§1.2 locally trivial deformations

- X: algebraic scheme / k

Def (deformations of X parametrized by S)

Fiber product. total scheme.

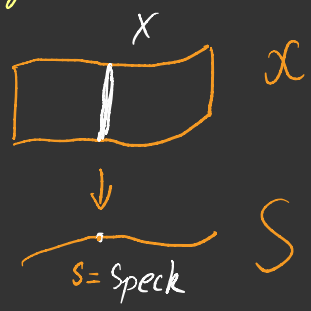
$$\begin{tikzcd} X = \mathcal{X} \times_S \operatorname{Spec} k \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r, hook] & S \end{tikzcd}$$

y:

π : flat & surjective

S : connected

parameter scheme



$\mathcal{A}$ = the category of local artinian **k**-algebras with residue field **k**

$\hat{\mathcal{A}}$ = the category of complete local noetherian **k**-algebras with residue field **k**

$\mathcal{A}^*$ = the category of local noetherian **k**-algebras with residue field **k**

$k[\varepsilon] := \dfrac{k[t]}{(t^2)}$ first-order infinitesimal neighborhood.

[Har77 II, Ex 2.8] 解释 $k[\varepsilon]$

2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

Rmk:

① give $\operatorname{Spec} \dfrac{k[t]}{(t^2)} \to X$ equivalent to give $x \in X$ with $k(x) \simeq k$ and $v \in T_x$

$$0 \to (t)/t^2 \to \frac{k[t]}{(t^2)} \to \frac{k[t]}{(t)} \to 0$$

$$\frac{k[t]}{(t)} \cong k$$

analoge element in m_x/m_x^2

② [Atiyah Thm 8.5 & 8.6

Theorem 8.5. *A ring A is Artin $\Leftrightarrow$ A is Noetherian and* $\dim A = 0$.

Proof. $\Rightarrow$: By (8.1) we have $\dim A = 0$. Let $\mathfrak{m}_i$ $(1 \leqslant i \leqslant n)$ be the distinct maximal ideals of A (8.3). Then $\prod_{i=1}^n \mathfrak{m}_i^k \subseteq (\bigcap_{i=1}^n \mathfrak{m}_i)^k = \mathfrak{N}^k = 0$. Hence by (6.11) A is Noetherian.

$\Leftarrow$: Since the zero ideal has a primary decomposition (7.13), A has only a finite number of minimal prime ideals, and these are all maximal since $\dim A = 0$. Hence $\mathfrak{N} = \bigcap_{i=1}^n \mathfrak{m}_i$ say; we have $\mathfrak{N}^k = 0$ by (7.15), hence $\prod_{i=1}^n \mathfrak{m}_i^k = 0$ as in the previous part of the proof. Hence by (6.11) A is an Artin ring. ■

If A is an Artin local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is the only prime ideal of A and therefore $\mathfrak{m}$ is the nilradical of A. Hence every element of $\mathfrak{m}$ is nilpotent, and $\mathfrak{m}$ itself is nilpotent. Every element of A is either a unit or is nilpotent. An example of such a ring is $\mathbf{Z}/(p^n)$, where p is prime and $n \geqslant 1$.

Proposition 8.6. *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal. Then exactly one of the following two statements is true:*

i) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ *for all n;*

ii) $\mathfrak{m}^n = 0$ *for some n, in which case A is an Artin local ring.*

usually use.

Proof. Suppose $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n. By Nakayama's lemma (2.6) we have $\mathfrak{m}^n = 0$. Let $\mathfrak{p}$ be any prime ideal of A. Then $\mathfrak{m}^n \subseteq \mathfrak{p}$, hence (taking radicals) $\mathfrak{m} = \mathfrak{p}$. Hence $\mathfrak{m}$ is the only prime ideal of A and therefore A is Artinian. ■

$$\frac{k[t]}{(t^2)} = k[\varepsilon] \in ob(A) \subseteq ob(A^*)$$

Artin Noether

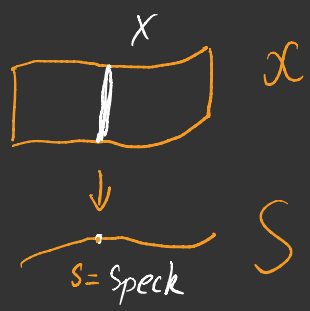
③

Fiber product.

total scheme.

$$\begin{array}{ccc} y: & X = \mathcal{X} \times_S \mathrm{Speck} & \to \mathcal{X} \\ & \downarrow & \downarrow \pi : \text{flat \& surjective} \\ & \mathrm{Speck} & \hookrightarrow S : \text{connected} \end{array}$$

parameter scheme



If $S = \mathrm{Spec}(A)$ affine, then $\mathcal{Y}$ is called

<u>local deformation</u>: $A \in \mathrm{ob}(A^*)$ noether

<u>infinitesimal deformation</u>: $A \in \mathrm{ob}(A)$ Artin unique prime ideal

