Real Analysis
Professor Rishi Vyas — Week 1
Shubhro Gupta — Due Tuesday, July 30, 2024

Collaborators: none

QUESTION A — 4 Points

Definition. *Let $x \in \mathbb{R}$. If $n \in \mathbb{N}$, we define $x^n := \overbrace{x \dots x}^{n \text{ times}}$. By convention, for $n = 0$ we interpret this as defining $x^0 := 1$.*

To Prove. Let $x, y, z \in \mathbb{R}$.

1. If 0' is an element of $\mathbb{R}$ such that $0' + x = x$ for all $x \in \mathbb{R}$, then $0' = 0$.

$$\begin{aligned} 0' + x &= x \\ 0' + x + (-x) &= x + (-x) \\ 0' + 0 &= 0 \qquad \text{(Additive Inverse (4))} \\ 0' &= 0 \\ &\square \end{aligned}$$

2. If 1' is an element of $\mathbb{R}$ such that $1' \cdot x = x$ for all $x \in \mathbb{R}$, then $1' = 1$.

$$\begin{aligned} 1' \cdot x &= x \\ 1' \cdot x \cdot x^{-1} &= x \cdot x^{-1} \\ 1' \cdot 1 &= 1 \qquad \text{(Multiplicative Inverse (8))} \\ 1' &= 1 \\ &\square \end{aligned}$$

3. $x \neq 0$ and $xy = xz$. Prove that $y = z$. Deduce that if $xy = 1$ then $y = x^{-1}$.

$$\begin{aligned} xy &= xz \\ ((x)^{-1} \cdot x)y &= ((x)^{-1} \cdot x)z \\ y &= z \\ &\square \end{aligned}$$

$$\begin{aligned} xy &= 1 \\ ((x)^{-1} \cdot x)y &= (x)^{-1} \cdot 1 \\ &\square \end{aligned}$$

4. $-0 = 0$

$$\begin{aligned} 0 &= 0 \\ 0 \times -1 &= 0 \times -1 \\ 0 &= -0 \end{aligned}$$

5. If $x \neq 0$, then $x^{-1} \neq 0$ and $\left(x^{-1}\right)^{-1} = x$.

$$\begin{aligned} \text{Suppose } x^{-1} &= 0 \\ x \cdot x^{-1} &= x \cdot 0 \\ 1 &= 0 \qquad \text{(Multiplicative inverse (8))} \end{aligned}$$

   $1 \neq 0$ so $x^{-1} \neq 0$.

6. $(-x) \times (-y) = xy$.

$$\begin{aligned} (-x) \times (-y) &= (-x)(-y) + 0y \\ &= (-x)(-y) + (x + (-x))y \\ &= (-x)(-y) + xy + (-x)y \qquad \text{(Distributive Law)} \\ &= ((-x)(-y) + (-x)y) + xy \\ &= (-x)(-y + y) + xy \\ &= (-x)(0) + xy \\ &= 0 + xy \\ (-x) \times (-y) &= xy \end{aligned}$$

7. If $x \neq 0$ and $n \in \mathbb{N}$, then $(-x)^{-1} = -\left(x^{-1}\right)$ and $\left(x^{-1}\right)^n = (x^n)^{-1}$.

8. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

$$\begin{aligned} \text{Suppose } xy &= 0 \\ x \cdot x^{-1} \cdot y &= 0 \cdot x^{-1} \\ 1 \cdot y &= 0 \cdot x^{-1} \\ y &= 0 \end{aligned}$$

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QUESTION **B**	3 Points

Definition. Let $x \in \mathbb{R}$, $x \neq 0$. If $n \in \mathbb{N}$,
To Prove. Let $x, y, z \in \mathbb{R}$.

1. If $x < y$ and $z > 0$, then $xz < yz$.

2. If $x < 0$ then $x^{-1} < 0$.

3. If $x, y \geq 0$ and $n \in \mathbb{Z}_{>0}$, prove that $x \leq y$ if and only if $x^n \leq y^n$. Deduce that $x < y$ if and only if $x^n < y^n$.

QUESTION **D**	4 Points

(a) Let $a, b \in \mathbb{R}$.

i. Consider the set $\{a, b\}$. Prove that the supremum of the set $\{a, b\}$ is equal to a if $a \geq b$, and is equal to b if $b \geq a$.

ii. Prove that
$$\sup(\{a, b\}) = \frac{a + b + |a - b|}{2}.$$

iii. Formulate and prove a variant of the above formula describing the infimum of the set $\{a, b\}$.

(b) Let $a_1, \dots, a_n$ be elements of $\mathbb{R}$. Prove that

$$\sup\left(\{a_1, \dots, a_n\}\right) = \sup\left(\{\sup\left(\{a_1, \dots, a_{n-1}\}\right), a_n\}\right).$$

Deduce that $\sup\left(\{a_1, \dots, a_n\}\right) \in \{a_1, \dots, a_n\}$.

Definition. *If $a_1, \dots, a_n$ are elements of $\mathbb{R}$, define the maximum of these elements by* $\max\{a_1, \dots, a_n\} := \sup\left(\{a_1, \dots, a_n\}\right)$. *From the previous question, we then observe that* $\max\{a_1, \dots, a_n\} \in \{a_1, \dots, a_n\}$. *We can define the minimum of a collection of elements in $\mathbb{R}$ in an analogous manner using inf instead of sup.*

Definition. *Let $S \subseteq \mathbb{R}$. We say that S is bounded if it is bounded above and bounded below.*

QUESTION **E**	3 Points

Let S be a subset of $\mathbb{R}$.

(a) Show that S is bounded if and only if there exists an $m \in \mathbb{R}$ such that $|x| \leq m$ for all $x \in S$.

(b) Deduce that the following three statements are equivalent:

i. The S is bounded.

ii. There exists $m \in \mathbb{R}$ such that $S \subseteq [-m, m]$.

iii. There exist $a, b \in \mathbb{R}$ with $a \leq b$ such that $S \subseteq [a, b]$.

Let $S \subseteq \mathbb{R}$. (1) Let $x, y \in \mathbb{R}$, and suppose x and y both satisfy the definition of an infimum of S. Show that $x = y$. (2) Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions: a.) The element x is a lower bound for S. b.) Given $\epsilon > 0$, there exists some $y \in S$ such that $y < x + \epsilon$. (3) Let $x \in \mathbb{R}$. Show that $x = \inf(S)$ if and only if it satisfies the following two conditions: a.) The element x is a lower bound for S. b.) Given $n \in \mathbb{Z}_{>0}$, there exists some $y \in S$ such that $y < x + \frac{1}{n}$.