A Geometric Algebra Primer

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Abstract. A brief introduction to Geometric Algebra with an emphasis on algorithms.

1. definitions

Definition 1.1 (Geometric Algebra). The *geometric algebra* $\mathrm{GA}(n)$ of *ambient rank* n is the quotient algebra $\mathbf{R}\{[1],[2],\ldots,[n]\}/I$, where $\mathbf{R}\{[1],[2],\ldots,[n]\}$ is the free (unital, associative, noncommutative) algebra over the real field $\mathbf{R}$ generated by noncommuting elements $[1],[2],\ldots,[n]$, and I is the (two-sided) ideal of $\mathbf{R}\{[1],[2],\ldots,[n]\}$ generated by the set of elements of the form $[i][i]-1$ and of the form $[i][j]-[j][i]$ when $i \neq j$.

The multiplication operator in $\mathrm{GA}(n)$ is called the *geometric product* and is written by juxtaposition: ab is the product of a and b, in that order. An (equivalence class of the) element of $\mathrm{GA}(n)$ of the form $[i_1][i_2]\cdots[i_m]$ is abbreviated $[i_1 i_2 \cdots i_m]$ and is called a *basis blade*. As a special case, the empty basis blade $[]$ is defined to be the multiplicative identity 1.

Example 1.2. A commonly-used geometric algebra in graphics applications is GA[3] of ambient rank 3. The algebra is generated by the elements [1], [2], and [3]. Clearly $[1][1] = [] = 1$, $[1][2] = -[2][1]$, and so on. From this one finds that $\mathrm{GA}(3)$ is an eight-dimensional vector space with basis $\{[],[1],[2],[3],[12],[13],[23],[123]\}$. A typical element of $\mathrm{GA}(3)$ is then equal to $s[]+a[1]+b[2]+c[3]+A[23]+B[13]+C[12]+S[123]$ where s, a, b, c, A, B, C, and S are real numbers.

We consider elements of $\mathrm{GA}(3)$ of the form $a[1]+b[2]+c[3]$ to be vectors (or rank one elements) and this three-dimensional vector space is the geometric entity $\mathrm{GA}(3)$ models. Elements of the form $s[]$, also just written s, are scalars (or rank zero elements), so in a geometric algebra, one can add a scalar to a vector. Elements of the form $A[23]+B[13]+C[12]$ are bivectors (rank two elements), which model oriented plane regions with an area magnitude.

Just as a vector has direction and magnitude, and one considers two vectors having the same direction and magnitude to be equal even if they have different locations in three-space, a bivector has a containing plane, one of two orientations on that plane, and an area, so two planar regions having the same area, contained in parallel or equal planes, and given the same orientation (which may be thought of as clockwise versus counterclockwise) are considered to be the same bivector.

Elements of the form $S[123]$ are considered volume elements (rank three elements), also called pseudoscalars because they have one degree of freedom in three-dimensional space. A volume element only has magnitude and orientation (positive or negative, which may be thought of as being a right-handed or left-handed coordinate system).

Addition of two elements expressed in the linear combination form mentioned above is done by combining like terms. If $x = 4 + 3[1] + 4[2] + 5[12]$ and $y = 3 + 2[1] + 3[2] + 4[12]$, then $x + y = 7 + 5[1] + 7[2] + 9[12]$.

Multiplication works like polynomial multiplication, except without commutativity, and using the rules of converting products of blades into the standard ones $[] = 1$, $[1]$, $[2]$, $[12]$, $[3]$, $[13]$, $[23]$, and $[123]$. So, using the same x and y as above, to find xy, one multiplies every term of x by every term of y, then reduces and combines like terms:

$$\begin{aligned}
xy = \quad & 12 + 8[1] + 12[2] + 16[12] \\
& + 9[1] + 6[1][1] + 9[1][2] + 12[1][12] \\
& + 12[2] + 8[2][1] + 12[2][2] + 16[2][12] \\
& + 15[12] + 10[12][1] + 15[12][2] + 20[12][12] \\
= \quad & 12 + 8[1] + 12[2] + 16[12] \\
& + 9[1] + 6 + 9[12] + 12[2] \\
& + 12[2] - 8[12] + 12 - 16[1] \\
& + 15[12] - 10[2] + 15[1] - 20 \\
= \quad & 10 + 16[1] + 26[2] + 32[12]
\end{aligned}$$

1.1. Standard Basis Blades. Generalizing from the example, we see that swapping adjacent unequal indices of a basis blade changes its sign and deleting pairs of adjacent, equal indices does not change the element. Thus, all basis blades in $\mathrm{GA}(n)$ are either plus or minus a *standard* basis blade $[i_1 i_2 \cdots i_m]$ $(m \le n)$ in which the indices are in increasing order and there are no duplicate indices. In this case, the *rank* of the basis blade is m. Any element of $\mathrm{GA}(n)$ can be written uniquely (up to ordering of terms) as a linear combination of standard basis blades (including $[]$, the rank-zero empty basis blade equal to 1 in the algebra), and the rank of that element is the maximum rank among the basis blades in the linear combination. The rank of the 0 element is defined to be $-\infty$. Thus, $\mathrm{GA}(n)$ is an algebra filtered by rank.

