The Arithmetic of Möbius Transformations in Dimension Four and Beyond

Taylor Dupuy
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The Basic Theory of Clifford-Bianchi Groups for Hyperbolic n-Space

Taylor Dupuy, Anton Hilado, Colin Ingalls, Adam Logan

Contents

FORD SPHERES IN THE CLIFFORD-BIANCHI SETTING

SPENCER BACKMAN, TAYLOR DUPUY, ANTON HILADO, VERONIKA POTTER

ABSTRACT. Let O be an order in $\left(\frac{-d_1,-d_2,\ldots,-d_{n-1}}{\mathbb{Q}}\right)$ with d_j positive square free integers. We show that the Ford Spheres $\mathcal{P}$ in $\mathcal{H}^{n+1}$ for Clifford-Bianchi groups $\mathrm{PSL}_2(O)$ are internally disjoint and integral. If we assume that O is Clifford-euclidean then $\mathcal{P}$ is also connected.

CONTENTS

other paper

Theorems

- New interesting analogs of Gaussian integers.
- Algorithms for fundamental domains
- Arithmeticity
- New "Clifford-Euclidean" examples

Theorems

In case of Clifford Euclidean orders packings are

- connected
- internally disj
- integral

Classical Story : PSL2(ZZ)

Higher Dimensions?

Hyperbolic Space and Clifford Algebras

Fundamental Domains

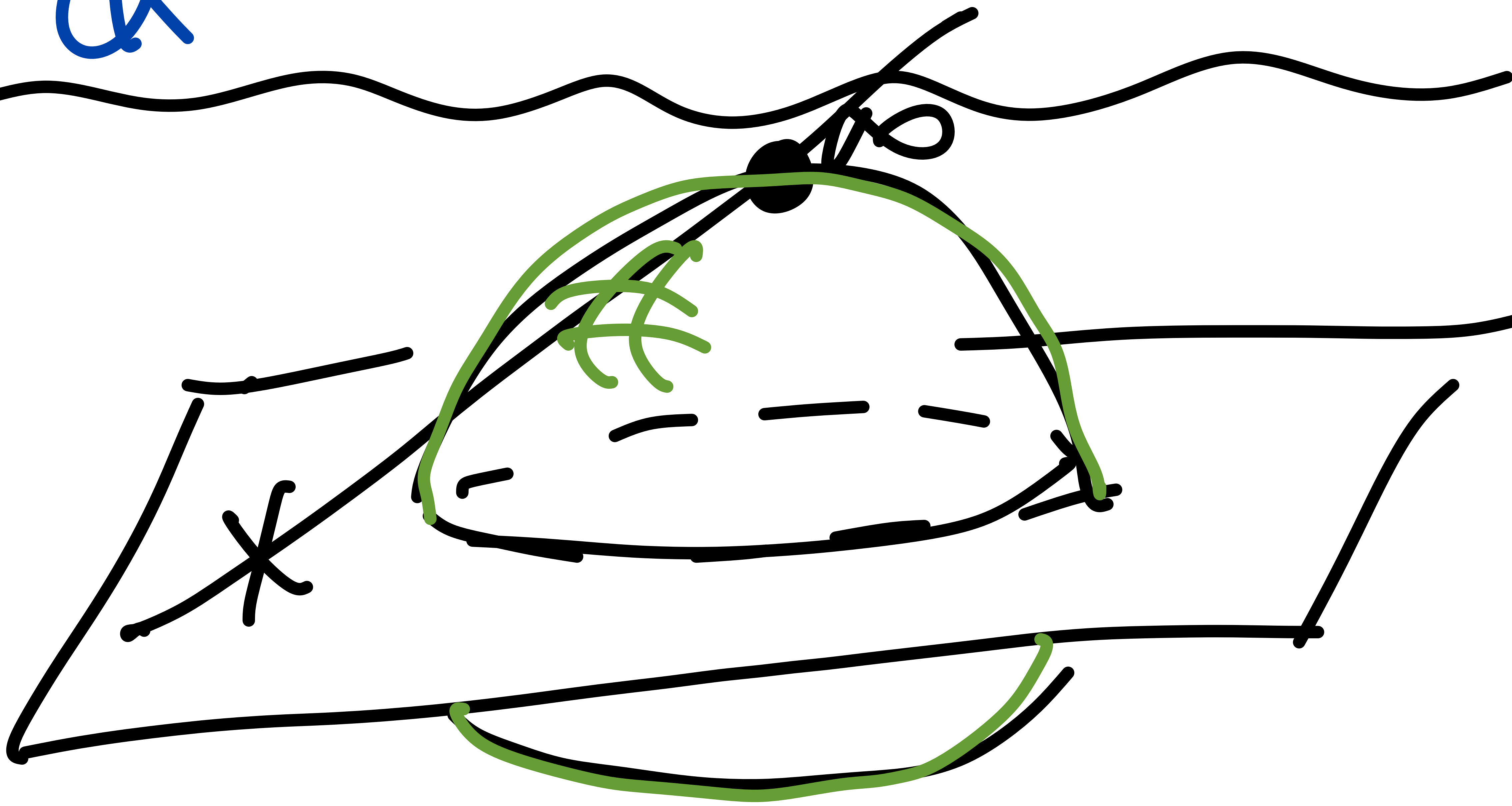
Classical Story : PSL2(ZZ)

Upper Half Space

$$\mathcal{H}^2 = \{x+iy : y>0\}$$

Riemann Sphere

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$



Recall:

$\mathrm{Aut}(\mathbb{P}^1) = PSL_2(\mathbb{C}) \longleftrightarrow$ Möbius Transformations $f(z) = \frac{az+b}{cz+d}$.

$\mathrm{Aut}(\mathcal{H}^2) = PSL_2(\mathbb{R})$

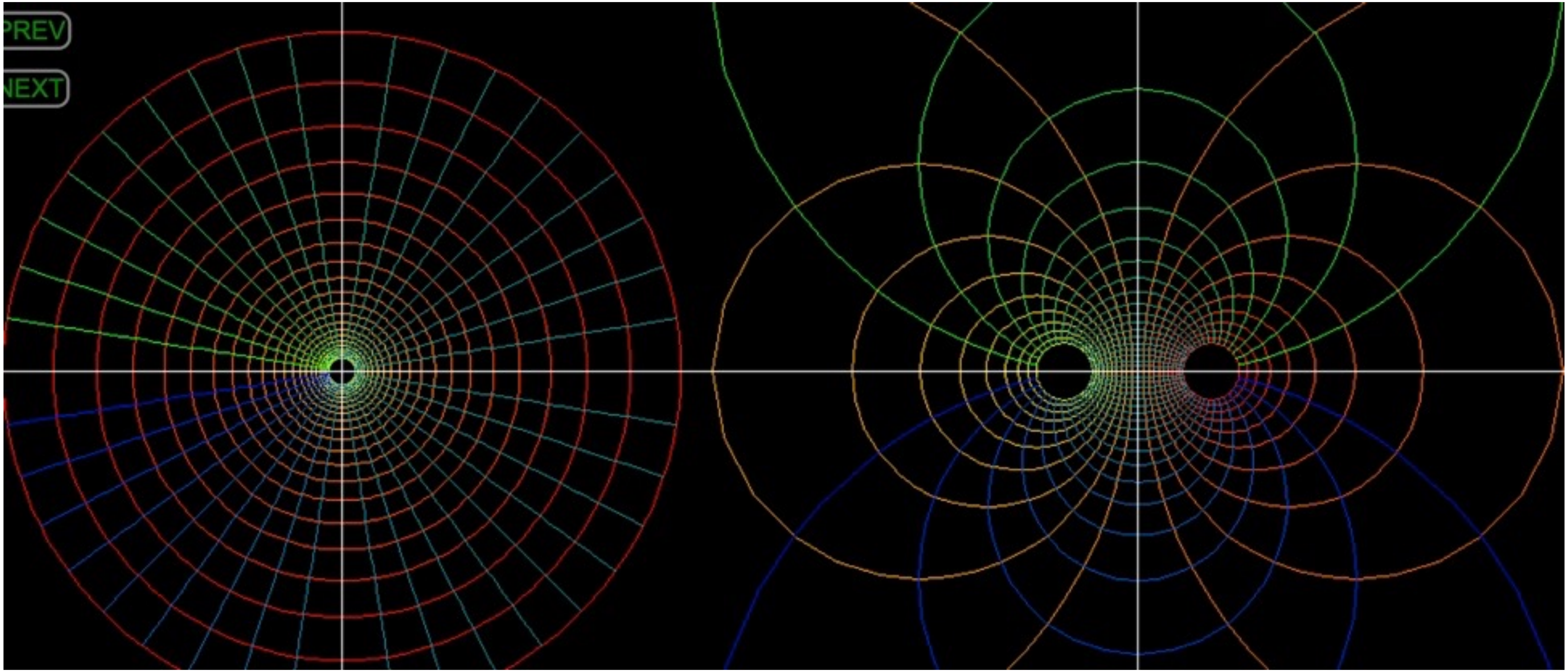
$\cup\!\mathrm{I}$

$PSL_2(\mathbb{Z}) = \langle S, T \rangle$,

$\boxed{\mathrm{orb}_{PSL_2(\mathbb{Z})}(\infty) = \mathbb{Q} \cup \{\infty\}}$

$S(z) = \frac{-1}{z}$, $T(z) = z+1$

$f(z) = (4z-1)/(4z+1)$



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The Carus Mathematical Monographs

Number Twenty-six

THE SENSUAL (quadratic) FORM

John H. Conway
Princeton University

Assisted by
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AFTERTHOUGHTS

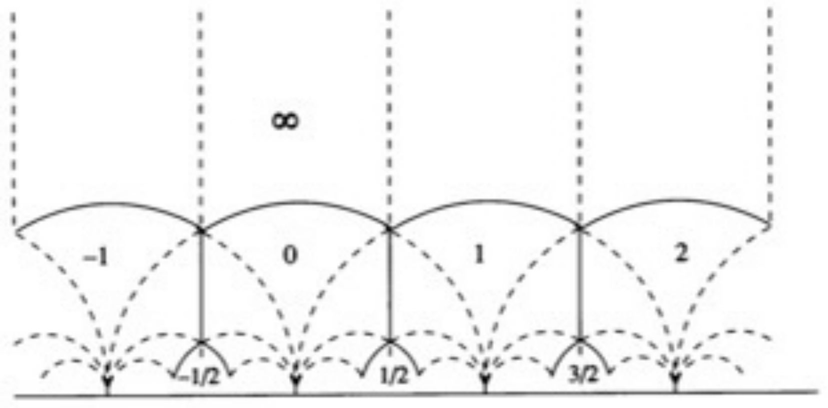
$\mathbf{PSL_2(Z)}$ and Farey Fractions

Introduction

The afterthoughts following our lectures will add more detail, introduce some related topics, or just put our ideas into some other context. We shall occasionally presume some knowledge of more standard treatments. The underlying subject of this lecture is the group $\mathbf{PSL_2(Z)}$, which can be regarded as the set of all maps

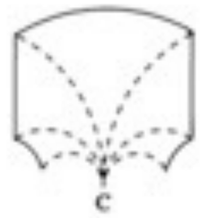
$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbf{Z}, ad-bc=1.$$

from the upper half-plane to itself. It is interesting to see how our topograph is drawn in the upper half plane $H = \{x+iy \mid y>0\}$.



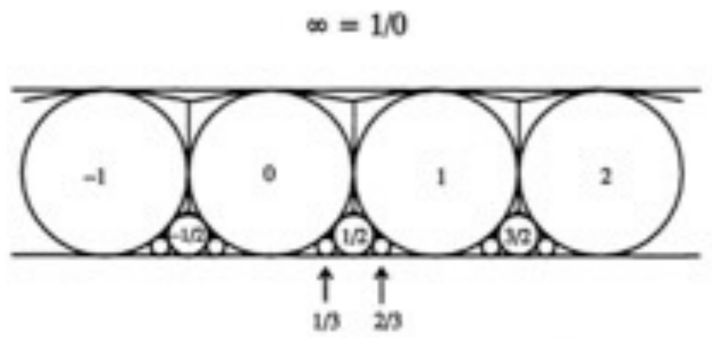
The picture shows H divided into fundamental regions for the group $\mathbf{PSL_2(Z)} = \Gamma$. The solid edges form a tree with three edges per vertex whose nodes and edges correspond to the superbases and bases for $\mathbf{Z}^+$. Each of the regions of our topograph consists of "fans" of fundamental regions.

We draw one such fan by itself:

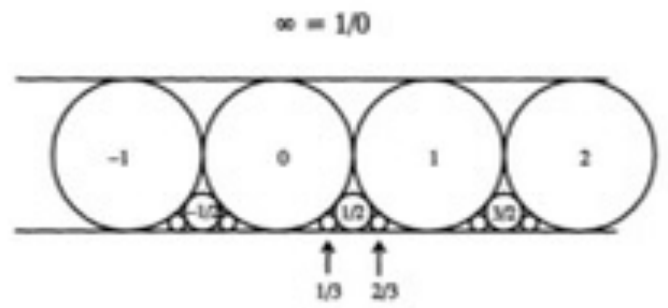


The fan labeled p/q is the face corresponding to the primitive vector (p,q) in $\mathbf{Z}^2$. It happens that the center C of this fan is the rational number p/q. Note that since $-p/-q$ is the same rational number as p/q, the primitive vectors $(-p,-q)$ and (p,q) automatically correspond to the same fan.

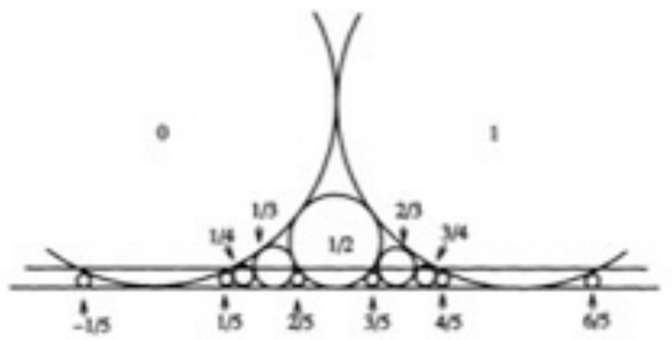
The geometry of this figure (which is really hyperbolic non-euclidean geometry) adds quite a bit to our knowledge. The fans have inscribed circles, which we now show:



and next we draw these circles by themselves:

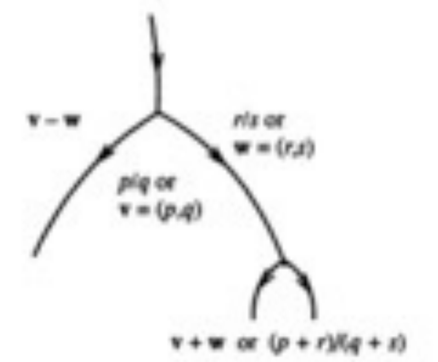


They are usually called *Ford circles*. The Ford "circle" for $\infty = 1/0$ is the horizontal line at height 1. The Ford circle for p/q is the circle of diameter $1/q^2$ in H that touches the real axis at p/q. The Farey series of order d consists of every rational number whose denominator is at most d. These correspond to the Ford circles that intersect any horizontal line L at height between $1/d^2$ and $1/(d+1)^2$. So for example from the line below, we get the Farey series of order 4, namely $\ldots \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \ldots$.



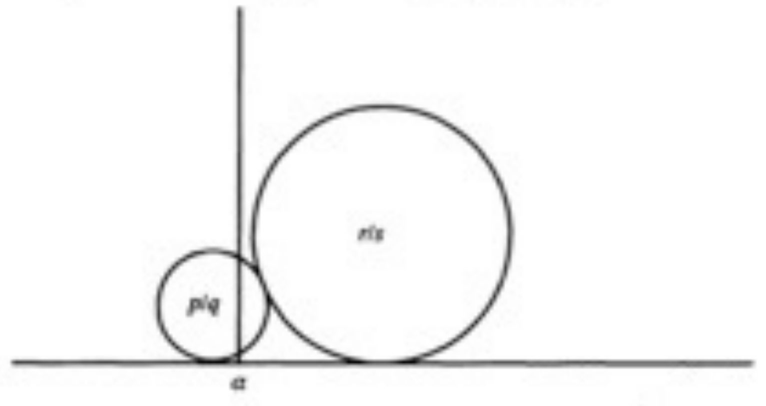
The "mediant" rule for Farey series tells us the first fraction that will appear between the adjacent Farey fractions p/q and r/s as the order is suitably increased; this is the *mediant fraction* $(p+r)/(q+s)$.

This mediant operation is more easily understood when one realizes that the fraction p/q really represents the vector (p,q). The situation is pictured below in the topograph.



This topograph shows that if the regions on either side of an edge have labels p/q and r/s, then those at the ends of that edge will have labels $(p \pm r)/(q \pm s)$. So the first fraction between p/q and r/s with larger denominator than those will indeed be $(p+r)/(q+s)$, their mediant.

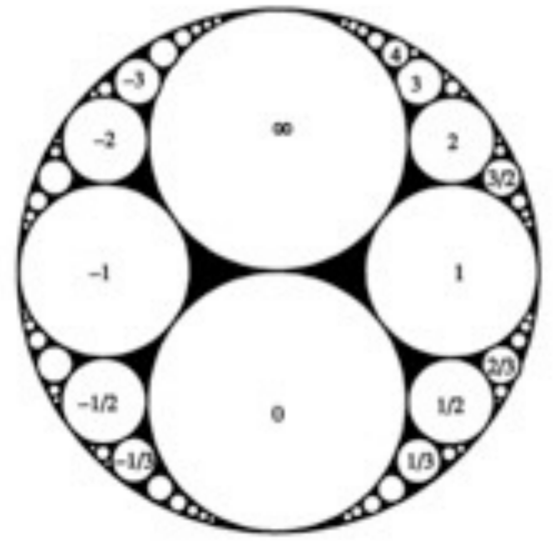
Some theorems of Diophantine approximation also become obvious. For example, for any irrational real number α, there are infinitely many rational numbers p/q for which $|\alpha - \frac{p}{q}| \leq 1/2q^2$.

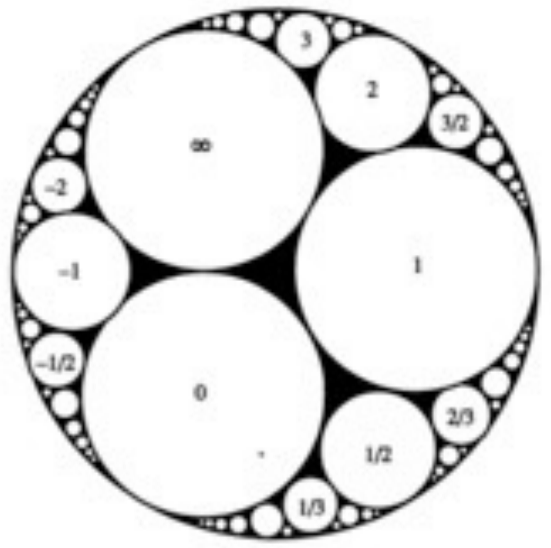


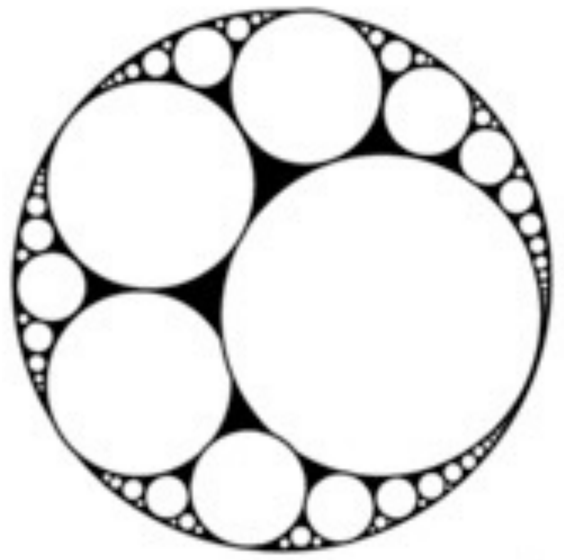
This is because if we take any two adjacent circles p/q and r/s whose tangent points with $\mathbf{R}$ lie on opposite sides of α then the vertical line through α must hit at least one of them. But if, as drawn in the figure, it hits the Ford circle for p/q, then $|\alpha - \frac{p}{q}| \leq 1/2q^2$.

The action of the groups $\mathbf{SL_2(Z)}$ and $\mathbf{GL_2(Z)}$ on the rationals becomes much easier to visualize if we apply the conformal map $z \mapsto \frac{z-i}{z+i}$ to change from the upper half-plane to the Poincaré disc, as in our next three figures. Now the Ford circles become circles tangent to the bounding disc. The group $\mathbf{GL_2(Z)}$ consists of all the symmetries of this figure, while $\mathbf{SL_2(Z)}$ consists only of the "rotational" ones.

The first of these figures exhibits the order 4 subgroup of $\mathbf{GL_2(Z)}$ generated by the operations $t \mapsto -t$ and $t \mapsto 1/t$, while the second one exhibits the order 6 subgroup generated by $t \mapsto 1/t$ and $t \mapsto 1-t$. In fact, $\mathbf{GL_2(Z)}$ is the free product of these two finite groups, amalgamated over their common subgroup of order 2. (If we pass to the rotation subgroups, we see how $\mathbf{SL_2(Z)}$ arises as the free product of groups of orders 2 and 3.)

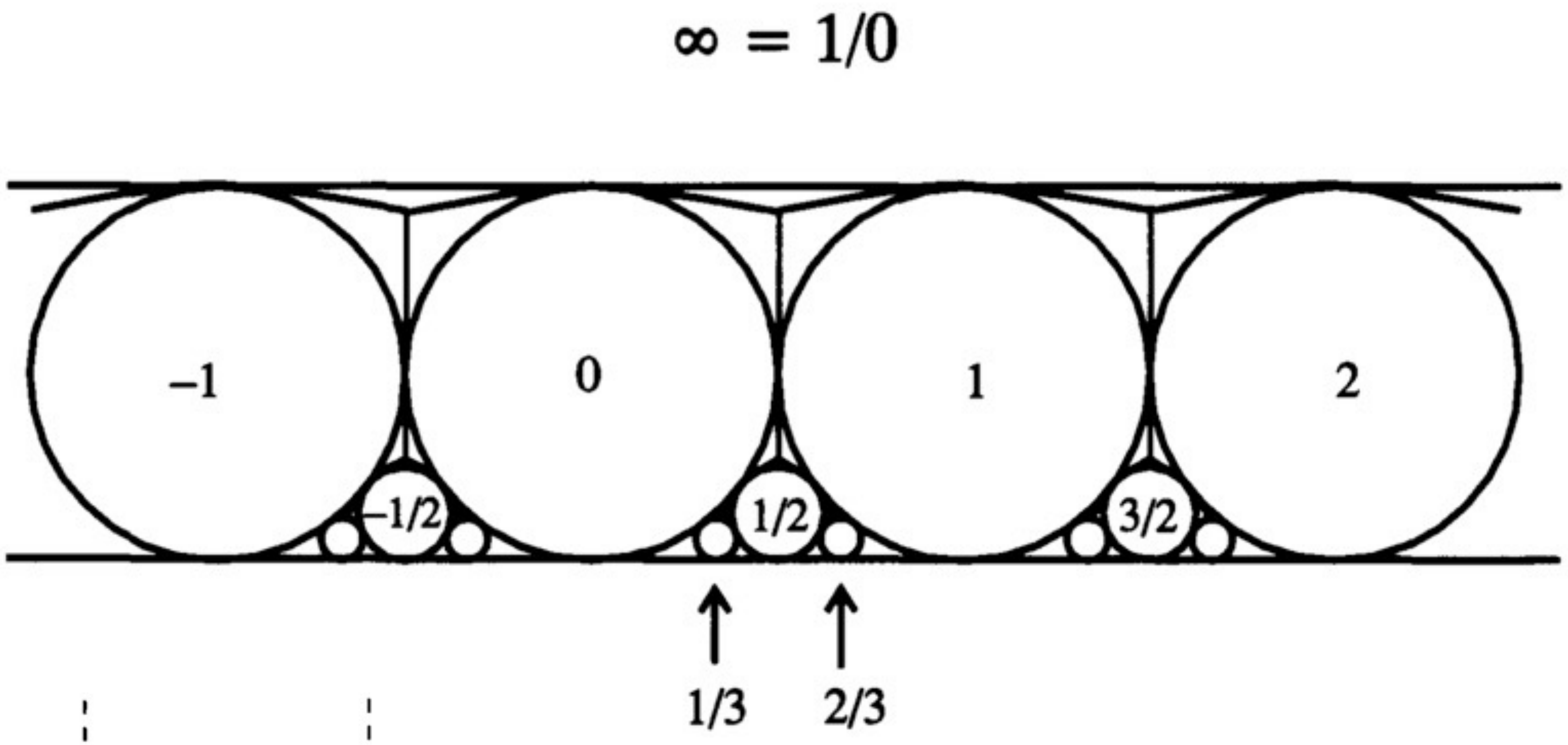


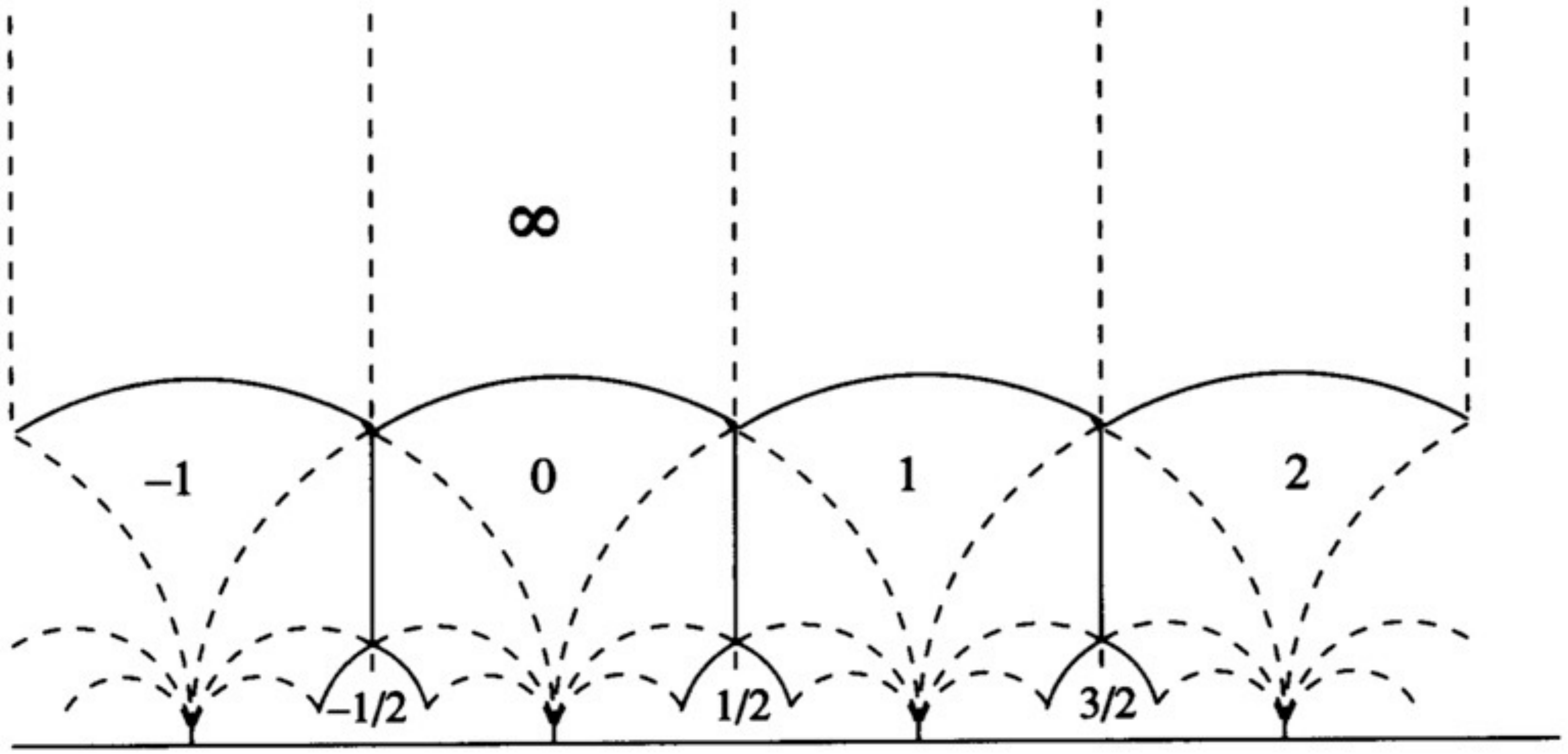




These two figures display some finite subgroups of $\mathbf{PGL_2(Z)}$ by Euclidean congruences. However, all the symmetries are represented by congruences in non-Euclidean geometry. In fact, $\mathbf{PGL_2(Z)}$ is the full symmetry group of a packing of the hyperbolic plane by circles of infinite radius (called *horocycles*). There is one horocyle for each rational number p/q, and it touches the boundary (identified with the real axis) at p/q. We have already seen this from three special viewpoints which exhibit particular subgroups; our last figure shows it from a viewpoint that has no special symmetry.

7 pages total.


