Beispiel gedämpftes Newton-Verfahren

Definition der Funktion:

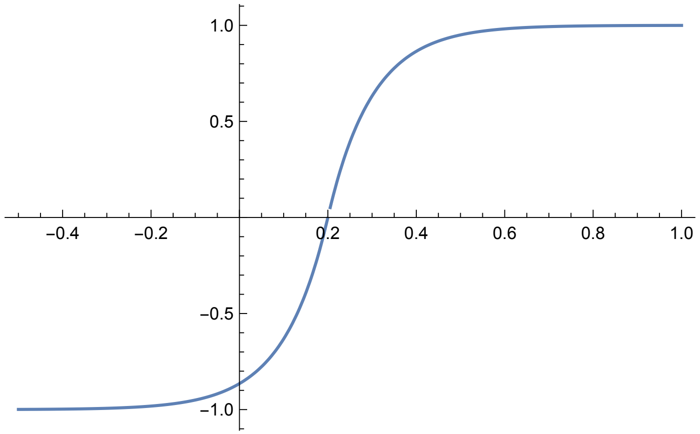
In[1]:=
```
f[x_, x0_, σ_] := Sign[x - x0] (1 - E^(-Abs[(x-x0)]/σ))
```

Lösung der Gleichung $f(x, x_0) = 0$ ist gegeben durch x_0. Für das Beispiel sei $x_0 = 0.2$, $\sigma_0 = 0.1$:

In[2]:=
```
x0 = 0.2; σ0 = 0.1;
```

In[3]:=
```
Plot[f[x, x0, σ0], {x, -.5, 1}, PlotRange → All]
```

Out[3]=



Ableitung von $f(x)$

Für $x \geq x_0$:

In[4]:= $\partial_{\mathbf{x}} - \left(1 - \mathbf{E}^{-\frac{\mathbf{x0-x}}{\sigma}}\right)$

Out[4]= $\dfrac{e^{-\frac{0.2-x}{\sigma}}}{\sigma}$

Für $x < x_0$:

In[5]:= $\partial_x \left(1 - E^{-\frac{x-x0}{\sigma}}\right)$

Out[5]= $\frac{e^{-\frac{-0.2+x}{\sigma}}}{\sigma}$

Somit folgt für alle x∈ℝ:

In[6]:= $df[x\_, x0\_, \sigma\_] := \frac{1}{\sigma} E^{-\frac{Abs[x-x0]}{\sigma}}$

Newton-Verfahren

Das Newton-Verfahren konvergiert nur auf einem kleinen Intervall um x_0.

In[7]:= `sol1 = NestList[# - (df[#, x0, σ0])^-1 f[#, x0, σ0] &, 0.33, 10]`

General::unfl : Underflow occurred in computation. ≫

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Out[7]= {0.33, 0.0630703, 0.356329, -0.0211203, 0.791548, -36.1818, 1.00991×10^{157}, Overflow[], Indeterminate, Indeterminate, Indeterminate}

In[8]:= `sol1 = Select[sol1, NumberQ[#] && Abs[#] < $MaxNumber &]`

Out[8]= {0.33, 0.0630703, 0.356329, -0.0211203, 0.791548, -36.1818, 1.00991×10^{157}}

Problem: der Funktionswert wird nicht kleiner:

In[9]:= `Abs[f[sol1, x0, σ0]]`

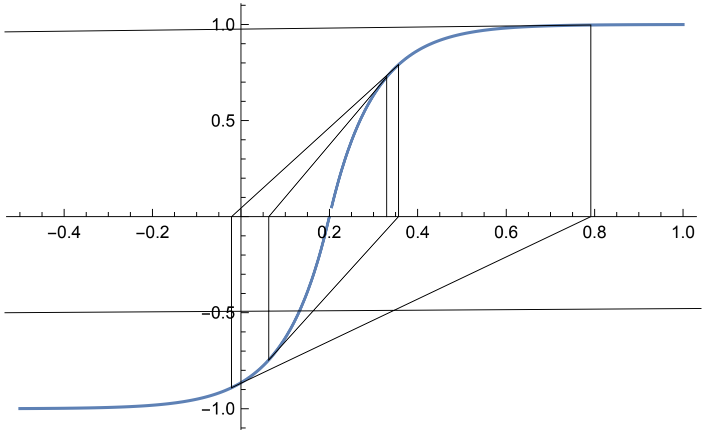
General::unfl : Underflow occurred in computation. ≫

Out[9]= {0.727468, 0.745714, 0.790554, 0.890431, 0.997303, 1., Underflow[] + 1}

In[10]:=
```
sol = sol1;
Show[Plot[f[x, x0, σ0], {x, -.5, 1}],
 Graphics[Line[Flatten[Transpose[Transpose /@ {{sol, Table[0, {Length[sol]}]}, {sol, f[sol, x0, σ0]}}], 1]]]]
```

General::unfl : Underflow occurred in computation. ≫

Out[11]=



Gedämpftes Newton-Verfahren

NewtonStep wird rekursiv mit dem Befehl NestList angewandt.