<u>first order deformation</u>: $A = k[\varepsilon] = \dfrac{k[t]}{(t^2)}$ unique prime ideal

$S|$ as k-vector space

Example: $\dfrac{k[t]}{(t^3)} \simeq k \oplus (t) \oplus t^2$ $\quad k \oplus (t)$

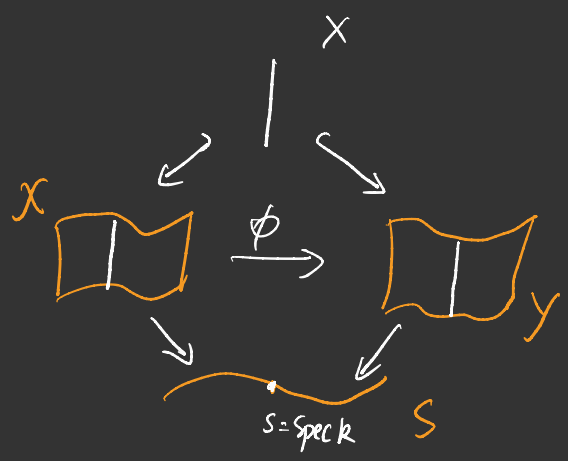
$R = k[x,y]$
$m = (x,y) \subseteq k[x,y]$
maximal ideal

$\dfrac{R_m}{m^3 R_m}$ local Artin k-algebra.

<u>Def</u>: another deformation

$$\begin{tikzcd}
X \times_S k = X \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & S
\end{tikzcd}$$

ξ



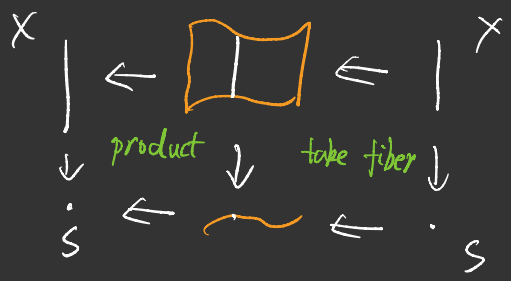
<u>isomorphism of $\mathcal{Y}$ with ξ</u>: $\exists$ S-isomorphism ϕ

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] & \\
\mathcal{X} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\
& S &
\end{tikzcd}$$

Def

η is called trivial family if it's isomorphic to product family

$$\begin{tikzcd}
X \arrow[d] & \overset{\mathcal{X}}{X \times_k S} \arrow[l] \arrow[d, "\pi"] & \pi^{-1}(s) = X \times_k S \times_S \operatorname{Spec} k \cong X \arrow[l] \arrow[d] \\
\operatorname{Spec} k & S \arrow[l] & \operatorname{Spec} k = s \arrow[l]
\end{tikzcd}$$



Fiber product.

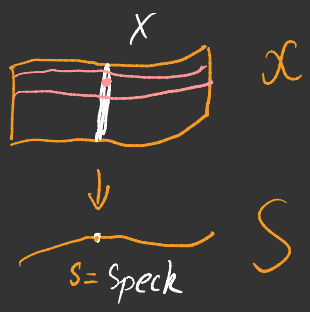
total scheme.

$$\eta : \begin{tikzcd}
X = \mathcal{X} \times_S \operatorname{Spec} k \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} k \arrow[r, hook] & S
\end{tikzcd}$$

π : flat & surjective

S : connected

parameter scheme



η is called locally trivial family (deformation) if $\forall x \in X$, $\exists$ open neighborhood $U_x \subset X$, such that

$$\begin{tikzcd} U_x \arrow[r] \arrow[d] & \mathcal{X}|_{U_x} \simeq U_x \times S \arrow[d] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

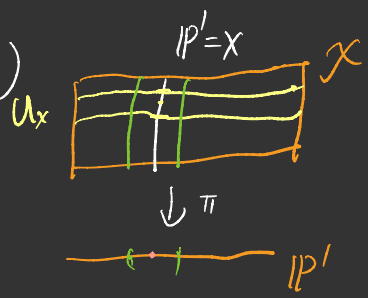
is trivial family/deformation of U_x.

Example: locally trivial but not trivial.

$$\mathcal{X} = \mathbb{F}_1 = \underline{\operatorname{Proj}}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$$

$$\downarrow \pi$$

$$S = \mathbb{P}^1$$



$\cdot$ $\mathcal{X} \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow \mathcal{X}$ non trivial.

Next time:

$S = \operatorname{Spec} A$, $A \in \operatorname{Ob}(\mathcal{A})$

Thm 1.2.4 every infinitesimal deformation of smooth affine scheme U_x is trivial.

