Real Analysis
Professor Rishi Vyas — Week 1
Shubhro Gupta — Due Monday, 18 August

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QUESTION **A** — 2 Points

To Show. The set $(-\infty, a)$ is open, i.e., $B(x, \delta) \subseteq (-\infty, a)$ for some $x \in (-\infty, a), \delta > 0$.
Proof. Let $x \in (-\infty, a)$, then $-\infty < x < a$.
Let $\delta = min(x + \infty, a - x)$.
This means $\delta \leq x + \infty$ and $\delta \leq a - x$.
We have to show that $B(x - \delta, x + \delta) \subseteq (-\infty, a)$.
From above $x - \delta \leq x - (x + \infty) = -\infty$ and $x + \delta \leq x + (a - x) = a$. (By subtracting x from both sides).
Thus, $B(x - \delta, x + \delta) \subseteq (-\infty, a)$.
Since for any $x \in (-\infty, a)$, we can find a $\delta > 0$ such that $B(x - \delta, x + \delta) \subseteq (-\infty, a)$, the set $(-\infty, a)$ is open. □

To Show. The set $(\infty, a]$ is not open, i.e., $B(x, \delta) \not\subseteq (\infty, a]$ for any $x \in (\infty, a], \delta > 0$.
Proof. Let $x \in (\infty, a]$, then $x \leq a < \infty$.
Assume $x = a$, and $B(x, \delta) \subseteq (\infty, a] \implies B(a - \delta, a + \delta) \subseteq (\infty, a]$.
This means $a + \delta \leq a \implies \delta \leq 0$.
But $\delta > 0$, hence $B(x, \delta) \not\subseteq (\infty, a]$.
Thus, for any $x \in (\infty, a]$, we cannot find a $\delta > 0$ such that $B(x, \delta) \subseteq (\infty, a]$. Hence, the set $(\infty, a]$ is not open. □

QUESTION **B** — 2 Points

To Determine. All $x \in \mathbb{R}$ that satisfies $|x + 3| + |x - 2| = 9$.
Solution.
<u>Case 1:</u> $x \geq 2$.
Then $|x + 3| = x + 3$ and $|x - 2| = x - 2$.
Thus, $x + 3 + x - 2 = 9 \implies 2x + 1 = 9 \implies x = 4$.

<u>Case 2:</u> $-3 \leq x < 2$.
Then $|x + 3| = x + 3$ and $|x - 2| = -x + 2$.
Thus, $x + 3 - x + 2 = 9 \implies 5 = 9$ which is not possible.

Thus, there are no solutions for $-3 \leq x < 2$. □

QUESTION **C**	2 Points

To Show. $\mathbb{Z}^c$ is open in $\mathbb{R}$.
Proof.
$\mathbb{Z}^c = \bigcup_{n\in\mathbb{Z}} (n, n+1)$.
Let $x \in \mathbb{Z}^c$, then $x \in (n, n+1)$ for some $n \in \mathbb{Z}$.
Let $\delta = min(x-n, n+1-x)$.
Then $x - \delta \leq x - (x-n) = n$ and $x + \delta \leq x + (n+1-x) = n+1$.
$B(x,\delta) = (x-\delta, x+\delta)$. Consider $x - \delta$. If $x - \delta = n$, then $x - \delta \in (n, n+1)$.
If $x - \delta < n$, then $x - \delta \in (n-1, n)$.
Similarly, $x + \delta \in (n, n+1)$ or $(n, n+1)$.

Therefore $B(x,\delta) = (x-\delta, x+\delta) \subseteq (n, n+1) \subseteq \mathbb{Z}^c$.

QUESTION **E**	2 Points

To Show. $\sup(S) \notin S$ for any non-empty, bounded above set $S \subseteq \mathbb{R}$.
Proof. Assume $\sup(S) \in S$, and $\sup(S) = x$.
Since S is open, $\exists \delta > 0$ such that $B(x,\delta) = (x-\delta, x+\delta) \subseteq S$.
Since x is the supremum of S, $\forall s \in S, s \leq x$.
$x < x + \delta$, since $\delta > 0$.
Also, $x + \delta \in S$, since $x + \delta \in B(x,\delta) \subseteq S$.
But $x + \delta > x$, which contradicts the fact that x is the supremum of S.
Thus, $\sup(S) \notin S$ for any non-empty, bounded above set $S \subseteq \mathbb{R}$. $\square$

QUESTION **F**	4 Points

To Find. Closure and limit of
(1) (0, 3] $\subseteq \mathbb{R}$
Solution.
<u>Closure</u> $\overline{(0,3]} = (0,3] \cup \{0\}$.
We have to show that $B(x,\delta) \cap (0,3] \neq \emptyset$ for any $x \in \overline{(0,3]}$ and $\delta > 0$.
Let $y = min\frac{0+\delta}{2}, 1.5$.
Since $0 < y < 1.5$, $y \in (0,3]$.
Taking $x = 0$, we have $B(0,\delta) \cap (0,3]$, we get $y \in B(0,\delta) \cap (0,3]$.
Thus, $\overline{(0,3]} = (0,3] \cup \{0\}$.

