Real Analysis
Professor Rishi Vyas — Week 1
Shubhro Gupta — Due Tuesday, July 30, 2024

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QUESTION **A** — 4 Points

Definition. *Let $x \in \mathbb{R}$. If $n \in \mathbb{N}$, we define $x^n := \overbrace{x \dots x}^{n \text{ times}}$. By convention, for $n = 0$ we interpret this as defining $x^0 := 1$.*

To Prove. Let $x, y, z \in \mathbb{R}$.

1. If 0' is an element of $\mathbb{R}$ such that $0' + x = x$ for all $x \in \mathbb{R}$, then $0' = 0$.

$$\begin{aligned} 0' + x &= x \\ 0' + x + (-x) &= x + (-x) \\ 0' + 0 &= 0 \qquad \text{(Additive Inverse (4))} \\ 0' &= 0 \end{aligned}$$

$\square$

2. If 1' is an element of $\mathbb{R}$ such that $1' \cdot x = x$ for all $x \in \mathbb{R}$, then $1' = 1$.

$$\begin{aligned} 1' \cdot x &= x \\ 1' \cdot x \cdot x^{-1} &= x \cdot x^{-1} \\ 1' \cdot 1 &= 1 \qquad \text{(Multiplicative Inverse (8))} \\ 1' &= 1 \end{aligned}$$

$\square$

3. $x \neq 0$ and $xy = xz$. Prove that $y = z$. Deduce that if $xy = 1$ then $y = x^{-1}$.

$$\begin{aligned} xy &= xz \\ ((x)^{-1} \cdot x)y &= ((x)^{-1} \cdot x)z \\ y &= z \end{aligned}$$

$\square$

$$\begin{aligned} xy &= 1 \\ ((x)^{-1} \cdot x)y &= (x)^{-1} \cdot 1 \end{aligned}$$

$\square$

4. $-0 = 0$

$$\begin{aligned} 0 &= 0 \\ 0 \times -1 &= 0 \times -1 \\ 0 &= -0 \end{aligned}$$

5. If $x \neq 0$, then $x^{-1} \neq 0$ and $\left(x^{-1}\right)^{-1} = x$.

$$\begin{aligned} \text{Suppose } x^{-1} &= 0 \\ x \cdot x^{-1} &= x \cdot 0 \\ 1 &= 0 \qquad \text{(Multiplicative inverse (8))} \end{aligned}$$

$1 \neq 0$ so $x^{-1} \neq 0$.

6. $(-x) \times (-y) = xy$.

$$\begin{aligned} (-x) \times (-y) &= (-x)(-y) + 0y \\ &= (-x)(-y) + (x + (-x))y \\ &= (-x)(-y) + xy + (-x)y \qquad \text{(Distributive Law)} \\ &= ((-x)(-y) + (-x)y) + xy \\ &= (-x)(-y + y) + xy \\ &= (-x)(0) + xy \\ &= 0 + xy \\ (-x) \times (-y) &= xy \end{aligned}$$

7. If $x \neq 0$ and $n \in \mathbb{N}$, then $(-x)^{-1} = -\left(x^{-1}\right)$ and $\left(x^{-1}\right)^{n} = (x^{n})^{-1}$.

8. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

$$\begin{aligned} \text{Suppose } xy &= 0 \\ x \cdot x^{-1} \cdot y &= 0 \cdot x^{-1} \\ 1 \cdot y &= 0 \cdot x^{-1} \\ y &= 0 \\ &※ \end{aligned}$$

QUESTION **B**	3 Points

Let $x, y, z \in \mathbb{R}$.

1. If $x < y$ and $z > 0$, then $xz < yz$.
   Proof.
   If $x < y$, then $y - x > 0$. Since $z > 0$, we can multiply both sides of the inequality by z to get $z(y - x) > 0$. This implies $zy - zx > 0$, which is equivalent to $xz < yz$. Thus, if $x < y$ and $z > 0$, then $xz < yz$.

2. If $x < 0$ then $x^{-1} < 0$.
   Proof.
   $\exists x \in \mathbb{R}$ such that $x < 0$. Since $x < 0$, we have $-x > 0$. Since $-x > 0$, we can take the reciprocal of both sides to get $(-x)^{-1} > 0$. This implies $x^{-1} > 0$. Thus, if $x < 0$, then $x^{-1} > 0$. $x < 0 \implies x - x < 0 - x \implies -x > 0$.