∞ = 1/0
–1
0
1
2
–1/2
1/2
3/2
1/3
2/3


∞
–1
0
1
2
–1/2
1/2
3/2

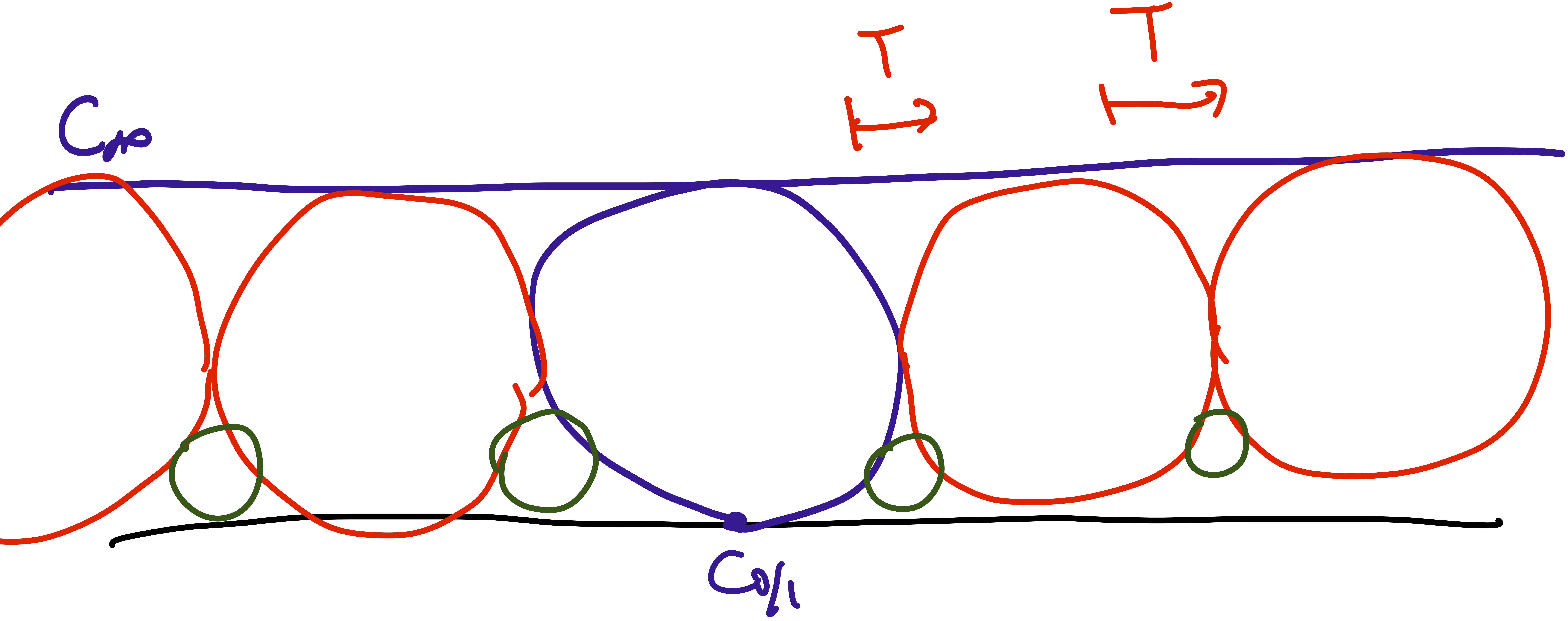
TWO WAYS
TO DEFINE
PACKING

Orbit under
$PSL_2(\mathbb{Z})$.

Normal Spheres

$\mathcal{H}^2 = \{ z \in \mathbb{C} : \operatorname{im}(z) > 0 \}$

Orbit under $PSL_2(\mathbb{Z})$.

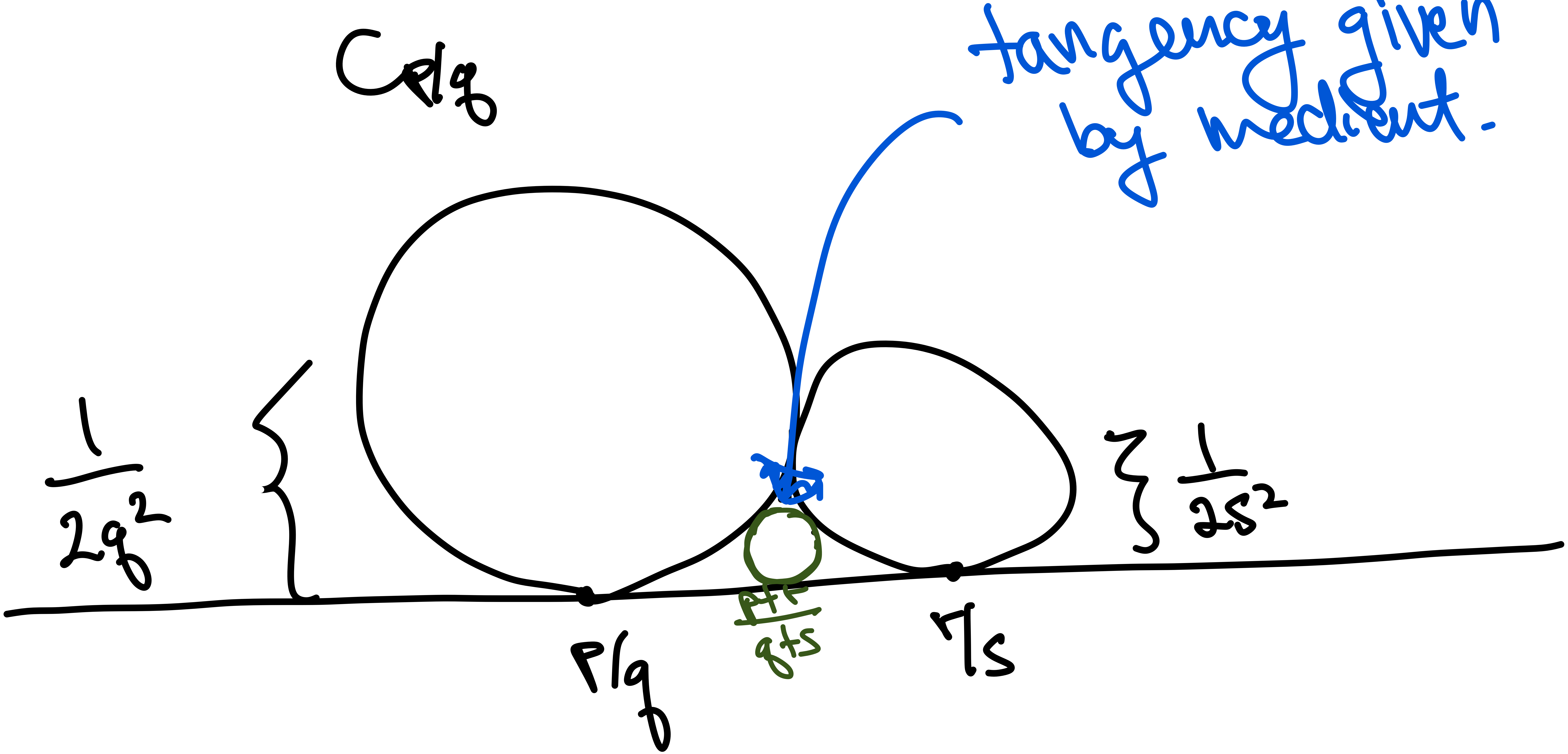


TWO WAYS TO DEFINE PACKING

→ Orbit under $PSL_2(\mathbb{Z})$.

→ Normal Spheres

Normal Spheres



Farey Fractions:

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$$

Farey Fractions:

level 2 $\quad \frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}$

level 3 $\quad \frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}$

level 4 $\quad \frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$

Farey Fractions:

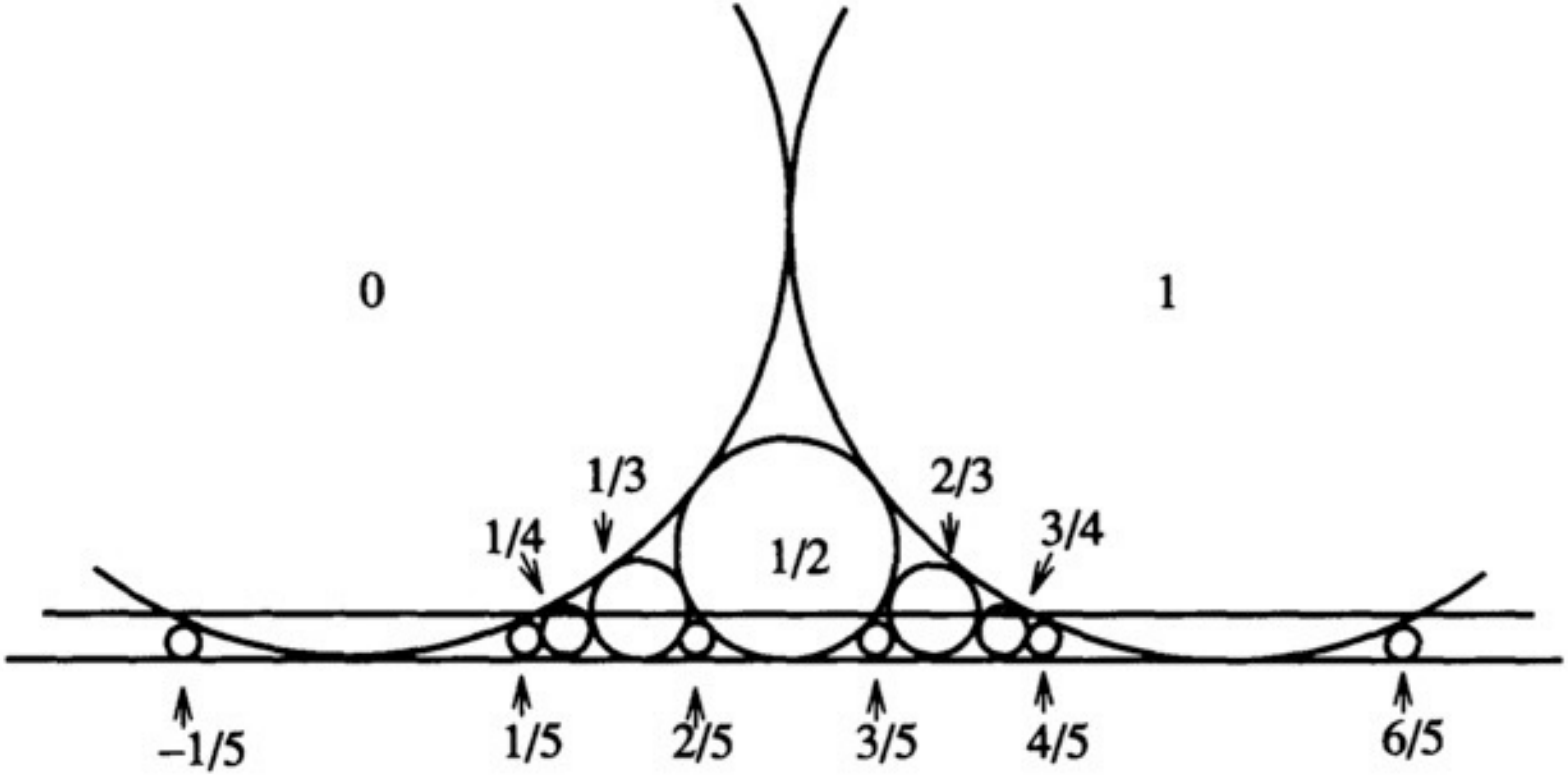
$$\frac{0+1}{1+3} = \frac{1}{4}$$

level 3 $\quad \boxed{\frac{0}{1} \quad \frac{1}{3}} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}$

level 4 $\quad \frac{0}{1} \quad \boxed{\frac{1}{4}} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$

mediant.

$\frac{0}{1}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{1}{1}$



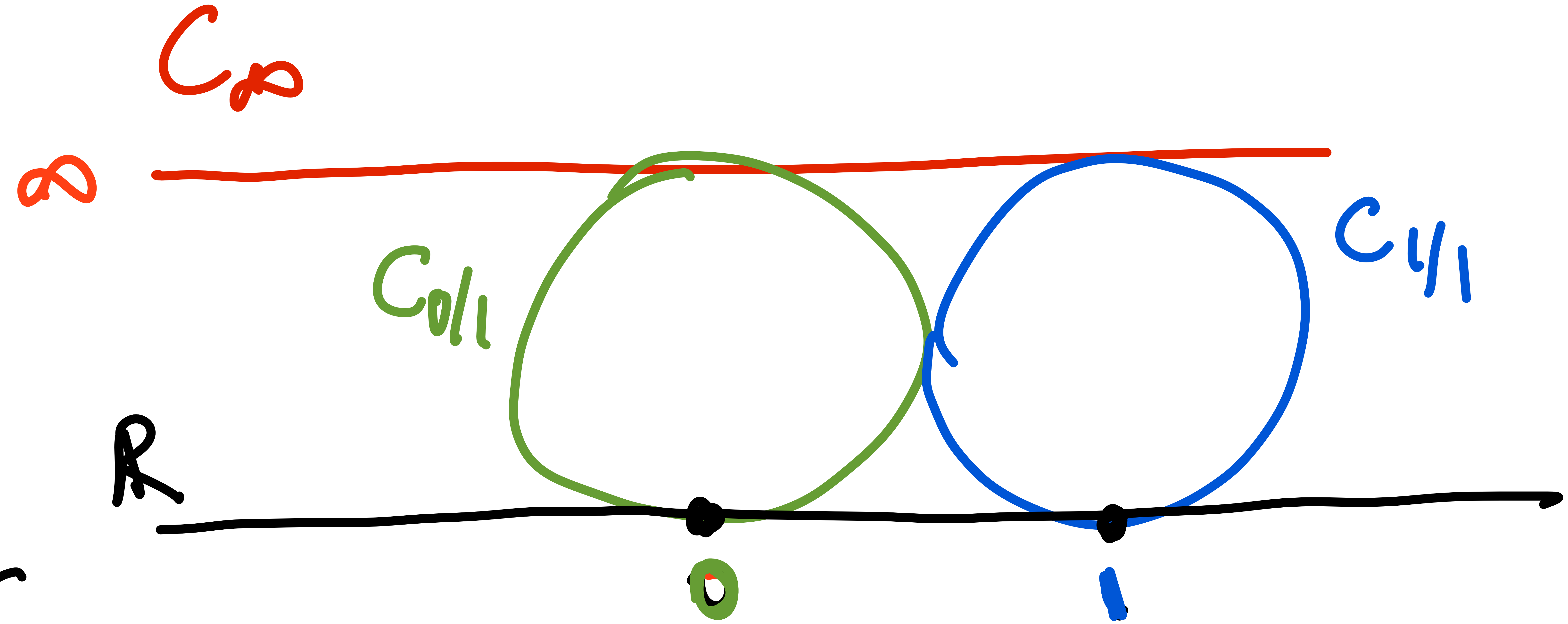
KEY IDEA:

- $\frac{p}{q} < \frac{r}{s}$ adjacent in $\mathcal{F}_n \Longleftrightarrow rq - ps = 1$

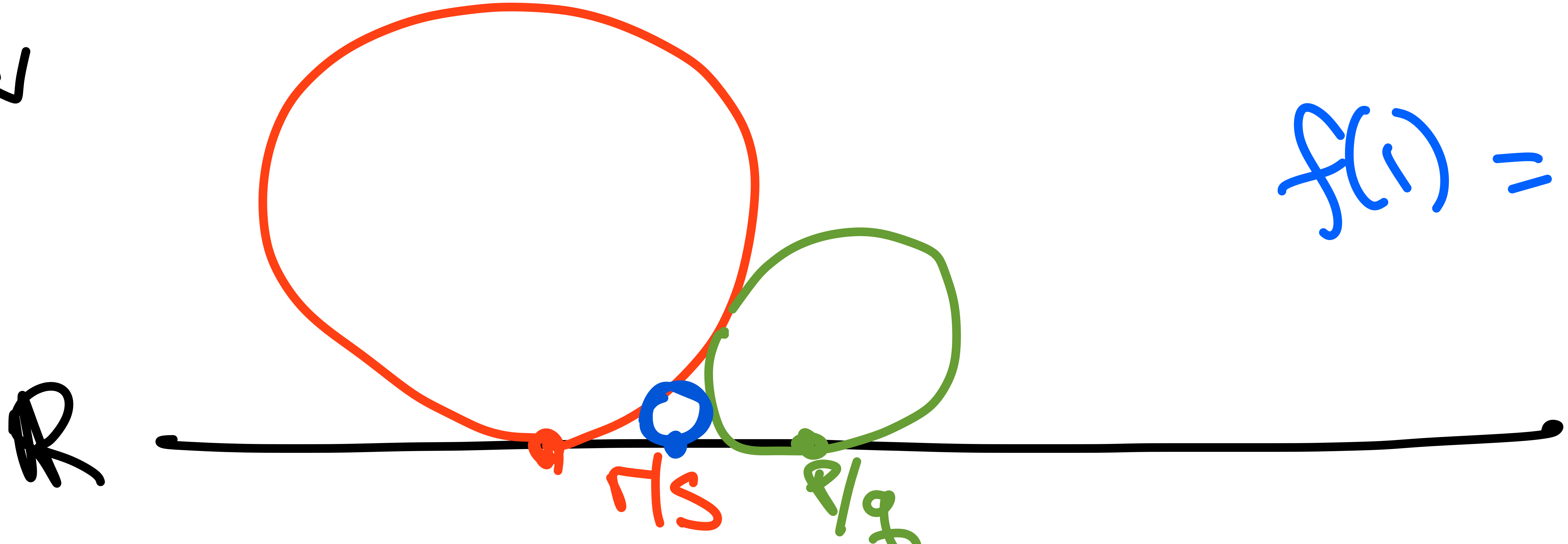
- $\begin{bmatrix} r & p \\ s & q \end{bmatrix} \in SL_2(\mathbb{Z})$, $\quad f(0) = \frac{p}{q}$

  $f(\infty) = \frac{r}{s}$

Generalizes to unimodularity



$$f(z) = \frac{rz+p}{sz+q}$$



$$f(1) = \frac{r+p}{s+q}$$

Summary

TWO WAYS TO DEFINE PACKING

Defn. The <u>Ford Packing</u> is

$$\mathcal{P} := \mathrm{ord}_{PSL_2(\mathbb{Z})}(C_{0/1})$$

Defn. The <u>Normal spheres</u> is the collection of

$$C_{p/q} = \left(\begin{array}{l}\text{circle tangent to } \partial\mathbb{H}^2 \text{ at} \\ \text{at } p/q \in \mathbb{Q} \text{ of radius} \\ \quad r = \frac{1}{2q^2}\end{array}\right)$$

Summary

Defn. The Ford Packing is
$$\mathcal{P} := \mathrm{orb}_{PSL_2(\mathbb{Z})}(C_{0/1})$$

Defn. The Normal Spheres is the collection of
$$C_{p/q} = \left(\begin{array}{l}\text{circle tangent to } \partial\mathbb{H}^2 \text{ at} \\ \text{at } p/q \in \mathbb{Q} \text{ of radius} \\ r = \frac{1}{2q^2}\end{array}\right)$$

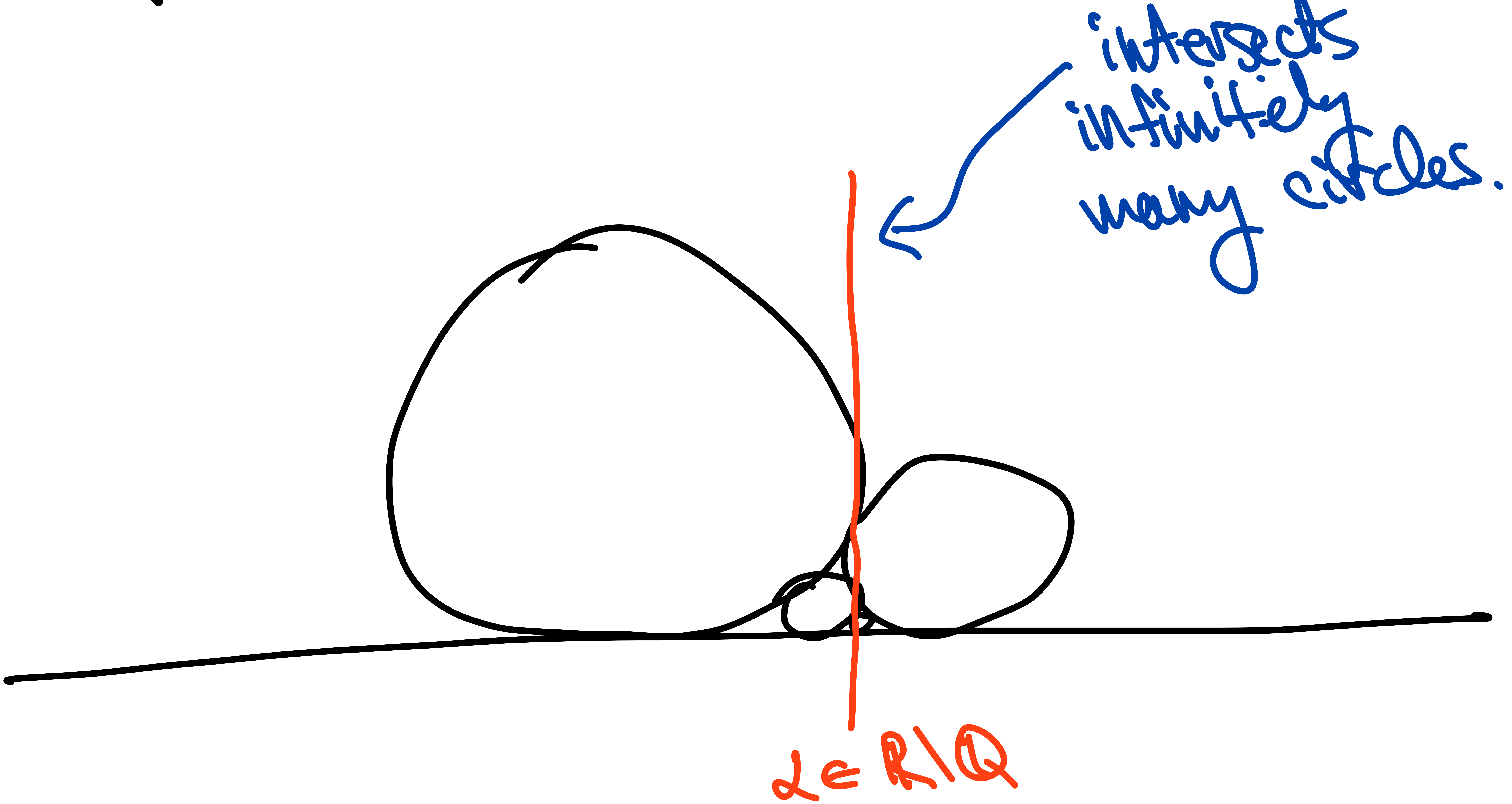
Theorem: These are the same.

Summary

1) <u>Theorem</u> : These are the same.

2) <u>Theorem</u>: Tangency is determined by Farey fraction adjacency.

Fun Application: Liouville's Theorem



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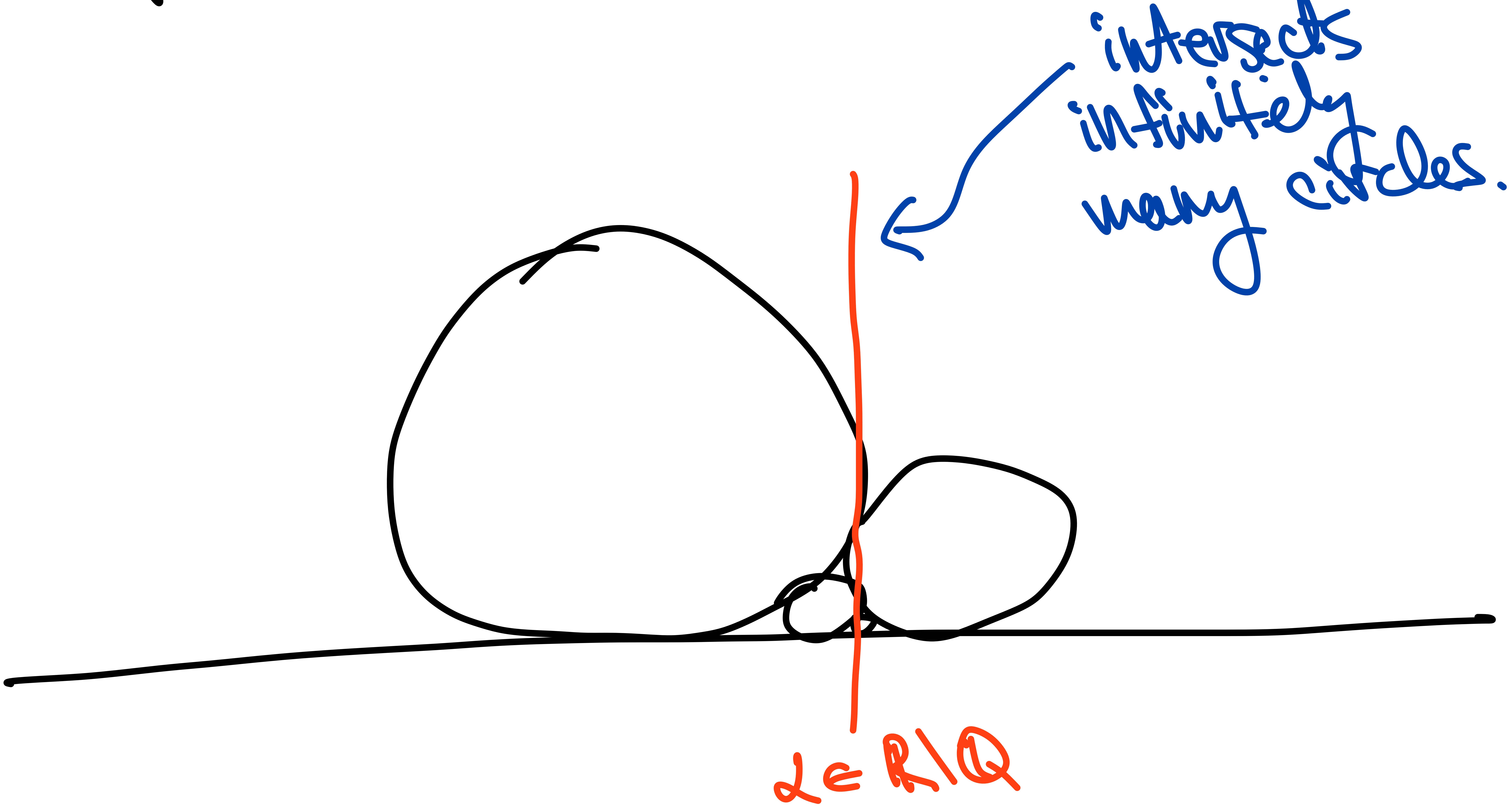


Fun Application: Liouville's Theorem

$$\boxed{\begin{array}{l}\text{Statement: } \forall \alpha \in \mathbb{R}\setminus\mathbb{Q},\ \exists \frac{p_i}{q_i} \in \mathbb{Q} \text{ with } p_i/q_i \to \alpha \text{ and} \\ \left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{2q_i^2}\end{array}}$$

~~Classical Story : PSL2(ZZ)~~

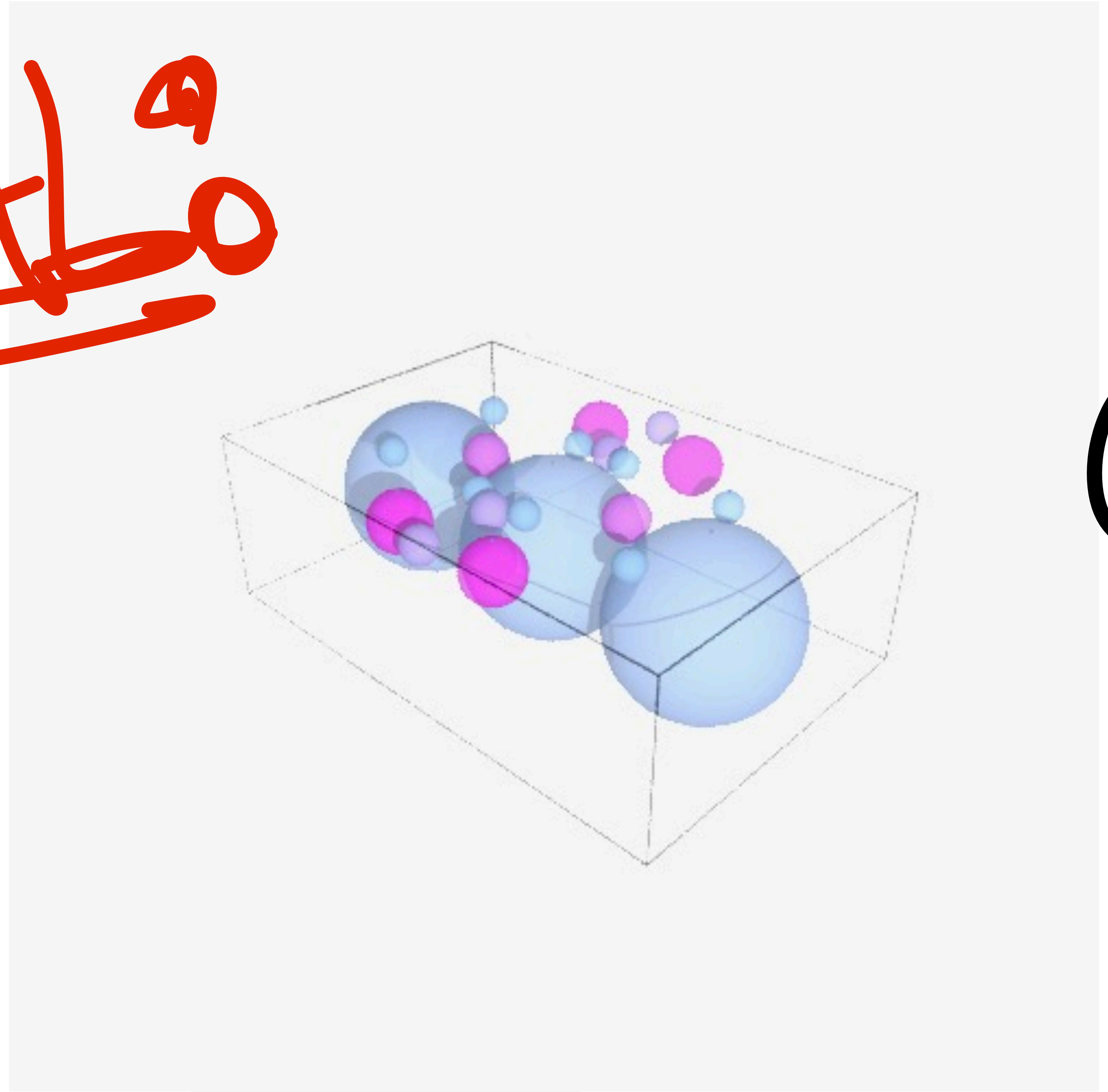
Higher Dimensions?

