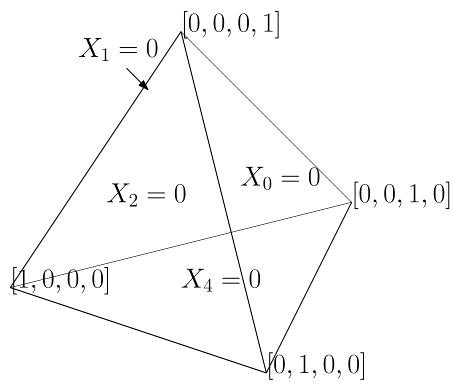

FIGURE 5. A picture of $\mathbb{P}^3$ with its standard decomposition.

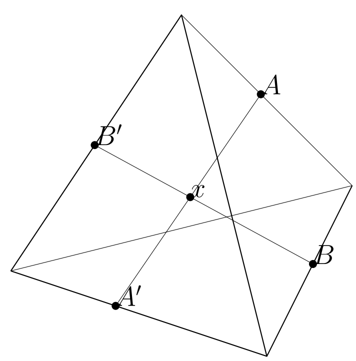

FIGURE 6. If there exists some point x such that $T_x X$ is equal $L(\Lambda', \Lambda, P)$ for all $\Lambda \in \lambda_2$ then we would have $\overline{AA'} = \overline{B'B}$. The situation looks very bad in this simple case.

Suppose that $M, N \in \lambda_2$ are not equal and let $L = L(M, M', x)$ and $K = L(N, N', x)$. Let A be the unique point where L intersects M and A' be the unique point where L intersects M'. Similarly define B and B'. If these are both tangent lines to some curve at a point $x \notin \lambda_1$ we have $L = K$. L intersects M at A and L also intersects N at B. M and N intersect in a unique point C. This means that M, N and L are contained in the unique plane π spanned by A, B and C. Since π is also the unique plane spanned by M and N, this means that $\pi \in \lambda_3$. But by hypothesis we supposed that x was not in any $\pi \in \lambda_3$ which is a contradiction.

4.16. **Enough to show there exists an admissible decomposition on the closed fiber.** Let R be a complete discrete valuation ring, and k be the residue field with characteristic p. Let X/R be a scheme. Let V and W be closed subschemes. Suppose $Z := V \cap W$ is non-empty. Then since Z is closed, it contains a closed point. Let $U = \mathrm{Spec}(A)$ be an affine open subset of Z containing P a closed point. This means $A/m_P = \kappa(P)$, where $\kappa(P)$ is the residue field of P; since P is closed, m_P is maximal and $\mathrm{char}(\kappa(P)) = p$. This means that we have a factorization

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & \kappa(P) \\ & A_0 \arrow[ur] & \end{tikzcd}$$

where $A_0 = A/p$. Since $P = \mathrm{Im}(\{P\} \to Z) = \mathrm{Im}(\{P\} \to Z_0 \hookrightarrow Z)$ we have $P \in Z_0$ (where we have let $\{P\} = \mathrm{Spec}\ \kappa(P)$). In other words every closed point in the intersection is contained in the special fiber of the intersection. In particular this tells us that that in order to show an intersection is empty that we need only show that the intersection of the special fibers is empty

$$X_0 \cap Y_0 = \emptyset \implies X \cap Y = \emptyset.$$

4.17. **Existence of admissible decompositions.** It remains to show that for every curve $X/\overline{\mathbb{F}}_p$ in $\mathbb{P}^N$ there exists some decomposition λ such that X does not intersect any $\Lambda' \in \lambda_{N+1-3}$. This can be done by the moving lemma and dimension counting. See [Ful94].

Recall that if X and W are subvarieties of $\mathbb{P}^N$ we say they intersect properly if

$$\dim(X \cap W) = \dim(W) + \dim(X) - N$$

where a negative dimension of intersection is interpreted as empty. Recall that for all W there exists a dense subset of $W' \sim_{rat} W$ such that the intersection is proper. This means if the dimension counting says that a general decomposition the centers of projections to projective planes doesn't intersect X then we can find such a decomposition.

Let W be the unions of the centers of projections to coordinate planes. $W = \bigcup_{\Lambda \in \lambda_3} \Lambda'$. Since W has dimension $N - 1 - 2$ and X has dimension 1 if W and X intersected properly $\dim(X \cap W) = (N - 1 - 2) + 1 - N = -2$ which would be empty, so by a moving lemma we can arrange so that X and W have an empty intersection.

This concludes the proof of the following theorem.

Theorem 4.25. *Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and $X \subset \mathbb{P}_R^N$ be a smooth irreducible curve of genus g. If $p \geq 6g - 5$ there exists an A_n structure on $J^1(X)_n$ for every $n \geq 1$.*

The embedding comes from the tricanonical embedding of our curve into projective space.

5. Cohomology classes coming from affine bundle structures

The aim of this section is to introduce the tools we will use in section 6 to prove Theorem 1.1 (on page 3). We first prove the following:

Theorem 5.1. *Let X be a scheme and let L be a line bundle on X. Let $l : \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$ be the canonical quotient. Let $[L] \in H^1(X, \mathcal{O}^\times)$ then*

$$H^1(X, \mathcal{O} \rtimes \mathcal{O}^\times)_{[L]} \cong H^1(X, L)$$

where $H^1(X, \mathcal{O} \rtimes \mathcal{O}^\times)_{[L]} = H^1(l)^{-1}([L])$ is the fiber over $[L]$ under the map l induces on cohomology.[4]

Theorem 5.2. *Let X be a scheme and let L be a line bundle on X. Let $r : \mathcal{O}^\times \ltimes \mathcal{O} \to \mathcal{O}^\times$ be the canonical quotient. Let $[L] \in H^1(X, \mathcal{O}^\times)$ then*

$$H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})_{[L]} \cong H^1(X, L^\vee)$$

where $H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})_{[L]} = H^1(r)^{-1}([L])$ is the fiber over $[L]$ under the map r induces on cohomology.

Remark 5.3. If $[a_{ij}, b_{ij}] \in H^1(X, F \rtimes G)$ we will call (a_{ij}) a (b_{ij})-**twisted cocycle** with values in F.

Sections 5.1 through 5.6 are devoted to proving the result above introducing and recalling the necessary definitions along the way. Note that an important consequence of the above is that

$$H^1(X, \mathrm{AL}_1) \cong \coprod_{[L] \in \mathrm{Pic}(X)} H^1(X, L).$$

Now, since $J^1(X)_0$ is an affine bundle for X a smooth curve and $\underline{\mathrm{Aut}}_{R_0}(\mathbb{A}^1_{R_0}) \cong \mathrm{AL}_1(\mathcal{O}_{X_0}) \cong \mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^\times$ where $a + bT$ identifies with (a, b) in the semi-direct product, Theorem 5.1 we can associate to $J^1(X)_0$ both a line bundle and a cohomology class with values in the line bundle.

Proposition 5.4. *For $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ a smooth curve we have $H^1(l)([J^1(X)_0]) = [F^*T_{X_0}] \in \mathrm{Pic}(X)$. If we let $\lambda : H^1(X_0, \mathcal{O} \rtimes \mathcal{O}^\times)_{F^*T_{X_0}} \to H^1(X_0, F^*T_{X_0})$ denote the isomorphism from theorem 5.1 then*

$$\lambda([J^1(X)_0]) = \mathrm{DI}_0(\delta),$$

the Deligne-Illusie class.

Sections 5.4 through sections 5.9 are devoted to proving these results.

Now if we are given an A_{n+1}-structure Σ_n on some affine bundle $J_n \to X_n$ how can we extract information from $[\Sigma_n] \in H^1(X_n, A_{n+1})$? One idea is to convert $[\Sigma_n]$ into Čech cohomology classes which take values in an abelian group via theorems 5.1 and 5.2; this is the approach of the present paper. To do this, we introduce the notion of twisted homomorphisms for sheaves of groups. If G acts on the left of F the set of such twisted homomorphisms will be denoted by $Z^1_{group}(G, F)$. Just as in the category of groups, a twisted homomorphism $\Phi \in Z^1_{group}(G, A)$ will

[4] The l stands for left, since this is the canonical for a semi-direct product coming from a left group action. The λ also stand for left. Later we will use r and ρ for similar situations in the case of right actions.

induce a morphism $f_\Phi : G \to A \rtimes \mathrm{Aut}(A)$. [5] Sections 5.10 and 5.11 defined the notion of a twisted homomorphism and section 5.11 gives examples crucial to the understanding section 6.

Finally, let $A \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_n})$ and suppose we are given some A-structure $[\Sigma] \in H^1(X_n, A)$ with property that $[\Sigma]_0 = [J^1(X)_0] \in H^1(X_0, \mathrm{AL}_1)$, also suppose we are given $\Phi \in Z^1_{group}(A_{n+1}, \mathcal{O}_{X_0})$; then we can constuct cohomology classes $\kappa(\Phi, [\Sigma_n]) \in H^1(X_0, \Omega^n)$ where $n \in \mathbb{Z}$ depends on what type of action A_{n+1} has on $\mathcal{O}_{X_0}$. This is outlined in sections **??** through 5.13 and used greatly in section 6.

5.1. **Sheafs of group actions.** Let G be a sheaf of groups and F be a sheaf of abelian groups on a topological space X. A **left action** is a morphism of sheaves of sets $\rho : G \times F \to F$ such that for all U the map $\rho_U : G(U) \times F(U) \to F(U)$ is a left group action.

5.1.1. *example.* Let X be a scheme and $\mathcal{O}$ be its structure sheaf. We define F by

$$F(U) = \mathcal{O}(U)[t].$$

Lemma 5.5. *When X is a Noetherian schemes F is a sheaf.*

Proof. This follows from the fact that $F = \varinjlim_n \mathcal{O}^{\oplus n}$ as a sheaf of abelian groups and that direct limit of sheaves is a sheaf when X is Noetherian. □

Let $G = \underline{\mathrm{Aut}}(\mathbb{A}^1)$. The right action of G on F can be given by applying an automorphism, $f(T) \mapsto f(\psi(T))$. There is also a generalization of the right action given by $\underline{\mathrm{Aut}}(\mathbb{A}^1)$ on $\mathcal{O}[T]$ that will be used: $\rho_n(\psi^*, f(T)) = f(\psi(T))\psi'(T)^n$. This action is a "pull-back of n-forms action" since $\psi * (f(T)dT^n) = f(\psi(T))d(\psi(T))^n = f(\psi(T))\psi'(T)^n dT^n$.

5.2. **Twisted Čech cocycles.** Suppose that G is a sheaf of groups acting on the left of a sheaf of abelian groups F. Suppose that $(g_{ij})_{(i,j)\in I\times I}$ is a Čech cocycle for G corresponding to a covering $\mathcal{U} = \{U_i\}_{i\in I}$. By a **(left)** (g_{ij})**- twisted cocyle with values in** F we will mean a collection of elements $a_{ij} \in F(U_{ij})$ such that

$$a_{ij} + g_{ij}a_{jk} + g_{ij}g_{jk}a_{ki} = 0.$$

For any groups (or sheaves of groups) G and H if G acts on the left (resp right) of H we can form the semi-direct product group $H \rtimes G$ (resp $H \ltimes G$) whose multiplication is defined by $(h_1, g_1) * (h_2, g_2)) = (h_1 g_1(h_2), g_1 g_2)$ (resp $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1^{g_2} h_2))$.

Let G act on the left of a sheaf of $\mathcal{O}$-modules F.[6]. If the pairs $(a_{ij}, g_{ij}) \in (F \rtimes G)(U_{ij})$ define a cocycle in $\check{Z}^1(\mathcal{U}, F \rtimes G)$ then

$$(a_{ij}, g_{ij})(a_{jk}, g_{jk})(a_{ki}, g_{ki}) = (a_{ij} + g_{ij}a_{jk} + g_{ij}g_{jk}a_{ki}, g_{ij}g_{jk}g_{ki}). \tag{5.1}$$

which means a_{ij} is a (g_{ij})-twisted cocycle. We will denote the collection of (g_{ij})-twisted cocycles by $\check{Z}^1(\mathcal{U}, F \rtimes G)_{(g_{ij})}$. This should be viewed as the fiber of the map $\check{Z}^1(l) : \check{Z}^1(\mathcal{U}, F \rtimes G) \to \check{Z}^1(\mathcal{U}, G)$ under the map induced on cocycles by the canonical projection $l : F \rtimes G$.

Remark 5.6. The last paragraph can be done for right twisted cocycles and right actions as well. This will be used in section 5.3.

---

[5] provided $\mathrm{Aut}(A)$ is a sheaf

[6] This hypothesis on F is used to ensure that $\underline{\mathrm{Aut}}(F)$ is in fact a sheaf

5.3. **Equivalence of left/right group action conventions.** In what follows the superscript 'op' denotes the opposite group.

Proposition 5.7. *Let F and L be sheaves of $\mathcal{O}_X$-modules where X is a scheme. Suppose that F be a locally L-trivial sheaf with $\{(U_i, \varphi_i)\}$ an L-trivializing cover (see section 3.4).*

(1) *There exists an isomorphism of groups*

$$S : (L \rtimes \underline{\mathrm{Aut}}(L))^{op} \to \underline{\mathrm{Aut}}(L)^{op} \ltimes L$$

given by $S(F,\varphi)^{op} = (\varphi^{op}, F)$.

(2) *The following diagram almost commutes,*

$$\begin{tikzcd} & \check{Z}^1(\mathcal{U}, F) \arrow[rr, "\alpha", "\sim"'] \arrow[d, "\beta", "\sim"'] & & \check{Z}^1(\mathcal{U}, L \rtimes \underline{\mathrm{Aut}}(L))_{(\varphi_{ij})} \arrow[dll, "\sim", "S'"'] \\ \check{Z}^1(\mathcal{U}, (\varphi_{ij}) \rtimes L))_{(\varphi_{ji}^{op})} & & & \end{tikzcd}$$

where

$$\begin{aligned} \alpha(s_{ij}) &:= (\varphi_i(s_{ij}), \varphi_{ij}) \\ \beta(s_{ij}) &:= (\varphi_{ji}^{op}, -\varphi_j(s_{ij})) \end{aligned}$$

and $S'(a,g) = S((a,g)^{-1})$.

Proof. First note that multiplication in $(L \rtimes G)^{op}$ is given by

$$(a,g) *^{op} (b,h) = (b,h)(a,g) = (b + ha, hg).$$

Second, note that multiplication in $G^{op} \ltimes L$ is given by

$$(g^{op}, a)(h^{op}, b) = (g^{op}h^{op}, ah^{op} + b) = ((hg)^{op}, ha + b)).$$

This proves the first part. It also tells us that there is an isomorphisms $L \rtimes G \to G^{op} \ltimes L$ given by $(a,g) \mapsto (a,g)^{-1} \mapsto S((a,g)^{-1})$ as a group and its opposite group are always isomorphic.

We first prove that α is an isomorphism. Let $a_{ij} = \varphi_i(s_{ij})$.

$$\begin{aligned} 0 &= s_{ij} + s_{jk} + s_{ki} \\ &= \varphi^{-1} a_{ij} + \varphi_j^{-1} a_{jk} + \varphi_k^{-1} a_{ki} \\ &\mapsto^{\varphi_i} a_{ij} + \varphi_i \varphi_j^{-1} a_{jk} + \varphi_i \varphi_k^{-1} a_{ki} \\ &= a_{ij} + \varphi_{ij} a_{jk} + \varphi_{ij}\varphi_{jk} a_{ki} \end{aligned}$$

which is exactly the condition for a left (φ_{ij})-twisted cocycle.

If $\rho : G \times L \to L$ is a left action we let $\rho^{op} : L \times G^{op} \to L$ be defined by $\rho^{op}(a, \psi) = \rho(\psi, a)$. One can check that this is indeed a right action for the opposite group. For example if φ_{ij} defines a cocycle for $G = \underline{\mathrm{Aut}}(L)$ then if we define

$$\psi_{ij} = \varphi_{ji}$$

we have

$$\begin{aligned}\psi_{ij} * \psi_{jk} * \psi_{ki} &= \psi_{jk}\psi_{ij} * \psi_{ki} \\ &= \psi_{ki}\psi_{jk}\psi_{ij} \\ &= \varphi_{ki}^{-1}\varphi_{jk}^{-1}\varphi_{ij}^{-1} \\ &= (\varphi_{ij}\varphi_{jk}\varphi_{ki})^{-1} = \mathrm{id}.\end{aligned}$$

We will use the notation $\rho^{op}(a,\psi) = a^{\psi}$ in what follows.

$$\begin{aligned}s_{ij} + s_{jk} + s_{ki} &= \varphi_j^{-1}(a_{ij}) + \varphi_k^{-1}(a_{jk}) + \varphi_i^{-1}(a_{ki}) \\ &\mapsto^{\varphi_i} \varphi_i\varphi_j^{-1}(a_{ij}) + \varphi_i\varphi_k^{-1}(a_{jk}) + a_{ki} \\ &= \varphi_i\varphi_k^{-1}\varphi_k\varphi_j^{-1}(a_{ij}) + \varphi_i\varphi_k^{-1}(a_{jk}) + a_{ki} \\ &= \varphi_{ik}(\varphi_{kj}a_{ij}) + \varphi_{ik}(a_{jk}) + a_{ki} \\ &= a_{ij}^{\psi_{jk}\psi_{ki}} + a_{jk}^{\psi_{ki}} + a_{ki}\end{aligned}$$

one can see that this is a right (ψ_{ij})-twisted cocycle since

$$(\psi_{ij}, a_{ij})(\psi_{jk}, a_{jk})(\psi_{ki}, a_{ki}) = (\psi_{ij}\psi_{jk}\psi_{ki}, a_{ij}^{\psi_{jk}\psi_{ki}} + a_{jk}^{\psi_{ki}} + a_{ki}).$$

□

5.4. **The Picard group.** To every invertible sheaf $\mathcal{L}$ on a scheme X we can associate a multiplicative cocycle. We denote the image $[\mathcal{L}] \in H^1(X, \mathcal{O}^\times)$.

Suppose that $\mathcal{L}$ is an invertible sheaf on X and let $\{U_i\}$ be a trivializing cover so that $\mathcal{L}(U_i) = \mathcal{O}(U_i)v_i$ where v_i is a generator of $\mathcal{L}(U_i)$. Since both v_i and v_j are generators of $\mathcal{L}(U_i \cap U_j)$ there exists some $m_{ji} \in \mathcal{O}(U_i \cap U_j)^\times$ such that

$$v_i = m_{ji}v_j \quad \text{[indexing is very important here!]}. \tag{5.2}$$

Let $\psi_i : \mathcal{L}(U_i) \to \mathcal{O}(U_i)$ be the trivialization defined by $\psi_i(av_i) = a \in OO(U_i)$. This gives $\psi_j \circ \psi_i^{-1}(a) = \psi_j(av_i) = \psi_j(am_{ji}v_j) = m_{ji}a$. The collection m_{ij} defines a cocycle for $\mathcal{O}^\times$ since

$$\begin{aligned}m_{ij}m_{jk}m_{ki} &= \psi_i\psi_j^{-1}(\psi_j\psi_k^{-1}(\psi_k\psi_i^{-1}(1)))) \\ &= 1.\end{aligned}$$

This allows us to define

$$[\mathcal{L}] := [m_{ij}] \in H^1(X, \mathcal{O}^\times) \tag{5.3}$$

where the m_{ij} are defined as in equation 5.2.

Remark 5.8. If we replace the invertible sheaf $\mathcal{L}$ with a locally free sheaf we get an analogous result with m_{ji} being matrices in equation 5.2. The general result gives $[\mathcal{F}] \in \check{H}^1(X, \mathrm{GL}_n(\mathcal{O}))$ in equation 5.3. We bring this up to since using the opposite convention $v_i = m_{ij}v_j$ in equation 5.2 makes the computation false for locally free modules although the indexing convention is irrelevant for invertible sheaves.

Proposition 5.9. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are invertible sheaves on a scheme X such that $[\mathcal{F}] = [m_{ij}]$ and $[\mathcal{G}] = [n_{ij}]$ then*

Duals: $[\mathcal{F}^\vee] = [1/m_{ij}]$
Tensor Powers: $[\mathcal{F} \otimes \mathcal{G}] = [m_{ij}n_{ij}]$

Frobenius: *Suppose that X_0 has characteristic p and F is the Frobenius on X_0. Then*

$$[F^*\mathcal{F}] = [m_{ij}^p]$$

where $F = F_{X_0}^*$.

Proof. Since $v_i = m_{ij}v_j$ we have $1 = v_i^* v_i = m_{ij} v_i^* v_j$ which implies that $v_i^* = v_j^*/m_{ij}$ or $m_{ij}^* = 1/m_{ij}$. The transition map between tensor powers is similar, $v_i \otimes \ldots \otimes v_i = m_{ij}^n v_j \otimes \cdots \otimes v_j$. A basis for $F^*\mathcal{F}$, consists of $F \circ v_i$ so transitions look like $F \circ v_i = F \circ (m_{ij}v_j) = m_{ij}^p F \circ v_j$. □

5.5. **The embedding lemma for cocycles.** What follows will use the conventions for the Picard group in section 5.4 on page 42.