<u>Limit</u> $\lim_{x\to 3}(0,3] = 3$.
We have to show that $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x-3| < \delta \implies |f(x) - 3| < \epsilon$.
Let $\delta = \epsilon$.
Then $0 < |x-3| < \delta \implies 0 < x-3 < \delta \implies 0 < x-3 < \epsilon \implies |f(x)-3| < \epsilon$.
Thus, $\lim_{x\to 3}(0,3] = 3$.

(2) $\mathbb{Z}$ Closure is $\mathbb{Z}$.
Solution.

(3) $\mathbb{Q} \cap (0,1)$ Closure is $[0,1]$.

QUESTION **G**	4 Points

Construct Example. Of an infinite susbet of $\mathbb{R}$ which is not an interval, and is neither closed or open.
Solution. Consider the set $S = \{1\} \cup (2, 3)$.
S is not an interval since it is not connected.
S is not closed since it does not contain its limit points.
S is not open since it does not contain all its boundary points.

Construct Example. Of a set $S \subseteq \mathbb{R}$ which is bounded and has exactly 2 limit points.
Solution. Consider the set S = S is bounded since $0 < S < 3$.
The limit points of S are 1 and 2, since

QUESTION **H**	3 Points

Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.
To Show. $x = \inf(S) \iff$ The element x is a lower bound and given $n \in \mathbb{N}, \exists y \in S$ such that $y < x + \frac{1}{n}$.
Proof.
<u>Case 1</u> $x = \inf(S)$.
Then x is a lower bound of S by definition.
If x is an infimum, then $\forall 1 > \epsilon > 0, \exists y \in S$ such that $x \leq y < x + \epsilon$.
Since $\epsilon > 0$, we can write $\epsilon = \frac{1}{n}$ for some $n \in \mathbb{N}$.
Thus, $\forall n \in \mathbb{N}, \exists y \in S$ such that $y < x + \frac{1}{n}$.

<u>Case 2</u> The element x is a lower bound and given $n \in \mathbb{N}, \exists y \in S$ such that $y < x + \frac{1}{n}$.
Let t be any lower bound of S.
Then $t \leq x$ since x is the greatest lower bound.
We have to show that $x \leq t$. Suppose $x > t$.
Then $x - t > 0$.
Let $\epsilon = x - t$.
Then $\epsilon > 0$.
Since $\epsilon > 0$, we can write $\epsilon = \frac{1}{n}$ for some $n \in \mathbb{N}$.
Thus, $\exists y \in S$ such that $y < x + \frac{1}{n} = t + \epsilon$. $y - \epsilon < t$. But $y - \epsilon < x$.
This contradicts the fact that x is the greatest lower bound.

QUESTION **I**	7 Points

(1) **To Draw.** A picture of the set $S := \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$.
Solution.
(1) The set S is the set of all possible differences between reciprocals of natural numbers.
(2) **To Show.** The supremum and infimum of S are 1 and 0 respectively.
Let $x = \frac{1}{n} - \frac{1}{m} \in S$.
Then $0 < x < 1$.
Thus, 0 is a lower bound of S.
Let $y \in S$, then $y = \frac{1}{n} - \frac{1}{m}$.
Then $y < 1$ for all $n, m \in \mathbb{N}$.
Thus, 1 is an upper bound of S.

(3) **To Show.** The set S is not open or closed. **Proof.**
Open Let $x \in S$, then $x = \frac{1}{n} - \frac{1}{m}$.
Let $\delta = min(\frac{1}{n}, \frac{1}{m})$.
Then $B(x, \delta) = (\frac{1}{n} - \delta, \frac{1}{m} + \delta)$.
Since $\delta < \frac{1}{n}$ and $\delta < \frac{1}{m}$, $B(x, \delta) \nsubseteq S$.
Thus, S is not open.

Closed Let $x \in S$, then $x = \frac{1}{n} - \frac{1}{m}$.
Let $\delta = min(\frac{1}{n}, \frac{1}{m})$.
Then $B(x, \delta) = (\frac{1}{n} - \delta, \frac{1}{m} + \delta)$.
Since $\delta < \frac{1}{n}$ and $\delta < \frac{1}{m}$, $B(x, \delta) \nsubseteq S$.
Thus, S is not closed.

(4) **To Calculate.** The Closure. **Solution.**
The closure of S is $\overline{S} = S \cup \{0, 1\}$.