Hyperbolic Space and Clifford Algebra

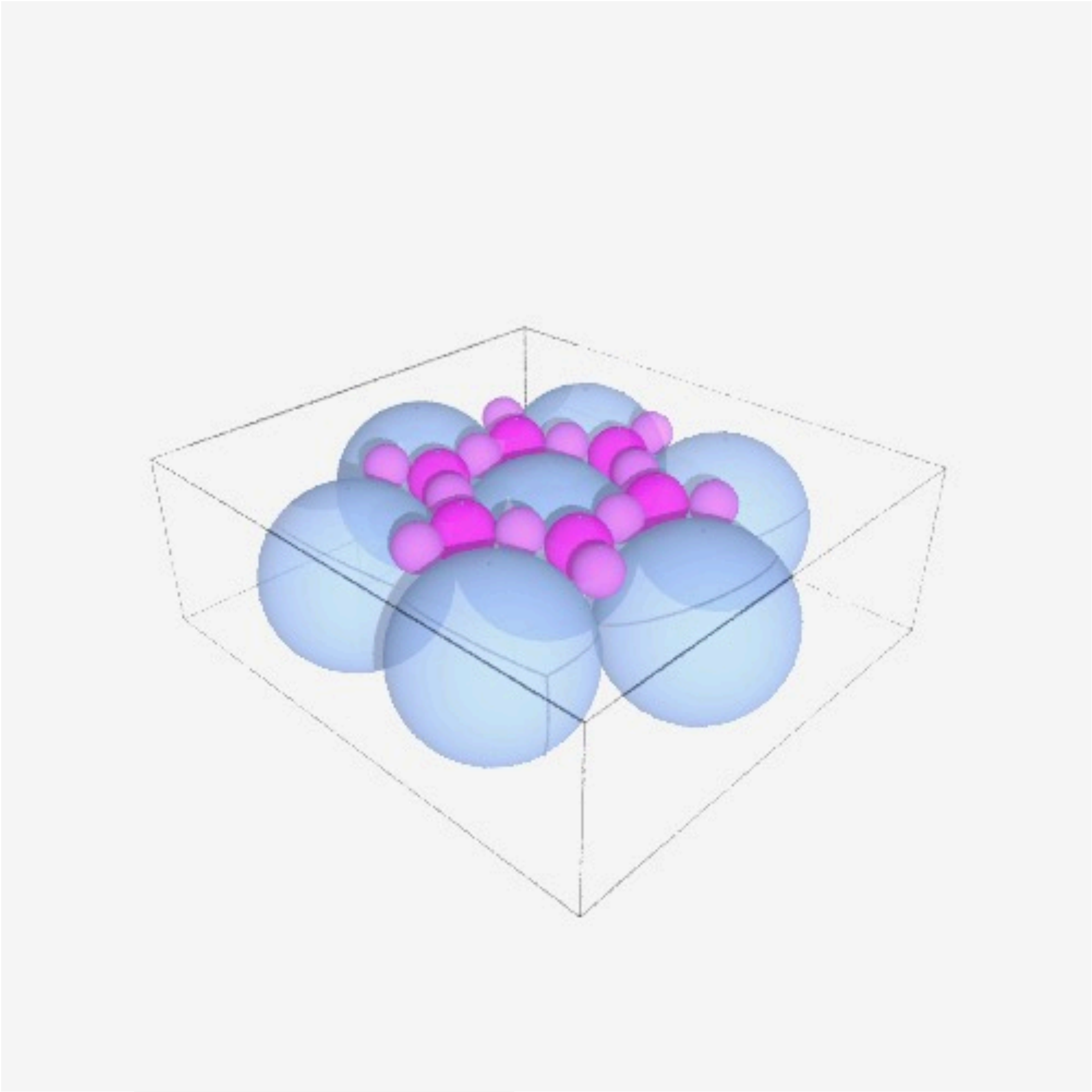
Fundamental Domain Algorithms

Higher Dimensions?

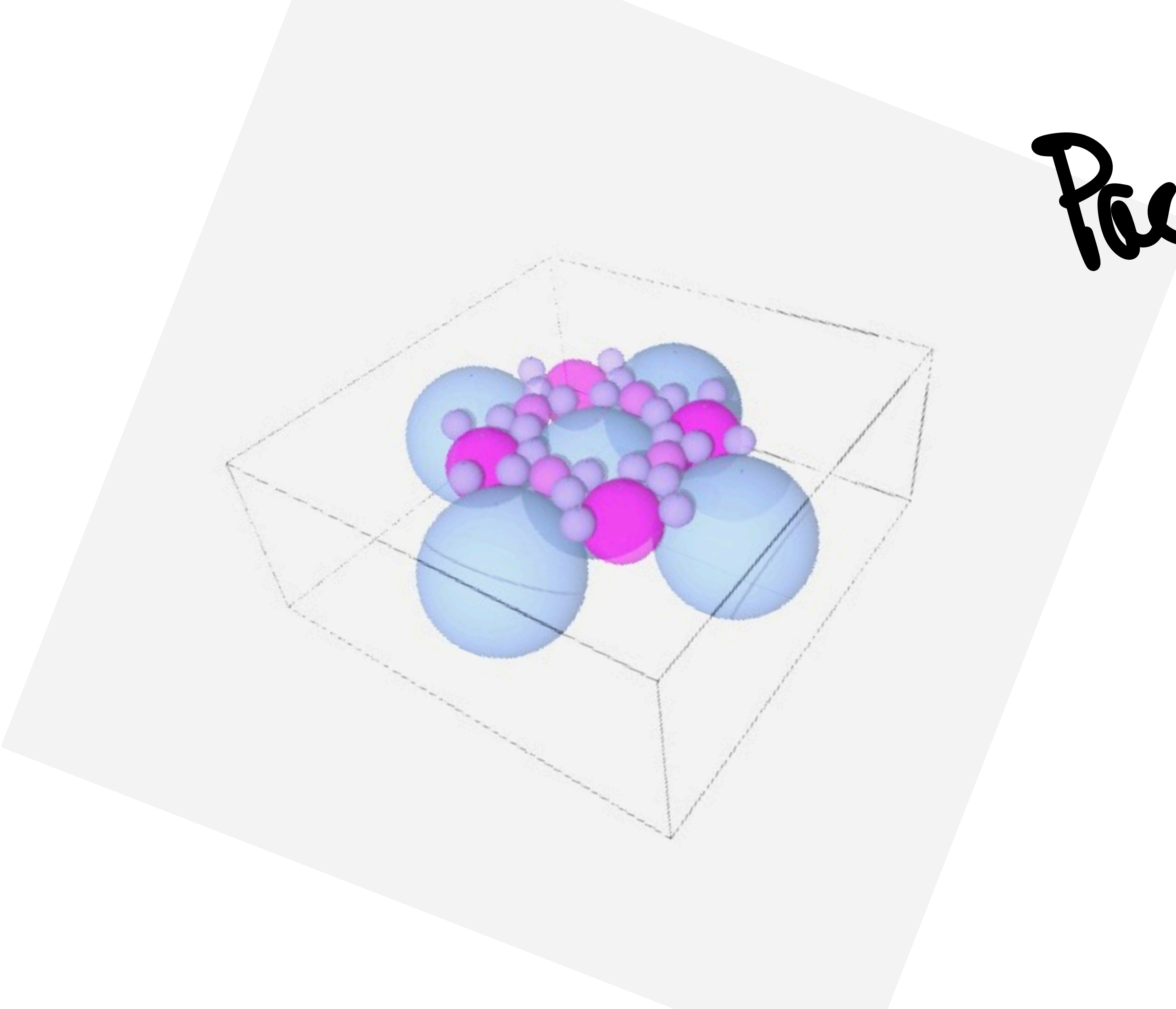
GOAL:



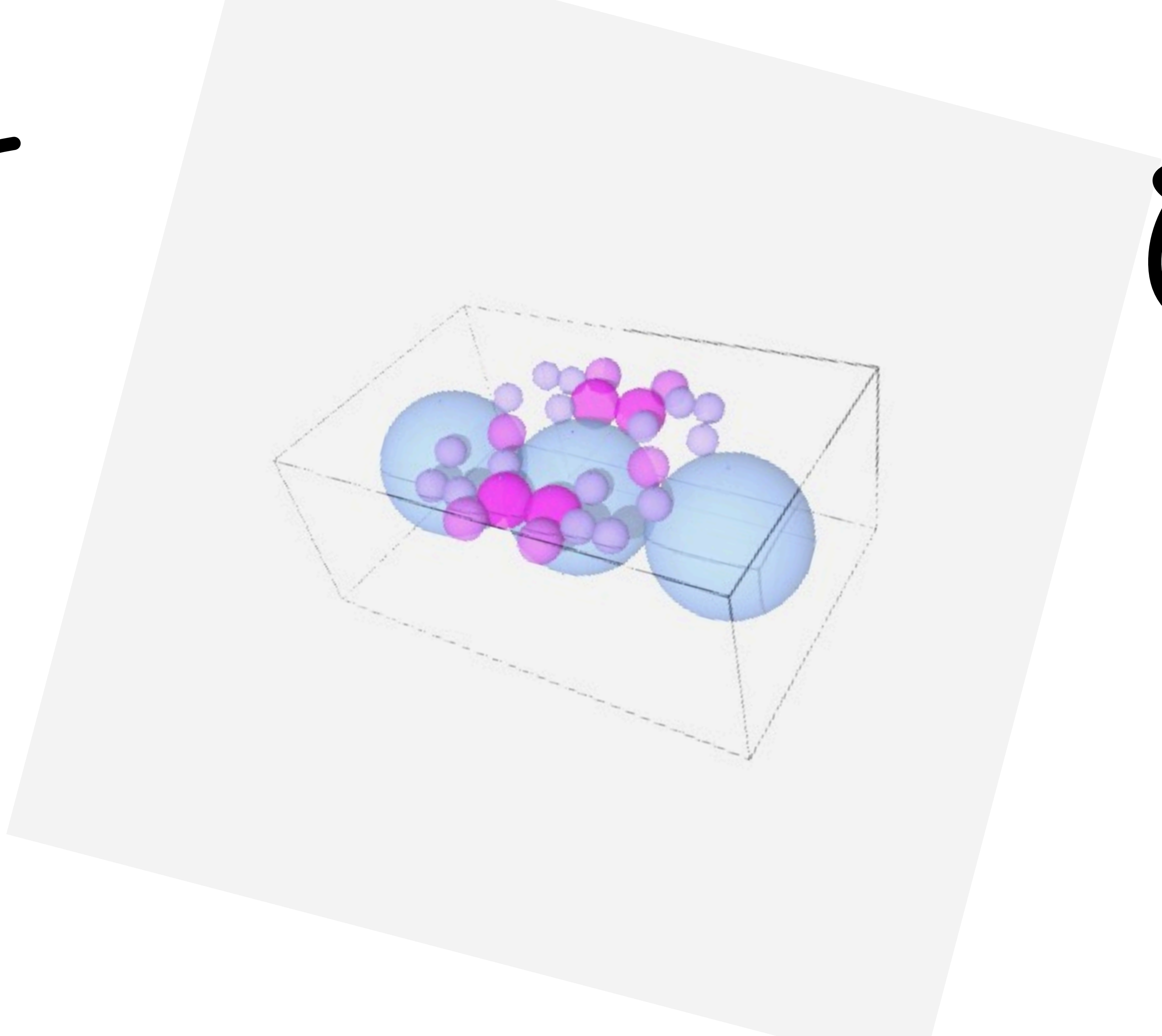
$\mathbb{Q}(\sqrt{-5})$



$\mathbb{Q}(\zeta_6)$



Packing for $\mathbb{Q}(i)$



$\mathbb{Q}(\sqrt{-11})$

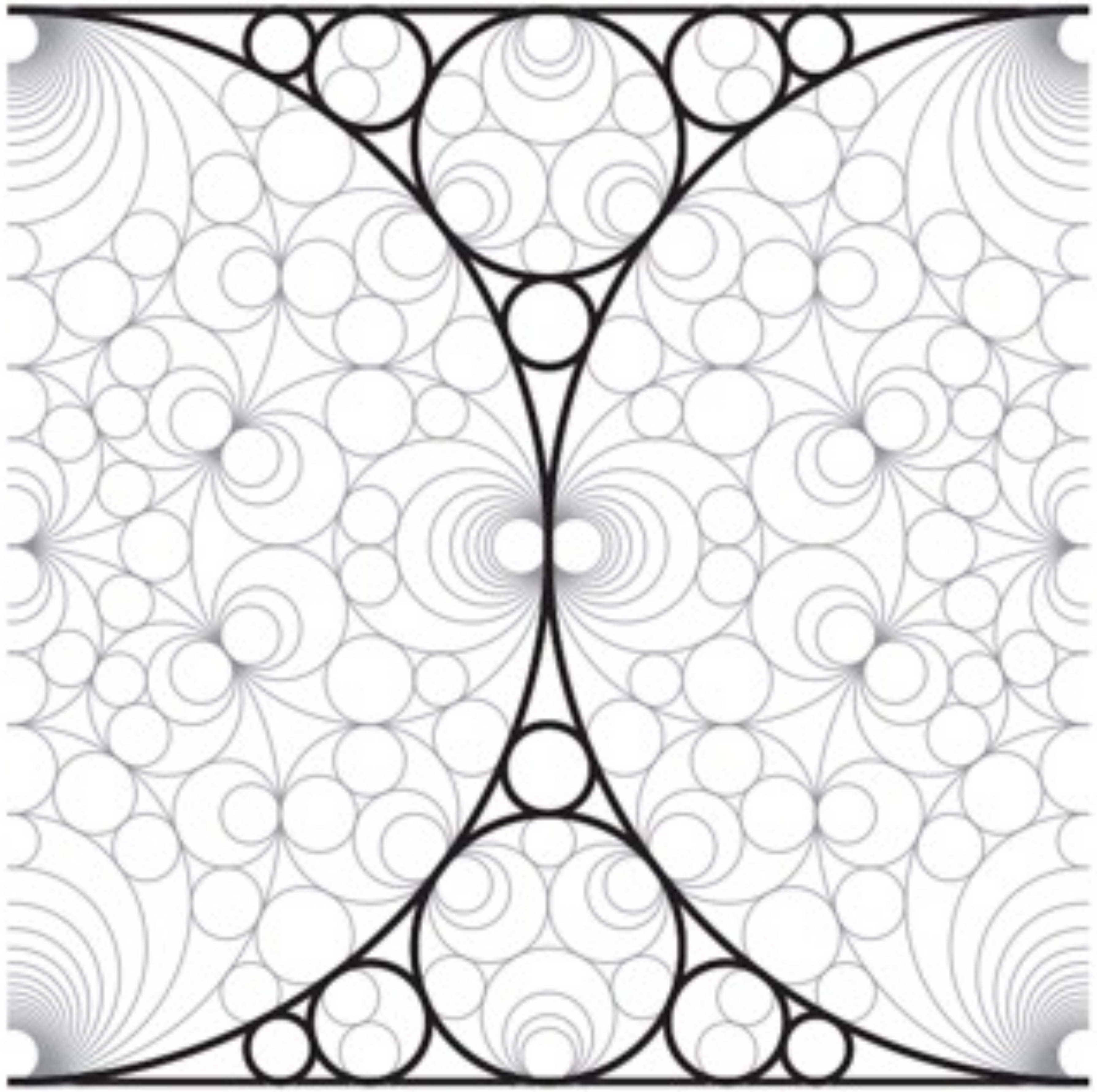
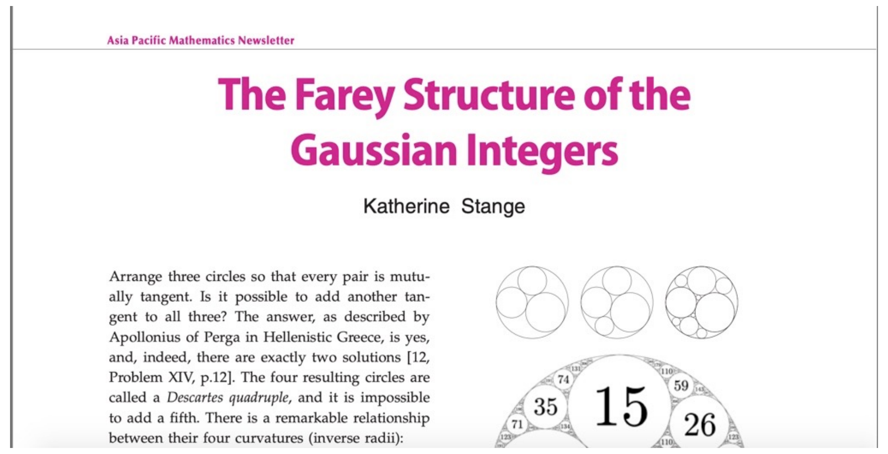


Fig. 4. The Schmidt arrangement of the Gaussian integers. The box between 0 and $1 + i$ is shown, including only circles with curvatures at most 20. The Schmidt arrangement is periodic under translation by $\mathbb{Z}[i]$. The Apollonian strip packing (which is bounded by two horizontal lines through 0 and i) is highlighted in black.



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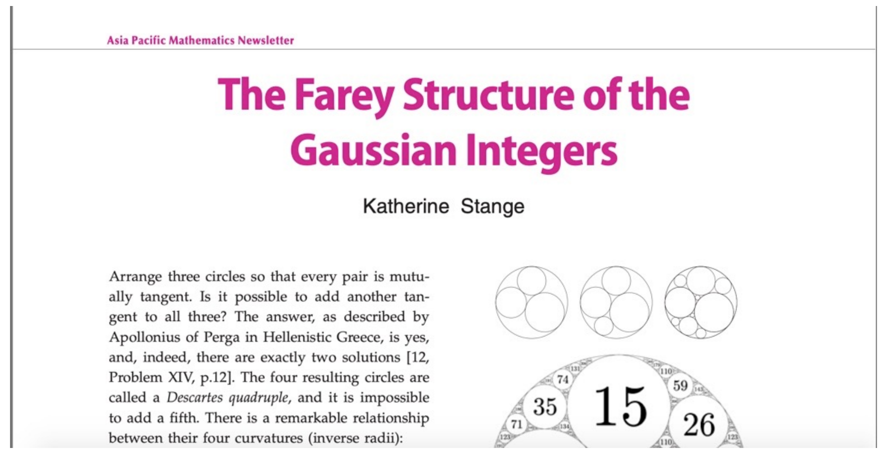
The Farey Structure of the Gaussian Integers

Katherine Stange

Arrange three circles so that every pair is mutually tangent. Is it possible to add another tangent to all three? The answer, as described by Apollonius of Perga in Hellenistic Greece, is yes, and, indeed, there are exactly two solutions [12, Problem XIV, p.12]. The four resulting circles are called a *Descartes quadruple*, and it is impossible to add a fifth. There is a remarkable relationship between their four curvatures (inverse radii):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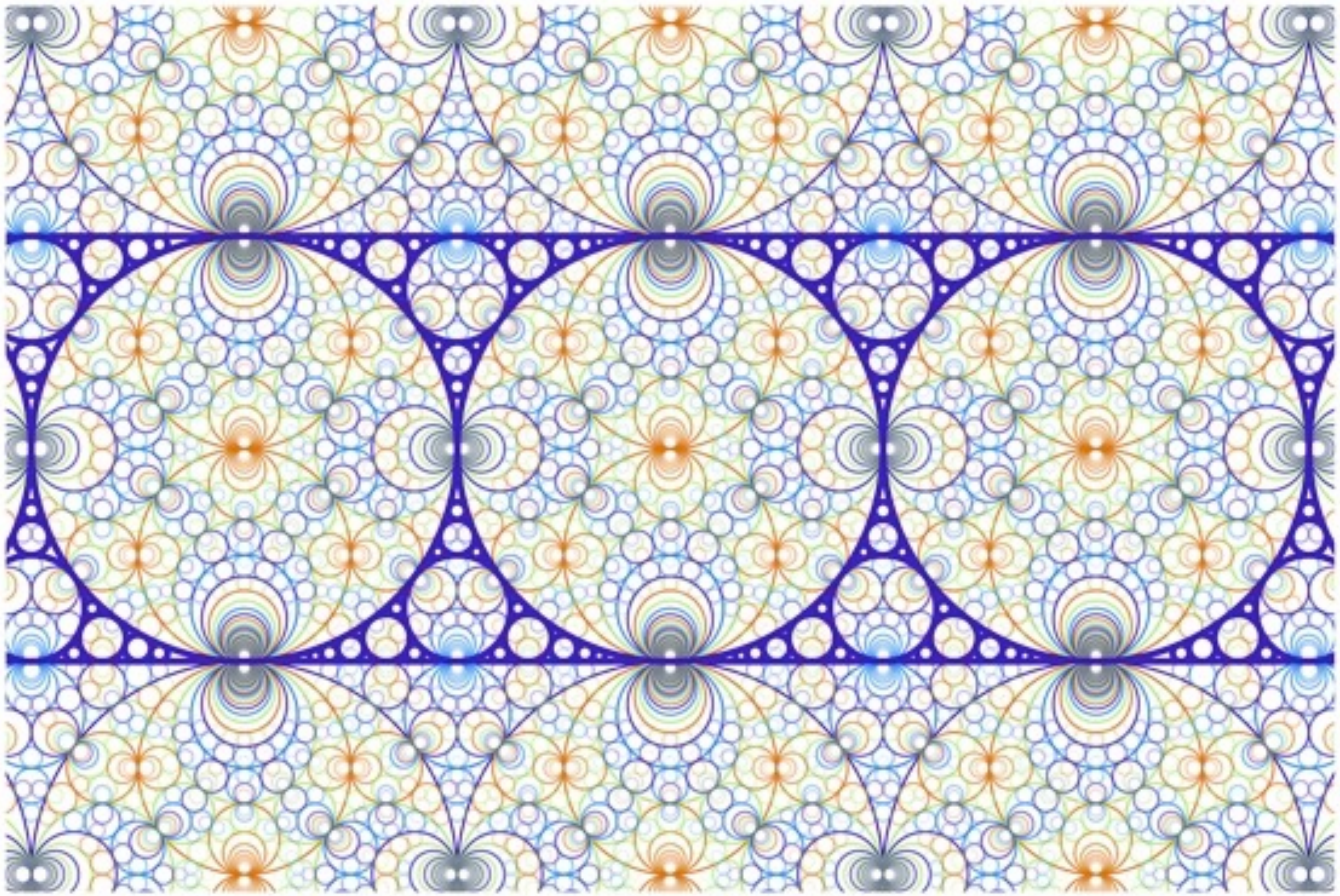
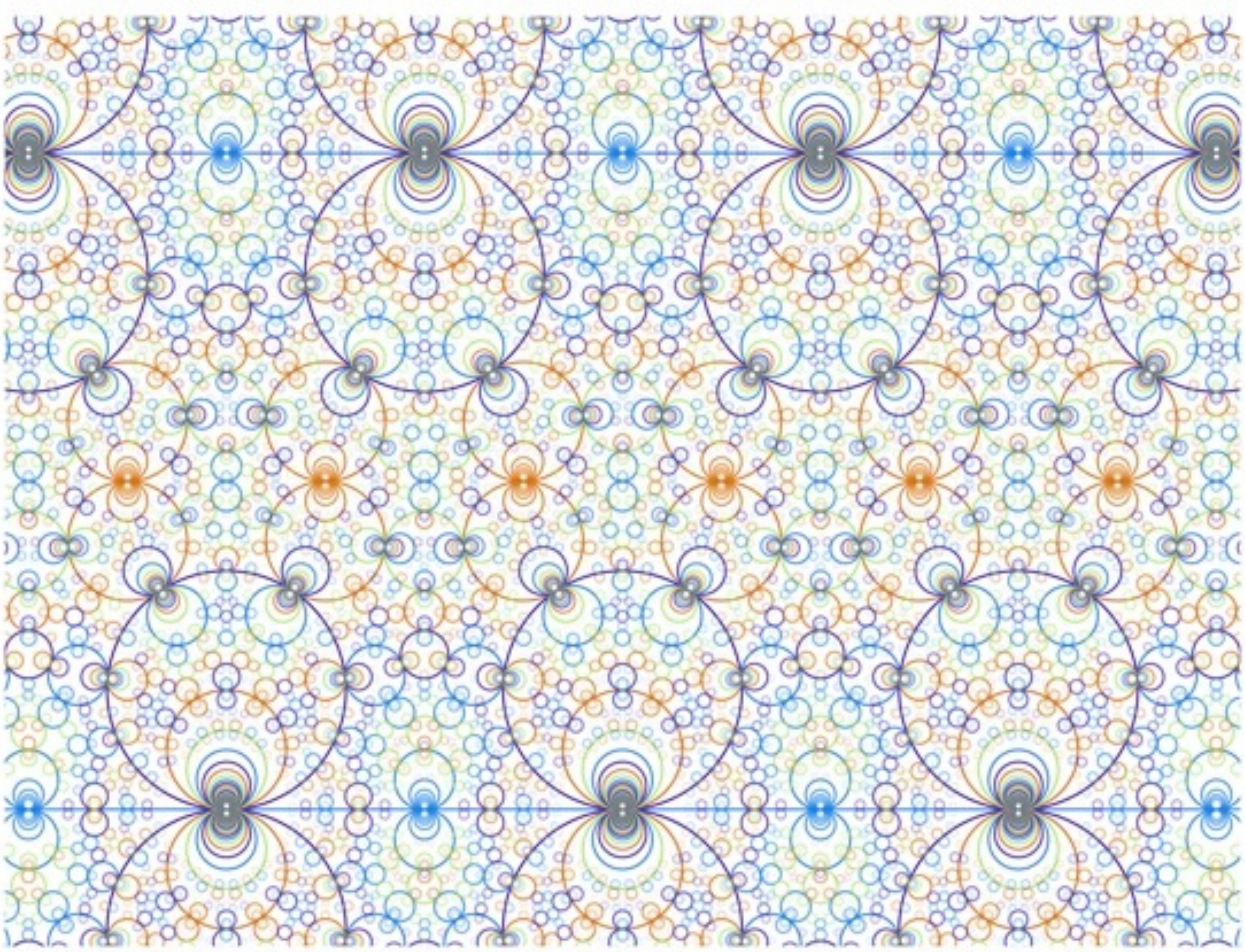
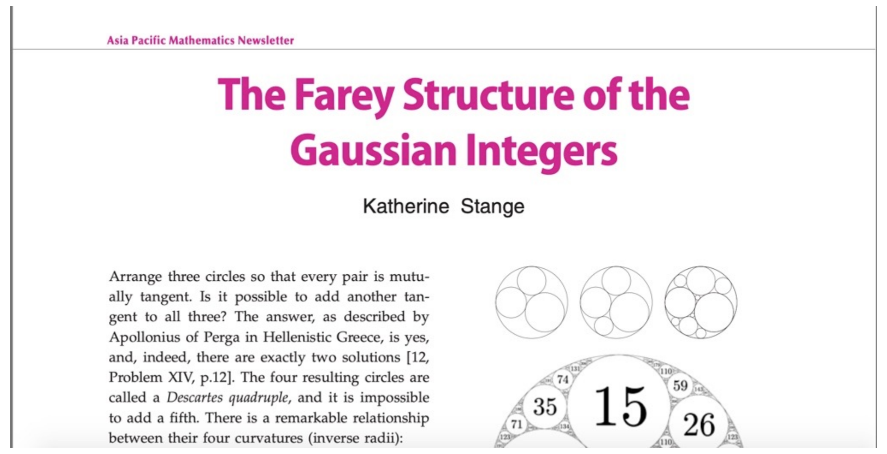


$\mathbb{Q}(i)$

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$\mathbb{Q}(\sqrt{-11})$.

General setup & Classification:

Kontorovich + Nakamura.

QUATERNION ORDERS AND SPHERE PACKINGS

ARSENIY SHEYDVASSER

ABSTRACT. We introduce an analog of Bianchi groups for rational quaternion algebras and use it to construct sphere packings that are analogs of the Apollonian circle packing known as integral crystallographic packings.

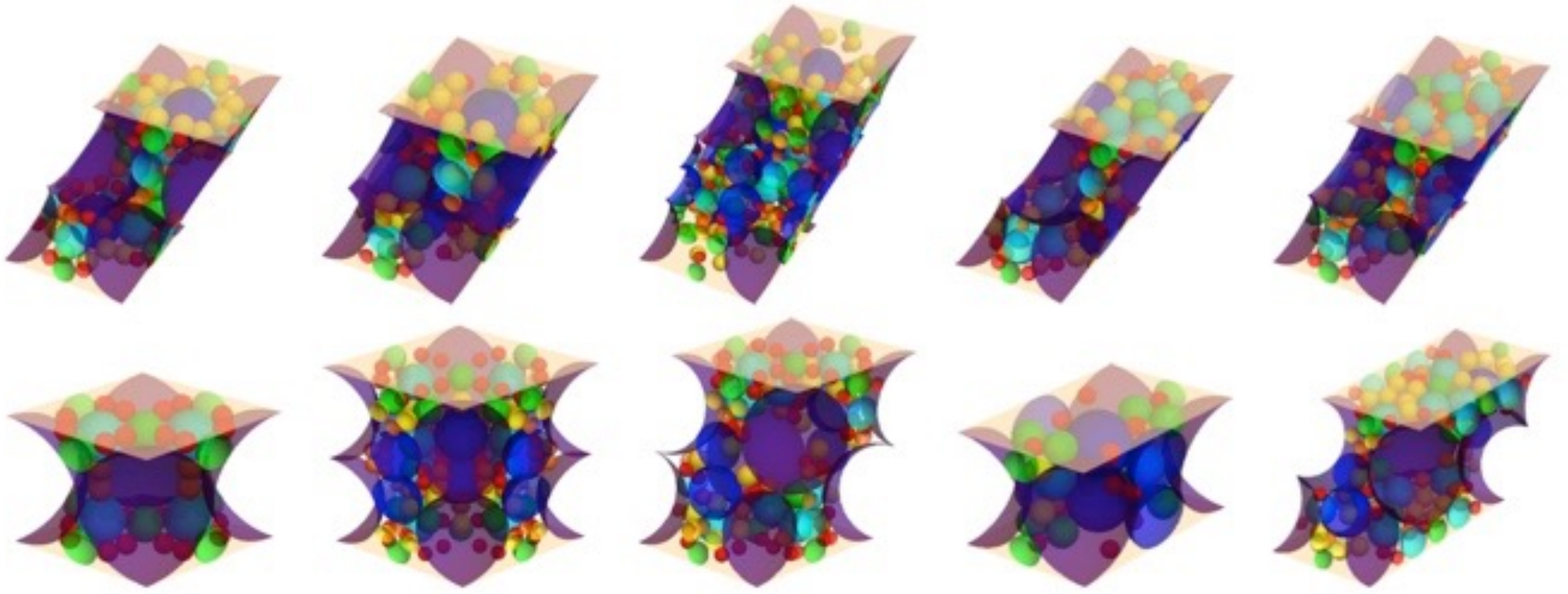


FIGURE 2. From left to right, and top to bottom: super-integral crystallographic packings corresponding to quaternion algebras $\left(\frac{-1,-6}{\mathbb{Q}}\right)$, $\left(\frac{-1,-7}{\mathbb{Q}}\right)$, $\left(\frac{-1,-10}{\mathbb{Q}}\right)$, $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, $\left(\frac{-2,-26}{\mathbb{Q}}\right)$, $\left(\frac{-3,-1}{\mathbb{Q}}\right)$, $\left(\frac{-3,-2}{\mathbb{Q}}\right)$, $\left(\frac{-3,-15}{\mathbb{Q}}\right)$, $\left(\frac{-7,-1}{\mathbb{Q}}\right)$, $\left(\frac{-11,-143}{\mathbb{Q}}\right)$.