Proposition 5.10. *Let $\mathcal{O}^\times$ act on the left of $\mathcal{O}$ in the standard way. Let F be an invertible sheaf of modules on a scheme X with $F(U_i) = \mathcal{O}(U_i)v_i$ and $m_{ij}v_i = v_j$ so that $[F] = [m_{ij}] \in H^1(X, \mathcal{O}^\times)$. There is a 1-1 correspondence between left (m_{ij})-twisted cocycles with values in $\mathcal{O}$ and cocycles with values in $\mathcal{F}$:*

$$\check{Z}^1(X, \mathcal{O} \rtimes \mathcal{O}^\times)_{(g_{ij})} \cong \check{Z}^1(\mathcal{U}, F)$$

Proof. Let be $\{U_i\}$ a trivializing cover and $\varphi_i : \mathcal{O}(U_i) \to F(U_i)$ be defined by $\varphi_i(f) = fv_i$. Note that since the a_{ij} are a twisted cocycle we have

$$a_{ij} + m_{ij}a_{jk} + m_{ij}m_{jk}a_{ki} = 0.$$

The collection of elements $s_{ij} = a_{ij}v_i$ or $s_{ij} = \varphi_i(a_{ij})$ defines a cocycle with values in F. Indeed

$$\begin{aligned} s_{ij} + s_{jk} + s_{ki} &= a_{ij}v_i + a_{jk}v_j + a_{ki}v_k \\ &= a_{ij}v_i + a_{jk}(m_{ij}v_i) + a_{ki}(m_{ij}m_{jk}v_i) \\ &= (a_{ij} + m_{ij}a_{jk} + m_{ij}m_{jk}a_{ki})v_i \\ &= 0. \end{aligned}$$

The argument is easily seen to be reversible. □

Remark 5.11.
- There exists a version of this for right cocycles where $\check{Z}^1(\mathcal{U}, \mathcal{O}^\times \ltimes \mathcal{O})_{(m_{ij})} \cong \check{Z}^1(\mathcal{U}, L^\vee)$.
- There exists a version of this for sheaves of $\mathcal{O}$-modules. If L is F-locally trivial with isomorphisms $\varphi_i : L(U_i) \to F(U_i)$ we have $\check{Z}^1(\mathcal{U}, L \ltimes \underline{\mathrm{Aut}}(L))_{(\varphi_{ij})} \cong \check{Z}^1(\mathcal{U}, L)$

5.6. **Well-definedness of the embedding lemma.** In this section we prove the following

Proposition 5.12. *If X is a scheme, F is a sheaf of $\mathcal{O}$-modules which is L-locally trivial then*

$$H^1(X, L \rtimes \mathrm{Aut}(L))_{[F]} \cong H^1(X, F)$$

where $H^1(X, L \times \mathrm{Aut}(L))_{[F]} = H^1(l)^{-1}([F])$, $[F] \in H^1(X, \underline{\mathrm{Aut}}(L))$ and $H^1(l)$ is the morphism induced on cohomology by the canonical projection $l : L \times \underline{\mathrm{Aut}}(L) \to \underline{\mathrm{Aut}}(L)$.

Remark 5.13.
(1) The case we will apply in this paper is when $L = \mathcal{O}$ and F is some line bundle on X.

(2) There is also a version for right actions which where there isomorphism is to the dual.

Let X be a scheme and let F and L be sheaves of $\mathcal{O}$-modules on X. Suppose that F is L-locally trivial with $\varphi_i : F(U_i) \to L(U_i)$. Define $\varphi_i \circ \varphi_j^{-1} = b_{ij}$ so that

$$[F] = [b_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(L)).$$

Let $(a_{ij}, b_{ij}), (a'_{ij}, b'_{ij}) \in \check{Z}^1(\mathcal{U}, L \ltimes \underline{\mathrm{Aut}}(L))$ such that

$$(a_{ij}, b_{ij}) = (a_i, b_i)(a'_{ij}.b'_{ij})(a_j, b_j)^{-1} = (a_i + b_i a'_{ij} - b_i b'_{ij} b_j^{-1} a_j, b_i b'_{ij} b_j)$$

so we have $[b_{ij}] = [b'_{ij}] = [F]$ in particular. Let F' have trivializations $\varphi'_i : F'(U_i) \to L(U_i)$ such that $\varphi'_i \circ (\varphi'_j)^{-1} = b'_{ij}$. Recall that from Proposition 3.5 we have an isomorphism of sheaves $\gamma : F' \to F$ defined by

$$\gamma(s) = \varphi_i^{-1} b_i \varphi'_i(s)$$

on local sections. Also recall that we have isomorphisms of fibers of cocycles

$$\begin{aligned} \alpha &: \check{Z}^1(\mathcal{U}, L \rtimes \underline{\mathrm{Aut}}(L))_{b_{ij}} \to \check{Z}^1(\mathcal{U}, F) \\ \alpha' &: \check{Z}^1(\mathcal{U}, L \rtimes \underline{\mathrm{Aut}}(L))_{b'_{ij}} \to \check{Z}^1(\mathcal{U}, F') \end{aligned}$$

defined by $\alpha(a_{ij}, b_{ij}) = \varphi_i^{-1}(a_{ij})$ and $\alpha'(a'_{ij}, b'_{ij}) = (\varphi'_i)^{-1}(a'_{ij})$.

Now define $\lambda : H^1(l)^{-1}([F]) \to \check{H}^1(X, F)$ by

$$\lambda(\eta) = \begin{cases} \alpha(a_{ij}, b_{ij}), & \eta = [a_{ij}, b_{ij}] \\ \gamma\alpha'(a'_{ij}, b'_{ij}), & b_{ij} = b_i b'_{ij} b_j^{-1} \end{cases} \tag{5.4}$$

where $H^1(l)^{-1}([F]) \subset \check{H}^1(X, L \ltimes \underline{\mathrm{Aut}}(L))$.

Theorem 5.14. *The map λ is well-defined and an isomorphism.*

Proof. Let $s_{ij} = \alpha(a_{ij}, b_{ij})$ and $s'_{ij} = \alpha'(a'_{ij}, b'_{ij})$ with

$$\begin{aligned} a_{ij} &= a_i + b_i a'_{ij} - b_i b'_{ij} b_j^{-1} a_j \\ b_{ij} &= b_i b'_{ij} b_j^{-1} \end{aligned}$$

we need to show that $\gamma(s'_{ij}) - s_{ij}$ is trivial in cohomology.

$$\begin{aligned} \gamma(s'_{ij}) - s_{ij} &= \varphi_i b_i \varphi'_i((\varphi'_i)^{-1} a'_{ij}) - \varphi_i^{-1}(a_{ij}) \\ &= \varphi_i^{-1} b_i a'_{ij} - \varphi_i^{-1}(a_i + b_i a'_{ij} - b_i b'_{ij} b_j a_j) \\ &= \varphi_i^{-1}(-a_i + b_i b'_{ij} b_j a_j) \\ &= \varphi_i^{-1}(-a_i + \varphi_{ij} a_j) \\ &= -\varphi_i^{-1} a_i + \varphi_j^{-1} a_j. \end{aligned}$$

Conversely, suppose that $[s_{ij}] = [t_{ij}]$ in $\check{H}^1(X, F)$. Write $t_{ij} = s_{ij} + s_i - s_j$

$$\begin{aligned} (\varphi_i(t_{ij}), \varphi_{ij}) &= (\varphi_i(s_{ij}) + \varphi_i(t_i) - \varphi_i(t_j), \varphi_{ij}) \\ &= (\varphi_i(t_i), \mathrm{id})(\varphi_i(s_{ij}), \varphi_{ij})(-\varphi_{ij}^{-1} \varphi_i(t_j), \mathrm{id}) \\ &= (\varphi_i(t_i), \mathrm{id})(\varphi_i(s_{ij}), \varphi_{ij})(-\varphi_j(t_j), \mathrm{id}) \end{aligned}$$

□

WARNING!!!: For right actions we have an isomorphism

$$\rho : H^1(r)^{-1}([F]) \to H^1(X, F^\vee)$$

where $r : \mathrm{Aut}(L) \ltimes L \to \mathrm{Aut}(L)$ is the obvious projection. This is a *huge* difference! Consider the following example: If F a pluricanonical sheaf of a curve of genus greater 2 than with $[F] = [m_{ij}]$ then a left twisted cocycle (a_{ij}, m_{ij}) will given an element of an large vector space while a right twisted cocycle (m_{ij}, a_{ij}) will be trivial. This corresponds to the fact that $H^1(X, F) = 0$ while $H^1(X, F^\vee)$ has lots of elements.

5.7. **The Frobenius tangent bundle.** Let C be a smooth curve over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and let $F = F_{C_0}^*$ in this section. The the morphism of sheaves $l : \mathrm{AL}_1(\mathcal{O}_{C_0}) = \mathcal{O}_{C_0} \rtimes \mathcal{O}_{C_0}^\times \to \mathcal{O}_{C_0}^\times$ defined by $(a_0, a_1) \mapsto a_1$ defines a morphism of sheaves.

Proposition 5.15. *Let C be a smooth curve over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ then*

$$H^1(l)([J^1(C)_0]) = [F^* T_{C_0}] \in \mathrm{Pic}(C_0). \tag{5.5}$$

Remark 5.16. Recall that we can write $[J^1(C)_0] = [g_{ij}(t)] = [a_0(g_{ij})(t) + a_1(g_{ij})(t)] \in H^1(C_0, \mathrm{Aut}_{R_0}(\mathbb{A}^1_{R_0}) = H^1(C_0, \mathrm{AL}_1)$. We will use this notation in what follows.

Proof. The problem reduces to the case when C is affine. Let

$$C = \mathrm{Spec}(R[x, y]/(f(x, y)))$$

be a smooth affine curve. Let $U_1 = D(f_x)$ and $U_2 = D(f_y)$. Note that $F \circ \frac{\partial}{\partial y} = v_1$ is a generator of $F^* T_{C_0}(U_1)$ and $F \circ \frac{\partial}{\partial x} = v_2$ is a generator of $F^* T_{C_0}(U_2)$. Using the convention we adopted in equation 5.2 we will have $v_1 = m_{21} v_2$ we need to show that $a_1(g_{12}(t)) = m_{12}$. In particular we claim that $m_{21} = \left(\frac{-f_y}{f_x}\right)^p$ and that $a_1(g_{12}(t)) = \left(\frac{-f_x}{f_y}\right)^p$.

To see this, note that $\Omega_{C_0/\overline{\mathbb{F}}_p}(U_1)$ is generated by $V_1 = dy$ and $\Omega_{C_0/\overline{\mathbb{F}}_p}(U_2)$ is generated by $V_2 = dx$—since $V_1 = dy = \frac{-f_x}{f_y} dx = \frac{-f_x}{f_y} V_2$ we have that the transition for $\Omega_{C_0/\overline{\mathbb{F}}_p}$ is given by $n_{21} = \frac{-f_x}{f_y}$. By the cocycle rules (Proposition 5.9) we have that $m_{21} = \left(\frac{-f_y}{f_x}\right)^p$.

It remains to compute $a_1(g_{12}(t)) = a_1(\beta_0)$. Recall that g_{12} is the powerseries obtained by expressing the étale coordinate on U_1 as a function of the ètale coordinate for U_2. Since y is the ètale coordinate on U_1 and x is the ètale coordinate on U_2 we will have $g_{12}(\dot{x}) = \dot{y}$. Using the equation

$$\begin{aligned} 0 = \delta f &= \frac{1}{p}[f^\phi(x^p, y^p) - f(x, y)^p] + \nabla f^\phi(x^p, y^p) \cdot (\dot{x}, \dot{y}) \\ &\quad + \frac{p}{2}[f^\phi_{xx}(x^p, y^p)\dot{x}^2 + 2 f^\phi_{xy}(x^p, y^p) + f^\phi_{yy}(x^p, y^p)\dot{y}^2] + O(p^2). \end{aligned}$$

We can solve for $\dot{y}$ in terms of $\dot{x}$ mod p to get $\dot{y} = \frac{-R}{f_y^p} - \left(\frac{f_x}{f_y}\right)^p \dot{x} \mod p$ where $R = \frac{f^\phi(x^p, y^p) - f(x, y)^p}{p}$. This implies $g_{12}(t) = \frac{-R}{f_y^p} - \left(\frac{f_x}{f_y}\right)^p \dot{x} \mod p$ and that $a_1(g_{12}(t)) = \left(\frac{-f_x}{f_y}\right)^p$, which is what we wanted to show. □

5.8. **The Deligne-Illusie class.** We will review this class and show its connection to $[J^1(X)_0]$.

Lemma 5.17. *Let X be a smooth scheme over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with $\dim_R(X) = d$*

(1) If $B \to A$ is a formally étale ring homomorphism then every lift of Frobenius on B can be extended to a lift of the Frobenius on A.

(2) If $U \subset X$ admits an étale map to $\mathbb{A}^d$ then there exists a lift of the Frobenius $\phi : \mathcal{O}(\widehat{U}) \to \mathcal{O}(\widehat{U})$.

proof idea. For part (2) note that $\widehat{\mathbb{A}}_R^d$ admits a lift of the Frobenius via the pth power map on its interminates, hence part (2) follows from part (1).

Part(1) uses the formally ètale property and the fact that p-derivations are equivalent to maps to rings of length two witt vectors. □

Let X be a scheme over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with $\dim_R(X) = d$. Furthermore, suppose that R admits a lift of the Frobenius $\delta : R \to R$. Let $\mathcal{U} = \{U_i\}$ be a cover of X by affine open sets which admit étale maps to $\mathbb{A}_R^d$. Then for each i there exists some lift of the p-derivation on R, $\delta_i : \mathcal{O}(U_i)\hat{} \to \mathcal{O}(U_i)\hat{}$ and the differences $\delta_i - \delta_j$ are Frobenius derivations mod p and hence define a cohomology class

$$\mathrm{DI}_0(\delta) := [\delta_i - \delta_j \mod p] \in H^1(C_0, F^* T_{C_0}).$$

The following Lemma of Raynaud may be interepreted as a non-vanishing result:

Lemma 5.18 (Raynaud ([Ray], Lemma I.5.1))**.** *Let X/R be a smooth curve. If X has positive Kodaira dimension (genus bigger than 2) then X does not admit a lift of the Frobenius.*

proof idea. In [Ray], the proof uses the fact that a lift of the Frobenius induces the Cartier Isomorphism on the de Rham complex degree considerations then show you that if the canonical sheaf has enough global sections the you get a contradiction. □

Remark 5.19. Interpreting things in terms of the Borger-Buium $\mathbb{F}_1$ philosophy we see that it is necessary for X/R to have genus less than one if X is to descend to $\mathbb{F}_1$.

5.9. **Recovery of the Deligne-Illusie class.**

Theorem 5.20. *Let L/X be a line bundle. Suppose that E is an L-torsor over a scheme X (not necessarily reduced). Suppose that E is Zariski locally trivial (meaning that there exists a cover of X by open sets U_i such that $E(U_i)$ has sections). Then*

(1) E has the natural structure of an affine bundle with an AL_1-structure. (Hence there exists some $[E] \in H^1(X, \mathrm{AL}_1)$ which determines the AL_1-structure on E up to isomorphism.)

(2) We have $H^1(l)([E]) = [L]$. (Hence there exists some $\lambda([E]) \in H_1(X, L)$, where $l : AL_1 = \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$ is the canonical projection for a left action, and λ is the correspondence between the fiber of cohomology of the semi-direct product and cohomology of a line bundle.)

(3) If $t_i \in E(U_i)$ then one can define $\eta([E]) = [t_i/t_j] \in H^1(X, L)$, which determines E as an L-torsor structure up to isomorphism.

(4) $\lambda([E]) = \eta([E])$.

Corollary 5.21. *If E/X is a torsor under some line bundle L/X, then the torsor structure under L is unique.*

Remark 5.22. This does not that that there cannot exist two distinct line bundles L_1 and L_2 such that E has a structure of both and L_1 and L_2 torsor although in the case when X is reduced this is not the case.

Partial Proof. We prove part 4 which shows the correspondence between cocycles obtained from the functor of point definition and the fiber-construction. Invertible sheafs are represented by $\mathrm{Spec}(S(L^\vee))$ so we know that a torsor under a line bundle L will be given by some gluing the disjoint union of $\mathrm{Spec}(S(L^\vee_{U_i}))$, where the subscript denotes the inverse image of L over the open set U_i of X. We also know that $\mathrm{Spec}(S(L^\vee_{U_i}))$ and $\mathrm{Spec}(S(L^\vee_{U_j}))$ are to be glued along two copies of $\mathrm{Spec}(S(L^\vee_{U_{ij}}))$ with the property that local sections of the product are determined by collections of local sections $(u_i)_i$ of L_{U_i}, with the property that $u_i = u_j + w_{ji}$ where w_{ji}. Here, $(w_{ij}) = (t_i/t_j)$ is the cocycle in $\check{Z}^1(\mathcal{U}, L)$ associated to the torsor struction on E.

Suppose that $L(U_i) = \mathcal{O}(U_i)v_i$. Let's write $w_{ij} = a_{ij}v_i$ so that cocycles w_{ij} with values in L correspond to the twisted cocycles $(a_{ij}, b_{ij}) \in \check{Z}^1(\mathcal{U}, AL_1)$. For line bundles the description of the physical sheaf is really simple: we have $L_{U_{ij}} \cong \mathrm{Spec}(\mathcal{O}(U_{ij}))[T_i, T_j]/(T_i - b_{ij}T_j)$ where local sections v_i of $L(U_i)$ are related by $b_{ij}v_i = v_j$. This makes the obvious guess that $\mathcal{O}(T_{U_{ij}})$ is isomorphic to $T'_{U_{ij}} = \mathcal{O}(U_{ij})[T_i, T_j]/(T_i - a_{ij} - b_{ij}T_j)$ (so that when $a_{ij} = 0$ we get the line bundle construction). Let E' be the scheme obtained by gluing the schemes $\mathrm{Spec}(\mathcal{O}(U_i)[T_i])$ along the subschemes $E'_{U_{ij}}$ then elements of $E(A)$ in this scheme correspond to points (c_i) where c_i is a point of $\mathbb{A}^1 \times U_i$, coming from $\mathcal{O}(U_i)$-linear maps from $\mathcal{O}_{U_i}[T_i]$ with the property that $c_i - a_{ij} - b_{ij}c_j = 0$ (in order for ring homomorphisms from $\mathcal{O}(U_{ij})[T_i, T_j]/(T_i - a_{ij} - b_{ij}T_j)$ to be well-defined). As we saw earlier, sections of E are in bijection with sections $u_i \in L(U_i)$ with the property that $u_i - u_j = w_{ji}$. Let $u_i = c_i v_i$ for each i. This gives $0 = (c_i - a_{ij} - b_{ij}c_j)v_i = u_i - u_j - w_{ij}$, so collections do indeed determine a local section.

Let $E'_{ij} = \mathrm{Spec}(\mathcal{O}(U_{ij})[T_i, T_j]/(T_i - a_{ij} - b_{ij}T_j))$, and let $\psi_i : E'_{ij} \to U_{ij} \times \mathbb{A}^1 = \mathrm{Spec}(\mathcal{O}(U_{ij})[T])$ be the map defined by $\psi_i^*(T) = T_i$. Then we have $\psi_{ij}^*(T) = (\psi_i\psi_j^{-1}) * (T) = (\psi_j^{-1})^*\psi_i^*(T) = \psi_j^{-1*}(T_i) = \psi_j^{-1*})(a_{ij} + b_{ij}T_j) = a_{ij} + b_{ij}T$. Note that this is exactly the correspondence that the map λ gives as well so we are done. □

We will now apply this theorem. Let C/R be a smooth curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with $\delta : R \to R$ is the unique p-derivation on R. Recall that $\underline{\mathrm{Aut}}(\mathbb{A}^1_{R_0}) = \mathrm{AL}_1$. Recall that

$$H^1(l)([J^1(C)_0]) = [F^*T_{C_0}] \in \mathrm{Pic}(C_0)$$

so by proposition 5.10 we can construct a Čech cocycle with values in $F^*T_{C_0}$. Also note that this implies that $J^1(C)_0$ has the structure of an algebraic $F^*T_{C_0}$-torsor. But we already know that $J^1(C_0)$ has the structure of a $F^*T_{C_0}$-torsor since the difference of two p-derivations is a derivation of the Frobenius so we have the following

Proposition 5.23 (construction from lifts equals construction from β_0)**.** *For C/R a smooth curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ we have*

$$\lambda([J^1(C)]_0) = \mathrm{DI}_0(\delta) \in H^1(C_0, F^*T_{C_0}).$$

5.10. **Twisted homomorphisms.** Let G be a sheaf of groups and A be a sheaf of abelian groups on a topological space X and let $\rho : G \times A \to G$ be a left action of G on A (see section 5.1 for a definition of actions of sheaves of groups). A **twisted homomorphism** is a morphism of sheaves of sets

$$\Phi : G \to A$$

such that for all $U \subset X$ open,

$$\Phi_U : G(U) \to A(U)$$

is a twisted homomorphism of groups meaning $\Phi(g_1 g_1) = \Phi(g_1) + g_1 \cdot \Phi(g_2)$. We will denote the abelian group of twisted homomorphisms by $Z^1_{\mathsf{Group}}(G, A)_\rho$. We will omit the subscript ρ when the action is obvious.

