Linear Algebra
Professor Shuchita Goyal — Week 1
Shubhro Gupta — Due October 10, 2024

Collaborators: none

Problem **1**	1.5 + 0.5 *Points*

To Show. If A is a square matrix of order $n \times n$, then there exists a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$.
Solution. A matrix B is symmetrical if $b_{ij} = b_{ji}$ for all i, j. A matrix C is skew-symmetric if $c_{ij} = -c_{ji}$ for all $i, j \in \mathbb{N}$.
Let $B = A + A^T$, then $(B)^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$. Which implies B is symmetric.
Similarly consider $C = A - A^T$, then $(C)^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -C$. Which implies C is skew-symmetric.
Adding B and C we get $B + C = A + A^T + A - A^T = 2A$. Dividing by 2 we get $A = \frac{1}{2}B + \frac{1}{2}C$. Since B is symmetric, $\frac{B}{2}$ is symmetric. Similarly $\frac{C}{2}$ is skew-symmetric as C is skew-symmetric.
Thus, we have shown that for any square matrix A, there exists a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$ $\square$.

To Show. B and C are unique.
Solution. Let B_1 and C_1 be symmetric and skew-symmetric matrices such that $A = B_1 + C_1$. Let B_2 and C_2 be another symmetric and skew-symmetric matrices such that $A = B_2 + C_2$. Then $B_1 + C_1 = B_2 + C_2 \implies B_1 - B_2 = C_2 - C_1$. Since B_1 and B_2 are symmetric, $B_1 - B_2$ is symmetric. Similarly, $C_2 - C_1$ is skew-symmetric. The only matrix which is both symmetric and skew-symmetric is the zero matrix. Thus, $B_1 = B_2$ and $C_1 = C_2$ $\square$.

Problem **2**	1.5 *Points*

To Do. Write the row vector $z = (3, 2) \in \mathbb{R}^{1\times 2}$ as a linear combination of $u, v, w \in \mathbb{R}^{1\times 2}$ where $u = (1, 1), v = (10, 7), w = (3, 13)$.
Solution. Linear combination of u, v, w is given by $(3, 2) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $\alpha \in \mathbb{R}$.
Writing it in matrix form, we get $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 10 & 3 \\ 1 & 7 & 13 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$. In augmented form we get,

$$\begin{bmatrix} 1 & 10 & 3 & | & 3 \\ 1 & 7 & 13 & | & 2 \end{bmatrix} \xrightarrow{L_2 - L1} \begin{bmatrix} 1 & 10 & 3 & | & 3 \\ 0 & -3 & 10 & | & -1 \end{bmatrix}$$

the augmented matrix is now in row echelon form (pivot 1 = 1, pivot 2 = -3). using back substitution in row$_2$, let $\alpha_3 = x$,then $-3\alpha_2 + 10x = -1 \implies \alpha_2 = \frac{10x+1}{3}$.

Substituting α_2 in row$_1$,

$$
\begin{aligned}
\alpha_1 + 10\left(\frac{10x+1}{3}\right) + 3x &= 3 \\
\alpha_1 + \frac{100}{3}x + \frac{10}{3} + 3x &= 3 \\
\alpha_1 + \frac{109}{3}x + \frac{10}{3} &= 3 \\
\alpha_1 + \frac{109x+10}{3} &= 3 \\
\alpha_1 &= 3 - \frac{109x+10}{3} \\
\alpha_1 &= \frac{9-109x-10}{3} \\
\alpha_1 &= \frac{-109x-1}{3}
\end{aligned}
$$

Thus, the row vector $z = (3, 2)$ can be written as a linear combination of u, v, w as $\begin{bmatrix} \frac{-109x-1}{3} & \frac{10x+1}{3}x & x \end{bmatrix}$. Putting $x = 0$, we get $\begin{bmatrix} \frac{-1}{3} & \frac{1}{3} & 0 \end{bmatrix}$.

Problem **3**	3 *Points*

To Show. Product of 2 upper triangular matrices is an upper triangular matrix.
Solution. A square matrix U is upper triangular if $u_{ij} = 0$ for all $i > j; i, j \in \mathbb{N}$. Let A and B be two upper triangular matrices of order $n \times n$. Let $C = AB$.
Let c_{ij} be the element at i^{th} row and j^{th} column of C. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Since A and B are upper triangular, $a_{ik} = 0$ for all $i > k$ and $b_{kj} = 0$ for all $j > k$.

We have to show that $c_{ij} = 0$ for all $i > j$.
To have a non-zero entry $[c_{ij}]$, both a_{ik} and b_{kj} must be non-zero.
For $i > j$:
Let us consider that $a_{ik} \neq 0$, then $i \leq k$. And similarly, $b_{kj} \neq 0$, then $k \leq j$. Combining both, we get $i \leq k \leq j$. But this is not possible as $i > j$. Thus, $c_{ij} = 0$ for all $i > j$ $\square$.

To Show. Show that the above is true for lower triangular matrices.
Solution. A square matrix L is lower triangular if $l_{ij} = 0$ for all $i < j$. Let A and B be two lower triangular matrices of order $n \times n$. Let $C = AB$.
Let c_{ij} be the element at i^{th} row and j^{th} column of C. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Since A and B are lower triangular, $a_{ik} = 0$ for all $i < k$ and $b_{kj} = 0$ for all $k < j$.

We have to show that $c_{ij} = 0$ for all $i < j$.
To have a non-zero entry $[c_{ij}]$, both a_{ik} and b_{kj} must be non-zero.
For $i < j$:
Let us consider that $a_{ik} \neq 0$, then $i \geq k$. And similarly, $b_{kj} \neq 0$, then $k \geq j$. Combining both, we get $i \geq k \geq j$. But this is not possible as $i < j$. Thus, $c_{ij} = 0$ for all $i < j$ $\square$.

Problem **4**	1+2 *Points*

To Show. trace$(A + B)$ = trace(A) + trace(B).
Solution. The trace of a matrix is the sum of the diagonal elements, for a square matrix A of order

$n \times n$, $\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$.

$$\begin{aligned}
\text{trace}(A+B) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\
&= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\
&= \text{trace}(A) + \text{trace}(B)
\end{aligned}$$

Thus, $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B) \qquad \square$.

To Show. $\text{trace}(AB) = \text{trace}(BA)$, for $A, B \in \mathbb{R}^{n \times n}$.
Solution. Let A and B be two square matrices of order $n \times n$. Diagonal entries of AB are given by $c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$. Similarly, diagonal entries of BA are given by $d_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}$.

$$\begin{aligned}
\text{trace}(AB) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} \\
&= \text{trace}(BA)
\end{aligned}$$

Thus, $\text{trace}(AB) = \text{trace}(BA) \qquad \square$.