Newtonsche Interpolationsformel

Ausgangslage

In[145]:= $f[x\_] := E^{Cos[x]^2}$

In[146]:=
```
pts = Table[{x, f[x]}, {x, 1, 5, 0.5}]
```

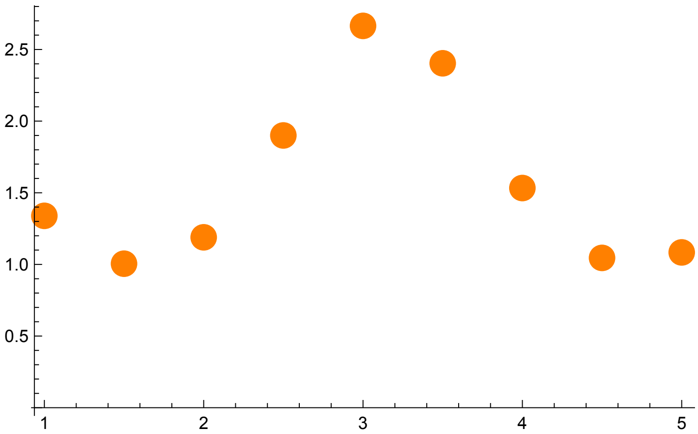
Out[146]=
```
{{1., 1.339}, {1.5, 1.00502}, {2., 1.18908}, {2.5, 1.89996}, {3., 2.66468},
 {3.5, 2.40356}, {4., 1.53304}, {4.5, 1.04544}, {5., 1.08379}}
```

In[147]:=
```
m = Length[pts]
```

Out[147]= 9

In[148]:=
```
plpts = ListPlot[pts, PlotStyle → {Orange, PointSize[.04]}]
```

Out[148]=



Polynom Interpolation

In[149]:=
```
polys[x_] = Table[InterpolatingPolynomial[pts[[ ;; k]], x], {k, 1, m}] // Simplify;
```

In[150]:=
```
polys[x] // Expand // TableForm
```

Out[150]//TableForm=

1.339

$2.00698 - 0.667977\, x$

$3.56113 - 3.25823\, x + 1.0361\, x^2$

$3.52606 - 3.18225\, x + 0.983498\, x^2 + 0.0116893\, x^3$

$1.11738 + 3.00004\, x - 4.71705\, x^2 + 2.25979\, x^3 - 0.321158\, x^4$

$1.86733 + 0.825189\, x - 2.30055\, x^2 + 0.968217\, x^3 + 0.0121523\, x^4 - 0.033331\, x^5$

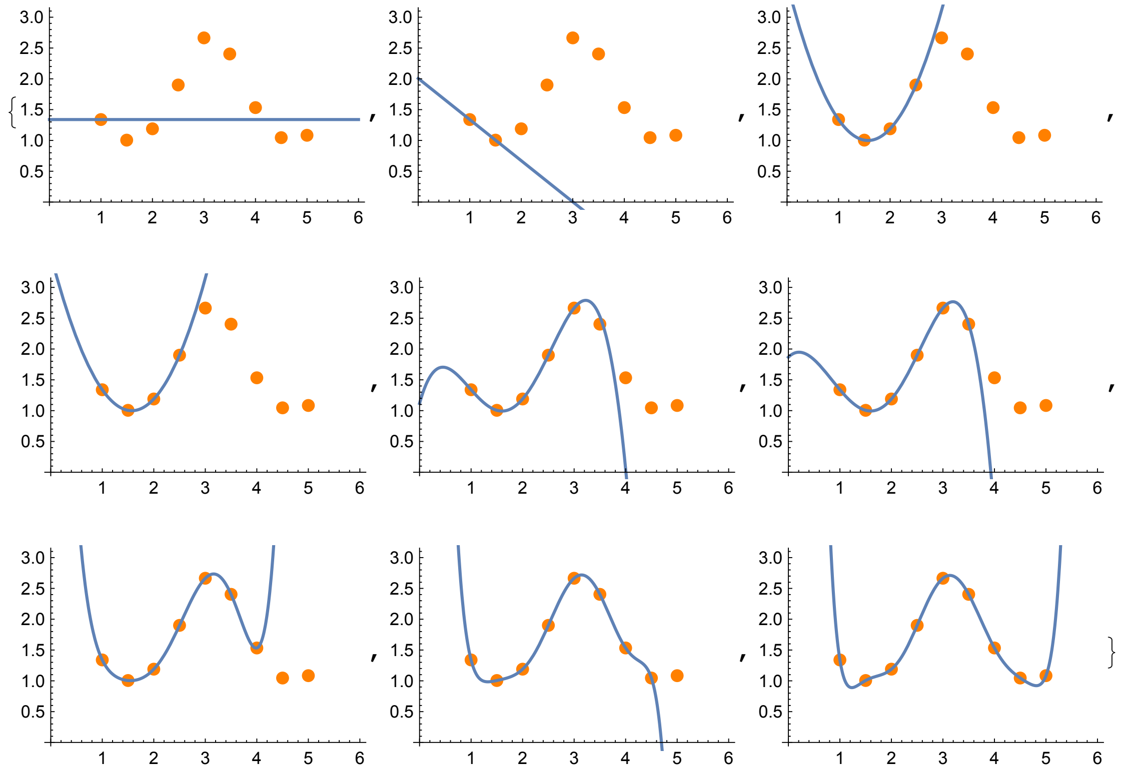
$17.4624 - 48.8562\, x + 60.8717\, x^2 - 40.2472\, x^3 + 14.617\, x^4 - 2.70677\, x^5 + 0.198032\, x^6$