Cor: If X smooth algebraic scheme, then every infinitesimal deformation of X is locally trivial.

[Har77 II. Ex 2.7, 2.8]

2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Ex 2.7

Suppose given $f: \operatorname{Spec} K \to X$ morphism
$p \mapsto x$

$f^{\#}_x: \mathcal{O}_{X,x} \to K$ is local homomorphism, i.e. $f_x^{\#\,-1}((0)) = m_x$

thus $k(x) = \mathcal{O}_{X,x}/m_x \hookrightarrow K$ is inclusion.

<u>Conversely</u>, given $x \in X$, $k(x) \hookrightarrow K$,

then define $f: \operatorname{Spec} K \to X$ as map, and define
$p \mapsto x$

$f^{\#}: \mathcal{O}_X \to f_* \mathcal{O}_{\operatorname{Spec} K}$ <u>check</u> it's local homomorphism.

$$\begin{cases} \mathcal{O}_X(U) \to 0 & \text{if } x \notin U \\ \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_x = k(x) \hookrightarrow K & \text{if } x \in U \end{cases}$$

2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of Spec $k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

$$T_x \simeq \mathrm{Hom}_{k(x)}(m_x/m_x^2, k(x))$$

$\frac{k[\varepsilon]}{(\varepsilon^2)}$ has unique prime ideal $\frac{(\varepsilon)}{(\varepsilon^2)} =: p. = (\bar{\varepsilon})$.

given $f: \mathrm{Spec}\, \frac{k[\varepsilon]}{(\varepsilon^2)} \to X$

$$p \mapsto x$$

then $f_x^{\#}: \mathcal{O}_{X,x} \to \frac{k[\varepsilon]}{(\varepsilon^2)}$ local ring homomorphism.

$f_x^{\#\,-1}(p) = m_x$, then define $\psi: m_x \to (\bar{\varepsilon}) \simeq k$

$$\begin{tikzcd}
m_x \arrow[r, "\psi"] \arrow[d] & (\bar{\varepsilon}) \simeq k \\
m_x/m_x^2 \arrow[ur, "\bar{\psi}"'] &
\end{tikzcd}$$

$b \mapsto \psi(b)$, $\bar{b} \mapsto \bar{\psi}$, $\bar{\varepsilon}$

Thus $\bar{\psi} \in \mathrm{Hom}_{k=k(x)}(m_x/m_x^2, \underset{k(x)}{k})$

tell us that $k(x) = \mathcal{O}_{X,x}/m_x \hookrightarrow \frac{\frac{k[\varepsilon]}{(\varepsilon^2)}}{\frac{(\varepsilon)}{(\varepsilon^2)}} \simeq k$

which implies $k(x) \simeq k$ by the following:

$$\begin{tikzcd} & X \arrow[d] \\ \operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)} \arrow[ur] & \operatorname{Spec} k \end{tikzcd} \qquad \begin{tikzcd} & k(x) \\ k \arrow[ur] & k \arrow[u, hook] \arrow[ul] \end{tikzcd}$$

<u>Conversely is similar.</u>

<u>Rmk</u>: X can be seem as representable functor

$X : (\text{Schs}) \longrightarrow (\text{Sets})$ represented by X :

$X(k) := X(\operatorname{Spec} k) = \operatorname{Hom}(\operatorname{Spec} k, X)$ all k-points in X

$X\left(k[\varepsilon]/(\varepsilon^2)\right) := \operatorname{Hom}\left(\operatorname{Spec} \frac{k[\varepsilon]}{(\varepsilon^2)}, X\right)$ set of k-point and tangent vector $v \in T_x$.

§1.2.2 Infinitesimal deformation of nonsingular affine scheme

<u>Lemma 1.2.3</u> suppose $Z_0 \subset Z$ noetherian closed subscheme defined by sheaf of nilpotent ideal $\underline{N \subset \mathcal{O}_Z}$. If Z_0 is affine, then Z is affine.

$N^r = 0$ for some $r \in \mathbb{N}$

Pf: Serre's affine criterion.

[Hart] III]

Theorem 3.7 (Serre [5]). *Let X be a noetherian scheme. Then the following conditions are equivalent:*

(i) *X is affine;*
(ii) *$H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and all $i > 0$;*
(iii) *$H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.*

suppose $\mathcal{F}$ is quasi-coherent sheaf on Z.

construct filtraction $0 = N^r\mathcal{F} \subset N^{r-1}\mathcal{F} \subset \cdots \subset N^2\mathcal{F} \subset N\mathcal{F} \subset \mathcal{F}$

exact sequence $0 \to \underset{0=}{N^r\mathcal{F}} \to N^{r-1}\mathcal{F} \xrightarrow{\simeq} \frac{N^{r-1}}{N^r}\mathcal{F} \to 0$

$\frac{N^{r-1}}{N^r}\mathcal{F}$ is $\mathcal{O}_X/N$ mod hence support on Z_0

thus $N^{r-1}\mathcal{F} \simeq \frac{N^{r-1}}{N^r}\mathcal{F}$ support on Z_0.

and $H^i(Z, N^{r-1}\mathcal{F}) = H^i(Z_0, N^{r-1}\mathcal{F}) = 0$.