Proposition 5.24. *Every twisted homomorphism $\Phi : G \to A$ twisted by a left action ρ induces a morphism of sheaves $G \to A \rtimes_\rho G$ given by $g \mapsto (\Phi(g), g)$.*

Proof. We will just check that the map respects the group multiplications

$$\begin{aligned}(\Phi(g), g) * (\Phi(h), h) &= (\Phi(g) + g \cdot \Phi(h), gh) \\ &= (\Phi(gh), gh).\end{aligned}$$

□

The main idea is that twisted homomorphism give us a way of producing twisted cocycles. The latter then give rise to sections of line bundles (See section 5.2 on page 40).

Corollary 5.25. *If $(g_{ij}) \in \check{Z}^1(\mathcal{U}, G)$ is a cocycle for G and $\Phi \in Z^1_{\mathsf{group}}(G, A)$ is a twisted homomorphism then $(\Phi(g_{ij})) \in \check{Z}^1((g_{ij}), G)$ defines a (g_{ij})-twisted cocycle with values in A.*

Remark 5.26. If $\Phi : G \to A$ is a twisted cocycle, $\ker(\Phi)$ is a subgroup of G.

5.11. **Examples of twisted homomorphisms.** In this section we give examples of $\Phi \in Z^1_{\mathsf{group}}(G, A)$.

5.11.1. *the derivative.* Here $G = \underline{\mathrm{Aut}}(\mathbb{A}^1)$ and $A = \mathcal{O}[T]^\times$ then $D : G \to A$ gives $D[f \circ g](T) = D[f](g(T))D[g](T)$ and it is a right cocycle where the action is composition.

5.11.2. *the schwarzian derivative.* A classical example of a right-cocycle $S : \underline{\mathrm{Aut}}(\mathbb{A}^1) \to \mathcal{O}_X[T]$ is the Schwarzian derivative

$$S[f](T) := \frac{f'''(T)}{f'(T)} - \frac{3}{2}\left(\frac{f''(T)}{f'(T)}\right)^2,$$

it satisfies $S[f \circ g] = S[f]^g + S[g]$ where $F^g(T) := F(g(T))g'(T)^2$.

5.11.3. *another schwarzian-like derivative.* Let $\tau : \mathrm{Aut}(\mathbb{A}^1) \to \mathcal{O}[T]$ defined by $\tau[f](T) = f''(T)/f'(T)$ one see that

$$\tau[f \circ g](T) = \tau[f](g(T))g'(T) + \tau[g](T).$$

the action is a right action ρ_1.

5.11.4. *the top-coefficient.* Let $d \geq 2$. Recall that $\tilde{\mathrm{A}}_d \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})$ consisting of automorphisms of degree less than or equal to d. Given $f \in \tilde{\mathrm{A}}_d \leq \mathrm{Aut}(\mathbb{A}^1_{R_1})$ we consider the degree d coefficient $a_d(f)$ of f. Recall that in equation 4.1 we wrote f as

$$f(T) = a_0 + a_1(T) + pF_f(T)$$

with $\mathrm{ord}_T(F_f) \geq 2$ which tells us that $a_d(f) = pa_d(F_f)$. So using the computation performed we see that

(5.6)

$$a_d(f \circ g) = a_d(F_{f\circ g}) = p(a_1(f)a_d(F_g) + pa_1(g)^d a_d(F_f)) = a_1(f)a_d(f) + a_1(g)^d a_d(g).$$

Now do the following:

(1) Divide $a_d(f)$ by p and reduce mod p. (Observe that this is just $a_d(F_f)$ mod p)
(2) Divide the above result by $a_1(f) \mod p$.

The result is a twisted homomorphism. For $f \in \tilde{\mathrm{A}}_d$ we define $\tau : \tilde{\mathrm{A}}_d \to R_1$ by

$$\tau(f) = \frac{a_d(F_f)}{a_1(f)} = \frac{a_d(f)/p}{a_1(f)}. \tag{5.7}$$

Using equation 5.6 we see that

$$\begin{aligned} \tau(f \circ g) &= \frac{a_d(f \circ g)/p}{a_1(f \circ g)} \\ &= \frac{a_1(f)a_d(F_g) + a_1(g)^d a_d(F_f)}{a_1(f)a_1(g)} \\ &= \frac{a_d(F_g)}{a_1(g)} + \frac{a_1(g)^{d-1}a_d(F_f)}{a_1(f)} \\ &= \tau(g) + a_1(g)^{d-1}\tau(f). \end{aligned}$$

This defined a right cocycle with an action of a $d-1$ power.

5.12. **New Čech cocycles from twisted homomorphisms.** Let $[\Sigma] \in \check{H}^1(X_n, A)$ be the class of some A-structure for some $A \leq \mathrm{Aut}(\mathbb{A}^1_{R_n})$, and let $\Phi : A \to \mathcal{O}_{X_0}$ let be a left (resp right) twisted homomorphism. If $f_\Phi : A \to \mathcal{O}_{X_0} \rtimes \mathcal{O}_{X_0}^\times$ (resp $A \to \mathcal{O}_{X_0}^\times \ltimes \mathcal{O}_{X_0}$) is the induced homomorphism one obtains a new class $H^1(f_\Phi)([\Sigma]) \in H^1(X_0, \mathcal{O} \rtimes \mathcal{O}^\times)$ (resp $H^1(X_0, \mathcal{O}^\times \ltimes \mathcal{O})$). We define

$$\kappa(\Phi, \Sigma) := \lambda(H^1(f_\Phi([\Sigma])) \in H^1(X_0, L)$$

where $[L] \in \mathrm{Pic}(X_0)$ is the natural projection of $H^1(f_\Phi)([\Sigma])$ induced by the canonical projection from the semi-direct product to $\mathcal{O}^\times$.

5.13. **Recovering the Deligne-Illusie class.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and C/R a smooth curve.

Proposition 5.27 (recovery of old classes)**.** *Let $\Phi \in Z^1_{\mathtt{group}}(A_2, \mathcal{O}_{C_0})$ with the action defined by $\rho(a + bT + pcT^2) = b \mod p \in \mathcal{O}_{C_0}^\times$ and Σ be any A_2-structure of $J^1(C)_1 \to C_1$. Then*

$$\kappa(\Phi, \Sigma) = [\delta_i - \delta_j] = \mathrm{DI}_0(\delta),$$

where $\mathrm{DI}_0(\delta)$ is the class introduced by Deligne and Illusie.

Remark 5.28. We didn't need to start with a cocycle for $[\Sigma] \in \check{H}^1(C_1, A_2)$. We could have very well started with $[J^1(C)_1] \in \check{H}^1(C_1, \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1}))$ or $[J^1(C)] \in \check{H}^1(\widehat{C}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1_R))$ and the same result.

6. Reduction of the structure group of the first jet space

Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. In this section prove the following

Theorem 6.1. *Let X/R be a smooth curve of genus $g \geq 2$. Suppose that for each $n \geq 1$, $J^1(X)_n$ admits an A_{n+1}-structure $\{\Sigma_n\}$ where for each $n \geq 2$ we have $\Sigma_n \otimes_{R_n} R_{n-1} = \Sigma_{n-1}$ then $J^1(X)$ has the structure of a torsor under a line bundle L.*

Remark 6.2. Note that the hypotheses of this theorem are satisfied by section 4.

Sections 6.2, 6.2 and 6.3 prove Theorem 6.1 for every A_2-structure—the heart of the whole argument can be seen readily in this simple case. This case will be the case case for induction. The idea is to use the right cocycle $\tau_2 : A_2 \to \mathcal{O}_{X_0}^\times \ltimes \mathcal{O}_{X_0}$ (section 5.11.4 on page 49) to construct

$$\kappa(\Sigma, \tau_2) \in H^1(X_0, \Omega_{X_0}^{\otimes p}).$$

Since $g(X_0) \geq 2$ we have $H^1(X_0, \Omega_{X_0}^{\otimes p}) = 0$ which implies that $\kappa(\tau_2, \Sigma) = 0$ and in turn shows that the structure group reduces. The reduction of the structure groups A_{n+1} for $n > 1$ employ a similar strategy.

Section 6.4 introduces a twisted homomorphism necessary for lifting mod p^3. Section 6.5 gives the first step of the reduction of the structure group mod p^3. Section 6.6 introduces a new twisted homomorphism for reducing the structure group mod p^3 again and section 6.7 gives another reduction of the structure group mod p^3 using this twisted homomorphism. Section 6.8 introduces the necessary objects to generalize the reduction procedure in the previous sections and section sec:reduction of the structure group in general uses these objects to reduce the structure group. Section 6.10 constucts $\mathrm{DI}_n(\delta)$ using what we just proved. In order to lift the Deligne-Illusie class completely we let $\widehat{\mathrm{DI}}(\delta) = \varprojlim \mathrm{DI}_n(\delta)$ and use Grothendieck's correspondence between Formal schemes and schemes when X admits an ample line bundle.

Throughout this section if $\sigma : H \to G$ is a morphism of sheaves of groups on a scheme X and $\eta \in H^1(X, G)$ we employ the notation

$$H^1(X, H)_\eta = H^1(\sigma)^{-1}(\eta)$$

for the fiber of classes in $H^1(X, H)$ which map to η under the induced map in cohomology.

6.1. **The top class, $\tilde{\mathrm{A}}_d$-structures and the reduction procedure.**

Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and X/R be a scheme. Consider the case mod p^2 where the $\tilde{\mathrm{A}}_d \subset \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})$ consisting of automorphisms of degree $\leq d$ form a group. How can one determine if an $\tilde{\mathrm{A}}_d$-structure Σ on an $\mathbb{A}^1_{R_1}$-bundle $f : J \to X$ is $\tilde{\mathrm{A}}_d$-equivalent to some $\tilde{\mathrm{A}}_{d-1}$-structure? Is there an effective way to determine if there exists $\tilde{\mathrm{A}}_d$-compatible trivializations (section 3.2 page 3.2) which have degree less than d? Yes, there is an effective way and it relies on the class

$$\kappa(\Sigma, \tau_d) \in \check{H}^1(X, L_\Sigma)$$

where $[L_\Sigma] = H^1(\pi)([\Sigma]) \in \mathrm{Pic}(X_0)$ and $\pi : \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1}) \to \mathcal{O}_{X_0}^\times$ is the canonical quotient (passing through the reduction mod p). In particular $\kappa(\Sigma, \tau_d)$ will vanish if the structure group can be reduced.

Remark 6.3. This approach is to studying reductions of structure groups is actually quite general:

- Cook up a cocycle that vanishes on your subgroup (recall that the kernel of cocycles are subgroups).
- Test the cohomology class associated to your structure
- If it doesn't vanish, it can't be reduced.

6.2. **Reductions of the structure group** mod p^2.

Proposition 6.4. *Let X/R be a smooth scheme where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Further assume that $\{(U_i, \psi_i)\}$ is an A_2-atlas for $J^1(X)_1 \to X_1$ inducing the structure Σ. If $\kappa(\Sigma, \tau_2) = 0$ then there exists an $\mathrm{AL}_1(\mathcal{O}_{X_1})$-atlas $\{(U_i, \psi_i')\}$ for $J^1(X)_1 \to X_1$.*

Proof. The transition maps take the form

$$\psi_{ij}(T) = a_{ij} + b_{ij}T + pc_{ij}T^2.$$

Let $f : A_2 \to \mathcal{O}_{X_0}^\times \ltimes \mathcal{O}_{X_0}$ be the group homomorphism defined by

$$f(a + bT + pcT^2) = (c/\overline{b}, \overline{b})$$

where $\overline{b}$ denotes the reduction of b mod p. Note that is group homomorphism induced by the cocycle τ_2. The image of ψ_{ij} under the map $H^1(f) : H^1(X_1, A_2) \to H^1(X_0, \mathcal{O}^\times \ltimes \mathcal{O})$ gives a right twisted cocycle

$$(c_{ij}/\overline{b}_{ij}, \overline{b}_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}^\times \rtimes \mathcal{O})$$

Recall that if $\varphi_i : \mathcal{O}(U_i) \to L(U_i)$ is a trivialization of a line bundle L with $\varphi_i(1) = v_i$ and $m_{ij}v_i = v_j$ then we say that $[m_{ij}] = [L]$ is the multiplicative cocycle associated to the line bundle in $\check{H}^1(\mathcal{O}^\times)$. If we let $r : \mathcal{O}^\times \ltimes \mathcal{O} \to \mathcal{O}^\times$ that we have

$$\rho : H^1(X_0, \mathcal{O}^\times \ltimes \mathcal{O}) \cong H^1(L^\vee).$$

Note that this is different from the left cocycle case where the map $l : \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$ induces the isomorphism

$$H^1(l)^{-1}([L]) \cong H^1(L).$$

Also recall that the isomorphism ρ between right twisted cocycles and the first cohomology of the dual bundle is given by

$$(\alpha_{ij}, m_{ij}) \mapsto s_{ij} = \varphi_j'(\alpha_{ij}).$$

where $\varphi_i' : \mathcal{O}(U_i) \to L^\vee(U_i)$ is given by $\varphi_j'(1) = v_j^*$ and $m_{ij}v_j^* = v_i^*$.

In our particular application $[\overline{b}_{ij}] = [F^*T_{X_0}]$ so that we have

$$\rho : H^1(r)^{-1}([F^*T_{X_0}]) \cong H^1(X_0, \Omega_{X_0}^{p+1}) = 0.$$

This tells us that we have

$$\rho(c_{ij}/\overline{b}_{ij}, \overline{b}_{ij}) = \varphi_j'(c_{ij}/\overline{b}_{ij}) = s_i - s_j = c_i v_i^* - c_j v_j^*$$

note that

$$(\varphi_j')^{-1}(c_i v_i^* - c_j v_j^*) = (\varphi_j')^{-1}(c_i \overline{b}_{ij} v_j^* - c_j v_j^*) = c_i \overline{b}_{ij} - c_j$$

where $\bar{b}_{ij}v_j^* = v_i^*$. In terms of twisted cocycles we have

$$(c_{ij}/\bar{b}_{ij}, \bar{b}_{ij}) = (c_i\bar{b}_{ij} - c_j, \bar{b}_{ij}) = (c_i, 1)(0, \bar{b}_{ij})(-c_j, 1) = (c_i, 1)(0, \bar{b}_{ij})(c_j, 1)^{-1}.$$

If we let $\psi_i(T) = T + pc_iT^2 \in A_2(U_i)$ then we have

$$f(\psi_{ij}) = f(\psi_i \circ b_{ij}T \circ \psi_j^{-1})$$

which tells us that

$$f(\psi_i^{-1} \circ \psi_{ij} \circ \psi_j \circ \frac{T}{b_{ij}}) = (1, 0).$$

This tells us that $\psi_i^{-1} \circ \psi_{ij} \circ \psi_j \circ \frac{T}{b_{ij}} \in \ker(f)$. Since

$$\ker(f) = \{a + (1 + pb')T : a \in \mathcal{O}_{X_1}, b \in \mathcal{O}_{X_2}^\times\} \leq \mathrm{AL}_1 \leq A_2$$

we have

$$\psi_i^{-1} \circ \psi_{ij} \circ \psi_j \circ \frac{T}{b_{ij}} = \gamma_{ij} \in \mathrm{AL}_1(U_{ij})$$

which means that

$$\psi_i^{-1} \circ \psi_{ij} \circ \psi_j = \gamma_{ij} \circ b_{ij}T \in \mathrm{AL}_1(U_{ij})$$

□

6.3. **Lift of the Deligne-Illusie mod p^2.** Suppose that X/R is a smooth curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. If $J^1(X)_1 \to X_1$ admits an A_2-structure then Proposition 6.4 shows that $J^1(X) \to X_1$ admits an $\mathrm{AL}_1(\mathcal{O}_{X_1})$ structure $\{(\psi_i, U_i)\}$ so that

$$\psi_{ij}(T) = a_{ij} + b_{ij}T.$$

The cocycle $[b_{ij}] \in H^1(X_1, \mathcal{O}^\times)$ gives rise to some L_1 where $[L_1] = [b_{ij}]$ is some wittfinitesimal deformation of the $F^*T_{X_0}$ the Frobenius tangent bundle since $L_1 \otimes_{R_1} R_0 = F^*T_{X_0}$. Similarly the left twisted cocycle $[a_{ij}, b_{ij}] \in Z^1(X_1, \mathcal{O}_{C_1} \rtimes \mathcal{O}_{C_1}^\times)_{[L_1]}$ by the isomorphism λ gives rise to a class

$$[\varphi_i(a_{ij})] = \mathrm{DI}_1(\delta) \in H^1(X_1, L_1)$$

which is a canonical lift of the Deligne-Illusie class. Note that Theorem 5.20 (page 46) implies that there cannot exists another L_1-torsor structure on $J^1(X)_1$.

Remark 6.5. It has not been ruled out that there exists another L_1'-torsor structure where L_1' is a distinct wittfinitesimal deformation of $F^*T_{X_0}$.

6.4. **The twisted homomorphism τ_3.** Let $f(T) = a_0 + a_1T + pa_2T^2 + p^2a_3T^3$ and $g(T) = b_0 + b_1T + pb_2T^2 + p^2b_3T^3$ be elements of A_3. We can see that

$$\begin{aligned} f(g(T)) &= a_0 + a_1(b_0 + b_1T + pb_2T^2 + p^2b_3T^3) \\ &\quad + pa_2(b_0 + b_1T + pb_2T^2)^2 + p^2a_3(b_0 + b_1T)^3 \\ &= a_0 + a_1b_0 + pa_2b_0^2 + pa_3b_0^3 \\ &\quad + (a_1b_1 + 2pa_2b_0b_1 + 3p^2a_3b_0^2b_1)T \\ &\quad + (pa_1b_2 + pa_2(b_1^2 + 2pb_0b_2) + 3p^2a_3b_0b_1^2)T^2 \\ &\quad + (p^2a_1b_3 + p^2a_3b_1^3)T^3 that \end{aligned}$$

We define $\tau_3 : A_3 \to \mathcal{O}_{C_0}$ by

$$\tau_3(a + bT + pcT^2 + p^2dT^3) = d/b \in \mathcal{O}_{C_0}.$$

This is a right cocycle since

$$\tau_3(f\circ g)=\frac{a_1b_3+a_3b_1^3}{a_1b_1}=\frac{a_3}{a_1}b_1^2+\frac{b_3}{b_1}=\tau_3(f)b_1^2+\tau_3(g).$$

Remark 6.6. In section 6.5 we use the fact that the action comes from the multiplication squared.

6.5. **First reduction** mod p^3**.** Let X/R be a smooth scheme where $R=\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Further assume that $\{(U_i,\psi_i)\}$ be an A_3-atlas for $J_2\to X_2$ and that $\psi_{ij}\otimes_{R_2}R_1\in\mathrm{AL}_1(\mathcal{O}_{X_1})$. Under these assumptions ψ_{ij} takes the form

$$\psi_{ij}(T)=a_{ij}+b_{ij}T+p^2c_{ij}T^2+p^2d_{ij}T^3\in A_3',$$

where A_3' is precisely the inverse image of the affine linear maps in A_2. Now consider the group homomorphism $f:A_3'\to\mathcal{O}_{X_0}^\times\rtimes\mathcal{O}_{X_0}$ induced by τ_3 given by

$$f(a+bT+p^2cT^2+p^2dT^3)=(\bar{b}^2,d/\bar{b}).$$

We know that $[f(\psi_{ij})]\in H^1(X_0,\mathcal{O}_{X_0}^\times\rtimes\mathcal{O}_{X_0})$ and since $[\bar{b}_{ij}]=[F^*T_{X_0}^{\otimes 2}]\in\mathrm{Pic}(X_0)$ we have a correspondence between right $(\bar{b}_{ij}^2)$-twisted cocycles and cocycles in $H^1(X_0,F^*T_{X_0}^{\otimes 2\vee})=H^1(X_0,\Omega^{2p+1})=0$. An argument similar to section 6.2 tells us that

$$f(\psi_{ij})=f(\psi_i\circ b_{ij}T\circ\psi_j^{-1})$$

where $\psi_i(T)=T+p^2d_iT$ and hence that $f(\psi_i^{-1}\circ\psi_{ij}\circ\psi_j\circ T/b_{ij})=(1,0)$ or that

$$\psi_i^{-1}\circ\psi_{ij}\circ\psi_j\circ T/b_{ij}\in\ker(f).$$

where $\ker(f)=\{a+(1+pb')T+p^2cT^2: a\in\mathcal{O}_{X_2},b'\in\mathcal{O}_{X_1},c\in\mathcal{O}_{X_0}\}\le A_3'$ then we have

$$\psi_i^{-1}\circ\psi_{ij}\circ\psi_j\circ T/b_{ij}=\gamma_{ij}$$

or that

$$\psi_i^{-1}\circ\psi_{ij}\circ\psi_j=\gamma_{ij}\circ b_{ij}T\in A_3''$$

where

$$A_3''=\{a+bT+p^2cT^2: a\in\mathcal{O}_{X_2},b\in\mathcal{O}_{X_2}^\times,c\in\mathcal{O}_{X_0}\}\le A_3'.$$

6.6. **A cocycle for** A_3''**.** Recall from section 6.5 that

$$A_3'':=\{a+bT+p^2cT^2: a\in\mathcal{O}_{X_2},b\in\mathcal{O}_{X_2},c\in\mathcal{O}_{X_0}\}\le A_3.$$

Let $f(T)=a_0+a_1T+p^2a_2T^2$ and $g(T)=b_0+b_1T+p^2b_2T^2$ be two elements of A_3''. We can see that

$$\begin{aligned}f(g(T)) &= a_0+a_1(b_0+b_1T+p^2b_2T^2)+p^2a_2(b_0+b_1T)^2\\ &= a_0+a_1b_0+p^2a_2b_0^2+(a_1b_1+2p^2a_2b_0b_1)T+p^2(a_1b_2+a_2b_1^2)T^2\end{aligned}$$

We define $\tau_2':A_3''\to\mathcal{O}_{C_0}$ by

$$\tau_2'(a+bT+p^2cT^2)=c/b\in\mathcal{O}_{C_0}.$$

This is a right cocycle since

$$\tau_2'(f\circ g)=\frac{a_1b_2+a_2b_1^2}{a_1b_1}=\frac{a_2}{a_1}b_1+\frac{b_2}{b_1}=\tau_2'(f)b_1+\tau_2'(g).$$

6.7. **Second reduction** $\mod p^3$. Let X/R be a smooth curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Suppose that $\{(U_i, \psi_i)\}$ is an A_3'' atlas for $\pi : J_2 \to X_2$ where we recall from section 6.5 that

$$A_3'' := \{a + bT + p^2cT^2 : a \in \mathcal{O}_{X_2}, b' \in \mathcal{O}_{X_1}, c \in \mathcal{O}_{X_0}\} \leq A_3.$$

by constructions similar to those of sections 6.2 and section 6.5 we will reduce the structure group further to $\mathrm{AL}_1(\mathcal{O}_{X_2})$.