$59.4704 - 193.184\, x + 264.494\, x^2 - 193.81\, x^3 + 81.7131\, x^4 - 19.7433\, x^5 + 2.53181\, x^6 - 0.13335$

$129.393 - 448.957\, x + 656.812\, x^2 - 524.735\, x^3 + 250.197\, x^4 - 72.9193\, x^5 + 12.7181\, x^6 - 1.2185$

In[151]:=
```
Table[Show[plpts, Plot[polys[x][[k]], {x, 0, 6}],
  PlotRange → {{0, 6}, {0, 3}}, AxesOrigin → {0, 0}], {k, 1, m}]
```

Out[151]=



In[152]:=
```
corrector[x_, n_] := (pts[[n, 2]] - polys[pts[[n, 1]]][[n - 1]]) /
                     (Product[pts[[n, 1]] - pts[[j, 1]], {j, 1, n - 1}])
  Product[x - pts[[j, 1]], {j, 1, n - 1}]
```

In[153]:=
```
Table[corrector[x, k + 1] // Expand, {k, 1, m - 1}] // TableForm
```

Out[153]//TableForm=

$0.667977 - 0.667977\,x$

$1.55415 - 2.59025\,x + 1.0361\,x^2$

$-0.0350678 + 0.0759803\,x - 0.0526017\,x^2 + 0.0116893\,x^3$

$-2.40868 + 6.18228\,x - 5.70055\,x^2 + 2.2481\,x^3 - 0.321158\,x^4$

$0.749947 - 2.17485\,x + 2.4165\,x^2 - 1.29158\,x^3 + 0.33331\,x^4 - 0.033331\,x^5$

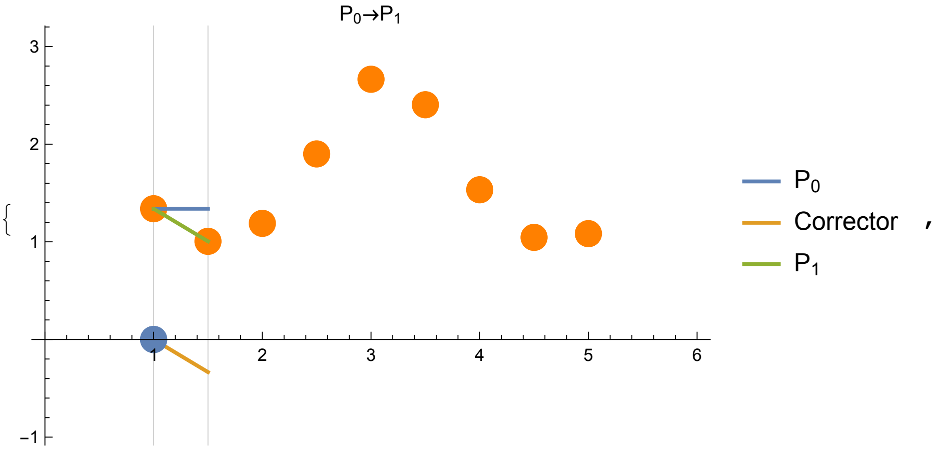
$15.595 - 49.6813\,x + 63.1723\,x^2 - 41.2155\,x^3 + 14.6049\,x^4 - 2.67344\,x^5 + 0.198032\,x^6$