A nonzero element of $\mathrm{GA}(n)$ that is a linear combination of standard basis blades of the same rank r is said to be *purely* of rank r. Rank 0 elements (that is, pure rank 0 elements along with the 0 element) are called *scalars*. Pure rank 1 elements (along with 0) are called *vectors*. Pure rank two elements (along with 0) are called *bivectors*, and so on. All elements are called *multivectors*. When the ambient rank is n, rank n elements (along with 0) are called *pseudoscalars*, rank $n-1$ elements (along with 0) are called *pseudovectors*, and so on.

The algebra $\mathrm{GA}(n)$ is generated, as a vector space, by the standard basis blades $[]$, $[1]$, $[2]$, $\ldots$, $[n]$, $[12]$, $\ldots$ $[123\ldots n]$, of which there are 2^n, so as a real vector space, $\mathrm{GA}(n)$ has dimension 2^n.

In addition to the geometric product, other products are defined in $\mathrm{GA}(n)$. The *inner product* is the symmetric component of the geometric product: $a \cdot b = \frac{ab+ba}{2}$.

The *outer product* is the antisymmetric component of the geometric product: $a \wedge b = \frac{ab-ba}{2}$. As the names imply, $a \cdot b = b \cdot a$, and $a \wedge b = -b \wedge a$. Then, $ab = a \cdot b + a \wedge b$.

1.2. Additional Operations. Note that for vectors, $a \cdot b$ is exactly the standard dot product, using $\{[1], [2], \ldots, [n]\}$ as the standard ordered basis. For vectors in GA(3), the wedge product has the same magnitude as the cross product, but is a bivector (pseudovector) rather than a vector. Also, the wedge product is defined in any number of dimensions, not just three, and for non-vector elements as well.

The *standard pseudoscalar* I of GA(n) is defined to be the rank-n standard basis blade $[123 \ldots n]$. Then, $I^2 = II$ must either be -1 or 1, so $I^4 = 1$ regardless of n. In particular, the subalgebra of GA(2) generated by $[]$ and I is isomorphic to the complex number field. In fact $I^2 = -1$ in GA(2) and the isomorphism maps 1 to $[]$ and $\imath$ to I. In addition, GA(3) contains a subalgebra isomorphic to the quanternion algebra. This subalgebra is generated by $[]$, $i = [23]$, $j = [13]$, and $k = [12]$, with $[]$ maping to 1 and i, j, and k mapping to the same-named elements of the quaternion ring.

The *norm* $|a|$ of an element $a \in \mathrm{GA}(n)$ is defined to be the square root of the sum of the squares of the coefficients of the standard basis blades making up the linear combination. Note for vectors this reduces to the usual norm for vectors.

The *reversion* $\tilde{a}$ of an element $a \in \mathrm{GA}(n)$ is formed from a by taking the linear combination of standard basis blades composing a, and then reversing the order of multiplication of the algebra generators in each term: $[i_{j_1}][i_{j_2}] \cdots [i_{j_m}] \to [i_{j_m}][i_{j_{m-1}}] \cdots [i_{j_1}]$. Equivalently, each term of a of pure rank r, where r is congruent to 2 or 3 modulo 4, is negated.

The *grade projection* $\mathrm{proj}_k(a)$ of an element $a \in \mathrm{GA}(n)$ is formed from a by deleting all terms of pure rank not equal to k.

The *left inverse* of $a \in \mathrm{GA}(n)$ is an element b satisfying $ba = 1 = []$ and the *right inverse* c likewise satisfies $ac = 1$. We use a^{-1} to represent an element that is both the left and right inverse of a. If $a\tilde{a}$ is a nonzero scalar, then $\frac{a}{a\tilde{a}}$ is well-defined by dividing each coefficient of a by the scalar denominator. In such a case, a does have a two-sided inverse and $a^{-1} = \frac{a}{a\tilde{a}}$.

The *dual* of an element a of GA(n) is aI^{-1}. Note that I^{-1} exists and is equal to I^3, which is either I or $-I$. Note that the dual of a scalar is a psuedoscalar, the dual of a vector is a pseudovector, and so on. The double dual of a is either a or $-a$.

Note that in GA(3), $a \wedge bI^{-1}$ is a vector, and is equal to the vector space cross product $a \times b$. Thus, we define a *generalized cross product* on any GA(n) to be $a \times b = a \wedge bI^{-1}$, the dual of the wedge product. In dimensions other than 3, the cross product of vectors is no longer a vector. However, in n dimensions, one can take the dual of the wedge product of $n-1$ vectors to give a vector which geometrically is similar to the cross product in three dimensions.

The *exponential* of $a \in \mathrm{GA}(n)$ is computed as follows. If $a = 0$, $e^0 = 1$ by definition. Otherwise, let $\theta = |a|$ and $b = a/\theta$. Then use DeMoivre's formula with b taking the role of the imaginary unit: $e^a = \cos\theta + b\sin\theta$. The inverse of this operation, a geometric algebraic logarithm, is difficult and multivalued, even more so than with complex numbers.