In[12]:=
```
NewtonStep[x_, x0_, σ0_] := Module[{λ, Cλ, s2, xn, sk, fk},
  λ = 1.;
  Cλ = 1 - λ/4;
  sk = df[x, x0, σ0]^-1 f[x, x0, σ0];
  xn = x - λ sk;
  s2 = df[x, x0, σ0]^-1 f[xn, x0, σ0];
  While[(Abs[s2] ≥ Cλ Abs[sk]) && (λ ≥ λmin),
   λ = λ / 2;
   Cλ = 1 - λ/4;
   xn = x - λ sk;
   s2 = df[x, x0, σ0]^-1 f[xn, x0, σ0];
  ];
  AppendTo[output, {newtonStep++, "λ=", NumberForm[λ, {8, 8}], "|f(x_k)| = ", Abs[f[xn, x0, σ0]],
    "|||f(x_k+1)||| = ", Abs[df[x, x0, σ0]^-1 f[xn, x0, σ0]], "|||f(x_k)||| = ", Abs[df[x, x0, σ0]^-1 f[x, x0, σ0]]}];
  xn
 ]
```

In[13]:=
```
λmin = 10^-3;
```

In den ersten drei Newton-Schritten wird die Schrittweite gedämpft, das Verfahren konvergiert.

In[14]:=
```
newtonStep = 1;
output = {};
sol2 = NestList[NewtonStep[#, x0, σ0] &, 1, 20]
```

Out[16]=
```
{1, -0.164046, 0.299821, 0.21415, 0.19895, 0.200006, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2}
```

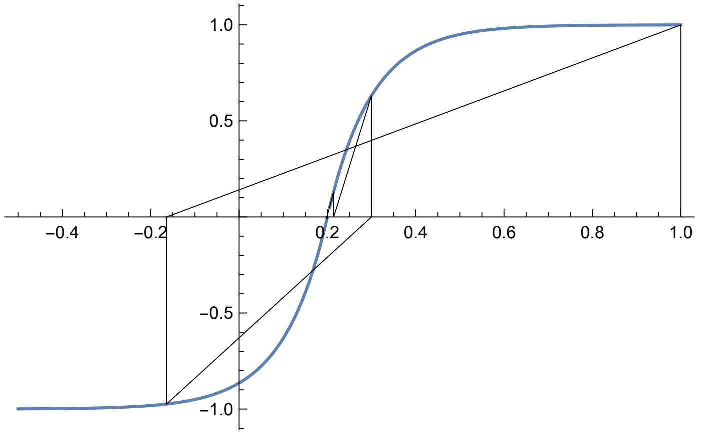
In[17]:=
```
output // TableForm
```

Out[17]//TableForm=

1	$\lambda=$	0.00390625	$\lvert f(x_k)\rvert$	=	0.97376	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	290.274	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 297.996
2	$\lambda=$	0.12500000	$\lvert f(x_k)\rvert$	=	0.631463	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	2.40647	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 3.71094
3	$\lambda=$	0.50000000	$\lvert f(x_k)\rvert$	=	0.131943	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.0358019	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.171343
4	$\lambda=$	1.00000000	$\lvert f(x_k)\rvert$	=	0.0104453	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.0012033	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.0151999
5	$\lambda=$	1.00000000	$\lvert f(x_k)\rvert$	=	0.0000553197	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	5.59037×10^{-6}	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.00105556
6	$\lambda=$	1.00000000	$\lvert f(x_k)\rvert$	=	1.53025×10^{-9}	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	1.53033×10^{-10}	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 5.53228×10^{-6}
7	$\lambda=$	1.00000000	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 1.53025×10^{-10}
8	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
9	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
10	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
11	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
12	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
13	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
14	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
15	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
16	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
17	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
18	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
19	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.
20	$\lambda=$	0.00097656	$\lvert f(x_k)\rvert$	=	0.	$\lvert\lvert\lvert f(x_{k+1})\rvert\lvert\lvert$	=	0.	$\lvert\lvert\lvert f(x_k)\rvert\lvert\lvert$ = 0.

In[18]:=
```
sol = sol2;
Show[Plot[f[x, x0, σ0], {x, -.5, 1}],
 Graphics[Line[Flatten[Transpose[Transpose /@ {{sol, Table[0, {Length[sol]}]}, {sol, f[sol, x0, σ0]}}], 1]]]]
```

Out[19]=



Vergleich des Konvergenzverhaltens

In[20]:=
```
ListLogPlot[Abs[{sol1 - x0, sol2 - x0}], PlotRange → {10^-15, 100}]
```

Out[20]=