Inductively $0 \to N^{r-1}\mathcal{F} - N^{r-2}\mathcal{F} \to \frac{N^{r-1}\mathcal{F}}{N^{r-2}\mathcal{F}} \to 0$

can prove $N^{r-1}F$ and $N^{r-1}F / N^{r-2}F$ support on Z_0.

use long exact sequence to prove $H^i(Z, N^{r-2}F) = 0$.

$\square$

Infinitesimal deformation of affine scheme.

lemma 1.2.3

$$\begin{tikzcd}
\operatorname{Spec} B_0 = X_0 \arrow[r] \arrow[d] & \mathcal{X} = \operatorname{Spec} B \arrow[d, "\pi : \text{flat, surjective}"] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec}(A)
\end{tikzcd}$$

$\operatorname{Spec} B_0 = \pi^{-1}(p)$

where $A \in \mathcal{A}$

$\mathcal{A}$ = the category of local artinian **k**-algebras with residue field **k**

$\hat{\mathcal{A}}$ = the category of complete local noetherian **k**-algebras with residue field **k**

$\mathcal{A}^*$ = the category of local noetherian **k**-algebras with residue field **k**

Locally

$$\begin{tikzcd}
B \otimes_A k = B_0 & B \arrow[l] \\
k \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

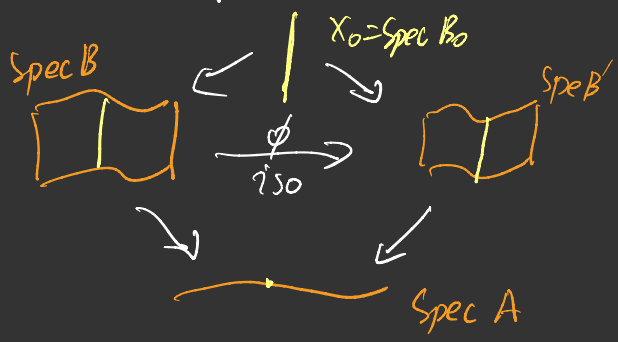
$B \otimes_A A/m = B / B \otimes_A m$

But $B \otimes m$ is nilpotent ideal in B, since m nilpotent by A Artian.

Lemma 1.2.3 $\Rightarrow$ $\mathcal{X}$ is affine.

isomorphism of deformation of $X_0 = \operatorname{Spec} B_0$ over A

$$\begin{tikzcd} & B_0 & \\ B \arrow[ur] \arrow[rr, "\varphi", "\text{iso}"'] & & B' \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$



- If $B' \simeq B_0 \otimes_k A$, then called B trivial.
- If all deformation of B_0 over A is trivial, then B_0 is called <u>rigid</u>.

<u>Thm 1.2.4.</u> every smooth B_0 k-algebra is rigid.

<u>Def</u>: (extension)

- $A \to R$ ring homomorphism.
- A-extension of R by I is exact sequence

$$0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

- φ: homomorphism of A-algebra.

$I \subset R'$ ideal and $\boxed{I^2 = 0}$, hence I is R-mod.

since $R \simeq R'/I$.

(small extension)

- K : field.
- R : local Artin K-alg residue field K.

exact sequence of K-mod: $0 \to I \to R' \xrightarrow{\varphi} R \to 0$

- $\boxed{m_{R'} \cdot I} = 0$, $m_{R'} := \varphi^{-1} m_R$

Example

$\mathrm{Spec}\, k[\varepsilon] \leftarrow \mathrm{Spec}\, k$

① $$0 \to \underset{I}{\frac{(t)}{(t^2)}} \to \frac{k[t]}{(t^2)} \to \underset{R}{\frac{k[t]}{(t)}} \to 0$$

is (small) k-extension of R by I.

② $$0 \to \frac{(x,y)^2}{(x,y)^3} \to \frac{k[x,y]}{(x,y)^3} \to \frac{k[x,y]}{(x,y)^2} \to 0.$$

is also small k-extension.

3-dim as k-vector space

③ $$0 \to \underbrace{\frac{(x^2)}{(x^3, x^2y, y^2)}}_{I} \to \underbrace{\frac{k[x,y]}{(x^3, x^2y, y^2)}}_{R'} \to \underbrace{\frac{k[x,y]}{(xy)^2}}_{R} \to 0$$

1-dim as k-vector space.

