ANNALS OF MATHEMATICS
Vol. 67, No. 3, May, 1958
Printed in Japan

ON DEFORMATIONS OF COMPLEX ANALYTIC STRUCTURES, II*

BY K. KODAIRA AND D. C. SPENCER

(Received November 7, 1957)

CHAPTER VI. SPECIAL FAMILIES OF COMPLEX STRUCTURES; OBSTRUCTION TO DEFORMATION

14. Examples

In this section we examine complex analytic families of some simple types of compact complex manifolds, e.g. hypersurfaces, complex tori, $\cdots$, and compute the numbers of moduli of these manifolds.

(α) *Projective space.* Let $V_o = P_n(\boldsymbol{C})$ be a complex projective space. We have $\dim H^1(V_o, \Theta_o) = 0$ (Bott [7]; see Lemma 14.1 below) and therefore, by Theorem 6.3, the family $\{V_o\}$ consisting of the single member V_o is complete. Hence the number of moduli $m(V_o)$ equals 0 and the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ holds.

(β) *Hypersurfaces in projective spaces.* Let $P_{n+1} = P_{n+1}(\boldsymbol{C})$ be the projective space of dimension $n+1$. We denote by ζ a point on P_{n+1} with the homogeneous coordinates $(\zeta_0, \zeta_1, \cdots, \zeta_{n+1})$. The number of monomials $\zeta_0^{h_0}\zeta_1^{h_1}\cdots\zeta_{n+1}^{h_{n+1}}$ of degree $h_0 + h_1 + \cdots + h_{n+1} = h$ in $n+2$ variables $\zeta_0, \zeta_1, \cdots, \zeta_{n+1}$ is given by

$$\mu + 1 = \binom{n+1+h}{h}. \tag{14.1}$$

Let $P_\mu = P_\mu(\boldsymbol{C})$. We associate with each point $t = (t_0, t_1, \cdots, t_\mu) \in P_\mu$ the homogeneous polynomial

$$t(\zeta) = t_0\zeta_0^h + t_1\zeta_0^{h-1}\zeta_1 + \cdots + t_\nu\zeta_0^{h_0}\cdots\zeta_{n+1}^{h_{n+1}} + \cdots + t_\mu\zeta_{n+1}^h$$

of degree h. The equation $t(\zeta) = 0$ defines a singular or multiple hypersurface on P_{n+1} if and only if t belongs to a proper (reducible) subvariety $\mathfrak{S}$ of P_μ. Now let $M = P_\mu - \mathfrak{S}$, and consider the non-singular submanifold

$$\mathscr{V}_{n,h} = \{(\zeta, t) \mid t(\zeta) = 0\}$$

of $P_{n+1} \times M$. Moreover let $\varpi : (\zeta, t) \to t$ be the projection of $\mathscr{V}_{n,h}$ onto

* This paper is a continuation of Part I of the same title which has appeared at the end of the preceding issue of this journal.

M and let $\Psi: (\zeta, t) \to \zeta$ be the projection of $\mathscr{V}_{n,h}$ onto P_{n+1}. Then $\mathscr{V}_{n,h}$ forms a complex analytic fibre space over M and Ψ maps each fibre $V_t = \varpi^{-1}(t)$ biregularly onto the non-singular hypersurface on P_{n+1} defined by $t(\zeta) = 0$. Thus $\mathscr{V}_{n,h} = \{V_t | t \in M\}$ *is the complex analytic family consisting of all non-singular hypersurfaces of order* h *on* P_{n+1}.

THEOREM 14.1. *The complex analytic family* $\mathscr{V}_{n,h}$ *of all non-singular hypersurfaces of order* h *on* P_{n+1} *is complete except for the following two special cases*: (i) $n = 1$ *and* $h \geqq 4$, (ii) $n = 2$ *and* $h = 4$.

PROOF. Let $\mathscr{V}^{(t)} = \{V_{t,s} | s \in N\}$ be a differentiable family of deformations $V_{t,s}$ of a non-singular hypersurface $V_t \in \mathscr{V}_{n,h}$ and let $\Phi_o = \Psi_t = \Psi | V_t$ be the inclusion map $V_t \to P_{n+1}$. It suffices to prove that

$$\Phi_o : V_{t,o} = V_t \longrightarrow P_{n+1}$$

can be extended to a differentiable map $\Phi: \mathscr{V}^{(t)} | U \to P_{n+1}$ which is holomorphic on each fibre $V_{t,s}$ provided that U is a sufficiently small neighborhood of o on N. In fact, if such an extension Φ exists, the image $\Phi(V_{t,s})$ of each fibre $V_{t,s}$ is obviously a non-singular hypersurface of the same order h and hence $\Phi(V_{t,s}) = V_{t_s}$, $t_s \in M \subset P_\mu$. The point t_s is determined by the linear equations

$$t_s(\zeta) = 0 \ , \qquad\qquad \text{for all } \zeta \in \Phi(V_{t,s}).$$

Hence $s \to t_s$ is a differentiable map. For any point $p \in V_{t,s} \subset \mathscr{V}^{(t)}$ we write $s = s(p)$. Then it is clear that

$$p \longrightarrow (\Phi(p), t_{s(p)}) (\in P_{n+1} \times M)$$

is a differentiable map of $\mathscr{V}^{(t)} | U$ into $\mathscr{V}_{n,h}$ which maps each fibre $V_{t,s}$ biregularly onto V_{t_s}. Hence $\mathscr{V}^{(t)} | U$ is the family induced from $\mathscr{V}_{n,h}$ by the map $s \to t_s$ of U into M.

To prove the existence of an extension Φ of $\Phi_o = \Psi_t$ we apply Theorems 13.2 and 13.3. Denote by E the complex line bundle on P_{n+1} defined by $\{e_{\lambda\nu}(\zeta)\}$, $e_{\lambda\nu}(\zeta) = \zeta_\nu / \zeta_\lambda$, and by $\Omega(E^l)$ the sheaf over P_{n+1} of germs of holomorphic sections of $E^l = E \otimes E \otimes \cdots \otimes E$ (l factors). Moreover let $E_t = \Psi_t^*(E)$ be the induced bundle over V_t. Since the bundle $[V_t]$ on P_{n+1} determined by the divisor V_t coincides with E_t^h, the canonical bundle K_t on V_t is given by $K_t = E_t^{h-n-2}$. We have the exact sequence

$$0 \longrightarrow \Omega(E^{l-h}) \longrightarrow \Omega(E^l) \xrightarrow{r_t} \Omega(E_t^l) \longrightarrow 0 \ ,$$

where r_t denotes the restriction map to V_t. It follows that

$$\longrightarrow H^q(P_{n+1}, \Omega(E^l)) \longrightarrow H^q(V_t, \Omega(E_t^l)) \longrightarrow H^{q+1}(P_{n+1}, \Omega(E^{l-h})) \longrightarrow \ ,$$

while $H^q(P_{n+1}, \Omega(E^l)) = 0$ for $1 \leqq q \leqq n$. Hence we obtain

$$H^q(V_t, \Omega(E_t^l)) = 0, \qquad \text{for } 1 \leqq q \leqq n-1. \tag{14.2}$$

(A) The case $n \geqq 3$. We have, by (14.2), $H^2(V_t, \Omega_t) = 0$ and

$$H^1(V_t, \Omega(E_t)) = 0 .$$

Hence, by Theorem 13.2, there exists an extension Φ of $\Phi_o = \Psi_t$.

(B) The case $n = 2$, $h \neq 4$. We have $H^1(V_t, \Omega(E_t)) = 0$ by (14.2), while, since $K_t = E_t^{h-4}$, $c(E_t) = -(h-4)^{-1}c_1$. Consequently, by Theorem 13.3, there exists an extension Φ of $\Phi_o = \Psi_t$.

(C) The case $n = 1$, $h < 4$. We have $H^2(V_t, \Omega_t) = 0$ and

$$H^1(V_t, \Omega(E_t)) \cong H^0(V_t, \Omega(K_t \otimes E_t^{-1})) = H^0(V_t, \Omega(E_t^{h-4})) = 0 .$$

Hence there exists an extension Φ of Φ_o by Theorem 13.2.

Now we determine the number of moduli of a hypersurface $V_t \in \mathscr{V}_{n,h}$, $n \geqq 2$, $h \geqq 2$, excluding the case $n = 2$, $h = 4$. The family $\mathscr{V}_{n,h}$ is complete but is not effectively parametrized. To find an effectively parametrized complete subfamily $\mathscr{V}_{n,h}|U'$, we apply the results of Section 12. Let Ξ be the sheaf of germs of holomorphic sections of the tangent bundle of P_{n+1}, Ξ_t the restriction of Ξ to V_t, and let Θ_t be the sheaf of germs of holomorphic sections of the tangent bundle of V_t. Since $[V_t] = E^h$, we have the exact sequences

$$0 \longrightarrow \Theta_t \longrightarrow \Xi_t \longrightarrow \Omega(E_t^h) \longrightarrow 0 , \tag{14.3}$$

$$0 \longrightarrow \Xi \otimes E^{-h} \longrightarrow \Xi \longrightarrow \Xi_t \longrightarrow 0 \tag{14.4}$$

(see Section 12, formulae (12.14), (12.15)).

LEMMA 14.1. *We have*

$$H^q(P_{n+1}, \Xi \otimes E^{-h}) = 0 \qquad \textit{for } q = 0, 1, 2, \tag{14.5}$$

$$H^1(P_{n+1}, \Xi) = 0 . \tag{14.6}$$

This lemma is an immediate consequence of the following theorem due to Bott [7]:

THEOREM (Bott). *The group $H^q(P_n, \Omega^p(E^k))$ vanishes except for the following three cases*: (i) $p = q$ *and* $k = 0$, (ii) $q = 0$ *and* $k > p$, (iii) $q = n$ *and* $k < p - n$.

In fact, we have, by the duality theorem,

$$H^q(P_{n+1}, \Xi \otimes E^{-l}) \cong H^{n+1-q}(P_{n+1}, \Omega^1(E^{l-n-2})) .$$

Hence (14.5) and (14.6) follow. On the other hand we have the isomorphism

$$\rho_{d,t} : (T_M)_t \cong H^0(V_t, \Omega(E_t^h)) ,$$

as was mentioned in Section 12 (compare (12.11)). Consequently we obtain from (12.17) and (12.4) the exact commutative diagram

$$\begin{tikzcd} & & 0 & 0 & & \\ 0 \arrow[r] & H^0(V_t, \Theta_t) \arrow[r] & H^0(V_t, \Xi_t) \arrow[r] \arrow[u] & H^0(V_t, \Omega(E_t^h)) \arrow[r] \arrow[u] & H^1(V_t, \Theta_t) \arrow[r] & 0 \\ & & H^0(P_{n+1}, \Xi) \arrow[u] & (T_M)_t \arrow[u, "\rho_{d,t}"] \arrow[ur, "\rho_t"'] & & \\ & & 0 \arrow[u] & 0 \arrow[u] & & \end{tikzcd} \tag{14.7}$$

The case $h = 2$. Let $V_1 \in \mathscr{V}_{n,2}$ be the hypersurface defined by

$$\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2 = 0 .$$

Since every hypersurface $V_t \in \mathscr{V}_{n,2}$ can be transformed into V_1 by a projective transformation acting on P_{n+1}, the family $\mathscr{V}_{n,2}$ is trivial, and hence, by Proposition 11.2, $m(V_t) = 0$. Moreover we infer from (14.7) that $H^1(V_t, \Theta_t) = \rho_t(T_t) = 0$.

The case $h \geqq 3$.

LEMMA 14.2. *We have*

$$H^0(V_t, \Theta_t) = 0 , \quad \textit{for } V_t \in \mathscr{V}_{n,h},\ n \geqq 2,\ h \geqq 3. \tag{14.8}$$

PROOF. By the duality theorem we have

$$H^0(V_t, \Theta_t) \cong H^n(V_t, \Omega^1(E_1^{h-n-2})) .$$

Hence, in case $h > n + 2$, our lemma follows immediately from Akizuki-Nakano's criterion (see Section 11, formula (11.12)). In case $h \leqq n + 2$, we reduce our lemma to Akizuki-Nakano's criterion with the help of the pair of exact sequences (compare Kodaira and Spencer [27], (3) and (4))

$$\cdots \longrightarrow H^{q-1}(V_t, \Omega^p(E_t^k)) \longrightarrow H^q(P_{n+1}, \Omega''^p(E^k)) \longrightarrow H^q(P_{n+1}, \Omega^p(E^k)) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^q(P_{n+1}, \Omega''^p(E^k)) \longrightarrow H^q(V_t, \Omega^{p-1}(E_t^{k-h})) \longrightarrow H^{q+1}(P_{n+1}, \Omega^p(E^{k-h})) \longrightarrow \cdots$$

It follows from the above theorem of Bott that

$$H^q(P_{n+1}, \Omega^p(E^k)) = 0, \quad H^{q+1}(P_{n+1}, \Omega^p(E^{k-h})) = 0$$

for $q = n + 2 - p$, $k = ph - n - 2$, provided that $2 \leqq p \leqq n$, $h \geqq 3$. Hence we infer from the above pair of exact sequences that

$$H^{n+1-p}(V_t, \Omega^p(E_t^{ph-n-2})) \longrightarrow H^{n+2-p}(P_{n+1}, \Omega''^p(E^{ph-n-2})) \longrightarrow 0 ,$$

$$H^{n+2-p}(P_{n+1}, \Omega''^p(E^{ph-n-2})) \longrightarrow H^{n+2-p}(V_t, \Omega^{p-1}(E_t^{ph-h-n-2})) \longrightarrow 0 .$$

Thus $H^{n+1-p}(V_t, \Omega^p(E_t^{ph-n-2})) = 0$ implies $H^{n+2-p}(V_t, \Omega^{p-1}(E_t^{ph-h-n-2})) = 0$, while we have $H^1(V_t, \Omega^n(E_t^{nh-n-2})) = 0$ by Akizuki-Nakano's criterion. Hence we obtain $H^n(V_t, \Omega^1(E_t^{h-n-2})) = 0$, q.e.d.

Let $\mathfrak{G}$ be the projective transformation group acting on P_{n+1}. Each element $\gamma \in \mathfrak{G}$ induces an analytic automorphism $t \to \gamma t$ of M defined by

$$(\gamma t)(\zeta) = t(\gamma^{-1}\zeta) , \qquad \zeta \in P_{n+1}.$$

Thus $\mathfrak{G}$ acts on M and this action induces a homomorphism $\mathfrak{g} \to (T_M)_t$ of the infinitesimal group $\mathfrak{g}$ of $\mathfrak{G}$ into $(T_M)_t$. Clearly we may identify $\mathfrak{g}$ with $H^0(P_{n+1}, \Xi)$ in a canonical manner. Moreover we get from (14.7) and (14.8) the exact commutative diagram :

$$(14.9) \qquad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 \longrightarrow & & H^0(V_t, \Xi_t) & \longrightarrow & H^0(V_t, \Omega(E_t^h)) & \longrightarrow & H^1(V_t, \Theta_t) \longrightarrow 0 . \\ & & \uparrow & & \rho_{d,t}\uparrow & \nearrow \rho_t & \\ & & H^0(P_{n+1}, \Xi) = \mathfrak{g} & \longrightarrow & (T_m)_t & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

With the help of this diagram we can show, in the same manner as in the proof of Theorem 12.4, the existence of an effectively parametrized complete subfamily $\mathscr{V}_{n,h}|U'$ containing any given fibre V_t. In fact, let $\mathfrak{U}$ be a small neighborhood of 1 on $\mathfrak{G}$ and let U'_t be a complex coordinate plane on M containing t, with $\dim U'_t = \dim M - \dim \mathfrak{G}$, which is *transverse* to the orbit $\mathfrak{U}\, t$. Then $U = \mathfrak{U}\, U'_t$ is a neighborhood of t on M and

$$h : (\zeta, \gamma t') \longrightarrow (\gamma^{-1}\zeta, t') , \qquad t' \in U'_t,$$

is a holomorphic map of $\mathscr{V}_{n,h}|U \subset P_{n+1} \times M$ onto $\mathscr{V}_{n,h}|U'$ which maps each fibre $V_{\gamma t'}$ biregularly onto $V_{t'}$. It follows that $\mathscr{V}_{n,h}|U'$ is complete, while (14.9) shows that $\mathscr{V}_{n,h}|U'$ is effectively parametrized. Hence we get

$$m(V_t) = \dim U' = \dim M - \dim \mathfrak{G} .$$

Moreover it follows from (14.9) that $m(V_t) = \dim H^1(V_t, \Theta_t)$. We note that $\dim \mathfrak{G} = (n+2)^2 - 1$ and, by (14.1), $\dim M = \mu = \binom{n+1+h}{h} - 1$. Thus we obtain

THEOREM 14.2. *Let V_t be an n-dimensional hypersurface of order h in $P_{n+1}(C)$, $n \geqq 2$, $h \geqq 2$. Except for the special case : $n = 2$, $h = 4$, the number $m(V_t)$ of moduli of V_t is given by*

$$(14.10)\quad m(V_t) = \dim H^1(V_t, \Theta_t) = \begin{cases} 0\,, & \text{for } h=2,\\ \binom{n+1+h}{h} - (n+2)^2\,, & \text{for } h \geqq 3. \end{cases}$$

We note that the *exceptional family* $\mathscr{V}_{2,4}$ is *not* complete. In fact it has been shown by Nakano that the surface $V_{t_0} \in \mathscr{V}_{2,4}$ defined by

$$\zeta_0^4 + \zeta_1^4 + \zeta_2^4 + \zeta_3^4 = 0$$

has an arbitrarily small deformation which is a *non-algebraic* Kähler surface. Moreover, for any member $V_t \in \mathscr{V}_{2,4}$, we have

$$\begin{cases} \dim H^1(V_t, \Theta_t) = 20\,,\\ \dim \rho_t((T_M)_t) = 19\,. \end{cases}$$

(γ) *Complex tori.* Let S be the space of $n \times n$ matrices $s = (s^\alpha_\beta)$ with $|\Im s| > 0$, where α denotes the row index and β the column index. For each matrix $s \in S$ we define an $n \times 2n$ matrix $\omega(s) = (\omega^\alpha_j(s))$ by

$$\omega^\alpha_j(s) = \begin{cases} \delta^\alpha_j\,, & \text{for } 1 \leqq j \leqq n,\\ s^\alpha_\beta\,, & \text{for } j = n+\beta,\ 1 \leqq \beta \leqq n. \end{cases}$$

Let C^n be the space of n complex variables $z = (z^1, \cdots, z^\alpha, \cdots, z^n)$ and let G be the discontinuous abelian group of analytic automorphisms of $C^n \times S$ generated by

$$g_j : (z, s) \longrightarrow (z + \omega_j(s), s)\,, \qquad j = 1, 2, \cdots, 2n,$$

where $\omega_j(s) = (\omega^1_j(s), \cdots, \omega^\alpha_j(s), \cdots, \omega^n_j(s))$ is the j-th column vector of $\omega(s)$. The factor space

$$\mathscr{B} = (C^n \times S)/G$$

is obviously a complex manifold and the canonical projection $C^n \times S \to S$ induces a regular map $\varpi : \mathscr{B} \to S$ such that $B_s = \varpi^{-1}(s)$ is a complex torus of complex dimension n with the periods $\omega_j(s)$ $(j = 1, 2, \cdots, 2n)$. Thus $\mathscr{B} = \{B_s \mid s \in S\}$ forms a complex analytic family of complex tori.

Let Θ_s be the sheaf of germs of holomorphic sections of the tangent bundle of B_s and let $H^{0,1}(\Theta_s)$ be the linear space of vector (0, 1)-forms $\sum_{\nu=1}^n c^\alpha_\nu d\bar{z}^\nu$ on B_s with constant coefficients c^α_ν. We have the canonical identification (Dolbeault isomorphism)

$$H^1(B_s, \Theta_s) = H^{0,1}(\Theta_s)\,.$$

For simplicity we write $\sum_\nu c^\alpha_\nu d\bar{z}^\nu$ in the form

$$c \cdot d\bar{z}, \quad c = (c^\alpha_\nu)\,.$$

Now let $(T_S)_s$ be the tangent space of S at s, where we write each element $u \in (T_S)_s$ as an $n \times n$ matrix $u = (u^\alpha_\beta)$ in an obvious manner. Then *the* *homomorphism*

$$\rho_s : (T_S)_s \longrightarrow H^{0,1}(\Theta_s) = H^1(B_s, \Theta_s)$$

is given by

(14.11) $$\rho_s(u) = u(s - \bar{s})^{-1} d\bar{z} \ .$$

This can be verified as follows : Let $\{\mathscr{U}_i\}$ be a sufficiently fine covering of $\mathscr{B} = (\boldsymbol{C}^n \times S)/G$ and let $(z_i^1, \cdots, z_i^\alpha, \cdots, z_i^n, s)$ be a system of local coordinates on $\mathscr{U}_i$ which is *induced from the given global euclidean system of coordinates on* $\boldsymbol{C}^n \times S$. Then we have

$$z_i^\alpha = z_k^\alpha + \sum_{j=1}^{2n} m_{ik}^j \omega_j^\alpha(s) \ , \qquad \text{on } \mathscr{U}_i \cap \mathscr{U}_k,$$

or

$$z_i^\alpha = z_k^\alpha + m_{ik}^\alpha + \sum_{\beta=1}^{n} m_{ik}^{n+\beta} s_\beta^\alpha \ , \qquad \text{on } \mathscr{U}_i \cap \mathscr{U}_k,$$

where the m_{ik}^j are rational integers. Hence, for $u \in (T_S)_s$, the image $\rho_s(u) \in H^1(B_s, \Theta_s)$ is represented by the 1-cocycle (compare Section 5)

$$\theta_{ik}^\alpha = \sum_{\beta=1}^{n} m_{ik}^{n+\beta} u_\beta^\alpha \ .$$

Let $(c_\beta^\alpha) = c = u(s - \bar{s})^{-1}$ and let

$$\psi_i^\alpha(z, \bar{z}) = \sum_{\beta=1}^{n} c_\beta^\alpha (\bar{z}_i^\beta - z_i^\beta) \ .$$

Then we have

(14.12) $$\psi_k^\alpha(z, \bar{z}) - \psi_i^\alpha(z, \bar{z}) = \theta_{ik}^\alpha \ .$$

Hence $\rho_s(u) \in H^1(B_s, \Theta_s) = H^{0,1}(\Theta_s)$ is given by

$$\rho_s(u) = \bar{\partial}\psi_i(z, \bar{z}) = c \cdot d\bar{z} \ .$$

Thus we obtain (14.11). This result shows that ρ_s gives an isomorphism

(14.13) $$\rho_s : (T_S)_s \cong H^1(B_s, \Theta_s) \ .$$

The completeness of the complex analytic family $\mathscr{B}$ can be verified as follows: Given a differentiable family $\mathscr{W} = \{W_t \mid t \in N\}$ of deformations W_t of a complex torus $W_o = B_s \in \mathscr{B}$, we consider the restriction $\mathscr{W}|U$ of $\mathscr{W}$ to a sufficiently small "spherical" neighborhood U of o on N. Obviously the fibre space $\mathscr{W}|U$ is differentiably trivial:

$$\mathscr{W}|U = X \times U \ ,$$

where X is the underlying differentiable torus on which the complex structures W_t are defined. By Theorem 3.1, $W_t \subset \mathscr{W}|U$ is a Kähler manifold. Hence the number of holomorphic 1-forms on W_t is independent of t, and therefore, by Theorem 2.2, (i), there exist n linearly independent holomorphic 1-forms $\Psi^\alpha(x, t)$, $\alpha = 1, 2, \cdots, n$, on each fibre W_t, $t \in U$, which, considered as $\mathfrak{F}$-forms on $\mathscr{W}|U = X \times U$, are differentiable in both variables x and t. Let $\{\mathfrak{c}_1, \cdots, \mathfrak{c}_j, \cdots, \mathfrak{c}_{2n}\}$ be a base of 1-cycles on X such that

$$\int_{\mathfrak{c}_j} dz^\alpha = \omega_j^\alpha(s) = \begin{cases} \delta_j^\alpha , & 1 \leqq j \leqq n, \\ s_\beta^\alpha , & j = n + \beta,\ 1 \leqq \beta \leqq n \end{cases}$$

with respect to the complex torus structure $W_o = B_s$. Since

$$dz^\alpha = \sum_{\beta=1}^n \varepsilon_\beta^\alpha \Psi^\beta(x, o) ,$$

where $\det(\varepsilon_\beta^\alpha) \neq 0$, the simultaneous linear equations

$$\sum_{\beta=1}^n \varepsilon_\beta^\alpha(t) \int_{\mathfrak{c}_j} \Psi^\beta(x, t) = \delta_j^\alpha , \qquad j = 1, \cdots, n,$$

have unique solutions $\varepsilon_\beta^\alpha(t)$ which depend differentiably on t and $\varepsilon_\beta^\alpha(o) = \varepsilon_\beta^\alpha$. Let

$$\Phi^\alpha(x, t) = \sum_{\beta=1}^n \varepsilon_\beta^\alpha(t) \Psi^\beta(x, t)$$

and let

$$s_\beta^\alpha(t) = \int_{\mathfrak{c}_{n+\beta}} \Phi^\alpha(x, t) , \qquad \text{for } 1 \leqq \beta \leqq n.$$

Then $t \to s(t) = (s_\beta^\alpha(t))$ is a differentiable map of U into S, $s(o) = s$, and

$$\int_{\mathfrak{c}_j} \Phi^\alpha(x, t) = \omega_j^\alpha(s(t)) , \qquad \text{for } 1 \leqq j \leqq 2n.$$

Letting

$$z^\alpha(x, t) = \int_{x_0}^x \Phi^\alpha(x, t) ,$$

where x_0 is the origin of the given system of complex coordinates z^α of $W_o = B_s$, we obtain therefore a differentiable map

$$h : (x, t) \longrightarrow (z^1(x, t), \cdots, z^\alpha(x, t), \cdots, z^n(x, t), s(t)) \bmod G$$

of $\mathscr{W}|U = X \times U$ into $\mathscr{B} = (C^n \times S)/G$ which maps each fibre $W_t \subset \mathscr{W}|U$ holomorphically into $B_{s(t)}$. Moreover, since $\Phi^\alpha(x, o) = dz^\alpha$, the restrictiction $h|W_o$ coincides with the inclusion map : $W_o = B_s \to B_s \subset \mathscr{B}$. Thus $h|W_o$ is biregular and therefore h maps each fibre $W_t \subset \mathscr{W}|U$ biregularly *onto* $B_{s(t)}$, provided that U is sufficiently small. Consequently $\mathscr{W}|U$ is a family induced from $\mathscr{B}$. Thus we infer that $\mathscr{B}$ is complete. Combining this with (14.13) we obtain the following

THEOREM 14.3. *The complex analytic family $\mathscr{B} = \{B_s \mid s \in S\}$ of complex tori B_s is complete and effectively parametrized. The number of moduli $m(B_s)$ of any complex torus B_s of complex dimension n is equal to*

n^2 and the equality

$$m(B_s) = \dim H^1(B_s, \Theta_s)$$

holds.