The main use for the exponential is in computing a *rotor*. If v and w are linearly-independent unit vectors (that is, $|v| = |w| = 1$) in the n-dimensional vector space whose basis is the rank-one standard blades, then vw is a bivector that represents an oriented plane in the n-dimensional space. A rotation about the origin in the direction from v to w along an angle *theta* is then modeled by the *rotor* r in $\mathrm{GA}(n)$ that is $r = e^{\frac{1}{2}vw\theta}$. Then if a is any element (vector or otherwise) of $\mathrm{GA}(n)$, its rotation is then $ra\tilde{r}$. If a is a vector, so is $ra\tilde{r}$, and this is an ordinary vector rotation. If a is a bivector representing an oriented plane, $ra\tilde{r}$ is the rotation of the plane. And so on.

The *meet* of two elements a and b in $\mathrm{GA}(n)$ is defined by $aI^{-1}b$, or the product of the dual of a with b.

2. Algorithms

An apparent way to represent a standard basis blade $a = [i_1 i_2 \ldots i_m]$ is via the integer $\mathrm{N}(a)$ defined to be $\sum_{k=1}^{m} 2^{i_k}$. Then, the product ab of $a = [i_1 i_2 \ldots i_m]$ and $b = [j_1 j_2 \ldots j_r]$ can be computed as follows:

(1) If $m = 0$, let c be the multivector b and return c
(2) If $r = 0$, let c be the multivector a and return c
(3) Let c be the blade for which $\mathrm{N}(c)$ is the bitwise exclusive or of $\mathrm{N}(a)$ and $\mathrm{N}(b)$.
(4) Let s be 1.
(5) Let p be the maximum of $\mathrm{N}(a)$ and $\mathrm{N}(b)$.
(6) Let d be m
(7) Let d be 1
(8) while $d \leq p$, do the following:
    (a) Compute the bitwise and of d and $\mathrm{N}(a)$.
    (b) If the result of the bitwise and is nonzero, then replace d with $d - 1$.
    (c) Compute the bitwise and of d and $\mathrm{N}(b)$.
    (d) If the result of the bitwise and is nonzero and if d is odd, then replace s with $-s$.
(9) Let c be the multivector that is the real coefficient s multiplied by the blade given by c, and return c.

A standard basis blade is simply a nonnegative integer in this representation. A choice needs to be made to represent a multivector, a real linear combination of these blades.

The *dense* representation of a multivector in $\mathrm{GA}(n)$ is an array A of dimension 2^n, thus having elements A_0, A_1, and so on up to A_{2^n-1}. Then, a multivector $\sum_{i=1}^{l} c_i b_i$, where the c_i are real numbrers and the b_i are standard basis blades in $\mathrm{GA}(n)$, is represented by the array A in which for $1 \leq i \leq l$, $A_{\mathrm{N}(b_i)} = c_i$, and all other elements of A are zero.

The dense representation requires little computation, but uses $O(2^n)$ memory per element of $\mathrm{GA}(n)$, so it is useful for low dimensionality, such as in representing the commonly-used $\mathrm{GA}(3)$.

For larger dimensions, a *sparse* representation is needed. Let D be a *dictionary*, algorithmically represented by a hash table in most cases, but mathematically works like the array: for any nonnegative integer i, D_i is a real number. For most i, this real number is zero and thus i has no entry in the hash table. Only when D_i is nonzero is there an entry in the hash table with key i and value D_i.

Then, a multivector $\sum_{i=1}^{l} c_i b_i$, where the c_i are real numbrers and the b_i are standard basis blades in $\mathrm{GA}(n)$, is represented by the dictionary D in which for

$1 \leq i \leq l$, $D_{\mathrm{N}(b_i)} = c_i$, and no other entries are present in D (so that the dictionary returns the default value of 0 for those).

The sparse representation requires a small amount of computation for hashing and, sometimes, rebuilding the table, but only requires $O(l)$ memory for an element of $\mathrm{GA}(n)$ having l nonzero terms. Thus, the sparse representation is most useful for large n when many elements are such that most standard basis blades have a zero coefficient.

In both cases, the addition procedure is evident and involves adding the values from one array or dictionary to the same-indexed values of the other.

The geometric product, however, requires an all-to-all mapping of array or dictionary elements. We illustrate the algorithm for the array case, with the dictionary case being nearly identical, just iterating over keys in the hash table instead of over all integers in the range from 0 to 2^{n-1}.

Suppose A and B are arrays representing multivectors in $\mathrm{GA}(n)$. We compute a new array C with 2^n entries representing the geometric product of the multivectors A and B.

(1) Initialize C to $C_0 = C_1 = \cdots = c_{2^{n-1}} = 0$.
(2) For each i from 0 to 2^{n-1} such that the coefficient A_i is nonzero
    (a) For each j from 0 to 2^{n-1} such that the coefficient B_j is nonzero
        (i) Let k be the integer that is the result of multiplying together the blades represented by i and j, and let s be the coefficient, 1 or -1, of the result.
        (ii) Let C_k be $C_k + A_i B_j s$, an ordinary sum and product of real numbers.
(3) Return the multivector C.