GUESS: Replace C with H!?

Quick Reminder:

Division Algebra.

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij \quad (ij = k)$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Hamilton's Quaternions

	$\mathbb{C}$	$\mathbb{H}$
Möbius Preserves spheres	✓	✓
Half Space $\cong$ Ball	✓	✓

$\mathrm{Aut}(\mathcal{H}^2) \cong PSL_2(\mathbb{R})$ $\quad$ $\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})$ $\Big)$ other facts

	$\mathbb{C}$	$\mathbb{H}$
Möbius Preserves Spheres	✓	✓
Half Space $\cong$ Ball	✓	✓
Automorph of Half Space	$\mathrm{Aut}(\mathcal{H}^2) \cong PSL_2(\mathbb{R})$	$\cancel{\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})}$ $\dim(\mathcal{H}^3) \neq \dim(\mathbb{H})$

	$\mathbb{C}$	$\mathbb{H}$
Automorph of Half Space	$\text{Aut}(\mathcal{H}^2) \cong PSL_2(\mathbb{R})$	~~$\text{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})$~~ $\dim(\mathcal{H}^3) \neq \dim(\mathbb{H})$

	$\mathbb{C}$	$\mathbb{H}$
Automorph of Half Space	$\mathrm{Aut}(\mathcal{H}^2) \cong PSL_2(\mathbb{R})$	~~$\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})$~~ $\dim(\mathcal{H}^3) \neq \dim(\mathbb{H})$ $\mathrm{Aut}(\mathcal{H}^4) \cong PSL_2(???)$

$$\mathrm{Aut}(\mathcal{H}^4) \cong PSL_2(???)$$

Already used in $\mathcal{H}^3$! $\mathrm{Aut}(\mathcal{H}^3) = PSL_2(\mathbb{C})$

subrings of $\mathbb{H}$: $\cancel{\mathbb{R}}, \cancel{\mathbb{C}}$

too small

WHAT DO WE DO?

WHAT DO WE DO?

$\cancel{\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})}$

$\dim(\mathcal{H}^3) \neq \dim(\mathcal{H})$

Turns out this is actually ok!!

$\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C})$

WHAT DO WE DO?

'Standard' Clifford Alg

explanation:

Clifford Vectors

$$\boxed{\mathcal{H}^n \subseteq V_n \subseteq \mathbb{C}_n}$$

Clifford-Möbius Trans.

$$\boxed{\mathrm{Aut}(\mathcal{H}^n) \cong PSL_2(\mathbb{C}_{n-1})}$$

~~**Classical Story : PSL2(ZZ)**~~

~~**Higher Dimensions?**~~

Hyperbolic Space and Clifford Algebras

Fundamental Domain Algorithms

Hyperbolic Space and Clifford Algebras

Clifford Algebras

Quadratic space:
$V \cong \mathbb{R}^n$, q = quad form

tensor alg:
$T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$

$$\mathrm{Clf}(V,q) = T(V)/I_q$$

Clifford ideal:

$$I_q = \langle v\otimes v + q(v) \mid v \in V \rangle$$

$\boxed{\text{Clifford Algebras}}$ $Clf(V,q) = T(V)/I_q$

$\boxed{\text{Clifford Vectors}}$ V an R-Module w/ basis $a_1, \dots, a_n$

$$\underbrace{Vec(V,q)}_{\text{subgroup of clifford vectors}} = \{ x_0 + x_1 a_1 + \dots + x_n a_n : x_j \in R \}$$

$$\mathrm{Clf}(V,q) = T(V)/I_q$$

$\boxed{\text{Clifford Algebras}}$

special case: $V = F^n$, $q(y_1, y_2, \ldots, y_n)$

$$= d_1 y_1^2 + d_2 y_2^2 + \cdots + d_n y_n^2$$

$$\rightsquigarrow \left(\frac{-d_1, -d_2, \ldots, -d_n}{F}\right) := \mathrm{Clf}(V,q)$$

Clifford Algebras

$$\mathrm{Clf}(V,q) = T(V)/I_q$$

special case: $V = F^n$, $q(y_1, y_2, \dots, y_n) = d_1 y_1^2 + d_2 y_2^2 + \dots + d_n y_n^2$

$$\rightsquigarrow \left(\frac{-d_1, -d_2, \dots, -d_n}{F}\right) := \mathrm{Clf}(V,q)$$

$$\left(\frac{-1}{\mathbb{R}}\right) = \mathbb{R}[i_1], \quad i_1^2 = -1$$

$$\cong \mathbb{C}$$

clifford vectors

$\mathbb{C}$

$$\left(\frac{-1,-1}{\mathbb{R}}\right) = \mathbb{R}[i_1, i_2], \quad i_1^2 = i_2^2 = -1, \quad i_1 i_2 = -i_2 i_1$$

$$\cong \mathbb{H}$$

clifford vectors

$$\mathbb{R} + \mathbb{R} i_1 + \mathbb{R} i_2$$

Clifford Algebras

$$\mathbb{C}_n = \text{Clifford Alg} = \left(\frac{\overbrace{-1,-1,\dots,-1}^{(n-1)\text{-times}}}{\mathbb{R}}\right) = \mathbb{R}[i_1, i_1, \dots, i_{n-1}],$$

$$i_a^2 = -1, \quad i_a i_b = -i_b i_a$$

- $\mathbb{C}_1 = \mathbb{R}$
- $\mathbb{C}_2 = \mathbb{C}$
- $\mathbb{C}_3 = \mathbb{H}$
- $\mathbb{C}_4 = 8$-dim'l algebra

Not $\mathbb{O}$!

Clifford Algebras

$$\mathbb{C}_n = \text{Clifford Alg} = \left(\frac{\overbrace{-1,-1,\dots,-1}^{(n-1)\text{-times}}}{\mathbb{R}}\right)$$

$$= \mathbb{R}[i_1, i_1, \dots, i_{n-1}], \qquad i_a^2 = -1, \quad i_a i_b = -i_b i_a$$

Clifford Vectors:

$$V_n := \text{Clifford Vectors of } \mathbb{C}_n.$$

$$\boxed{\dim_{\mathbb{R}}(V_n) = n.}$$

Ahlfors

Normalize:

- $\dim_{\mathbb{R}}(V_n) = n$
- $V_n \subseteq \mathbb{C}_n$

$\mathcal{H}^n \subseteq V_n \subseteq \mathbb{C}_n$	$\mathrm{Aut}(\mathcal{H}^n) \cong PSL_2(\mathbb{C}_{n-1})$
$\mathcal{H}^2 \subseteq V_2 \subseteq \mathbb{C}_2 = \mathbb{C}$	$\mathrm{Aut}(\mathcal{H}^2) \cong PSL_2(\mathbb{C}_1) = PSL_2(\mathbb{R})$
$\mathcal{H}^3 \subseteq V_3 \subseteq \mathbb{C}_3 = \mathbb{H}$	$\mathrm{Aut}(\mathcal{H}^3) \cong PSL_2(\mathbb{C}_2) = PSL_2(\mathbb{C})$
$\mathcal{H}^4 \subseteq V_4 \subseteq \mathbb{C}_4$	$\mathrm{Aut}(\mathcal{H}^4) \cong PSL_2(\mathbb{C}_3) = PSL_2(\mathbb{H})$.

History

Vahlen 1902

Maass 1949

Ahlfors 1984

Elstrodt - Grunewald - Mennicke 1987, 1988, 1990

Maclachlan - Waterman - Weiland 1989

Vulakh 1993, 1995, 1999

..

Krausser 2010, 2013, 2015

General Idea:

$$\mathcal{H}^2 \subseteq \mathbb{C}$$

Satake $\mathcal{H}^{2,*} = \mathcal{H}^2 \cup \mathbb{Q} \cup \{\infty\}$

$$PSL_2(\mathbb{C})$$
$$\cup\!\!\mid$$
$$PSL_2(\mathbb{R})$$
$$\cup\!\!\mid$$
$$PSL_2(\mathbb{Z})$$

$$\mathcal{H}^{n+1} \subseteq \mathbb{C}_{n+1}$$

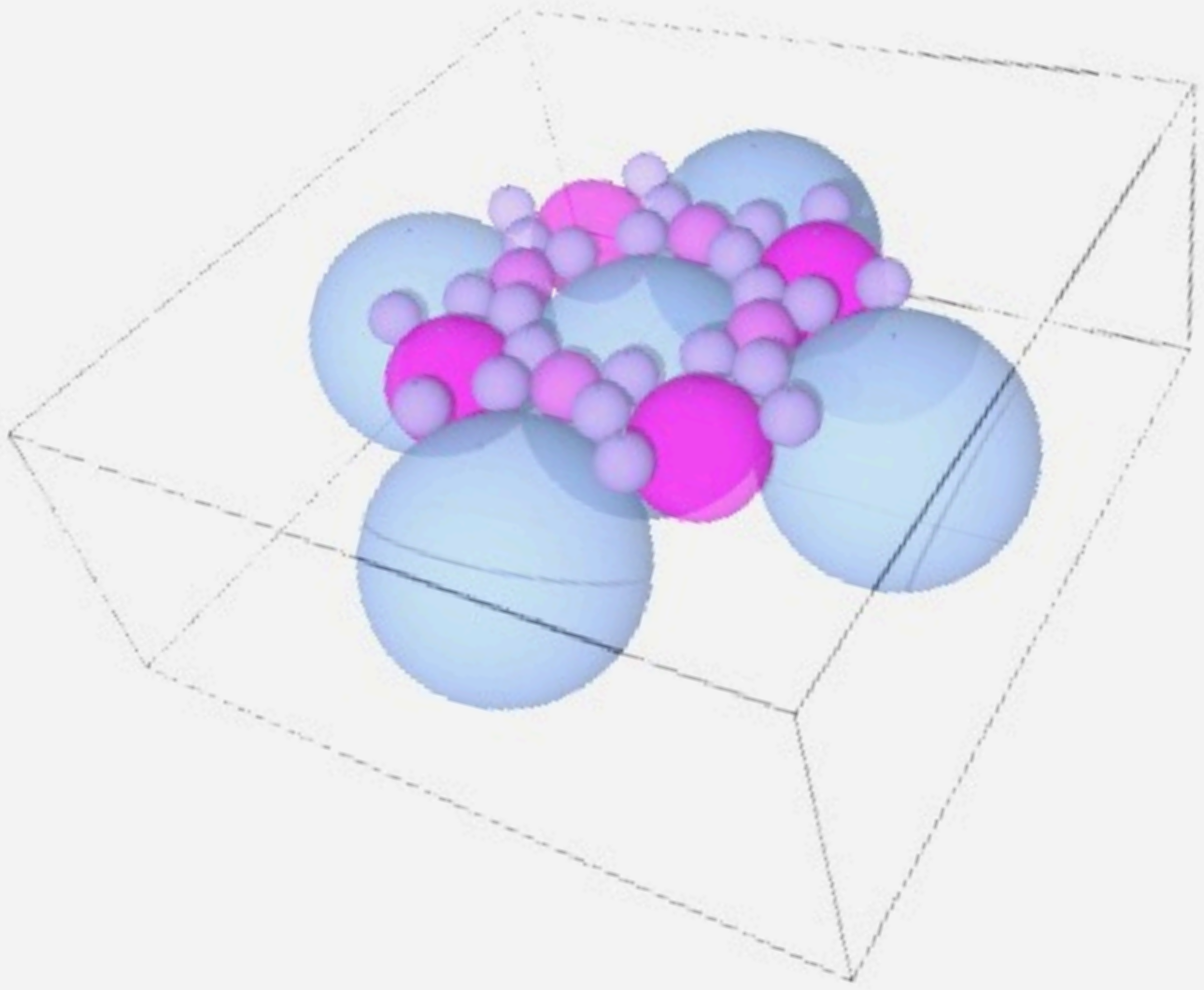
$$\mathcal{H}^{n+1,*} = \mathcal{H}^{n+1} \cup \mathrm{Vec}(K) \cup \{\infty\}$$

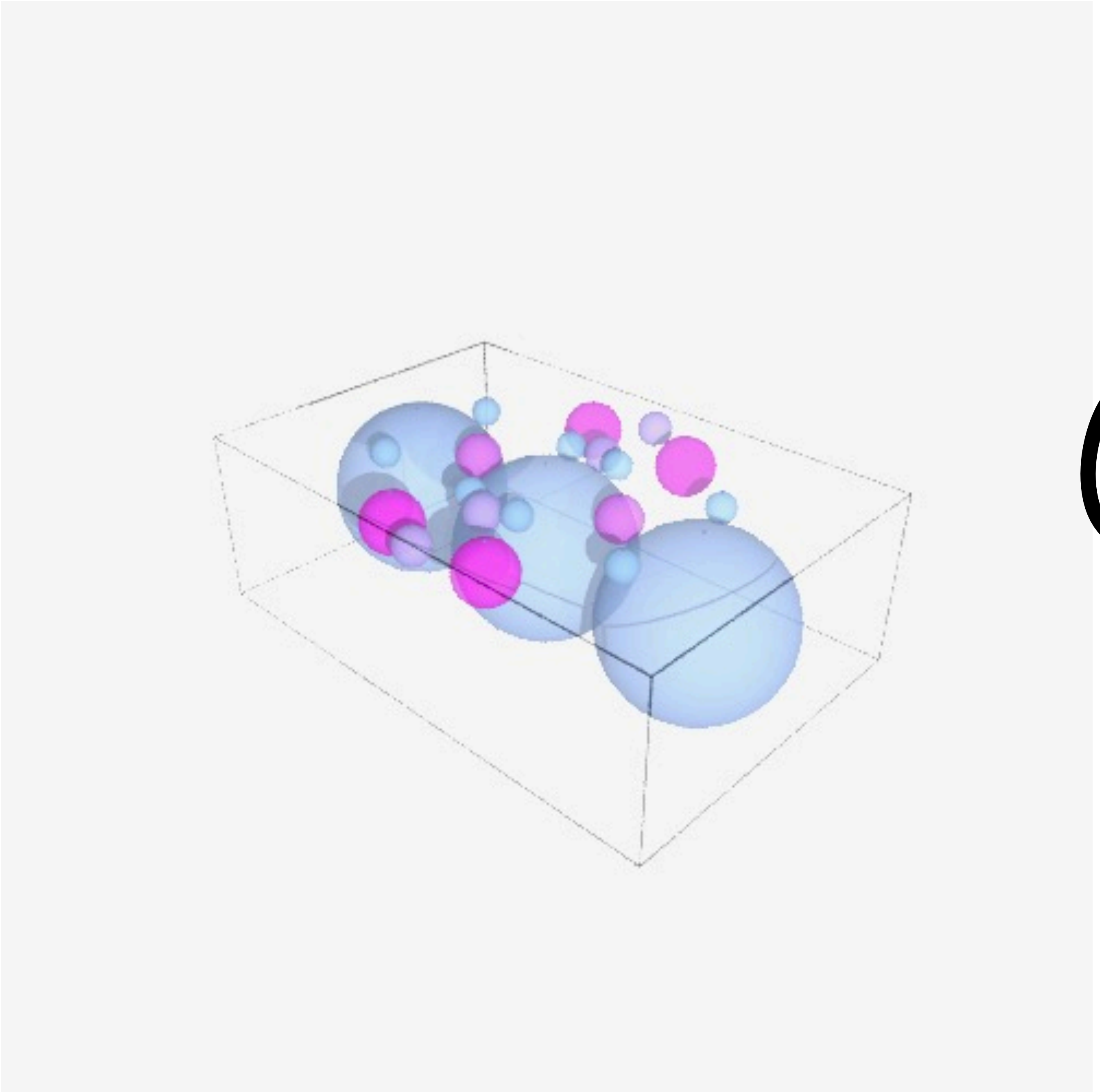
$$PSL_2(\mathbb{C}_{n+1})$$
$$\cup\!\!\mid$$
$$PSL_2(\mathbb{C}_n)$$
$$\cup\!\!\mid$$
$$PSL_2(\mathcal{O})$$

Normal Spheres for $\mathbb{Q}(i)$

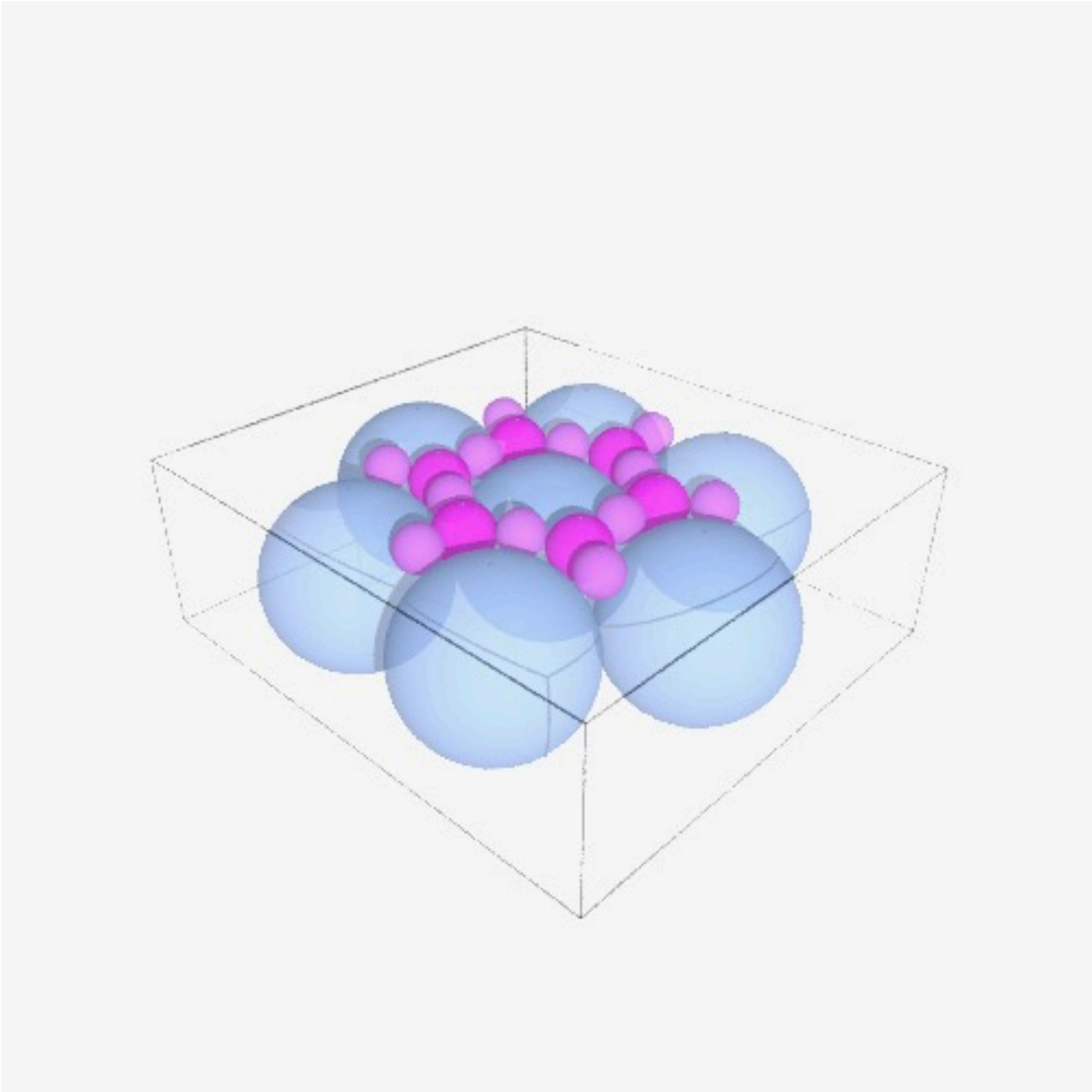
joint with:

