

TEST 2 — Math 264 — Fall 2010

November 2, 2010

Show all of your work. Remember to use English sentences where necessary. If you don't know how to do a problem explain what you do and don't know to get as many points as possible.

1. Compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions

(a) $u(x, y) = e^{xy}$.

(b) $u(x, y) = \sin(xy)y$.

(a) $u_x = ye^{xy}$
 $u_y = xe^{xy}$

(b) $u_x = y^2 \cos(xy)$
 $u_y = \cos(xy)y^2 + \sin(xy)y$
 $\sin(xy) + xy \cos(y)$

2. Show that the function $u(x, t) = f(x - ct) + g(x + ct)$ satisfies the wave equation $u_{tt} - c^2 u_{xx} = 0$.

$u_{xx} = f''(x - ct) + g''(x + ct)$

$u_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct)$

$\Rightarrow u_{tt} - c^2 u_{xx} = [f''(x - ct) + g''(x + ct)] - c^2 [f''(x - ct) + g''(x + ct)]$
 $= 0$

two variables

3. (a) How can you show that a limit of a function in TWO doesn't exist?

(b) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^3 + y^3}$ doesn't exist.

(a) Approach the point along two different curves & get two different values. +4

(b) Let $\vec{r}_1(t) = (t, t)$, $\vec{r}_2(t) = (t, 0)$ &

$$f(x, y) = \frac{x^2 y}{x^3 + y^3}.$$

$$f(\vec{r}_1(t)) = \frac{t^3}{t^3 + t^3} = \frac{1}{2} \rightarrow \frac{1}{2} \text{ as } t \rightarrow 0.$$

$$f(\vec{r}_2(t)) = \frac{t^2(0)}{t^3 + (0)^3} = 0 \rightarrow 0 \text{ as } t \rightarrow 0$$

Since $\frac{1}{2} \neq 0$ the limit doesn't exist.

4. Let $f(x, y)$ be a function of two variables. Using the chain rule show that $f_r = \frac{f_x x + f_y y}{\sqrt{x^2 + y^2}}$, where $x = r \cos(\theta)$ and $y = r \sin(\theta)$

$$f_r = f_x x_r + f_y y_r$$

$$= f_x \frac{x}{\sqrt{x^2 + y^2}} + f_y \frac{y}{\sqrt{x^2 + y^2}}$$

$$= \frac{f_x x + f_y y}{\sqrt{x^2 + y^2}} \quad +5$$

SIDE WORK:

$$x_r = \cos(\theta)$$

$$= \frac{x}{r}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}$$

$$y_r = \sin(\theta)$$

$$= \frac{y}{r}$$

$$= \frac{y}{\sqrt{x^2 + y^2}}$$

5. Find the plane tangent to the graph of the following functions at the specified point.

(a) $f(x, y) = e^{xy} + x^2$, $P = (0, 1, 1)$.

(b) $g(x, y) = x^3 + y^3$, $Q = (-1, 1, 0)$.

(a) $f_x = ye^{xy} \Rightarrow f_x(0, 1) = 1$

$f_y = xe^{xy} \Rightarrow f_y(0, 1) = 0$

+5

$z = f(0, 1) + \nabla f(0, 1) \cdot (x-0, y-1)$

$\Rightarrow \boxed{z = 1 + (1, 0) \cdot (x, y-1)} \text{ or } z = x+1$

(b) $g_x = 3x^2 \Rightarrow g_x(-1, 1) = 3$

$g_y = 3y^2 \Rightarrow g_y(-1, 1) = 3$

+5

$z = g(-1, 1) + \nabla g(-1, 1) \cdot (x+1, y-1)$

$\Rightarrow \boxed{z = 0 + (3, 3) \cdot (x+1, y-1)}$

6. Find the plane tangent to the surface at the specified point.

or $z = 3(x+1) + 3(y-1)$

(a) $x^4 + y^2 + z^2 = 1$, $P = (1, 0, 0)$

(b) $z - x^3 - y^3 = 0$, $Q = (-1, 1, 0)$. (Hint: compare to part b of the previous problem)

(a) Let $g(x, y, z) = x^4 + y^2 + z^2$. $g_x = 4x^3$, $g_y = 2y$
 $g_z = 2z$. $\Rightarrow \nabla g(1, 0, 0) = (4, 0, 0)$.

$\nabla g(1, 0, 0) \cdot (x-1, y-0, z-0) = 0$ (Formula for tangent plane)

$\Rightarrow \boxed{4(x-1) = 0 \text{ is the tangent plane}}$

(b) setting $f(x, y) = x^3 + y^3$ and finding the tangent plane of the graph of this function is exactly the same problem, so

$\boxed{z = 3(x+1) + 3(y-1)}$

7. Find the critical points of $f(x, y) = -(x^2 - 1)^2 - y^2$ and classify them.

CRIT PTS: $\nabla f = 0 \Rightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} -2(x^2 - 1)(2x) = 0 \\ -2y = 0 \end{cases}$

$\Rightarrow x = \pm 1$ or $x = 0$ with $y = 0$. So the critical points are $(1, 0)$, $(-1, 0)$ & $(0, 0)$.