We constructed the family $\mathscr{B}$ from complex tori whose period matrices $\omega = (\omega_j^\alpha)$ are normalized in such a way that $\omega_j^\alpha = \delta_j^\alpha (1 \leqq j \leqq n)$. Our next remark is that, by applying a similar procedure to complex tori with *arbitrary* period matrices, we obtain an effectively parametrized complete complex analytic family $\mathscr{C}$ of complex tori which is larger than $\mathscr{B}$. (Cf. Calabi [8]; Calabi's treatment has been simplified by R. Bott (unpublished). See also Blanchard [6, pp. 14 and 28] where a complex analytic family of 2-tori over $P_1(\boldsymbol{C})$ is constructed. Blanchard's family is the restriction of $\mathscr{C}$ (in the case $n = 2$) to a curve $P_1(\boldsymbol{C})$ in the base space.)

In fact, let L be the space of all $n \times 2n$ matrices $\omega = (\omega_j^\alpha)$ with $(\sqrt{-1})^n \det \left(\frac{\omega}{\bar{\omega}}\right) > 0$ and let G be the discontinuous abelian group of analytic automorphisms of $\boldsymbol{C}^n \times L$ generated by

$$g_j : (z, \omega) \longrightarrow (z + \omega_j, \omega) , \qquad j = 1, 2, \cdots , 2n.$$

Then $\mathfrak{L} = (\boldsymbol{C}^n \times L)/G$ is a complex analytic manifold and the canonical projection $\boldsymbol{C}^n \times L \rightarrow L$ induces a regular map $\varpi : \mathfrak{L} \rightarrow L$ such that $\varpi^{-1}(\omega)$ is a complex torus with the period matrix ω. Thus $\mathfrak{L}$ forms a complex analytic family over L, but this $\mathfrak{L}$ is not effectively parametrized. Now let $\mathfrak{G} = GL(n, \boldsymbol{C})$. Clearly $\mathfrak{G}$ acts on $\boldsymbol{C}^n$ and L from the left in a canonical manner and $M = L/\mathfrak{G}$ is a complex manifold. We define the action of $\tau \in \mathfrak{G}$ on $\boldsymbol{C}^n \times L$ by

$$\tau : (z, \omega) \longrightarrow (z\tau, \tau\omega) .$$

Since this τ commutes with g_j, τ induces an analytic automorphism of $\mathfrak{L} = (\boldsymbol{C}^n \times L)/G$ which we denote by the same symbol τ. Thus $\mathfrak{G}$ becomes a group of analytic automorphisms acting on $\mathfrak{L}$ and the factor space

$$\mathscr{C} = \mathfrak{L}/\mathfrak{G}$$

is a complex analytic fibre space over $M = L/\mathfrak{G}$ whose fibres are complex tori, where the canonical projection $\varpi : \mathscr{C} \rightarrow M$ is induced from $\varpi : \mathfrak{L} \rightarrow L$. We note that

$$\mathscr{E} = (\boldsymbol{C}^n \times L)/\mathfrak{G}$$

is a complex vector bundle over $M = L/\mathfrak{G}$ and $g_j : \boldsymbol{C}^n \times L \rightarrow \boldsymbol{C}^n \times L$ induces a fibre-preserving analytic automorphism of $\mathscr{E}$ which we denote by the same symbol g_j. Thus G can be considered as a discontinuous group of fibre-preserving analytic automorphisms of $\mathscr{E}$ and we obtain

$$\mathscr{C} = \mathscr{E}/G .$$

It is easy to see that M is covered by $\binom{2n}{n}$ coordinate neighborhoods $S_{j_1 j_2 \cdots j_n}$, $1 \leqq j_1 < \cdots < j_n \leqq 2n$, which are analytically homeomorphic to S and the restrictions $\mathscr{C} | S_{j_1 j_2 \cdots j_n}$ are analytically equivalent to the complex analytic family $\mathscr{B}$ over S. In fact, L is covered by $\binom{2n}{n}$ open subsets

$$L_{j_1 \cdots j_n} = \{\omega \mid \det (\omega^{\alpha}_{j_\beta})_{\alpha, \beta = 1, \cdots, n} \neq 0\}$$

which are $\mathfrak{G}$-invariant, and therefore M is covered by

$$S_{j_1 \cdots j_n} = L_{j_1 \cdots j_n}/\mathfrak{G} .$$

We consider one of $L_{j_1 \cdots j_n}$, say $L_{12 \cdots n}$. It is clear that $(s, \tau) \to \tau\omega(s)$ is a biregular map of $S \times \mathfrak{G}$ onto $L_{12 \cdots n}$. Hence $S_{12 \cdots n} = L_{12 \cdots n}/\mathfrak{G}$ is analytically homeomorphic to S. Moreover $(z, s, \tau) \to (\tau z, \tau\omega(s))$ is a biregular map of $\boldsymbol{C}^n \times S \times \mathfrak{G}$ onto $\boldsymbol{C}^n \times L_{12 \cdots n}$ and the diagram

$$\begin{array}{ccc} (z,\ s,\ \tau) & \longrightarrow & (\tau z,\ \tau\omega(s)) \\ {\scriptstyle g_j}\downarrow & & \downarrow{\scriptstyle g_j} \\ (z + \omega_j(s),\ s,\ \tau) & \longrightarrow & (\tau z + \tau\omega_j(s),\ \tau\omega(s)) \end{array}$$

is commutative. Hence this map induces a biregular map of $\mathscr{B} \times \mathfrak{G}$ onto $\mathfrak{L} | L_{12 \cdots n}$ and therefore $\mathscr{C} | S_{12 \cdots n} = (\mathfrak{L} | L_{12 \cdots n})/\mathfrak{G}$ is analytically equivalent to $\mathscr{B}$. Thus $\mathscr{C}$ is covered by $\binom{2n}{n}$ open subfamilies

$$\mathscr{C} | S_{j_1 \cdots j_n}$$

each of which is analytically equivalent to the complete family $\mathscr{B}$. Hence $\mathscr{C}$ *is complete*. We denote a point on M by t and the corresponding fibre $\varpi^{-1}(t)$ by C_t. Moreover we identify S with $S_{12 \cdots n}$ and $\mathscr{B}$ with $\mathscr{C} | S_{12 \cdots n}$.

Now let $\Gamma = SL(2n, \boldsymbol{R})$ and let γ be an arbitrary element of Γ. The linear transformation $\omega \to \omega\gamma$ of L induces an analytic automorphism $t \to t\gamma$ of $M = L/\mathfrak{G}$. In case s and $s\gamma$ both belong to $S = S_{12 \cdots n} \subset M$, the explicit form of $s\gamma$ is given by

$$s\gamma = (\gamma_{11} + s\gamma_{21})^{-1}(\gamma_{12} + s\gamma_{22}) , \qquad \text{where } \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} .$$

Clearly Γ acts transitively on M. Hence, denoting by Γ_t the subgroup of Γ consisting of all γ such that $t\gamma = t$, we have

$$M = \Gamma/\Gamma_t .$$

Let $\Delta = SL(2n, \boldsymbol{Z})$. Δ is a discrete subgroup of Γ of the first kind (a theorem of Minkowski; cf. Siegel [35], p. 675). It is clear that *two fibres C_t and C_u of $\mathscr{C}$ are analytically homeomorphic if and only if $u = t\delta$, $\delta \in \Delta$.*

The case $n = 1$. It is well known that Δ is a *discontinuous* group of automorphisms of M and the factor space M/Δ, which is analytically homeomorphic to $\boldsymbol{C}$, is the *space of moduli* of elliptic curves in the classical sense.

The case $n \geqq 2$. A general theorem of Siegel [35] tells us that Δ is *not* a discontinuous group of automorphisms of M. Moreover a slight modification of Siegel's proof of the theorem leads to the following result: *Every non-empty open subset U of M contains a point t such that $t\Delta \cap U$ is an infinite set of points.* In fact this can be proved as follows: Take a point $u \in U$ and a small "spherical" neighborhood N of 1 on Γ such that $uN \subset U$. Since the subgroup Γ_u is *not* compact, we can find an infinite sequence $\{\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots\}$ of elements $\gamma_n \in \Gamma_u$ such that $\gamma_n N \cap \gamma_m N$ is empty for $n \neq m$. Let $P_n = \gamma_n N\Delta$, $Q_n = P_n \cup P_{n+1} \cup P_{n+2} \cup \cdots$, and let $Q = \cap_n Q_n$. Then, denoting by μ the right invariant Haar measure on Γ, we have $\mu(Q) > 0$, and, for each $\kappa \in Q$, there exist an infinite subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ and a sequence $\{\delta_k\}$, $\delta_k \in \Delta$, such that $\kappa\delta_k \in \gamma_{n_k} N$ (see Siegel [35], p. 681). It follows that $\delta_k \neq \delta_h$ for $k \neq h$ and that $u\kappa\delta_k \in uN \subset U$. Hence, if $u\kappa\Delta \cap U$ is a finite set of points, $u\kappa$ satisfies $u\kappa\delta = u\kappa$ for infinitely many elements $\delta = \delta_k\delta_h^{-1}$ of Δ. It suffices therefore to verify that, for each element $\delta \neq \pm 1$ of Δ, the subset $\Sigma_\delta = \{\gamma \mid u\gamma\delta = u\gamma\}$ of Γ has the measure $\mu(\Sigma_\delta) = 0$. Let $M_\delta = \{t \mid t\delta = t\}$. $S \cap M\delta$ is the *proper* subvariety of S determined by the equation

$$s\delta_{21}s + \delta_{11}s - s\delta_{22} - \delta_{12} = 0 \ ,$$

provided that $\delta \neq \pm 1$. Hence $M\delta$ is a bunch of proper subvarieties of M and therefore $\Sigma_\delta = \{\gamma \mid u\gamma \in M_\delta\}$ is also a bunch of proper subvarieties of Γ. Consequently we obtain $\mu(\Sigma_\delta) = 0$.

The above result shows that the factor space M/Δ is not a Hausdorff space; thus the natural construction of the space of moduli of complex tori fails. Moreover, *for any non-empty open subset U of M, it is impossible to find any differentiable family consisting of all biregularly distinct complex tori belonging to $\mathscr{C} \mid U$.* In fact, let $\mathscr{V} \xrightarrow{\varpi} \Lambda$ be a differentiable family of biregularly distinct complex tori. For each point $\lambda \in \Lambda$ there exists a neighborhood N_λ of λ on Λ such that $\mathscr{V} \mid N_\lambda$ is induced from $\mathscr{C}$ by a differentiable map $f_\lambda : N_\lambda \to M$. Since Λ is connected, paracompact and locally compact, Λ is covered by a countable sequence of compact sets $K_1, K_2, \cdots, K_j, \cdots$, each of which is contained in one of N_λ, say N_{λ_j}. Let $f_j = f_{\lambda_j}$. Then $f_j(\lambda') \notin f_j(\lambda'')\Delta$ for $\lambda' \neq \lambda''$, since, by hypothesis, $\varpi^{-1}(\lambda')$ and $\varpi^{-1}(\lambda'')$ are biregularly distinct. Hence, by the above result, the compact subset $f_j(K_j)$ contains no open subset of M, and consequently $\cup_j f_j(K_j)\Delta \not\supset U$. This proves that there exists a fibre C_t, $t \in U$, which is biregularly equivalent to no fibre of $\mathscr{V}$.

This result indicates that the number of moduli of complex tori cannot be defined as the dimension of the parameter manifold of the family of all biregularly distinct complex tori (compare Section 11).

(δ) *Hypersurfaces on abelian varieties.* Let $\mathscr{B} = \{B_s | s \in S\}$ be the complex analytic family of complex tori introduced in (γ) above and let H be the space of $n \times n$ *symmetric* matrices $w = (w_{\alpha\beta})$ whose imaginary parts $\Im w$ are *positive definite*. Given a diagonal matrix

$$\Delta = \begin{pmatrix} e_1 & & & & 0 \\ & \ddots & & & \\ & & e_\alpha & & \\ & & & \ddots & \\ 0 & & & & e_n \end{pmatrix}$$

whose diagonal elements e_α are *positive* integers, such that $e_{\alpha+1} \equiv 0\ (e_\alpha)$, we consider the subfamily $\mathscr{A}_\Delta = \mathscr{B} \,|\Delta^{-1}H$ of $\mathscr{B}$, where $\Delta^{-1}H$ denotes the subspace of S consisting of all matrices of the form $\Delta^{-1}w$, $w \in H$. Letting $A_w = B_{\Delta^{-1}w}$, we have

$$\mathscr{A}_\Delta = \{A_w | w \in H\} .$$

We note that the period matrix $\omega(A^{-1}w)$ of $A_w \in \mathscr{A}_\Delta$ is a *Riemann matrix* written in a normal form. Thus every member A_w of $\mathscr{A}_\Delta$ is an *abelian variety.*

Obviously we have

$$\mathscr{A}_\Delta = (\boldsymbol{C}^n \times H)/G_\Delta ,$$

where G_Δ is the discontinuous abelian group of analytic automorphisms of $\boldsymbol{C}^n \times H$ generated by

$$g_j : (z, w) \longrightarrow (z + \omega_j(\Delta^{-1}w), w) , \qquad j = 1, \cdots, 2n.$$

We extend each automorphism g_j of $\boldsymbol{C}^n \times H$ to an automorphism $\hat{g}_j$ of $\boldsymbol{C}^n \times H \times \boldsymbol{C}$ which is defined by

$$\hat{g}_j : (z, w, \zeta) \longrightarrow (z + \omega_j(\Delta^{-1}w), w, f_j(z, w)\zeta),$$

where

$$f_j(z, w) = \begin{cases} 1\,, & \text{for } 1 \leqq j \leqq n, \\ e^{-2\pi i e_\alpha z^\alpha}\,, & \text{for } j = n+\alpha,\ 1 \leqq \alpha \leqq n. \end{cases} \tag{14.14}$$

Since

$$f_k(z + \omega_j(\Delta^{-1}w), w)\cdot f_j(z, w) = f_j(z + \omega_k(\Delta^{-1}w), w)\cdot f_k(z, w)\,,$$

we have $\hat{g}_j\hat{g}_k = \hat{g}_k\hat{g}_j$, and therefore $\hat{g}_1, \cdots, \hat{g}_j, \cdots, \hat{g}_{2n}$ generate a discontinuous group $\hat{G}_\Delta \cong G_\Delta$ of analytic automorphisms of $\boldsymbol{C}^n \times H \times \boldsymbol{C}$. Hence

$$\mathscr{E} = (\boldsymbol{C}^n \times H \times \boldsymbol{C})/\hat{G}_\Delta$$

is a complex manifold. Moreover the canonical projection $\boldsymbol{C}^n \times H \times \boldsymbol{C} \to \boldsymbol{C}^n \times H$ induces a regular map $\mathscr{E} \to \mathscr{A}_\Delta$ and thus $\mathscr{E}$ is a complex line bundle over $\mathscr{A}_\Delta$. Let E_w be the restriction of $\mathscr{E}$ to A_w. Then the *characteristic* class $c(E_w)$ of E_w is represented by the d-closed (1, 1)-form

$$\gamma = i\textstyle\sum_{\alpha,\beta=1}^n \gamma_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta\,, \qquad (\gamma_{\alpha\beta}) = \dfrac{1}{2}\Delta(\Im w)^{-1}\Delta. \tag{14.15}$$

This can be verified as follows : Let

$$a(z) = \exp\{-\pi\textstyle\sum_{\alpha,\beta}\gamma_{\alpha\beta}(z^\alpha - \bar{z}^\alpha)(z^\beta - \bar{z}^\beta - e_\beta^{-1}[w_{\beta\beta} - \bar{w}_{\beta\beta}])\}\,.$$

We infer readily that

$$\frac{a(z + \omega_j(\Delta^{-1}w))}{a(z)} = |f_j(z, w)|^2\,.$$

Hence the representative of $c(E_w)$ is given by

$$\frac{i}{2\pi}\partial\bar{\partial}\log a(z) = i\textstyle\sum\gamma_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta$$

(see Kodaira [24], Lemma 2.1). The above result shows that $c(E_w)$ *is positive*. The A_w, $w \in H$, may be regarded as complex structures defined on one and the same differentiable torus X whose real euclidean coordinates $(x^1, \cdots, x^i, \cdots, x^{2n})$ are related to the complex euclidean coordinates z^α on A_w by

$$z^\alpha \equiv \textstyle\sum_{j=1}^{2n} x^j\omega_j^\alpha(\Delta^{-1}w) \pmod{G_\Delta}\,.$$

It is obvious that

$$\gamma = \textstyle\sum_{\beta=1}^n e_\beta dx^\beta \wedge dx^{n+\beta}\,. \tag{14.16}$$

Thus the characteristic class $c(E_w)$ of E_w is independent of w, as it should be.

For each vector $a = (a_1, \cdots, a_\alpha, \cdots, a_n)$ whose components a_α are rational integers such that $0 \leqq a_\alpha \leqq e_\alpha - 1$, the *theta function* $\theta_a(z, w)$ is defined as follows :

$$\theta_a(z, w) = \sum_l c_a(l, w) \cdot \exp 2\pi i \sum_{\alpha=1}^{n}(l_\alpha e_\alpha + a_\alpha) z^\alpha ,$$

where Σ_l denotes the sum extended over all vectors $l = (l_1, \cdots, l_\alpha, \cdots, l_n)$, $l_\alpha \in \boldsymbol{Z}$, and where

$$c_a(l, w) = \exp \pi i \left\{ \sum_{\alpha,\beta=1}^{n} w_{\alpha\beta} l_\alpha \left(l_\beta + \frac{2a_\beta}{e_\beta} \right) - \sum_{\alpha=1}^{n} w_{\alpha\alpha} l_\alpha \right\}$$

(see Siegel [36], Chap. VII ; Krazer [30]). Then $\theta_a(z, w)$ is a holomorphic function on $\boldsymbol{C}^n \times H$ satisfying the functional equations

$$\theta_a(z + \omega_j(\Delta^{-1}w), w) = f_j(z, w) \cdot \theta_a(z, w) , \qquad j = 1, 2, \cdots, 2n, \tag{14.17}$$

i.e. $\theta_a(z, w)$ *is a holomorphic section of the complex line bundle* $\mathscr{E}$ *over* $\mathscr{A}_\Delta$. For each fixed w, $\theta_a(z, w)$ is therefore a holomorphic section of the complex line bundle E_w over A_w. Let $e = e_1 e_2 \cdots e_n$. Then *these* e *holomorphic sections* $\theta_a(\cdots, w)$, $0 \leqq a_\alpha \leqq e_\alpha - 1$, *are linearly independent on* A_w, *and form a base of* $H^0(A_w, \Omega(E_w))$ (see Siegel, loc. cit.). Thus an arbitrary element of $H^0(A_w, \Omega(E_w))$ is written in the form $\Sigma_a \lambda_a \theta_a(\cdots, w)$. We consider the coefficients $(\cdots, \lambda_a, \cdots)$ as the homogeneous coordinates of a point λ in a projective space P_{e-1} of dimension $e - 1$, and, for each point $t = (w, \lambda) \in H \times P_{e-1}$, we denote by V_t the divisor on A_w of the holomorphic section $\Sigma_a \lambda_a \theta_a(z, w)$ of E_w.

Now let M be the open subset of $H \times P_{e-1}$ consisting of all points t for which the V_t are non-singular prime divisors, i.e. irreducible non-singular subvarieties of A_w of codimension 1, and let $\mathscr{A}_\Delta \times P_{e-1}|M$ be the open subset $\cup_{(w,\lambda)\in M} A_w \times \lambda$ of $\mathscr{A}_\Delta \times P_{e-1}$. Then the holomorphic equation

$$\sum_a \lambda_a \theta_a(z, w) = 0$$

defines a non-singular submanifold $\mathscr{V}$ of $\mathscr{A}_\Delta \times P_{e-1}|M$ of codimension 1, and moreover, letting $\varpi : \mathscr{V} \to M$ be the regular map induced by the canonical projection $\mathscr{A}_\Delta \times P_{e-1} \to H \times P_{e-1}$, we have $\varpi^{-1}(t) = V_t$. Thus $\mathscr{V} = \{V_t \mid t \in M\}$ is a complex analytic family, provided that M is not empty. We note that M *is not empty* in general. In fact, if, for examle, $e_1 \geqq 2$, then each complete linear system $|E_w| = \{V_{(w,\lambda)} \mid \lambda \in P_{e-1}\}$ has neither fixed component nor base point and therefore general members $V_{(w,\lambda)}$ of $|E_w|$ are non-singular prime divisors. Thus, in this case, $w \times P_{e-1} - M$ is a bunch of proper subvarieties of $w \times P_{e-1}$ for each point $w \in H$. In what follows we assume that $n \geqq 3$.

Now we show that the complex analytic family $\mathscr{V} = \{V_t \mid t \in M\}$ is effectively parametrized. Let Ξ_w be the sheaf of germs of holomorphic sections of the tangent bundle of A_w and let $r_t(\Xi_w)$ be the restriction of Ξ_w to $V_t \subset A_w$. Clearly the sheaf Θ_t of germs of holomorphic sections of the tangent bundle of V_t is a subsheaf of $r_t(\Xi_w)$. Let $\iota_t : \Theta_t \to r_t(\Xi_w)$ be the inclusion map. Moreover let $p: M \to H$ be the canonical projection. Then we have the commutative diagram

$$\begin{array}{ccc} H^1(V_t, \Theta_t) & \xrightarrow{\iota_t^*} & H^1(V_t, r_t(\Xi_w)) \\ \uparrow \rho_t & & \uparrow r_t^* \\ & & H^1(A_w, \Xi_w) \\ & & \nearrow \rho_w \\ (T_M)_t & \xrightarrow{p} & (T_H)_w \end{array}$$

where $w = p(t)$. Denote by Ω_w the sheaf over A_w of germs of holomorphic functions. We have

$$\Xi_w \cong \Omega_w \oplus \Omega_w \oplus \cdots \oplus \Omega_w \ (n \text{ terms})$$

while $c(E_w) > 0$. Hence we get

$$H^q(A_w, \Omega(E_w)) = 0 , \qquad \text{for } q \geqq 1,$$
$$H^q(A_w, \Xi_w \otimes E_w^{-1}) = 0 , \qquad \text{for } q \leqq n-1.$$

We have $[V_t] = E_w$. Hence the restriction $E_t = r_t(E_w)$ of E_w to V_t coincides with the canonical bundle on V_t and therefore the first Chern class $c_1 = -c(E_t)$ of V_t is negative. Hence, by Nakano's theorem, we get

$$H^0(V_t, \Theta_t) = 0 .$$

Combining these results with the diagrams (12.17) and (12.4) we obtain the exact commutative diagram (14.18), which is shown on the following page, where $w = p(t)$ and $(T_P)_t$ denotes the subspace of $(T_M)_t$ consisting of tangent vectors along the subspace $w \times P_{e-1} \cap M$ and where μ^* is a special case of the map μ_t^* in (12.17). We omit here the subscript t for simplicity.