Recall the right group cocycle $\tau_2' : A_3'' \to \mathcal{O}_{C_0}$ defined by

$$\tau_2'(a + bT + p^2cT^2) = c/b.$$

Let $f : A_3'' \to \mathcal{O}_{C_0}^\times \rtimes \mathcal{O}_{C_0}$ be the map induced by τ_2' which is defined by

$$f(a + bT + p^2cT^2) = (\bar{b}, c/\bar{b})$$

where $\bar{b}$ is the reduction mod p of b.

Now since $\{(U_i, \psi_i)\}$ is an A_3'' atlast we can write

$$\psi_{ij}(T) = a_{ij} + b_{ij}T + p^2c_{ij}T^2.$$

If we apply f to $\psi_{ij} \in H^1(X_2, A_3'')$ we have

$$f(\psi_{ij}) \in H^1(X_0, \mathcal{O}^\times \rtimes \mathcal{O})$$

which is a right cocycle. Since $[\bar{b}_{ij}] = [F^*T_{X_0}]$ under the correspondence $\phi_i(1) = v_i \in F^*T_{X_0}(U_i)$ with $\bar{b}_{ij}v_i = v_j$ the isomophisms ρ is between between right twisted cocycles $[\bar{b}_{ij}, \alpha_{ij}]$ and $H^1(X_0, F^*T_{X_0}^\vee) = H^1(X_0, \Omega^{p+1}) = 0$. Hence we have that there exists some collection $\psi_i(T) = T + p^2c_iT^2$ such that

$$f(\psi_{ij}) = f(\psi_i \circ b_{ij}T \circ \psi_j^{-1})$$

which means

$$\psi_i^{-1} \circ \psi_{ij} \circ \psi_j \circ T/b_{ij} = \gamma_{ij}$$

where $\gamma_{ij} \in \ker(f)(U_{ij}) = \{a + (1 + pb')T : a \in \mathcal{O}_{X_2}(U_{ij}), b' \in \mathcal{O}_{X_1}(U_{ij})\} \subset \mathrm{AL}_1(\mathcal{O}_{C_2}(U_{ij})$ which implies that

$$\psi_i^{-1} \circ \psi_{ij} \circ \psi_j = \gamma_{ij} \circ b_{ij}T \in \mathrm{AL}_1(\mathcal{O}_{X_2}).$$

This completes the reduction of the structure group mod p^3.

6.8. **Players in the reduction of the structure group mod p^{n+1}.** Define $A_{n,m}' := \{a_0 + a_1T + p^{n-1}a_2T^2 + \cdots + p^{n-1}a_mT^m : a_0 \in \mathcal{O}_{X_n}, a_1 \in \mathcal{O}_{X_n}^\times, a_i \in \mathcal{O}_{X_n}\}$ let

$$\begin{aligned} f(T) &= a_0 + a_1T + p^{n-1}a_2T^2 + p^{n-1}a_3T^3 + \cdots + p^{n-1}a_mT^m \\ g(T) &= b_0 + b_1T + p^{n-1}b_2T^2 + p^{n-1}b_3T^3 + \cdots + p^{n-1}b_mT^m \end{aligned}$$

then we have

$$\begin{aligned} f(g(T)) &= a_0 + a_1(b_0 + b_1T + p^{n-1}\sum_{k=2}^m b_kT^k) + p^{n-1}\sum_{k=2}^m a_k(b_0 + b_1T)^k \\ &= \text{(lower order terms) } + p^{n-1}(a_1b_m + b_1^m a_n)T^m \qquad (6.1) \end{aligned}$$

Let's define $\tau_{n,m}' : A_{n,m}' \to \mathcal{O}_{X_0}$ by

$$\tau_{n,m}'(a_0 + a_1T + p^{n-1}\sum_{k=2}^m a_kT^k) := a_m/a_1 \in \mathcal{O}_{X_0}.$$

From equation 6.1 we see that

$$\tau'_{n,m}(f \circ g) = \tau'_{n,m}(f) b_1^{m-1} + \tau'_{n,m}(g),$$

which shows that $\tau'_{n,m}$ is a right cocycle at least of monoids. It will turn out to be a right cocycle of groups once we prove that $A'_{n,m}$ is a group for $m \leq n$.

Proposition 6.7. *$A'_{n,m}$ and A'_n are groups.*

Proof. First let $\pi_n : A_n \to A_{n-1}$ be the reduction mod p^{n-1} map. We see that A'_n is a group because

$$A'_n = \pi_n^{-1}(\mathrm{AL}_1(\mathcal{O}_{X_{n-1}})),$$

and the inverse image of every group is a group. We already showed that $A'_{n,m}$ is closed under composition.

We define the subgroup

$$\{p^{n-1}a_0 + (1+p^{n-1}a_1)T + p^{n-1}a_2T^2 + \cdots + p^{n-1}a_mT^n\} = N_{n,m} \leq N_n := \ker(\pi_n : A_n \to A_{n-1})$$

which are actually an abelian. A simple computation can show that

$$N_n \cong \mathcal{O}_{X_0}^{\oplus n+1}$$

where the isomorphism is given by

$$p^{n-1}c_0 + (1+p^{n-1}c_1)T + p^{n-1}c_2T^2 + \cdots + p^{n-1}c_nT^n \mapsto (c_0, c_1, c_2, \ldots, c_n).$$

From this fact is it then clear that $N_{n,m}$ is a subgroup of N_n such that

$$N_{n,m} \cong \mathcal{O}_{X_0}^{\oplus m+1}.$$

It is easy to see thatn $A'_{n,m}$ is the subgroup generated by $\mathrm{AL}_1(\mathcal{O}_{X_n})$ and $N_{n,m}$. $\square$

6.9. **Reduction of the structure group mod p^{n+1}.** Let X/R be a smooth curve of genus $g \geq 2$ where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Suppose that $\pi : J_n \to X_n$ has an a collection of trivializations $\{(U_i, \psi_i)\}$ where $\psi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{A}^1_{R_n}$ that is an A_n-atlas i.e. such that

$$\psi_{ij} \in A_n(U_{ij}).$$

We claim that there exists a reduction of the structure group to $\mathrm{AL}_1(\mathcal{O}_{X_n}) = \{a_0 + a_1T : a \in \mathcal{O}_{X_n}, b \in \mathcal{O}_{X_n}^\times\}$. In other word that there exists a collection of trivializations $\{(\psi'_i, U_i)\}$ of $\pi : J_n \to X_n$ such that

$$\psi'_{ij} \in \mathrm{AL}_1(\mathcal{O}_{X_n}(U_{ij})).$$

The proof is very much in the styles of sections 6.2, 6.5, 6.7 where we proved this for X_1 and X_2.

The proof is by induction on n so we can suppose without loss of generality that the trivializing cover $\{(\psi_i, U_i)\}$ gives an $A'_n = \pi_n^{-1}(\mathrm{AL}_1(\mathcal{O}_{X_{n-1}}))$ structure on $\pi_n : J_n \to X_n$ since by inductive hypothesis we can assume that $\{(\psi_i, U_i)\} \otimes_{R_n} R_{n-1}$ is an $\mathrm{AL}_1(\mathcal{O}_{X_{n-1}})$-atlas for $\pi_{n-1} : J_{n-1} \to X_{n-1}$.

The strategy is to reduce that A'_n-atlas $\Sigma_n = \{(\psi_i, U_i)\}$ to an $A'_{n,n-1}$-atlas $\{(\psi_i^{(n-1)}, U_i)\}$ then to a $A'_{n,n-2}$-atlas and so on until we get to an $A'_{n,1} = \mathrm{AL}_1(\mathcal{O}_{X_n})$-atlas.

Lemma 6.8. *If $2 \leq m \leq n$ then any $A'_{n,m}$-atlas $\{(\psi_i, U_i)\}$ for $\pi : J_n \to X_n$ admits a reduction to an $A'_{n,m-1}$-atlas. Equivalently there exists some $f_i \in A'_{n,m}(U_i)$ such that*

$$f_i \circ \psi_{ij} \circ f_j^{-1} \in A'_{n,m}$$

Proof. Recall the cocycle $\tau'_{n,m} : A_{n,m} \to \mathcal{O}_{X_0}$ from section 6.8 defined by

$$\tau'_{n,m}(a_0 + a_1 T + p^{n-1} a_2 T^2 + \cdots + p^{n-1} a_m T^m) = a_m / a_1.$$

Using this cocycle define the map $f_{n,m} : A_{n,m'} \to \mathcal{O}_{X_0}^\times \ltimes \mathcal{O}_{X_0}$ by

$$f_{n,m}(a_0 + a_1 T + p^{n-1} a_2 T^2 + \cdots + p^{n-1} a_m T^m) = (\overline{a}_1^{m-1}, a_m).$$

This map applied to ψ_{ij} gives a right twisted cohomology class

$$[f_{n,m}(\psi_{ij})] \in H^1(X_0, \mathcal{O}^\times \ltimes \mathcal{O}).$$

We again recall the conventions for the embedding lemma since this is what we want to apply: suppose that L is an invertible sheaf with trivializations $\psi_i : \mathcal{O}(U_i) \to L(U_i)$ where $\psi_i(1) = v_i$ and such that $m_{ij} v_i = v_j$, then the Picard isomorphism between the Picard group of invertible sheaf modulo isomorphism and $H^1(\mathcal{O}^\times)$ is given by

$$[L] \mapsto [m_{ij}].$$

The embedding lemma tells us that if $[L] = [m_{ij}]$ with trivializations as above, then a left twisted cocycle $[\alpha_{ij}, m_{ij}] \in H^1(l)^{-1}([m_{ij}]) \subset H^1(\mathcal{O} \rtimes \mathcal{O}^\times)$ is isomorphic to $H^1(X, L)$ via the map

$$[\alpha_{ij}, m_{ij}] \mapsto s_{ij} := \varphi_i(\alpha_{ij}),$$

where $l : \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$. The alternative version of the embedding lemma tells us that if we have a right twisted cocycle with trivializations $\varphi'_i : \mathcal{O}(U_i) \to L^\vee(U_i)$ given by $\varphi'_i(1) = v_i^*$ such that $\frac{1}{m_{ij}} v_i^* = v_j^*$ there is a similar isomorphism of $\rho : H^1(X, \mathcal{O}^\times \ltimes \mathcal{O})_{[L]} \cong H^1(X, L^\vee)$ given by

$$\rho[m_{ij}, \alpha_{ij}] = [\varphi'_j(\alpha_{ij})].$$

We note that $f_{n,m}(\psi_{ij})$ a right cocycle whose multiplicative component is $[\overline{a}_1(\psi_{ij})^{m-1}] = [F^* T_{X_0}^{\otimes m-1}] \in H^1(X_0, \mathcal{O}^\times)$. The target under the isomorphism ρ will then be $H^1(X_0, \Omega^{\otimes p(m-1)+1})$ which is trivial since $m \geq 2$. This tells us that

$$\varphi'_j(a_m(\psi_{ij})/\overline{a}_1(\psi_{ij}) = s_{ij} = s_i - s_j = c_i v_i^* - c_j v_j^* = (c_i \overline{a}_1(\psi_{ij}) - c_j) v_j^*,$$

or that

$$a_m(\psi_{ij})/\overline{a}_1(\psi_{ij}) = c_i \overline{a}_1(\psi_{ij}) - c_j.$$

In terms of right twisted cocycles this means that

$$\begin{aligned} f_{n,m}(\psi_{ij}) &= (\overline{a}_1(\psi_{ij}), c_i \overline{a}_1(\psi_{ij} - c_j)) \\ &= (c_i, 1)(\overline{a}_1(\psi_{ij}), 0)(0, c_j)^{-1} \\ &= f_{n,m}(g_i \circ a_1(\psi_{ij}) T \circ g_j^{-1}) \end{aligned}$$

where $g_i(T) := T + p^{n-1} c_i T^m$ where we use the same symbol c_i to denote any lift of the c_i from the previous equation. This implies that

$$g_i^{-1} \circ \psi_{ij} \circ g_j \circ T / a_1(\psi_{ij}) = \gamma_{ij}$$

where $\gamma_{ij} \in \ker(f_{n,m}) = \{b_0 + (1 + p b'_1) T + \cdots + p^{n-1} b_{m-1} T^{m-1}\} \subset A'_{n,m-1}$ and hence

$$g_i^{-1} \circ \psi_{ij} \circ g_j = \gamma_{ij} \circ T a_1(\psi_{ij}) \in A'_{n,m-1}.$$

The f_i and as listed in proposition are $f_i = g_i^{-1}$. $\square$

This completes the proof of Theorem 6.1.

Remark 6.9. (1) Note that the trivializations
$$\psi_i' := f_i \circ \psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^1_{R_n}$$
give and $A'_{n,m-1}$-atlas to $J_n \to X_n$.

(2) Note that
$$\psi'_{ij} \equiv \psi_{ij} \mod p^{n-1}$$
which means that at the end of the procedure when we obtain the $\mathrm{AL}_1(\mathcal{O}_{X_n})$-atlas for $J_n \to X_n$ it reduces to the $\mathrm{AL}_1(\mathcal{O}_{X_{n-1}})$-atlas of $J_{n-1} \to X_{n-1}$ atlas that we started with at the beginning of the proof

6.10. **Lifts of the Frobenius tangent bundle and Deligne-Illusie.** Let X/R be a smooth curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. In section 6.9 we showed that if $J_n \to X_n$ admits an A_n-structure then it admits a natural AL_1-structure by a sequence of reductions to smaller groups
$$A_n \supset A'_n = A'_{n,n} \supset A'_{n,n-1} \supset \cdots \supset A'_{n,1} = \mathrm{AL}_1(\mathcal{O}_{X_n}).$$

In particular we showed there exists an AL_1-atlas $\{(\psi_i, U_i)\}$ of $J_n \to X_n$ such that
$$\psi_{ij}(T) = a_{ij} + b_{ij}T$$
where $a_{ij} \in \mathcal{O}_{X_n}(U_{ij})$ and $b_{ij} \in \mathcal{O}_{X_n}^\times$. The isomorphism $\mathrm{AL}_1(\mathcal{O}_{X_n}) \to \mathcal{O}_{X_n} \rtimes \mathcal{O}_{X_n}^\times$ given by
$$a + bT \mapsto (a, b)$$
induces an isomorphisms if affine linear cocycles with left twisted cocycles,
$$H^1(X_n, \mathrm{AL}_1) \cong H^1(X_n, \mathcal{O} \rtimes \mathcal{O}^\times).$$
Now since b_{ij} is a multiplicative cocycle we have $[L_n] = [b_{ij}]$ where L_n is some invertible sheaf with trivializations $\varphi_i : \mathcal{O}_{X_n}(U_i) \to L_n(U_i)$ where $\varphi_i(1) = v_i$ and $b_{ij}v_i = v_j$ and since
$$[b_{ij}] \otimes_{R_n} R_0 = [F^* T_{X_0}]$$
we know that L_n is some n-th order wittfinitesimal deformation of the Frobenius tangent bundle.

Let $\pi : \mathcal{O} \rtimes \mathcal{O}^\times \to \mathcal{O}^\times$ be the natural projection. The left embedding isomorphism
$$\lambda : \pi^{-1}([L_n]) \subset H^1(X_n, \mathcal{O} \rtimes \mathcal{O}^\times) \to H^1(X_n, L_n)$$
tells us that
$$\mathrm{DI}_n(\delta) := \lambda[\psi_{ij}] \in H^1(X_n, L_n)$$
is an nth order wittfinitesimal lift of the Deligne-Illusie class.

7. Multiple A_2-structures on jet spaces of elliptic curves mod p^2

Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. The goal of this section is to prove and explain the following surprising fact:

Theorem 7.1 (Multiple Structures on Jet Spaces of Elliptic Curves)**.** *If $E/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is a smooth projective elliptic curve with $p > 6g - 5$ then $J^1(E)_1$ admits at least two distinct A_2-structures—one coming from the group structure and one coming from étale projections.*

This section can be thought of in four separate pieces:

- Section 7.1 proves the existence of $\Sigma_{elliptic}$

- Section 7.4 computes an A_2-atlas on $J^1(E)_1$ that defined the A_2-structure Σ_{plane}. Sections 7.2 and 7.3 play supporting roles.
- Section 7.9 computes $\kappa(\Sigma,\tau_2) \in H^1(E_0, \Omega_{E_0}^{\otimes p})$. Sections 7.5,7.6, 7.7 and 7.8 should be viewed as supporting material for this.
- The last several sections show that $\kappa(\Sigma,\tau_2) \neq 0$ by taking a particular $V \in H^0(E_0, \Omega_{X_0}^{-p+1})$ and showing $\langle \kappa(\Sigma,\tau_2), V\rangle \neq 0$ in $H^1(E_0,\Omega)$. The sections 7.10, 7.13. 7.11 and 7.12 prepare for this result.

7.1. **Elliptic curves: $\Sigma_{elliptic}$.** Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. In this section we follow Buium ([Bui94a], section 7.2).

Proposition 7.2. *There exist a trivializing cover $\mathcal{U}_{elliptic} = \{(\psi_i, U_i)\}_{i\in I}$ of E where $\psi_i : J^1(U_i) \to \widehat{U}_i \hat{\times} \widehat{\mathbb{G}}_a$ and $\psi_{ij} : \widehat{U}_{ij} \hat{\times} \widehat{\mathbb{G}}_a \to \widehat{U}_{ij} \hat{\times} \widehat{\mathbb{G}}_a$ are given by affine linear polynomials. In other words the map $\mathcal{O}(U_{ij})[t]^{\frown} \to \mathcal{O}(U_{ij})[t]^{\frown}$ is given by*

$$t \mapsto a_{ij} + t.$$

Let $E \to \mathrm{Spec}(R)$ be a smooth elliptic curve. Since $J^1(-)$ is a functor and E is a group scheme then $J^1(E)$ is a group-formal-scheme (the diagrams that make E a group-scheme and the functorality of $J^1(-)$ give $J^1(E)$ a group-formal-scheme). It also turns out that $\pi : J^1(E) \to \widehat{E}$ is a morphism of group formal schemes [Bui94a]. The idea of the above proposition is to use the group structure of $J^1(E)$ to get $\pi^{-1}(U) = J^1(U) \to \widehat{U} \hat{\times} \widehat{\mathbb{A}}^1$ via the group law.

Theorem 7.3. *Let E be an elliptic curve over* $\mathrm{Spec}(R)$ *and let $N^1 := \ker(J^1(E) \to \widehat{E})$. We have the isomorphism of formal group schemes*

$$N^1 \cong \widehat{\mathbb{G}}_a \tag{7.1}$$

There exists an open cover $\{U_i\}$ of E such that there exists local sections $s_i : \widehat{U}_i \to J^1(U_i)$, such that these local sections induce isomorphisms $J^1(U_i) \to \widehat{U}_i \hat{\times} N^1$ which when combined with Theorem 7.3 give $\psi_i : J^1(U_i) \to \widehat{U}_i \times \widehat{\mathbb{G}}_a$.