Normal Spheres for $\mathbb{Q}(i)$

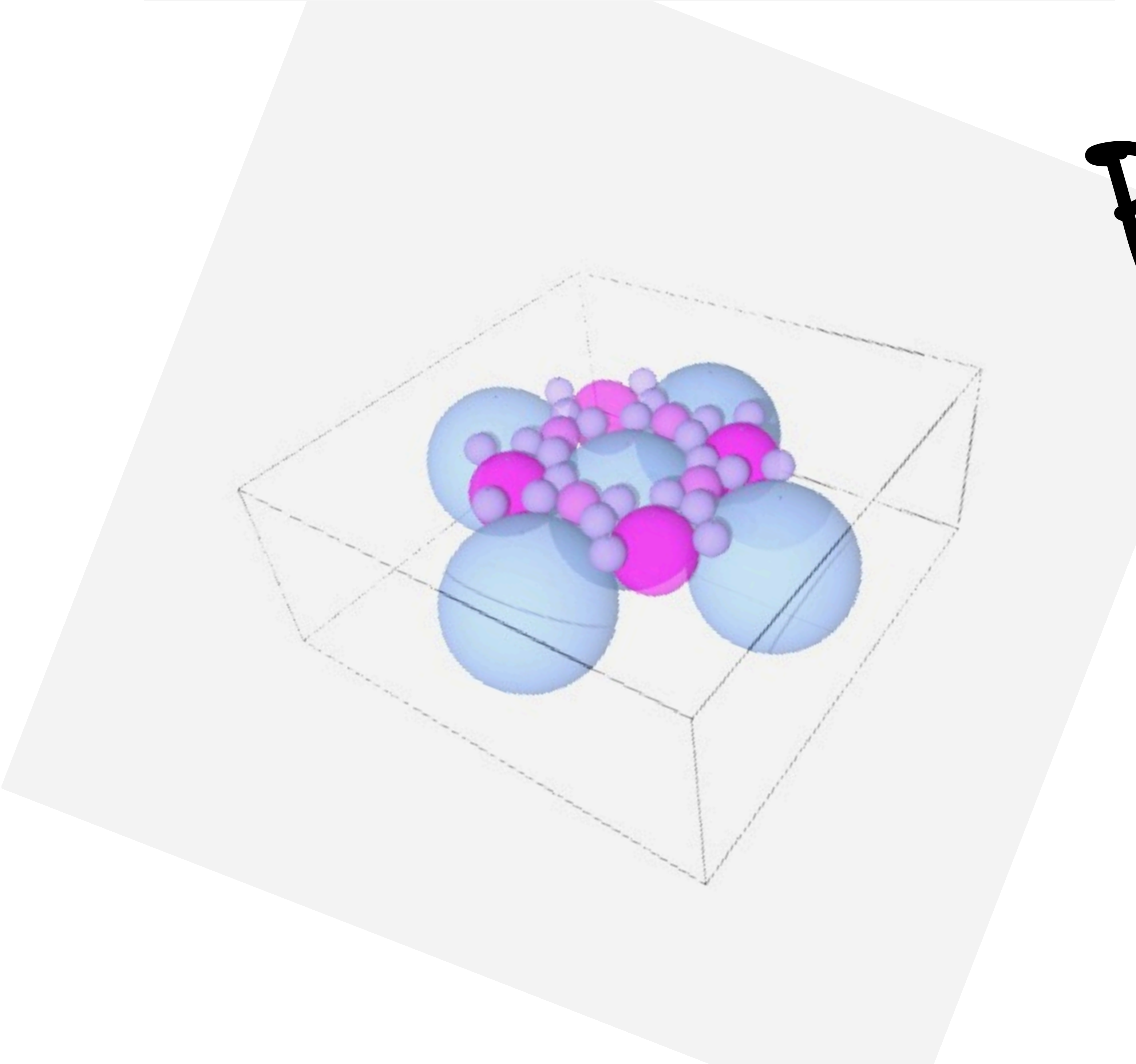




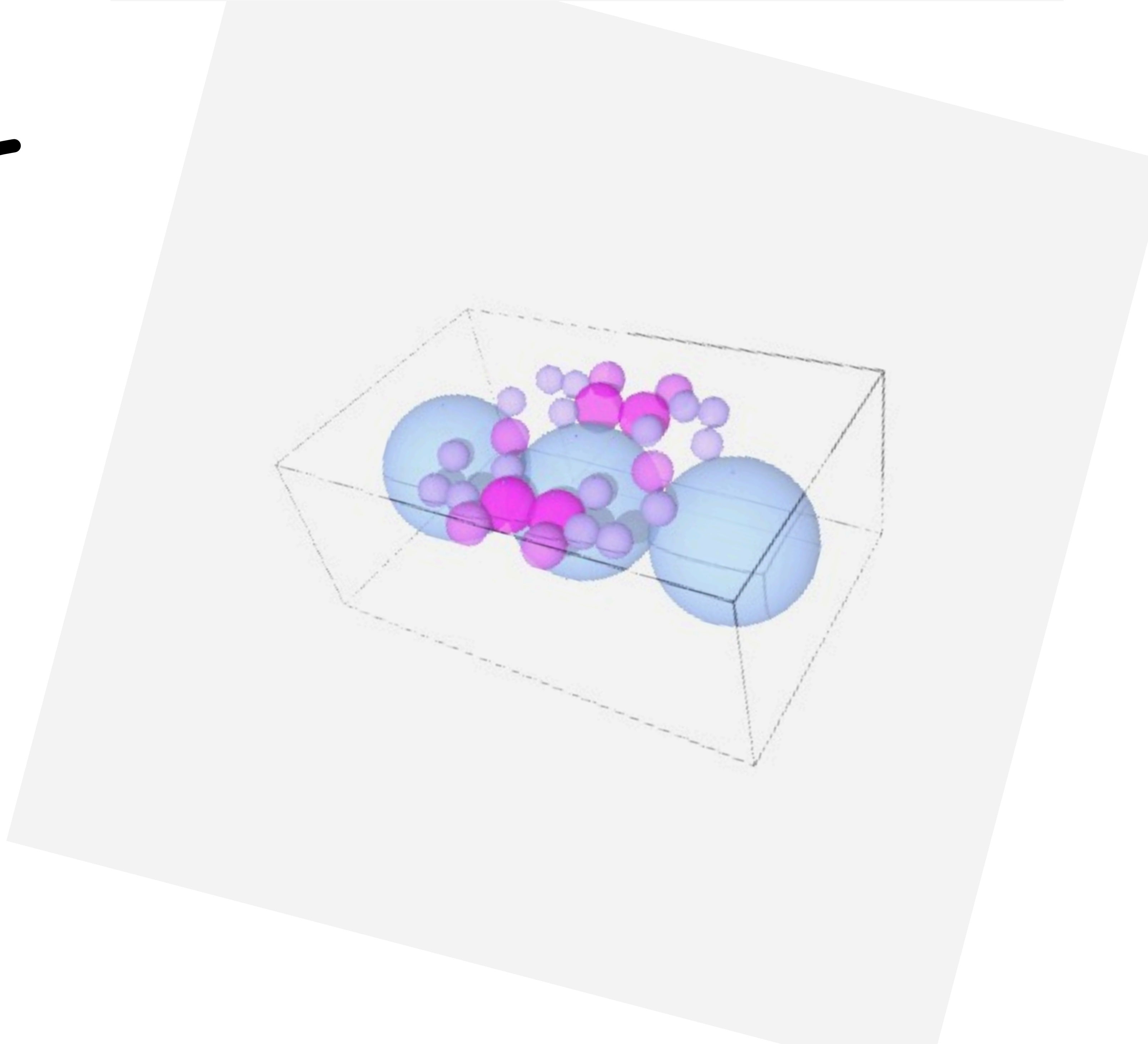
$\mathbb{Q}(\sqrt{-5})$



$\mathbb{Q}(\zeta_6)$

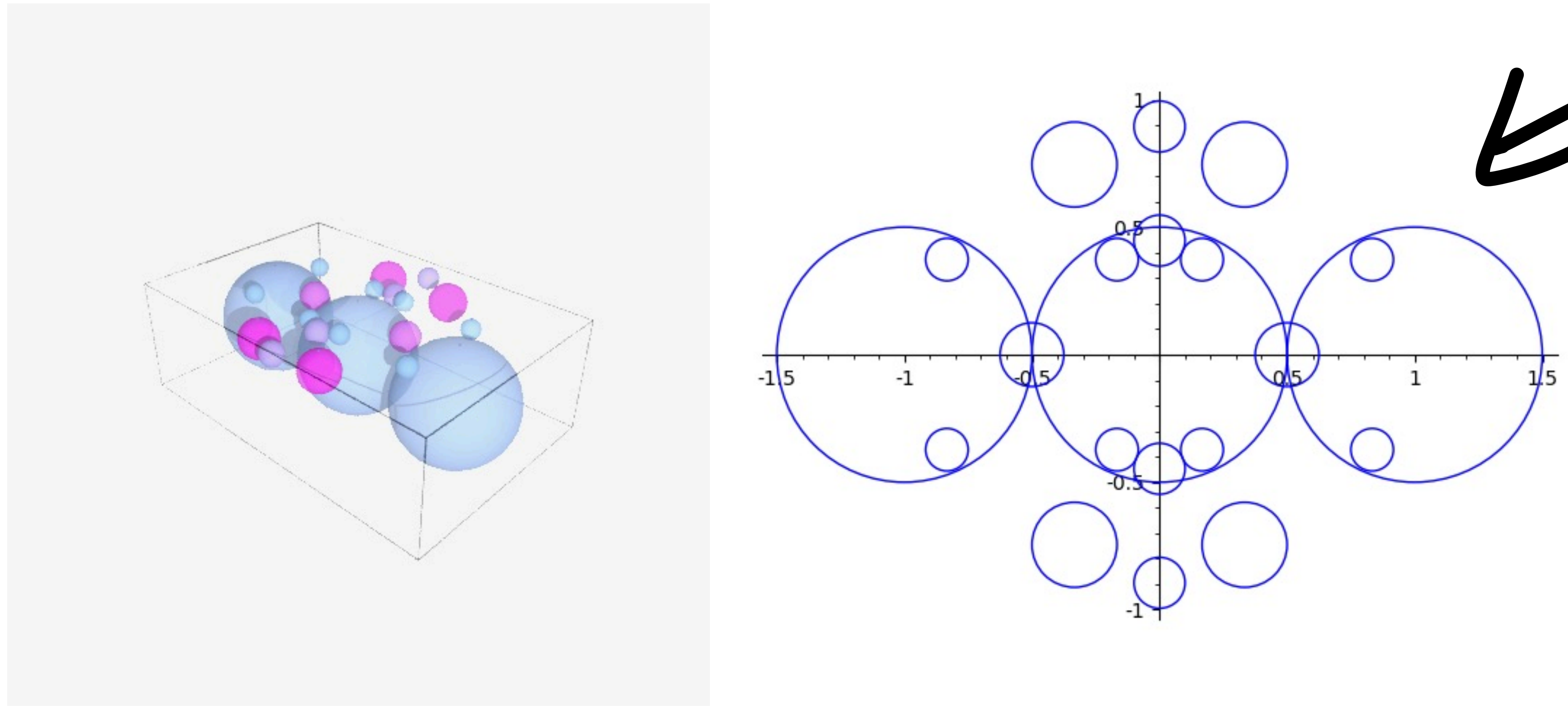


Packing for $\mathbb{Q}(i)$

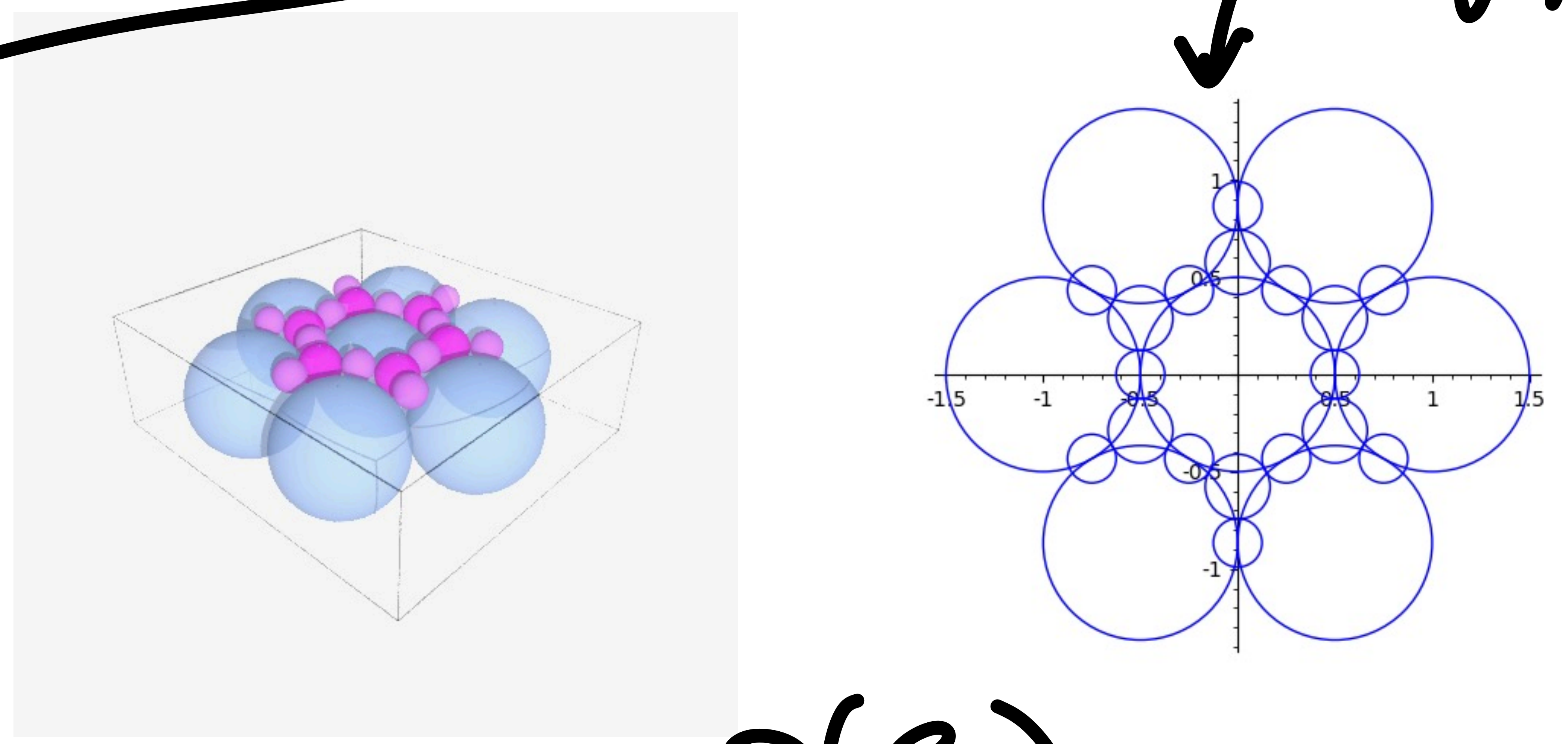


$\mathbb{Q}(\sqrt{-11})$

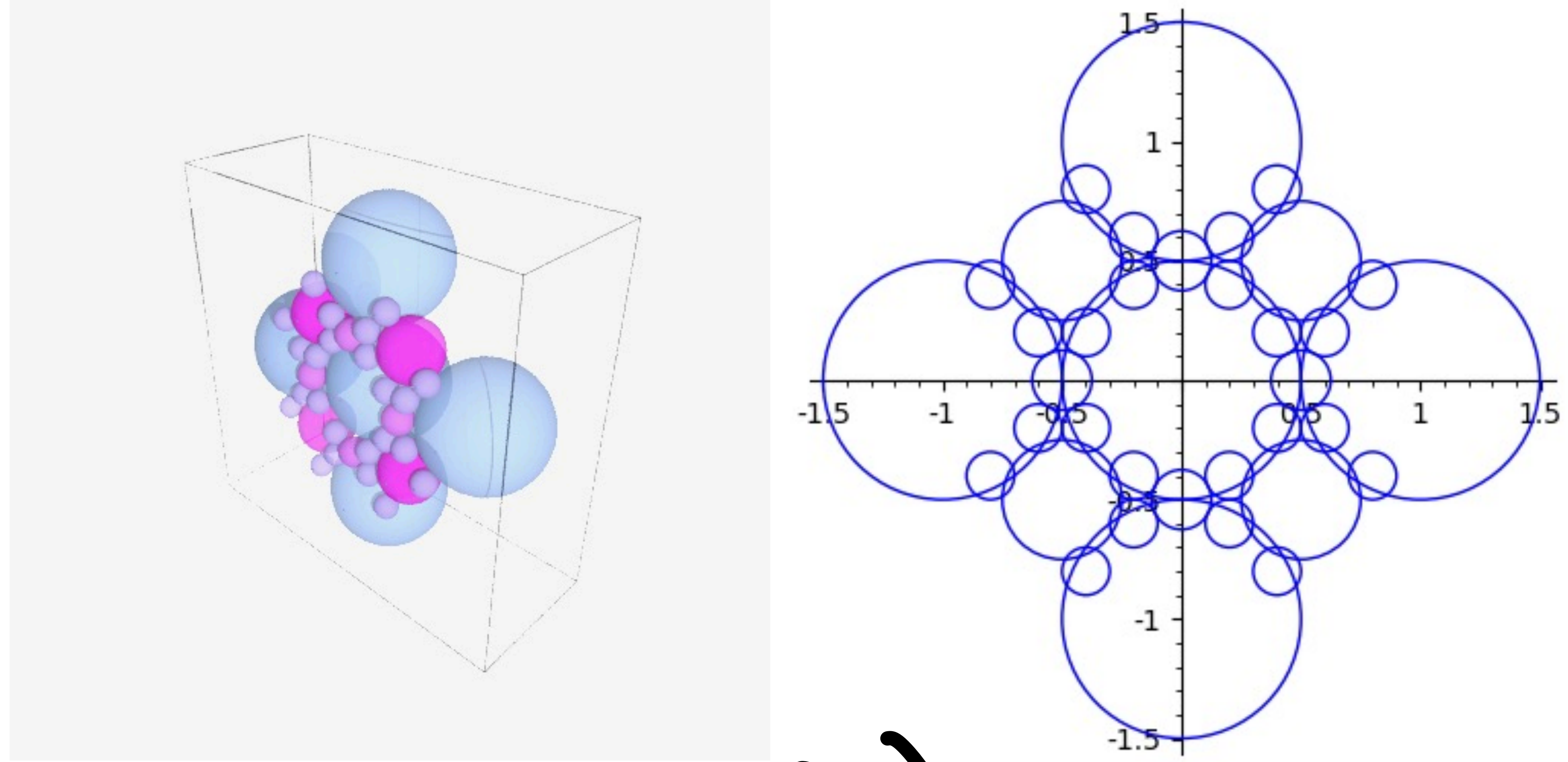
$\partial \mathbb{H}^3$



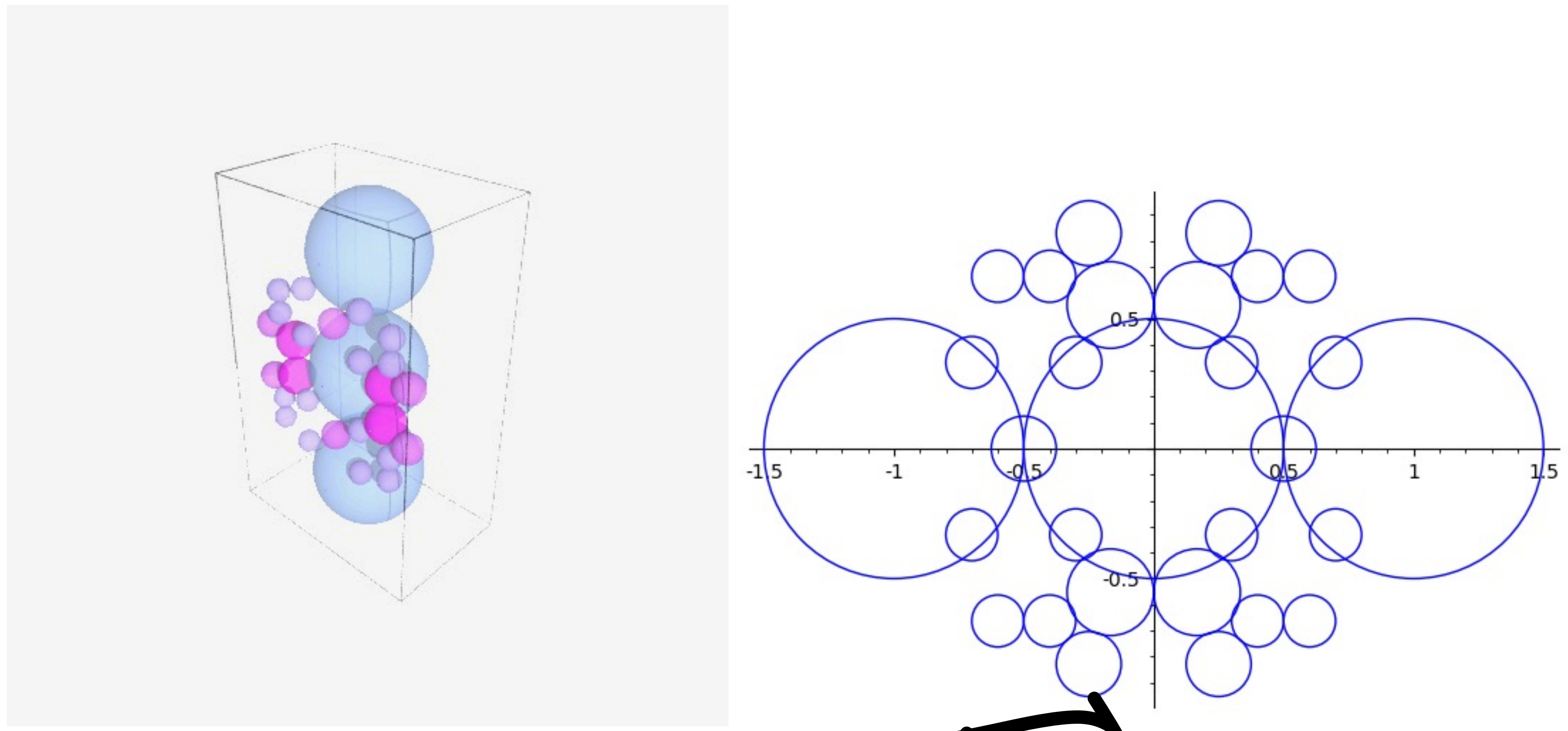
$\mathbb{Q}(\sqrt{-5})$



$\mathbb{Q}(\zeta_3)$



$\mathbb{Q}(i)$



$\mathbb{Q}(\sqrt{-11})$

Classical Case:

$$\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \underbrace{\mathcal{H}^2 \cup \overbrace{\mathbb{Q} \cup \{\infty\}}^{\text{cusps}}}_{\text{Satake compactification}}$$

Bianchi Case:

$$\mathrm{PSL}_2(\mathbb{Z}[i]) \curvearrowright \underbrace{\mathcal{H}^3 \cup \overbrace{\mathbb{Q}(i) \cup \{\infty\}}^{\text{cusps}}}_{\text{Satake compactification}}$$

$$PSL_2(\mathbb{Z}) \curvearrowright \mathcal{H}^2 \cup \mathbb{Q} \cup \{\infty\}$$

$$PSL_2(\mathbb{Z}[i]) \curvearrowright \mathcal{H}^3 \cup \mathbb{Q}(i) \cup \{\infty\}$$

NEW CASE

$$PSL_2(\text{Hurwitz Quaternions}) \curvearrowright \mathcal{H}^4 \cup \underbrace{(\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j \cup \{\infty\})}_{\text{cusps}}$$

$$O_3 = \mathbb{Z}\left[i, j, \frac{1+i+j+ij}{2}\right]$$

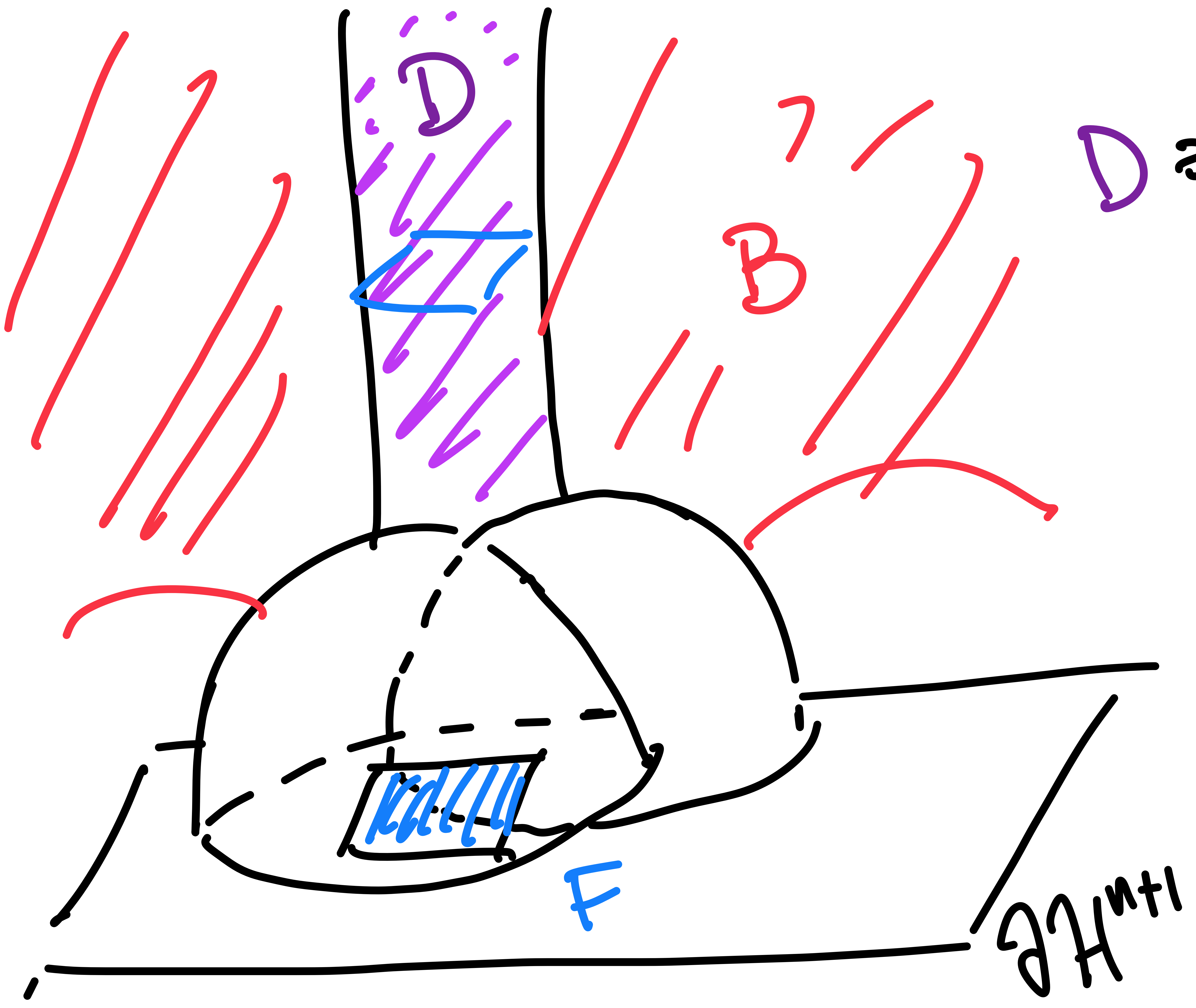
$$PSL_2\left(\begin{smallmatrix}\text{Hurwitz}\\\text{Quaternions}\end{smallmatrix}\right) \curvearrowright \mathbb{H}^4 \cup (\underbrace{\mathbb{Q}+\mathbb{Q}i+\mathbb{Q}j}_{\text{Clifford Vectors}} \cup \{\infty\})$$

KEY OBSERVATION

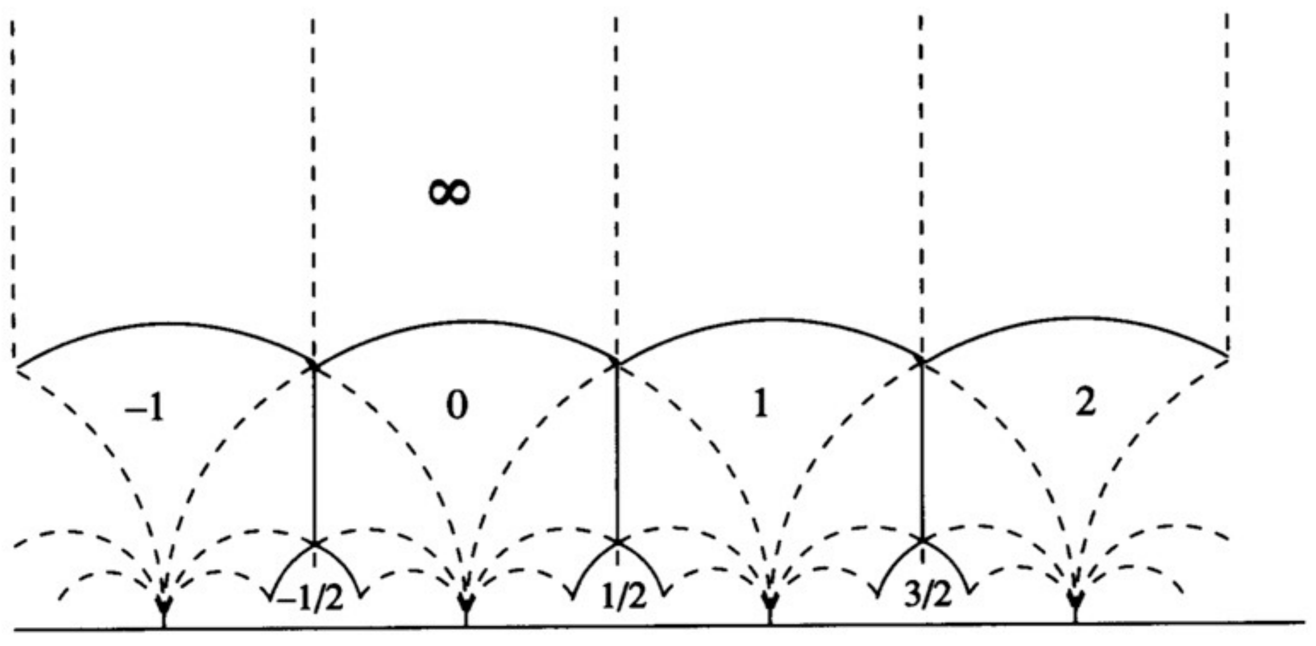
General Setup:

$$PSL_2\left(\begin{smallmatrix}\text{Order in}\\\text{Clifford Algebra}\end{smallmatrix}\right) \curvearrowright \underbrace{\mathbb{H}^{n+1} \cup \mathrm{Vec}(K) \cup \{\infty\}}_{\text{Satake compactification}}$$

Fundamental Domains



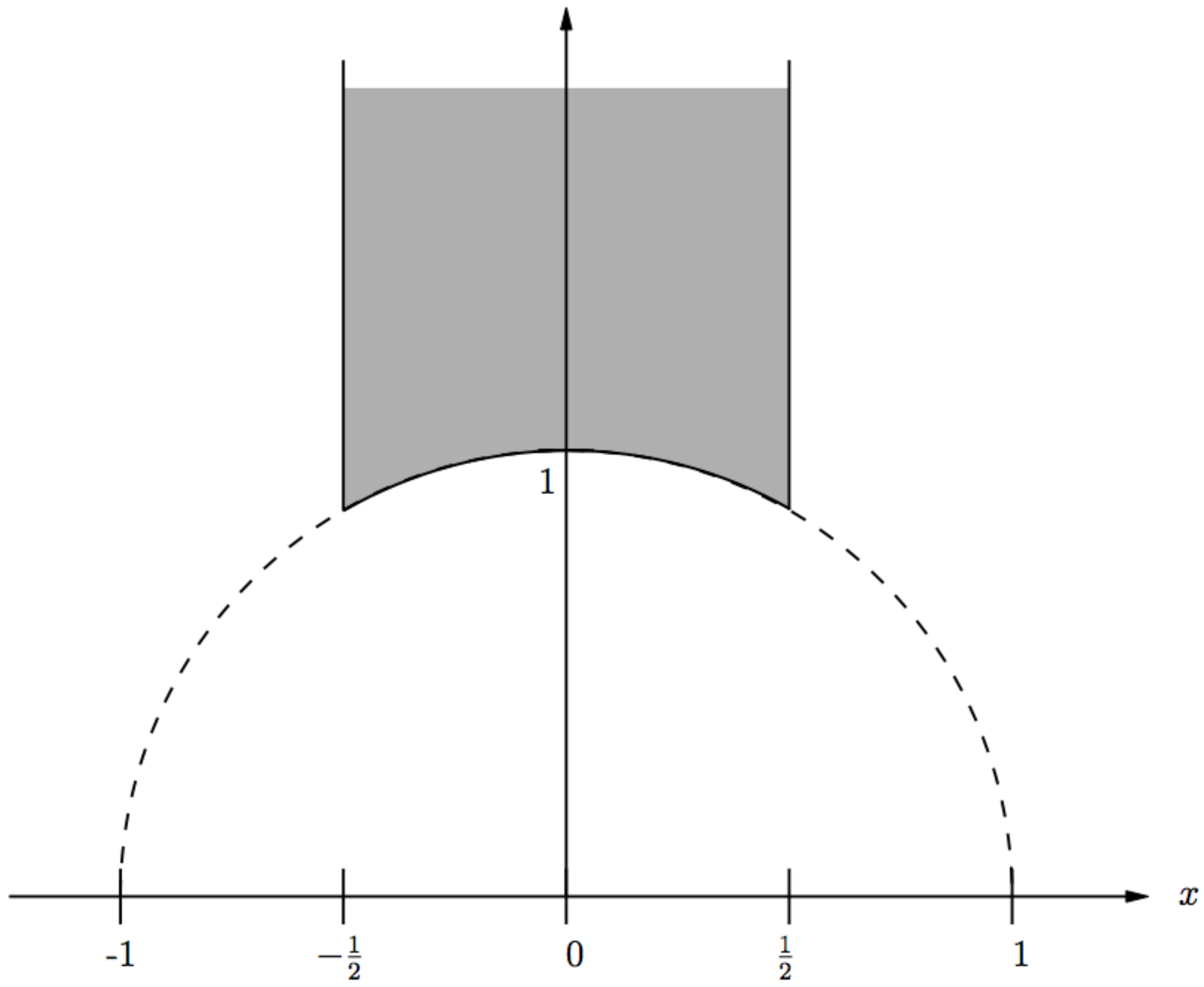
$$D \ni \underbrace{\underbrace{x_0 + i_1 x_1 + \cdots + i_{n-1} x_{n-1}}_{\in F} + i_n x_n}_{\in B}$$



27

$\mathcal{O}_1 = \mathbb{Z}$

$PSL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2$



A fundamental domain for the action of $SL(2,\mathbb{Z})$ acting on t
gion). The cusp is at $y \to \infty$.