LEMMA 14.3. *Let $u \in (T_P)_t$ and let $\xi \in H^0(A_w, \Xi_w)$. If $\rho_{d,t}(u) = \mu^* r_t^*(\xi)$, then we have $\xi = 0$ and $u = 0$.*

PROOF (cf. Kodaira [22], §11). Let $t = (w, \lambda)$, $\lambda = (\lambda_0, \cdots, \lambda_a, \cdots)$, where at least one component, say λ_0, is not equal to zero. We write u in the form

$$u = \sum_{a \neq 0} u_a \frac{\partial}{\partial \lambda_a}$$

in terms of the inhomogeneous coordinates $(1, \cdots, \lambda_a, \cdots)$ of P_{e-1}. Now V_t is defined by the equation

$$\sum_a \lambda_a \theta_a(z, w) = 0 .$$

(14.18)
$$\begin{tikzcd}[column sep=small]
 & & & & & H^2(A_w, \Omega_w) & \\
0 \arrow[r] & H^0(V_t, r_t(\Xi_w)) \arrow[r, "\mu^*"] & H^0(V_t, \Omega(E_t)) \arrow[r, "\delta^*"] & H^1(V_t, \Theta_t) \arrow[r, "\iota_t^*"] & H^1(V_t, r_t(\Xi_w)) \arrow[r, "\mu^*"] & H^1(V_t, \Omega(E_t)) \arrow[r] \arrow[u, "\delta^*"', "\cong"] & \cdots \\
 & H^0(A_w, \Xi_w) \arrow[u, "r_t^*"', "\cong"] & & & H^1(A_w, \Xi_w) \arrow[u, "r_t^*"', "\cong"] & & \\
 & 0 \arrow[r] & (T_P)_t \arrow[r] \arrow[uu, "\rho_{d,t}"] & (T_M)_t \arrow[r, "p"] \arrow[uu, "\rho_t"] & (T_H)_w \arrow[r] \arrow[u, "\rho_w"] & 0 &
\end{tikzcd}$$

We have, by our definition of $\rho_{d,t}$ (Section 12),

(14.19) $$\rho_{d,t}(u) = -\sum_{a\neq 0} u_a r_t(\theta_a) ,$$

where $r_t(\theta_a)$ denotes the restriction to V_t of the holomorphic section $\theta_a = \theta_a(z, w)$ of E. Let $\xi = \sum_{\alpha=1}^{n} \xi^\alpha\, \partial/\partial z^\alpha$. It follows from (14.17) that

(14.20) $$\sum_a \lambda_a \xi\theta_a(z + \omega_j, w) = f_j \cdot \sum_a \lambda_a \xi\theta_a(z, w) + \xi f_j \cdot \sum_a \lambda_a \theta_a(z, w) ,$$

where $\omega_j = \omega_j(\Delta^{-1}w)$ and $f_j = f_j(z, w)$. We have therefore

$$r_t(\textstyle\sum_a \lambda_a \xi\theta_a(z + \omega_j, w)) = r_t f_j \cdot r_t(\sum_a \lambda_a \xi\theta_a(z, w)) .$$

This shows that $r_t(\sum_a \lambda_a \xi\theta_a)$ is a holomorphic section of E_t over V_t. By our definition of the map μ^*, we have

$$\mu^* r_t^*(\xi) = r_t(\textstyle\sum_a \lambda_a \xi\theta_a) .$$

Our hypothesis $\rho_{d,t}(u) = \mu^* r_t^*(\xi)$ implies therefore that

$$\textstyle\sum_a \lambda_a \xi\theta_a(z, w) + \sum_{a\neq 0} u_a\theta_a(z, w) = 0 , \qquad \text{for } z \in V_t.$$

It follows that

$$h(z) = \frac{\sum_a \lambda_a \xi\theta_a(z, w) + \sum_{a\neq 0} u_a\theta_a(z, w)}{\sum_a \lambda_a\theta_a(z, w)} \tag{14.21}$$

is a holomorphic function on $C^n \times w$. Moreover we infer from (14.17) and (14.20) that $h(z)$ satisfies the functional equation

$$h(z + \omega_j(\Delta^{-1}w)) = h(z) + \frac{\xi f_j(z, w)}{f_j(z, w)} ,$$

where, by (14.14),

$$\frac{\xi f_j(z, w)}{f_j(z, w)} = \begin{cases} 0 , & \text{for } 1 \leqq j \leqq n, \\ -2\pi i e_\beta \xi^\beta , & \text{for } j = n + \beta,\ 1 \leqq \beta \leqq n. \end{cases}$$

Thus $h(z)$ is an additive holomorphic function on A_w and hence

$$h(z) = h_0 + \textstyle\sum_{\alpha=1}^n h_\alpha \cdot z^\alpha .$$

We have therefore

$$\textstyle\sum_{\alpha=1}^n h_\alpha \omega_j^\alpha(\Delta^{-1}w) = \begin{cases} 0 , & \text{for } 1 \leqq j \leqq n, \\ -2\pi i e_\beta \xi^\beta , & \text{for } j = n + \beta,\ 1 \leqq \beta \leqq n. \end{cases}$$

It follows that $h_\alpha = 0$ and $\xi^\beta = 0$. Hence, by (14.21),

$$h_0\theta_0(z, w) + \textstyle\sum_{a=0}(h_0\lambda_a - u_a)\theta_a(z, w) = 0$$

and therefore $h_0 = 0$, $u_a = 0$, q.e.d.

Now it is easy to see that $\rho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is injective. Suppose that $\rho_t(u) = 0$ for an element $u \in (T_M)_t$. Then we infer from (14.18) that $\rho_w p(u) = 0$, while, as was shown in (γ) above, ρ_w is injective. Hence $p(u) = 0$ and therefore $u \in (T_P)_t$. Moreover $\delta^*\rho_{d,t}(u) = \rho_t(u) = 0$ and therefore there exists $\xi \in H^0(A_w, \Xi_w)$ such that $\rho_{d,t}(u) = \mu^* r_t^*(\xi)$. Consequently, by the above lemma, we obtain $u = 0$. Thus $\rho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is injective, i.e. $\mathscr{V} = \{V_t \mid t \in M\}$ is effectively parametrized.

We note that

(14.22) $$H^0(V_t, \Omega(E_t)) = \rho_{d,t}(T_P)_t \oplus \mu^* H^0(V_t, r_t(\Xi_w)) .$$

In fact the above Lemma 14.3 shows that $\rho_{d,t}(T_P)_t$ and $\mu^* H^0(V_t, r_t(\Xi_w)$ are linearly independent and that

$$\dim \rho_{d,t}(T_P)_t = \dim (T_P)_t = e - 1 ,$$
$$\dim \mu^* H^0(V_t, r_t(\Xi_w)) = \dim H^0(A_w, \Xi_w) = n .$$

On the other hand, since $H^1(A_w, \Omega(E_w)) = 0$, we have

$$0 \to H^0(A_w, \Omega_w) \to H^0(A_w, \Omega(E_w)) \to H^0(V_t, \Omega(E_t)) \to H^1(A_w, \Omega_w) \to 0 .$$

It follows that

$$\dim H^0(V_t, \Omega(E_t)) = e - 1 + n .$$

Hence we obtain (14.22).

Next, we prove that $\rho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is surjective. In view of (14.18), it suffices for this purpose to show that the sequence

(14.23) $$0 \to (T_H)_w \xrightarrow{\rho_w} H^1(A_w, \Xi_w) \xrightarrow{\delta^* \mu^* r_t^*} H^2(A_w, \Omega_w) \to 0$$

is exact. In fact, given an arbitrary element $\theta \in H^1(V_t, \Theta_t)$, we infer from (14.18) that there exists $v \in (T_H)_w$ satisfying $r_t^* \rho_w(v) = \iota_t^*(\theta)$, since (14.23) implies the exactness of the sequence

$$0 \to (T_H)_w \xrightarrow{r_t^* \rho_w} H^1(V_t, r_t(\Xi_w)) \xrightarrow{\mu^*} H^1(V_t, \Omega(E_t)) \to 0 .$$

Take $u \in (T_M)_t$ such that $p(u) = v$. Then, by (14.18), we have $\iota_t^* \rho_t(u) = \iota_t^*(\theta)$ and therefore

$$\theta - \rho_t(u) \in \delta^* H^0(V_t, \Omega(E_t)) ,$$

while, by (14.22), $\delta^* H^0(V_t, \Omega(E_t)) = \delta^* \rho_{d,t}(T_P)_t$. Hence there exists $u_1 \in (T_P)_t \subset (T_M)_t$ such that

$$\theta - \rho_t(u) = \delta^* \rho_{d,t}(u_1) = \rho_t(u_1) .$$

Consequently we obtain $\theta = \rho_t(u + u_1) \in \rho_t(T_M)_t$.

Let $H_w^{0,1}(\Xi_w)$ be the linear space of vector (0, 1)-forms on A_w with con stant coefficients and let $H_w^{0,2}$ be the linear space of (0, 2)-forms on A with constant coefficients. We note that $H^1(A_w, \Xi_w) = H_w^{0,1}(\Xi_w)$, $H^2(A_w, \Omega_w = H_w^{0,2}$.

LEMMA 14.4. *The homomorphism* $\delta^* \mu^* r_t^* : H_w^{0,1}(\Xi_w) \to H_w^{0,2}$ *is given by*

$$\delta^* \mu^* r_t^* : c \cdot d\bar{z} \to \pi \sum_{\alpha,\beta} \sum_{\nu=1}^{n} (c_\alpha^\nu \gamma_{\nu\beta} - \gamma_{\alpha\nu} c_\beta^\nu) d\bar{z}^\alpha \wedge d\bar{z}^\beta ,$$

where $c = (c_\beta^\alpha)$ *and* $(\gamma_{\alpha\beta}) = \dfrac{1}{2} \Delta (\Im w)^{-1} \Delta$.

PROOF. Let $\{U_i\}$ be a finite covering of A_w by coordinate neighbor hoods U_i and let $(z_i^1, \cdots, z_i^\alpha, \cdots, z_i^n)$ be the system of local coordinates on U_i such that

$$z_i^\alpha = z_k^\alpha + m_{ik}^\alpha + \sum_{\beta=1}^n m_{ik}^{n+\beta} e_\alpha^{-1} w_\beta^\alpha$$

(see (γ) above). Letting

$$\psi_i^\alpha = \sum_{\beta=1}^n c_\beta^\alpha (\bar{z}_i^\beta - z_i^\beta) ,$$

we have $\bar{\partial}\psi_i^\alpha = \sum_\beta c_\beta^\alpha d\bar{z}^\beta$. Hence $\sum_\beta c_\beta^\alpha d\bar{z}^\beta \in H_w^{0,1}(\Xi_w) = H^1(A_w, \Xi_w)$ is represented by the 1-cocycle $\xi = (\xi_{ik}^\alpha)$, $\xi_{ik}^\alpha = \psi_k^\alpha - \psi_i^\alpha$. The divisor V_t of A_w is determined in each U_i by the holomorphic function

$$R_i(z) = \sum_a \lambda_a \theta_a(z, w) ,$$

where $t = (w, \lambda)$, and thus the complex line bundle E_w on A_w is determined by the system $\{f_{ik}(z)\}$, $f_{ik}(z) = R_i(z)/R_k(z)$. In terms of the corresponding fibre coordinate ζ_i on $E_w | U_i$, we denote holomorphic sections

$$z \longrightarrow (z, \zeta_i(z))$$

of $E_w | U_i$ simply by $\zeta_i(z)$. By our definition of the homomorphism μ^*, $\mu^* r_t^*(\xi)$ is represented by the 1-cocycle

$$r_t\left(\sum_{\alpha=1}^n \xi_{ik}^\alpha \frac{\partial R_i}{\partial z^\alpha}\right)$$

which is the restriction to V_t of the 1-cochain

$$\sum_\alpha \xi_{ik}^\alpha \frac{\partial R_i}{\partial z^\alpha}$$

on the nerve of the covering $\{U_i\}$ with coefficients in $\Omega(E_w)$, where

$$\sum \xi_{ik}^\alpha \frac{\partial R_i}{\partial z^\alpha}$$

refers to the fibre coordinate ζ_i. In the exact sequence

$$0 \longrightarrow \Omega_w \longrightarrow \Omega(E_w) \xrightarrow{r_t} \Omega(E_t) \longrightarrow 0$$

the map $\Omega_w \to \Omega(E_w)$ has the form $\varphi \to R_i(z)\varphi$ for $z \in U_i$. Hence, by the definition of δ^*, $\eta = \delta^* \mu^* r_t^*(\xi)$ is represented by the 2-cocycle

$$\eta_{ikl} = \sum_{\alpha=1}^n (\xi_{ik}^\alpha R_i^{-1} \partial_\alpha R_i + \xi_{kl}^\alpha R_k^{-1} \partial_\alpha R_k + \xi_{li}^\alpha R_l^{-1} \partial_\alpha R_l)$$

where $\partial_\alpha = \partial / \partial z^\alpha$. Since $\xi_{ik}^\alpha = \psi_k^\alpha - \psi_i^\alpha$, we have

$$\eta_{ikl} = \tau_{ik} + \tau_{kl} + \tau_{li}, \quad \tau_{ik} = \sum_\alpha \psi_k^\alpha \cdot (\partial_\alpha \log R_i - \partial_\alpha \log R_k) .$$

Hence η is represented by the 1-cocycle

$$\sigma_{ik} = \bar{\partial}\tau_{ik} = \sum_\alpha \bar{\partial}\psi^\alpha \cdot (\partial_\alpha \log R_i - \partial_\alpha \log R_k)$$

on the nerve of the covering $\{U_i\}$ with coefficients in the sheaf of germs of $\bar\partial$-closed (0, 1)-forms, where $\bar\partial\psi^\alpha = \bar\partial\psi^\alpha_k$ is independent of k. By (14.17), we have

$$\partial_\alpha \log \theta_a(z+\omega_j, w) - \partial_\alpha \log \theta_a(z, w) = \begin{cases} 0, & 1 \leqq j \leqq n, \\ -2\pi i e_\beta \delta^\alpha_\beta, & j = n+\beta, 1 \leqq \beta \leqq n, \end{cases}$$

where $\omega_j = \omega_j(\Delta^{-1}w)$. It follows that

$$\partial_\alpha \log R_i(z) - \partial_\alpha \log R_k(z) = -2\pi i m^{n+\alpha}_{ik} e_\alpha ,$$

while, letting

$$\varphi_{i\alpha} = 2\pi \textstyle\sum_\beta \gamma_{\alpha\beta}(\bar z^\beta_i - z^\beta_i) ,$$

we have

$$\varphi_{i\alpha} - \varphi_{k\alpha} = 2\pi \textstyle\sum_\nu \gamma_{\alpha\nu} e^{-1}_\nu (\bar w^\nu_\beta - w^\nu_\beta) m^{n+\beta}_{ik} = -2\pi i m^{n+\alpha}_{ik} e_\alpha .$$

Hence we get

$$\sigma_{ik} = \textstyle\sum_\alpha \bar\partial\psi^\alpha \cdot \varphi_{i\alpha} - \sum_\alpha \bar\partial\psi^\alpha \cdot \varphi_{k\alpha}$$

and therefore η is represented by the $\bar\partial$-closed (0, 2)-form

$$-\bar\partial(\textstyle\sum_\alpha \bar\partial\psi^\alpha \cdot \varphi_{i\alpha}) = \sum_\alpha \bar\partial\psi^\alpha \wedge \bar\partial\varphi_\alpha = \pi \sum_{\alpha,\beta}\sum_\nu (c^\nu_\alpha \gamma_{\nu\beta} - \gamma_{\alpha\nu} c^\nu_\beta) d\bar z^\alpha \wedge d\bar z^\beta ,$$

q.e.d.

Now it is easy to verify the exactness of the sequence (14.23). Let $v \in (T_H)_w$. By (14.11), we have

$$\rho_w(v) = \Delta^{-1} v (w - \bar w)^{-1} \Delta d\bar z . \tag{14.24}$$

Corresponding to this, an arbitrary element $c \cdot d\bar z$ of $H^{0,1}_w(\Xi_w)$ may be written in the form

$$c \cdot d\bar z = \Delta^{-1} b (w - \bar w)^{-1} \Delta d\bar z ,$$

where $b = (b_{\alpha\beta})$ is an $n \times n$ matrix. We have

$$c = \Delta^{-1} b (2i\Im w)^{-1} \Delta = -i \Delta^{-1} b \Delta^{-1} \gamma$$

and therefore

$$\pi \textstyle\sum_{\alpha,\beta,\nu} (c^\nu_\alpha \gamma_{\nu\beta} - \gamma_{\alpha\nu} c^\nu_\beta) d\bar z^\alpha \wedge d\bar z^\beta = i\pi \sum_{\alpha,\beta} (b_{\alpha\beta} - b_{\beta\alpha}) d\bar z'^\alpha \wedge d\bar z'^\beta ,$$

where

$$z'^\alpha = \textstyle\sum^n_{\beta=1} e^{-1}_\alpha \gamma_{\alpha\beta} z^\beta .$$

Combining this with the above Lemma 14.4, we infer that

$$\delta^* \mu^* r^*_t : H^1(A_w, \Xi_w) \longrightarrow H^2(A_w, \Omega_w)$$

is surjective and that $\Delta^{-1} b (w - \bar w)^{-1} \Delta d\bar z$ belongs to the kernel of $\delta^*\mu^* r^*_t$ if and only if b is symmetric, while, by (14. 24), $\rho_w : (T_H)_w \to H^1(A_w, \Xi_w)$

is injective and, since $(T_H)_w$ consists of all $n \times n$ symmetric matrices, $\Delta^{-1}b(w-\bar{w})^{-1}\Delta d\bar{z}$ belongs to $\rho_w(T_H)_w$ if and only if b is symmetric. Hence the sequence (14. 23) is exact. Thus we obtain

PROPOSITION 14.1. *We have the isomorphism*

$$\rho_t : (T_M)_t \cong H^1(V_t, \Theta_t) . \tag{14.25}$$

Our complex analytic family $\mathscr{V}$ is complete, as one expects in view of the above Proposition 14.1. In fact, the completeness of $\mathscr{V}$ can be verified as follows: Let V_t be any member of $\mathscr{V}$. By our construction, V_t is a hypersurface on A_w, $w=p(t)$, and the complex line bundle $E_w = [V_t]$ has the characteristic class $c(E_w) > 0$. It follows that the restrictions $\Phi_t^\alpha = r_t(dz^\alpha)$ of dz^α, $\alpha = 1, 2, \cdots, n$, form a base of holomorphic 1-forms on V_t. Clearly the inclusion map $\iota_t : V_t \to A_w$ is written in the form

$$\iota_t : y \longrightarrow \iota_t(y) \equiv \left(\int_{y_0}^{y}\Phi_t^1, \cdots, \int_{y_0}^{y}\Phi_t^\alpha, \cdots, \int_{y_0}^{y}\Phi_t^n\right) \bmod \omega(\Delta^{-1}w) ,$$

where y denotes a point on V_t. Moreover we can choose a Betti base $\{\mathfrak{z}_1, \cdots, \mathfrak{z}_j, \cdots, \mathfrak{z}_{2n}\}$ of 1-cycles on V_t such that

$$\int_{\mathfrak{z}_j}\Phi_t^\alpha = \omega_j^\alpha(\Delta^{-1}w)$$

(a theorem of Lefschetz). Now, letting $\mathscr{W} = \{W_\mu \mid \mu \in N\}$ be a differentiable family of deformations W_μ of $V_t = W_o$, we consider the restriction $\mathscr{W} \mid U$ of $\mathscr{W}$ to a small neighborhood U of o on N. $\mathscr{W} \mid U$ is differentially trivial, i.e.

$$\mathscr{W} \mid U = Y \times U ,$$

where Y is the underlying differentiable manifold on which the complex structures W_μ are defined.

(i) There exists a differentiable map $\mu \to s(\mu)$ of U into S and a differentiable map Λ of $\mathscr{W} \mid U$ into $\mathscr{B} = \{B_s \mid s \in S\}$ which maps each fibre W_μ biregularly into $B_{s(\mu)}$, provided that U is sufficiently small.

PROOF. By Theorem 3.1, W_μ is a Kähler manifold for $\mu \in U$. Hence the number of linearly independent holomorphic 1-forms on W_μ is independent of μ and therefore, by Theorem 2.2, (i), there exist n linearly independent holomorphic 1-forms $\Psi^\alpha(y, \mu)$, $\alpha = 1, 2, \cdots, n$, on W_μ which are differentiable in both variables y and μ, where we use (y, μ) as the differentiable coordinates of points on $\mathscr{W} \mid U = Y \times U$. Since

$$\int_{\mathfrak{z}_j}\Phi_t^\alpha = \delta_j^\alpha , \qquad \text{for } 1 \leqq j \leqq n,$$

the simultaneous linear equations

$$\sum_{\beta=1}^{n} \varepsilon_\beta^\alpha(\mu)\int_{\mathfrak{d}_j} \Psi^\beta(y, \mu) = \delta_j^\alpha , \qquad j = 1, \cdots, n,$$

have unique solutions $\varepsilon_\beta^\alpha(\mu)$ which depend differentiably on μ. Let

$$\Phi^\alpha(y, \mu) = \sum_{\beta=1}^{n} \varepsilon_\beta^\alpha(\mu)\Psi^\beta(y, \mu)$$

and let

$$s_\beta^\alpha(\mu) = \int_{\mathfrak{d}_{n+\beta}} \Phi^\alpha(y, \mu) , \qquad \text{for } 1 \leqq \beta \leqq n.$$

Clearly we have $\Phi^\gamma(\quad , o) = \Phi_t^\alpha$ and $\mu \to s(\mu) = (s_\beta^\alpha(\mu))$ is a differentiable map of U into S such that $s(o) = \Delta^{-1}w$. Letting

$$z^\alpha(y, \mu) = \int_{y_0}^{y} \Phi^\alpha(y, \mu) ,$$

we obtain therefore a differentiable map

$$\Lambda : (y, \mu) \to (z^1(y, \mu), \cdots, z^\alpha(y, \mu), \cdots, z^n(y, \mu), s(\mu)) \bmod G$$

of $\mathscr{W} | U = Y \times U$ into $\mathscr{B} = (\boldsymbol{C}^n \times S)/G$ which maps each fibre W_μ biregularly into $B_{s(\mu)}$. Clearly the image $\Lambda(W_\mu)$ is a non-singular prime divisor on $B_{s(\mu)}$ and $\Lambda(W_o) = \iota_t(V_t) = V_t \subset B_{s(o)} = A_w$.

(ii) $s(\mu) \in \Delta^{-1}H$.

PROOF. Let $X = \boldsymbol{R}^{2n}/\boldsymbol{Z}^{2n}$ be the underlying differentiable torus on which the complex tori $B_{s(\mu)}$ are defined. The euclidean complex coordinates (z^α) on $B_{s(\mu)}$ are related to the euclidean real coordinates (x^j) on X by

$$z^\alpha = x^\alpha + \sum_{\beta=1}^{n} x^{n+\beta} s_\beta^\alpha(\mu) . \tag{14.26}$$

Now let $F_\mu = [\Lambda(W_\mu)]$ be the complex line bundle over $B_{s(\mu)}$ determined by $\Lambda(W_\mu)$. Since $\Lambda(W_\mu) \sim \Lambda(W_o) = V_t$ on X, we have $c(F_\mu) = c(E_w)$. Hence, by (14.16), $c(F_\mu)$ is represented by the "harmonic" 2-form

$$\sum_{\beta=1}^{n} e_\beta dx^\beta \wedge dx^{n+\beta} ,$$

while this 2-form must be of type (1, 1) with respect to the complex structure $B_{s(\mu)}$. Thus we get

$$\sum_{\beta=1}^{n} e_\beta dx^\beta \wedge dx^{n+\beta} = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta ,$$

where $g_{\alpha\beta}$ are constants. Inserting (14.26) in the right hand side of this equality, we obtain

$$g_{\alpha\beta} - g_{\beta\alpha} = 0 ,$$

$$\sum_\nu \bar{s}_\alpha^\nu(\mu) g_{\beta\nu} - \sum_\nu s_\alpha^\nu(\mu) g_{\nu\beta} = \delta_{\alpha\beta} e_\beta ,$$

$$\sum_{\lambda,\nu} s_\alpha^\lambda(\mu) g_{\lambda\nu} \bar{s}_\beta^\nu(\mu) - \sum_{\lambda,\nu} s_\beta^\lambda(\mu) g_{\lambda\nu} \bar{s}_\alpha^\nu(\mu) = 0 ,$$

or, in terms of matrix notation,

$$g = g', \quad \bar{s}(\mu)'g' - s(\mu)'g = \Delta, \quad s(\mu)'g\bar{s}(\mu) = \bar{s}(\mu)'g's(\mu) ,$$

where′ denotes the transposed matrix. It follows that

$$\Delta s(\mu) = \bar{s}(\mu)'g's(\mu) - s(\mu)'gs(\mu)$$

is symmetric, while, since $\Im \Delta s(o) = \Im w > 0$, it is obvious that $\Im \Delta s(\mu) > 0$ for $\mu \in U$. Thus we infer that $\Delta s(\mu) \in H$, q.e.d.