Lemma 7.4. *Let E/R be a smooth elliptic curve where $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$,*

1. *If $U \subset E$ is an open subset which admits a section $s : \widehat{U} \to J^1(U)$ then there exists some $\psi : J^1(U) \cong \widehat{U} \hat{\times} \widehat{\mathbb{G}}_{add}$ associated to s.*
2. *$\psi \in \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_n})(U)$ affine linear if and only if ψ is affine linear on the level of points; That is for all $S \in \mathsf{CRing}_{R_n}$, the map $\psi : (U_n \times \mathbb{A}^1_{R_n})(S) \to (U_n \times \mathbb{A}^1_{R_n})(S)$ is affine linear.*
3. *If E/R admits a cover $\mathcal{U} = \{U_i\}$ by affine open subset and sections $s_i : \widehat{U}_i \to J^1(U_i)$ then the associated ψ_i give an* AL_1*-atlas.*

Proof. It is enough to show the bijection on the level of points: for $P \in J^1(U_i)(S)$ where $S \in \widehat{\mathsf{CRing}}_R$ we have

$$P = s_i(\pi(P)) +_{J^1(E)} (P -_{J^1(E)} s_i(\pi(P)) = A +_{J^1(E)} B$$

where $A \in s_i(\widehat{E}(S))$ and $B \in N^1(S)$. It is easy to check that this is a bijection which shows the natural bijection of sets

$$J^1(U_i)(S) \leftrightarrow (\widehat{U} \hat{\times} N^1)(S).$$

Composing with the isomorphism in Theorem 7.3 and using Yoneda gives the result.

Suppose that $\psi \in \mathrm{Aut}(\widehat{\mathbb{A}}^1)(U)$ is affine linear: $\psi(t) = a + bt$. Take $P \in (\widehat{U}\hat{\times}\widehat{\mathbb{A}}^1)(S)$ since

$$\mathsf{ForSch}(\mathrm{Spf}(S), \widehat{U}\hat{\times}\widehat{\mathbb{A}}^1) = \widehat{\mathsf{CRing}}_R(\mathcal{O}(U)[t]\hat{\ }, S),$$

where on the right hand side the map is given by $t \mapsto P$ we have

$$\psi : P \mapsto a + bP,$$

or $\psi : (t \mapsto P) \mapsto (t \mapsto a + bP)$. This shows that the map is affine linear on the level of rings.

Conversely suppose that the map is affine linear on the level of points. This means for all $n \geq 0$ and all $(A, B) \in (U_n \times \mathbb{A}^1_{R_n})(S)$ we have $\psi_n(A, B) = (A, a(A) + b(A)B)$. Recall that for all $n \geq 0$ and all $S \in \mathsf{CRing}_{R_n}$ we have $(U_n \times \mathbb{A}^1_{R_n})(S) = \mathrm{Sch}_{R_n}(\mathrm{Spec}(S), U_n \times \mathbb{A}^1_{R_n}) = \mathsf{CRing}_{R_n}(\mathcal{O}(U_n)[t], S)$. If we let $S = \mathcal{O}(U_n)[t]$ then $\mathrm{id}_{\mathcal{O}(U_n)[t]} \mapsto \psi_n$ tells us that $\psi_n(t)$ is affine linear for each $n \geq 1$ which shows ψ is affine linear.

It remains to show that for all $n \geq 0$ the maps $\psi_{ij} : U_n \times \mathbb{A}^1_{R_n} \to U_n \times \mathbb{A}^1_{R_n}$ are affine linear on the level of points. Let $P = (A, B) \in (U_n \times \mathbb{A}^1_{R_n})(S)$ so that $A \in U_n(S)$ and $B \in \mathbb{A}^1_{R_n}(S)$, $\beta : \mathbb{A}^1 \to N^1$ be the isomorphism in theorem 7.3, and ι be the inclusion map $\iota : N^1 \subset J^1(E)$. We have that $\psi_j^{-1} = (s_j \times \iota) \circ (\mathrm{id} \times \beta)$ so that

$$\psi_j^{-1}(A, B) = s_j(A) +_{J^1} \beta(B).$$

Similarly for $P \in J^1(U_{ij})(S)$, we saw that

$$\psi_i(P) = (\pi(P), \beta^{-1}(P -_{J^1} s_i\pi(P))).$$

This means that

$$\begin{aligned} \psi_{ij}(A, B) &= (A, \beta^{-1}(s_j(A) +_{J^1} \beta(B) -_{J^1} s_i(A))) \\ &= (A, \beta^{-1}s_{ji}(A) +_{\mathbb{A}^1} B). \end{aligned}$$

which shows that ψ_{ij} is affine linear on the level of points. $\square$

Theorem 7.5 (Vanishing Theorem)**.** *Let E over $S = \mathrm{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$ be a smooth elliptic curve and let $\Sigma_{elliptic}$ be the A_2-structure on the first arithmetic just space mod p^2, $J^1(E)_1$, induced by the group structure on $J^1(E)_1$ then*

$$\kappa(\Sigma_{elliptic}, \tau_2) = 0 \in H^1(E_0, F^*_{E_0}T_{E_0}).$$

Proof. $\tau_2(a_{ij} + b_{ij}T) = 0$, since the transitions are affine linear and $\tau_2(a + bT + pcT^2) = c/b \mod p$. $\square$

The remainder of this section is devoted to proving

Theorem 7.6 (Non-vanishing structure)**.** *Let E/R be a smooth projective elliptic curve and Σ_{plane} be the A_2 structure on $J^1(E)_1$ coming from étale trivializations*

$$\kappa(\Sigma_{plane}, \tau_2) \neq 0$$

where Σ_{plane} is computed in section 4

Remark 7.7. Section 9 given an explicit construction of Σ_{plane} for $E \subset \mathbb{P}^2_R$.

7.2. **Conventions for trivializing line bundles.** For a line bundle L with trivializations $\varphi_j : \mathcal{O}(U_j) \to L(U_j)$ given by $\varphi_j(1) = v_j$ with $m_{ij} v_i = v_j$ we associate the cohomology class

$$[F] = [m_{ij}] \in H^1(X, \mathcal{O}).$$

Note that if $\varphi_{ij} := \varphi_i^{-1} \circ \varphi_j$ then

$$\varphi_{ij}(1) = \varphi_i^{-1}(\varphi_j(1)) = \varphi_i^{-1}(v_j) = \varphi_i^{-1}(m_{ij} v_i) = m_{ij}.$$

Also recall that given $[m_{ij}, a_{ij}] \in H^1(X, \mathcal{O}^\times \rtimes \mathcal{O})$, and $\varphi_i' : \mathcal{O}(U_i) \to L^\vee(U_i)$, where $\varphi_i'(1) = v_i^*$, where $v_i^*(v_i) = 1$, we have $m_{ji} v_i^* = v_j^*$ and $m_{ji} = 1/m_{ij}$. The isomorphism $\rho : H^1(X\mathcal{O}^\times \ltimes \mathcal{O}) \cong H^1(X, L)$ is given by $[m_{ij}, a_{ij}] \mapsto [s_{ij}] := [\varphi_j'(a_{ij})] \in H^1(X, F^\vee)$. We can see this since

$$\begin{aligned} s_{ij} + s_{jk} + s_{ki} &= \varphi_j'(a_{ij}) + \varphi_k'(a_{jk}) + \varphi_i'(a_{ki}) \\ &= a_{ij} v_j^* + a_{jk} v_k^* + a_{ki} v_i^* \\ &= a_{ij} m_{ji} v_i^* + a_{jk} m_{ki} v_i^* + a_{ki} v_i^* \\ &= (a_{ij} m_{jk} m_{ki} + a_{jk} m_{ki} + a_{ki}) v_i^* \\ &= 0. \end{aligned}$$

7.3. **Presentations of elliptic curves.** $E = \frac{U \sqcup V}{\sim}$, $U = \mathrm{Spec}\ k[x,y]/(y^2 - f(x))$, $V = \mathrm{Spec}\ k[u,v]/(v^2 - f(1/u)u^4)$, $k = \overline{\mathbb{F}}_p$, $u = 1/x$, $v = y/x^2$, $f(x) = x^4 + ax^3 + bx + c$ $U_1, U_3 \subset U$ and $U_2 \subset V$,

$$\begin{array}{r|l|} U_1 = D(f'(x)) & y \text{ étale parameter} \\ U_2 = D(v) & u \text{ étale parameter} \\ U_3 = D(y) & x \text{ étale parameter} \end{array}$$

$E = U_1 \cup U_2$

7.4. **Transition map computations.** On each of these open sets we have $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^1$, $i = 1, 2, 3$ which induce $\psi_{ij}^* : \mathcal{O}(U_{ij})[T] \to \mathcal{O}(U_{ij})[T]$, $\psi_{ij}^*(T) = \psi_{ij}(T)$, where

$$\begin{aligned} \psi_1^* &: \mathcal{O}(U_1)[T] \to \mathcal{O}(U_1)[\dot{y}], \\ \psi_2^* &: \mathcal{O}(U_2)[T] \to \mathcal{O}(U_2)[\dot{u}], \\ \psi_3^* &: \mathcal{O}(U_3)[T] \to \mathcal{O}(U_3)[\dot{x}]. \end{aligned}$$

We want to compute $\psi_{21}(T)$. Note that $\psi_{21}^* = (\psi_2 \circ \psi_1^{-1})^* = (\psi_1^{-1})^* \psi_2^*$, and since

$$\mathcal{O}(U_{21})[T] \xrightarrow{\psi_2^*} \mathcal{O}_{21}[\dot{u}] = \mathcal{O}(U_{21})[\dot{y}] \xrightarrow{(\psi_1^{-1})^*} \mathcal{O}(U_{21})[T]$$

we have

$$\psi_{21} = \text{“}\dot{u} \text{ as a polynomial in } \dot{y}\text{”}$$

To compute ψ_{21} we will compute

$$\begin{aligned} \psi_{23} &= \text{“}\dot{u} \text{ as a polynomial in } \dot{x}\text{”} \\ \psi_{31} &= \text{“}\dot{x} \text{ as a polynomial in } \dot{y}\text{”} \end{aligned}$$

To get $\dot{u}$ as polynomial in $\dot{x}$ and then $\dot{x}$ as a polynomial in $\dot{y}$, so that $\psi_{21} = \psi_{23} \circ \psi_{31}$.

Since $u = 1/x$, we have

$$\begin{aligned}\dot{u} &= \frac{-\dot{x}}{x^p\phi(x)} \\ &= \frac{-\dot{x}}{x^p(x^p + p\dot{x})} \\ &= \frac{-\dot{x}}{x^{2p}} \sum_{j\geq 0} \left(\frac{-p\dot{x}}{x^p}\right)^j \\ &\equiv \frac{-\dot{x}}{x^{2p}} \left(1 - \frac{p\dot{x}}{x^p}\right) \mod p^2\end{aligned}$$

We next compute ψ_{31}. Since $y^2 = f(x)$ we have

$$\begin{aligned}2y^p\dot{y} + p\dot{y}^2 &= \frac{f^\phi(x^p + p\dot{x}) - f(x)^p}{p} \\ &= \frac{f^\phi(x^p) - f(x)^p}{p} + f^{\phi\prime}(x^p)\dot{x} + \frac{1}{2}f^{\phi\prime\prime}(x^p)\dot{x}^2 + \frac{1}{6}f^{\phi\prime\prime\prime}(x^p)\dot{x}^3 + \frac{1}{24}f^{\phi\prime\prime\prime\prime}(x^p)\dot{x}^4, \\ &\equiv r + f^{\phi\prime}(x^p)\dot{x} + \frac{p}{2}f^{\phi\prime\prime}(x^p)\dot{x}^2 \mod p^2, \qquad r := \frac{f^\phi(x^p) - f(x)^p}{p}\end{aligned}$$

Where $f^\phi(T) = T^4 + \phi(a)T^3 + \phi(b)T + \phi(c)$ which gives

$$\dot{x} \equiv (2y^p\dot{y} - r)/f^{\phi\prime}(x^p) \mod p,$$

Writing

$$\dot{x} \equiv (2y^p\dot{y} - r)/f^{\phi\prime}(x^p) + p\dot{x}_1 \mod p^2$$

, and solving for $\dot{x}_1$

$$\begin{aligned}& 2y^p\dot{y} + p\dot{y}^2 \equiv r + f^{\phi\prime}(x^p)((2y^p\dot{y} - r)/f^{\phi\prime}(x^p) + p\dot{x}_1) + \frac{p}{2}f^{\phi\prime\prime}(x^p)\left((2y^p\dot{y} - r)/f^{\phi\prime}(x^p) + p\dot{x}_1\right)^2 \mod p \\ \implies \quad & \dot{y}^2 \equiv f^{\phi\prime}(x^p)\dot{x}_1 + \frac{f^{\phi\prime\prime}(x^p)}{2}\left(\frac{2y^p\dot{y} - r}{f^{\phi\prime}(x^p)}\right)^2 \mod p \\ \implies \quad & \dot{x}_1 \equiv \frac{1}{f^{\phi\prime}(x^p)}\left[\dot{y}^2 - \frac{f^{\phi\prime\prime}(x^p)}{2}\left(\frac{2y^p\dot{y} - r}{f^{\phi\prime}(x^p)}\right)^2\right] \mod p \\ \implies \quad & \dot{x} \equiv \frac{2y^p\dot{y} - r}{f^{\phi\prime}(x^p)} + \frac{p}{f^{\phi\prime}(x^p)}\left[\dot{y}^2 - \frac{f^{\phi\prime\prime}(x^p)}{2}\left(\frac{2y^p\dot{y} - r}{f^{\phi\prime}(x^p)}\right)^2\right] \mod p^2.\end{aligned}$$

This means

$$\psi_{23}(T) = \frac{-T}{x^{2p}}\left(1 - \frac{pT}{x^p}\right) \mod p^2 \tag{7.2}$$

$$\psi_{31}(T) = \frac{2y^pT - r}{f^{\phi\prime}(x^p)} + \frac{p}{f^{\phi\prime}(x^p)}\left[T^2 - \frac{f^{\phi\prime\prime}(x^p)}{2}\left(\frac{2y^pT - r}{f^{\phi\prime}(x^p)}\right)^2\right] \mod p^2 \tag{7.3}$$

7.5. **Multiplicative part of transition maps.** THe multiplicative part is defined by

$$\begin{aligned}m(\psi_{ij}(T)) &= m(a_{ij} + b_{ij}T + pc_{ij}T) = \bar{b}_{ij} \\ \bar{b}_{ij} &:= b_{ij} \mod p\end{aligned}$$

This implies that

$$\begin{aligned} m(\psi_{23}) &= -1/x^{2p} \\ m(\psi_{31}) &= 2y^p/(f')^p \end{aligned}$$

Compare to section 7.6 where we computed the transition maps for differentials.

7.6. **Trivializations of the cotangent sheaf.** Let $L = \Omega_E$ and $\varphi_i' : \mathcal{O}(U_i) \to \Omega_E(U_i)$, given by $\varphi_i(1) = v_i$ where $\Omega_E(U_i) = \mathcal{O}(U_i)v_i$ with

$$\begin{aligned} \varphi_1'(1) &= dy, \\ \varphi_2'(1) &= du, \\ \varphi_3'(1) &= dx. \end{aligned}$$

Since $y^2 = f(x)$, we have $2ydy = f'(x)dx$, and since $x = 1/u$ we have $dx = -du/u^2$. This gives

$$\begin{aligned} \varphi_{23}'(1) &= (\varphi_2')^{-1}(dx) = -1/u^2, \\ \varphi_{31}'(1) &= (\varphi_3')^{-1}(dy) = \frac{f'(x)}{2y}, \\ \varphi_{21}'(1) &= \varphi_{23}'(1)\varphi_{31}'(1) = \frac{-1}{u^2} \cdot \frac{f'(x)}{2y} = \frac{-x^2 f'(x)}{2y}. \end{aligned}$$

7.7. **Transitions for twisted cocycles.** Using the relations

$$(m_{ik}, \tau_{ik}) = (m_{ij}, \tau_{ij})(m_{jk}, \tau_{jk}) = (m_{ij}m_{jk}, \tau_{ij}m_{jk} + \tau_{jk})$$

we get

$$\tau_{ik} = \tau_{ij}m_{jk} + \tau_{jk},$$

which will simplify computations considerably.

7.8. **Computing τ_2.** Let $\tau = \tau_2$ and recal that

$$\tau(\psi_{ij}) = \tau(a_{ij} + b_{ij}T + pc_{ij}T^2) := c_{ij}/\overline{b}_{ij}.$$

This gives

$$\begin{aligned} \tau(\psi_{23}) &= \frac{1}{x^{3p}}/m(\psi_{23}) = \frac{-1}{x^p}, \\ \tau(\psi_{31}) &= \frac{1}{f^{\phi\prime}}\left[1 - \frac{f^{\phi\prime\prime}}{2}\left(\frac{2y^p}{f^{\phi\prime}}\right)^2\right]/m(\psi_{31}) \\ &= \frac{(f')^p}{2y^p} \cdot \frac{1}{f^{\phi\prime}}\left[1 - \frac{f^{\phi\prime\prime}}{2}\left(\frac{2y^p}{f^{\phi\prime}}\right)^2\right] \\ &= \left[\frac{1}{2y} - 2y\frac{f''}{(f')^2}\right]^p. \end{aligned}$$

Using the formula $\tau_{ik} = \tau_{ij} m_{jk} + \tau_{jk}$ from section 7.7 with $i = 2, k = 1, j = 3$ we have

$$\begin{aligned}
\tau(\psi_{21}) &= \tau_{23} m_{31} + \tau_{31} \\
&= \frac{-1}{x^p} \cdot \frac{2y^p}{(f')^p} + \left[\frac{1}{2y} - 2y \frac{f''}{(f')^2} \right]^p \\
&= \left[\frac{-yf'}{x} + \frac{1}{2y} - \frac{2yf''}{(f')^2} \right]^p
\end{aligned}$$

7.9. **Formula for embedded cocycles.** We will now apply the isomorphism from section 7.2. Let $\varphi'_j : \mathcal{O}(U_j) \to \Omega^p_E$, and $s_{ij} := \varphi'_j(\tau_{ij})$ be as above. In this section we compute

$$\varphi'_1(\tau_{21}).$$

From section 7.6 we have

$$U_1 = D(f'(x)), \quad \varphi'_1(1) = dy^{\otimes p}$$

and

$$s_{21} := \varphi'_1(\tau_{21}) \quad = \quad \left[\frac{-yf'}{x} + \frac{1}{2y} - \frac{2yf''}{(f')^2} \right]^p dy^{\otimes^p}$$

so that

$$[s_{21}] \in H^1(E_0, \omega^p).$$

7.10. **Global section of the Frobenius tangent bundle.** We will use the presentation

$$E = U \cup V$$

where $U = \mathrm{Spec}(k[x,y]/(y^2 - f(x))$ and $V = \mathrm{Spec}(k[u,v]/(v^2 - f(1/u)u^4))$, with $f(x) = x^4 + ax^3 + bx + c$. Consider the differential $\omega := dx/y = 2dy/f'(x)$ defined on U. The relation

$$dx/y = (-1/u^2 du)/(v/u^2)^{-1} = -du/v,$$

shows that ω is defined on V by symmetry. Since $\omega^* := (dx/y)^* = y\partial_x$ we have

$$y\partial_x = y(dx)^* = y(2ydy/f'(x))^* = \frac{f'(x)}{2}\partial_y$$

is defined on U and since

$$y\partial_x = y(dx)^* = y(-du/u^2)^* = -yu^2\partial_u = -v\partial_u$$

shows that it is defined on V as well symmetry (this step breaks down for Kummer curves $y^e = f(x)$ with a two chart model where $v = y/x^{\deg(f)/e}$ and $u = 1/x$ when $e > 2$ or $\deg(f) > 4$—see section 9 for more about computations with superelliptic curves.). This shows

$$\begin{aligned}
& y\partial_x \in H^0(E, T_E) \\
\implies \quad & (y\partial_x)^{\otimes p-1} \in H^0(E, T_E^{\otimes p-1})
\end{aligned}$$

7.11. **Pairing prodedure.** The cup product gives

$$\eta_{21} \smile (y\partial_x)^{\otimes p-1} \in H^1(E,\Omega)$$

where

$$(y\partial_x)^{\otimes p-1} = (\frac{f'(x)}{2}\partial_y)^{\otimes p-1}.$$

Let

$$\begin{aligned} f(x) &= (x-s_1)(x-s_2)(x-s_3)(x-s_4) \\ f'(x) &= 4(x-r_1)(x-r_2)(x-r_3) \end{aligned}$$

where $s_1, s_2, s_3, s_4, r_1, r_2$ and r_3 are subject to the relations induced by $\frac{d}{dx}[(x-s_1)(x-s_2)(x-s_3)(x-s_4)] = 4(x-r_1)(x-r_2)(x-r_3)$.

$$\begin{aligned} \eta_{21} \smile (y\partial_x)^{\otimes p-1} &= \left[\frac{-yf'}{x} + \frac{1}{2y} - \frac{2yf''}{(f')^2}\right]^p dy^{\otimes p} \smile (\frac{f'(x)}{2}\partial_y)^{\otimes p-1} \\ &= \frac{1}{2^{p-1}}\left[\frac{-yf'}{x} + \frac{1}{2y} - \frac{2yf''}{(f')^2}\right]^p (f')^{p-1}dy \\ &= \frac{1}{2^{p-1}}\left[\frac{-y(f')^2}{x} + \frac{f'}{2y} - \frac{2yf''}{f'}\right]^p \frac{dy}{f'} \\ &= \frac{1}{2}\left[\frac{-y(f')^2}{x} + \frac{f'}{2y} - \frac{2yf''}{f'}\right]^p \frac{dx}{y} \\ &= \frac{1}{2}\left[\frac{-(f')^2}{x}y + \frac{1}{2}\frac{f'}{f}y - 2\frac{f''}{f'}y\right]^p y\frac{dx}{f} \\ &= \frac{1}{2}\left[\frac{-(f')^2}{x} + \frac{1}{2}\frac{f'}{f} - 2\frac{f''}{f'}\right]^p f^{l+1}\frac{dx}{f}, \qquad 2l+1 = p \\ &= \frac{1}{2}\left[\frac{-(f')^2}{x} + \frac{1}{2}\frac{f'}{f} - 2\frac{f''}{f'}\right]^p f^l dx, \qquad 2l+1 = p \end{aligned}$$

So that

$$\begin{aligned} \eta_{21} \smile (y\partial_x)^{\otimes p-1} &= A + B + C, \\ A &= -\frac{1}{2}\frac{(f')^{2p}f^l}{x^p}dx, \\ B &= \frac{f^l}{4}\sum_{i=1}^4 \frac{1}{(x-s_i)^p}dx, \\ C &= -f^l\sum_{i=1}^3 \frac{1}{(x-r_i)^p}dx. \end{aligned}$$

We will now analyze each piece separately.