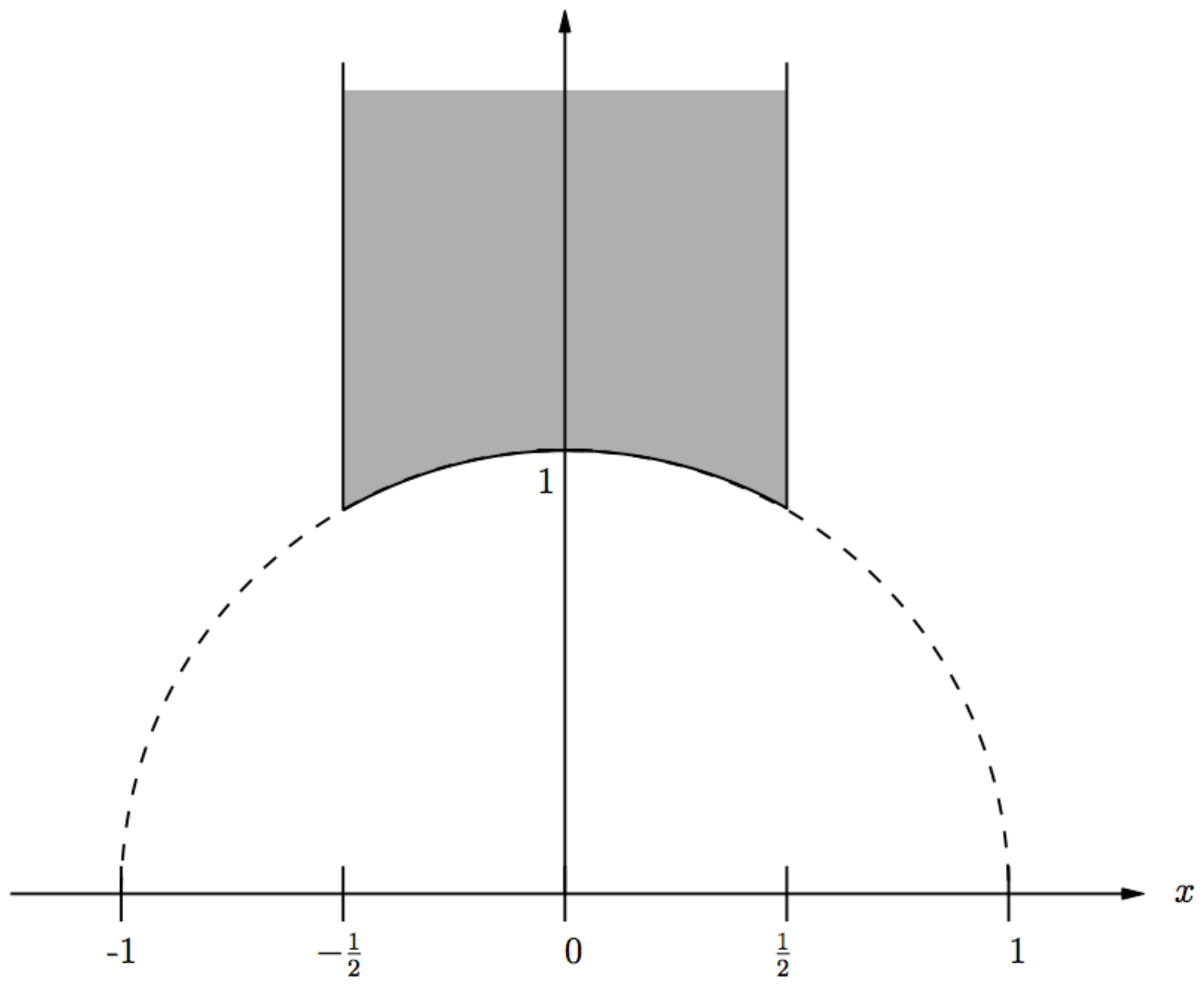
$\mathcal{O}_1 = \mathbb{Z}$

$\mathcal{O}_2 = \mathbb{Z}[i]$

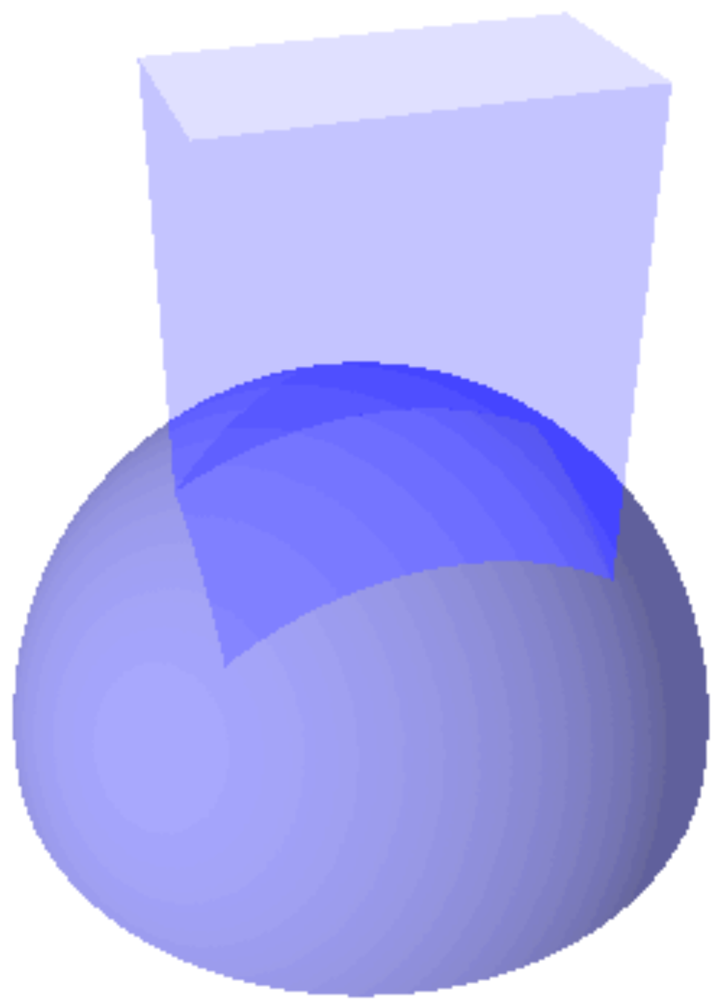
$i^2 = -1$

$PSL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2$

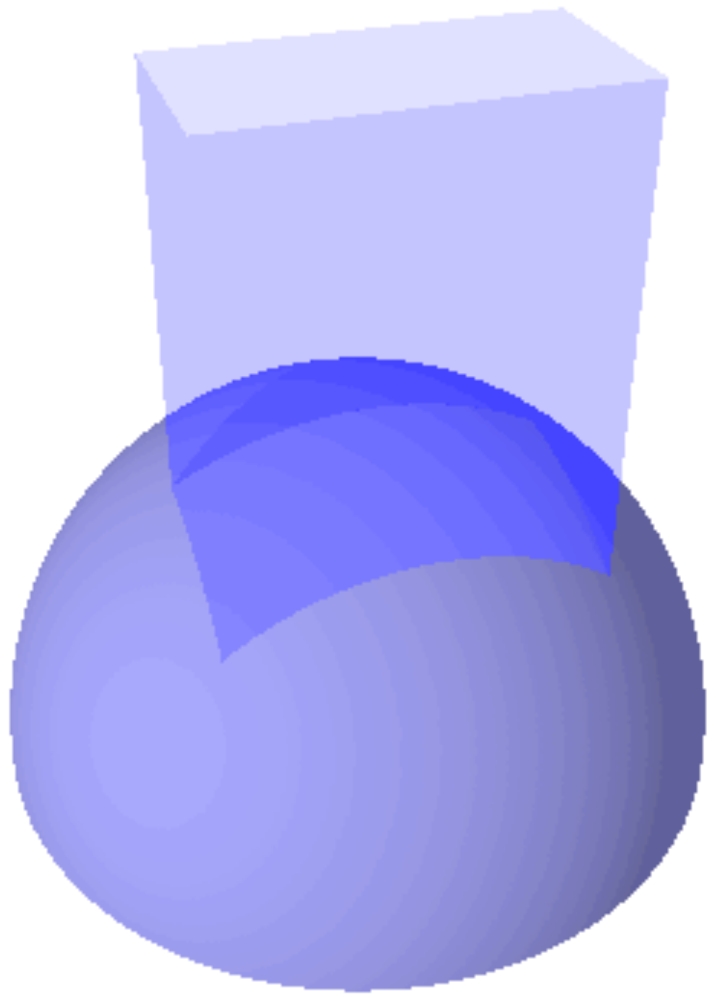
$PSL_2(\mathbb{Z}[i]) \curvearrowright \mathbb{H}^3$

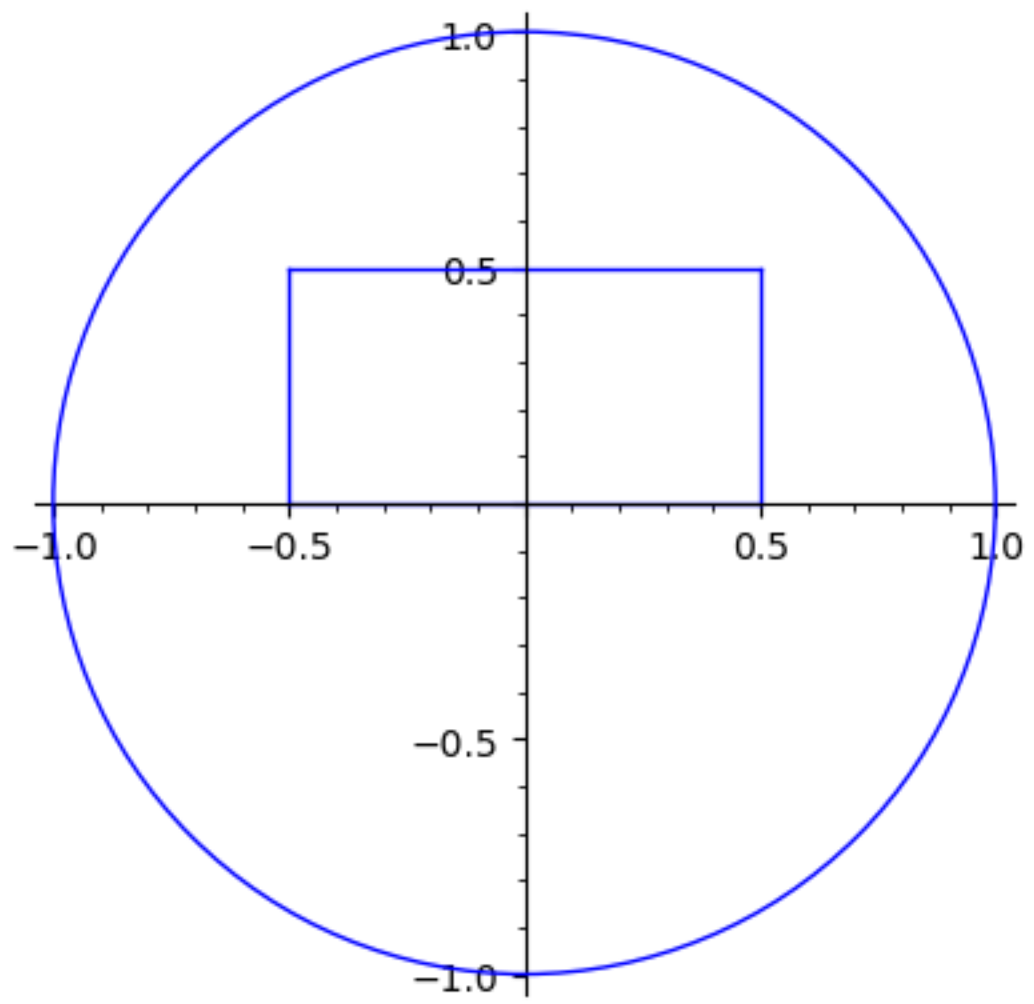


A *fundamental domain for the action of $SL(2,\mathbb{Z})$ acting on t*
gion). The cusp is at $y \to \infty$.

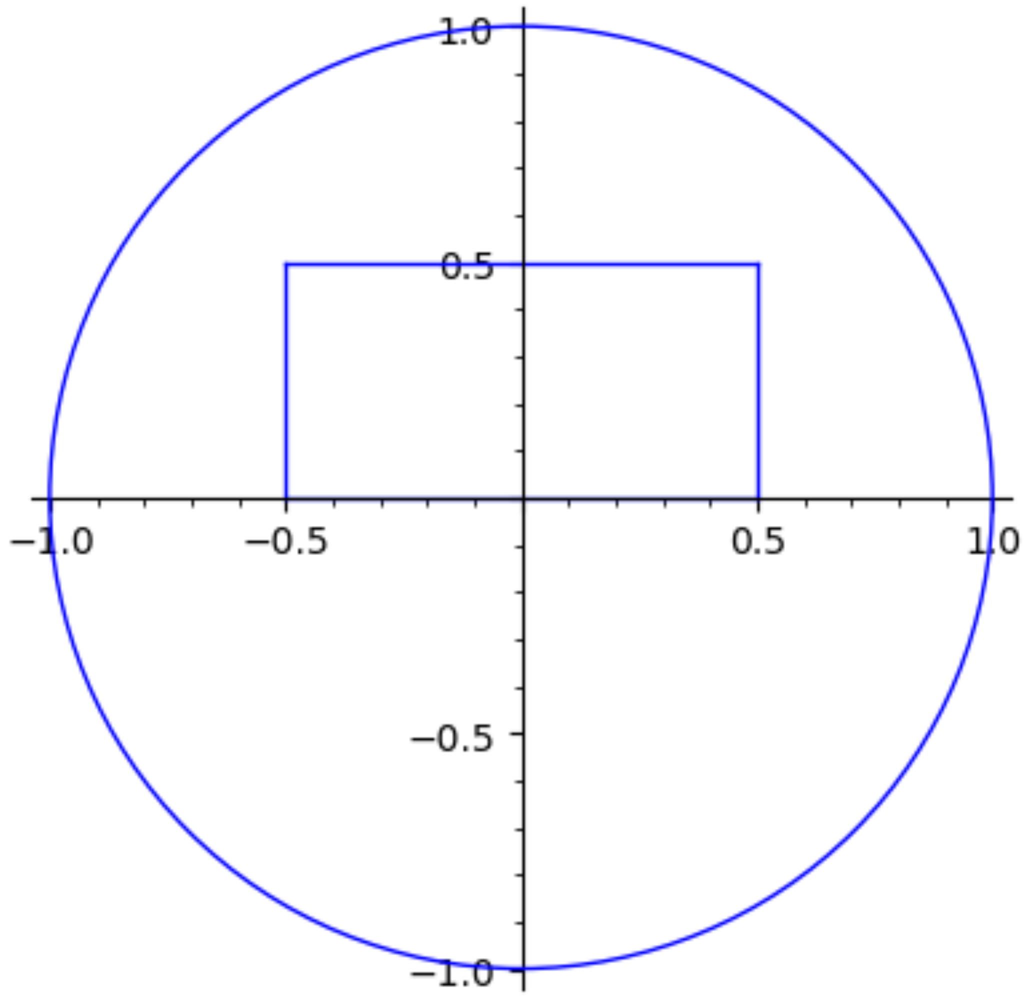


$PSL_2(\mathbb{Z}[i]) \curvearrowright \mathcal{H}^3$

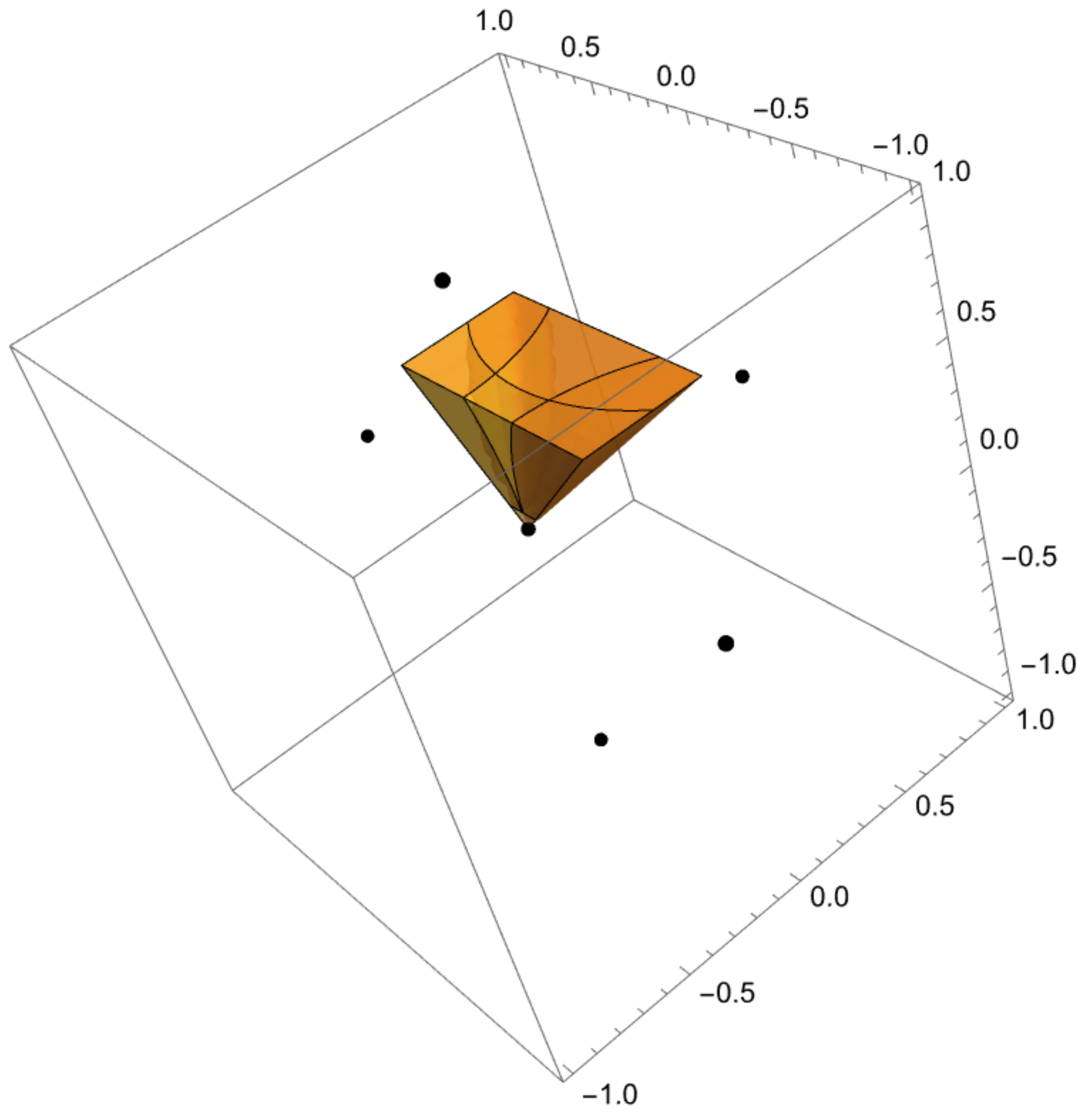




$PSL_2(\mathbb{Z}[i]) \curvearrowright \mathcal{H}^3$



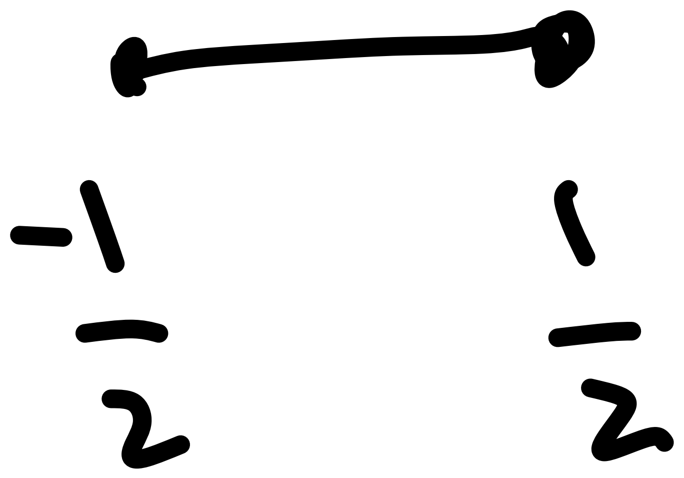
$PSL_2(\mathcal{O}_3) \curvearrowright \mathcal{H}^4$



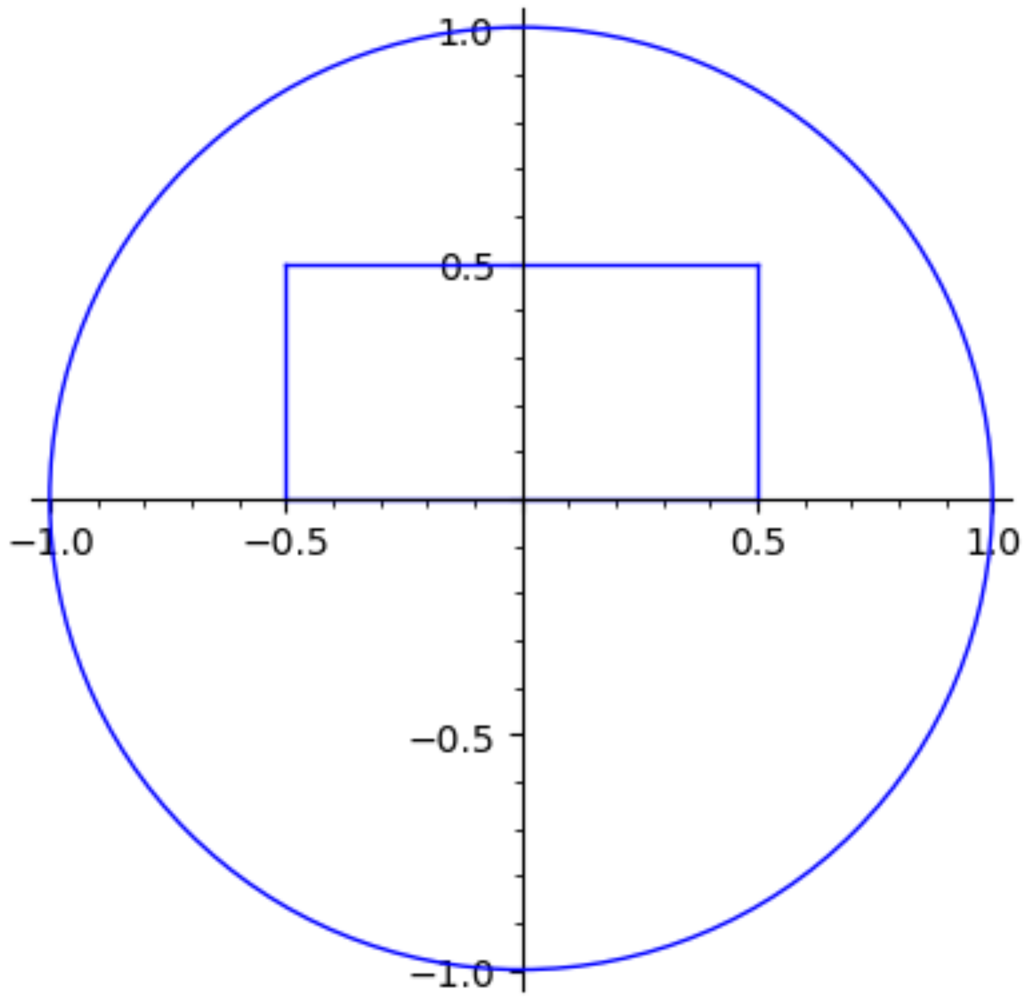
$$O_4 = \mathbb{Z}[i_1, i_2, i_3, \zeta, \alpha], \quad \zeta = (1 + i_1 + i_2 + i_3)/2$$

$$\alpha = (i_1 - i_{12} - i_{23} - i_{31})/2$$

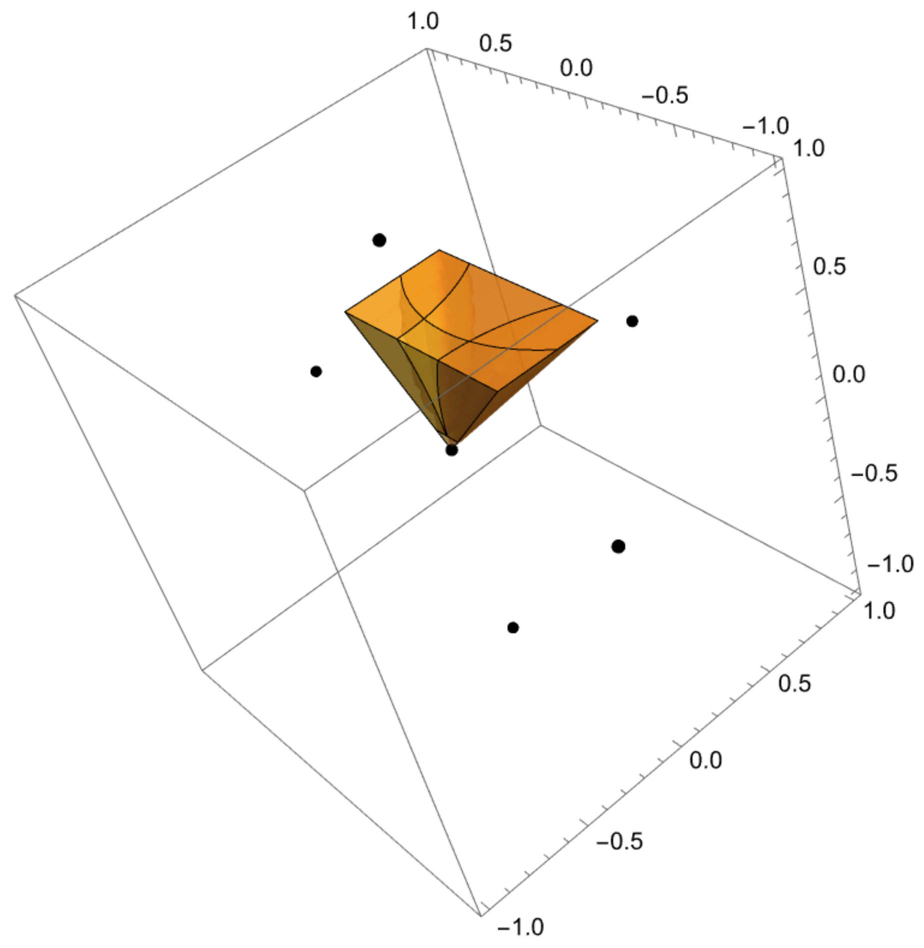
$\mathcal{O}_1 = \mathbb{Z}$



$\mathcal{O}_2 = \mathbb{Z}[i]$



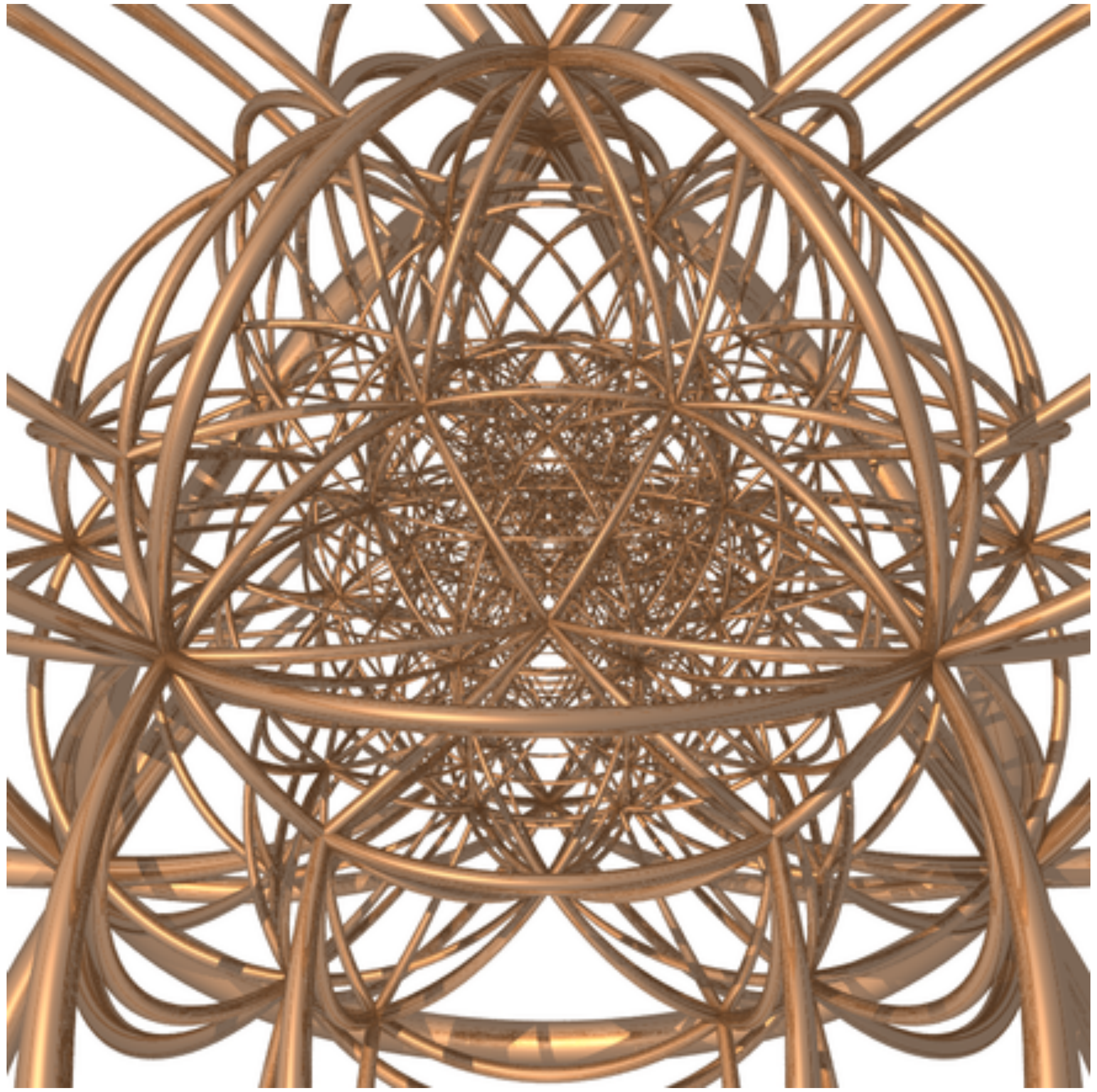
$\mathcal{O}_3 = \mathbb{Z}[i, j, \delta]$



$\mathcal{O}_4 = \mathbb{Z}[i_1, i_2, i_3, \xi, \alpha]$

???
. . .