(iii) Let $w(\mu) = \Delta s(\mu)$. Obviously $B_{s(\mu)} = A_{w(\mu)}$ and $c(F_\mu) = c(E_{w(\mu)})$. Hence there exists a translation $\tau_\mu : z^\alpha \to z^\alpha + \tau_\mu^\alpha$ acting on $A_{w(\mu)}$ such that $\tau_\mu^* E_{w(\mu)} = F_\mu$, where $\tau_\mu^* E_{w(\mu)}$ is the line bundle induced from $E_{w(\mu)}$ by the map $\tau_\mu : A_{w(\mu)} \to A_{w(\mu)}$. In fact, since E_w is the complex line bundle over A_w defined by the "multipliers" $f_j(z, w)$ (see (14.14)), $\tau^* E_w \otimes E_w^{-1}$ is defined by the multipliers

$$\frac{f_j(z + \tau, w)}{f_j(z, w)} = \begin{cases} 1 , & \text{for } 1 \leqq j \leqq n, \\ e^{-2\pi i e_\alpha \tau^\alpha} , & \text{for } j = n + \alpha,\ 1 \leqq \alpha \leqq n. \end{cases}$$

This shows that $\tau \to \tau^* E_w \otimes E_w^{-1}$ is a homomorphism of A_w *onto* its Picard variety $\mathfrak{P}_w$. It follows that $F_\mu = \tau_\mu^* E_{w(\mu)}$, since $F_\mu \otimes E_{w(\mu)}^{-1} \in \mathfrak{P}_{w(\mu)}$. Moreover the τ_μ^α depend differentiably on μ and $\tau_o^\alpha = 0$. Now we have $[\tau_\mu \Lambda(W_\mu)] = E_{w(\mu)}$ and therefore $\tau_\mu \Lambda(W_\mu)$ is a member of $\mathscr{V}$:

$$\tau_\mu \Lambda(W_\mu) = V_{t(\mu)} , \quad t(\mu) = (\lambda(\mu), w(\mu)) \in M .$$

Clearly $\mu \to t(\mu)$ is a differentiable map of U into M, $t(o) = t$; and

$$(y, \mu) \longrightarrow \tau_\mu \Lambda(y, \mu)$$

is a differentiable map of $\mathscr{W} | U = Y \times U$ into $\mathscr{V} \subset \mathscr{A}_\Delta \subset \mathscr{B}$ which maps each fibre W_μ biregularly onto $V_{t(\mu)}$. Thus $\mathscr{W} | U$ is a family induced from $\mathscr{V}$. Hence *$\mathscr{V}$ is complete.*

Now, since $\dim_C M = e - 1 + \frac{1}{2} n(n + 1)$, we obtain $m(V_t) = e - 1 + \frac{1}{2} n(n + 1)$. Replacing n by $n + 1$, we formulate our results in the following

THEOREM 14.4. *The complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ of non-singular hypersurfaces V_t of dimension n on abelian varieties of dimension $n + 1$ is effectively parametrized and complete. The number of moduli $m(V_t)$ of $V_t \in \mathscr{V}$ is given by*

(14.27) $$m(V_t) = e - 1 + \frac{1}{2}(n + 1)(n + 2) .$$

Moreover the equality

(14.28) $$m(V_t) = \dim H^1(V_t, \Theta_t)$$

holds.

The above formula (14.27) shows that our number of moduli $m(V_t)$ coincides with the classical one (see Andreotti [3], pp. 28-32). We note that, given any non-singular hypersurface V on an abelian variety A with $c([V]) > 0$, we can find a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ of the type described above containing a member $V_t \subset A_w$, $w = p(t)$, such that there is a biregular map of A onto A_w which maps $V \subset A$ onto $V_t \subset A_w$. Hence the above formulas (14.27) and (14.28) hold for V.

We conclude this section with a remark on Noether's formula. Noether verified his formula with respect to several examples of regular surfaces with ordinary singularities in 3-space $P_3(\boldsymbol{C})$. On the other hand, Noether's formula fails for non-singular surfaces of order > 5 in $P_3(\boldsymbol{C})$. In fact, letting V_t be a non-singular surface in $P_3(\boldsymbol{C})$ of order $h > 5$, we have, by Theorem 14.2,

$$m(V_t) = \dim H^1(V_t, \Theta_t) = \frac{1}{6}(h+1)(h+2)(h+3) - 16 ,$$

while

$$10(p_a + 1) - 2c_1^2 = \frac{10}{6}(h-1)(h-2)(h-3) + 10 - 2h(h-4)^2 .$$

Clearly we have

$$m(V_t) > 10(p_a + 1) - 2c_1^2 ;$$

moreover $10(p_a + 1) - 2c_1^2$ is negative for large h. Obviously $H^0(V_t, \Theta_t) = 0$ and therefore, by (11.7),

$$\dim H^2(V_t, \Theta_t) = \frac{1}{2}h^3 - 5h^2 + \frac{31}{2}h - 15 .$$

Now we consider a non-singular surface V_t on an abelian variety A_w of dimension 3 with $c(V_t) > 0$. This surface V_t is irregular and therefore does not satisfy Noether's hypothesis; but still it is interesting to compute $10(p_a + 1) - 2c_1^2$. By Theorem 14.4 we have

$$m(V_t) = \dim H^1(V_t, \Theta_t) = e + 5 ,$$

while $p_a = e - 1$, $c_1^2 = 6e$, and therefore

$$10(p_a + 1) - 2c_1^2 = -2e .$$

Hence we get

$$\dim H^2(V_t, \Theta_t) = 3e + 5 .$$

15. Deformations of Hopf manifolds

Let M be the complex manifold consisting of 2×2 matrices $t = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix}$ such that

$$(15.1) \qquad |\alpha + \delta| > 3, \quad |(\alpha - \delta)^2 + 4\beta\gamma| < 1 ,$$

and consider each matrix t as the linear transformation

$$t : z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \longrightarrow tz = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

acting on the space $\boldsymbol{C} \times \boldsymbol{C}$ of two complex variables (z^1, z^2). The eigenvalues of t are given by

$$\sigma \pm \sqrt{\Delta}, \quad \text{where } \sigma = \frac{1}{2}(\alpha + \delta),\ \Delta = \frac{1}{4}(\alpha - \delta)^2 + \beta\gamma .$$

The condition (15.1) implies that

$$(15.2) \qquad |\sigma \pm \sqrt{\Delta}| > 1 .$$

Now let $\boldsymbol{W} = \boldsymbol{C} \times \boldsymbol{C} - (0, 0)$ and define an analytic automorphism η of $\boldsymbol{W} \times M$ by

$$\eta : (z, t) \longrightarrow (tz, t)$$

Then it follows from (15.2) that the infinite cyclic group $G = \{\eta^m | m \in \boldsymbol{Z}\}$ is a properly discontinuous group of analytic automorphisms without fixed point of $W \times M$ and therefore the factor space

$$\mathscr{V} = (W \times M)/G$$

is a complex manifold. We denote by p the canonical map of $W \times M$ onto $\mathscr{V}$. Clearly the canonical projection $W \times M \to M$ induces a regular map $\varpi : \mathscr{V} \to M$ such that the commutative diagram

$$\begin{tikzcd} W \times M \arrow[rr, "p"] \arrow[dr] & & \mathscr{V} \arrow[dl, "\varpi"] \\ & M & \end{tikzcd}$$

holds and the triple $(\mathscr{V}, M, \varpi)$ defines a complex analytic family of complex manifolds. We have

$$V_t = \varpi^{-1}(t) = W/G_t ,$$

where G_t is the infinite cyclic group generated by t. We call V_t a Hopf manifold (compare Hopf [19]). We note that W is simply connected and therefore W is the universal covering of V_t. Let t, s be two points on M.

THEOREM 15.1. *V_t is analytically homeomorphic to V_s if and only if there exists a matrix $u \in GL(2, \boldsymbol{C})$ such that $t = u^{-1}su$, and if $t = u^{-1}su$, $p(z, t) \to p(uz, s)$ gives a biregular map of V_t onto V_s.*

PROOF. It is clear that $p(z, t) \to p(uz, s)$ is a biregular map of V_t onto V_s if $t = u^{-1}su$. Suppose conversely that there exists a biregular map f of V_t onto V_s. Then f induces a biregular map $\tilde{f}$ of the universal covering $W = \tilde{V}_t$ of V_t onto $W = \tilde{V}_s$ such that the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\tilde{f}} & W \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s^a} \\ W & \xrightarrow{\tilde{f}} & W \end{array}$$

holds, where s^a is one of the generators of G_s, i.e. $s^a = s$ or s^{-1}. Let $\tilde{f}(z) = (\tilde{f}^1(z), \tilde{f}^2(z))$. By a theorem of Hartogs, the holomorphic functions $\tilde{f}^\nu(z)$ on $W = \boldsymbol{C} \times \boldsymbol{C} - (0, 0)$ are holomorphic on the whole space $\boldsymbol{C} \times \boldsymbol{C}$ and hence

$$\tilde{f}^\nu(z) = \sum_{m,n=0}^{\infty} c^\nu_{mn}(z^1)^m(z^2)^n .$$

Now, letting

$$u = \begin{pmatrix} c^1_{10} & c^1_{01} \\ c^2_{10} & c^2_{01} \end{pmatrix},$$

we infer from $s^a\tilde{f}(z) = \tilde{f}(tz)$ that $s^a u = ut$. Thus we get $t = u^{-1}s^a u = u^{-1}su$ or $u^{-1}s^{-1}u$, but, in view of (15.2), $t \neq u^{-1}s^{-1}u$. Hence we obtain $t = u^{-1}su$, q.e.d.

$t \to (\sigma, \Delta) = (1/2(\alpha + \delta), 1/4(\alpha - \delta)^2 + \beta\gamma)$ is a holomorphic map of M onto the domain $\{(\sigma, \Delta) \mid |\sigma| > 3/2, |\Delta| < 1/4\}$ and therefore defines a structure of analytic fibre space on M. The fibre

$$M_{\sigma,\Delta} = \{t \mid \alpha + \delta = 2\sigma, (\alpha - \delta)^2 + 4\beta\gamma = 4\Delta\}$$

is a quadratic surface which is non-singular if $\Delta \neq 0$. In case $\Delta = 0$, $M_{\sigma,0}$ has exactly one singular point $\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ and thus $M'_{\sigma,0} = M_{\sigma,0} - \sigma$ is non-singular. Moreover we have

$$t = u \begin{pmatrix} \sigma + \sqrt{\Delta} & 0 \\ 0 & \sigma - \sqrt{\Delta} \end{pmatrix} u^{-1}, \qquad \text{for } t \in M_{\sigma,\Delta},\ \Delta \neq 0, \tag{15.3}$$

and

$$(15.4) \qquad t = u\begin{pmatrix}\sigma & 1\\ 0 & \sigma\end{pmatrix}u^{-1}, \qquad \text{for } t \in M'_{\sigma,0}.$$

The matrix u in (15.3) or (15.4) is given for example by

$$u = \begin{pmatrix}\tau + \sqrt{\Delta} & \beta\\ \gamma & -\tau - \sqrt{\Delta}\end{pmatrix} \text{ or } \begin{pmatrix}b\beta + c\tau & c\\ -b\tau + c\gamma & b\end{pmatrix},$$

where $\tau = 1/2(\alpha - \delta)$ and b, c are arbitrary numbers such that $b^2\beta + c^2\gamma \neq 0$. Combining (15.3), (15.4) with Theorem 15.1, we infer that the restriction $\mathscr{V}|M_{\sigma,\Delta}$ of $\mathscr{V}$ to $M_{\sigma,\Delta}$, $\Delta \neq 0$, and $\mathscr{V}|M'_{\sigma,0}$ are locally complex analytically trivial. The restriction $\mathscr{V}|M_{\sigma,0}$ is not a differentiable family in the sense of Definition 1.1, since $M_{\sigma,0}$ has a singular point σ. But we can transform $\mathscr{V}|M_{\sigma,0}$ into a complex analytic family by introducing "uniformization variables" on $M_{\sigma,0}$. In fact, $M_{\sigma,0}$ consists of all matrices of the form

$$(15.5) \qquad t_\sigma(\zeta) = \begin{pmatrix}\sigma + \zeta_1\zeta_2 & \zeta_1^2\\ -\zeta_2^2 & \sigma - \zeta_1\zeta_2\end{pmatrix}, \quad \zeta = (\zeta_1, \zeta_2) \in \boldsymbol{C} \times \boldsymbol{C}.$$

Hence, using ζ as "uniformization variables" on $M_{\sigma,0}$, we obtain from $\mathscr{V}|M_{\sigma,0}$ a complex analytic family $\mathscr{V}^*_\sigma = \{V_{t_\sigma(\zeta)} | \zeta \in \boldsymbol{C} \times \boldsymbol{C}\}$, where the structure of complex analytic fibre space of $\mathscr{V}^*_\sigma$ is induced from $\mathscr{V}$ by the regular map $\zeta \to t_\sigma(\zeta)$ of $\boldsymbol{C} \times \boldsymbol{C}$ into M. Since $\zeta \to t_\sigma(\zeta)$ is locally biregular between $\boldsymbol{C} \times \boldsymbol{C} - (0, 0)$ and $M'_{\sigma,0}$, it follows from the above result that $\mathscr{V}^*_\sigma$ is analytically trivial in a neighborhood of each point $\zeta \neq 0$, while $\mathscr{V}^*_\sigma$ is not differentiably trivial in a neighborhood of 0, since $V_{t_\sigma(0)} = V_\sigma$ is not analytically homeomorphic to $V_{t_\sigma(\zeta)}$, $\zeta \neq 0$.

Letting U be a small "spherical" neighborhood on M, we determine the explicit form of the homomorphism

$$\rho_U : T_M(U) \longrightarrow H^1(\mathscr{V}|U, \Theta) .$$

Let $S = \{z \mid |z^1|^2 + |z^2|^2 = 1\}$ be the unit spherical surface on W, R_t the ring domain on W bounded by two surfaces S and tS, $N(S)$ a neighborhood of S, and let

$$\mathscr{W}_1 = N(S) \times U, \quad \mathscr{W}_2 = \cup_{t \in U} R_t \times t, \quad \mathscr{W}_3 = \eta\, \mathscr{W}_1 .$$

Then, letting $\mathscr{U}_i = p(\mathscr{W}_i)$, we have

$$\mathscr{V}|U = \mathscr{U}_1 \cup \mathscr{U}_2 \cup \mathscr{U}_3 .$$

Since $\mathscr{W}_i \subset \boldsymbol{C}^2 \times U$, we may define the maps $h_i : \mathscr{U}_i \to \boldsymbol{C}^2 \times U$ simply by $h_i = p^{-1}: \mathscr{U}_i \to \mathscr{W}_i$ (see Definition 1.1) and use $(z_i, t) \in \mathscr{W}_i$ as the local coordinates of a point $p(z_i, t)$ on $\mathscr{U}_i$. The transformations $h_{ik}(z_k, t)$ are given therefore by

$$h_{12}(z_2, t) = z_2, \quad h_{23}(z_3, t) = z_3, \quad h_{31}(z_1, t) = tz_1$$

(compare Section 5). Let $v : t \to v_t$ be an element of $T_M(U)$ and let

$$\theta_{ik}(p) = v_t \cdot h_{ik}(z_k, t) , \qquad \text{where } p = p(z_k, t).$$

We have

$$(15.6) \qquad \begin{cases} \theta_{12}(p) = \theta_{23}(p) = 0 , \\ \theta_{31}(p) = v_t z_1 , & \text{where } p = p(z_1, t), \end{cases}$$

and the 1-cocycle $\{\theta_{ik}\}$ thus obtained determines the cohomology class $\theta_U = \rho_U(v)$ in $H^1(\mathscr{V}|U, \Theta)$. By Proposition 2.1, we have the isomorphism

$$\iota : H^1(\mathscr{V}|U, \Theta) \cong H^{0,1}_{\bar\partial}(\mathfrak{F}) ,$$

where $H^{0,1}_{\bar\partial}(\mathfrak{F})$ denotes the $\bar\partial$-cohomology of $\mathfrak{F}$-forms of type (0, 1) on $\mathscr{V}|U$. The $\bar\partial$-closed $\mathfrak{F}$-form φ representing $\iota\theta_U$ is given by

$$(15.7) \qquad \varphi = \bar\partial\lambda_i , \quad \lambda_k - \lambda_i = \theta_{ik} ,$$

where λ_i are differentiable sections of $\mathfrak{F}$ over $\mathscr{U}_i$. φ induces a $\bar\partial$-closed $\mathfrak{F}$-form $\tilde\varphi = \begin{pmatrix}\tilde\varphi^1 \\ \tilde\varphi^2\end{pmatrix}$ on $W \times U$ which is *G-invariant* in the sense that

$$t\tilde\varphi(z, t) - \tilde\varphi(tz, t) = 0 .$$

λ_i induces a differentiable section $\tilde\lambda_i$ of $\mathfrak{F}$ on $\mathscr{W}_i$. It follows from (15.6) and (15.7) that $\tilde\lambda = \tilde\lambda_1 = \tilde\lambda_2 = \tilde\lambda_3$ is a differentiable section of $\mathfrak{F}$ on $\mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3$ and that

$$\tilde\varphi = \bar\partial\tilde\lambda, \quad t\tilde\lambda(z_1, t) - \tilde\lambda(z_3, t) = v_t z_1 , \qquad \text{where } z_3 = tz_1.$$

This shows that $\tilde\lambda$ can be extended to a global differentiable section $\tilde\lambda(z, t)$ of $\mathfrak{F}$ on $W \times U$ satisfying

$$(15.8) \qquad \tilde\varphi = \bar\partial\tilde\lambda , \quad t\tilde\lambda(z, t) - \tilde\lambda(tz, t) = v_t z .$$

Now if $\varphi = \bar\partial\mu$ on $\mathscr{V}|U$, then μ induces $\tilde\mu$ on $W \times U$ satisfying

$$(15.9) \qquad \tilde\varphi = \bar\partial\tilde\mu , \quad t\tilde\mu(z, t) - \tilde\mu(tz, t) = 0 .$$

Hence the difference $w(z, t) = \tilde\lambda(z, t) - \tilde\mu(z, t)$ is *holomorphic* in z and satisfies

$$(15.10) \qquad tw(z, t) - w(tz, t) = v_t z .$$

Conversely if there exists $w(z, t)$ on $W \times U$ which is holomorphic in z and satisfies (15.10), then $\tilde\mu(z, t) = \tilde\lambda(z, t) - w(z, t)$ satisfies (15.9) and therefore $\tilde\mu$ determines μ on $\mathscr{V}|U$ such that $\varphi = \bar\partial\mu$. Suppose that such $w(z, t)$ exists. By a theorem of Hartogs $w^\nu(z, t)$ is an entire function in z and hence it can be expanded into a power series

$$w^{\nu}(z,\ t) = \sum_{m,n=0}^{\infty} w^{\nu}_{mn}(t)(z^1)^m(z^2)^n\ , \qquad \nu = 1,\ 2,$$

where each coefficient

$$w^{\nu}_{mn}(t) = \left(\frac{1}{2\pi i}\right)^2 \cdot \iint_{|z^1|=|z^2|=1} \frac{w^{\nu}(z,\ t)dz^1dz^2}{(z^1)^{m+1}(z^2)^{n+1}}$$

is differentiable in t. Letting

$$w_t = \begin{pmatrix} w^1_{10}(t) & w^1_{01}(t) \\ w^2_{10}(t) & w^2_{01}(t) \end{pmatrix},$$

we obtain from (15.10) the equality

$$v_t = tw_t - w_t t\ . \tag{15.11}$$

Conversely if there exists a differentiable vector field $w : t \to w_t$ on U satisfying (15.11), then $w(z,\ t) = w_t z$ satisfies (15.10). For simplicity we denote the vector field $t \to (tw_t - w_t t)$ by $k(w)$. Then the above result can be stated as follows:

THEOREM 15.2. *The kernel of the homomorphism*

$$\rho : T(U) \to H^1(\mathscr{V}|U,\ \Theta)$$

is the linear subspace $k(T(U))$ of $T(U)$.

It is easy to verify that, in case U contains no "scalar" $\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$, $k(T(U))$ consists of all differentiable vector fields v on U such that v_t is tangent to the surface $M_{\sigma,\Delta} \ni t$ at each point $t \in U$. Thus $\mathscr{V}$ is 2-trivial at each point t on M which is not a scalar.

By restricting the above argument to one point t on M, we obtain the corresponding theorem for the homomorphism $\rho_t : (T_M)_t \to H^1(V_t,\ \Theta_t)$.

THEOREM 15.3. *The kernel of the homomorphism ρ_t is the linear subspace $k_t((T_M)_t)$ of $(T_M)_t$ consisting of all vectors of the form $k_t(w_t) = tw_t - w_t t$.*

In case t is not a scalar, $k_t((T_M)_t)$ coincides with the tangent space of the surface $M_{\sigma,\Delta} \ni t$ at t, while, for a scalar $\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$, $k_\sigma((T_M)_\sigma) = 0$. We have

$$\dim \rho_t((T_M)_t) = \begin{cases} 4 & \text{if } t = \sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \\ 2 & \text{otherwise.} \end{cases} \tag{15.12}$$

THEOREM 15.4. $\rho_t((T_M)_t) = H^1(V_t,\ \Theta_t)$.

PROOF. In case $t = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, this equality can be verified as follows: Let φ be an arbitrary $\bar{\partial}$-closed vector form of type (0, 1) on V_t. φ induces a $\bar{\partial}$-closed vector form $\tilde{\varphi}$ on W which is G_t-invariant in the sense that

$$t\tilde{\varphi}(z) - \tilde{\varphi}(tz) = 0 . \tag{15.13}$$

In view of (15.8), it is sufficient for our purpose to show that there exists a differentiable vector field $\tilde{\lambda}$ on W and an element $v_t \in (T_M)_t$ satisfying

$$\tilde{\varphi} = \bar{\partial}\tilde{\lambda} , \quad t\tilde{\lambda}(z) - \tilde{\lambda}(tz) = v_t z . \tag{15.14}$$

First we determine $H^1(W, \Theta)$, where Θ is the sheaf of germs of holomorphic vector fields on W. Let $\boldsymbol{C}^* = \boldsymbol{C} - 0$. Then W is covered by two Stein manifolds $W_1 = \boldsymbol{C} \times \boldsymbol{C}^*$, $W_2 = \boldsymbol{C}^* \times \boldsymbol{C}$. Letting N be the nerve of the covering $\{W_1, W_2\}$ of W, we get

$$H^1(W, \Theta) \cong H^1(N, \Theta) \cong \{\theta_{12}\}/\{\theta_2 - \theta_1\}$$

where θ_{12} are holomorphic vector fields on $W_1 \cap W_2 = \boldsymbol{C}^* \times \boldsymbol{C}^*$ and θ_1, θ_2 are holomorphic vector fields on W_1, W_2. Expanding θ_{12}, θ_1, θ_2 into power series in z^1, z^2, we infer that each element of $H(W, \Theta)$ has a unique representative $\theta_{12}(z)$ of the form

$$\theta_{12}(z) = \sum_{m,n=1}^{\infty} c_{mn} (z^1)^{-m}(z^2)^{-n} ,$$

where c_{mn} are constant vectors. Now $\tilde{\varphi}$ corresponds to an element of $H^1(W, \Theta)$ whose representative θ_{12} is related to $\tilde{\varphi}$ by

$$\tilde{\varphi} = \bar{\partial}\lambda_1 = \bar{\partial}\lambda_2 , \quad \lambda_2 - \lambda_1 = \theta_{12} ,$$

where λ_1, λ_2 are differentiable vector fields on W_1, W_2, respectively. Since W_1, W_2 are invariant under the map $z \to tz$, we infer from (15.13) that

$$t\theta_{12}(z) - \theta_{12}(tz) = 0 ,$$

or

$$\alpha c^1_{mn} - \alpha^{-m}\delta^{-n}c^1_{mn} = 0 , \quad \delta c^2_{mn} - \alpha^{-m}\delta^{-n}c^2_{mn} = 0 ,$$

and hence $\theta_{12}(z) = 0$. Thus $\lambda = \lambda_1 = \lambda_2$ is a differentiable vector field on W and satisfies

$$\tilde{\varphi} = \bar{\partial}\lambda .$$

It follows from (15.13) that

$$\bar{\partial}(t\lambda(z) - \lambda(tz)) = 0 .$$

Hence $t\lambda(z) - \lambda(tz)$ is holomorphic on W and therefore holomorphic on $\boldsymbol{C} \times \boldsymbol{C}$. Let

$$t\lambda(z) - \lambda(tz) = \sum_{m,n=0}^{\infty} a_{mn} (z^1)^m(z^2)^n ,$$

where a_{mn} are constant vectors. Let

$$\mu(z) = \sum_{\substack{m+n\neq 1\\ m,n\geqq 0}} b_{mn}\,(z^1)^m(z^2)^n\ ,$$

where b_{mn} are constant vectors defined by the equations

$$t\cdot b_{mn} - \alpha^m\delta^n b_{mn} = a_{mn}\ .$$

Then we have

$$t\mu(z) - \mu(tz) = \sum_{\substack{m+n\neq 1\\ m,n\geqq 0}} a_{mn}\,(z^1)^m(z^2)^n\ .$$

Setting

$$\tilde\lambda(z) = \lambda(z) - \mu(z)\ ,\quad v_t z = a_{10}z^1 + a_{01}z^2\ ,$$

we obtain therefore (15.14). Thus we see that $\rho_t(T_M)_t = H^1(V_t, \Theta_t)$ for $t = \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$ (the case $\alpha = \delta$ included).