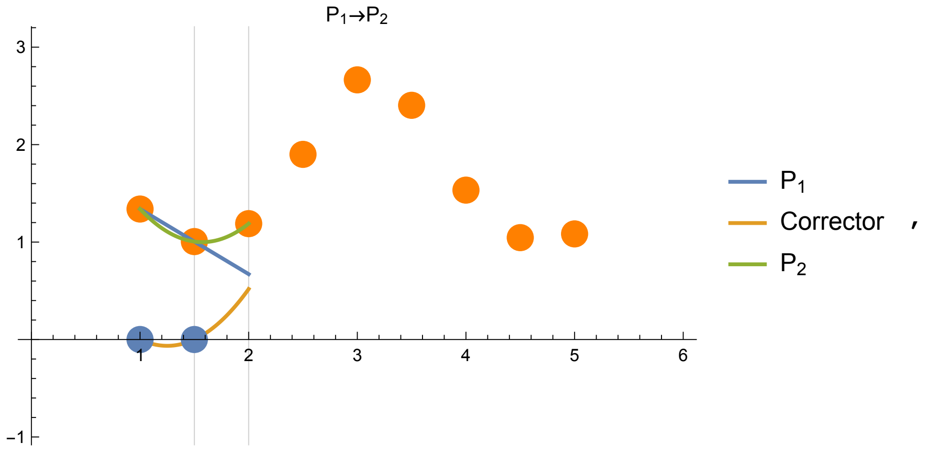
$42.008 - 144.327\,x + 203.622\,x^2 - 153.563\,x^3 + 67.0961\,x^4 - 17.0366\,x^5 + 2.33378\,x^6 - 0.13335$

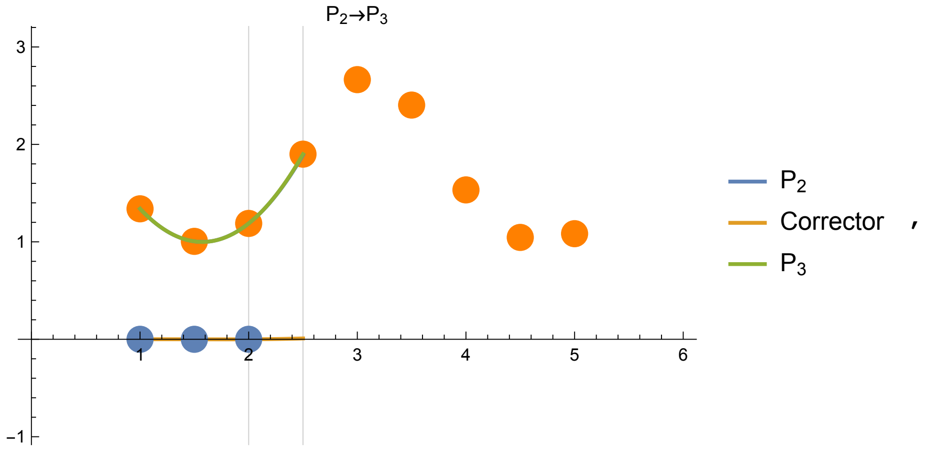
$69.923 - 255.774\,x + 392.318\,x^2 - 330.926\,x^3 + 168.484\,x^4 - 53.176\,x^5 + 10.1863\,x^6 - 1.08522\,x$