$m_{R'} = (x, y)$

$m_{R'} \cdot I = 0$

which is small k-extension.

<u>Thm 1.2.4.</u> every smooth B_0 k-algebra is rigid.

<u>Step 1</u>: first order case: where $k[\varepsilon] \simeq k[t]/(t^2)$

given deformation

$$\mathcal{J}: \quad \begin{array}{ccc} B & \longrightarrow & B_0 \text{ smooth.} \\ \uparrow f & & \uparrow \\ k[\varepsilon] & \longrightarrow & k = k[\varepsilon]/(\varepsilon^2) \end{array}$$

trivial

$$\mathcal{J}: \quad \begin{array}{ccc} B_0 \otimes_k k[\varepsilon] & \longrightarrow & B_0 \\ \uparrow & & \uparrow \\ k[\varepsilon] & \longrightarrow & k \simeq k[\varepsilon]/(\varepsilon^2) \end{array}$$

$$\begin{tikzcd}
 & B_0 & \\
B \arrow[ur] \arrow[rr, "\phi ?"] & & B_0 \otimes_k k[\varepsilon] \simeq B_0[\varepsilon] \arrow[ul] \\
 & k[\varepsilon] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

k-extension

① $0 \to \underbrace{B_0 \otimes (\varepsilon)}_{I^2=0} \to B_0[\varepsilon] \to B_0 \to 0$ is obtained by

$0 \to (\varepsilon) \to k[\varepsilon] \to k \simeq k[\varepsilon]/(\varepsilon) \to 0$ tensor B_0

Theorem C.9. *Let $f : R \to B$ be an e.f.t. ring homomorphism. Then the following conditions are equivalent:*

(i) For every extension of R-algebras:

$$0 \to I \to A \to A' \to 0 \tag{C.3}$$

the map

$$\mathrm{Hom}_{R-alg}(B, A) \to \mathrm{Hom}_{R-alg}(B, A')$$

is surjective.

(ii) If $B = P/J$, where $P = S^{-1}R[X_1, \ldots, X_d]$, $S \subset R[X_1, \ldots, X_d]$ is a multiplicative system and $J \subset P$ is an ideal, the conormal sequence

$$0 \to J/J^2 \xrightarrow{\delta} \Omega_{P/R} \otimes_P B \to \Omega_{B/R} \to 0$$

is split exact. In particular, J/J^2 and $\Omega_{B/R}$ are finitely generated projective B-modules.

(iii) B is a smooth R-algebra.

(iv) (Jacobian criterion of smoothness) If P and J are as in (ii) the map

$$(J/J^2) \otimes_B K(p) \xrightarrow{\delta \otimes_B K(p)} \Omega_{P/R} \otimes_P K(p) \qquad \text{where } K(p) = B_p/m_{B_p}$$

is injective for every prime ideal $p \subset B$.

ϕ exist by Thm C.9.

Lemma A.4. *Let R be a ring, I an ideal and $f : F \to G$ a homomorphism of R-modules with G flat. Assume that one of the following conditions is satisfied:*

(a) I is nilpotent.

(b) R is noetherian, I is contained in the Jacobson radical of R and F and G are finitely generated.

If the induced homomorphism $F/IF \to G/IG$ is an isomorphism, then f is an isomorphism.

lemma A.4 $\Rightarrow$ ϕ is isomorphism.

Indeed, $\phi : B \longrightarrow B_0[\varepsilon]$ as $k[\varepsilon]$-mod

$I = (\varepsilon) \subset k[\varepsilon]$

$$\tilde{\phi} : B/IB \longrightarrow B_0[\varepsilon]/I B_0[\varepsilon] \quad \text{is isomorphism}$$

$$B_0 \cong B \otimes \frac{k[\varepsilon]}{(\varepsilon)} \cong$$

$$I B_0[\varepsilon] = B_0[\varepsilon] \otimes \frac{k[\varepsilon]}{(\varepsilon)} \cong B_0$$

hence ϕ is isomorphism.

$\square$

<u>Prop</u>: $\varphi : A \to B$ k-alg homomorphism, isomorphism as k-mod.

Then φ is ring isomorphism.

$\varphi^{-1} : B \longrightarrow A$ is ring homomorphism.

$b = \varphi(a) \to a$

$\square$

Last time

Thm 1.2.4 every smooth k-alg is rigid.

i.e. every infinitesimal deformation of smooth affine scheme is trivial deformation.

Cor every infinitesimal deformation of smooth algebraic scheme is locally trivial.