7.12. **Traces.** Let $\omega = [\omega_{ij}] \in H^1(X,\Omega)$. Here is how we compute traces:

For a fixed index l where $(\omega_{ij})_{(i,j)\in I\times I} \in H^1(\mathcal{U},\Omega)$ we define

$$\mathrm{Res}_a(l,\omega) := \mathrm{Res}_a(\omega_{il}), \text{ if } a \in U_i.$$

where $\mathrm{Res}_a(\omega)$ is computed by writing ω as $\omega = f(t)dt$ where t is a local parameter at a and then taking the coefficient of t^{-1}. We claim that this definition is well-defined (i.e. is independent of the open set U_i containing a). Suppose $a \in U_j$ as well

$$\mathrm{Res}_a(\omega_{il}) - \mathrm{Res}_a(\omega_{jl}) = \mathrm{Res}(\omega_{il} + \omega_{lj}) = \mathrm{Res}_a(\omega_{ij}) = 0$$

since ω_{ij} regular on $U_i \cap U_j$. This proves $\mathrm{Res}_a(l, \omega)$ is well defined.

Pick some arbitrary index l and defined

$$\mathrm{Tr}(\omega) := \sum_a \mathrm{Res}_a(l, \omega).$$

We claim that the definition is independent of the index l. Choose another index k, then

$$\sum_a \mathrm{Res}_a(\omega_{i_a l}) - \sum_a \mathrm{Res}_a(\omega_{i_a k}) = -\sum_a \mathrm{Res}_a(\omega_{l i_a} + \omega_{i_a k}) = -\sum_a \mathrm{Res}_a(\omega_{lk}) = 0$$

where i_a is any index such that $a \in U_{i_a}$. It is a theorem that the trace map induces an isomorphism $H^1(X, \Omega) \cong K$ if X is a curve over K.

7.13. **Trace after pairing.** Again let $E = U_1 \cup U_2$

We will pick the index 2,

$$\mathrm{Tr}(\eta) = \sum_a \mathrm{Res}_a(2, \eta) = \sum_{a \in U_1 \setminus U_2} \mathrm{Res}_a(\eta_{21})$$

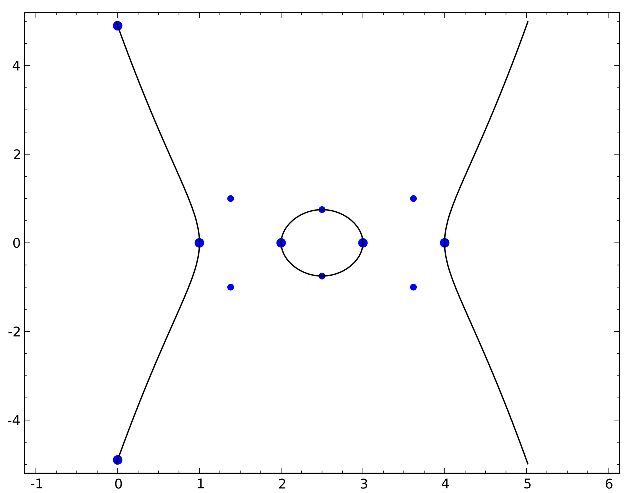

FIGURE 7.

The curve pictured is $U : y^2 = f(x) = (x-1)(x-2)(x-3)(x-4)$. The set $U_1 = D(f'(x))$ is the curve with the small points removed (the points not on the curve represent complex points). The larger dots represent the set $U_1 \setminus U_2$ where $U_2 = D(v)$ where $v^2 = (1-u)(1-2u)(1-3u)(1-4u)$ defined the chart at infinity with $u = 1/x$ and $v = y/x^2$—the points include the zeros of $f(1/u)u^4$ and the points above zero in $\mathbb{P}^1$.

From figure 7.13 we see that

$$U_1 \setminus U_2 = \{(0, \pm\sqrt{s_1 s_2 s_3 s_4}), (s_1, 0), (s_2, 0), (s_3, 0), (s_4, 0)\}$$

We will analyze terms A B and C separately

7.13.1. We first see that

$$\mathrm{Tr}(C) = 0$$

since C has no residues on $U_1 \setminus U_2$.

7.13.2. In this section we compute $\mathrm{Tr}(A)$. We have $2A = (f')^{2p} f^l / x^p$

$$\begin{aligned} Ax^p/2^{4p-1} &= \frac{(f')^{2p} f^l}{4^{2p}} \\ &= (x-r_1)^{2p}(x-r_2)^{2p}(x-r_3)^{2p}(x-s_1)^l(x-s_2)^l(x-s_3)^p(x-s_4)^l \\ &= \sum_{r=0}^{3N} \alpha_r x^r, \end{aligned}$$

where

$$\begin{aligned} &\alpha_r \\ = &\sum_{a+b+c+d+e+f+g=r} \binom{2p}{a}\binom{2p}{b}\binom{2p}{c}\binom{l}{d}\binom{l}{e}\binom{l}{f}\binom{l}{g}(-r_1)^{2p-a}(-r_2)^{2p-b}(-r_3)^{2p-c}(-s_1)^{l-d}(-s_2)^{l-e}(-s_3) \end{aligned}$$

and in the sum $a, b, c \leq 2p$ and $d, e, f, g \leq l$.

$$\mathrm{Tr}(A) = 2^{4p-1}\alpha_{p-1}.$$

Note that $3 \cdot 2p + 4 \cdot l - (p-1) = 6p + 2(p-1) - (p-1) = 7p - 1$ which is even so we can factor out all the negatives from α_{p-1}. In addition, since we are working in characteristic p and $a, b, c \leq 2p$ we must have and $\binom{2p}{n} = 0 \mod p$ only for $n \neq 0, p, 2p$. This means only when $a = b = c = 0$ do we get a contribution to the sum in α_{p-1} so

$$\alpha_{p-1} = (r_1 r_2 r_3)^{2p} \sum_{d+e+f+g=p-1} \binom{l}{e}\binom{l}{f}\binom{l}{g}(s_1)^{l-d}(s_2)^{l-e}(s_3)^{l-f}(s_4)^{l-g}.$$

7.13.3. In this section we compute

$$\mathrm{Tr}(B).$$

Writing

$$B = B_1 + B_2 + B_3 + B_4$$

where

$$B_i = \frac{f(x)^l}{4}\frac{1}{(x-s_i)^p}$$

we have

$$\begin{aligned}
4B_1 &= \frac{(x-s_1)^l(x-s_2)^l(x-s_3)^l(x-s_4)^l}{(x-s_1)^p} \\
&= \frac{(x-s_2)^l(x-s_3)^l(x-s_4)^l}{(x-s_1)^{l+1}} \\
&= \frac{(x-s_1+s_1-s_2)^l(x-s_1+s_1-s_3)^l(x-s_1+s_1-s_4)^l}{(x-s_1)^{l+1}} \\
4B_1(x-s_1)^{l+1} &= (x-s_2)^l(x-s_3)^l(x-s_4)^l \\
&= (x-s_1+s_1-s_2)^l(x-s_1+s_1-s_3)^l(x-s_1+s_1-s_4)^l \\
&= \sum_{i=0}^{l}\sum_{j=0}^{l}\sum_{k=0}^{l}\binom{l}{i}\binom{l}{k}\binom{l}{k}(s_1-s_2)^{l-i}(s_1-s_3)^{l-j}(s_1-s_4)^{l-k}(x-s_1)^{i+j+k} \\
&= \sum_{r=0}^{3l}\beta_{1,j}(x-s_1)^j, \\
\beta_{1,j} &= \sum_{i+j+k=r}\binom{l}{i}\binom{l}{k}\binom{l}{k}(s_1-s_2)^{l-i}(s_1-s_3)^{l-j}(s_1-s_4)^{l-k}
\end{aligned}$$

If $j=l$ then we have $a=l-i, b=l-j, c=l-k$. Then $i+j+k=l$ gives $l-a+l-b+l-c=l$ or $2l=a+b+c$ so

$$\beta_{1,l} = \sum_{a+b+c=2l}\binom{l}{a}\binom{l}{b}\binom{l}{c}(s_1-s_2)^a(s_1-s_3)^b(s_1-s_4)^c$$

gives

$$\begin{aligned}
\mathrm{Tr}(B_1) &= \beta_{1,l}/4 = \frac{1}{4}\sum_{a+b+c=2l}\binom{l}{a}\binom{l}{b}\binom{l}{c}(s_1-s_2)^a(s_1-s_3)^b(s_1-s_4)^c \\
\mathrm{Tr}(B) &= \frac{1}{4}\sum_{a+b+c=2l}\binom{l}{a}\binom{l}{b}\binom{l}{c}\big\{(s_1-s_2)^a(s_1-s_3)^b(s_1-s_4)^c+(s_2-s_1)^a(s_2-s_3)^b(s_2-s_4)^c \\
&\quad +(s_3-s_1)^a(s_3-s_2)^b(s_3-s_4)^c+(s_4-s_1)^a(s_4-s_2)^b(s_4-s_3)^c\big\}
\end{aligned}$$

7.13.4. *altogether.*

$$\begin{aligned}
\mathrm{Tr}(\eta) &= \mathrm{Tr}(A+B+C) \\
&= 2^{4p-1}(r_1r_2r_3)^{2p}\sum_{d+e+f+g=p-1}\binom{l}{d}\binom{l}{e}\binom{l}{f}\binom{l}{g}(s_1)^{l-d}(s_2)^{l-e}(s_3)^{l-f}(s_4)^{l-g} \\
&\quad +\frac{1}{4}\sum_{a+b+c=2l}\binom{l}{a}\binom{l}{b}\binom{l}{c}\big\{(s_1-s_2)^a(s_1-s_3)^b(s_1-s_4)^c+(s_2-s_1)^a(s_2-s_3)^b(s_2-s_4)^c \\
&\quad +(s_3-s_1)^a(s_3-s_2)^b(s_3-s_4)^c+(s_4-s_1)^a(s_4-s_2)^b(s_4-s_3)^c\big\}
\end{aligned}$$

Note that this expression lives in $M=\overline{\mathbb{F}}_p[s_1,s_2,s_3,s_2,r_1,r_2,r_3]/I$ where I is the ideal generated by the relations induced by demanding $\frac{d}{dx}[(x-s_1)(x-s_2)(x-s_3)(x-s_4)]=4(x-r_1)(x-r_2)(x-r_3)$. We will show that the image of $\mathrm{Tr}(\eta)\in M$ under a non-zero morphism is non-zero. Here we will set $s_1=s_2=s$ and $s_3=s_4=t$.

This implies that $r_1 = s$, $r_2 = t$ and $r_2 = \frac{s+t}{2}$. This means that

$$\mathrm{Tr}(\eta) \mapsto 2^{4p-1} s^{2p} t^{2p} \left(\frac{s+t}{2}\right)^{2p} \sum_{d+e+f+g=p-1} \binom{l}{d}\binom{l}{e}\binom{l}{f}\binom{l}{g} s^{2l-e-d} t^{2l-f-g}$$

If we look at the term where $e+d = p-1$ then we must have $f = g = 0$ and hence a contribution to the highest degree term possible in t. But since $e+d = p-1$ and $e, d \leq l = (p-1)/2$ we must have $e = d = (p-1)/2 - l$ which gives the expression

$$2^{4p-1} s^{2p} t^{2p} \left(\frac{s+t}{2}\right)^{2p} t^{p-1}$$

which contributes a non-zero coefficient to $s^{2p} t^{5p-1}$. Since no other terms can contribute we must have that $\mathrm{Tr}(\eta)$ is non-zero.

Remark 7.8. (1) This concludes the proof of Theorem 7.1.

(2) The proof is not at all special. In fact, if one takes the heuristic that $\mathrm{Tr}(\eta) \in M = \overline{\mathbb{F}}_p[s_1, s_2, s_3, s_4, r_1, r_2, r_3]/I$ is some non-zero random element then one could should be able to pick any basis M as an $\overline{\mathbb{F}}_p$-vector space and try to show that its coefficient is non-zero.

8. A Study of Multiple Structures

This purpose of this section is to explain Theorem 7.1 (page 57). The fact that multiple A_2-structures exist on the affine bundle $J^1(E)_1 \to E_1$ is unexpected and in wondering if this was even possible we set out to provide some examples of this type of behavior in other settings. This section contains our results.

In section 8.1 we prove The map from $H^1(E, \tilde{\mathrm{A}}_d) \to H^1(E, \tilde{\mathrm{A}}_{d+1})$ is injective in the category of pointed sets (although there some d where it is not injective in the category of sets). This in some sense shows that theorem 7.1 is very close to being false since $H^1(\iota)[\Sigma_{elliptic}] = H^1(\iota)[\Sigma_{plane}] = [J^1(X)_1] \in H^1(X_1, \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})$ where ι is the inclusion the subgroup $A_2 \subset \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})$. In section 8.3 we give a setup for computions which are similar to our cohomology computation where in sections 8.3 and 8.4 we give examples where this toy multiple-structures phenomena exist.

8.1. **Toy examples of multiple structures.**

Let's fix some notation. For $H \leq G$ be sheaves of groups on X. We will let G/H denote the presheaf of sets defined by

$$G/H(U) = G(U)/H(U)$$

and we will let (G/H) denote the sheaf associated to the presheaf.

Let (X, p) and (Y, q) be pointed sets. A map $f : (X, p) \to (Y, q)$ is **injective** if and only if $f^{-1}(q) = \{p\}$. Note that injectivity as a map of sets is enough to show a morphism of pointed sets is injective (but not conversely).

Lemma 8.1. *The map $\check{H}^1(X, H) \to \check{H}^1(X, G)$ is injective if and only if the presheaf G/H is actually a sheaf.*

Proof. Let f be the natural map $f : \check{H}^1(X, H) \to \check{H}^1(X, G)$ and suppose it is injective. Let $\{U_i\}$ be a cover of X and suppose that $g_i \in G_i := G(U_i)$ have the property that $[g_i] \sim [g_j] \mod H_{ij}$. This means that $g_i g_j^{-1} = h_{ij} \in H_{ij}$. Injectivity means that for all $\eta = [h_{ij}]$ we have $f(\eta) = 1$ if and only if $\eta = 1$. Equivalently, if there exists some $g_i \in G_i$ such that $h_{ij} = g_i g_j^{-1}$ then there exists some $h_i = H_i$ such that $h_{ij} = h_i h_j^{-1}$. So we have $g_i g_j^{-1} = h_i h_j^{-1}$ which implies that

$g_i' := h_i^{-1} g_i \in G_i$ have the property that $g_i' = g_j'$ on G_{ij}. Since G is a sheaf there exists some $g' \in G(X)$ such that $g'|_{U_i} = g_i'$. So $[g'] \in G/H(X)$ is the unique lift of the collection $[g_i]$ so G/H is a sheaf.

Conversely, suppose that the presheaf G/H is actually a sheaf and that $f([h_{ij}]) = 1$. This means that $h_{ij} = g_i g_j^{-1}$ for some collection of g_i. The collection $[g_i] \in G/H(U_i)$ agree on intersections and since G/H is a sheaf there exists a unique $[g] \in G/H(X)$ such that $[g]|_{U_i} = [g_i]$. This means $g_i = h_i g^{-1}$ and we have $h_{ij} = g_i g_j^{-1} = h_i g^{-1} g h_j^{-1} = h_i h_j^{-1}$ which shows that $[h_{ij}]$ was trivial to begin with. $\square$

Remark 8.2. This result was "derived" by considering the pretend long exact sequences associated to the short exact sequence $1 \to H \to G \to G/H \to 1$. This may work in some appropriate category but it is not the category of pointed sets and it is not the category of sets with group actions.

8.2. **A proposition about non-abelianess of the multiple structures phenomena.** Suppose that $A = \{(\psi_i, U_i)\}_{i \in I}$ and $A' = \{(\psi_j', U_j')\}_{j \in J}$ are two A_2-atlases for a morphism of schemes $f : J \to E$ over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}/p^2\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ amd let the associated A_2-structures be denoted by Σ and Σ' respectively. These two A_2-structures allow us to define β and β' in $\check{H}^1(E, A_2)$ which may be distinct. We can also consider the images of β and β' in $\check{H}^1(E, A_d)$ for some $d \geq 2$. Although the $\{\psi_i\}$ and $\{\psi_j'\}$ may not be A_2-compatible they are certainly A_d compatible for some d, when working with a finite cover, in fact, $d = \max_{i,j} \deg(\psi_i \circ \psi_j')$.

Lemma 8.3. *Let $X/\overline{\mathbb{F}}_p$ be a proper scheme with $X = U \cup V$ with $U \cap V \neq \emptyset$. For all $d \geq 3$, the presheaf A_d/A_{d-1} is actually a sheaf.*

Proof. The idea is to apply Lemma 8.1 and show that $\check{H}(\mathcal{U}, H) \hookrightarrow \check{H}^1(\mathcal{U}, G)$. Let $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and let $X = U \cup V$ and let

$$\begin{aligned} f(T) &= a_0 + a_1 T + p a_2 T^2 + p a_3 T^3 + \cdots + p a_d T^d \in A_d(U), \\ g(T) &= b_0 + b_1 T + p b_2 T^2 + p b_3 T^3 + \cdots + p b_d T^d \in A_d(V), \end{aligned}$$

then

$$f(g(T)) = (a_0 + a_1 b_0 + p a_2 b_0^2 + \cdots + p a_d b_0^d) + \cdots + (a_d b_1^d + a_1 b_d) T^d$$

In order for $f(g(T)) \in A_{d-1}(U \cap V)$ we need

$$a_d b_1^d + a_1 b_d = 0 \iff -a_d/a_1 = b_d/b_1^d = \lambda$$

Note that $\lambda \in \mathcal{O}(U_0) \cap \mathcal{O}(V_0)$ which makes $\lambda \in \mathcal{O}(X_0) = \overline{\mathbb{F}}_p$

We claim that for all $f \in A_d(U)$ where there exists a $g \in A_d(V)$ with $f(g(T)) \in A_{d-1}(U \cap V)$ there exists some $h = c_0 + c_1 T + p c_2 T^2 + \cdots + p c_d T^d \in A_d(R/p^2) = A_d(X)$ such that $f(h(T)) \in A_{d-1}(U)$.

We just need to find a solution $\lambda = c_d/c_1^d$. To do this let c_1 be any unit in R/pR and take $c_d = c_1^d * \lambda$. $\square$

This result implies the following:

$$\eta = 1 \in H^1(\{U, V\}, A_d) \text{ and } \eta \in H^1(\{U, V\}, A_{d-1}) \implies \eta = 1 \in H^1(\{U, V\}, A_{d-1}).$$

The contrapositive is something like the injectivity result we are interested in: if a cohomology class is nontrivial in $\check{H}^1(X, A_{d-1})$ then it is nontrivial in $\check{H}^1(X, A_d)$.

Corollary 8.4. *Suppose that X/R_1 is proper and can be covered by two affine open subset with nonempty intersection*

(1) *For all* $d \geq 3$ *the map*
$$\check{H}^1(X, A_{d-1}) \to \check{H}^1(X, A_d)$$
is an injective morphism of pointed sets.

(2) *For all* $d > 2$ *the map*
$$\check{H}^1(X, A_2) \to \check{H}^1(X, A_d)$$
is an injective morphism of pointed sets.

Observe that this does not contradict our main result but comes very close. Since $\check{H}^1(\{U, V\}, A_d)$ is not a group it is possible for the map
$$\check{H}^1(\{U, V\}, A_2) \to \check{H}^1(\{U, V\}, A_d)$$
to be injective as a map of pointed sets but not injective as a map of sets.

We have show that it is the possibile to have $\eta \neq \eta' \in H^1(\{U, V\}, A_2)$ (coming from the two A_2 structures on elliptic curves) having an equal image in $H^1(\{U, V\}, A_d)$ for some $d >> 2$.

Our situation for elliptic curves is such that $[\Sigma_{plane}] \neq [\Sigma_{elliptic}]$ are nontrivial in $\check{H}^1(X, A_2)$ with $f([\Sigma_{plane}]) = f([\Sigma_{elliptic}])$ where $f : \check{H}^1(X, A_2) \to \check{H}^1(X, A_d)$ for some large d.

