POSITIVITY AND LIFTS OF THE FROBENIUS

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Abstract. If $X/\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is a smooth projective scheme with positive Kodaira dimension then X does not have a lift of the p-Frobenius.

1. Introduction

This paper is motivated by J. Borger's work on Λ-schemes [Bor09]. A Λ-scheme is a scheme $X/\mathbb{Z}$ with a commuting family of lifts of the p-Frobenius for each prime p and according to Borger the lifts should be viewed as descent data a category of "schemes over the field with one element" (think Weil-Descent).

Throughout this paper p will be a fixed prime and $W = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ will the p-adic completion of the maximally unramified extensions of the p-adic integers. Recall that it is the unique p-adically complete discrete valuation ring that it is unramified over $\mathbb{Z}_p$ and has residue field $\overline{\mathbb{F}}_p$. An alternative description of W is $W = W(\overline{\mathbb{F}}_p)$ the ring of p-typical Witt vectors.

In this paper we show that flat projective schemes X/W of positive Kodaira dimension cannot have a lift of the Frobenius. We do this by generalizing a result of Raynaud's to higher dimensions.

Let's fix the following notation: If A is a flat W-algebra then we write

$$A \mod p^{n+1} = A_n = A/p^{n+1} = A/p^{n+1}A$$

If X/W is a flat scheme then we use the similar notation

$$X_n = X \mod p^{n+1} = X \otimes_W W_n.$$

If A is a W-algebra, a **lift of the Frobenius** is a ring endomorphisms $\phi : A \to A$ with the property that for all $a \in A$ we have

$$\phi(a) \equiv a^p \mod p.$$

The ring W has a unique lift of the Frobenius ϕ. If we present W as

$$W = \mathbb{Z}_p[\zeta : \zeta^n = 1, p \nmid n]^{\widehat{}}, \tag{1.1}$$

then $\phi(\zeta) = \zeta^p$ for all roots of unity and $\phi|_{\mathbb{Z}_p} = \mathrm{id}$

Every scheme $X_0/\overline{\mathbb{F}}_p$ admits sheaf endomorphism $F^* : \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}$ called the **absolute Frobenius** such that $F^*(s) = s^p$ for a local section $s \in \mathcal{O}_{X_0}$. This sheaf endomorphism defines a unique morphisms of schemes $F : X_0 \to X_0$ which is constant on the underlying topological space and acts as stated on local sections of the structure sheaf. If X/W is a flat scheme which reduces to X_0 mod p we call a morphism of sheaves $\phi_X : X \to X$ a **lift of the Absolute Frobenius** provided it is equal to the absolute Frobenius mod p and agrees with the unique lift of the Frobenius ϕ on W. We discuss lifts of the Frobenius further in section 2.1.

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This paper generalizes what is known in the case of (smooth projective) curves which can be described by the table below:

(1.2)

genus	$g(C)=0$	$g(C)=1$	$g(C)\geq 2$
Has Lift?	Always	Sometimes	Never

For the first column $C=\mathbb{P}^1_W$ and we can construct the lift by taking p-th powers on the coordinates and patching the map together. The elliptic curve case is essentially the theory of the canonical lifts due to Serre and Tate and is reviewed in [Kat81]. The hyperbolic case is due to Raynaud and is a Cartier operator argument. Naively, one can reformute the 1.2 in terms of Kodaira dimension (see section 2.3) and ask if a similar table holds. In the case of curves $g=0$ if and only if $\kappa(C)<0$, $g=1$ if and only if $\kappa(C)=0$ and $g\geq 2$ if and only if $\kappa(C)=1$. In higher dimensions one finds:

(1.3)

Kodaira Dimension	$\kappa(X)<0$	$\kappa(X)=0$	$\kappa(X)>0$
Has Lift?	Rarely	Very Rarely	Never

This main result of this paper is a modest modernization of Raynaud's Lemma which appears to be missing in the literature.

Theorem 1.1 (Generlized Raynaud Lemma)**.** *Let $W=\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ and X/W be a smooth projective scheme. If X has positive Kodaira dimension then X does not admit a lift of the Frobenius.*

The strategy of the above version of Raynaud's Lemma ([Ray83] Lemma I.5.1) is roughly the same as the original: Supposing a lift of the Frobenius, cook-up a map between the de Rham complexes and show that it is non-zero using the theory of the Cartier operator. Next, using the positivity, show that such a non-zero map can't exist giving a contradiction.