Now we consider an arbitrary point $t = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in M$. In view of (15.12) it suffices to show that

$$\dim H^1(V_t, \Theta_t) = 2\ , \qquad\qquad \text{for } t \neq \sigma = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}.$$

This is obvious if $\Delta = \frac{1}{4}(\alpha-\delta)^2 + \beta\gamma \neq 0$, since t is equivalent to

$$\begin{pmatrix}\sigma + \sqrt{\Delta} & 0\\ 0 & \sigma - \sqrt{\Delta}\end{pmatrix}.$$

If $\Delta = 0$, t is equivalent to $\begin{pmatrix}\sigma & 1\\ 0 & \sigma\end{pmatrix}$. Thus it suffices to prove

$$\dim H^1(V_t, \Theta_t) = 2$$

for $t = \begin{pmatrix}\sigma & 1\\ 0 & \sigma\end{pmatrix}$. For this purpose we first compute

$$h^{p,q}(V_t) = \dim H^q(V_t, \Omega_t^p)$$

for arbitrary $t \in M$, where Ω_t^p is the sheaf of germs of holomorphic p-forms on V_t. We have

$$\sum_{p,q}(-1)^{p+q}h^{p,q}(V_t) = \textit{Euler number of } V_t = 0,$$
$$h^{p,q}(V_t) = h^{2-p,2-q}(V_t)\ ,$$

while it is easy to verify that

$$h^{0,0}(V_t) = 1\ ,\quad h^{1,0}(V_t) = h^{2,0}(V_t) = 0\ .$$

It follows that

$$h^{2,2}(V_t) = 1\ ,\quad h^{0,2}(V_t) = h^{1,2}(V_t) = 0\ ,$$

$$h^{2,1}(V_t) = h^{0,1}(V_t)\ , \quad h^{1,1}(V_t) = 2h^{0,1}(V_t) - 2\ .$$

In case $t = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, we can determine $H^1(V_t, \Omega_t^0)$ in the same manner as $H^1(V_t, \Theta_t)$ and obtain $h^{0,1}(V_t) = \dim H^1(V_t, \Omega_t^0) = 1$. It follows that $h^{0,1}(V_t) = 1$ for $t \in M_{\sigma,\Delta}$, $\Delta \neq 0$. In case $t \in M_{\sigma,0} - \sigma$, we get, by the principle of upper semi-continuity (Theorem 2.1),

$$1 \leqq h^{0,1}(V_t) \leqq h^{0,1}(V_\sigma) = 1\ ,$$

since $h^{0,1}(V_t)$ is constant on $M_{\sigma,0} - \sigma$. Thus we get

$$h^{2,1}(V_t) = h^{0,1}(V_t) = 1\ , \quad h^{1,1}(V_t) = 0\ .$$

Now we come back to V_t with $t = \begin{pmatrix} \sigma & 1 \\ 0 & \sigma \end{pmatrix}$. The meromorphic 2-form $(z^2)^{-2}dz^1 \wedge dz^2$ on W is G_t-invariant and hence it induces a meromorphic 2-form on V_t. It follows that the canonical bundle K_t on V_t is given by

$$K_t = [-2C]\ ,$$

where C is the curve on V_t defined by $z^2 = 0$. We note that C is an elliptic curve. By (11.10) we have

$$\dim H^q(V_t, \Theta_t) = \dim H^{2-q}(V_t, \Omega^1(K_t))\ .$$

With the help of a recurrence formula (Kodaira and Spencer [27], formula (14)), we get

$$\begin{aligned} \textstyle\sum_{q=0}^2 (-1)^q \dim H^q(V_t, \Omega_t^1([-2C])) &= \textstyle\sum_{q=0}^2 (-1)^q \dim H^q(V_t, \Omega_t^1) \\ &= h^{1,0}(V_t) - h^{1,1}(V_t) + h^{1,2}(V_t) = 0\ . \end{aligned}$$

On the other hand it is easy to verify that the sheaf $\Omega^1(K)$ on W has no G_σ-invariant holomorphic section. Hence $H^0(V_\sigma, \Omega^1(K_\sigma)) = 0$ and therefore, by the principle of upper semi-continuity,

$$\dim H^0(V_t, \Omega^1(K_t)) \leqq \dim H^0(V_\sigma, \Omega^1(K_\sigma)) = 0\ .$$

Consequently we obtain

$$\dim H^1(V_t, \Theta_t) = \dim H^0(V_t, \Theta_t)\ .$$

Now $H^0(V_t, \Theta_t)$ is isomorphic to the linear space of holomorphic vector fields θ on W satisfying $\theta(tz) = t\theta(z)$. Expanding $\theta(z)$ into power series

$$\theta(z) = \textstyle\sum_{m,n=0}^\infty c_{mn}\,(z^1)^m(z^2)^n\ ,$$

we have

$$\textstyle\sum tc_{mn}\,(z^1)^m(z^2)^n = \sum c_{mn}\,(\sigma z^1 + z^2)^m(\sigma z^2)^n\ .$$

It follows that $c_{mn} = 0$ for $m + n \neq 1$ and

$$c_{10}^2 = 0 , \quad c_{10}^1 = c_{01}^2 .$$

Thus we see that

$$\dim H^1(V_t, \Theta_t) = \dim H^0(V_t, \Theta_t) = 2 ,$$

q.e.d.

Letting

$$t(s) = t_\sigma(s, 0) = \begin{pmatrix} \sigma & s^2 \\ 0 & \sigma \end{pmatrix} ,$$

we form a one-parameter subfamily $\mathcal{V}^* = \{V_{t(s)} | s \in \boldsymbol{C}\}$ of $\mathcal{V}_\sigma^*$. Since $V_{t(0)} = V_\sigma$ is not analytically homeomorphic to $V_{t(s)}$, $s \neq 0$, *$\mathcal{V}^*$ is not locally differentiably trivial in a neighborhood of $0 \in \boldsymbol{C}$, whereas its infinitesimal deformation $\rho_s^*(u_s)$ vanishes for all $s \in \boldsymbol{C}$ for an arbitrary differentiable vector field $u : s \to u_s$ on $\boldsymbol{C}$.* In fact, we have

$$\rho_s^*(u_s) = u_s \cdot \rho_{t(s)}(t'(s)) , \quad \text{where } t'(s) = \frac{d}{ds} t(s) = \begin{pmatrix} 0 & 2s \\ 0 & 0 \end{pmatrix}$$

and $\rho_{t(s)}(t'(s)) = 0$ for $s \neq 0$, since $t'(s)$ is a tangent vector of $M'_{\sigma,0}$ at $t(s)$, while $t'(0) = 0$ for $s = 0$. Thus $\rho_s^*(u_s) = 0$ for all $s \in \boldsymbol{C}$. Now denoting by $T(\boldsymbol{C})$ the linear space of differentiable vector fields on $\boldsymbol{C}$, we show that the image $\rho^*(T(\boldsymbol{C}))$ of $T(\boldsymbol{C})$ in $H^1(\mathcal{V}^*, \Theta)$ is isomorphic to $\boldsymbol{C}$. Applying the above argument to $\mathcal{V}^*$ we infer that $\rho(u) = 0$ if and only if there exists a differentiable map : $s \to w_s$ of $\boldsymbol{C}$ into $GL(2, \boldsymbol{C})$ such that

$$t(s)w_s - w_s t(s) = u_s t'(s) .$$

It follows that $\rho(u) = 0$ if and only if u_s/s is differentiable in s. Hence we obtain $\rho^*(T(\boldsymbol{C})) \cong \boldsymbol{C}$.

Since all $V_{t(s)}$, $s \neq 0$, are analytically homeomorphic to $V_{t(1)}$, we may say that, when s goes to zero, the complex structure of $V_{t(s)}$ jumps from the structure of $V_{t(1)}$ to that of $V_{t(0)}$. This phenomenon is not unusual for manifolds of complex dimension $\geqq 2$.

The family $\mathcal{V}^*$ shows the impossibility of defining in general a non-trivial "distance" between two complex-analytic structures V', V'' on the same compact differentiable manifold when their complex dimension exceeds 1. We recall that in the case where V', V'' are curves (complex dimension 1) Teichmüller [38] (compare also Ahlfors [1]) has constructed a non-negative distance $D(V', V'')$ which satisfies the following conditions: (1) $D(V', V''') \leqq D(V', V'') + D(V'', V''')$; (2) $D(V', V'') = 0$ if and only if V' and V'' belong to one and the same trivial differentiable family ; (3) if the complex structure tensor J' of V' converges to the structure tensor J'' of V'', then $D(V', V'')$ tends to zero. The family $\mathcal{V}^*$ constructed above shows that a distance function satisfying (1) – (3)

cannot exist in general for manifolds of complex dimension greater than 1. In fact, if $D(V', V'')$ were such a function, then for the family $\mathscr{V}^*$ we would have: (a) $D(V_{t(s)}, V_{t(1)}) = 0$ for $s \neq 0$ (by (2)); (b) $D(V_{t(0)}, V_{t(s)})$ tends to zero as s approaches zero (by (3)). This leads to a contradiction since (by (1)) $D(V_{t(0)}, V_{t(1)}) \leqq D(V_{t(0)}, V_{t(s)}) + D(V_{t(s)}, V_{t(1)}) = D(V_{t(0)}, V_{t(s)})$ for $s \neq 0$, and $D(V_{t(0)}, V_{t(s)})$ can be made arbitrarily small by choosing $|s|$ sufficiently small. Hence $D(V_{t(0)}, V_{t(1)}) = 0$ which implies (by (2)) that $V_{t(0)}$ and $V_{t(1)}$ are analytically equivalent — a contradiction.

Combined with Proposition 11.3, the above Theorem 15.4 shows that, *if* the number of moduli $m(V_t)$ of a Hopf manifold $V_t \in \mathscr{V}$ is defined, $m(V_t) = \dim H^1(V_t, \Theta_t)$. It would seem to be reasonable to conjecture that $m(V_t)$ is defined in case t is not a scalar. On the other hand, we infer readily that $m(V_\sigma)$ cannot be defined in case σ is a scalar. In fact, if there exists an effectively parametrized complete complex analytic family $\mathscr{W} \to N$ of deformations of V_σ, then we would have

$$\dim N = m(V_\sigma) = \dim H^1(V_\sigma, \Theta_\sigma) = 4 ,$$

while, since $\mathscr{W}$ contains V_t for all $t \in U$, U being a sufficiently small neighborhood of σ on M, we would get, by choosing non-scalar t in U,

$$\dim N = m(V_t) = \dim H^1(V_t, \Theta_t) = 2 ,$$

but this is a contradiction. Moreover the structure of the family $\mathscr{V}$ would seem to indicate that, not only does our definition of number of moduli fail for V_σ, but the concept of the number of moduli of V_σ does not make sense.

16. Obstruction to deformation of complex structure

Let V_o be a compact complex-analytic manifold and let Θ_o denote the sheaf of germs of holomorphic sections of the tangent bundle of V_o. We have seen (Proposition 6.2) that any deformation space in $H^1(V_o, \Theta_o)$ is an abelian Lie algebra, and we recall (Definition 6.5) that a class $\theta_o \in H^1(V_o, \Theta_o)$ is obstructed if $[\theta_o, \theta_o] \neq 0$ in $H^2(V_o, \Theta_o)$. The question arises: is there a V_o for which obstructed classes in $H^1(V_o, \Theta_o)$ exist? The answer is that there are such V_o, at least for complex dimension exceeding 2. In fact, take $V_o = T_q \times P_1(\boldsymbol{C})$ where $P_1(\boldsymbol{C})$ is the complex projective line and T_q is an arbitrary complex q-torus, $q \geqq 2$. Then, as we shall show, $\dim_{\boldsymbol{C}} H^1(V_o, \Theta_o) = q^2 + 3q$ and, for each point $p \in P_2(\boldsymbol{C})$, there is a deformation space D_p, $\dim_{\boldsymbol{C}} D_p = q^2 + q$, which is a *maximal* abelian Lie algebra in $H^1(V_o, \Theta_o)$ and $\bigcap_{p \in P_2(\boldsymbol{C})} D_p = H^1(T, \Theta_T)$ where

$T = T_q$ and Θ_T is the sheaf of germs of holomorphic vector fields on T. Thus $\cap D_p$ is the space $H^1(T, \Theta_T)$ of infinitesimal deformations of T, $\dim_C H^1(T, \Theta_T) = q^2$ (compare Section 14). The vector space $H^1(V_o, \Theta_o)$ is spanned (over $\boldsymbol{C}$) by the deformation spaces D_p, $p \in P_2(\boldsymbol{C})$, but the sum of two classes belonging to different deformation spaces is obstructed in general.

We begin by discussing a slightly more general example, namely $V_o = T_q \times P_n(\boldsymbol{C})$ where $P_n(\boldsymbol{C})$ is complex projective space of arbitrary complex dimension n. For simplicity write $P = P_n(\boldsymbol{C})$, $T = T_q$, and let Θ_o, Θ_P, Θ_T denote respectively the sheaves of germs of holomorphic vector fields on V_o, P, T. Furthermore, let Ω_P, Ω_T be respectively the sheaves of germs of holomorphic functions on P, T. Since $H^1(P, \Theta_P) = 0$ and $H^1(P, \Omega_P) = 0$, we have (Künneth formula !)

$$(16.1) \qquad H^1(V_o, \Theta_o) \cong H^0(P, \Theta_P) \otimes H^1(T, \Omega_T) \oplus H^1(T, \Theta_T) ,$$

where the tensor product in the right member is over $\boldsymbol{C}$. In fact, this formula may be proved by representing the cohomology in terms of harmonic forms (compare Proposition 2.1).

Denote the inhomogeneous coordinates on P by $(1, \zeta^1, \cdots, \zeta^\alpha, \cdots, \zeta^n)$; then we have :

LEMMA 16.1. *Every element of $H^0(P, \Theta_P)$, expressed in terms of the inhomogeneous coordinates of P, has the form $\varphi(\zeta) = (\varphi^1(\zeta), \cdots, \varphi^\alpha(\zeta), \cdots, \varphi^n(\zeta))$ where*

$$\varphi^\alpha(\zeta) = \sum_{\lambda=1}^n (c_\lambda^0 \zeta^\alpha \zeta^\lambda + c_\lambda^\alpha \zeta^\lambda) + c_0^\alpha .$$

Here c_λ^0, c_λ^α, c_0^α are complex constants.

PROOF. An arbitrary projective transformation $h : P \to P$, referred to the inhomogeneous coordinates, sends $\zeta = (\zeta^1, \cdots, \zeta^n)$ into

$$h(\zeta) = (h^1(\zeta), \cdots, h^n(\zeta))$$

where

$$h^\alpha(\zeta) = \frac{\sum_\lambda a_\lambda^\alpha \zeta^\lambda + a_0^\alpha}{\sum_\lambda b_\lambda^0 \zeta^\lambda + b_0^0} , \qquad a_\lambda^\alpha,\ a_0^\alpha,\ b_\lambda^0,\ b_0^0 \in \boldsymbol{C}.$$

Given $\varphi \in H^0(P, \Theta_P)$, let $h = h(\zeta, s) = (h^1(\zeta, s), \cdots, h^n(\zeta, s)) = \exp(s\varphi)(\zeta)$. Then $\partial h^\alpha/\partial s|_{s=0} = \varphi^\alpha(\zeta)$ and since, for each s, $\zeta \to h(\zeta, s)$ is a projective transformation, we infer immediately that $\varphi^\alpha(\zeta)$ has the form stated, q.e.d.

Next, $T = \boldsymbol{C}^q/\mathscr{G}$ where $\mathscr{G}$ is the free abelian group generated by non-degenerate periods $\omega_j = (\omega_{j1}, \cdots, \omega_{jq})$, $j = 1, 2, \cdots, 2q$, and where, if $(z_1, \cdots, z_q)$ are the euclidean coordinates of $\boldsymbol{C}^q$, $\omega_j \in \mathscr{G}$ operates on

C^q by sending $z=(z_1, \cdots, z_q)$ into $z+\omega_j=(z_1+\omega_{j1}, \cdots, z_q+\omega_{jq})$. On T there is the euclidean Kähler metric $ds^2=\sum_{\alpha=1}^q |dz_\alpha|^2$ and $H^1(T, \Omega_T)$ is isomorphic to the space of harmonic forms on T of type (0, 1) where, as basis for these harmonic forms, we can choose $d\bar{z}_1, \cdots, d\bar{z}_q$. Any element of $H^0(P, \Theta_P)\otimes H^1(T, \Omega_T)$ then has a unique representative of the form

$$\psi=\sum_{k=1}^q \varphi_k \cdot d\bar{z}_k \tag{16.2}$$

where $\varphi_\alpha \in H^0(P, \Theta_P)$, $\alpha=1, \cdots, q$. Finally (compare Theorem 14.3)

$$\dim_C H^1(T, \Theta_T)=q^2 .$$

Thus

$$\dim_C H^1(V_o, \Theta_o)=(n+1)^2 q+q(q-1) . \tag{16.3}$$

Given an element $\psi \in H^0(P, \Theta_P)\otimes H^1(T, \Omega_T)$, which we may assume is of the form (16.2), then $[\psi, \psi] \in H^0(P, \Theta_P)\otimes H^2(T, \Omega_T) \subset H^2(V_o, \Theta_o)$. In fact,

$$[\psi, \psi]=\sum_{j<k} (\varphi_j \cdot \varphi_k - \varphi_k \cdot \varphi_j) d\bar{z}_j \wedge d\bar{z}_k$$

where

$$\varphi_j \cdot \varphi_k=((\varphi_j \cdot \varphi_k)^1, \cdots, (\varphi_j \cdot \varphi_k)^n), \ (\varphi_j \cdot \varphi_k)^\alpha=\sum_{\beta=1}^n \varphi_j^\beta \frac{\partial}{\partial \zeta^\beta} \varphi_k^\alpha .$$

Clearly

$$H^0(P, \Theta_P)\otimes H^2(T, \Omega_T) \longrightarrow H^2(V_o, \Theta_o)$$

is injective ; hence $[\psi, \psi] \neq 0$ *in general.* This is true even in the case $n=1$. In that case

$$\psi=\sum_{k=1}^q (a_k \zeta^2+b_k \zeta+c_k) d\bar{z}_k \tag{16.4}$$

and

$$[\psi, \psi]=\sum_{j<k} \{(b_j a_k-b_k a_j)\zeta^2+2(c_j a_k-c_k a_j)\zeta+(c_j b_k-c_k b_j)\} d\bar{z}_j \wedge d\bar{z}_k .$$

Thus $[\psi, \psi]=0$ if and only if the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ & \cdots & \\ a_q & b_q & c_q \end{pmatrix} \tag{16.5}$$

has rank not exceeding 1. Next, consider two elements ψ, ψ' belonging to $H^0(P, \Theta_P)\otimes H^1(T, \Omega_T)$ which are represented by

$$\psi=\sum_{k=1}^q (a_k \zeta^2+b_k \zeta+c_k) d\bar{z}_k, \quad \psi'=\sum_{k=1}^q (a'_k \zeta^2+b'_k \zeta+c'_k) d\bar{z}_k .$$

Then $[\psi, \psi]=0$, $[\psi, \psi']=0$ and $[\psi', \psi']=0$ if and only if

(16.6)
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ & \cdots & \\ a_q & b_q & c_q \\ a_1' & b_1' & c_1' \\ & \cdots & \\ a_q' & b_q' & c_q' \end{pmatrix}$$

has rank at most equal to 1. Finally, for $\psi \in H^0(P, \Theta_P) \otimes H^1(T, \Omega_T)$, $\tau \in H^1(T, \Theta_T)$, we have $[\psi, \tau] = 0$.

Assume henceforth that $n = 1$, i.e. $P = P_1(\boldsymbol{C})$. In view of the above remarks there corresponds, to each point $p \in P_2(\boldsymbol{C})$, a maximal abelian Lie algebra $D_p \subset H^1(V_o, \Theta_o)$, where

(16.7)
$$\dim_C D_p = q^2 + q .$$

Namely, let (a, b, c) be the homogeneous coordinates of the point $p \in P_2(\boldsymbol{C})$ and assume, for example, that $a \neq 0$. To each point $r \in \boldsymbol{C}^q$, $r = (\rho_1, \cdots, \rho_q)$, let $\psi(p, r)$ be the element (16.4) which corresponds to the matrix (16.5) whose k-th row is $a_k = \rho_k \cdot 1$, $b_k = \rho_k(b/a)$, $c_k = \rho_k(c/a)$. Then

(16.8)
$$D_p = \{\psi(p, r) \mid r \in \boldsymbol{C}^q\} \oplus H^1(T, \Theta_T) .$$

We now show that each D_p is a deformation space. This is accomplished by constructing complex-analytic fibre bundles over T with fibre $P_1(\boldsymbol{C})$ in the following manner.

We have $T = \boldsymbol{C}^q/\mathscr{G}$ where $\mathscr{G}$ is the free abelian group generated by $\omega_j = (\omega_{j1}, \cdots, \omega_{jq})$, $j = 1, 2, \cdots, 2q$. Given an element $\psi(p, r)$, $p \in P_2(\boldsymbol{C})$, $r \in \boldsymbol{C}^q$, let $t: \mathscr{G} \to PGL(1, \boldsymbol{C})$ be the representation of the abelian group $\mathscr{G}$ in the complex projective group $PGL(1, \boldsymbol{C})$ which is defined by

$$\omega_j \longrightarrow g_j = \exp\left(\textstyle\sum_{k=1}^q \bar{\omega}_{jk}\varphi_k\right)$$

where φ_k is the coefficient of $d\bar{z}_k$ in the expression (16.4) for $\psi(p, r)$. For each ω_j we define $\gamma_j : \boldsymbol{C}^q \times P_1(\boldsymbol{C}) \to \boldsymbol{C}^q \times P_1(\boldsymbol{C})$ by $\gamma_j : (z, \zeta) \to (z + \omega_j, g_j\zeta)$. Then $\gamma_1, \cdots, \gamma_{2q}$ generate a group $\Gamma_t \cong \mathscr{G}$ which operates in $\boldsymbol{C}^q \times P_1(\boldsymbol{C})$ and

(16.9)
$$V_t = (\boldsymbol{C}^q \times P_1(\boldsymbol{C}))/\Gamma_t$$

is a complex-analytic bundle over T. Since t is determined by the pair (p, r), we may consider $t = t(p, r)$ as a point of the space $M = P_2(\boldsymbol{C}) \times \boldsymbol{C}^q$. Thus $\{V_t \mid t \in M\}$ forms a complex analytic family of deformations of $V_o = T \times P_1(\boldsymbol{C})$. We note that $V_{t(p,0)} = V_o$ for all $p \in P_2(\boldsymbol{C})$.

To prove that D_p is a deformation space, it is sufficient to show that the subspace $\{\psi(p, r) | r \in \boldsymbol{C}^q\}$ of D_p is contained in the image $\rho_{t(p,0)}((T_M)_{t(p,0)})$ since $H^1(T, \Theta_T)$ is the deformation space for the base space T of the projective line bundles V_t. Given $\psi(p, r) \in D_p$, let $t(s) = t(p, sr)$ for $-1 < s < +1$, where $sr = (s\rho_1, \cdots, s\rho_q)$, and let $u \in (T_M)_{t(p,0)}$ be the tangent of the arc $t(s)$ at $t(0) = t(p, 0)$. We compute $\theta_o = \rho_{t(p,0)}(u) \in H^1(V_o, \Theta_o)$. For each $t(s)$ we have the representation :

$$\omega_j \longrightarrow g_j(s) = \exp\left(s \cdot \sum_{k=1}^q \bar{\omega}_{jk}\varphi_k\right) ,$$

where φ_k is the coefficient of $d\bar{z}_k$ in $\psi(p, r)$. Applying d/ds at $s = 0$, we obtain an *additive* representation :

$$\omega_j \longrightarrow \left(\frac{dg_j(s)}{ds}\right)_{s=0} = \sum_{k=1}^q \bar{\omega}_{jk}\varphi_k .$$

The infinitesimal deformation $\theta_o = \rho_{t(p,0)}(u)$ is obtained as follows : Take a differentiable function $z \to \varphi(z) \in H^0(P, \Theta_P)$ on $\boldsymbol{C}^q$ such that

$$\varphi(z + \omega_j) - \varphi(z) = \sum_{k=1}^q \bar{\omega}_{jk}\varphi_k . \tag{16.10}$$

Then θ_o is the $\bar{\partial}$-cohomology class of $\bar{\partial}\varphi$. Clearly

$$\varphi(z) = \sum_{k=1}^q \bar{z}_k\varphi_k$$

satisfies (16.10) and hence

$$\theta_o = \sum_{k=1}^q \varphi_k \cdot d\bar{z}_k = \psi(p, r) ,$$

q.e.d.

REMARK. The example $V_o = T_q \times P_1(\boldsymbol{C})$ is an algebraic manifold if and only if T_q is an abelian variety, but V_o is a Kähler manifold no matter how T_q is chosen. However, there exists a V_o which is not a Kähler manifold (*a fortiori* not algebraic) and which has much the same properties. Such a V_o may be obtained, for example, by replacing $P_1(\boldsymbol{C})$ by a Hopf manifold, of the type denoted by the letter σ in Section 15, which corresponds to the matrix

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

We denote such a Hopf manifold again by P, and we denote (as in Section 15) the coordinates of $\boldsymbol{C}^2$ by $\zeta = (\zeta^1, \zeta^2)$. Then, as is easily verified, every element of $H^0(P, \Theta_P)$ has the form $\varphi(\zeta) = (\varphi^1(\zeta), \varphi^2(\zeta))$ where $\varphi^1(\zeta) = a\zeta^1 + b\zeta^2$, $\varphi^2(\zeta) = c\zeta^1 + d\zeta^2$, $a, b, c, d \in \boldsymbol{C}$. An arbitrary element of $H^0(P, \Theta_P) \otimes H^1(T, \Omega_T)$ has the form

$$\psi = \sum_{k=1}^q \varphi_k \cdot d\bar{z}_k$$

where $\varphi_k \in H^0(P, \Theta_P)$. It is easily seen that there exist obstructed elements ψ (i.e. ψ with $[\psi, \psi] \neq 0$ in $H^2(V_o, \Theta_o)$).

The only Hopf manifolds of Section 15 which, substituted for $P_1(\boldsymbol{C})$, give manifolds V_o with the desired properties are those of type σ; for these are the only Hopf manifolds with a sufficiently large group of analytic automorphisms to produce obstructed elements by the above argument.

