ANALYSIS 1

MATH 201

ASSIGNMENT 1

Remark. *This assignment is due on **Tuesday July 30** by **23:59** and will count towards your final grade as described in the course grading policy. Late submissions will be handled in accordance with course policies. When writing your arguments, you can use any result that has been proved in the class, mentioned on a previous assignment, or mentioned on the this assignment in a question preceding the one you are considering as long as you explicitly state the result that you are using. The number of points each question is worth is mentioned in square brackets at the beginning of the question. Your final score for this assignment will be scaled as described in the course policies.*

Definition. *Let $x \in \mathbb{R}$. If $n \in \mathbb{N}$, we define $x^n := \overbrace{x \ldots x}^{n \text{ times}}$. By convention, for $n = 0$ we interpret this as defining $x^0 := 1$.*

Question A. [4] Let $x, y, z \in \mathbb{R}$. Prove the following statements.

(1) If $0'$ is an element of $\mathbb{R}$ such that $0' + x = x$ for all $x \in \mathbb{R}$, then $0' = 0$.
(2) If $1'$ is an element of $\mathbb{R}$ such that $1'x = x$ for all $x \in \mathbb{R}$, then $1' = 1$.
(3) Suppose $x \neq 0$ and $xy = xz$. Prove that $y = z$. Deduce that if $xy = 1$ then $y = x^{-1}$.
(4) $-0 = 0$
(5) If $x \neq 0$, then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
(6) $(-x) \times (-y) = xy$.
(7) If $x \neq 0$ and $n \in \mathbb{N}$, then $(-x)^{-1} = -(x^{-1})$ and $(x^{-1})^n = (x^n)^{-1}$.
(8) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

Definition. *Let $x \in \mathbb{R}$, $x \neq 0$. If $n \in \mathbb{N}$, we define $x^{-n} := (x^{-1})^n$ (by the previous question, this is the same as $(x^n)^{-1}$). We have thus defined x^m for any $x \neq 0$ and $m \in \mathbb{Z}$.*

Question B. [3] Let $x, y, z \in \mathbb{R}$. Prove the following statements:

(1) If $x < y$ and $z > 0$, then $xz < yz$.
(2) If $x < 0$ then $x^{-1} < 0$.
(3) If $x, y \geq 0$ and $n \in \mathbb{Z}_{>0}$, prove that $x \leq y$ if and only if $x^n \leq y^n$. Deduce that $x < y$ if and only if $x^n < y^n$.

Question C. [4]

(1) Show that $\mathbb{Q}$ is neither bounded below nor bounded above.
(2) Determine the supremums and infimums of the following sets. Substantiate your claims with a proof.
   a.) $(-1, 0]$
   b.) $(1, 2)$
   c.) $\{(-1)^n/n \mid n \in \mathbb{Z}_{>0}\}$

Question D. [4]

(1) Let $a, b \in \mathbb{R}$.
  a.) Consider the set $\{a, b\}$. Prove that the supremum of the set $\{a, b\}$ is equal to a if $a \geq b$, and is equal to b if $b \geq a$.
  b.) Prove that
$$\sup(\{a, b\}) = \frac{a + b + |a - b|}{2}.$$
  c.) Formulate and prove a variant of the above formula describing the infimum of the set $\{a, b\}$.

(2) Let $a_1, \dots, a_n$ be elements of $\mathbb{R}$. Prove that
$$\sup(\{a_1, \dots, a_n\}) = \sup(\{\sup(\{a_1, \dots, a_{n-1}\}), a_n\}).$$
Deduce that $\sup(\{a_1, \dots, a_n\}) \in \{a_1, \dots, a_n\}$.

Definition. *If* $a_1, \dots, a_n$ *are elements of* $\mathbb{R}$, *define the* maximum *of these elements by* $\max\{a_1, \dots, a_n\} := \sup(\{a_1, \dots, a_n\})$. *From the previous question, we then observe that* $\max\{a_1, \dots, a_n\} \in \{a_1, \dots, a_n\}$. *We can define the minimum of a collection of elements in* $\mathbb{R}$ *in an analogous manner using* inf *instead of* sup.

Definition. *Let* $S \subseteq \mathbb{R}$. *We say that* S *is* bounded *if it is bounded above and bounded below.*

Question E. [3] Let S be a subset of $\mathbb{R}$.

(1) Show that S is bounded if and only if there exists an $m \in \mathbb{R}$ such that $|x| \leq m$ for all $x \in S$.

(2) Deduce that the following three statements are equivalent:
  a.) The S is bounded.
  b.) There exists $m \in \mathbb{R}$ such that $S \subseteq [-m, m]$.
  c.) There exist $a, b \in \mathbb{R}$ with $a \leq b$ such that $S \subseteq [a, b]$.

Question F. [4] Let $S \subseteq \mathbb{R}$.

(1) Let $x, y \in \mathbb{R}$, and suppose x and y both satisfy the definition of an infimum of S. Show that $x = y$.

(2) Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions:
  a.) The element x is a lower bound for S.
  b.) Given $\epsilon > 0$, there exists some $y \in S$ such that $y < x + \epsilon$.

(3) Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions:
  a.) The element x is a lower bound for S.
  b.) Given $n \in \mathbb{Z}_{>0}$, there exists some $y \in S$ such that $y < x + \frac{1}{n}$.

Question G. [4] Prove that in the presence of properties (1) to (16), property (17) and property $(17)_L$ are equivalent. Deduce that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

Question H. [2] An element $z \in \mathbb{R}$ is said to be *irrational* if $z \notin \mathbb{Q}$. Let $x, y \in \mathbb{R}$ such that $x < y$. Show that there exists an irrational number z such that $x < z < y$.

Question I. [4] For $x \geq 0$ in $\mathbb{R}$ and $n > 0$ in $\mathbb{N}$, we denote the unique $y \in \mathbb{R}$ such that $y \geq 0$ and $y^n = x$ (as constructed in class) by $x^{\frac{1}{n}}$. Prove the following statements:

(1) Let $a, b \in \mathbb{R}$ with $a, b, \geq 0$, and let $n \in \mathbb{N}$ such that $n > 0$. Show that $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

(2) Let $a \in \mathbb{R}$ with $a > 0$, and let $n \in \mathbb{N}$ such that $n > 0$. Show that $(a^{\frac{1}{n}})^{-1} = (a^{-1})^{\frac{1}{n}}$.
(3) Let $a \in \mathbb{R}$ with $a > 0$, let $n \in \mathbb{N}$ such that $n > 0$, and let $m \in \mathbb{Z}$. Show that $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$.
(4) Let $a \in \mathbb{R}$ with $a > 0$, let $n, n' \in \mathbb{N}$ such that $n, n' > 0$, let $m, m' \in \mathbb{Z}$, and suppose $mn' = nm'$. Show that $(a^{\frac{1}{n}})^m = (a^{\frac{1}{n'}})^{m'}$.

Definition. *Let $a \in \mathbb{R}$, $a > 0$, and let $r \in \mathbb{Q}$. We can write $r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n > 0$. We* define *$a^r := (a^{\frac{1}{n}})^m$. By the previous question, this is the same as $(a^m)^{\frac{1}{n}}$, and is well-defined.*