“Linear” Arithmetic Differential Equations

Taylor Dupuy
University of New Mexico

origins

- Relative Mordell-Lang (characteristic 0)
- ~> p-derivations
- ~> Hrushovski's Relative Manin-Mumford (characteristic p)

Seek a category:

$\mathsf{Sch}_{\mathbf{F}_1} = $ "Category of Schemes over the field with one element"

$$\mathrm{Spec}(\mathbf{Z})$$

$$\overline{\mathrm{Spec}(\mathbf{Z})_{\mathbf{F}_1}}$$

Arakelov Theory (compactification)

Finding an appropriate category
to view as deeper base

Part 1: Introduction to Wittferential Algebra

Part 2: Linear Arithmetic Differential Equations

PART I

Wifferential Algebra

What is a p-derivation?

Fermat's Little Theorem

$\forall n \in \mathbb{Z}, \forall p$ prime

$$n \equiv n^p \mod p$$

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The Frobenius

$$F : A/p \mapsto A/p$$
$$a \mapsto a^p$$

Properties of p-derivations

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non-linear

Always an A-algebra

Defn. (Buium, Joyal 90's)

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$\forall a, b \in A$

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EXERCISE:

For $\delta_p : \mathbb{Z} \to \mathbb{Z}$

defined by $\delta_p(n) = \frac{n-n^p}{p}$

show $\delta_p(p^m) = p^{m-1} \cdot$ (unit mod p)

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Idea: order of vanishing is "bumped down"

$$\delta_t = \frac{d}{dt}$$

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Example:

$$\delta_p(p) = \frac{p - p^p}{p}$$
$$= 1 - p^{p-1}$$

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derivations $\longleftrightarrow$ ring homomorphisms

$\delta : A \to A$ $\quad$ $f : A \to A[\varepsilon]/\langle \varepsilon^2 \rangle$

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"Wittferentiation"

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"infinitesimals"

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Funky Addition

$$(a_0, a_1) +_W (b_0, b_1) = (a_0 + b_0, a_1 + a_1 + p \textstyle\sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} a_0^{p-j} b_0^j)$$

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exercise: $W_1(\mathbb{F}_p) = \mathbb{Z}/p^2$

Analogies

Dual Numbers

$$D_1(A) = A[t]/\langle t^2 \rangle$$

Truncated Witt Vectors

$$W_1(A)$$

Power Series

$$D(A) = A[[t]]$$

Witt Vectors

$$W(A)$$

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$$(a_0, a_1, a_2, \ldots) +_W (b_0, b_1, b_2, \ldots) = (s_0(a, b), s_1(a, b), s_2(a, b), \ldots)$$

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mult polyn

$W(A) \cong_{sets} A^{\mathbb{N}}$

Witt polynomials

$$w_0(a) = a_0$$
$$w_1(a) = a_0^p + pa_1$$
$$w_2(a) = a_0^{p^2} + pa_1^p + p^2 a_2$$
$$w_3(a) = a_0^{p^3} + pa_1^{p^2} + p^2 a_2^p + p^3 a_3$$
$$\vdots$$

Defining property of Witt vectors:

The map $W(A) \to A^{\mathbb{N}}$ defined by

$$(a_0, a_1, a_2, \ldots) \mapsto (w_0(a), w_1(a), w_2(a), \ldots)$$

is a ring homomorphism.

Compute some of the addition polynomials

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$$w_0(a) + w_0(b) = w_0(s)$$
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$$a_0 + b_0 = s_0$$
$$(a_0^p + pa_1) + (b_0^p + pb_1) = s_0^p + ps_1$$
$$(a_0^{p^2} + pa_1^p + p^2 a_2) + (b_0^{p^2} + pb_1^p + p^2 b_2) = s_0^{p^2} + ps_1 + p^2 s_2$$
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Exercise. $W(\mathbb{F}_p) = \mathbb{Z}_p$

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$\phi : A \to B$

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lifts of the Frobenius $\approx$ p-derivations

$\delta_p : A \to B$ + rules

$\phi(a) := a^p + p\delta_p(a)$

$$\delta_p(a) := \frac{\phi(a) - a^p}{p}$$

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p-torsion free

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Summary

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The poor man's model companion:

$$\boxed{R = \widehat{\mathbf{Z}}_p^{\mathrm{ur}}}$$

Properties:

1) unramified CDVR, residue field $\overline{\mathbf{F}}_p$

2) $\widehat{\mathbf{Z}}_p^{\mathrm{ur}} = \mathbf{Z}_p[\zeta; \zeta^n = 1, p \nmid n]\widehat{\ }$

3) unique lift of the Frobenius: $\exists! \phi : R \to R$

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R WILL ALWAYS MEAN THIS RING IN THIS TALK

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PART 2

Linear Arithmetic Differential Equations

Simplest Possible Equation:

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Remarks.