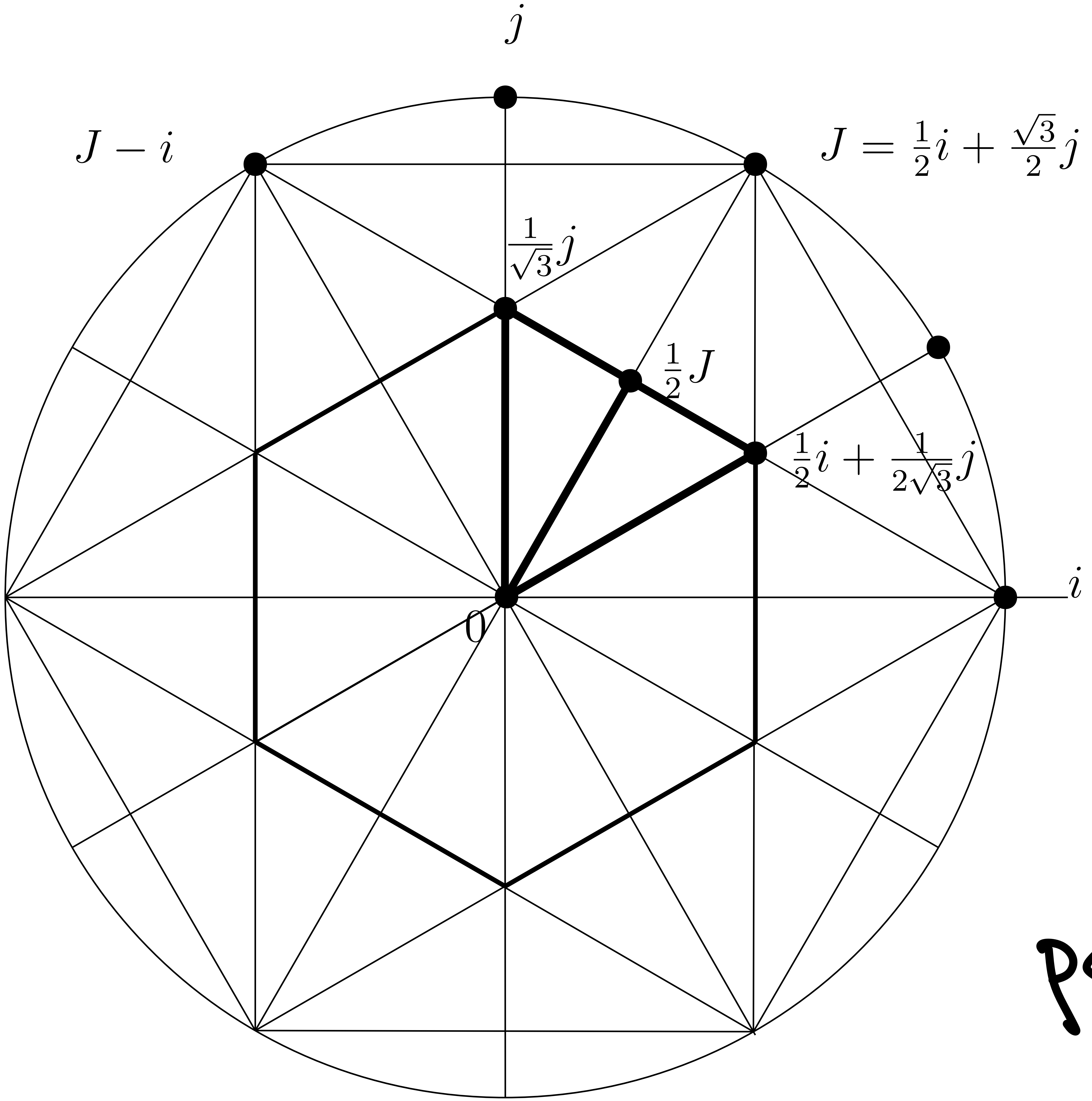
$$O_4 = \mathbb{Z}[i_1, i_2, i_3, \zeta, \alpha]$$



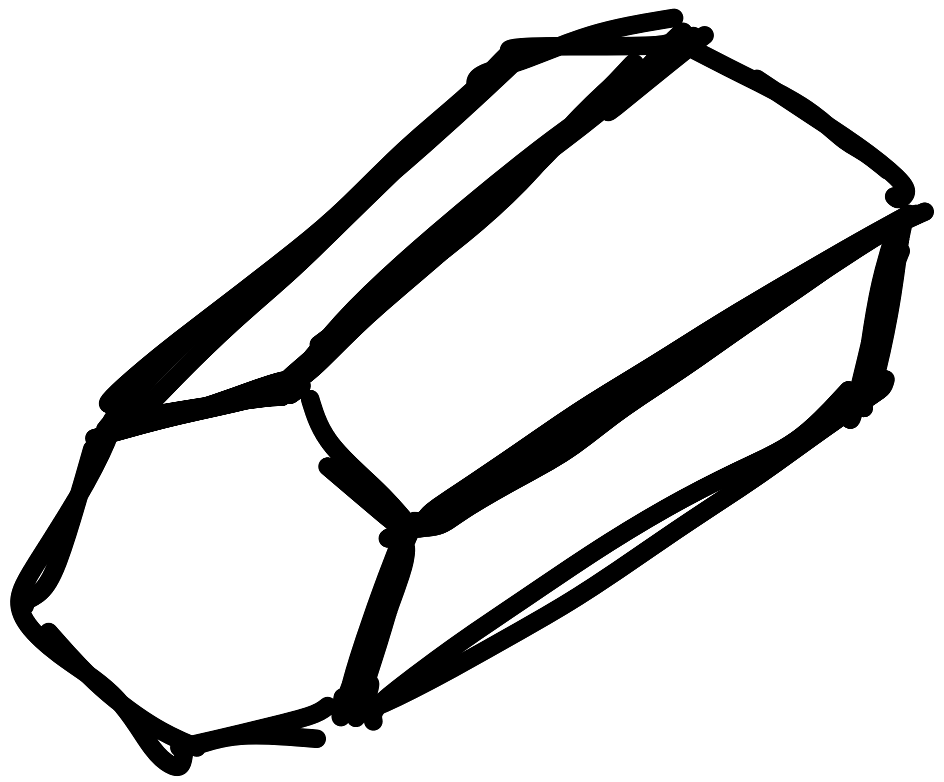
24-cell

4-copies

[from Wikipedia: projection of 1-cells of "24-cell" into 3D]



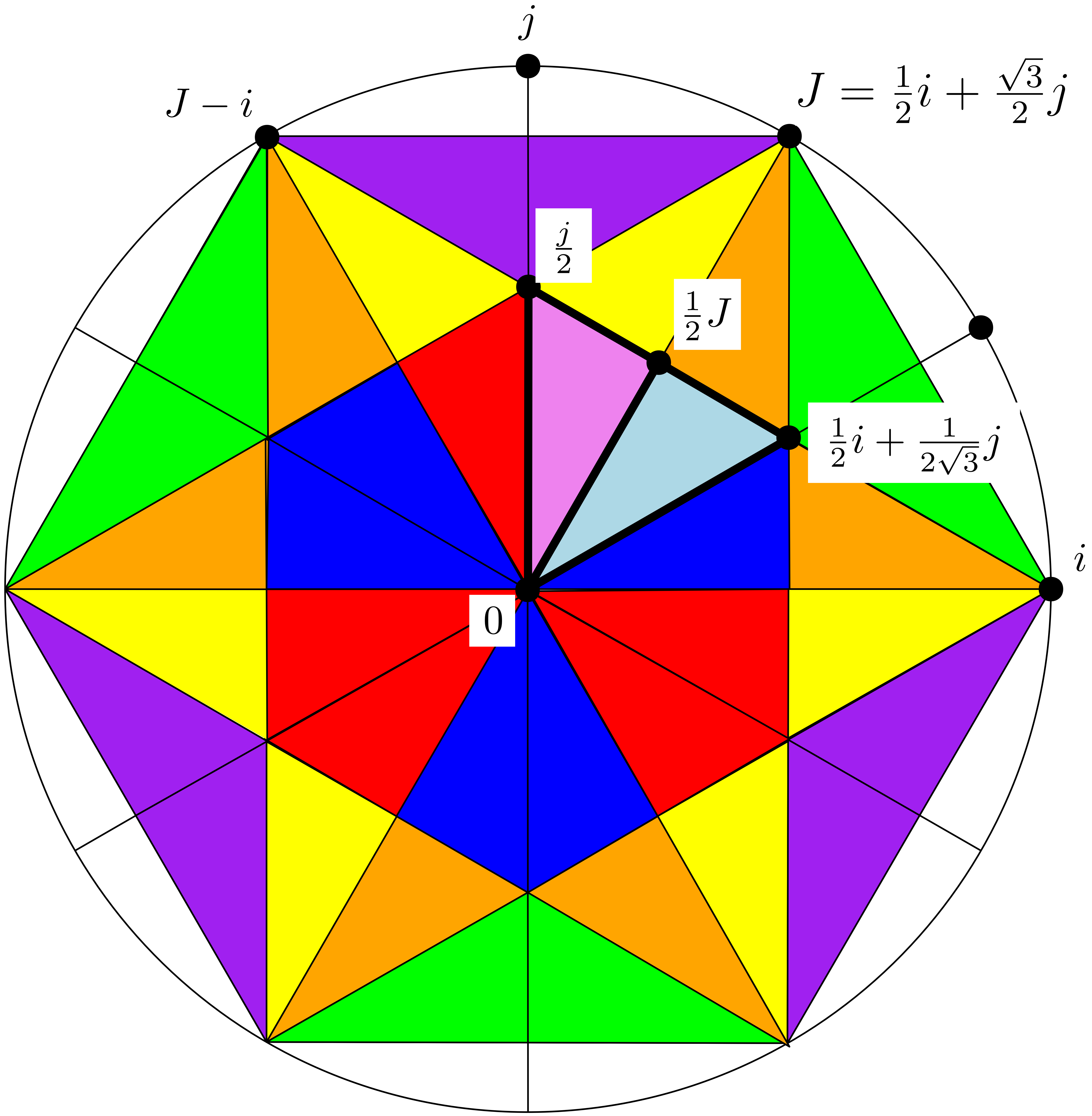
$\Theta(-1,-3)_2$



Allan Wrench

$PSL_2(\mathcal{O}_3) \curvearrowright \mathcal{H}^4$

$\mathcal{O}(-1,-3)_2$



What are these Groups?

$$PSL_2(\text{Order in Clifford Algebra}) \curvearrowright \mathbb{Z}^{n+1} \cup \mathrm{Vec}(K) \cup \{\infty\}$$

$$SL_2(\mathbb{C}_n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{C}_n^{\times} \\ ad^{*} - bc^{*} = 1 \\ a^{*}c,\ b^{*}d \in \mathbb{R} \end{array} \right\}$$

Clifford Group

Official Definition

Various Conjugations

$$\mathbb{C}_n^{\times} = (\text{Clifford Group})$$

$$(i_1+i_2)i_3 \qquad (1+i_1+2i_3)(1-i_1+7i_3)(2i_1+1)$$

Generated by Clifford Vectors

$$x\bar{x} = |x|^2, \qquad \bar{x} = (x')^{*} = (x^{*})'$$

reversal

parity

Ask that the group behaves as you want it

Definition:

$$GL_2(\mathbb{C}_n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}_n) : \forall m \geq n \; \forall x \in V_m, \; (ax+b)(cx+d)^{-1} \in V_m \right\}$$

Theorem: This gives the weird definition

Proof Idea:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = g$$

$$g(0) = \lim_{x \to 0} (ax+b)(cx+d)^{-1} = bd^{-1}$$

$$g(\infty) = \lim_{x \to \infty} (ax+b)(cx+d)^{-1} = ac^{-1}$$

FORCES RATIOS TO BE VECTORS!

Proof Idea:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = g$$

$g^{-1}(\infty) = ???$

$$\underbrace{(ax+b)(cx+d)^{-1}}$$

$cx+d=0 \iff x = -c^{-1}d$

More vector ratios forced!

Key observations

Clifford Vectors

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a^{-1}b$

$c^{-1}d$

ac^{-1} $\quad$ bd^{-1}

Key observations

Clifford Vectors

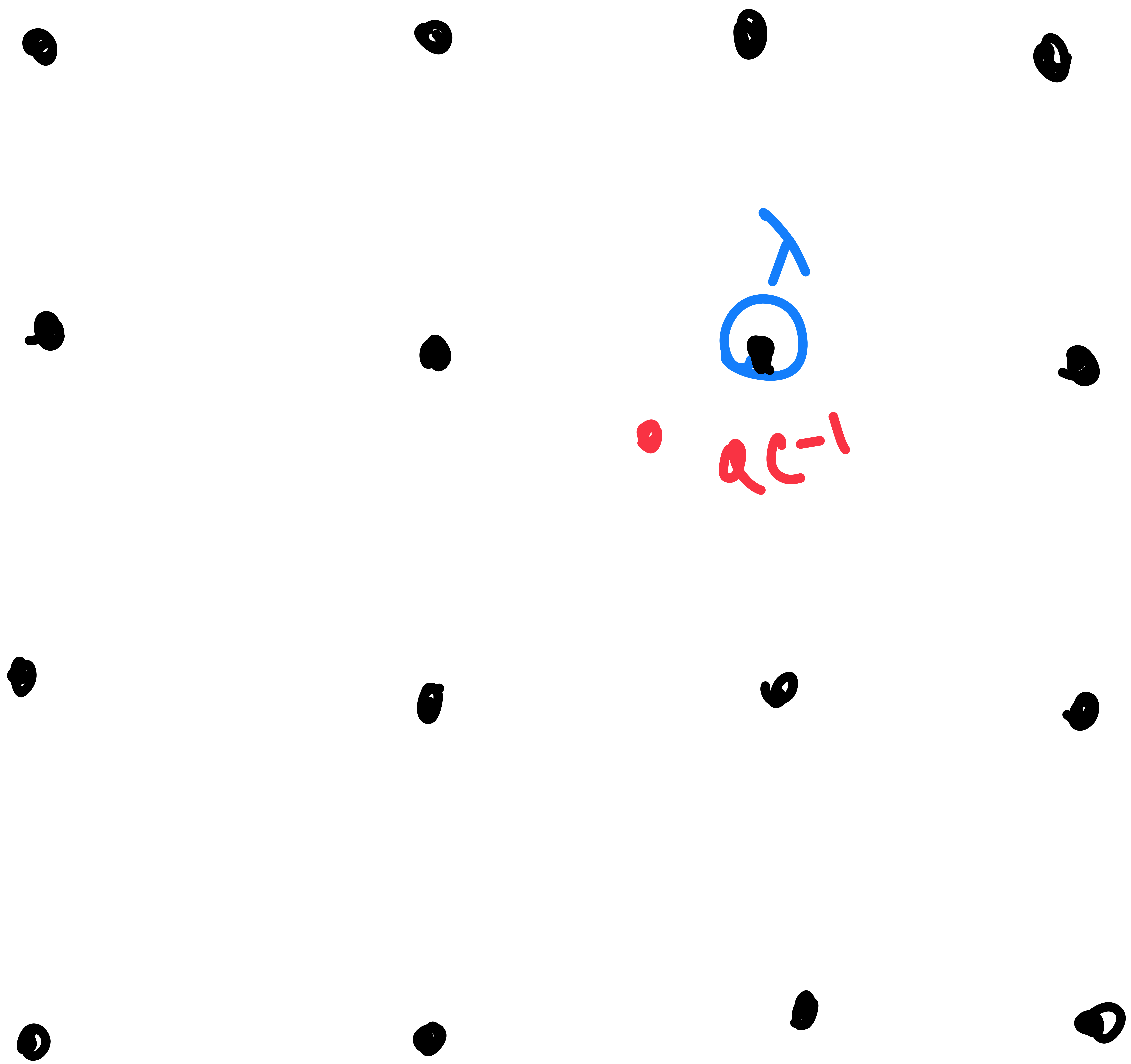
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a^{-1}b \\ c^{-1}d \end{matrix}$$

$$ac^{-1} \qquad bd^{-1}$$

Euclidean Algorithm

Euclidean Algorithm

$\mathrm{Vec}(\Theta) = \Lambda$



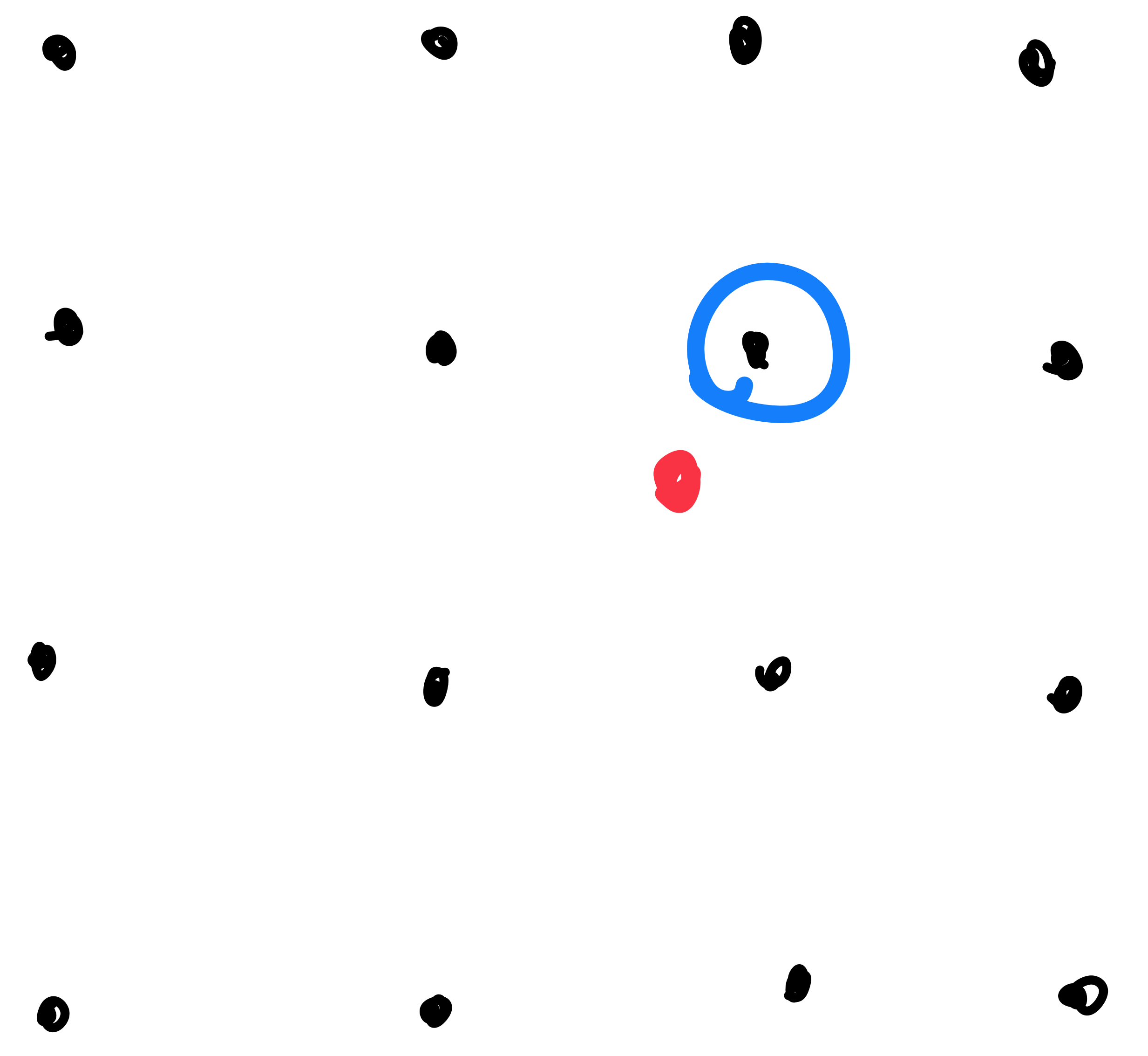
Euclidean Algorithm

$\mathrm{Vec}(\mathcal{O}) = \Lambda$

$$|\lambda - ac^{-1}| < 1$$

$$\rightsquigarrow |\underbrace{a - \lambda c}_{r}| < |c|$$

$$a = \lambda c + r$$



Euclidean Algorithm

Definition of $\mathcal{O}$ being Clifford-Euclidean:

Let $a, c \in \mathcal{O}^{\Delta}$ such that $ac^{-1} \in \mathrm{Vec}(K)$

$\exists \lambda \in \mathrm{Vec}(\mathcal{O})$, $\exists r \in \mathcal{O}^{\Delta}$ such that

$$a = \lambda c + r$$

where $|a - \lambda c| < |c|$

[In general, replace $|x|$ with $N(x)$ for some Norm function.]

Analogs of Gaussian Integers

Analogs of Gaussian Integers

Computed in Magma

The simplest order are maximal $\mathcal{O}$ with

$$\mathbb{Z}[i_1, \dots, i_{n-1}] \subseteq \mathcal{O} \subseteq \mathbb{Q}(i_1, \dots, i_{n-1})$$

$\underline{n=1}$: $\mathbb{Z}$

$\underline{n=2}$: $\mathbb{Z}[i]$

$\underline{n=3}$: $\mathcal{O}_3 = $ (Hurwitz Quaternions)

$$= \mathbb{Z}\left[i, j, \frac{1+i+j+ij}{2}\right]$$

$$\mathrm{Vec}(\mathcal{O}_3) \cong \mathbb{Z}^3$$

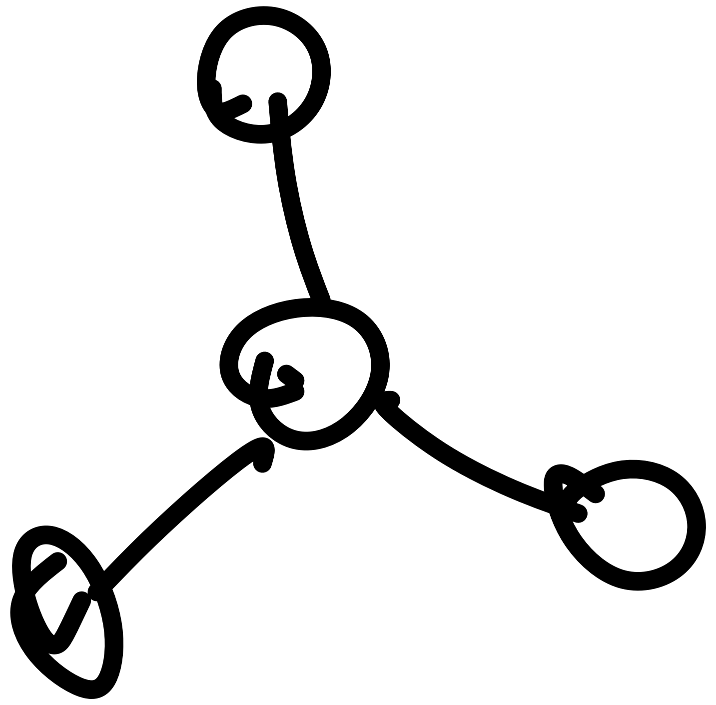
<u>$n=4$</u>: $O_4 = \mathbb{Z}[i_1, i_2, i_3, \zeta, \alpha]$, $\zeta = (1+i_1+i_2+i_3)/2$

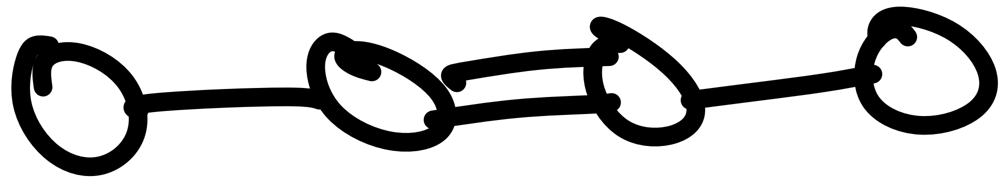
(new!) $\alpha = (i_1 - i_{12} - i_{23} - i_{31})/2$

$$\mathrm{Vec}(O_4) \cong \tfrac{1}{2} D_4 \cong \mathbb{Z}^4 + \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\mathbb{Z}$$

Interesting Root Lattice.

Has D_4 and F_4 root systems:





(not simply laced)

$O_4^\times >$ (clifford group)

acts on $\mathrm{Vec}(O_4)$

$O_4^\times = \begin{pmatrix}\text{clifford}\\ \text{group}\end{pmatrix}$ acts on $\mathrm{Vec}(O_4)$

given $u \in O_4^\times$, forces $|u| = 1$.

$\pi_u(x) = u x u^* \leftarrow$ rotation.

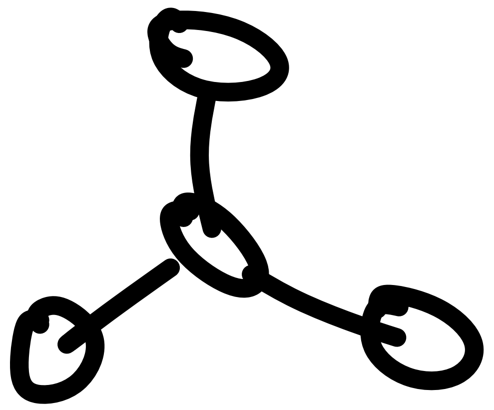
not vector

$O_4^\times = \langle i_1, i_2, i_3, \zeta, \alpha \rangle$

orientation pres

$u \in \mathrm{Vec}(O_4)^\times = \mathrm{Vec}(O_4) \cap O_4^\times \rightsquigarrow \mathrm{Weyl}(D_4)^+$

$\pi_\alpha = \begin{pmatrix}\text{triality}\\ \text{map}\end{pmatrix} = \Bigg($ permutation of



$\Bigg)$

UPSHOT: The group $\mathcal{O}_4^\times$ gives interesting outer actions.

$\underline{n=5}$: no longer unique order

$$\left\{\begin{array}{l} O_{5,!} \quad \text{odd ball} \\ \\ O_{5,j},\ j=0,1,2,3,4 \quad \leftarrow \text{all similar} \end{array}\right.$$

$$\frac{1+i_1+i_2+i_1 i_2}{2}$$

$$\frac{1+i_1+i_2+i_3}{2} \quad 11110$$

$$\frac{i_1+i_2+i_3+i_4}{2} \quad 01111$$

$$\frac{1+i_1+i_3+i_4}{2} \quad 11011$$

$$\frac{1+i_2+i_3+i_4}{2} \quad 10111$$

Orders Associated To Codes:

Let C be a doubly even code in $\mathbb{F}_2^n$.

$$\mathcal{O}_C := \left(\text{unique maximal order containing } \mathbb{Z}\left[\frac{I \cdot c}{2} : c \in C\right]\right)$$

+

$$\mathrm{Vec}(\mathcal{O}_C) = \Lambda_C = (\text{lattice assoc to code})$$

(Conjectural Construction)

example: O_4 1111 code, $wt(1111)=4$
$d=1$

example The five dimensional cases

- odd ball $\Leftrightarrow$ trivial code
- (other orders) $\Leftrightarrow$ 7-dim'l $[5,1,4]$ codes

01111, 10111, 11011, 11101, 11110

spans of these in $\mathbb{F}_2^5$

example: $C = H(8,4)$ Hamming Code

$\Theta_{H(8,4)}$ = (order assoc to $H(8,4)$)

$\mathrm{Vec}(\Theta_{H(8,4)}) = E_8$

Qstn.

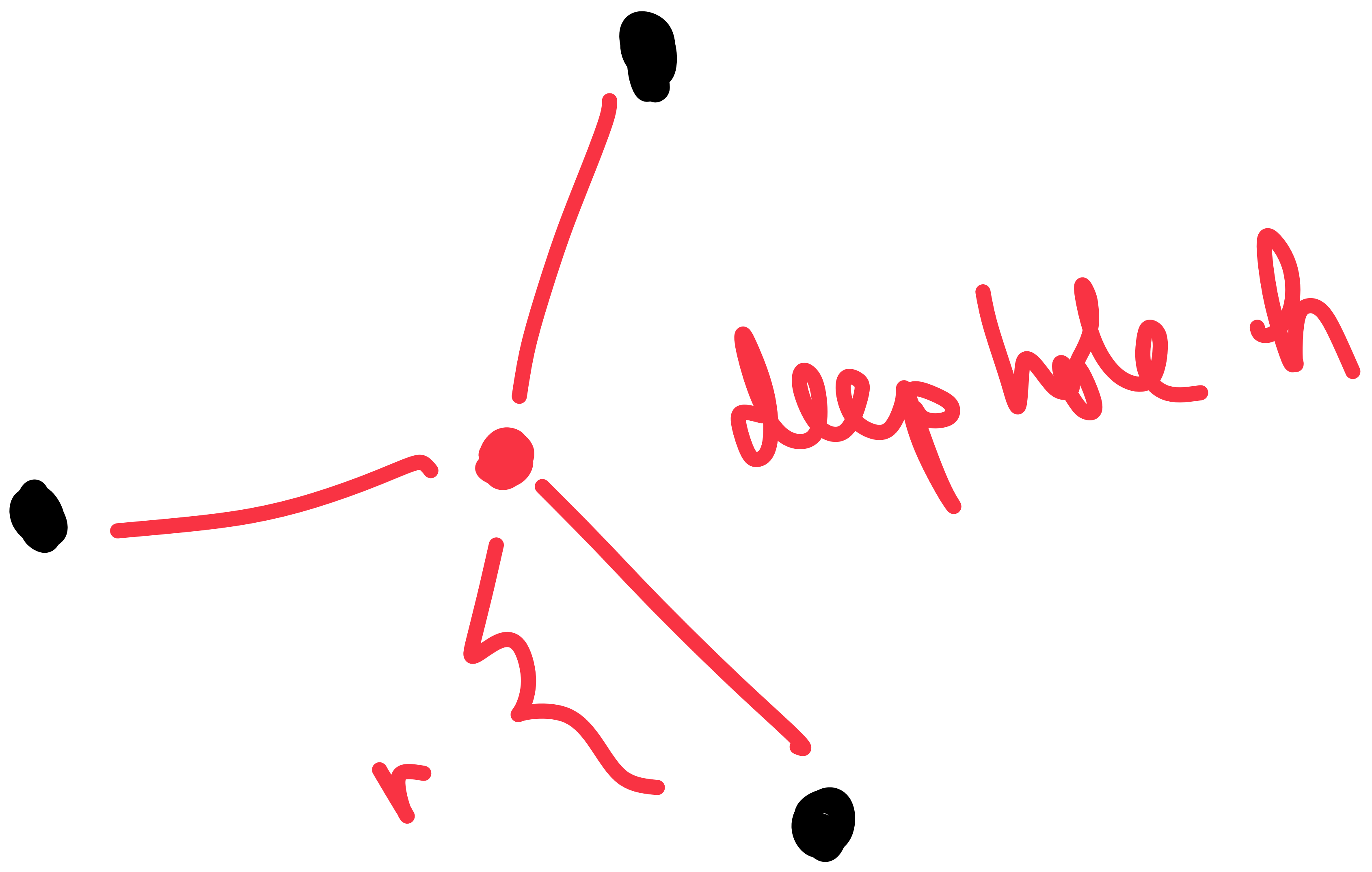
1) G_{24} Golay code $\rightsquigarrow$ Λ_{24} Leech lattice

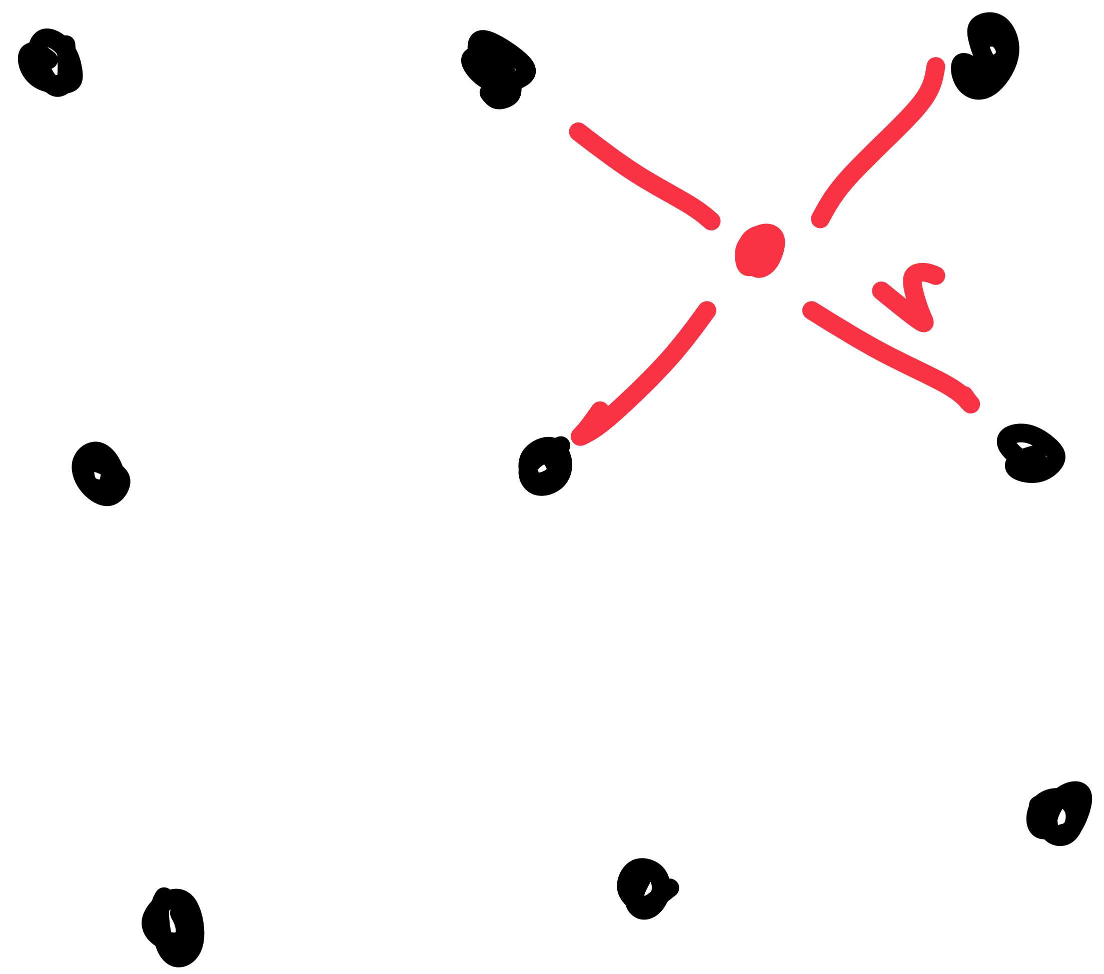
(Does there exist an order)

2) What about other Neimeier lattices?