Goal:

Proposition 1.2.9. *Let X be an algebraic variety. There is a 1–1 correspondence:*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of first order} \\ \text{locally trivial deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

called the Kodaira–Spencer correspondence, *where* $T_X = Hom(\Omega^1_X, \mathcal{O}_X) = Der_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X)$, *such that* $\kappa(\xi) = 0$ *if and only if* ξ *is the trivial deformation class. In particular if X is nonsingular then κ is a 1–1 correspondence*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{first-order deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

$\mathcal{A}$ = the category of local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$

$\hat{\mathcal{A}}$ = the category of complete local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$

$\mathcal{A}^*$ = the category of local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$

$A \in \mathcal{A}$

$$\begin{tikzcd} \mathrm{I} \arrow[r] \arrow[d] & \mathrm{I} \arrow[d] \\ \cdot \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\text{flat, surj}"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

$A/m = k \twoheadleftarrow A$

First order : $A = k[\varepsilon]$

Lemma 1.2.6. *Let B_0 be a* $\mathbf{k}$*-algebra, and*

$$e : 0 \to (t) \to \tilde{A} \to A \to 0$$

$\tilde{A}/(t)$ as k-mod exact sequence.

a small extension in $\mathcal{A}$. Then there is a canonical isomorphism of groups:

$$\left\{ \begin{array}{c} \text{automorphisms of the trivial deformation } B_0 \otimes_{\mathbf{k}} \tilde{A} \\ \text{inducing the identity on } B_0 \otimes_{\mathbf{k}} A \end{array} \right\} \to \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

In particular the group on the left is abelian.

T_{B_0}

$A \to B$

M

<u>small extension</u> : $t \cdot m_{\tilde{A}} = 0 \Rightarrow (t)$ is A-mod

$\Downarrow$ $t \in m_{\tilde{A}}$

$t^2 = 0$

Spec B $=$ $\mathrm{Hom}(k[\varepsilon], X)$

$x \mapsto v$

<u>Example</u> : <u>small extension</u>

$$0 \to (t) \to \frac{k[t]}{(t^2)} \to \frac{k[t]}{(t)} \simeq k \to 0$$

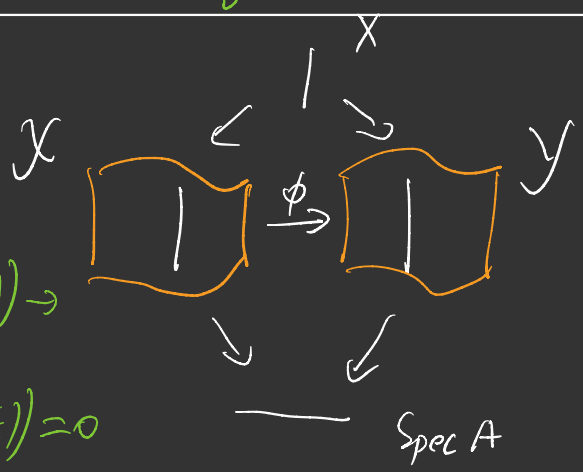
$k[\varepsilon]$ $\quad$ B $\quad$ k

$\mathrm{Ext}^1_k(B, (t)) = 0$

then

$0 \to \mathrm{Hom}(B, (t)) \to \mathrm{Hom}(B, k[\varepsilon]) \to$

$\mathrm{Hom}(B, k) \to \mathrm{Ext}^1_k(B, (t)) = 0$



proof of lem 1.2.6

Lem 1.11 $0 \to I \to R' \xrightarrow{\varphi} R \to 0$ A-extension

i.e. exact sequence of A-mod

$$\begin{tikzcd} & R' \arrow[r, "\varphi"] & R \\ B \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] \arrow[ur, "\varphi f_1 = \varphi f_2"'] & & \end{tikzcd}$$

$f_2 - f_1$

If $\exists$ $f_1, f_2 : B \to R'$ A-algebra homomorphism, such that $\varphi \circ f_1 = \varphi \circ f_2$, then $g := f_2 - f_1 : B \to I$ is $\boxed{\text{A-der.}}$ ☆

i.e. $g \in \mathrm{Der}_A(B, I)$

<u>proof</u>: I is $R = R'/I$ -mod since $I^2 = 0$ in extension.

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_A(B, I) & \to & \mathrm{Hom}_A(B, R') & \to & \mathrm{Hom}_A(B, R) & \to \mathrm{Ext}^1(B, I) \\ & f_2 - f_1 & \mapsto & f_2 - f_1 & \mapsto & \varphi(f_2 - f_1) = 0 & \end{array}$$

Moreover I have B-mod structure by $B \xrightarrow{f_i} R' \xrightarrow{\varphi} R$

$$B \times I \longrightarrow I$$

$$(b, x) \mapsto \varphi \circ f_i(b)\, x =: b \cdot x$$

<u>check</u> $g = f_2 - f_1 : B \to I$ is A-derivation.