17. Obstruction to deformation of complex fibre bundles

Let V_o be a compact complex-analytic manifold and let B_o be a complex fibre bundle over V_o. An element $\sigma \in H^1(V_o, \Sigma_o)$ is obstructed if $[\sigma, \sigma] \neq 0$ in $H^2(V_o, \Sigma_o)$ (compare Section 7). We have already exhibited a V_o with obstructed elements in $H^1(V_o, \Theta_o)$ for the deformation of the complex structure of V_o. Hence we exhibit here obstructions for elements $\sigma \in H^1(V_o, \Sigma_o)$ satisfying $\kappa_o(\sigma) = 0$ which are the images of obstructed elements in $H^1(V_o, \Xi_o)$ (see the infinitesimal deformation diagram of Section 7).

Assume that $q \geqq 2$, $n \geqq 3$, take $V_o = T_q$ where T_q is a complex q-torus, and take over V_o the trivial vector bundle $B_o = V_o \times \boldsymbol{C}^n$. In the case of a trivial bundle the sequence (9.2) splits, as can be seen immediately from (9.3). Hence the corresponding sequence of sheaves splits and $H^r(V_o, \Sigma_o) = H^r(V_o, \Theta_o) \oplus H^r(V_o, \Xi_o)$. We plainly have

$$H^1(V_o, \Xi_o) \cong \mathfrak{gl}(n, \boldsymbol{C}) \otimes_{\boldsymbol{C}} H^1(V_o, \Omega_o) \tag{17.1}$$

where $\mathfrak{gl}(n, \boldsymbol{C})$ denotes the Lie algebra of $GL(n, \boldsymbol{C})$; hence, in particular

$$\dim_{\boldsymbol{C}} H^1(V_o, \Xi_o) = qn^2 \,. \tag{17.2}$$

Now $V_o = \boldsymbol{C}^q / \mathscr{G}$ where $\mathscr{G}$ is a free abelian subgroup of $\boldsymbol{C}^q$ generated by non-degenerate periods ω_j, $j = 1, \cdots, 2q$, and $H^1(V_o, \Omega_o)$ is isomorphic to the space of harmonic forms of type (0, 1) generated by $d\bar{z}_1, \cdots, d\bar{z}_q$ (compare Section 16). Hence we have

$$H^1(V_o, \Xi_o) = \mathfrak{gl}(n, \boldsymbol{C}) d\bar{z}_1 \oplus \cdots \oplus \mathfrak{gl}(n, \boldsymbol{C}) d\bar{z}_q \,,$$

where we replace for simplicity the isomorphism by the identity. Similarly we have

$$H^2(V_o, \Xi_o) = \mathfrak{gl}(n, \boldsymbol{C}) d\bar{z}_1 \wedge d\bar{z}_2 \oplus \cdots \oplus \mathfrak{gl}(n, \boldsymbol{C}) d\bar{z}_\alpha \wedge d\bar{z}_\beta \oplus \cdots$$

Thus an arbitrary element ψ of $H^1(V_o, \Xi_o)$ is written in the form

$$\psi = \sum_{\alpha=1}^{q} C_\alpha d\bar{z}_\alpha \,, \qquad C_\alpha \in \mathfrak{gl}(n, \boldsymbol{C}),$$

and

$$[\psi, \psi] = \sum_{\alpha<\beta} (C_\alpha C_\beta - C_\beta C_\alpha) d\bar{z}_\alpha \wedge d\bar{z}_\beta \,.$$

It is clear that *a general element* $\psi \in H^1(V_o, \Xi_o)$ *is obstructed*, i.e.

$[\psi, \psi] \neq 0$ in $H^2(V_o, \Xi_o)$.

For each maximal abelian Lie subalgebra $\mathfrak{a} \subset \mathfrak{gl}(n, \boldsymbol{C})$, we define a subalgebra $D(\mathfrak{a})$ of $H^1(V_o, \Xi_o)$ by

$$D(\mathfrak{a}) = \mathfrak{a}\, d\bar{z}_1 \oplus \cdots \oplus \mathfrak{a}\, d\bar{z}_q \ .$$

Then $D(\mathfrak{a})$ *is a bundle deformation space which is a maximal abelian Lie subalgebra of* $H^1(V_o, \Xi_o)$. *Moreover* $\cap_{\mathfrak{a}} D(\mathfrak{a}) \cong \boldsymbol{C}^q$ and *the vector space* $H^1(V_o, \Xi_o)$ *is spanned* (*over* $\boldsymbol{C}$) *by the bundle deformation spaces* $D(\mathfrak{a})$. To prove this it suffices to show that $D(\mathfrak{a})$ is a bundle deformation space. For this purpose we denote by N the subspace of $H^1(V_o, \Xi_o)$ consisting of all elements t satisfying $[t, t] = 0$ and for each element

$$t = \sum_{\alpha=1}^{q} C_\alpha d\bar{z}_\alpha$$

of N we consider the group $\Gamma_t \cong \mathscr{G}$ of analytic automorphisms of $\boldsymbol{C}^q \times \boldsymbol{C}^n$ generated by

$$\gamma_j : (z, \zeta) \longrightarrow (z + \omega_j, \quad \exp\,(\textstyle\sum_{\alpha=1}^{q} \omega_{j\alpha} C_\alpha) \cdot \zeta) \ , \qquad j = 1, 2, \cdots, 2q.$$

Then the factor space

$$B_t = (\boldsymbol{C}^q \times \boldsymbol{C}^n)/\Gamma_t$$

is a complex-analytic vector bundle over V_o. We observe that N is a "quadratic cone" in the euclidean space $H^1(V_o, \Xi_o)$ and that

$$\dim_C N = qn + n(n-1) \ .$$

Given a linear subspace $D(\mathfrak{a}) \subset N$, we take a line $t(s) = s\psi$, $-\infty < s < +\infty$, $\psi \in D(\mathfrak{a})$, and compute the infinitesimal deformation $\xi_o \in H^1(V_o, \Xi_o)$ of the one-parameter family $\{B_{t(s)} | -\infty < s < +\infty\}$ at $s = 0$. Let

$$\psi = \sum_{\alpha=1}^{q} C_\alpha d\bar{z}_\alpha \ .$$

$\mathrm{B}_{t(s)}$ is the vector bundle over $V_o = \boldsymbol{C}^q/\mathscr{G}$ determined by the representation

$$\omega_j \longrightarrow \exp\,\{s \cdot \textstyle\sum_{\alpha=1}^{q} \bar{\omega}_{j\alpha} C_\alpha\} \ .$$

Applying $\partial/\partial s$, we obtain an *additive* representation

$$\omega_j \longrightarrow \sum_{\alpha=1}^{q} \bar{\omega}_{j\alpha} C_\alpha \ .$$

The infinitesimal deformation ξ_o is obtained by taking a differentiable map $z \longrightarrow \varphi(z)$ of $\boldsymbol{C}^q$ into $\mathfrak{gl}(n, \boldsymbol{C})$ which satisfies

$$\text{(17.3)} \qquad \varphi(z + \omega_j) - \varphi(z) = \sum_{\alpha=1}^{q} \bar{\omega}_{j\alpha} C_\alpha$$

and applying $\bar{\partial}$ to $\varphi(z) : \xi_o = \bar{\partial}\varphi(z)$. Since

$$\varphi(z) = \sum_{\alpha=1}^{q} C_\alpha \bar{z}_\alpha$$

clearly satisfies (17.3), we have

$$\xi_o = \sum_{\alpha=1}^{q} C_\alpha d\bar{z}_\alpha = \psi \ .$$

Thus every element ψ of $D(\mathfrak{a})$ is an infinitesimal deformation of the differentiable family $\{B_t | t \in D(\mathfrak{a})\}$ at $t = 0$ and therefore $D(\mathfrak{a})$ is a bundle deformation space.

CHAPTER VII. COMPLEX ANALYTIC FAMILIES OF COMPLEX STRUCTURES

18. Differential and complex analytic geometries of deformations

The theory of differentiable families of complex manifolds expounded in Sections 1–10 may be called the "differential geometry of deformations of complex manifolds". It is more natural to attempt to construct the corresponding theory for complex analytic families of complex manifolds which utilizes only holomorphic maps. Such a theory will be called the "analytic geometry of deformations of complex manifolds". It is easy to see that many concepts in the differential geometry of deformations have counterparts in the analytic geometry. We have already mentioned in Section 1 the concept of "complex analytic triviality" which is the counterpart of "differentiable triviality" (see Definition 1.3). In this section we discuss some elementary relations between the differential geometry and the analytic geometry of deformations. In particular, we prove that "complex analytic triviality" coincides with "differentiable triviality".

Let $\mathscr{V} \xrightarrow{\varpi} M$ be a complex analytic family of compact complex manifolds. We denote by the symbols $\mathfrak{F}$, $\mathfrak{E}$, $\mathfrak{D}$, O, Θ, $\cdots$ the counterparts in analytic geometry of the concepts in differential geometry denoted by the same symbols $\mathfrak{F}$, $\mathfrak{E}$, $\mathfrak{D}$, O, Θ, $\cdots$. These symbols used in the original sense of differential geometry will be distinguished by affixing a subscript F (which signifies "along the fibres"). Thus $\mathfrak{D}$ denotes the sheaf over $\mathscr{V}$ of germs of holomorphic functions, while $\mathfrak{D}_F$ denotes the sheaf over $\mathscr{V}$ of germs of differentiable functions which are holomorphic "along the fibres" of $\mathscr{V}$. Similarly, if $\mathscr{B} \rightarrow \mathscr{V}$ is a holomorphic vector bundle over $\mathscr{V}$, $\mathfrak{D}(\mathscr{B})$ is the sheaf over $\mathscr{V}$ of germs of holomorphic sections of $\mathscr{B}$ and $\mathfrak{D}_F(\mathscr{B})$ is the sheaf over $\mathscr{V}$ of germs of differentiable sections of $\mathscr{B}$ which are holomorphic along the fibres of $\mathscr{V}$. The bundle $\mathfrak{E}$ is the holomorphic tangent bundle of $\mathscr{V}$, $\mathfrak{F}$ the sub-bundle of $\mathfrak{G}$ of tangent vectors along the fibres of $\mathscr{V}$, $\Theta = \mathfrak{O}(\mathfrak{F})$, T_M the sheaf over M of germs of holomorphic sections of the (holomorphic) tangent bundle of M, and T is the sheaf over $\mathscr{V}$ induced from T_M by the map $\varpi: \mathscr{V} \to M$. Moreover, Π denotes the subsheaf of $\mathfrak{O}(\mathfrak{G})$ consisting of germs of holomorphic vector fields on $\mathscr{V}$ whose "horizontal components" are constant along each fibre of $\mathscr{V}$. We have the fundamental sequence

$$0 \longrightarrow \Theta \longrightarrow \Pi \longrightarrow T \longrightarrow 0 .$$

For any open subset U on M, ρ_U denotes the homomorphism

$$\delta^* : T_M(U) = H^0(\mathscr{V} \mid U, T) \longrightarrow H^1(\mathscr{V} \mid U, \Theta) .$$

Passed to the limit, ρ_U gives the homomorphism $\rho: T_M \to \mathscr{H}^1(\Theta)$. As was mentioned in Section 5, Theorem 5.1 remains valid for analytic geometry, namely: *The complex analytic family $\mathscr{V} \to M$ is locally complex-analytically trivial if and only if the map $\rho: T_M \to \mathscr{H}^1(\Theta)$ vanishes.*

Now we examine the relation between $\mathscr{H}^1(\Theta)$ and $\mathscr{H}^1(\Theta_F)$ where $\Theta_F = \mathfrak{O}_F(\mathfrak{F})$ or, more generally, the relation between $H^1(\mathscr{V} \mid U, \mathfrak{O}(\mathscr{B}))$ and $H^1(\mathscr{V} \mid U, \mathfrak{O}_F(\mathscr{B}))$. In the notation of Section 2 we write $\mathfrak{G}^*(r, s) = (\wedge^r \mathfrak{G}^*) \wedge (\wedge^s \overline{\mathfrak{G}}^*)$, $\mathfrak{F}^*(r, s) = (\wedge^r \mathfrak{F}^*) \wedge (\wedge^s \overline{\mathfrak{F}}^*)$. The differentiable sections of $\mathfrak{G}^*(r, s)$ will be called differential forms of type (r, s) on $\mathscr{V}$, while the differentiable sections of $\mathfrak{F}^*(r, s)$ will be called differential forms of type (r, s) along the fibres of $\mathscr{V}$. Similarly, if $\mathscr{B} \to \mathscr{V}$ is a complex-analytic vector bundle over $\mathscr{V}$, then the differentiable sections of $\mathscr{B} \otimes \mathfrak{G}^*(r, s)$, $\mathscr{B} \otimes \mathfrak{F}^*(r, s)$ will be called respectively $\mathscr{B}$-forms of type (r, s) and $\mathscr{B}$-forms of type (r, s) along the fibres of $\mathscr{V}$. We have the exact sequence of complex-analytic vector bundles over $\mathscr{V}$

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{G}/\mathfrak{F} \longrightarrow 0$$

and therefore, by dualization,

$$0 \longrightarrow (\mathfrak{G}/\mathfrak{F})^* \longrightarrow \mathfrak{G}^* \longrightarrow \mathfrak{F}^* \longrightarrow 0 .$$

Hence there is a natural projection $F(r, s)$ of the space of differential forms of type (r, s) onto the space of differential forms of type (r, s) along the fibres of $\mathscr{V}$. Taking the direct sum over all (r, s), we obtain a projection F of forms on $\mathscr{V}$ onto the space of forms along the fibres of $\mathscr{V}$. Given any complex-analytic vector bundle $\mathscr{B} \to \mathscr{V}$, this induces a projection, which we shall also denote by the letter F, of the space of $\mathscr{B}$-forms onto the space of $\mathscr{B}$-forms along the fibres of $\mathscr{V}$. The exterior differential d operating on the differential forms on $\mathscr{V}$ splits in the usual way, namely $d = \partial + \bar{\partial}$ where ∂ is an operator of type $(1, 0)$, $\bar{\partial}$

an operator of type (0, 1), and $\bar\partial$ operates also on $\mathscr{B}$-forms on $\mathscr{V}$. The operator $\bar\partial$ along the fibres of $\mathscr{V}$ defined in Section 2 will be denoted by $\bar\partial_F$. We have

$$F\circ\bar\partial=\bar\partial_F\circ F\,. \tag{18.1}$$

Let $\mathfrak{D}(\mathscr{B})$ be the sheaf over $\mathscr{V}$ of germs ξ of $\bar\partial$-closed $\mathscr{B}$-forms of type (0, 1) satisfying $F\xi=0$. Then we have the exact sequence

$$0\longrightarrow\mathfrak{D}(\mathscr{B})\xrightarrow{\ i\ }\mathfrak{D}_F(\mathscr{B})\xrightarrow{\ \bar\partial\ }\mathfrak{D}(\mathscr{B})\longrightarrow 0 \tag{18.2}$$

where i is the inclusion map. In fact, if ψ is a germ of $\mathfrak{D}_F(\mathscr{B})$, we have, by (18.1), $F\bar\partial\psi=\bar\partial_F\psi=0$; hence $\bar\partial\psi\in\mathfrak{D}(\mathscr{B})$. Conversely, if $\xi\in\mathfrak{D}(\mathscr{B})$, then there exists a germ ψ of differentiable section of $\mathscr{B}$ such that $\bar\partial\psi=\xi$. Since $\bar\partial_F\psi=F\bar\partial\psi=F\xi=0$, ψ belongs to $\mathfrak{D}_F(\mathscr{B})$. Obviously $\psi\in\mathfrak{D}_F(\mathscr{B})$ belongs to the subsheaf $\mathfrak{D}(\mathscr{B})$ if and only if $\bar\partial\psi=0$. We obtain from (18.2) the exact cohomology sequence

$$\begin{aligned}\cdots\longrightarrow H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))&\xrightarrow{\ \bar\partial\ }H^0(\mathscr{V}|U,\mathfrak{D}(\mathscr{B}))\\ \xrightarrow{\ \delta^*\ }H^1(\mathscr{V}|U,\mathfrak{D}(\mathscr{B}))&\xrightarrow{\ i^*\ }H^1(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))\longrightarrow\cdots\end{aligned} \tag{18.3}$$

Let o be an arbitrary point on M and let U be a sufficiently small "spherical" neighborhood of o on M covered by a single system of local coordinates $(t^1,\cdots,t^\alpha,\cdots,t^m)$ with the center o. Moreover, let $B_t=r_t(\mathscr{B})$ be the restriction of $\mathscr{B}$ to $V_t=\varpi^{-1}(t)$ and let $\Omega(B_t)$ be the sheaf over V_t of germs of holomorphic sections of B_t. If $\psi\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))$ and if $f=f(t)$ is a differentiable function on U, then the product $f\psi\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))$ is defined in an obvious manner and

$$r_t(f\psi)=f(t)\cdot r_t(\psi)\,,\qquad r_t(\psi)\in H^0(V_t,\Omega(B_t)).$$

Clearly an arbitrary element $\xi\in H^0(\mathscr{V}|U,\mathfrak{D}(\mathscr{B}))$ has the form

$$\xi=\textstyle\sum_{\alpha=1}^m\xi_\alpha d\bar t^\alpha,\qquad \xi_\alpha\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B})).$$

In particular, if $\psi\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))$, $\bar\partial\psi$ can be written in the form

$$\bar\partial\psi=\textstyle\sum_{\alpha=1}^m\bar\partial_\alpha\psi\cdot d\bar t^\alpha,\quad \bar\partial_\alpha\psi\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B})).$$

Thus the *partial derivatives* $\bar\partial_\alpha\psi=\bar\partial\psi/\partial\bar t^\alpha$ of $\psi\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))$ are defined.

Now assume that $\dim H^0(V_t,\Omega(B_t))=h$ is independent of $t\in M$. By Theorem 2.2, (i), we can find $\psi_1,\cdots,\psi_\nu,\cdots,\psi_h\in H^0(\mathscr{V}|U,\mathfrak{D}_F(\mathscr{B}))$

such that, for each $t \in U$, the restrictions $r_t(\psi_1), \cdots, r_t(\psi_h)$ form a base of $H^0(V_t, \Omega(B_t))$. Clearly an arbitrary element $\psi \in H^0(\mathscr{V}|U, \mathfrak{D}_F(\mathscr{B}))$ is written uniquely in the form

$$\psi = \sum_{\nu=1}^{h} x_\nu \psi_\nu ,$$

where the x_ν are differentiable functions on U. Let

$$\bar{\partial}_\alpha \psi_\nu = \sum_{\lambda=1}^{h} c_{\lambda\nu\alpha} \psi_\lambda$$

where the $c_{\lambda\nu\alpha}$ are differentiable functions on U. Then the partial derivatives $\bar{\partial}_\alpha \psi$ of $\psi = \sum_\nu x_\nu \psi_\nu$ are given by

$$\bar{\partial}_\alpha \psi = \sum_{\lambda=1}^{h} (\bar{\partial}_\alpha x_\lambda + \sum_\nu c_{\lambda\nu\alpha} x_\nu) \psi_\lambda$$

where $\bar{\partial}_\alpha x_\lambda = \partial x_\lambda / \partial \bar{t}^\alpha$. Hence $\psi = \sum x_\nu \psi_\nu$ *belongs to* $H^0(\mathscr{V}|U, \mathfrak{D}(\mathscr{B}))$ *if and only if* $x_1, \cdots, x_\nu, \cdots, x_h$ *satisfy the simultaneous differential equations*

$$\bar{\partial}_\alpha x_\lambda + \sum_{\lambda=1}^{h} c_{\lambda\nu\alpha} x_\nu = 0 , \qquad \lambda = 1, \cdots, h \, ; \, \alpha = 1, \cdots, m. \tag{8.4}$$

We infer from $\bar{\partial}_\beta \bar{\partial}_\alpha \psi = \bar{\partial}_\alpha \bar{\partial}_\beta \psi$ that the coefficients $c_{\lambda\nu\alpha}$ of (18.4) satisfy the *integrabilility condition*

$$\bar{\partial}_\alpha c_{\lambda\nu\beta} - \bar{\partial}_\beta c_{\lambda\nu\alpha} + \sum_\mu (c_{\lambda\mu\alpha} c_{\mu\nu\beta} - c_{\lambda\mu\beta} c_{\mu\nu\alpha}) = 0 . \tag{18.5}$$

PROPOSITION. 18.1. *On a sufficiently small neighborhood* $U_\varepsilon = \{t \,|\, |t^\alpha| < \varepsilon (1 \leqq \alpha \leqq m)\}$ *of* 0, *the simultaneous differential equations*

$$\bar{\partial}_\alpha x_\lambda + \sum_{\nu=1}^{h} c_{\lambda\nu\alpha} x_\nu = 0, \;\; \lambda = 1, \cdots, h \, ; \, \alpha = 1, \cdots , m,$$

have h sets of solutions $x_{1\mu} = x_{1\mu}(t), \cdots, x_{\lambda\mu} = x_{\lambda\mu}(t), \cdots, x_{h\mu} = x_{h\mu}(t)$ $(\mu = 1, 2, \cdots, h)$ *of class* C^∞ *satisfying the initial conditions* : $x_{\lambda\mu}(0) = \delta_{\lambda\mu}$.

A proof of this proposition will be given in Section 19 below.

THEOREM 18.1. *Assume that* $\dim H^0(V_t, \Omega(B_t)) = h$ *is independent of* t. *Let U be a sufficiently small neighborhood of an arbitrary point o on M. Then there exist h elements* $\varphi_1, \cdots, \varphi_h \in H^0(\mathscr{V}|U, \mathfrak{D}(\mathscr{B}))$ *such that, for each* $t \in U$, *the restrictions* $r_t(\varphi_1), \cdots, r_t(\varphi_h)$ *form a base of* $H^0(V_t, \Omega(B_t))$.

PROOF. Let $x_{\lambda\mu}(t)$ be the differentiable functions given in the above Proposition 18.1. Clearly the $h \times h$ matrix $(x_{\lambda\mu}(t))$ is non-singular for $t \in U$, provided that U is sufficiently small. Now let $\varphi_\mu = \sum_{\lambda=1}^{h} x_{\lambda\mu} \psi_\lambda$. Then $\varphi_1, \cdots, \varphi_\mu, \cdots, \varphi_h$ belong to $H^0(\mathscr{V}|U, \mathfrak{D}(\mathscr{B}))$ and $r_t(\varphi_1), \cdots, r_t(\varphi_h)$ form a base of $H^0(V_t, \Omega(B_t))$, q. e. d.

The above Theorem 18.1 shows that, in case $\dim H^0(V_t, \Omega(B_t))$ is independent of t, $\bigcup_{t\in M} H^0(V_t, \Omega(B_t))$ forms a holomorphic vector bundle over M in a canonical manner and $H^0(\mathscr{V}, \mathfrak{O}(\mathscr{B}))$ is isomorphic to the space of holomorphic sections of the bundle $\bigcup_{t\in M} H^0(V_t, \Omega(B_t))$ (compare Proposition 2.7).

PROPOSITION 18.2. *If* $\dim H^0(V_t, \Omega(B_t))$ *is independent of* $t \in M$, *then*

$$i^* : H^1(\mathscr{V}|U, \mathfrak{O}(\mathscr{B})) \longrightarrow H^1(\mathscr{V}|U, \mathfrak{O}_F(\mathscr{B}))$$

is injective, provided that U is a sufficiently small spherical neighborhood on M.

PROOF. In view of (18.3) it suffices to show that

$$\bar{\partial} : H^0(\mathscr{V}|U, \mathfrak{O}_F(\mathscr{B})) \longrightarrow H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{B}))$$

is *surjective*. Let $\varphi_1, \cdots, \varphi_h$ be elements of $H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{B}))$ given by Theorem 18.1. Then an arbitrary element $\xi \in H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{B}))$ is written in the form

$$\xi = \sum_{\alpha=1}^{m} \sum_{\nu=1}^{h} g_{\nu\alpha}\varphi_\nu d\bar{t}^\alpha$$

where the $g_{\nu\alpha}$ are differentiable functions on U. Since

$$0 = \bar{\partial}\xi = \sum_{\alpha,\beta} \sum_{\nu} \frac{1}{2}(\bar{\partial}_\alpha g_{\nu\beta} - \bar{\partial}_\beta g_{\nu\alpha})\varphi_\nu d\bar{t}^\alpha \wedge d\bar{t}^\beta,$$

we have

$$\bar{\partial}_\alpha g_{\nu\beta} - \bar{\partial}_\beta g_{\nu\alpha} = 0 ,$$

or

$$\bar{\partial}(\sum_\alpha g_{\nu\alpha} d\bar{t}^\alpha) = 0 .$$

By hypothesis U is "spherical" and therefore U is a domain of holomorphy. Hence there exist differentiable functions f_ν on U such that $g_{\nu\alpha} = \bar{\partial}_\alpha f_\nu$. Let $\varphi = \sum f_\nu \varphi_\nu$. Clearly $\varphi \in H^0(\mathscr{V}|U, \mathfrak{O}_F(\mathscr{B}))$ and $\xi = \bar{\partial}\varphi$, q. e. d.