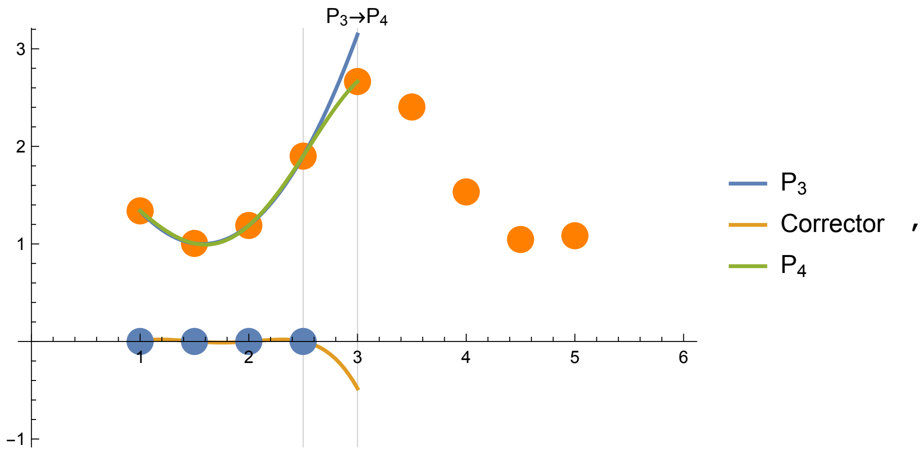
In[154]:=
```
Table[Show[plpts,
  Plot[{polys[x][[k]], corrector[x, k + 1], polys[x][[k]] + corrector[x, k + 1]},
   {x, pts[[1, 1]], pts[[k + 1, 1]]}, PlotStyle → Thick,
   PlotLegends → {ToString[Subscript["P", k - 1], TraditionalForm],
     "Corrector", ToString[Subscript["P", k], TraditionalForm]}],
  ListPlot[Table[{pts[[j, 1]], 0}, {j, 1, k}], PlotStyle → PointSize[.04]],
  PlotRange → {{0, 6}, {-1, 3}}, AxesOrigin → {0, 0},
  GridLines → {{pts[[k, 1]], pts[[k + 1, 1]]}, None}, ImageSize → Medium,
  PlotLabel → ToString[Subscript["P", k - 1], TraditionalForm] <>
    "→" <> ToString[Subscript["P", k], TraditionalForm]], {k, 1, m - 1}]
```