① g is A-linear.

② $g(1) = f_2(1) - f_1(1) = 0$

③ product rule:

$$
\begin{aligned}
g(b_1 b_2) &= f_2(b_1 b_2) - f_1(b_1 b_2) \\
&= f_2(b_1) f_2(b_2) - f_1(b_1) f_1(b_2) \\
&= \underline{f_2(b_1)} f_2(b_2) - \underline{f_2(b_1)} f_1(b_2) + f_2(b_1) \underline{f_1(b_2)} - f_1(b_1) \underline{f_1(b_2)} \\
\text{in } R' \quad &= \underbrace{f_2(b_1)}_{\in R'} \cdot \underbrace{g(b_2)}_{\in I} + \underbrace{f_1(b_2)}_{\in R'} \cdot \underbrace{g(b_1)}_{\in I} \\
&= \underbrace{\varphi \circ f_2(b_1)}_{\in R} \cdot g(b_2) + \underbrace{\varphi \circ f_1(b_2)}_{\in R} g(b_1) \\
&= b_1 \cdot g(b_2) + b_2 \cdot g(b_1).
\end{aligned}
$$

$\square$

proof of lem 1.2.6:

$$0 \to (t) \to \widetilde{A} \to A \to 0$$

$0 \to (t) \to k[\varepsilon] \to k \to 0$ Imagine.

$B_0 \otimes _$: as k-mod.

$$0 \to B_0 \otimes_k (t) \to B_0 \otimes_k \widetilde{A} \xrightarrow{\varphi} B_0 \otimes_k A \to 0.$$

$$\begin{tikzcd}
B_0 \otimes_k (t) & B_0 \otimes_k \widetilde{A} \arrow[l, "\theta - id"'] \arrow[r, "\varphi"] & B_0 \otimes_k A \\
& B_0 \otimes_k \widetilde{A} \arrow[u, "id,\ \theta\ \text{:automorphism}"] \arrow[ur, "\varphi \circ \theta = \varphi \circ id"'] &
\end{tikzcd}$$

Lem 1.11

thus $\theta - id : B_0 \otimes_k \widetilde{A} \longrightarrow B_0 \otimes_k (t)$

$$x \mapsto t \cdot dx$$

where $d: B_0 \otimes_k \tilde{A} \longrightarrow B_0$ is $\tilde{A}$-derivation.

$x \mapsto \underline{dx}$ notation

Therefore $\theta(x) = x + t\,dx$

$$d \in \mathrm{Der}_{\tilde{A}}(B_0 \otimes_k \tilde{A}, B_0) = \mathrm{Hom}_{B_0 \otimes \tilde{A}}(\Omega_{B_0 \otimes_k \tilde{A}/\tilde{A}}, B_0)$$

$$\begin{cases} = \mathrm{Hom}_{B_0 \otimes \tilde{A}}(\Omega_{B_0/k} \otimes_{B_0} (B_0 \otimes_k \tilde{A}), B_0) \\ = \mathrm{Hom}_{B_0}(\Omega_{B_0/k}, B_0) \otimes_{B_0} (B_0 \otimes_k \tilde{A}) \\ \text{as set} \quad = \mathrm{Hom}_{B_0}(\Omega_{B_0/k}, B_0) \end{cases}$$

on book

$$= \mathrm{Der}_k(B_0, B_0)$$

$$\begin{tikzcd} B_0 & \longrightarrow & B_0 \otimes_k \tilde{A} \\ k \arrow[u] & \longrightarrow & \tilde{A} \arrow[u] \end{tikzcd}$$

<u>Define</u>: $\left\{ \begin{array}{l} \text{automorphism of trivial deformation} \\ \text{of } B_0 \otimes \tilde{A} \text{ induced identity on } B_0 \otimes A \end{array} \right\} \longrightarrow \mathrm{Der}_k(B_0, B_0)$

$$\sigma \longmapsto \delta$$

$$\theta \longmapsto d$$

satisfy $\theta(x) = x + t \cdot dx$

$$\sigma \circ \theta \mapsto d + \delta$$

by the following: $\begin{cases} \theta(x) = x + t\,dx \\ \sigma(x) = x + t\,\delta x \end{cases}$

$$\sigma \circ \theta(x) = \theta(x) + t\,\delta(\theta(x))$$
$$= x + t\,dx + t(\delta x + t\,\delta dx)$$
$$= x + t(dx + \delta x) + \cancel{t^2 \delta dx} \to 0 \quad \text{since } t^2 = 0$$

$$\theta x = x + t\,dx$$

Lemma 1.2.6. *Let B_0 be a* $\mathbf{k}$*-algebra, and* $0 \to (t) \to k[\varepsilon] \to k \to 0$

$$e : 0 \to (t) \to \tilde{A} \to A \to 0$$

$U_{ij} = \operatorname{Spec} B_0$

a small extension in $\mathcal{A}$. Then there is a canonical isomorphism of groups:

$$\left\{ \begin{array}{c} \text{automorphisms of the trivial deformation } B_0 \otimes_{\mathbf{k}} \tilde{A} \\ \text{inducing the identity on } B_0 \otimes_{\mathbf{k}} A \end{array} \right\} \to \operatorname{Der}_{\mathbf{k}}(B_0, B_0)$$

In particular the group on the left is abelian.