ANALYSIS OF CRIT PTS: $H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -4(x^2 - 1) & 0 \\ 0 & -2 \end{bmatrix}$

(x, y)	$\det H(x_0, y_0)$	$f_{xx}(x_0, y_0)$	conclusion
$(1, 0)$	$16 > 0$	-8	concave down \Rightarrow max
$(-1, 0)$	$16 > 0$	-8	concave down \Rightarrow max
$(0, 0)$	$-8 < 0$	not needed	saddle

8. Find the maximal volume for a rectangle inscribed in the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

for you want to optimize:

$V(x, y, z) = 8xyz$, $x \geq 0, y \geq 0, z \geq 0$

(solid is symmetric in xy, yz & xz planes)

Constraint:

$x^2 + 2y^2 + 3z^2 = 1$

$\begin{cases} \nabla V = \lambda \nabla g \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases} \Rightarrow \begin{cases} 8yz = \lambda 2x \\ 8xz = \lambda 4y \\ 8xy = \lambda 6z \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases}$

$\Rightarrow \begin{cases} 8xyz = \lambda 2x^2 \\ 8xyz = \lambda 4y^2 \\ 8xyz = \lambda 6z^2 \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases} \Rightarrow 3\left(\frac{8xyz}{2\lambda}\right) = 1 \Rightarrow 8xyz = \frac{2\lambda}{3}$

for whatever λ is.

$\Rightarrow \lambda 2x^2 = \frac{2\lambda}{3} \Rightarrow \lambda(2x^2 - \frac{2}{3}) = 0, \lambda \neq 0 \Rightarrow 2x^2 - \frac{2}{3} = 0$
 $\Rightarrow x = \pm \frac{1}{\sqrt{3}}$, similarly $4y^2 - \frac{2}{3} = 0$ & $6z^2 - \frac{2}{3} = 0$

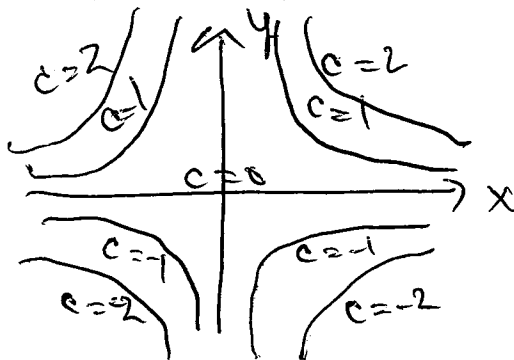
$$\Rightarrow y = +\sqrt{\frac{1}{6}}, z = +\sqrt{\frac{1}{9}} = \frac{1}{3}, \quad +4$$

$$8xyz = 8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{3}\right) = \frac{8}{9\sqrt{2}}. //$$

9. (a) Plot the level curves of the function $f(x, y) = x^2 y$.
 (b) What do the level surfaces of the function $g(x, y, z) = x^2 + 4y^2 + z^2$ look like? (A one sentence answer is acceptable)

a) $x^2 y = c \Rightarrow y = \frac{c}{x^2}$

+5



(b) They look like ellipsoids of various volumes.

+5

10. Suppose that $f(x, y)$ is a differentiable function and $C \in \mathbb{R}$ is some real number. If $\vec{r}(t)$ is differentiable and parametrizes the level set $\{(x, y) : f(x, y) = C\}$. Show that for all t in the domain, $\vec{r}'(t)$ and $(\nabla f)(\vec{r}(t))$ are perpendicular.

$$f(\vec{r}(t)) = C$$

$$\Rightarrow \frac{d}{dt} [f(\vec{r}(t))] = 0$$

+5

(since the derivative of a constant is zero)

$$\Rightarrow \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = 0$$

(By the chain rule)

+5

11. (EXTRA CREDIT)

- (a) For $\vec{r}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by $\vec{r}(s, t) = (x(s, t), y(s, t))$ what is the Jacobian Matrix of \vec{r} ?
 (b) How does it provide a way to view the chain rule for several variables as a "chain rule".
 (Write the pair of equations that we call the chain rule as a single matrix equation involving the Jacobian).
 (c) Show that the Hessian Matrix is the Jacobian of the gradient.

(a)

$$(\mathbb{D} \vec{r})(s, t) = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \quad (+3)$$

(b) If $g(s, t) = f(x(s, t), y(s, t))$ we have

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad (+4)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

this can be written as a matrix equation

$$\begin{aligned} (\nabla g)(s, t) &= \left(\frac{\partial g}{\partial s}, \frac{\partial g}{\partial t} \right) \\ &= \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}, \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \end{aligned}$$

$$= (\nabla f)(x(s, t), y(s, t)) \underbrace{\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}}_{\text{Matrix Mult}} (\mathbb{D} \vec{r})(s, t)$$

So

$$\boxed{\nabla g(s, t) = \nabla f(x(s, t), y(s, t)) (\mathbb{D} \vec{r})(s, t)}$$

(C) The Hessian is $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.
 The gradient is $\nabla f = (f_x, f_y)$. If we let
 $u = f_x$ & $v = f_y$ we have

$$D(\nabla f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \quad +3$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} [f_x] & \frac{\partial}{\partial y} [f_x] \\ \frac{\partial}{\partial x} [f_y] & \frac{\partial}{\partial y} [f_y] \end{bmatrix}$$

$$= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = H(x,y). //$$

Other Extra Credit:

1. Prove $P \neq NP$ ⁺¹
2. Prove Riemann Hypoth ⁺¹
3. Prove Mass Gap for Yang-Mills ⁺¹
4. Prove Birch & Swinnerton-Dyer Conjecture ⁺¹
5. Show existence of solution to Navier-Stokes equations ⁺¹
6. Hodge Conjecture ⁺¹

7. Poincare Conjecture ⁺¹