Discrete Mathematics
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Bonus Assignment
Due 4 September, 2024

Collaborators: none

Exercise **1** *Points* 3

Given. $f : \mathbb{N} \to \mathbb{N}$, such that, $\forall m, n \in \mathbb{N}, m < n \to f(m) < f(n)$. $f(10) = 10$
To Prove. $f(i) = i$ for $i \leq 10, n \in \mathbb{N}$.
Proof by Induction.
Base Case. $f(10) = 10$ (Given).
Inductive Hypothesis. Assume that $f(i) = i$ for $2 \leq i \leq 10$. We need to show that $f(i-1) = i-1$.
Inductive Step. Since f is strictly increasing, $f(i-1) < f(i)$, and $f(i) = i$ (by inductive hypothesis). Therefore, $f(i-1) < i$. Since $f : \mathbb{N} \to \mathbb{N}$, $f(i-1)$ is a natural number.

The only natural number less than i that $f(i-1)$ is equal, while still satisfying the condition $f(i-1) < i$, is $i-1$.

- $\hookrightarrow$ If $f(i-1) < i-1$, then it would not be a strictly increasing function. Since for some $m < i-1$, $f(m)$ has to be smaller than $f(i-1)$, which is not possible (there aren't enough distinct natural numbers to assign to $f(m)$ for $m < i$–1 while still maintaining the strictly increasing property).
- $\hookrightarrow$ If $f(i-1) > i-1$, then $f(i-1) \geq i$, which is a contradiction to the inductive hypothesis and the fact that $f(i-1) < i$.

Therefore, $f(i-1) = i-1$, and starting from $f(10) = 10$, we get $f(9) = 9$, $f(8) = 8$, and so on, till $f(1) = 1$. $\square$

Exercise **2** *Points* 3

Given. $f : \mathbb{N} \to \mathbb{N}$, such that, $\forall m, n \in \mathbb{N}, m > n \to f(m) < f(n)$.
To Prove. $\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq k \to f(n) = 0$.
Proof.
The given condition states that n increases $f(n)$ decreases. As $f(x) : \mathbb{N} \to \mathbb{N}$, and is strictly decreasing, it cannot keep decreasing forever while remaining $\mathbb{N}$, i.e., $n < 0$, which is bounded below by 0, and the sequence must reach 0 at some point.

Let k be the smallest natural number such that $f(k) = 0$. Then, for any $n > k$, $f(n) < f(k) = 0$, and the only $\mathbb{N}$ less than 0 is 0. Therefore, $f(n) = 0$ for all $n \geq k$. $\square$

It's not a strictly decreasing function and is a bit weird as the condition $m > n \to f(m) < f(n)$ can't really be fulfilled, since $f(n) = 0$ always after some point. So we didn't even need to prove anything as the premise (if f is strictly decreasing $\to \cdots$) of the condition itself is not true.

Exercise **3**	*Points* 3

Given. $\exists e \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + e = e$. e_1 and e_2 are two such integers such that $x + e_1 = x + e_2$ for all $x \in \mathbb{Z}$.
$x + e = e \to x = 0$, which implies that the only value that x can take is 0. The statement seems incorrect because no integer e can satisfy the condition $x + e = e$ for all integers x unless the set of integers is restricted to a single value, which contradicts the definition of $\forall x \in \mathbb{Z}$.

We need to redefine the problem statement. That is $x + e = x$ for all $x, e \in \mathbb{Z}$, and e_1 and e_2 are two such integers such that $x + e_1 = x + e_2$ for all $x \in \mathbb{Z}$.

To Prove. e is unique, $x + e_1 = x + e_2 \to e_1 = e_2$. $x = 0$.
Proof.
Uniqueness. Let e_1 and e_2 be two such integers such that $x + e_1 = x + e_2$ for all $x \in \mathbb{Z}$. Subtracting x from both sides, we get $e_1 = e_2$. Therefore, e is unique. $\square$
Existence. Let $x = 0$, then $0 + e_1 = 0 + e_2 \to e_1 = e_2$. Therefore, $e = 0$. $\square$

I don't understand this question at all, how is e unique? It's not unique, it's 0. Did we not need to prove this question as well? Since even here the premise itself is not true.

Exercise **4**	*Points* 3

Given. A relation $\sim$ defined on $\mathbb{Z}$ as $\mathbb{N} \times \mathbb{N}$, such that $(m_1, n_1) \sim (m_2, n_2) \iff m_1 + n_2 = m_2 + n_1$.
To Prove. $\sim$ is an equivalence relation.
Proof
Reflexive. To show its reflexive, we need to show that $(m, n) \sim (m, n)$. $(m_1, n_1) \sim (m_1, n_1) \iff m_1 + n_1 = m_1 + n_1$. Since $m_1 + n_1 = m_1 + n_1$, $\sim$ is reflexive. $\square$

Symmetric. To show its symmetric, we need to show that $(m_1, n_1) \sim (m_2, n_2) \to (m_2, n_2) \sim (m_1, n_1)$. If $(m_1, n_1) \sim (m_2, n_2)$, then $m_1 + n_2 = m_2 + n_1$. Therefore, $m_2 + n_1 = m_1 + n_2$, which implies $(m_2, n_2) \sim (m_1, n_1)$. $\square$

Tansitive. To show its transitive, we need to show that $(m_1, n_1) \sim (m_2, n_2)$ and $(m_2, n_2) \sim (m_3, n_3) \to (m_1, n_1) \sim (m_3, n_3)$. If $(m_1, n_1) \sim (m_2, n_2)$, then $m_1 + n_2 = m_2 + n_1$. If $(m_2, n_2) \sim (m_3, n_3)$, then $m_2 + n_3 = m_3 + n_2$. Adding the two equations, we get $m_1 + n_2 + m_2 + n_3 = m_2 + n_1 + m_3 + n_2 \to m_1 + n_3 = m_3 + n_1$. Therefore, $(m_1, n_1) \sim (m_3, n_3)$. $\square$