$x \mapsto \delta(x)(x^{(p)})^{-1}$ almost a cocycle

Theorem (existence and uniqueness)

$\alpha \in \mathfrak{gl}_n(R)$, $u_0 \in \mathrm{GL}_n(R)$

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \mod p \end{cases}$$

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$$u = \lim_{n \to \infty} f^n(u_0)$$

Theorem (coeffs in CDVR)

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$\mathcal{O}$ = Complete Discrete Valuation Subring

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O})$$

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$$\phi^{\nu+1}(u) = \phi^\nu(\epsilon u^{(p)})$$

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$$\phi^{\nu+1}(u) = \phi^\nu(\epsilon u^{(p)})$$

$$\phi(\phi^\nu(u)) = \epsilon(\phi^\nu(u))^{(p)}$$

Theorem (coeffs in valuation delta subring)

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \mod p \end{cases}$$

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$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

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$$\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a) < \infty$$

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general principle:

$$\varphi/F \implies \text{fixed points of } \varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a) \subset \mathbf{A}^n(F^a)$$

$\implies$ columns are in finite algebraic extensions

Lemma. (Fornaes and Sibonay)

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$\Longrightarrow Y$ zero dimensional in $\mathbf{P}^n$

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homogeneity gives rest of finiteness.

Galois Theory

$\mathcal{O} \subset R$, $\alpha \in \mathfrak{gl}_n(\mathcal{O})$

$$\delta u = \alpha u^{(p)}$$

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Galois Group
=
$$\{c \in \mathrm{GL}_n(\mathcal{O}) : \exists \sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]), \sigma \circ \delta = \delta \circ \sigma, \sigma(u) = uc\}$$
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Galois Group of $\delta u = \alpha u^{(p)}$

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Alternative Descriptions:

1) $\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$

$$\Gamma_{u/\mathcal{O}} \cong G_{u/\mathcal{O}}$$

2) $$0 \to I_{u/\mathcal{O}} \to \mathcal{O}[x, 1/\det(x)] \to \mathcal{O}[u] \to 0$$

$$\mathrm{Stab}_{\mathrm{GL}_n(R)}(I_{u/\mathcal{O}}) \cong G_{u/\mathcal{O}}$$

$$\delta u = \alpha u^{(p)}$$

Question: For what $c \in \mathrm{GL}_n(R)$ do we get

$$\delta(uc) = \alpha(uc)^{(p)} \text{ ?}$$

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Main Example:

T = maximal torus of diagonals

W = permutation matrices

$N = WT = TW$

G^δ = matrices with roots of unity entries

$N^\delta = T^\delta W = WT^\delta$ = permutation matrices with roots of unity entries

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Main Example:

subgroup of $\mathrm{GL}_n(\mathbf{F}_1^a)$

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Theorem.

$\mathcal{O} \subset R$ delta subring

$\mathcal{O} \subset R^{\phi^\nu}$ $u \in \mathcal{O} \implies G_{u/\mathcal{O}}$ finite

$$\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$$

$$= G_{u/\mathcal{O}}$$

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$$= G_{u/\mathcal{O}}$$

finite to begin with

Theorem A.

$\exists \Omega \subset \mathbf{Q}^2$ "thin set"

$\forall \alpha \in \mathbf{Z}^2 \setminus \Omega$, $\exists u \in \mathrm{GL}_n(R)$ $\quad \delta u = \alpha u^{(p)}$

$G_{u/\mathcal{O}}$ = finite group containing W

Theorem B.

$$X = \{u \in \mathrm{GL}_n(R); u \equiv 1 \mod p\}$$

= ball around identity

$\exists \Omega \subset X$ of the second category

$\forall u \in X \setminus \Omega$, $\forall \mathcal{O} \subset R$ δ-closed subring

1) $\delta(u)(u^{(p)})^{-1} \in \mathfrak{gl}_n(\mathcal{O})$

2) $R^\delta \subset \mathcal{O}$

$$\Longrightarrow \quad G_{u/\mathcal{O}} = N^\delta$$

Theorem C.

$$\exists \Omega \subset \mathrm{GL}_n(K^a)$$ Zariski closed

$$\forall u \in \mathrm{GL}_n(R) \setminus \Omega$$

$$u' = \alpha u^{(p)}$$

$$\mathcal{O} \ni \alpha \quad \delta\text{-closed subring of } R$$

$$\dim((Z \cdot G_{u/\mathcal{O}})^{\mathrm{Zar}}) \leq n$$

Theorem B.

$$X = \{u \in \mathrm{GL}_n(R); u \equiv 1 \mod p\}$$

= ball around identity

$\exists \Omega \subset X$ of the second category

$\forall u \in X \setminus \Omega$, $\forall \mathcal{O} \subset R$ δ-closed subring

1) $\delta(u)(u^{(p)})^{-1} \in \mathfrak{gl}_n(\mathcal{O})$
2) $R^\delta \subset \mathcal{O}$

$\Longrightarrow \quad G_{u/\mathcal{O}} = N^\delta$

- Talk about proof of theorem B if time allows.

$$\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$$

Z

N