8.3. **A helpful example.** We will be concerned with 6 groups $G_1, G_2, G_{12}, H_1, H_2, H_{12}$ be groups such that

$$\begin{aligned} G_1, G_2 &\leq G_{12}, \\ H_1, H_2 &\leq H_{12}, \\ H_i \leq G_i, H_{12} &\leq G_{12} & i = 1, 2 \\ G_i \cap H_{ij} &= H_i, & i = 1, 2 \quad j = 1, 2 \end{aligned} \tag{8.1}$$

(8.2)

We consider $G_1 \times G_2$ to act on G_{12} via $((g_1, g_2), g_{12}) \mapsto g_1 g_{12} g_2^{-1}$ and similarly for H_{12} and $H_1 \times H_2$.

Problem: Find an example of six groups as above where the inclusion
$$\mathrm{Orb}_{H_1 \times H_2}(h_{12}) < \mathrm{Orb}_{G_1, G_2}(h_{12}) \cap H_{12}.$$
is strict for some $h_{12} \in H_{12}$.

Remark 8.5. Fixing G_1, G_2, G_{12} and H_{12} fixes the problem. Since $G_1 \cap H_{12} = H_1$ and $G_2 \cap H_{12} = H_2$.

Let $X = U_1 \cup U_2$ be a cover of a scheme $X/\mathbb{C}$. The six groups

$G_{12} = A_d(U_{12}), G_1 = A_d(U_1), G_2 = A_d(U_2), H_{12} = A_2(U_{12}), H_1 = A_2(U_1), H_2 = A_2(U_2)$

satisfy this properties (8.1). In Čech cohomology we have which
$$\check{H}^1(\mathcal{U}, G) = G_1 \backslash G_{12} / G_2.$$
$$\check{H}^1(\mathcal{U}, H) = H_1 \backslash H_{12} / H_2$$
Then the property
$$\mathrm{Orb}_{H1 \times H2}(h12) < \mathrm{Orb}_{G_1 \times G_2}(h_{12}) \cap H_{12}$$
is equivalent to the existence of h_{12} and h'_{12} in H_{12} such that $[h_{12}] \neq [h'_{12}]$ in $\check{H}^1(\mathcal{U}, H)$ and $[h_{12}] = [h'_{12}]$ in $\check{H}^1(\mathcal{U}, G)$.

8.4. **Another helpful example.** The following example was suggested to me by Buium. Let $R < S$ be commutative rings

- $G_{12} = \mathrm{SL}_2(S)$
- $H_{12} = \mathrm{SL}_2(R)$
- $G_1 = \{\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in S\}$
- $G_2 = \{\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} : s \in S\}$
- $H_1 = \{\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} : r \in R\}$
- $H_2 = \{\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} : r \in R\}$

Let $h \in H_{12}$. We let $[h]_H$ denote the class of h in $H_1 \backslash H_{12}/H_2$ and $[h]_G$ denote the class of h in $G_1 \backslash G_{12}/G_2$. We claim that we can find some R and S such that the map $\pi : H_1 \backslash H_{12}/H_2 \to G_1 \backslash G_{12}/G_2$ has the property that

(1) $\pi^{-1}(\text{ trivial class}_G) = \text{trivial class}_H$
(2) There exists a class $[h]$ such that $\#\pi^{-1}(\pi([h]_H) > 1$

The following proves the first part. Suppose that $h \in H_{12}$, as above, has

$$\pi([h]_H) = [1_G]$$

then $[h]_H = [1_G]$

If $\pi([h]) = 1$ there exists some $U = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ in the upper and lower triangular matrices such that

$$h = UL = \begin{bmatrix} 1+ab & a \\ b & 1 \end{bmatrix}$$

which implies that $a \in R$ and $b \in R$ which shows that h is in $G_1 \setminus G_{12}/G_2$ then h is trivial in $H_1 \setminus H_{12}/H_2$. In other words the map π is injective on the level of pointed sets. Let

Let

$$U = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in G_1,$$
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H_{12},$$
$$L = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \in G_2.$$

Let

$$B := UAL.$$

B has entries

$$\begin{aligned} B_{11} &= a + by + x(c+dy), \\ B_{12} &= b + dx, \\ B_{21} &= c + dy, \\ B_{22} &= d. \end{aligned} \tag{8.3}$$

We now specialize to the case $R = \mathcal{O}_K$,
$S = K$ where $K/\mathbb{Q}$ is a number field and $\mathcal{O}_K$ is its ring of integers. If we let $x = 1/d$ and $y = 1/d$ then we need

$$b/d + c/d + d/d^2 = (b + c + 1)/d \in \mathcal{O}_K$$

to have $B \in H_{12}$. So, any solution (a, b, c, d) of

$$\begin{cases} ad - bc = 1 \\ c + b + 1 \in d\mathcal{O}_K \end{cases} \tag{8.4}$$

gives a matrix A such that $UAL \in H_{12} = \mathrm{GL}_2(\mathcal{O}_K)$ where

$$U = \begin{bmatrix} 1 & 1/d \\ 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 1/d & 1 \end{bmatrix}.$$

Specialize to the case were d is a prime and consider the equations above mod d:

$$\begin{aligned} bc &\equiv -1 \mod d \\ b + c &\equiv -1 \mod d \end{aligned}$$

which gives $b(-1-b) \equiv -1 \implies -b^2 - b + 1 \equiv 0 \mod d$ or $b^2 + b - 1 = 0 \mod d$. If $b = (-1-\sqrt{5})/2 \in \mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{5})$ so that $\mathcal{O}_K = R = \mathbb{Z}[b]/\langle b^2 + b - 1\rangle = \mathbb{Z}[(-1+\sqrt{5})/2]$ we want d is a prime that splits completely in K then this congruence holds. If we let

$$\begin{aligned} b &= (-1-\sqrt{5})/2, \\ c &= (-1+\sqrt{5})/2 - dc', \\ a &= -c'(1+\sqrt{5})/2, \\ d &= \text{ a large prime} \end{aligned}$$

where $c' \in \mathbb{Z}$ we can check that $ad - bc = 1$ and $c + b + 1 = dc' \in d\mathcal{O}_K$ This then gives as in equation 8.3,

$$\begin{aligned} B_{11} &= a + by + cx + dxy = a + (b + c + 1)/d = a + c', \\ B_{12} &= b + dx = b + 1, \\ B_{21} &= c + dy = c + 1, \\ B_{22} &= d, \end{aligned}$$

So $B \in H_{12} = \mathrm{SL}_2(\mathcal{O}_K)$.

Here is what we have now: we can choose d and c' so that

$$B' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H_{12}$$

and

$$B = \begin{bmatrix} a + c' & b + 1 \\ c + 1 & d \end{bmatrix} \in H_{12}$$

and it are not in the same equivalence class in $H_1 \backslash H_{12}/H_2$.

Suppose that $B = UB'L$ where U and L have coefficients in $\mathcal{O}_K$. Any two members of the same equivalence class are related by equation 8.4. In particular note that if d is a particularly large number then B_{21} differs from $A_{21} = c$ by dy, a very large number. But we note that $|B_{21} - B'_{21}| = 1$, which is a contradiction.

9. Appendix: superelliptic curve computations

In this section we

- Compute étale morphisms which give the A_2-structure Σ_{super} on the $J^1(C)_1$ for C a superelliptic curve,
- Compute a representative of the right twisted cocycle $\tau_2(\Sigma_{super}) \in \check{H}^1(E_0, \mathcal{O} \ltimes \mathcal{O}^\times)$ as a (m_{ij})-twisted cocycle where $[m_{ij}] = F^* T_{C_0}$.

Section 9.5 sets up a nice cover on which we will do computations. Section 9.6 computes transition maps with respect to this cover and section 9.7 gives an explicit presentation of the class associated to Σ_{super} on a superelliptic curve. Section 9.8 computes the pairing of $[\Sigma_{super}]$ with τ_2 to give $\kappa(\Sigma_{super}, \tau_2)$.

9.1. **Superelliptic curves.** Let $K = \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ and $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. A Curve C/K is **superelliptic** if there exists some n such that $\mathbb{Z}/n\mathbb{Z} = G \hookrightarrow \mathrm{Aut}(C)$ such that $C/G \cong \mathbb{P}^1$. If C/K is superelliptic then there exists some $y \in k(C)$ and $e \geq 0$ such that $y^e = F(x) \in K(\mathbb{P}^1) = K(x)$. This means that C/K is birational to $V(e^N - F(x)) \subset$. Let $F \in \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[x]$ be a polynomial of degree d with $p \nmid d$ such that $(F, F') = (1)$ and let $e \mid d$. One can construct a smooth proper for a superellipic curve C/R as follows: first let $\widetilde{F}(t) = t^d F(1/t)$ and define two affine schemes

$$U = \mathrm{Spec}(R[x,y]/(y^e - F(x))), \quad V = \mathrm{Spec}(R[u,v]/(v^e - \widetilde{F}(u))).$$

and glue U and V using $(u, v) = (x^{-1}, yx^{-d/e})$ on the set $D(u) \sim D(x)$, so that

$$C_R := \frac{U \sqcup V}{\sim}. \tag{9.1}$$

When its reduction mod p smooth, the resulting scheme C_R is a smooth proper model of C/K.

Remark 9.1. The standard projective plane model $\mathrm{Proj}(R[X,Y,Z]/(Y^e Z^{d-e} - Z^d F(X/Z)) \subset \mathbb{P}^2_R$ can be singular at infinity.

From now on the term "superelliptic curve" will refer to a model $C = C_R/R$ as above. If C/R is smooth superelliptic then the genus of C is

$$g = \frac{(d-2)(e-1)}{2}.$$

This is just an application of Riemann-Hurwitz applied to that map $C \to \mathbb{P}^1_R$ induced by y. The ramified points are correspond to the zeros of $F(x)$; they each have ramification degree e and since there are d of them we have $2g - 2 = e(-2) + d(e-1)$ which gives the result.

Remark 9.2.
1. Note that if C is plane then $g(C) = \binom{n-1}{2}$ for some n. So C can be plane when $d = e$. We will consider this special case later.
2. There exist superelliptic curves of every genus.
3. The collection of superelliptic curves contains all hyperelliptic curves.

9.2. **A_2-structures for superelliptic curves.** In this section $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Let C/R be a superelliptic curve as in the previous subsection. In what follows we will let $U_1 = D(y)$, $U_2 = D(F'(x))$, $V_1 = D(v)$ and $V_2 = D(\tilde{F}'(u))$ which have étale maps to $\mathbb{A}^1$ induced by y, x, v and u respectively.

Proposition 9.3. *The transitions on the trivializing cover $\mathcal{U} = \{U_1, U_2, V_1, V_2\}$ of C give an A_2-atlas. For every $X, Y \in \mathcal{U}$ with canonical trivializations $\psi_X : \pi^{-1}(X) \to \widehat{X} \hat{\times} \widehat{\mathbb{A}}^1_R$ and $\psi_Y : \pi^{-1}(Y) \to \widehat{Y} \hat{\times} \widehat{\mathbb{A}}^1_R$ we have*

$$(\psi_X \circ \psi_Y^{-1}) \otimes_R R_1 \in A_2(X \cap Y). \tag{9.2}$$

Proof. By Lemma 4.8 we know that $\psi_{U_1U_2}$ and $\psi_{V_1V_2}$ are in A_2. Observe that $\{\psi_{U_1V_1}, \psi_{U_1U_2}, \psi_{V_1V_2}\}$ is a spanning set for our transitions in the sense of section 4.4. It remains to show $\psi_{U_1V_1} \otimes R_1 \in A_2(U_1 \cap V_1)$. To do this we need to show that $\dot{x}$ and $\dot{u}$ are related by a degree two polynomial mod p^2. Since $u = 1/x$ this means $\dot{u} = \dot{x}/x^p(x^p + p\dot{x}) = \dot{x}/x^{2p}(1 + p\dot{x}/x^p)$ or

$$\begin{aligned} \dot{u} &= \frac{\dot{x}}{x^{2p}} \sum_{j \geq 0} (-1)^j (p\frac{\dot{x}}{x^p})^k \\ &\equiv \frac{\dot{x}}{x^{2p}}(1 - p\frac{\dot{x}}{x^p}) \mod p^2 \end{aligned}$$

which has degree two.

□

Σ_{super} is defined to be the A_2-structure containing the trivializations $(\psi_{U_1}.U_1)$,$(\psi_{U_2}.U_2)$,(ψ_{V_1}, V_1) and (ψ_{V_2}, V_2).

9.3. **Projective plane curves: Σ_{plane}.** Let $C \subset \mathbb{P}^2_R$ be a smooth plane curve over Spec(R) given by

$$C = \mathrm{Proj}(R[X, Y, Z]/(F(X, Y, Z))).$$

We will cover C by the standard affine opens C_1, C_2 and C_3 obtained by removing plane sections corresponding to X,Y and Z respectively. We will then cover each C_i by open sets U_i, V_i;

$$C_i = U_i \cup V_i$$

Where both U_i and V_i admit an étale map to $\mathbb{A}^1$. The curve C_1 will have coordinated $x = X/Z, y = Y/Z$, and its defining equation is $f(x, y) = F(x, y, 1) = 0$. The cover will consist of $U_1 = \{\partial f/\partial x \neq 0\}$ and $V_1 = \{\partial f/\partial y \neq 0\}$ and on these open sets y and x will be the étale coordinates respectively. We similarly define C_2 with coordinates $u = X/Y$ and $v = Z/Y$ and equation $g(u, v) = F(u, 1, v) = 0$ so that v is an étale coordinate on U_2 and u is an étale coordinate on V_2 and define C_3 with coordinates $s = Y/X$ and $t = Z/X$ with $h(s, t) = F(1, s, t) = 0$ and U_3 so that t is étale and V_3 so that s is étale.

Proposition 9.4. *Let C be a smooth plane curve with trivializing cover $\mathcal{U} = \{U_1, V_1, U_2, V_2, U_3, V_3\}$ as above. For each $U, V \in \mathcal{U}$ the trivializations $\psi_U : J^1(U) \to \widehat{U} \times \widehat{\mathbb{A}}^1$ and $\psi_V : J^1(V) \to \widehat{V} \times \widehat{\mathbb{A}}^1$ are A_2 compatible.*

Proof. We want to show that the trivializations over the various open sets in $\mathcal{U}$ give rise to a A_2-structure. That is for U and V let $\psi_{UV} = \psi_U \circ \psi_V^{-1}$. We need to show that for every U and V in $\mathcal{U}$ that

$$\psi_{UV} \otimes_R R_1 \in A_2(U \cap V).$$

The amounts to showing that for any two distinct U and V in $\mathcal{U}$ with étale coordinates ξ and η (from x, y, u, v, s, t) respectively we can write $\dot{\eta} = H(\dot{\xi}) \mod p^2$ in $\mathcal{O}^1(U \cap V)_1$ where H is a polynoimal of the form $a + b\dot{\xi} + pc\dot{\xi}^2$ for $a, b \in \mathcal{O}_{C_1}(U \cap V)$ and $c \in \mathcal{O}_{C_0}(U \cap V)$.

From Lemma 4.8 we have that $\psi_{U_iV_i} \in A_2(U_i \cap V_i)$. To complete a spanning set it is sufficient to show $\psi_{V_1V_3}$ and $\psi_{U_1U_2}$ are in A_2:

- The étale coordinates on V_1 and V_3 are x and t respectively. Since $t = 1/x$ we have
$$\begin{aligned} \dot{t} &= \delta(1/x) \\ &= -(\dot{x}/x^{2p})(1 - p\dot{x}/x^p) \\ &= -\dot{x}/x^{2p} + p\dot{x}^2/x^{3p} \end{aligned}$$
which implies $\psi_{V_3V_1} \in A_2(V_1 \cap V_3)$.
- The étale coordinates on U_1 and U_2 are y and v and $y = 1/v$. The computation is the same as in the previous bullet.

□

9.4. **Superelliptic curves as plane curves.** Consider the affine plane curve defined by
$$y^d - f_d(x) = 0$$
where f_d is a polynomial of degree d. We have discussed two ways to to get a proper model for this curve

- As a projective curve
$$\mathrm{Proj}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[X,Y,Z]/(Y^d - f_d(X/Z)Z^d))$$
- As a superelliptic curve
$$\frac{U \sqcup V}{\sim},$$
with $U = \mathrm{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[x,y]/(y^d - f_d(x)))$, $V = \mathrm{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[u,v]/(v^d - f_d(1/u)u^d)$ and $x = 1/u$ and $y = v/u^{e/d} = v/u$ on $U \cap V$.

The sets U and V identify with standard affine open sets of $\mathrm{Proj}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[X,Y,Z]/(Y^d - f_d(X/Z)Z^d))$ and the gluing is exactly the gluing of the two affine open subsets of the plane curve. The atlas Σ_{plane} was induced by transitions between six charts where partial derivatives of local equations are not vanishing. Setting $X = 1$ in the homogeneous polynomial $Y^d - f_d(X/Z)Z^d$ gives the an affine piece
$$Y^d - f_d(1/X)Z^d = 0.$$
We called this chart C_3 in section 9.3. The superelliptic chart obtained by setting $x = 1/u$ and $y = v/u$ gives
$$V : v^d - f_d(1/u)u^d = 0.$$
This shows that $V = C_3$. We also know trivially that $C_1 = U$. This shows the following

Lemma 9.5. *With the notation as above*
$$\mathrm{Proj}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[X,Y,Z]/(Y^d - f_d(X/Z)Z^d)) \cong \frac{U \sqcup V}{\sim}$$
as schemes over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$.

In particular, the atlas for superelliptic curves is a sub-atlas for plane curves $\mathcal{U}_{super} \subset \mathcal{U}_{plane}$ so Σ_{super} is the same if a superelliptic curve is viewed as a plane curve or as an abstract curves obtained by a gluing procedure.

9.5. **Covers for computing.** The general idea is to find a nice cover of C by two open sets to make Čech cohomology computations easy in the future. Recall that our superelliptic curves are obtained by gluing two affine open subsets together. We will be using the same notation as in section 9.

Proposition 9.6. *Let $C/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ be a superelliptic curve such that $F(0), F'(0), \widetilde{F}(0), \widetilde{F}'(0)$ are all nonzero then $\{D(F'(x)), D(\widetilde{F}'(u)\}$, $\{D(F'(x)), D(v)\}$, $\{D(y), D(\widetilde{F}'(u))\}$ are all trivializing covers for $J^1(C) \to \widehat{C}$.*

Proof. It is enough to prove two open sets cover $C_0 = C \mod p$. Indeed, if U, V cover C_0 then they cover C by properness of $C \to \mathrm{Spec}(\widehat{\mathbb{Z}}_p^{\mathrm{ur}})$ so without loss of generality we work with $C_0(\overline{\mathbb{F}}_p)$ as a variety over $\overline{\mathbb{F}}_p$.

It is helpful to think of $U = V(y^e - F(x)) \subset \mathbb{A}^1(\overline{\mathbb{F}}_p)$ as the "main part" of a complete curve and $V = V(v^e - \widetilde{F}(u)) \subset \mathbb{A}^2(\overline{\mathbb{F}}_p)$ as the "chart at infinity' that completes U. Here $(U_0 \setminus V_0)(\overline{\mathbb{F}}_p)$ will contain only a finite set of points. For a collection $\{U', V'\}$ (we will drop the functor of points notation from now on) to form an open cover we need to check two things

(1) V' contains the points at infinity:
$$(\text{points at } \infty) = V' \setminus (U' \cap V').$$
(2) V' contains the points deleted from U', where
$$(\text{points deleted from } U' \ ) = U \setminus U'.$$

We will first find the points at infinity and the points deleted from U in each case. First note that
$$(\text{points at } \infty) = \{Q_j = (0, b_j) = (0, \zeta_e^j \sqrt[e]{\widetilde{F}(0)}), \text{ for } j = 0, 1, \ldots, e-1\}.$$
This is because $U \cap V$ is the subset of V where $u \neq 0$, so $V \setminus (U \cap V)$ we have $u = 0$, which means $v^e = \widetilde{F}(0)$.

We claim that both $D(v)$ and $D(\widetilde{F}'(u))$ contain all of the Q_j's. We let $Q_j = (u_j, v_j)$ in uv-coordinates as above. If $Q_j \notin D(\widetilde{F}'(u))$ then $\widetilde{F}'(0) = 0$ — but this was assumed not to be the case when we defined our superelliptic curves (see 9). If $Q_j \notin D(v)$ then $v_j = 0$ which means $\widetilde{F}(0) = 0$ a contradiction.

It remains to check the deleted points. If our cover is using $U' = D(F'(x))$ then there are $e(d-1)$ deleted points:
$$(\text{points deleted from } U \ ) = \{P_{ij} = (x_i, y_{ij}) = (x_i, \zeta_e^j \sqrt[e]{F(x_i)})\},$$
where $x_1, \ldots, x_{d-1}$ are the $d-1$ solutions of $F'(x) = 0$. Since $P_{ij} = (u_i, v_{ij}) = (x_i^{-1}, \zeta_e^j \sqrt[e]{F(x_i)} x_i^{-d/e})$ and our assumptions give $x_i \neq 0$ (since 0 isn't a root of $F'(x)$) we have have $P_{ij} \in V$. We claim that for all i and j, $P_{ij} \in D(v) \cap D(\widetilde{F}'(u))$.