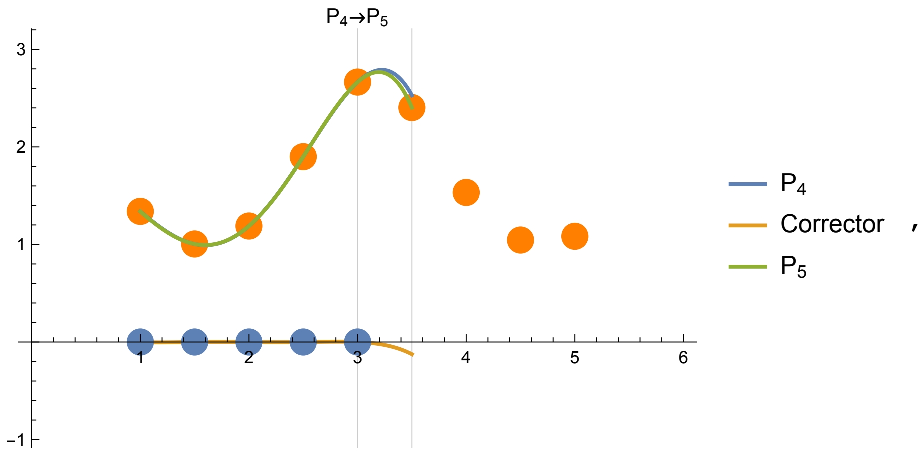
Out[154]= {

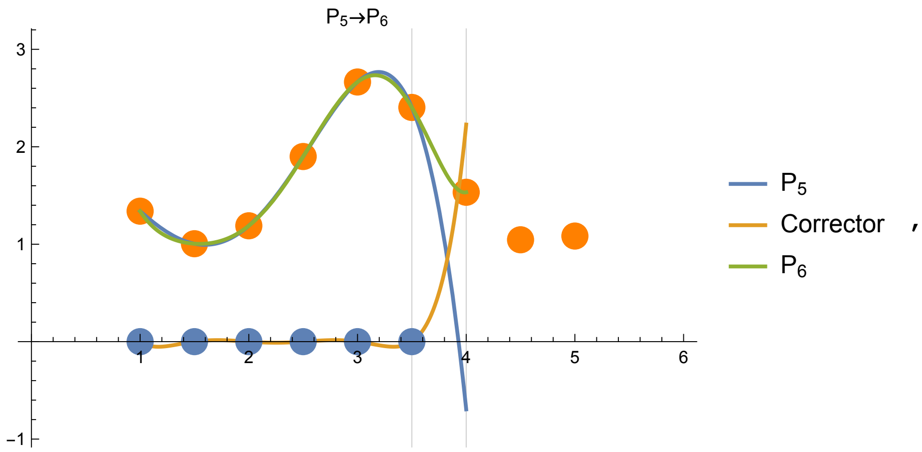

P0→P1
P0
Corrector
P1
3
2
1
−1
1
2
3
4
5
6

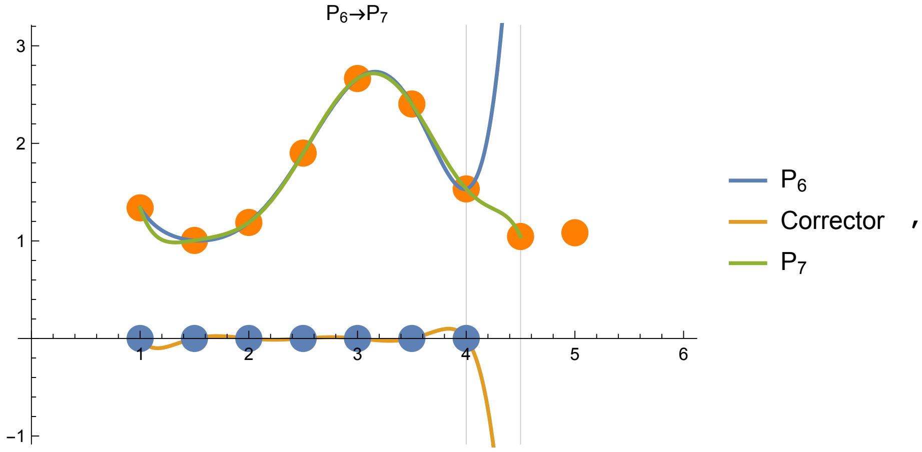

P1→P2
P1
Corrector
P2
3
2
1
−1
1
2
3
4
5
6

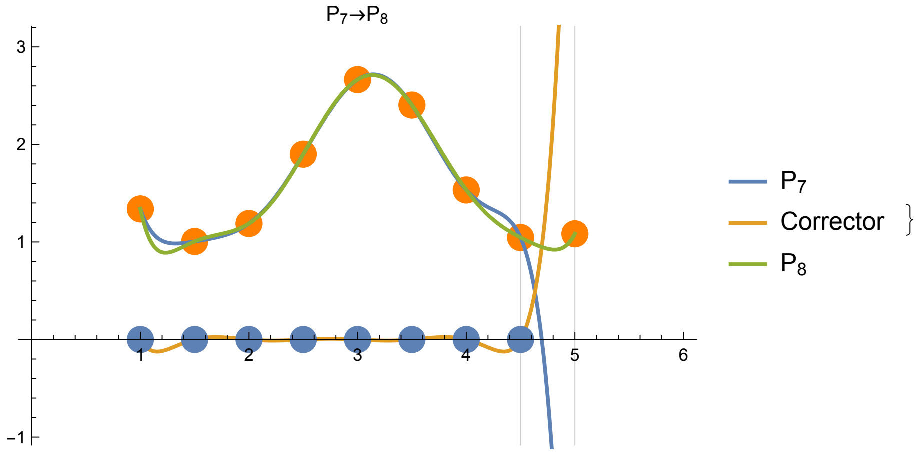

P2→P3
P2
Corrector
P3
3
2
1
−1
1
2
3
4
5
6


P3→P4
P3
Corrector
P4


P4→P5
P4
Corrector
P5


P5→P6
P5
Corrector
P6





Kontrolle P_k + Corrector $= P_{k+1}$

Differenz sollte Null sein:

In[155]:=
```
Table[polys[x][[k]] + corrector[x, k + 1] - polys[x][[k + 1]], {k, 1, m - 1}]
```