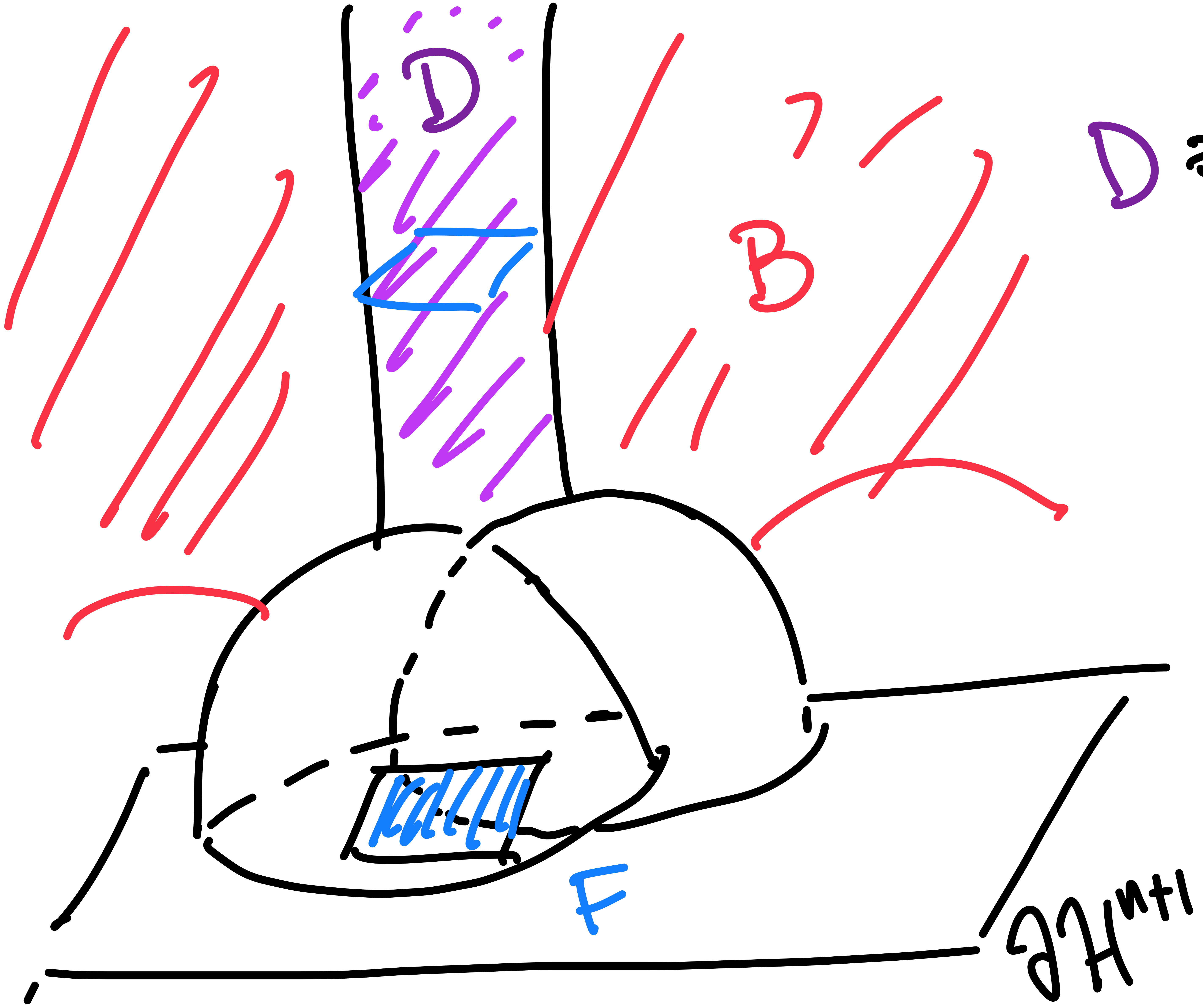
3) Connection to Bott periodicity?

Covering Radius





Analogs of Gaussian Integers



$$D \ni \underbrace{\underbrace{x_0 + i_1 x_1 + \cdots + i_{n-1} x_n}_{\in F} + i_n x_n}_{\in B}$$

Fundamental Domain Algorithms

Step 1: Give a "proof by moduli interpretation" that B has unique representatives.

Step 2: Use description of B to get bounding spheres.

Fundamental Domain Algorithms

<u>Step 1</u>: Give a "proof by moduli interpretation" that B has unique representatives.

$$
\begin{tikzcd}
\mathbb{Z}^3 \arrow[r] & \left(\text{Positive Definite Hermitian Forms}\right) / \mathbb{R}^\times_{>0} \\
B \arrow[u, phantom, "\cup" description] \arrow[r] & \left(\text{forms with minimal proper value at } (1,0)\right) \arrow[u, phantom, "\cup" description]
\end{tikzcd}
$$

"bubble domain"
(representatives of max'd height)

[There is a similar story for Clifford Hermitian Matrices]

Fundamental Domain Algorithms

Unimodular: $(\mu, \lambda) \in \mathcal{O}^2$ right unimodular iff

$$\exists \begin{bmatrix} \alpha & x \\ \mu & \lambda \end{bmatrix} \in SL_2(\mathcal{O})$$

Bubble: $S_{(\lambda,\mu)} = S(\mu^{-1}\lambda)$ $\quad |x - \mu^{-1}\lambda| = 1/|\mu|$

Bubble Domain:

$$\{ x \in \mathcal{H}^{n+1} : \forall (\lambda,\mu),\ |x - \mu^{-1}\lambda| \geq 1/|\mu| \}$$

$$SL_2(\mathbb{C}_n) \cong \mathrm{Spin}_{1,n+1}(\mathbb{R}) \longrightarrow SO_{1,n+1}(\mathbb{R})^0$$

$$SL_2(Clf_q) \cong Clf(Q)_+$$

isom of $\mathbb{Z}$-group schemes.

Generalization:

$$\left[\begin{array}{l} \mathcal{H}^{n+1} = \{ x \in V_{n+1} : x_n > 0 \} \\ ds = \dfrac{d|x|}{x_n} \end{array}\right]$$ Clifford Uniformization of Hyperbolic Space.

Amazing Construction: $PSL_2(\mathbb{C}_n) \circlearrowright \mathcal{H}^{n+1}$ by

$$x \longmapsto (ax+b)(cx+d)^{-1}$$

History: Vahlen, Maass, Ahlfors, Waterman, Maclachlan-Weiland-Waterman, Elstrodt-Grunewald-Mennicke, Krausshar, Vulakh.

<u>Basic Idea</u>: Demand $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}_n)$ such that $\forall x \in V_m$ with $m \geq n$ that $(ax+b)(cx+d)^{-1} \in V_m$.

$\rightsquigarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $ac^{-1} \in V_n$, $bd^{-1} \in V_n$ (+ other conds)

$$\Delta = ad^* - bc^* \in \mathbb{R}$$

$$\rightsquigarrow GL_2(\mathbb{C}_n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{C}_n^\times \\ c^*a,\ d^*b \in V_n,\ \Delta \in \mathbb{R} \end{array} \right\}$$

$$\cup\!\!\!\mid$$

$SL_2(\mathbb{C}_n)$ ← WLOG transformations can be taken here.

Using a good theory of Weil Restriction, for q an integral quad form get

$$Sl_2(\mathbb{H}_q) \leftarrow \mathbb{Z}\text{-group scheme}$$
$$\cong$$
$$Spin_Q$$

} Spin group of some other quadratic form (comes from integral Bott periodicity)

$\rightsquigarrow$ exact sequence of $\mathbb{Z}$-group schemes (fppf sheaves)

$$1 \to \mu_2 \to \mathrm{Spin}_Q \to \underbrace{SO_Q}_{\mathbb{Z}\text{-form of } SO_{1,n+1}} \to 1$$

$SO_Q(\mathbb{R}) = SO_{1,n+1}(\mathbb{R})$.

UPSHOT: Given $\mathcal{O} \subseteq K = Clf(\mathbb{Q}^{q-1}, q)$ $(\mathcal{O} = \mathcal{O}^*)$
we have $SL_2(\mathcal{O})$ is an
an arithmetic subgroup of $SO_Q(\mathbb{R})$.

<u>Arithmetic Subgroups:</u>

- Means $\simeq G(\mathbb{Z})$, G $\mathbb{Z}$-linear alg group.
- Gives Borel-Harish-Chandra, $\Gamma = SL_2(\mathcal{O})$ discrete, finite covol
(∃ fundamental doms)

Exact Seq of $\mathbb{Z}$-Group Schemes

$$1 \to \mu_2 \to Spin_Q \to SO_Q \to 1$$

$$1 \to \mu_2(\mathbb{Z}) \to Spin_Q(\mathbb{Z}) \to SO_Q(\mathbb{Z}) \to H^1(Spec(\mathbb{Z}), \mu_2)$$

Allows us to prove $SL_2(\mathcal{O})$ is a genuine Arithmetic Group

Sheaves of groups on $Spec(\mathbb{Z})_{fppf}$

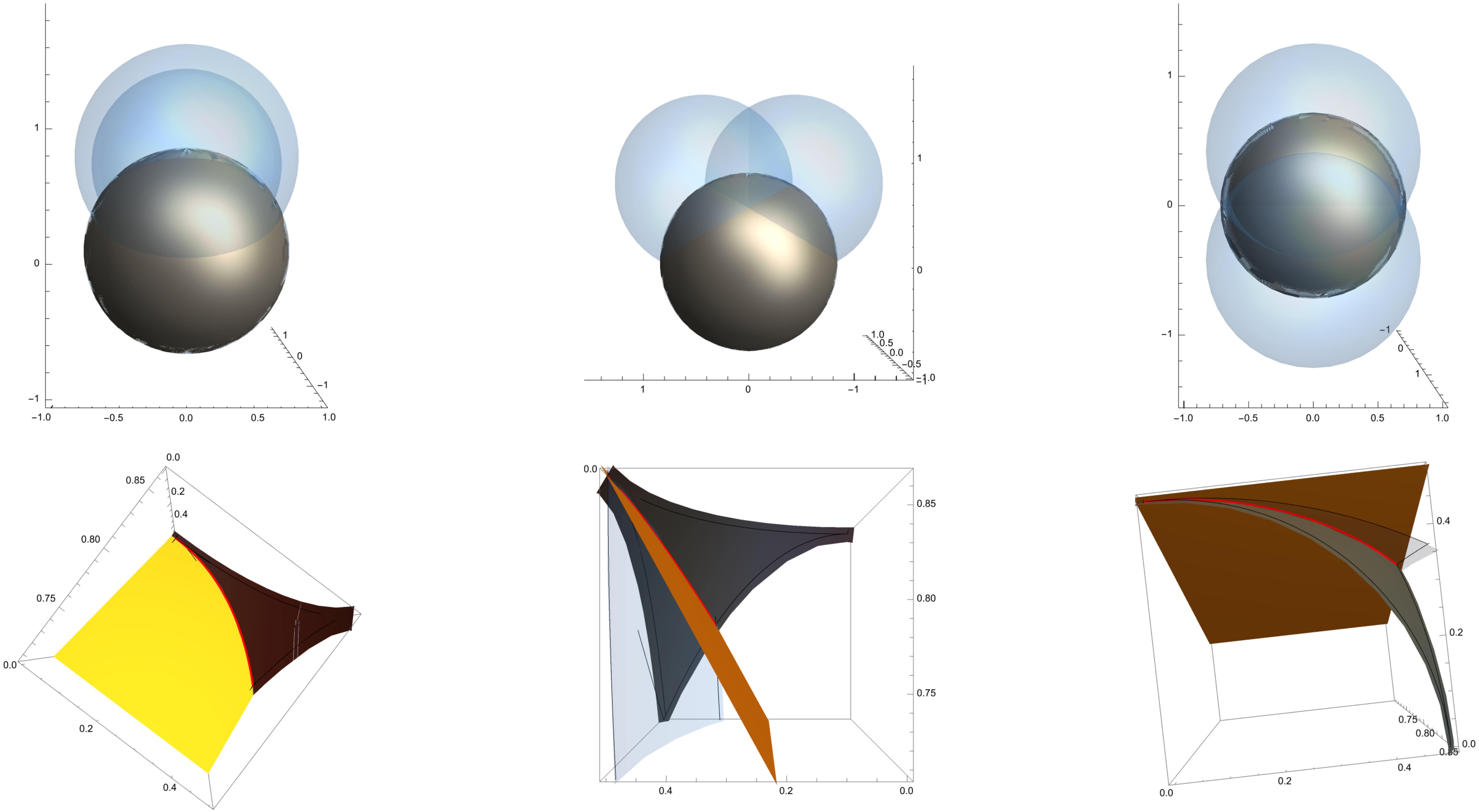
Technicalities of Groups

UPSHOT: Fancy general theorems like Borel-Harish Chandra apply.

Borel-Harish Chandra: $\Gamma \leq G(\mathbb{R})$ arithmetic
$\Rightarrow$ Γ discrete and finite covolume.

$\rightsquigarrow$ General (ugly) theory of fundamental domains.

Technicalities of Groups


1
0
−1
−1.0 −0.5 0.0 0.5 1.0
1
0
−1
1 0 −1
1.0 0.5 0.0 −0.5 −1.0
1
0
−1
−1 0 1
−1.0 −0.5 0.0 0.5 1.0
0.0
0.85
0.2
0.4
0.80
0.75
0.0
0.2
0.4
0.0
0.85
0.80
0.75
0.4 0.2 0.0
0.4
0.2
0.75
0.80
0.85
0.0
0.0 0.2 0.4

THANK YOU!

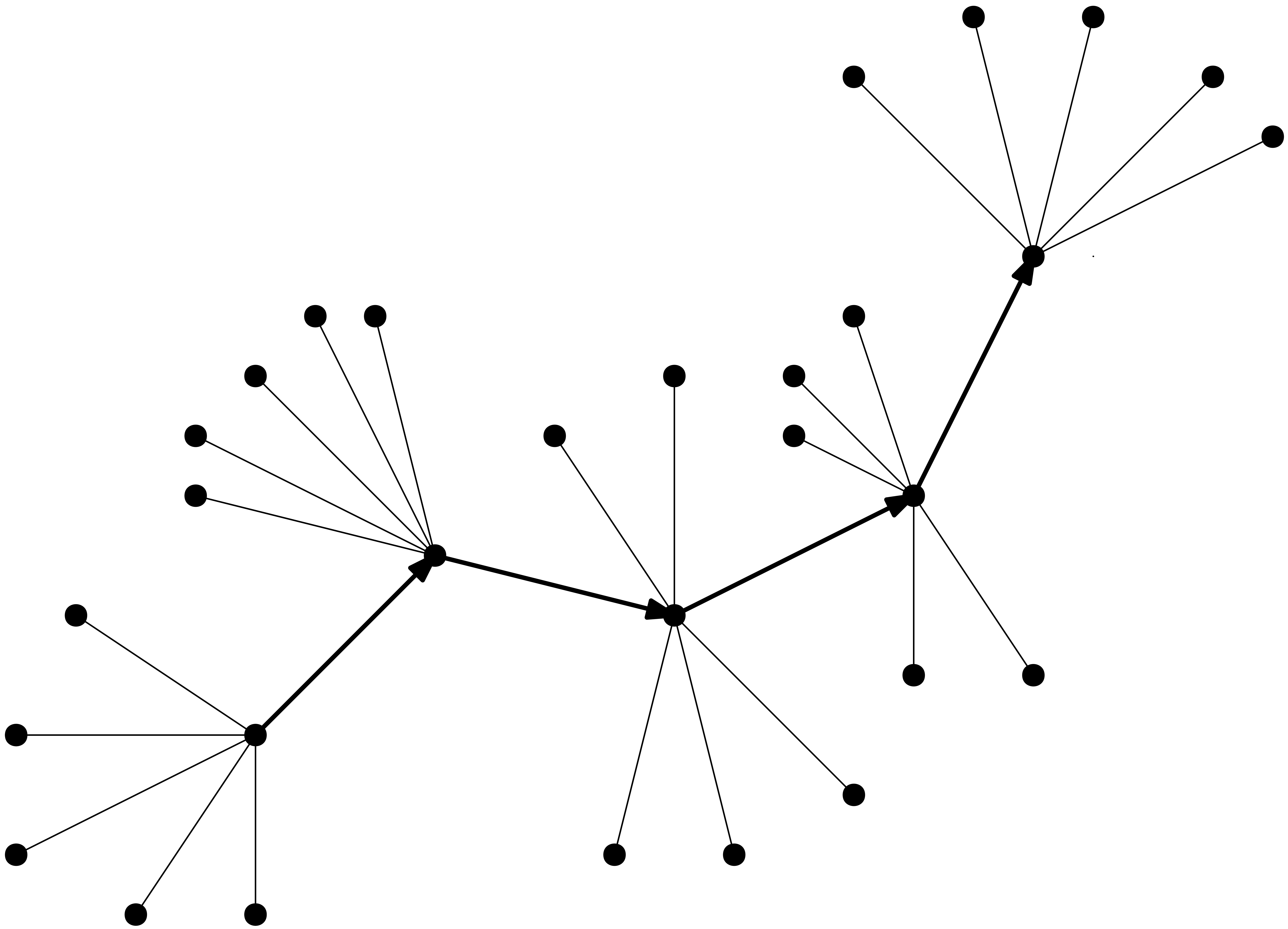
- What can you do with this?

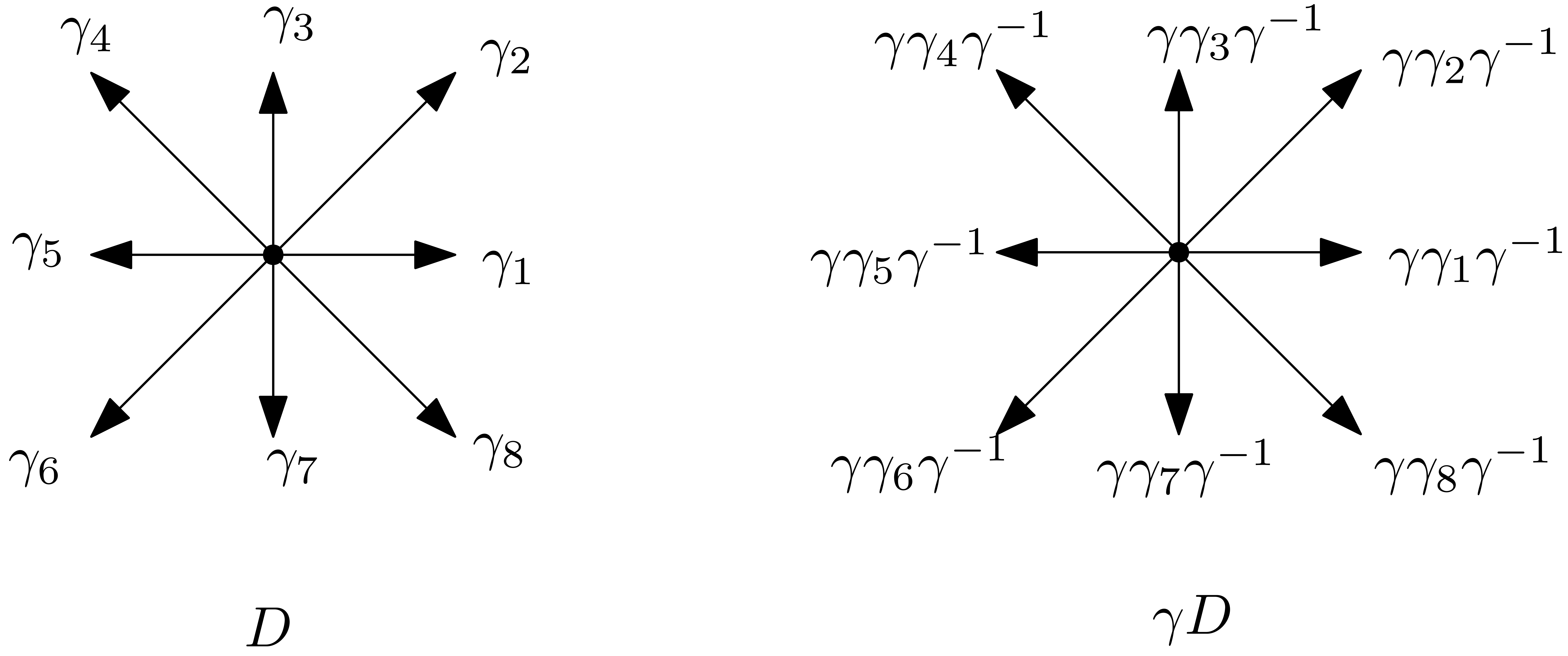
Explicit Algorithms For Fundamental Domains

1) Abstract description of fundamental domain

2) Extra hypotheses

→ generators & relations for groups




γ_1
γ_2
γ_3
γ_4
γ_5
γ_6
γ_7
γ_8
D
$\gamma\gamma_1\gamma^{-1}$
$\gamma\gamma_2\gamma^{-1}$
$\gamma\gamma_3\gamma^{-1}$
$\gamma\gamma_4\gamma^{-1}$
$\gamma\gamma_5\gamma^{-1}$
$\gamma\gamma_6\gamma^{-1}$
$\gamma\gamma_7\gamma^{-1}$
$\gamma\gamma_8\gamma^{-1}$
γD

$$q_A(u,v) = |\xi u|^2 + |\bar{\xi} u + v|^2$$

Satake Compactifications

Given $\Gamma = SL_2(\mathcal{O}) \curvearrowright \mathcal{H}^{n+1}$ we define the partial

Satake Compactification

$$\mathcal{H}^{n+1,\text{Satake}} = \mathcal{H}^{n+1} \cup \underbrace{\mathrm{Vec}(K)}_{\substack{\text{rational Clifford} \\ \text{Alg}}} \cup \{\infty\}$$

$$\mathcal{H}^{n+1,\text{Satake}} = \mathcal{H}^{n+1} \cup \underbrace{\mathrm{Vec}(K)}_{\text{rational Clifford Alg}} \cup \{\infty\}$$

$$\mathcal{Y}(\Gamma) = \left[\Gamma \backslash \mathcal{H}^{n+1}\right], \quad \mathcal{X}(\Gamma) = \left[\Gamma \backslash \mathcal{H}^{n+1,\text{Satake}}\right]$$

orbifold

our analog of Shimura Curves.

$\boxed{\text{Bott Periodicity}}$

examples:

$\mathbb{C}_1 = \mathbb{R}$

$\mathbb{C}_2 = \mathbb{C} = \mathbb{R}[i]$, $i = \sqrt{-1}$

$\mathbb{C}_3 = \mathbb{H} = \mathbb{R}[i,j] = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij$

(we call $ij = k$)

$\mathbb{C}_4 =$ (some ring has zero divisors)

$\cong \mathbb{H} \oplus \mathbb{H}$.

Bott Periodicity

$\mathbb{C}_1 = \mathbb{R}$

$\mathbb{C}_2 = \mathbb{C}$

$\mathbb{C}_3 = \mathbb{H}$

$\mathbb{C}_4 = \mathbb{H} \oplus \mathbb{H}$

$\mathbb{C}_5 = M_2(\mathbb{H})$

$\mathbb{C}_6 = M_4(\mathbb{C})$

$\mathbb{C}_7 = M_8(\mathbb{R})$

$\mathbb{C}_8 = M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$

$\mathbb{C}_9 = M_{16}(\mathbb{R})$

$$\boxed{\text{Bott Periodicity}}$$

$$\mathbb{C}_{n+8} \cong M_{16}(\mathbb{C}_n)$$

$$\mathbb{C}_1 = \mathbb{R}$$

$$\mathbb{C}_9 = M_{16}(\mathbb{R})$$

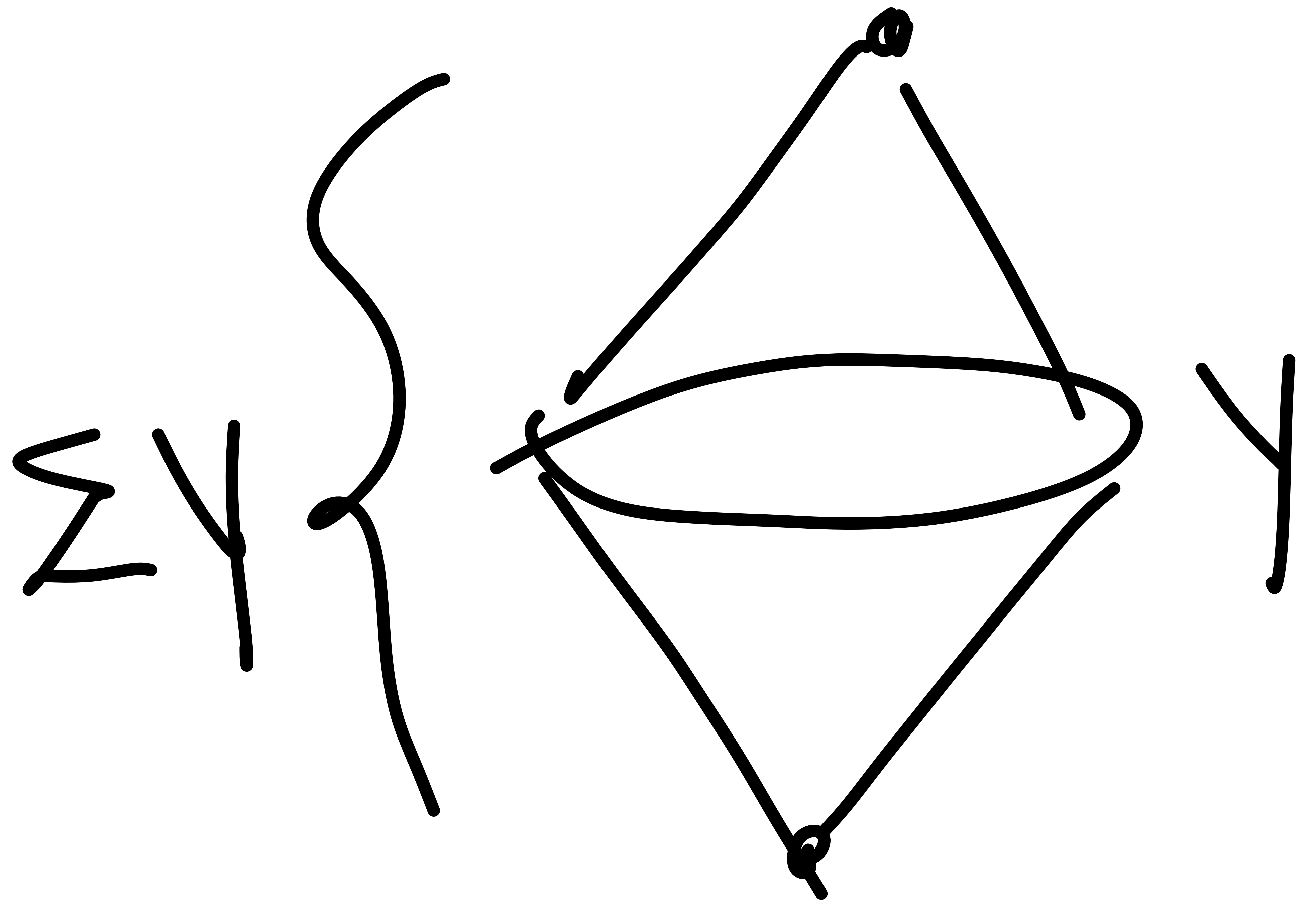
$\boxed{\text{Real K-Theory}}$

$KO^0(Y)$ = Groth Group of Real Vector Bundles

$$KO^n(Y) = KO^0(\Sigma^n Y)$$

$$= [Y, X_n^\infty]$$

$\underbrace{(X_n^\infty)_{n\geq 0}}_{\text{spectrum}}$, $\Omega X_n^\infty \simeq X_{n+1}^\infty$



suspension

vector space

$$\underbrace{X_n(W)}_{\text{space of } \mathbb{C}_n\text{-structures on } W} = G_n / G_{n+1}$$

$$X_n(W) \cong X_{n+8}(W^{\oplus 16})$$

$\mathbb{C}_{n+8}$

Morita Equivalence:

$$\mathrm{Mod}_{\mathbb{C}_n} \cong \mathrm{Mod}_{M_{16}(\mathbb{C}_n)}$$

$$W \longmapsto W^{\oplus 16}$$

$\dim(W) \to \infty$

$$X_n^{\infty} = G_n^{\infty} / G_{n+1}^{\infty}$$

Analogs of Complex Conj:

- parity involution: $x \mapsto x'$ induced by $i_a \mapsto -i_a$, $0 \le a < n$

$$(\alpha\beta)' = \alpha'\beta'$$

- reversal/transpose involution: $x \mapsto x^*$

$$(\alpha\beta)^* = \beta^*\alpha^*$$ reverses multiplication order

E.g. $(i_1 i_2 + i_3 + 1)^* = i_2 i_1 + i_3 + 1$

- Clifford Conjugation: $x \longmapsto \bar{x}$

$$\bar{x} = (x')^* = (x^*)'.$$

- Relationship to Absolute Value: $x = \sum_{S \subseteq \{1,\dots,n-1\}} x_S i_S$

$$|x|^2 = \sum_S x_S^2 = \mathrm{Re}(x\bar{x})$$

0-component.

if $x \in V_n$ then $|x|^2 = x\bar{x}$.