Note that
$$\begin{aligned} \frac{d}{du}[\widetilde{F}(u)] &= \frac{d}{du}[u^d F(1/u)] \\ &= du^{d-1} F(1/u) + u^{d-2} F'(1/u). \end{aligned} \tag{9.3}$$

- If $P_{ij} \notin D(\widetilde{F}'(u))$ then $\widetilde{F}'(x_i^{-1}) = 0$, which implies that $\widetilde{F}'(1/x_i) = dx_i^{d+1} y_{ij}^e$ since x_i is a root of $F'(x)$, which is a contradiction.
- If $P_{ij} \notin D(v)$ then $v_{ij} = 0$ we see that $F(x_i)$ needs to be equal to zero which is false since F doesn't have multiple roots.

This shows that $\{D(F'(x)), D(\widetilde{F}'(u)\}$ and $\{D(F'(x)), D(v)\}$ both form a cover of C.

We now move on to the case where our cover is using $U' = D(y)$. In this case our deleted points are $U \setminus D(y)$ which have $y = 0$ which means $0 = F(x)$. This gives:

$$(\text{points deleted from } U\ ) = \{T_j = (a_j, 0)\}_{j=1}^d.$$

If T_j not in $D(\widetilde{F}'(u))$ then $\widetilde{F}'(u_j) = 0$, where $u_j = 1/a_j$. But $\widetilde{F}'(u_j) = -a_j^{-d+2}F(1/a_j) = 0$ since a_j is a root of F. But because $a_j \neq 0$ and F has no repeated roots this can't be true. $\square$

9.6. **Transition computations.** Now consider a cover consisting of $U_1 = D(F'(x))$ and $U_2 = D(y)$ be as in the covering lemma (Lemma **??**). On the open sets $U_1 \cap U_2 \subset C$ both x and y give formally étale maps to $\widehat{\mathbb{A}}^1$ and the trivialization lemma (Lemma 4.2) we have

$$\mathcal{O}(U_1 \cap U_2)[\dot{y}]^{\hat{}} = \mathcal{O}^1(\widehat{U}_1 \cap \widehat{U}_2) = \mathcal{O}(U_1 \cap U_2)[\dot{x}]^{\hat{}}.$$

The following lemma decribes how $\dot{x}$ and $\dot{y}$ are related on $U_1 \cap U_2$. In what follows $R := \frac{F^\phi(x^p) - F(x)^p}{p}$ and $F^\phi(x) := \sum_{j=0}^d \phi(a_j)x^j$ for $F(x) = \sum_{i=0}^d c_i x^i$ where $\phi(a) = x^p + p\delta(a)$.

Lemma 9.7 (affine transitions)**.**

$$\dot{x} \equiv a + b\dot{y} + pc\dot{y}^2 \mod p^2 \tag{9.4}$$

where

$$\begin{aligned} a &= \frac{-R}{F^{\phi\prime}(x^p)} + p\frac{F^{\phi\prime\prime}(x^p)^2R^2}{2F^{\phi\prime}(x^p)^3} \\ b &= \frac{ey^{p(e-1)}}{F^{\phi\prime}(x^p)} - 2pey^{p(e-1)}\frac{F^{\phi\prime\prime}(x^p)R}{F^{\phi\prime}(x^p)^2} \\ c &= \frac{1}{2F^{\phi\prime}(x^p)}\left(e(e-1)y^{p(e-2)} + e^2y^{2p(e-1)}\frac{F^{\phi\prime\prime}(x^p)}{F^{\phi\prime}(x^p)^2}\right) \end{aligned}$$

We provide two proofs of the above formulas.

Proof using Quadratic Formula. Applying δ to the both sides of the equation $y^e = F(x)$ gives

$$ey^{p(e-1)}\dot{y} + p\frac{(e-1)e}{2}y^{p(e-2)}\dot{y}^2 \equiv R + F^{\phi\prime}(x^p)\dot{x} + p\frac{F^{\phi\prime\prime}(x^p)}{2}\dot{x}^2 \mod p^2. \tag{9.5}$$

Let

$$\begin{aligned} A &= pF^{\phi\prime\prime}(x^p)/2, \\ B &= F^{\phi\prime}(x^p), \\ C &= R - ey^{p(e-1)}\dot{y} - p\frac{(e-1)e}{2}y^{p(e-2)}\dot{y}^2. \end{aligned} \tag{9.6}$$

We can apply the quadratic formula to solve for $\dot{x}$

$$\dot{x} = \frac{2C}{B}\left[\sum_{j=0}^\infty \binom{1/2}{j+1}\left(-\frac{4AC}{B^2}\right)^j\right]$$

Note that B is invertible in our ring so the above formula makes sense as an element of $\mathcal{O}^1(U_1 \cap U_2)\widehat{\ }$, also note that $p | \frac{4AC}{B^2}$ so the series above makes sense and

$$\dot{x} \equiv \frac{2C}{B}\left[\binom{1/2}{1} - \binom{1/2}{2}\frac{4AC}{B^2}\right] \mod p^2$$

since $\binom{1/2}{1} = 1/2$ and $\binom{1/2}{2} = -1/8$,

$$\begin{aligned}
\dot{x} &\equiv \frac{C}{B}\left(1 + \frac{AC}{B^2}\right) \mod p^2 \\
&= \frac{R - ey^{p(e-1)}\dot{y} - p\frac{(e-1)e}{2}y^{p(e-2)}\dot{y}^2}{(F^{\phi\prime}(x^p))}\left(1 + \frac{(pF^{\phi\prime\prime}(x^p)/2)\cdot(R - ey^{p(e-1)}\dot{y} - p\frac{(e-1)e}{2}y^{p(e-2)}\dot{y}^2)}{(F^{\phi\prime}(x^p))^2}\right)
\end{aligned}$$

which after simplification gives our result. □

Remark 9.8. Note that we can obtain a full series solution for $\dot{x}$ in terms of $\dot{y}$ using this approach.

Proof in Hensel Style . Supposing that $\dot{x} = \dot{x}_0 + p\dot{x}_1 \mod p^2$ and substituting into equation 9.5 we get

(9.7)

$$ey^{p(e-1)}\dot{y} + p\frac{(e-1)e}{2}y^{p(e-2)}\dot{y}^2 = R + F^{\phi\prime}(x^p)\dot{x}_0 + p\left(F^{\phi\prime}(x^p)\dot{x}_1 + \frac{F^{\phi\prime\prime}(x^p)}{2}\dot{x}_0^2\right)$$

The above equation mod p yields,

$$\dot{x}_0 = \frac{R - ey^{p(e-1)}\dot{y}}{F^{\phi\prime}(x^p)} \mod p \tag{9.8}$$

Making this substitution into equation 9.7 gives us

$$p\left(\frac{e(e-1)}{2}y^{p(e-2)} - \dot{x}_1F^{\phi\prime}(x^p) + \frac{F^{\phi\prime\prime}(x^p)}{2}\dot{x}_0^2\right) = 0 \mod p^2,$$

which is equivalent to

$$\frac{e(e-1)}{2}y^{p(e-2)} - \dot{x}_1F^{\phi\prime}(x^p) + \frac{F^{\phi\prime\prime}(x^p)}{2}\dot{x}_0^2 = 0 \mod p.$$

Solving this equation for $\dot{x}_1$ gives

$$\dot{x}_1 = \left(e(e-1)y^{p(e-2)}\dot{y}^2 + F^{\phi\prime\prime}(x^p)\dot{x}_0^2\right)/2F^{\phi\prime}(x^p). \tag{9.9}$$

writing $\dot{x} = \dot{x}_0 + p\dot{x}_1$ and writing the result as $a + b\dot{y} + pc\dot{y}^2$ gives us

$$\begin{aligned}
a &= \frac{-R}{-F^{\phi\prime}(x^p)} + p\frac{F^{\phi\prime\prime}(x^p)}{2F^{\phi\prime}(x^p)}\frac{R^2}{F^{\phi}(x^p)^2} \\
b &= \frac{ey^{p(e-1)}}{F^{\phi\prime}(x^p)} + p\frac{F^{\phi\prime\prime}(x^p)}{F^{\phi\prime}(x^p)}\cdot\left(\frac{-R}{F^{\phi\prime}(x^p)ey^{p(e-1)}}\right) \\
c &= \frac{1}{2F^{\phi\prime}(x^p)}\left(e(e-1)y^{p(e-2)} + F^{\phi\prime\prime}(x^p)e^2y^{2p(e-1)}/F^{\phi\prime}(x^p)^2\right)
\end{aligned}$$

which after simplification gives our result. □

Remark 9.9.
- Observe that no p-derivations appear in the formula for C. This informs us that c is not "feeling" deformations in the "arithmetic direction"/lift direction.

- With a minor bit of effort one can obtain a full series (which we will not use) for $\dot{y}$ in terms of $\dot{x}$:

$$\begin{aligned}
\dot{y} &= \frac{y^p}{p}\sum_{l=0}^{\infty} B_l p^l \dot{x}^l, \\
B_l &= \sum_{j=1}^{\infty} \binom{1/e}{j} b_{lj} \\
b_{lj} &= \sum_{|\alpha|=l} a_\alpha, \\
& \alpha = (\alpha_1, \ldots, \alpha_j), \quad a_\alpha = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_j} \\
\left(\frac{\delta f}{f^p}\right)^j &= \sum_{l=0}^{\infty} \left(\sum_{|\alpha|=l} a_\alpha \right) p^l x^l
\end{aligned}$$

9.7. **The class associated to the Σ_{super}.**

Lemma 9.10. *Let C be a smooth superelliptic curve over $R = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, $W_1 = D(F'(x))$ and $W_2 = D(v)$. The transition map $\psi_{21} \in \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_1})(W_{12})$ is induced by*

$$\psi_{21}(T) \equiv a_{21} + b_{21}T + pc_{21}T^2 \mod p^2 \tag{9.10}$$

where

$$\begin{aligned}
a_{21} &= u^{2p}a - u^{3p}pa^2 \\
b_{21} &= u^{2p}b + 2pu^{3p}ab \\
c_{21} &= u^{2p}c - b^2u^{3p}
\end{aligned}$$

where a, b and c are the values computed in Lemma 9.7.

Proof. By Proposition 9.7 we have that $\dot{x} = a + b\dot{y} + pc\dot{y}^2 \mod p^2$ for some a, b and c. Since $x = 1/u$ we have

$$\dot{x} = \frac{-\dot{u}}{u^p(u^p + p\dot{u})}. \tag{9.11}$$

This gives

$$\begin{aligned}
\frac{-\dot{u}}{u^p(u^p+p\dot{u})} = a + b\dot{y} + pc\dot{y}^2 \mod p^2 \quad &\Longrightarrow \quad -\dot{u} = (u^{2p} + pu^p\dot{u})(a + b\dot{y} + pc\dot{y}^2) \mod p^2 \\
&\Longrightarrow \quad -[1 + pu^p(a + b\dot{y} + pc\dot{y}^2)]\dot{u} = u^{2p}(a + b\dot{y} + pc\dot{y}^2) \mod p^2 \\
&\Longrightarrow \quad \dot{u} = -\frac{u^{2p}(a + b\dot{y} + pc\dot{y}^2)}{1 + pu^p(a + b\dot{y} + pc\dot{y}^2)} \mod p^2.
\end{aligned}$$

Simplifying the final expression gives our result

$$\begin{aligned}
\frac{u^{2p}(a + b\dot{y} + pc\dot{y}^2)}{1 + pu^p(a + b\dot{y} + pc\dot{y}^2)} &= \frac{u^{2p}(a + b\dot{y} + pc\dot{y}^2)}{1 + pu^p(a + b\dot{y})} \\
&= [u^{2p}(a + b\dot{y} + pc\dot{y}^2)]\sum_{k=0}^{\infty}(-pu^p(a + b\dot{y}))^k \\
&= [u^{2p}(a + b\dot{y} + pc\dot{y}^2)](1 - pu^p(a + b\dot{y})) \\
&= (u^{2p}a - pu^{3p}a^2) + (u^{2p}b - 2pu^{3p}ab)\dot{y} + p(u^{2p}c - b^2u^{3p})\dot{y}^2
\end{aligned}$$

□

We now have

$$[\Sigma_{super}] = [a_{21} + b_{21}t + pc_{21}t^2] \in \check{H}^1(C_1, A_2).$$

9.8. **Explicit pairing part 1: getting a twisted cocycle.** Using proposition 9.10 one can show the following,

Proposition 9.11. *The class τ_2 is defined by*
(9.12)

$$\tau_2(\Sigma_{super})_{21} = \tau_2(a_{21}+b_{21}T+pc_{21}T^2) = \left[\frac{e-1}{y} + \frac{ey^{e-1}}{F'(x)}\left(\frac{F''(x)}{2F'(x)} - \frac{1}{x}\right)\right]^p \in \check{Z}^1(C_0, \mathcal{O}_{C_0} \ltimes \mathcal{O}_{C_0}^\times)$$

defines a twisted cocycle with values in $\mathcal{O}_{C_0}$ twisted by $\mathcal{O}_{C_0}^\times$ acting by multiplication.

Proof. From Proposition 9.10 and Lemma 9.7, where we used that $x = 1/u$ on $U \cap V$ we have

$$\begin{aligned} b_{21} &\equiv u^{2p}b \mod p \\ &\equiv \frac{ey^{p(e-1)}}{x^{2p}F^{\phi\prime}(x^p)} \mod p \\ &\equiv \left(\frac{ey^{e-1}}{x^2F'(x)}\right)^p \mod p. \end{aligned}$$

We also have

$$\begin{aligned} c_{21} &\equiv u^{2p}c - u^{3p}b^2 \\ &\equiv \frac{1}{x^{2p}}\left\{\frac{1}{2F^{\phi\prime}(x^p)}\left[e(e-1)y^{p(e-2)} + e^2y^{2p(e-1)}\frac{F^{\phi\prime\prime}(x^p)}{F^{\phi\prime}(x^p)^2}\right] - \frac{1}{x^p}\left[\frac{ey^{p(e-1)}}{F^{\phi\prime}(x^p)}\right]^2\right\} \\ &\equiv \frac{1}{x^{2p}}\left\{\left[\frac{e(e-1)y^{p(e-2)}}{2F^{\phi\prime}(x^p)} + \frac{e^2y^{2p(e-1)}}{F^{\phi\prime}(x^p)^2}\left(\frac{F^{\phi\prime\prime}(x^p)}{2F^{\phi\prime}(x^p)} - \frac{1}{x^p}\right)\right]\right\} \\ &\equiv \left\{\frac{1}{x^2}\left[\frac{e(e-1)y^{e-2}}{2F'(x)} + \frac{e^2y^{2(e-1)}}{F'(x)^2}\left(\frac{F''(x)}{2F'(x)} - \frac{1}{x}\right)\right]\right\}^p. \end{aligned}$$

This means that

$$\begin{aligned} \tau_2(a_{21} + b_{21}T + pc_{21}T^2) = \frac{c_{21}}{b_{21}} &= \left(\frac{x^2F'(x)}{ey^{e-1}}\right)^p \cdot \left\{\frac{1}{x^2}\left[\frac{e(e-1)y^{e-2}}{2F'(x)^2} + \frac{e^2y^{2(e-1)}}{F'(x)}\left(\frac{F''(x)}{2F'(x)} - \frac{1}{x}\right)\right]\right\}^p \\ &= \left[\frac{e-1}{y} + \frac{ey^{e-1}}{F'(x)}\left(\frac{F''(x)}{2F'(x)} - \frac{1}{x}\right)\right]^p. \end{aligned}$$

□

Remark 9.12. If we want to rationalize the denominator with respect to $y = \sqrt[e]{F(X)}$, this becomes
(9.13)

$$\tau_2(a_{21} + b_{21}T + pc_{21}T^2) = \left[\frac{2(e-1)F'(x)^2 + eF(x)(F''(x)x - 2F''(x))}{2xF'(x)^2F(x)}y^{e-1}\right]^p$$

References

[Bae] J. Baez. Torsors made easy. *http://math.ucr.edu/home/baez/torsors.html*.

[Bor09] J. Borger. Lambda-rings and the field with one element. *Arxiv preprint arxiv:0906.3146*, 2009.

[Bor11] J. Borger. The basic geometry of witt vectors. ii: Spaces. *Mathematische Annalen*, 351(4):877–933, 2011.

[BP09] A. Buium and B. Poonen. Independence of points on elliptic curves arising from special points on modular and shimura curves, i: Global results. *Duke Mathematical Journal*, 147(1):181–191, 2009.

[Bui] A. Buium. Fields of definition of algebraic varieties in characteristic zero. *Compositio Math.*, 61.

[Bui92] A. Buium. Intersections in jet spaces and a conjecture of s. lang. *The Annals of Mathematics*, 136(3):557–567, 1992.

[Bui94a] A. Buium. *Arithmetic Differential Equations*. Hermann, Paris, Frace, 1994.

[Bui94b] A. Buium. *Differential Algebra and Diophantine Geometry*. Hermann, Paris, Frace, 1994.

[Bui95] A. Buium. Differential characters of abelian varieties over p-adic fields. *Invent. Math.*, 122(2):309–340, 1995.

[Bui96] A. Buium. Geometry of p-jets. *Duke Mathematical Journal*, 82(2):349–367, 1996.

[Bui09] A. Buium. Geometry of differential polynomial functions, i: Algebraic groups. *American Journal of Mathematics*, 115(6):1385–1444, 2009.

[BW05] J. Borger and B. Wieland. Plethystic algebra. *Advances in Mathematics*, 194(2):246–283, 2005.

[CC11] A. Connes and C. Consani. On the notion of geometry over ff1. *J. Algebraic Geom*, 20:525–557, 2011.

[CC12] A. Connes and C. Consani. The universal thickening of the field of real numbers. *Arxiv preprint arXiv:1202.4377*, 2012.

[Con07] B. Conrad. *Various Approaches to p-adic geometry*. Arizona Winter School Lecture Notes, 2007.

[Den91] C. Deninger. On the γ-factors attached to motives. *Inventiones Mathematicae*, 104(1):245–261, 1991.

[Den92] C. Deninger. Local l-factors of motives and regularized determinants. *Inventiones mathematicae*, 107(1):135–150, 1992.

[Den94] C. Deninger. Motivic l-functions and regularized determinants. . *Proc. Symp. Pure Math.,*, 55(1):707–743, 1994.

[Dur07] Durov. *New Approach to Arakelov Geometry*. PhD thesis, 2007.

[Fre09] Fresán. Compactification projective de spec z (d'apres durov). Master's thesis, Max Plank Institute for Mathematics, 2009.

[Ful94] W. Fulton. *Intersection Theory, 2nd Edition*. Springer-Verlag, Berlin, 1994.

[Hul01] C. Hulburt. isogeny covariant differential modular forms modulo p. *Compos. Math.*, 128(1):17–34, 2001.

[Hur98] Chris Hurlburt. *Isogeny Invariant Differential Modular Forms*. PhD thesis, University of New Mexico, Albuquerque, United States, 1998.

[Hur03] C. Hurlburt. Zero loci of differential modular forms. *J. Number Theory.*, 98(1):47–54, 2003.

[Joy85] Joyal. -anneaux et vecteurs de witt. *C.R. Acad. Sci. Canada*, 7(3):177–182, 1985.

[Ked09] K. Kedlaya. Some new directions in p-adic hodge theory. *Journal de thèorie des nombres de Bordeaux*, 21(2):285–300, 2009.

[Kur92] N. Kurokawa. Multiple zeta functions: an example. *Adv. Stud. Pure Math*, 21:219–226, 1992.

[Lor] O. Lorscheid. The geometry of blueprints. part ii: Tits-weyl models of algebraic groups. *arXiv:1201.1324v1*.

[Lor12] O. Lorscheid. The geometry of blueprints. part i. *Advances*, 299(3):1804–1846, 2012.

[Mana] Y. Manin. Cyclotomy and the field with one element. *arXiv:0809.1566v1 [math.AG]*.

[Manb] Y. Manin. Lectures on zeta functions and motives (according to deninger and kurakowa). *Columbia University Number Theory Seminar*, 228.

[PD] L. Illusie P. Deligne. Revêtements modulo p^2 et d'ecomposition de complexe de de rham. *Invent. Math.*

[PL09] J.L. Peña and O. Lorscheid. Mapping f_1-land: An overview of geometries over the field with one element. *Arxiv preprint arXiv:0909.0069*, 2009.

[Ray] M. Raynaud. *Around the Mordell conjecture for function fields and a conjecture of Serge Lang*, volume 1016, pages 1–19. Springer.

[Sou04] C. Soulé. Les variétés sur le corps à un élément. *Mosc. Math. J*, 4(1):217–244, 2004.

[Tit] J. Tits. Sur les analogues algébriques des groups semi-simples complexes. *Colloque d'algèbre superérieure*, tenu á Bruxelles de 19 au 22 dècembre 1956.

[Toe] Vaquiè Toen. Au-dessous de spec z. *Journal of K-Theory*, 3.

[Vak] R. Vakil. A beginners guide to jet bundles from the point of view of algebraic geometry. *math.stanford.edu/ vakil/files/jets.pdf*.

[Yal] Bora Yalkinoglu. On arithmetic models and functoriality of bost-connes systems. with an appendix by sergey neshveyev. *Inventiones Mathematicae*, pages 1–43. 10.1007/s00222-012-0396-1.

Department of Mathematics and Statistics, MSC01 1115, 1 University of New Mexico, Albuquerque, New Mexico, 87131-0001

E-mail address: `dupuy@math.unm.edu`