3. If $x, y \geq 0$ and $n \in \mathbb{Z}_{>0}$, prove that $x \leq y$ if and only if $x^n \leq y^n$. Deduce that $x < y$ if and only if $x^n < y^n$.
   To Prove. $x, y \geq 0$ and $n \in \mathbb{Z}_{>0}$, then $x \leq y \iff x^n \leq y^n$. Case I, $x \leq y$

QUESTION **C**	4 Points

1. Show that $\mathbb{Q}$ is neither bounded above nor bounded below.

2. Determine the supremum and infimum of the following sets. Substantiate your claims with proofs:
   Definition. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.
   The element x is the *supremum* of S if $s \leq x, \forall s \in S$ and for any upper bound y of S, $x \leq y$.
   The element x is the *infimum* of S if $s \geq x, \forall s \in S$ and for any lower bound y of S, $x \geq y$.

   (a) (-1, 0] $= \{x \in \mathbb{R}, -1 < x \leq 0\}$
   To Prove. Supremum is 0, Infimum is -1.
   Proof. 0 is an upper bound by the definition of the set. Let y be an upper bound for (-1, 0], since $0 \in (-1, 0] \implies 0 \leq y$. So, 0 is the supremum.
   -1 is a lower bound by the definition of the set. To show that -1 is the greatest lower bound, lets assume M is the infimum and $M > -1$ (M cannot be less than -1 since -1 is already a lower bound, and any number lesser than -1 cannot be the infimum).
   Let x be the midpoint between M and -1, $x = \frac{M+(-1)}{2} = \frac{M-1}{2} = M - (\frac{1+M}{2})$. This implies $M > x$, and we have to show that $x > -1$.
   But let's assume $x \leq -1$, then

$$\begin{aligned} M - (\frac{1+M}{2}) &\leq -1 \\ M - 1 &\leq -2 \\ M &\leq -1 \end{aligned}$$

   But we assumed $M > -1$, thus by contradiction, $x > -1$. This implies $M > x > -1$, which contradicts the assumption that M is the infimum. Thus, -1 is the infimum.

   (b) (1, 2) $= \{x \in \mathbb{R}, 1 < x < 2\}$
   To Prove. Supremum is 2, Infimum is 1.
   Proof. 2 is an upper bound by the definition of the set. To show that 2 is the least upper bound, let's assume M is the supremum and $M < 2$ (M cannot be greater than 2 since 2 is already an upper bound, and any number greater than 2 cannot be the supremum).
   Let x be the midpoint between M and 2, $x = \frac{M+2}{2}$. This implies $M < x$, and we have to show that $x < 2$.
   But let's assume $x \geq 2$, then

$$\begin{aligned} \frac{M+2}{2} &\geq 2 \\ M + 2 &\geq 4 \\ M &\geq 2 \end{aligned}$$

   But we assumed $M < 2$, thus by contradiction, $x < 2$. This implies $2 > x > M$, which contradicts the assumption that M is the supremum. Thus, 2 is the supremum.
   □

   1 is a lower bound by the definition of the set. To show that 1 is the greatest lower bound, lets assume M is the infimum and $M < 1$ (M cannot be greater than 1 since 1 is already a lower bound, and any number greater than 1 cannot be the infimum).
   Let x be the midpoint between M and 1, $x = \frac{M+1}{2}$. This implies $M < x$, and we have to

show that $x < 1$.
But let's assume $x \geq 1$, then

$$\begin{aligned}
\frac{M+1}{2} &\geq 1 \\
M+1 &\geq 2 \\
M &\geq 1
\end{aligned}$$

But we assumed $M < 1$, thus by contradiction, $x < 1$. This implies $1 > x > M$, which contradicts the assumption that M is the infimum. Thus, 1 is the infimum. $\square$

QUESTION **D**	4 Points

1. Let $a, b \in \mathbb{R}$

   (a) Consider the set $\{a, b\}$. Prove that the supremum of the set $\{a, b\}$, is equal to a if if $a \geq b$ and is equal to b of $b \geq a$.
   Solution.
   Case I, Let $a, b \in \mathbb{R}$ and $a \geq b$.
   $\forall x \in \{a, b\}, x \leq a$ therefore a is an upper bound.
   Let y be an upper bound for $\{a, b\}$, since $a \in \{a, b\} \implies a \leq y$. So, a is the supremum.

   Case II, Let $a, b \in \mathbb{R}$ and $b \geq a$.
   $\forall x \in \{a, b\}, x \leq b$ therefore b is an upper bound.
   Let y be an upper bound for $\{a, b\}$, since $b \in \{a, b\} \implies b \leq y$. So, b is the supremum.