THEOREM 18.2. *If a complex analytic family $\mathscr{V} \to M$ of compact complex manifolds is locally differentiably trivial, then $\mathscr{V} \to M$ is locally complex-analytically trivial.*

PROOF. It suffices to show that the homomorphism $\rho : T_M \to \mathscr{H}^1(\Theta)$ vanishes. By hypothesis $\dim H^0(V_t, \Theta_t)$ is independent of $t \in M$. Hence, applying Proposition 18.2 to $\Theta = \mathfrak{O}(\mathfrak{F})$ and $\Theta_F = \mathfrak{O}_F(\mathfrak{F})$, we infer that $i^*: H^1(\mathscr{V}|U, \Theta) \longrightarrow H^1(\mathscr{V}|U, \Theta_F)$ is injective for any sufficiently small spherical domain U on M. This implies that $i^* : \mathscr{H}^1(\Theta) \longrightarrow \mathscr{H}^1(\Theta_F)$ is injective. On the other hand, we have the commutative diagram

$$\begin{tikzcd} & T_M \arrow[dl, "\rho"'] \arrow[dr, "\rho_F"] & \\ \mathscr{H}^1(\Theta) \arrow[rr, "i^*"] & & \mathscr{H}^1(\Theta_F) \end{tikzcd}$$

where ρ_F denotes the map ρ in the differential geometry of deformations. In fact, for any germ $\tilde{v} \in T_M$, $\rho(\tilde{v})$ and $\rho_F(\tilde{v})$ are represented by the same 1-cocycle (see (5.4)). By hypothesis $\rho_F = i^* \circ \rho$ vanishes (see Theorem 5.1). Consequently ρ vanishes, q. e. d.

As an immediate consequence of Theorems 6.2 and 18.2 we have :

THEOREM 18.3. *Let $\mathscr{V} \rightarrow M$ be a complex analytic fibre space without singular fibres. If each fibre $V_t = \varpi^{-1}(t)$ of $\mathscr{V}$ is compact, connected and satisfies $H^1(V_t, \Theta_t) = 0$, then $\mathscr{V} \rightarrow M$ is a complex analytic fibre bundle.*

For example, any complex analytic fibre space with fibre $P_n(\boldsymbol{C})$ (which has no singular fibre) is a complex analytic fibre bundle whose structure group is the projective transformation group acting on $P_n(\boldsymbol{C})$. A classical theorem of Noether [32] and Enriques [13] concerning an algebraic surface possessing a family of ∞^1 rational curves implies in particular that any algebraic fibre space (without singular fibre) of projective lines over an algebraic curve is a projective line bundle. The above result generalizes this to complex analytic fibre spaces with fibre $P_n(\boldsymbol{C})$ over arbitrary base manifolds.

We introduced in Section 1 the concept of " completeness " of a differentiable family (see Definition 1.7). The meaning of the corresponding concept " complex analytic completeness " in the complex analytic geometry of deformations is quite clear. Namely a complex analytic family $\mathscr{V} \xrightarrow{\varpi} M$ with compact fibres is called complex analytically complete if, for any complex analytic family $\mathscr{V}^{(t)} = \{V_{t,s} | s \in N\}$ of deformations of any fibre $V_{t,o} = \varpi^{-1}(t)$ of $\mathscr{V}$, there exists a holomorphic map $s \rightarrow t(s)$, $t(o) = t$, of a neighborhood U of o on N into M such that $\{V_{t,s} | s \in U\}$ coincides with the complex analytic family induced from $\mathscr{V}$ by the map $s \rightarrow t(s)$.

We note that, if $\mathscr{V} \rightarrow M$ is an effectively parametrized complex analytic family which is complete in the complex analytic sense, each point $t \in M$ has a neighborhood U such that $\mathscr{V} | U$ is uniquely determined by the fibre V_t up to equivalence in the complex analytic sense. This can be easily proved in the same way as Proposition 11.1.

PROPOSITION 18.3. *Let* $\mathscr{V} \xrightarrow{\varpi} M$ *and* $\mathscr{W} \xrightarrow{\pi} N$ *be complex analytic families with compact fibres, let* $f : N \longrightarrow M$ *be a differentiable map of* N *into* M *and let* $g : \mathscr{W} \longrightarrow \mathscr{V}$ *be a differentiable map of* $\mathscr{W}$ *into* $\mathscr{V}$ *which maps each fibre* $W_s = \pi^{-1}(s)$, $s \in N$, *biregularly onto* $V_{f(s)} = \varpi^{-1}(f(s))$ *in the complex analytic sense. If* $\mathscr{V} \xrightarrow{\varpi} M$ *is effectively parametrized and if each fibre* $V_t = \varpi^{-1}(t)$, $t \in M$, *admits no continuous group of analytic automorphisms, then the maps* $f : N \rightarrow M$ *and* $g : \mathscr{W} \longrightarrow \mathscr{V}$ *are both holomorphic.*

PROOF. It can be verified, in the same manner as in the proof of Proposition 11.1, that $f : N \rightarrow M$ is holomorphic. Now let U be a sufficiently small neighborhood of an arbitrary point on N covered by a system of holomorphic coordinates $(s^1, \cdots, s^\nu, \cdots, s^l)$, let $\{\mathscr{W}_i\}$ be a finite covering of $\mathscr{W}|U$ by coordinate neighborhoods $\mathscr{W}_i$, and let $(\zeta_i^1, \cdots, \zeta_i^n, s^1, \cdots, s^l)$ be a system of holomorphic coordinates on $\mathscr{W}_i$ such that

$$\pi : (\zeta_i^1, \cdots, \zeta_i^n, s^1, \cdots, s^l) \rightarrow (s^1, \cdots, s^l) .$$

We may assume that $f(U)$ is contained in a neighborhood on M covered by a system of holomorphic coordinates $(t^1, \cdots, t^m)$ and that $g(\mathscr{W}_i)$ is contained in a neighborhood $\mathscr{V}_i \subset \mathscr{V}$ covered by a system of holomorphic coordinates $(z_i^1, \cdots, z_i^n, t^1, \cdots, t^m)$ such that

$$\varpi : (z_i^1, \cdots, z_i^n, t^1, \cdots, t^m) \rightarrow (t^1, \cdots, t^m) .$$

In terms of the local coordinates, the map $g : \mathscr{W} \rightarrow \mathscr{V}$ is represented in the form

$$\begin{cases} z_i^\alpha = g_i^\alpha(\zeta_i, s) , \\ t = f(s) , \end{cases}$$

where $g_i^\alpha(\zeta_i, s)$ are differentiable in (ζ_i, s), holomorphic in ζ_i, and $f(s)$ is holomorphic in s. We define

$$\theta_{i\bar{\nu}}^\alpha(z_i, s) = \frac{\partial}{\partial \bar{s}^\nu} g_i^\alpha(\zeta_i, s) , \qquad \text{where } z_i^\alpha = g_i^\alpha(\zeta_i, s),$$

and consider, for each fixed $s \in U$,

$$\theta_{i\bar{\nu}}(z_i, s) = (\theta_{i\bar{\nu}}^1(z_i, s), \cdots, \theta_{i\bar{\nu}}^\alpha(z_i, s), \cdots, \theta_{i\bar{\nu}}^n(z_i, s))$$

as a holomorphic vector field on $V_{f(s)} \cap g(\mathscr{W}_i)$. We have

$$z_i^\alpha = h_{ik}^\alpha(z_k, t) , \qquad \text{on } \mathscr{V}_i \cap \mathscr{V}_k,$$

$$\zeta_i^\alpha = \eta_{ik}^\alpha(\zeta_k, s) , \qquad \text{on } \mathscr{W}_i \cap \mathscr{W}_k,$$

where $h_{ik}^\alpha(z_k, t)$ and $\eta_{ik}^\alpha(\zeta_k, s)$ are holomorphic functions in z_k, t and ζ_k, s, respectively. Moreover we have

$$g_i^\alpha(\eta_{ik}(\zeta_k, s), s) = h_{ik}^\alpha(g_k(\zeta_k, s), f(s)) .$$

By applying $\partial/\partial\bar{s}^{\nu}$, we obtain from this the equality

$$\theta^{\alpha}_{i\bar{\nu}}(z_i, s) = \sum_{\beta} \frac{\partial z^{\alpha}_i}{\partial z^{\beta}_k} \cdot \theta^{\beta}_{k\bar{\nu}}(z_k, s) , \quad \text{where } z^{\alpha}_i = h^{\alpha}_{ik}(z_k, f(s)),$$

or

$$\theta_{i\bar{\nu}}(z_i, s) = \theta_{k\bar{\nu}}(z_k, s) , \quad \text{on } V_{f(s)} \cap g(\mathscr{W}_i) \cap g(\mathscr{W}_k).$$

This shows that $\theta_{\bar{\nu}}(s) = \theta_{i\bar{\nu}}(\cdots, s) = \theta_{k\bar{\nu}}(\cdots, s)$ is a global holomorphic vector field on $V_{f(s)}$. Hence, by hypothesis, $\theta_{\bar{\nu}}$ vanishes identically, and therefore

$$\frac{\partial}{\partial\bar{s}^{\nu}} g^{\alpha}_i(\zeta_i, s) = \theta^{\alpha}_{i\bar{\nu}}(z_i, s) = 0.$$

Thus g is a holomorphic map, q. e. d.

THEOREM 18.4. *Let $\mathscr{V} \xrightarrow{\varpi} M$ be an effectively parametrized complex-analytic family whose fibres $V_t = \varpi^{-1}(t)$, $t \in M$, are compact and admit no continuous group of analytic automorphisms. If $\mathscr{V} \xrightarrow{\varpi} M$ is differentiably complete, then $\mathscr{V} \xrightarrow{\varpi} M$ is complex analytically complete.*

PROOF. Let $\mathscr{W} = \{V_{t,s} | s \in N\}$ be a complex analytic family of deformations $V_{t,s}$ of $V_t = V_{t,o} \in \mathscr{V}$. By hypothesis there exist a neighborhood U of o on N and a differentiable map $f : U \to M$ such that $\mathscr{W}|U$ coincides with the differentiable family induced from $\mathscr{V} \to M$ by the map f. This means that there exists a differentiable map $g : \mathscr{W}|U \to \mathscr{V}$ which maps each fibre $V_{t,s}$ biregularly onto $V_{f(s)}$ in the complex analytic sense. Now, by the above Proposition 18.3, f and g are holomorphic. Thus $\mathscr{W}|U$ is the complex analytic family induced from $\mathscr{V}$ by the holomorphic map $f : U \to M$ in the complex analytic sense, q. e. d.

THEOREM 18.5. *The complex analytic family $\mathscr{V}_{n,h}$ of all non-singular hypersurfaces of order h on $P_{n+1}(C)$, $n \geqq 2$, is complex analytically complete except for the special case : $n = 2$, $h = 4$.*

PROOF. (i) The case $h = 2$. Since, by Theorem 14.2, the complex structure of $V_t \in \mathscr{V}_{n,2}$ is rigid, any complex analytic family of "small" deformations of $V_t \in \mathscr{V}_{n,2}$ is differentiably trivial, and therefore, by Theorem 18.2, is complex analytically trivial. Hence $\mathscr{V}_{n,2}$ is complex analytically complete.

(ii) The case $h \geqq 3$. For any hypersurface $V_t \in \mathscr{V}_{n,h}$, there exists an effectively parametrized complex analytic subfamily $\mathscr{V}_{n,h}$ which is differentiably complete (see Section 14, (γ)) while, by Lemma 14.2, every hypersurface $V_t \in \mathscr{V}_{n,h}$ admits no continuous group of analytic automorphisms. Hence, by the above Theorem 18.4, $\mathscr{V}_{n,h}|U'$ is complex analytically complete and therefore $\mathscr{V}_{n,h}$ is complex analytically complete.

THEOREM 18.6. *The complex analytic family of complex tori is complex analytically complete.*

PROOF. It suffices to notice that, in the proof of Theorem 14.3, we can choose, with the help of Theorem 18.1, the 1-forms $\Psi^{\alpha}(x, t)$ such that $\Psi^{\alpha}(x, t)$ are holomorphic on $\mathscr{W}|U$ in the full sense, provided that the given family of deformations of a torus $B_s \in \mathscr{B}$ is complex analytic.

19. Partial differential equations in $\bar{\partial}$

Let $f(z, u) = f(z, u_1, \cdots, u_l)$ be a function of class C^∞ of a complex variable z and real variables $u_1, \cdots, u_l$ defined on the domain : $|z| < \varepsilon$, $u \in U$, where $0 < \varepsilon < 1$ and U is a domain on the space of real variables $(u_1, \cdots, u_l)$. For simplicity we write ∂_u^k for any partial derivation

$$\frac{\partial^k}{\partial u_1^{k_1} \partial u_2^{k_2} \cdots \partial u_l^{k_l}}, \qquad (k_1 + \cdots + k_l = k)$$

of order k and ∂_z, $\bar{\partial}_z$ for $\partial/\partial z$, $\partial/\partial \bar{z}$, respectively.

LEMMA 19.1. *If $f(z, u)$ is of class C^∞ and if $f(z, u)$ and its partial derivatives $\partial_u^k f(z, u)$ are bounded for $|z| < \varepsilon$, $u \in U$, then*

$$g(z, u) = \frac{1}{2\pi i} \iint_{|\zeta| < \varepsilon} \frac{f(\zeta, u)}{\zeta - z} d\zeta d\bar{\zeta}$$

is also a function of class C^∞ and $g(z, u)$, $\partial_u^k g(z, u)$ are bounded for $|z| < \varepsilon$, $u \in U$. Moreover we have

$$\bar{\partial}_z g(z, u) = f(z, u), \tag{19.1}$$

$$\partial_z g(z, u) = \frac{1}{2\pi i} \iint_{|\zeta| < \varepsilon} \frac{f(\zeta, u) - f(z, u)}{(\zeta - z)^2} d\zeta d\bar{\zeta}, \tag{19.2}$$

$$\partial_u^k g(z, u) = \frac{1}{2\pi i} \iint_{|\zeta| < \varepsilon} \frac{\partial_u^k f(\zeta, u)}{\zeta - z} d\zeta d\bar{\zeta}. \tag{19.3}$$

PROOF. It is well known that, for each fized u, $g(z, u)$ is a function of class C^∞ in z and satisfies (19.1). To verify that $g(z, u)$ is of class C^∞ in both variables z and u, we take ε_0, ε_1, ε_2, $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$, and functions $\rho_1(\zeta)$, $\rho_2(\zeta)$ of class C^∞ such that

$$\rho_1(\zeta) + \rho_2(\zeta) = 1, \ |\rho_1(\zeta)| \leqq 1, \ |\rho_2(\zeta)| \leqq 1,$$
$$\rho_1(\zeta) = 0 \text{ for } |\zeta| \geqq \varepsilon_2, \ \rho_2(\zeta) = 0 \text{ for } |\zeta| \leqq \varepsilon_1,$$

and let

$$f_1(\zeta, u) = \rho_1(\zeta) f(\zeta, u), \ f_2(\zeta, u) = \rho_2(\zeta) f(\zeta, u).$$

Assume that $|z| < \varepsilon_0$. The function

$$g_1(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon}\frac{f_1(\zeta, u)}{\zeta - z}d\zeta d\bar{\zeta} = \frac{1}{2\pi i}\iint\frac{f_1(\zeta + z, u)}{\zeta}d\zeta d\bar{\zeta}$$

is of class C^∞ in z and u. In fact, the partial derivative $\partial_z^p\bar{\partial}_z^q\partial_u^k g_1(z, u)$ is given by the integral

$$\frac{1}{2\pi i}\iint\partial_z^p\bar{\partial}_z^q\partial_u^k f_1(\zeta + z, u)\frac{d\zeta d\bar{\zeta}}{\zeta}$$

which converges absolutely and uniformly for $|z| < \varepsilon_0$, $u \in U_1 \subset U$, U_1 being an arbitrary open subset of U with compact closure contained in U. Next,

$$g_2(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon}\frac{f_2(\zeta, u)}{\zeta - z}d\zeta d\bar{\zeta}$$

is also a function of class C^∞ in z and u, $|z| < \varepsilon_0$, $u \in U$, since, for arbitrary p, q, k, the partial derivative

$$\partial_z^p\bar{\partial}_z^q\partial_u^k\left(\frac{f_2(\zeta, u)}{\zeta - z}\right) = \begin{cases} 0 & (q \geqq 1) \\ \dfrac{p!\rho_2(\zeta)\partial_u^k f(\zeta, u)}{(\zeta - z)^{p+1}}, & (q = 0)\end{cases}$$

is continuous and bounded for $|z| < \varepsilon_0$, $|\zeta| < \varepsilon$, $u \in U$. Hence $g(z, u) = g_1(z, u) + g_2(z, u)$ is a function of class C^∞ in z and u, $|z| < \varepsilon_0$, $u \in U$, where ε_0 is an arbitrary number $< \varepsilon$. Consequently $g(z, u)$ is of class C^∞ for $|z| < \varepsilon$, $u \in U$. Moreover we have

$$\partial_u^k g(z, u) = \partial_u^k g_1(z, u) + \partial_u^k g_2(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon}\frac{\partial_u^k f(\zeta, u)}{\zeta - z}d\zeta d\bar{\zeta}.$$

We obtain from this

$$|\partial_u^k g(z, u)| < 4\varepsilon \sup_{|\zeta|<\varepsilon}|\partial_u^k f(\zeta, u)|,$$

since

$$\frac{1}{2\pi}\iint_{|\zeta|<\varepsilon}\frac{|d\zeta d\bar{\zeta}|}{|\zeta - z|} < 4\varepsilon.$$

Thus $\partial_u^k g(z, u)$ is bounded. We infer readily that

$$g(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon}\frac{f(\zeta, u) - f(z, u)}{\zeta - z}d\zeta d\bar{\zeta} + \bar{z}f(z, u).$$

This leads immediately to the formula (19.2), q.e.d.

We fix a positive integer h and, for any $h \times h$ matrix $X = (x_{\lambda\nu})_{\lambda,\nu=1,\ldots,h}$, we denote by $|X|$ the *norm* of X: $|X| = \sqrt{\sum_{\lambda,\nu}|x_{\lambda\nu}|^2}$. Now let $\mathscr{S}$ be the space of matrix-valued functions

$$X(z, u) = (x_{\lambda\nu}(z, u))_{\lambda,\nu=1,\cdots,h}$$

of class C^∞ in z and u defined on the domain: $|z| < \varepsilon$, $u \in U$ such that $|X(z, u)|$ and $|\partial_u^k X(z, u)|$ are bounded. Given a function $C(z, u) \in \mathscr{S}$ such that $\partial_z C(z, u) \in \mathscr{S}$, $\bar{\partial}_z C(z, u) \in \mathscr{S}$, we consider the differential equation

$$\bar{\partial}_z X(z, u) = C(z, u)X(z, u) \ . \tag{19.4}$$

By hypothesis there exist positive constants $a, a^{(1)}, \cdots, a^{(k)}, \cdots$, such that

$$|C(z, u)| < a, \quad |\partial_z C(z, u)| + |\bar{\partial}_z C(z, u)| < a \ , \tag{19.5}$$

$$|\partial_u^k C(z, u)| < a^{(k)}, \quad |\partial_z \partial_u^k C(z, u)| + |\bar{\partial}_z \partial_u^k C(z, u)| < a^{(k)} \ . \tag{19.6}$$

To prove the existence of a solution $X(z, u)$ of (19.4) satisfying the initial condition: $X(0, u) = 1$, we introduce the integral transformation

$$\mathscr{I}_z(C)\colon X(z, u) \to Y(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon} \frac{zC(\zeta, u)X(\zeta, u)}{\zeta(\zeta - z)} d\zeta d\bar{\zeta} \ .$$

We note that

$$\begin{aligned} Y(z, u) = & \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon} \frac{(CX)(\zeta, u)}{\zeta - z}\, d\zeta d\bar{\zeta} \\ & - \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon} \frac{(CX)(\zeta, u)}{\zeta}\, d\zeta d\bar{\zeta} \ , \end{aligned} \tag{19.7}$$

where $(CX)(\zeta, u) = C(\zeta, u)X(\zeta, u)$.

LEMMA 19.2. *If $X(z, u)$ belongs to $\mathscr{S}$, then $Y(z, u) = (\mathscr{I}_z(C)X)(z, u)$ belongs to $\mathscr{S}$ and satisfies*

$$\bar{\partial}_z Y(z, u) = C(z, u)X(z, u), \quad Y(0, u) = 0 \ , \tag{19.8}$$

$$\partial_z Y(z, u) = \frac{1}{2\pi i}\iint_{|\zeta|<\varepsilon} \frac{C(\zeta, u)X(\zeta, u) - C(z, u)Y(z, u)}{(\zeta - z)^2} d\zeta d\bar{\zeta} \ . \tag{19.9}$$

If, moreover,

$$X(0, u) = 0 \ , \quad |\partial_z X(z, u)| + |\bar{\partial}_z X(z, u)| < b \ , \tag{19.10}$$

then $Y(z, u)$ satisfies

$$|\partial_z Y(z, u)| + |\bar{\partial}_z Y(z, u)| < 5\varepsilon ab \ . \tag{19.11}$$

PROOF. We infer from Lemma 19.1 that $Y(z, u)$ belongs to $\mathscr{S}$ and satisfies (19.8) and (19.9). Now we assume (19.10). Since $|\partial_z X(z, u)| +$

$|\bar{\partial}_z X(z, u)| < b$ implies that

$$|X(\zeta, u) - X(z, u)| < b|\zeta - z| ,$$

we have

$$|X(z, u)| < b|z| < b\varepsilon$$

and therefore

$$|\bar{\partial}_z Y(z, u)| \leqq |C(z, u)| \cdot |X(z, u)| < \varepsilon ab .$$

Moreover we have

$$|C(\zeta, u)X(\zeta, u) - C(z, u)X(z, u)| < 2ab|\zeta - z|$$

and consequently, by (19.9),

$$|\partial_z Y(z, u)| \leqq \frac{ab}{\pi} \iint_{|\zeta|<\varepsilon} \frac{|d\zeta d\bar{\zeta}|}{|\zeta - z|} < 4\varepsilon ab .$$

Hence we obtain (19.11), q.e.d.

In what follows we assume that

$$5\varepsilon a < 1 . \tag{19.12}$$

Let $E_0(z, u) = 1$ and let

$$E_n(z, u) = (\mathscr{I}_z(C)^n E_0)(z, u) , \qquad n = 1, 2, \cdots .$$

By Lemma 19.2, $E_n(z, u)$ belongs to $\mathscr{S}$ and satisfies

$$\bar{\partial}_z E_n(z, u) = C(z, u)E_{n-1}(z, u) , \quad E_n(0, u) = 0 . \tag{19.13}$$

LEMMA 19.3. *The series* $\sum_{n=0}^{\infty} E_n(z, u)$, $\sum_{n=0}^{\infty} \partial_z E_n(z, u)$, $\sum_{n=0}^{\infty} \bar{\partial}_z E_n(z, u)$ *converge absolutely and uniformly for* $|z| < \varepsilon$, $u \in U$.

PROOF. We have

$$|\partial_z E_1(z, u)| + |\bar{\partial}_z E_1(z, u)| < 5a .$$

Hence, by Lemma 19.2, we obtain

$$|\partial_z E_n(z, u)| + |\bar{\partial}_z E_n(z, u)| < (5\varepsilon a)^{n-1} 5a$$

and consequently

$$|E_n(z, u)| < (5\varepsilon a)^n ,$$

q.e.d.

Letting

$$E(z, u) = \sum_{n=0}^{\infty} E_n(z, u) ,$$

we infer therefore that $E(z, u)$ *and* $\partial_z E(z, u)$, $\bar{\partial}_z E(z, u)$ *are continuous in* z *and* u and that

$$(19.14) \qquad \bar{\partial}_z E(z, u) = C(z, u)E(z, u), \quad E(0, u) = 1 .$$

To prove that $E(z, u)$ is a function of class C^∞ in z and u, we need the following lemma.

LEMMA 19.4. *The series*

$$\sum_{n=0}^{\infty} \partial_u^k E_n(z, u), \quad \sum_{n=0}^{\infty} \partial_z \partial_u^k E_n(z, u), \quad \sum_{n=0}^{\infty} \bar{\partial}_z \partial_u^k E_n(z, u)$$

converge absolutely and uniformly for $|z| < \varepsilon$, $u \in U$.