Proposition 1.2.9. *Let X be an algebraic variety. There is a 1–1 correspondence:*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of first order} \\ \text{locally trivial deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

called the Kodaira–Spencer correspondence, *where* $T_X = Hom(\Omega^1_X, \mathcal{O}_X) = Der_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X)$, *such that* $\kappa(\xi) = 0$ *if and only if* ξ *is the trivial deformation class. In particular if* X *is nonsingular then* κ *is a 1–1 correspondence*

$$\kappa : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{first-order deformations of } X \end{array} \right\} \to H^1(X, T_X)$$

proof: Locally trivial:

$$\begin{tikzcd} X|_{U_i} \arrow[r, hook] \arrow[d] & \mathcal{X}|_{U_i} \arrow[d] \arrow[r, "\sim"] & U_i \times_k \operatorname{Spec} k[\varepsilon] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon] & \end{tikzcd}$$

trivial.

choose affine open cover $\{U_i\}$ of X, such that $\mathcal{X}|_{U_i}$ trivial.

Then $\theta_i : U_i \times \operatorname{Spec} k[\varepsilon] \to \mathcal{X}|_{U_i}$

$\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times \operatorname{Spec} k[\varepsilon] \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$

is automorphism.

Lem 1.2.6 $\Rightarrow$ $\theta_{ij} \mapsto d_{ij} \in \Gamma(U_{ij}, T_X|_{U_{ij}})$

check $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1_{U_{ijk} \times \operatorname{Spec} k[\varepsilon]}$

$\Rightarrow$ $d_{ij} + d_{jk} - d_{ik} = 0$

thus $\{d_{ij}\} \in H^1(X, T_X)$ čech 1-cocycle.

Suppose $\Phi : X \longrightarrow X'$ isomorphism of 1-order deform.

$\exists$ $\alpha_i : U_i \times \operatorname{Spec} k[\varepsilon] \xrightarrow{\theta_i} X|_{U_i} \xrightarrow{\Phi|_{U_i}} X'|_{U_i} \xrightarrow{\theta_i'^{-1}} U_i \times \operatorname{Spec} k[\varepsilon]$

Then $\theta_i' \circ \alpha_i = \Phi|_{U_i} \circ \theta_i$

Thus $(\theta_i' \circ \alpha_i)^{-1}(\theta_j' \alpha_j) = \theta_i^{-1} \underline{\Phi|_{U_{ij}}^{-1} \Phi|_{U_{ij}}} \theta_j = \theta_i^{-1}\theta_j$

$= \alpha_i^{-1}(\theta_i'^{-1}\theta_j')\alpha_j$

$= \alpha_i^{-1}\theta_{ij}'\alpha_j$

$= \theta_{ij}$

$$\xRightarrow[\text{Lem 1.2.6}]{} \begin{cases} \theta_{ij} \mapsto d_{ij} \\ \theta_{ij}' \mapsto d_{ij}' \\ \alpha_i \mapsto a_i \in \Gamma(U_i, T_X) \\ \alpha_j \mapsto a_j \in \Gamma(U_j, T_X) \end{cases} \Rightarrow \quad d_{ij}' + \underbrace{\overset{d\{a_i\}}{\overset{\shortparallel}{a_j - a_i}}}_{\text{coboundary}} = d_{ij}$$

$\{a_i\}_{i \in I} \in$

Therefore $\{d_{ij}\} = \{d_{ij}'\}$ in $H^1(X, T_X)$. $\square$

trivial $\mathcal{X} = X \times \operatorname{Spec} k[\varepsilon] \Leftarrow X \times \operatorname{Spec} k = X$

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{(t)}{(t^2)} \otimes_k B \arrow[r] & \dfrac{k[t]}{(t^2)} \otimes_k B \arrow[r] & \dfrac{k[t]}{(t)} \otimes_k B = k \arrow[r] & 0 \\
 & & \dfrac{k[t]}{(t^2)} \otimes B \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{X} = X \times k[\varepsilon] \arrow[loop above] & & X \\
X \arrow[r] \arrow[d] & \arrow[d] & & \arrow[d] & \\
k \arrow[r] & k[\varepsilon] \arrow[r] & & k &
\end{tikzcd}$$