   (b) **To Prove.**
   $$\sup(\{a, b\}) = \frac{a + b + |a - b|}{2}.$$
   Solution.
   Case I, $a \geq b$
   From QD.1(a), we know that the supremum of the set $\{a, b\}$ is a
   $$\begin{aligned} \sup(\{a, b\}) &= a \\ \sup(\{a, b\}) &= a + b - b \\ \sup(\{a, b\}) &= \frac{2a + b - b}{2} \\ \text{Since } a \geq b, a - b = |a - b| \\ \sup(\{a, b\}) &= \frac{a + b + |a - b|}{2} \\ &\square \end{aligned}$$

   Case II, $b \geq a$
   From QD.1(a), we know that the supremum of the set $\{a, b\}$ is b
   $$\begin{aligned} \sup(\{a, b\}) &= b \\ \sup(\{a, b\}) &= a + b - a \\ \sup(\{a, b\}) &= \frac{a + b - (a - b)}{2} \\ \text{Since } a \leq b, a - b \leq 0 &\implies |a - b| = -(a - b) \\ \sup(\{a, b\}) &= \frac{a + b + |a - b|}{2} \\ &\square \end{aligned}$$

   Thus for both cases, the formula $\sup(\{a, b\}) = \frac{a+b+|a-b|}{2}$ works. Hence proved.

(c) Formulate and prove a variant of the above formula describing the infimum of the set $\{a, b\}$.
Solution.
Case I, $a \geq b$
Using the same principle from QD.1(a)

$$\inf(\{a, b\}) = b$$
$$\inf(\{a, b\}) = b + a - a$$
$$\inf(\{a, b\}) = \frac{b + b + a - a}{2}$$
$$\inf(\{a, b\}) = \frac{a + b - a + b}{2}$$
$$\inf(\{a, b\}) = \frac{a + b - (a - b)}{2}$$
$$\text{Since } a \geq b, a - b = |a - b|$$
$$\inf(\{a, b\}) = \frac{a + b + |a - b|}{2}$$

□

Case II, $b \geq a$
Using the same principle from QD.1(a)

$$\inf(\{a, a\}) = a$$
$$\inf(\{a, b\}) = a + b - b$$
$$\inf(\{a, b\}) = \frac{a + b - b + a}{2}$$
$$\inf(\{a, b\}) = \frac{b + b - (b - b)}{2}$$
$$\text{Since } a \geq b, a - b = |a - b|$$
$$\inf(\{a, b\}) = \frac{a + b + |a - b|}{2}$$

□

2. Let $a_1, \cdots, a_n$ be elements of $\mathbb{R}$. Prove that

$$\sup(\{a_1, \ldots, a_n\}) = \sup(\{\sup(\{a_1, \ldots, a_{n-1}\}), a_n\}).$$

Deduce that $\sup(\{a_1, \ldots, a_n\}) \in \{a_1, \ldots, a_n\}$.
Solution.
Case I, Let $a_1, \cdots, a_n$ be elements of $\mathbb{R}$ and $a_1 \geq a_2 \geq \cdots \geq a_n$.

QUESTION **E**	3 Points

Let S be a subset of $\mathbb{R}$.

1. Show that S is bounded if and only if there exists an $m \in \mathbb{R}$ such that $|x| \leq m$ for all $x \in S$.
   To Prove. S is bounded $\iff$ $\exists m \in \mathbb{R}$ such that $|x| \leq \forall x \in S$.
   Proof. Let $S \subseteq \mathbb{R}$ be a bounded set From $E(1), \exists m \in \mathbb{R}$ such that $|x| \leq m \forall x \in S$

$$\begin{aligned} &\forall x \in S, x \leqslant m \text{ and } -x \leqslant m \\ &\forall x \in S, -m \leqslant x \text{ and } x \leqslant m \\ &\forall x \in S, -m \leqslant x \leqslant m \\ &\forall x \in S, S \in [-m, m] \\ &S \leq [-m, m] \quad [a \neq b] \end{aligned}$$

   Assume $\exists m \in \mathbb{R}$ sit $S \subseteq [-m, m]$

$$\Rightarrow \forall x \in S, -m \leqslant x \leqslant m$$

   $\Rightarrow \forall x \in S, -m \leqslant x \Rightarrow -m$ is a lower bound $\Rightarrow \forall \in S, x \leqslant m \Rightarrow m$ is an upper bound of S. Since S has a lower bound and an upper bound, S is bounded in $\mathbb{R}$.

$$[b \Rightarrow a]$$

2. Deduce that the following three statements are equivalent:
   (a) The S is bounded.
   (b) There exists $m \in \mathbb{R}$ such that $S \subseteq [-m, m]$.
   (c) There exist $a, b \in \mathbb{R}$ with $a \leq b$ such that $S \subseteq [a, b]$.