Remark 1.2. It is natural to ask whether or not for every Fano variety admits a lift of the Frobenius. The answer to this question is negative and examples can be found in [PS89]. Here they show that the only flag variety which admits a lift of the Frobenius is projective space.

1.1. **Plan of the Paper.** In section 2 we provide the necessary background and in section 3 we give the proof.

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2. Background Necessary for the Proof

Section 2.1 reviews the notions of absolute and relative Frobenius morphisms, section 2.2 recalls the Cartier operator, and section 2.3 recalls the notion of Kodaira dimension.

2.1. **Lifts of the Frobenius.** Let $S_0=\mathrm{Spec}(\overline{\mathbb{F}}_p)$ and F_{S_0} denote the absolute Frobenius on S_0. If $u_0: X_0\to S_0$ is a morphism of schemes we have the following commutative diagram which defines the scheme $X_0^{(p)}$ and the maps F_{X_0/S_0} and φ_0

$$\begin{tikzcd} X_0 \arrow[dr, dashed, "F_{X_0/S_0}"] \arrow[drr, bend left, "F_{X_0}"] \arrow[ddr, bend right, "u_0"'] & & \\ & X_0^{(p)} = X_0 \times_{S_0, F_{S_0}} S_0 \arrow[r, "\varphi_0"] \arrow[d, "u_0^{(p)}"] & X_0 \arrow[d, "u_0"] \\ & S_0 \arrow[r, "F_{S_0}"] & S_0 \end{tikzcd} \tag{2.1}$$

The morphism F_{X_0/S_0} is called a **relative Frobenius**. For all $f \in \mathcal{O}_{X_0}$ and all $c \in \overline{\mathbb{F}}_p$ we have

$$\begin{aligned} \varphi_0^*(f) &= f \otimes 1 \\ \varphi_0^*(cf) &= c^p \varphi_0^*(f) \end{aligned}$$

Commutivity of the diagram implies that

$$F^*_{X_0/S_0}(\varphi_0^* f) = F^*_{X_0/S_0}(f \otimes 1) = F^*_{X_0}(f) = f^p. \tag{2.2}$$

Commutivity also says that unlike F_{X_0} the map $F^*_{X_0/S_0} : \mathcal{O}_{X_0^{(p)}} \to \mathcal{O}_{X_0}$ is a morphism of varieties over $\overline{\mathbb{F}}_p$. For all $c \in \overline{\mathbb{F}}_p$ and $f \in \mathcal{O}_{X_0}$

$$\begin{aligned} c \cdot \varphi_0^*(f) &= f \otimes c \\ F^*_{X_0/S_0}(c \cdot \varphi_0^*(f)) &= cf^p \end{aligned}$$

We employ the notation

$$f^{(p)} := \varphi_0^*(f) = f \otimes 1 \tag{2.3}$$

so that for all $f \in \mathcal{O}_{X_0}$ and all $c \in \overline{\mathbb{F}}_p$

$$\begin{aligned} (cf)^{(p)} &= c^p f^{(p)} \\ F^*_{X_0/S_0}(f^{(p)}) &= f^p \end{aligned}$$

Suppose now that X/W is flat with reduction mod p equal to X_0/S_0 then a **lift of the absolute Frobenius** is a map $\phi_X : X \to X$ whose reduction mod p if F_X.

Let S be a lift of S_0 and let $\phi : S \to S$ lifting the absolute Frobenius on S. Let $X^\phi = X \times_{S,\phi} S$ be the pullback of X by ϕ with $\varphi : X^{(\phi)} \to X$ given by the first projection. We call ϕ_X a **lift of the absolute Frobenius** (compatible with the lift ϕ on S) and $\phi_{X/S}$ a **lift of the relative Frobenius** on X (compatible with the lift ϕ on S) if we have the following diagram:

$$\begin{tikzcd} X \arrow[dr, "\phi_{X/S}"] \arrow[drr, bend left, "\phi_X"] \arrow[ddr, bend right, "u"'] & & \\ & X^{(\phi)} \arrow[r, "\varphi"] \arrow[d, "u^{(\phi)}"] & X \arrow[d, "u"] \\ & S \arrow[r, "\phi"] & S \end{tikzcd} \tag{2.4}$$

Note from the above diagram that lift of the absolute Frobenius on X exists if and only if a lift of the relative Frobenius exists.

2.2. **Cartier Operator.** In what follows we let $\underline{H}^\bullet$ denote the cohomology which takes values in sheaves. Also if F is a sheaf on a scheme X then the notation "$s \in F$" will mean "$s \in F(U)$ for some open subset U of Xs".