Out[155]=
```
{-0.667977 - 0.667977 (-1. + x) + 0.667977 x,
 -1.55415 + 1.0361 (-1.5 + x) (-1. + x) + 2.59025 x - 1.0361 x^2,
 0.0350678 + 0.0116893 (-2. + x) (-1.5 + x) (-1. + x) - 0.0759803 x + 0.0526017 x^2 -
  0.0116893 x^3, 2.40868 - 0.321158 (-2.5 + x) (-2. + x) (-1.5 + x) (-1. + x) -
  6.18228 x + 5.70055 x^2 - 2.2481 x^3 + 0.321158 x^4,
 -0.749947 - 0.033331 (-3. + x) (-2.5 + x) (-2. + x) (-1.5 + x) (-1. + x) +
  2.17485 x - 2.4165 x^2 + 1.29158 x^3 - 0.33331 x^4 + 0.033331 x^5,
 -15.595 + 0.198032 (-3.5 + x) (-3. + x) (-2.5 + x) (-2. + x) (-1.5 + x) (-1. + x) +
  49.6813 x - 63.1723 x^2 + 41.2155 x^3 - 14.6049 x^4 + 2.67344 x^5 - 0.198032 x^6, -42.008 -
  0.133359 (-4. + x) (-3.5 + x) (-3. + x) (-2.5 + x) (-2. + x) (-1.5 + x) (-1. + x) +
  144.327 x - 203.622 x^2 + 153.563 x^3 - 67.0961 x^4 + 17.0366 x^5 - 2.33378 x^6 + 0.133359 x^7,
 -69.923 + 0.0493284 (-4.5 + x) (-4. + x) (-3.5 + x) (-3. + x) (-2.5 + x)
   (-2. + x) (-1.5 + x) (-1. + x) + 255.774 x - 392.318 x^2 + 330.926 x^3 -
  168.484 x^4 + 53.176 x^5 - 10.1863 x^6 + 1.08522 x^7 - 0.0493284 x^8}
```

In[156]:= `% // Simplify`

Out[156]= $\{0., 2.22045\times10^{-16}+4.44089\times10^{-16}\,x,$
$-6.21725\times10^{-15}+1.27259\times10^{-14}\,x-8.6528\times10^{-15}\,x^2+1.92381\times10^{-15}\,x^3,$
$8.88178\times10^{-16}\,(-2.+1.\,x)^3\,(-1.+1.\,x), -7.77156\times10^{-16}+6.21725\times10^{-15}\,x-$
$1.15463\times10^{-14}\,x^2+8.88178\times10^{-15}\,x^3-2.94209\times10^{-15}\,x^4+3.53884\times10^{-16}\,x^5,$
$-1.1191\times10^{-13}+3.41061\times10^{-13}\,x-4.12115\times10^{-13}\,x^2+2.55795\times10^{-13}\,x^3-$
$8.70415\times10^{-14}\,x^4+1.5099\times10^{-14}\,x^5-1.05471\times10^{-15}\,x^6,$
$5.81935\times10^{-12}-2.00089\times10^{-11}\,x+2.82228\times10^{-11}\,x^2-2.13163\times10^{-11}\,x^3+$
$9.2939\times10^{-12}\,x^4-2.36611\times10^{-12}\,x^5+3.23297\times10^{-13}\,x^6-1.8513\times10^{-14}\,x^7,$
$-1.32616\times10^{-10}+4.85073\times10^{-10}\,x-7.44194\times10^{-10}\,x^2+$
$6.27836\times10^{-10}\,x^3-3.19631\times10^{-10}\,x^4+1.00897\times10^{-10}\,x^5-$
$1.93321\times10^{-11}\,x^6+2.05969\times10^{-12}\,x^7-9.36265\times10^{-14}\,x^8\}$

Bis auf den numerischen Fehler passt das:

In[157]:= `% // Chop`

Out[157]= $\{0, 0, 0, 0, 0, 0, 0, -1.32616\times10^{-10}+4.85073\times10^{-10}\,x-$
$7.44194\times10^{-10}\,x^2+6.27836\times10^{-10}\,x^3-3.19631\times10^{-10}\,x^4+1.00897\times10^{-10}\,x^5\}$