QUESTION **F**	4 Points

1. Let $x, y \in \mathbb{R}$, and suppose x and y both satisfy the definition of an infimum of S. Show that $x = y$.
   To Prove. If $x, y \in \mathbb{R}$ and both satisfy the definition of an infimum of S, then $x = y$.
   Definition. Let $S \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is the infimum of S if $x \leq s \forall s \in S$ and for any lower bound y of S, $x \geq y$.
   Proof. Suppose x and y are the infimum of S.
   Since y is an infimum of S, y is a lower bound of S. Therefore, by the property of infimum, $y \leq x$ because x is the infimum.
   Since x is an infimum of S, x is a lower bound of S. Therefore, by the property of infimum, $x \geq y$ because y is the infimum.
   By the property of antisymmetry, $x \geq y$ and $y \geq x$ implies $x = y$. Thus, $x = y$.
   $\square$

2. Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions:

   (a) The element x is a lower bound for S.

   (b) Given $\epsilon > 0$, there exists some $y \in S$ such that $y < x + \epsilon$.
   To Prove. $x = \inf(S) \iff x$ is a lower bound for S and $\forall \epsilon > 0, \exists y \in S$ such that $y < x + \epsilon$.
   Solution.
   (a) It's already in the definition of infimum that x has to be a lower bound, so no need to prove that.
   (b) If such y does not exist, $\forall y \in S, y \geq x + \epsilon \implies x + \epsilon$ is a lower bound of S. $x + \epsilon$ is a lower bound $> x \implies x$ is not infimum.
   If such y exists, then $x + \epsilon$ is not a lower bound of S. Thus, x is the infimum of S.

3. Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions:

   (a) The element x is a lower bound for S.

   (b) Given $n \in \mathbb{Z}_{>0}$, there exists some $y \in S$ such that $y < x + \frac{1}{n}$.

QUESTION **G**	4 Points

Prove that in the presence of properties (1) to (16), property (17) and property $(17)_L$ are equivalent. Deduce that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

To Prove. If A and B are subsets of $\mathbb{R}$ such that both A and B are non-empty and $a \leq b$ for all $a \in A$ and $b \in B$, there exists $s \in \mathbb{R}$ such that $a \leq s \leq b$ for all $a \in A$ and $b \in B \iff$ every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

Proof.
Assuming $17 \implies 17_L$
Property $(17)_L$ states that every non-empty bounded below subset of $\mathbb{R}$ has an infimum. Let $S \subseteq \mathbb{R}$ be a non-empty bounded below subset. Define $B := \{x \in \mathbb{R} \mid x \leq s \text{ for all } s \in S\}$, the set of all lower bounds of S. Since S is bounded below, B is non-empty.
By property (17), there exists $t \in \mathbb{R}$ such that $t \leq s$ for all $s \in S$ and $b \leq t$ for all $b \in B$. Thus, t is a lower bound for S (since $t \leq s$ for all $s \in S$) and t is greater than or equal to any other lower bound of S (since $b \leq t$ for all $b \in B$). Therefore, t is the infimum of S.
We conclude that property $(17)_L$ holds.

Assuming $17_L \implies 17$
Assume property $(17)_L$ holds. We need to show that property (17) also holds.
Property (17) states that given non-empty subsets A and B of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$, there exists $t \in \mathbb{R}$ such that $a \leq t \leq b$ for all $a \in A$ and $b \in B$.
Let A and B be non-empty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$. This condition implies that every element of B is a lower bound for A, so A is bounded below. By property $(17)_L$, A has an infimum, say t. Since t is a lower bound for A, we have $t \leq a$ for all $a \in A$. Moreover, since t is the greatest lower bound of A, and every element of B is a lower bound for A, we have $b \leq t$ for all $b \in B$. Thus, there exists $t \in \mathbb{R}$ such that $a \leq t \leq b$ for all $a \in A$ and $b \in B$.
We conclude that property (17) holds.

Therefore $17 \iff 17_L$. $\square$