Theorem 2.1 (Cartier). *Let X_0/S_0 be a smooth projective scheme where $S_0/\overline{\mathbb{F}}_p$. There exists a unique isomorphism of graded $\mathcal{O}_{X_0^{(p)}}$-algebras*

$$C^{-1}_{X_0/S_0} : \bigoplus_i \Omega^i_{X^{(p)}/S} \to \bigoplus_i \underline{H}^i(F_{X_0/S_0 *}\Omega^\bullet_{X_0/S_0})$$

such that for every $\xi, \eta \in \bigoplus_i \Omega^i_{X^{(p)}/S}$ and $f \in \mathcal{O}_{X^{(p)}}$ we have

1. $C^{-1}_{X_0/S_0}(1) = 1$
2. $C^{-1}_{X_0/S_0}(\xi \wedge \eta) = C^{-1}_{X_0/S_0}(\xi) \wedge C^{-1}_{X_0/S_0}(\eta)$
3. $C^{-1}_{X_0/S_0}(d\varphi^* f) = [f^{p-1}df]$ *in* $\underline{H}^1(F_{X_0/\overline{\mathbb{F}}_p *}\Omega^\bullet_{X_0/\overline{\mathbb{F}}_p})$

and $C^{-1}_{X_0/S_0}$ is an isomorphism.

For a proof of this theorem see [Kat70] Theorem (7.2).

Remark 2.2. The map $C^{-1}_{X_0/S_0}$ in the above proposition is called the **inverse of the Cartier operator**. The **Cartier operator** $C_{X_0/S_0} : \bigoplus_i \underline{H}^i(F_{X/S*}\Omega^\bullet_{X/S}) \to \bigoplus_i \Omega^i_{X_0^{(p)}/S_0}$ is the inverse of $C^{-1}_{X_0/S_0}$.

2.3. **Kodaira Dimension.** Let X/W be a smooth scheme. Then X admits a canonical sheaf ω and one can define the *canonical ring* as

$$R(X, \omega) := \bigoplus_{n \geq 0} H^0(X, \omega^{\otimes n}) \tag{2.5}$$

which is a graded W-module.

One should recall that for every n the module $H^0(X, \omega^{\otimes n})$ is free and finite rank since W is local. We should recall that for every X there exists some polynomial $P_X(t)$ which takes integral values and some integer $N \geq 0$ such that for all $n > N$

$$h^0(X, \omega^{\otimes n}) = P_X(n).$$

Here $h^0(X, \omega^{\otimes n}) = \mathrm{rk}_W H^0(X, \omega^{\otimes n})$.

The **Kodaira dimension** (introduced by Iitaka) of X is defined to be the dimension of the projective scheme associated to canonical ring:

$$\kappa(X) = \dim_W \mathrm{Proj}\ R(X, \omega). \tag{2.6}$$

Equivalently, one can define the Kodaira dimension to be the projective dimension of $R(X, \omega)$.

Remark 2.3. We will find it convenient to use divisorial notation in place of sheaf notation in places.

- In this case K will denote a (fixed representative of) the canonical divisor and $\omega \cong \mathcal{O}(K)$.
- We will also use the notation $H^i(X, D) := H^i(X, \mathcal{O}(D))$.
- One should also recall that $\mathcal{O}(nD) \cong \mathcal{O}(D)^{\otimes m}$ for integers m where $L^{\otimes -1}$ will denote the dual of a sheaf of modules L.

Let X/L be a smooth schemes over a field L of characteristic zero. The condition $\kappa(X) > 0$ is equivalent to the canonical divisor K being rationally equivalent to a non-trivial divisor with non-negative $\mathbb{Q}$-coefficients

Here is how to see this: by definition of positive Kodaira dimension there exists some m and some non-zero $s \in H^0(X, mK)$. By definition of $\mathcal{O}(mK)$ we have $D := (s) + mK \geq 0$ which proves that $\frac{1}{m}D \sim K$ as $\mathbb{Q}$-divisors.

3. Proof of Main Theorem

The aim of this section is to prove theorem 1.1. Here we suppose that X/S is a smooth projective scheme where $S = \operatorname{Spec}(W)$ has a lift of the Frobenius ϕ_S and $\phi_{X/S} : X \to X^{(\phi)}$ is a lift ϕ_X of the Frobenius (compatible with the lift of the Frobenius on S).

If for local sections x we define $\delta(x) = \frac{\phi^*(x) - x^p}{p}$ we can write

$$\phi_{X/S}^*(x) = x^p + p\delta(x). \tag{3.1}$$

Note that since X is flat multiplication by p is injective and division by p makes sense. On differentials dx we have

$$\phi_{X/S}^* dx = px^{p-1}dx + pd\delta(x) \tag{3.2}$$

which implies that elements in the image of the map $\phi_{X/S}^*\Omega_{X^{(\phi)}/S} \to \Omega_{X/S}$ are divisible by p. We extend this map multiplicatively: For $\omega = dx_1 \wedge \cdots \wedge dx_r$ a local section of the rth wedge power of the module of Kahler differentials we have

$$\phi_{X/S}^*(dx_1 \wedge \cdots \wedge dx_r) \in p^r\Omega_{X/S} \tag{3.3}$$

at each stage if we divide by the appropriate power of p (p^r for r-forms) we get a morphism of complexes $\tau^\bullet : \Omega^\bullet_{X^{(\phi)}/S} \to (\phi_{X/S})_*\Omega^\bullet_{X/S}$. Note that $\tau^1(dx) = x^{p-1}dx + d\delta(x)$ which a lift of the inverse of the Cartier operator (in the sense that it is a morphism of modules that reduces to the $C^{-1}_{X_0/S_0}$).

Observe that the image of τ is closed. If we let $\pi^\bullet$ denote the map which takes closed forms to their class in cohomology we have

$$C^\bullet_{X_0/S_0} \circ (\pi^\bullet \circ \tau^\bullet)_0 = \text{identity}, \tag{3.4}$$

where $C^\bullet_{X_0/S_0}$ is the Cartier isomorphism. Equation 3.4 just implies that $(\pi^\bullet \circ \tau^\bullet)_0$ is just the inverse of the Cartier isomorphism defined in section 2.2. This implies that $(\pi^\bullet \circ \tau^\bullet)_0$ is non-zero which implies that $\pi^\bullet \circ \tau^\bullet$ is non-zero which implies that $\tau^\bullet$ is non-zero. Define $(\tau')^i$ to be the adjoint of τ^i for each i. By adjointness we also have that $(\tau')^\bullet : \phi_{X/S}^*\Omega^\bullet_{X^{(\phi)}/S} \to \Omega^\bullet_{X/S}$ is nonzero; in particular since τ^i is nonzero for each $0 \leq i \leq \dim_S(X)$ we have $\tau^{i\prime}$ being non-zero for each $0 \leq i \leq \dim_S(X)$. We should also mention that since $(\pi^\bullet \circ \tau^\bullet)_0 = \pi_0^\bullet \circ \tau_0^\bullet)$ we also have that $\tau_0'^\bullet$ is nonzero in each degree. In degree $m = \dim(X)$ we have a non-zero map

$$\tau_0^m : F^*_{X_0/S_0}\omega_{X_0^{(p)}/S_0} \to \omega_{X_0/S_0}. \tag{3.5}$$

We claim that such a non-zero map can't exist. In terms of divisors these line bundles are

$$F^*_{X_0/S_0}\omega_{X_0^{(p)}/S_0} \cong \omega^{\otimes p}_{X_0/S_0} \cong \mathcal{O}(pK_{X_0}), \quad \omega_{X_0/S_0} \cong \mathcal{O}(K_{X_0}),$$

and $\tau_0^m \neq 0$ implies

$$\mathsf{Mod}_{X_0}(\mathcal{O}(pK_{X_0}), \mathcal{O}(K_{X_0})) \cong \mathsf{Mod}_{X_0}(\mathcal{O}, \mathcal{O}((1-p)K_{X_0})) \cong H^0(X_0, (1-p)K_{X_0})$$

is nonzero (here Mod_X denotes the category of $\mathcal{O}_X$-modules and $\mathsf{Mod}_X(A,B)$ denotes the morphisms of $A \to B$ of $\mathcal{O}_X$-modules). By flatness of X/W we have $h^0(X_0,(1-p)K_{X_0}) = h^0(X_L,(1-p)K_{X_L})$ where $L = \mathrm{Frac}(W) = \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ and X_L is the generic fiber of X/W. Since $\kappa(X_L) > 0$ is equivalent to $K_{X_L} \geq 0$ as a $\mathbb{Q}$-divisor we have $h^0(X_L,(1-p)K_{X_L}) = 0$ which gives a contradiction.

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