PROOF. We apply induction on k. In case $k = 0$, our lemma is reduced to Lemma 19.3. Assume therefore that our lemma is proved for $k \leqq j - 1$ and let

$$\beta_n^{(j)} = \sup_{z,u}(|\partial_z \partial_u^j E_n(z, u)| + |\bar{\partial}_z \partial_u^j E_n(z, u)|) .$$

Since $E_n(0, u) = 0$ ($n \geqq 1$), we have $\partial_u^j E_n(0, u) = 0$ and consequently

$$|\partial_u^j E_n(z, u)| \leqq \varepsilon \beta_n^{(j)} .$$

It suffices therefore to show that

$$\sum_{n=1}^{\infty} \beta_n^{(j)} < + \infty .$$

For simplicity we write $C, E_n, \cdots$ for $C(z, u), E_n(z, U), \cdots$. It is clear that $\partial_u^j(CE_n)$ is written in the form

$$(19.15) \qquad \partial_u^j(CE_n) = C \cdot \partial_u^j E_n + \sum_{k=0}^{j-1} \sum_{\partial_u^k} w(\partial_u^j, \partial_u^k) \partial_u^{j-k} C \cdot \partial_u^k E_n ,$$

where the sum $\sum_{\partial_u^k}$ is extended over all partial derivations ∂_u^k of order k and $w(\partial_u^j, \partial_u^k)$ are non-negative integers depending only on the pairs $\partial_u^j, \partial_u^k$. We note that

$$\sum_{\partial_u^k} w(\partial_u^j, \partial_u^k) = \binom{j}{k} .$$

Now, since $E_{n+1}(z, u) = (\mathscr{I}_z(C)E_n)(z, u)$, we have, by (19.3) and (19.7),

$$\partial_u^j E_{n+1}(z, u) = \frac{1}{2\pi i} \iint_{|\zeta| < \varepsilon} \frac{z \partial_u^j \{C(\zeta, u)E_n(\zeta, u)\}}{\zeta(\zeta - z)} d\zeta d\bar{\zeta} .$$

Inserting (19.15) in the right hand side of this formula, we obtain

$$(19.16) \qquad \partial_u^j E_{n+1} = \mathscr{I}_z(C) \partial_u^j E_n + \sum_{k=0}^{j-1} \sum w(\partial_u^j, \partial_u^k) \mathscr{I}_z(\partial_u^{j-k} C) \partial_u^k E_n .$$

By hypothesis there exist convergent series $\sum_{n=0}^{\infty} b_n^{(k)}$ of positive constants $b_n^{(k)}$ for $k \leqq j - 1$ such that

$$|\partial_z \partial_u^k E_n(z, u)| + |\bar{\partial}_z \partial_u^k E_n(z, u)| < b_n^{(k)} ,$$

while $\partial_u^k E_n(0, u) = 0$. Hence, using Lemma 19.2, we derive from (19.16) the inequality

$$\beta_{n+1}^{(j)} \leqq 5\varepsilon a \beta_n^{(j)} + \sum_{k=0}^{j-1} \binom{j}{k} 5\varepsilon a^{(j-k)} b_n^{(k)} ,$$

where $a^{(j-k)}$ is defined in (19.6). This yields

$$\beta_{n+1}^{(j)} \leqq (5\varepsilon a)^n \beta_1^{(j)} + \sum_{k=0}^{j-1} \binom{j}{k} 5\varepsilon a^{(j-k)} \sum_{i=1}^{n} (5\varepsilon a)^{n-i} b_i^{(k)} .$$

Consequently we obtain

$$\sum_{n=1}^{\infty} \beta_n^{(j)} \leqq \frac{1}{1 - 5\varepsilon a} \cdot \{\beta_1^{(j)} + \sum_{k=0}^{j-1} \binom{j}{k} 5\varepsilon a^{(j-k)} \sum_{n=1}^{\infty} b_n^{(k)}\} .$$

On the other hand, we infer readily that $\beta_1^{(j)} < 5a^{(j)}$. Hence we get

$$\sum_{n=1}^{\infty} \beta_n^{(j)} < + \infty ,$$

q.e.d.

It follows from (19.14) that $E(z, u)$ is a weak solution of the elliptic differential equation

$$\partial_z \bar{\partial}_z E(z, u) - C(z, u) \cdot \partial_z E(z, u) - \partial_z C(z, u) \cdot E(z, u) = 0 .$$

with respect to the variable z whose coefficients $C(z, u)$, $\partial_z C(z, u)$ are functions of class C^∞ in z and u, while the above Lemma 19.4 implies that all partial derivatives $\partial_u^k E(z, u)$ of $E(z, u)$ with respect to u exist and are continuous in z and u. We infer from this readily that $E(z, u)$ is a function of class C^∞ in both variables z and u. In fact, given arbitrary points z_0 and u_0, $|z_0| < \varepsilon$, $u_0 \in U$, and an arbitrary integer $\nu > 0$, we take a sufficiently small number $\delta > 0$. Then, for $|z - z_0| < \delta$, $|u - u_0| < \delta$, $E(z, u)$ satisfies an integral equation

$$E(z, u) + \iint_{|\zeta|<\varepsilon} P(z, \zeta, u) E(\zeta, u) d\zeta d\bar{\zeta} = 0 , \tag{19.17}$$

where the kernel $P(z, \zeta, u)$ is a function of class C^ν in z, u, ζ defined for $|z - z_0| < \delta$, $|\zeta| < \varepsilon$, $|u - u_0| < \delta$, such that $P(z, \zeta, u) = 0$ for $|\zeta| > \varepsilon - \delta$ (see Kodaira and Spencer [28], pp. 144–145; cf. also Baily [5], pp. 868–885). Since $P(z, \zeta, u)E(\zeta, u)$ admits partial derivatives with respect to z and u up to the order ν which are continuous in z, ζ, u and vanishes for $|\zeta| > \varepsilon - \delta$, we infer from (19.17) that $E(z, u)$ is a function of class C^ν in z and u, where ν is an arbitrary positive integer. Thus we conclude that $E(z, u)$ is of class C^∞ with respect to z and u. We denote $E(z, u)$ by $\mathscr{E}_z(C)(z, u)$:

$$\mathscr{E}_z(C)(z, u) = \sum_{n=0}^{\infty} (\mathscr{I}_z(C)^n E_0)(z, u), \qquad (E_0 = 1) . \tag{19.18}$$

Now let

$$C_\alpha(t) = C_\alpha(t_1, \cdots, t_l), \qquad \alpha = 1, 2, \cdots, m ,$$

be matrix-valued functions of class C^∞ in l complex variables $t_1, \cdots, t_l$ (whose values are $h \times h$ matrices) defined on the domain: $|t_\beta| < 1$ $(1 \leqq \beta \leqq l)$ and let

$$\bar{\nabla}_\alpha = \bar{\partial}_\alpha - C_\alpha(t), \qquad \bar{\partial}_\alpha = \partial/\partial\bar{t}_\alpha .$$

We consider the system of differential equations

$$\bar{\nabla}_\alpha X(t) = 0, \qquad \alpha = 1, 2, \cdots, m ,$$

where $m \leqq l$. We assume that this system satisfies the integrability condition:

$$\bar{\nabla}_\alpha\bar{\nabla}_\beta - \bar{\nabla}_\beta\bar{\nabla}_\alpha = 0, \qquad \alpha, \beta = 1, 2, \cdots, m . \tag{19.19}$$

PROPOSITION 19.1[1]. *For a sufficiently small number $\varepsilon > 0$, there exists a matrix-valued function $X(t)$ of class C^∞ in t defined on the domain: $|t_\beta| < \varepsilon$ $(1 \leqq \beta \leqq l)$ satisfying*

$$\bar{\nabla}_\alpha X(t) = 0 \ (\alpha = 1, 2, \cdots, m), \ X(0) = 1 . \tag{19.20}$$

PROOF. We apply induction on m. (i) The case $m = 1$. Let

$$z = t_1, u_1 = \Re t_2, u_2 = \Im t_2, \cdots, u_{2l-2} = \Im t_l,$$
$$C_1(z, u) = C_1(z, u_1 + iu_2, \cdots, u_{2l-3} + iu_{2l-2}).$$

Then, for sufficiently small $\varepsilon > 0$, $C_1(z, u)$ satisfies (19.5), (19.6) and (19.12) for $|z| < \varepsilon$, $|u_j| < \varepsilon$ $(1 \leqq j \leqq 2l - 2)$. Hence, letting $X(t) = \mathscr{E}_z(C_1)(z, u)$, we obtain a solution $X(t)$ of (19.20). (ii) Assume that our proposition is proved for the system of $m - 1$ differential equations $\bar{\nabla}_\alpha X(t) = 0$ $(\alpha = 2, \cdots, m)$. We have therefore a function $F(t)$ of class C^∞ defined for $|t_\beta| < \varepsilon'$ $(1 \leqq \beta \leqq l)$ satisfying

$$\bar{\nabla}_\alpha F(t) = 0 \ (\alpha = 2, 3, \cdots, m), \quad F(0) = 1 .$$

Take a positive number $\varepsilon'' < \varepsilon'$ such that $F(t)$ is non-singular for $|t_\beta| < \varepsilon''$ $(1 \leqq \beta \leqq l)$, and define $C(t)$ by

$$\bar{\nabla}_1 F(t) = -F(t)C(t), \qquad |t_\beta| < \varepsilon'' .$$

Since for each α, $2 \leqq \alpha \leqq m$,

$$0 = \bar{\nabla}_1\bar{\nabla}_\alpha F(t) = \bar{\nabla}_\alpha\bar{\nabla}_1 F(t) = -\bar{\nabla}_\alpha(F(t)C(t)) = -F(t)\bar{\partial}_\alpha C(t),$$

[1] Professor L. Nirenberg has called our attention to the fact that Proposition 19.1 can be deduced, by a device originally due to E. Cartan, from the main theorem of his paper on a complex Frobenius theorem which will appear in the proceedings of the Conference on the Theory of Analytic Functions held at the Institute for Advanced Study, September 1–14, 1957.

we have

$$\bar{\partial}_\alpha C(t) = 0, \qquad \alpha = 2, 3, \cdots, m . \tag{19.21}$$

Now let

$$X(t) = F(t)Y(t) .$$

Then we have

$$\bar{\nabla}_1 X(t) = F(t)\{\bar{\partial}_1 Y(t) - C(t)Y(t)\} ,$$

$$\bar{\nabla}_\alpha X(t) = F(t) \cdot \bar{\partial}_\alpha Y(t), \qquad \alpha = 2, \cdots, m .$$

Hence the system of differential equations (19.20) is reduced to

$$\begin{cases} \bar{\partial}_1 Y(t) = C(t)Y(t), \quad Y(0) = 1 , \\ \bar{\partial}_\alpha Y(t) = 0, \qquad \alpha = 2, \cdots, m . \end{cases} \tag{19.22}$$

Let $z = t_1$, $u_1 = \Re t_2$, $u_2 = \Im t_2, \cdots, u_{2l-2} = \Im t_l$. For sufficiently small $\varepsilon > 0$, $C(z, u_1 + iu_2, \cdots, u_{2l-3} + iu_{2l-2})$ satisfies (19.5), (19.6), (19.12) for $|z| < \varepsilon$, $|u_j| < \varepsilon$. Hence $Y(t) = \mathscr{E}_z(C)(z, u)$ is a function of class C^∞ for $|z| < \varepsilon$, $|u_j| < \varepsilon$ and satisfies $\bar{\partial}_1 Y(t) = C(t)Y(t)$, $Y(0) = 1$. Moreover $Y(t)$ satisfies $\bar{\partial}_\alpha Y(t) = 0$ for $\alpha = 2, 3, \cdots, m$. In fact, by Lemma 19.4 we have

$$\bar{\partial}_\alpha Y(t) = \sum_{n=0}^{\infty} \bar{\partial}_\alpha E_n(z, u) ,$$

while, by (19.16) and (19.21),

$$\bar{\partial}_\alpha E_n(z, u) = \mathscr{I}_z(C)(\bar{\partial}_\alpha E_{n-1})(z, u) ,$$

and therefore

$$\bar{\partial}_\alpha E_n(z, u) = \mathscr{I}_z(C)^n(\bar{\partial}_\alpha E_0)(z, u) = 0 .$$

Thus we infer that $Y(t) = \mathscr{E}_z(C)(z, u)$ satisfies (19.22) and consequently $X(t) = F(t) \cdot \mathscr{E}_z(C)(z, u)$ satisfies (19.20), q.e.d.

Now it is clear that Proposition 18.1 is equivalent to the above Proposition 19.1.

Chapter VIII. Arbitrary deformations ; problems

20. Arbitrary deformations of projective plane and 2-torus

In Section 14, (α), we pointed out that the complex analytic structure of projective n-space $P_n(\boldsymbol{C})$ is *rigid* under any small deformation and, in Section 14, (γ), we showed that any small deformation of a complex n-torus is a complex n-torus (whose complex structure is generally different). The question arises: do these statements remain valid for arbitrarily large deformations? We are able to answer this question only for the case $n = 2$; then the answer is affirmative, namely :

THEOREM 20.1. *An arbitrary deformation of the projective plane is the projective plane, i.e. if V is c-homotopic to the projective plane, then V is the projective plane.*

THEOREM 20.2. *An arbitrary deformation of a complex torus of complex dimension 2 is a complex torus, i.e. if V is c-homotophic to a 2-torus, then V is a 2-torus.*

In connection with Theorem 20.1 we remark that Hirzebruch and Kodaira [18] have proved that any Kähler manifold of odd complex dimension n whose underlying differentiable structure is that of $P_n(\boldsymbol{C})$ is (complex analytically homeomorphic to) $P_n(\boldsymbol{C})$. Moreover, it is an immediate consequence of their paper that a Kähler manifold of arbitrary complex dimension n whose complex structure is a *deformation* of $P_n(\boldsymbol{C})$ is $P_n(\boldsymbol{C})$.

PROOF OF THEOREM 20.1. Let $\mathscr{V} = \{V_t \mid -1 < t < 1\}$ be a one-parameter differentiable family of compact complex analytic surfaces. Since $P_2(\boldsymbol{C})$ is rigid, it suffices to show that, *if V_t is $P_2(\boldsymbol{C})$ for $t > 0$, V_0 is also* $P_2(\boldsymbol{C})$. If V_0 is an algebraic surface, then, by the above remark, V_0 is $P_2(\boldsymbol{C})$, while, if there exists on V_0 an irreducible curve C with the intersection multiplicity $(C\cdot C) > 0$, V_0 is an algebraic surface (Kodaira [25], Theorem 3.3). Thus it is sufficient to prove the existence of an irreducible curve C with $(C\cdot C) > 0$ on V_0.

Let X be the underlying differentiable manifold of V_t. By hypothesis $H^2(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$. It follows that V_t has no holomorphic 2-form. In fact, if φ_t were a holomorphic 2-form on V_t which does not vanish identically, we would have $\overline{\varphi}_t = a\varphi_t + d\psi$, where a is a constant, and therefore

$$0 < \int_{V_t} \varphi_t \wedge \overline{\varphi}_t = \int_{V_t} d(\psi \wedge \varphi_t) = 0 \ ,$$

a contradiction ! By the duality theorem we obtain therefore

$$\dim H^2(V_t, \Omega_t) = 0 \ , \qquad -1 < t < 1. \tag{20.1}$$

For $t > 0$ let E_t be the complex line bundle over V_t defined by a line on $V_t = P_2(\boldsymbol{C})$. The characteristic class $c = c(E_t)$ is obviously a generator of $H^2(X, \boldsymbol{Z})$. Let $U = \{t \mid -\varepsilon < t < \varepsilon\}$ be a small neighborhood of 0. Then, using (20.1), we infer from Proposition 13.1 that there exists a complex line bundle $\mathscr{F} \in H^1(\mathscr{V} \mid U, \mathfrak{O}^*)$ such that $\delta^*(\mathscr{F}) = c$ (compare (13.5)). Let $F_t = \mathscr{F} \mid V_t$. Clearly we have $F_t = E_t$ for $0 < t < \varepsilon$. Now, by Theorem 2.1 (upper semi-continuity), we have

$$\dim H^0(V_0, \Omega(F_0)) \geqq \dim H^0(V_0, \Omega(E_t)) = 3 \ , \qquad t > 0.$$

Hence the complete linear system $|F_0|$ contains a divisor $D > 0$. Let $D = \sum_k m_k C_k$ where C_k are irreducible curves on V_0. Since the homology class of D is dual to the generator c, one of C_k, say C_1, is not homologous to zero on X and theorefore $(C_1 \cdot C_1) > 0$, q.e.d.

PROOF OF THEOREM 20.2. Letting $\mathscr{V} = \{V_t \mid -1 < t < 1\}$ be a differentiable family of complex structures such that V_t is a complex torus of complex dimension 2 for $t > 0$, we prove that V_0 is also a complex torus. Let $\mathscr{V}$ have a quasi-hermitian metric (Definition 2.4). Using the theory of harmonic forms we readily verify that there exists a sequence $\{t_\mu\}$, $t_\mu \to 0$, $t_\mu > 0$, and closed holomorphic 1-forms φ_μ^1, φ_μ^2 on V_{t_μ} such that φ_μ^α converges uniformly to a limit φ_0^α, where φ_0^α, $\alpha = 1, 2$, are closed holomorphic 1-forms on V_0 which are linearly independent.

Let $\{\mathfrak{z}_j \mid j = 1, 2, 3, 4\}$ be a basis for the 1-cycles on the underlying real torus X, and let

$$\omega_j^\alpha(t_\mu) = \int_{\mathfrak{z}_j} \varphi_\mu^\alpha .$$

Clearly

$$\omega_j^\alpha(t_\mu) \to \omega_j^\alpha(0) = \int_{\mathfrak{z}_j} \varphi_0^\alpha .$$

It suffices to show that

$$D = \begin{vmatrix} \omega_1^1 & \omega_2^1 & \omega_3^1 & \omega_4^1 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \overline{\omega}_1^1 & \overline{\omega}_2^1 & \overline{\omega}_3^1 & \overline{\omega}_4^1 \\ \overline{\omega}_1^2 & \overline{\omega}_2^2 & \overline{\omega}_3^2 & \overline{\omega}_4^2 \end{vmatrix} \neq 0 \qquad \text{for } \omega_j^\alpha = \omega_j^\alpha(0).$$

Since $X = \mathfrak{z}_1 \times \mathfrak{z}_2 \times \mathfrak{z}_3 \times \mathfrak{z}_4$ we have

$$D = \int_X \varphi_0^1 \wedge \varphi_0^2 \wedge \overline{\varphi}_0^1 \wedge \overline{\varphi}_0^2 . \tag{20.2}$$

If V_0 is a Kähler surface, then φ_0^α, $\overline{\varphi}_0^\alpha$ are harmonic. Hence, if $D = 0$, then there exists a non-trivial linear relation $\sum (a_\alpha \varphi_0^\alpha + b_\alpha \overline{\varphi}_0^\alpha) = 0$ which implies that $\sum a_\alpha \varphi_0^\alpha = 0$, $\sum b_\alpha \overline{\varphi}_0^\alpha = 0$, in contradiction of the linear independence of φ_0^1, φ_0^2. Our theorem is thus proved in this case.

Suppose that $D = 0$. Then, by (20.2), $\varphi_0^1 \wedge \varphi_0^2 = 0$; hence $\varphi_0^1 = f \cdot \varphi_0^2$ where f is a non-constant meromorphic function on V_0. If V_0 is algebraic, therefore in particular a Kähler manifold, then V_0 is a complex torus. Assume that V_0 is not algebraic. Then V_0 is a fibre space of elliptic curves over a curve Δ and every meromorphic function on V_0 is constant along each fibre (Kodaira [25], § 4). In particular, f is induced from a meromorphic function on Δ which we denote by the same symbol f. We have $0 = d\varphi_0^1 = df \wedge \varphi_0^2$ and hence $\varphi_0^2 = gdf$, $\varphi_0^1 = fgdf$, where g is a meromorphic function on Δ. Thus φ_0^1, φ_0^2 are induced from holomorphic 1-forms on Δ which we denote by ψ^1, ψ^2 respectively.

Let $\{\gamma_i\}$ be a basis for the 1-cycles on Δ and let π: $V_0 \to \Delta$ be the canonical projection. For each γ_i, we construct an arc α_i in V_0 such that $\pi: \alpha_i \to \gamma_i$ is a homeomorphism except at the end points of α_i and we join these two end points by an arc l_i on the fibre. Then $\mathfrak{y}_i = \alpha_i + l_i$ is a 1-cycle on V_0 and $\pi(\mathfrak{y}_i) = \gamma_i$. Hence the $\mathfrak{y}_i$ must be homologically independent. It follows that $b^1(\Delta) \leqq b^1(V_0) = 4$ where b^1 denotes the first Betti number. In particular, the genus of the curve Δ cannot exceed 2. But Δ has two independent holomorphic 1-forms, namely ψ^1, ψ^2; therefore the genus of Δ is equal to 2 and $b^1(\Delta) = 4$. Now we have

$$\mathfrak{y}_i \sim \sum m_{ij}\mathfrak{z}_j$$

on V_0, where $\sim$ denotes homological equivalence. It follows that the periods

$$\pi_i^\alpha = \int_{\gamma_i} \psi^\alpha$$

are given by

$$\pi_i^\alpha = \sum_{j=1}^4 m_{ij}\omega_j^\alpha(0) ,$$

while

$$\det\begin{pmatrix} \pi_i^\alpha \\ \overline{\pi_i^\alpha} \end{pmatrix} \neq 0 .$$

This contradicts with $D = 0$. Thus the assumption that V_0 is not algebraic is false, q.e.d.

21. Deformation of a single coordinate domain

For completeness, we incorporate here remarks which illustrate the fundamental difference between complex dimension 1 and higher dimensions as reflected by Hartogs' theorem.

Let $\mathscr{V} = \{V_t | t \in M\}$ be a differentiable family of deformations V_t of complex structures of a complex manifold V_0 (not necessarily compact). If N is a sufficiently small neighborhood in M of the point o, then $\mathscr{V} | N = \{V_t | t \in N\}$ is differentiably homeomorphic to $X \times N$ where X is the underlying differentiable manifold of V_t. Let Y denote an open subset of X whose closure $\overline{Y}$ is compact and let $\mathscr{W}$ be the subdomain of $\mathscr{V} | N$ obtained by removal of the set $\overline{Y} \times N$. Clearly $\mathscr{W}$ is the differentiable family of the open subsets $W_t = V_t - \overline{Y} \times t$ of V_t: $\mathscr{W} = \{W_t | t \in N\}$.

PROPOSITION 21.1. *Suppose that $\mathscr{W}$ is trivial and that $\bar{Y} \times N$ is contained in a domain $\mathscr{U} \subset \mathscr{V}$ which is covered by a single coordinate system $(z^1, \cdots, z^n, t)$ and has the form*

$$\mathscr{U} = \{(z, t) \mid |z^1| < 1, \cdots, |z^n| < 1, t \in N\} .$$

Then, for a sufficiently small neighborhood $U \subset N$ of o, the differentiable family $\mathscr{V} \mid U$ is trivial, provided that $\dim_C V_t \geqq 2$.

PROOF. Let

$$\mathscr{U}^{(c)} = \{(z, t) \mid |z^1| < c, \cdots, |z^n| < c, t \in U\} ,$$

where $0 < c < 1$. By hypothesis there exists a differentiable map $f\colon \mathscr{W} \to W_o$ whose restriction f_t to each fibre W_t, $t \in N$, is a biregular holomorphic map of W_t onto W_o. Obviously we may assume that $f_o\colon W_o \to W_o$ is the identity map. Hence we can choose U and b, c, $0 < b < c < 1$, such that $\mathscr{U}^{(b)} \supset \bar{Y} \times U$ and

$$f(\mathscr{U}^{(c)} \cap \mathscr{W}) \subset \mathscr{U} \cap V_o = \{z \mid |z^1| < 1, \cdots, |z^n| < 1\} .$$

For each $t \in U$, the map f_t restricted to $\mathscr{U}^{(c)} \cap W_t$ is represented therefore in the form

$$(z, t) \to f_t(z) = (f_t^1(z), \cdots, f_t^\alpha(z), \cdots, f_t^n(z)) .$$

By the theorem of Hartogs, each holomorphic function $f_t^\alpha(z)$ on $\mathscr{U}^{(c)} \cap W_t$ can be extended to a holomorphic function $h_t^\alpha(z)$ defined on $\mathscr{U}^{(c)} \cap V_t$. Moreover, since $h_t^\alpha(z)$ is given by the Cauchy integral

$$\frac{1}{(2\pi i)^n} \iint \cdots \int \frac{f_t^\alpha(\zeta) d\zeta^1 d\zeta^2 \cdots d\zeta^n}{(\zeta^1 - z^1)(\zeta^2 - z^2) \cdots (\zeta^n - z^n)}$$

over the distinguished boundary : $|\zeta^1| = |\zeta^2| = \cdots = |\zeta^n| = c_1$, where $b < c_1 < c$, $h_t^\alpha(z)$ depends differentiably on t. It follows that the map $f_t\colon W_t \to W_o$, $t \in U$, may be prolonged into a regular holomorphic map $h_t\colon V_t \to V_o$ depending differentiably on t. Since $f_t^{-1}\colon W_o \to W_t$ may also be prolonged into a regular map $V_o \to V_t$, we infer readily that h_t is biregular, q.e.d.

PROPOSITION 21.2. *Suppose that $\mathscr{W}$ is trivial and that $\bar{Y}$ is contained in a coordinate domain $D_o \subset V_o$ of the form: $D_o = \{z \mid |z^1| < 1, \cdots, |z^n| < 1\}$. Then $\mathscr{V} \mid U$ is trivial for a sufficiently small neighborhood U of o, provided that* $\dim_C V_t \geqq 2$.

PROOF. By hypothesis there exists a differentiable map $f\colon \mathscr{W} \to W_o$ whose restriction f_t to W_t is a biregular holomorphic map of W_t onto W_o such that $f_o\colon W_o \to W_o$ is the identity map.

Define $R_t = f_t^{-1}(D_o \cap W_o)$, $t \in U$; then R_t is covered by a single system of holomorphic coordinates, namely by the coordinates $z^1, \cdots, z^n$ of D_o. Let B_t correspond to the boundary of D_o in the map h_t and denote by D_t that subdomain of V_t bounded by B_t which contains R_t. We regard D_o as a bounded subdomain of euclidean n-space $\boldsymbol{C}^n$ with coordinates $z^1, \cdots, z^n$; then D_t, $t \in U$, may be regarded as a deformation of D_o whose complex structure near the boundary coincides with that of D_o.

Let $\mathscr{G}$ be the free abelian group generated by non-degenerate periods $\omega_j = (\omega_{j1}, \cdots, \omega_{jn})$, $j = 1, \cdots, 2n$, and let $\mathscr{G}$ operate on $\boldsymbol{C}^n$ in the usual way. By proper choice of the periods ω_j, we may assume that the closure of D_o is contained in a fundamental domain of $\mathscr{G}$. Replacing D_o by D_t, $t \in U$, we obtain a family of deformations A_t of the torus $A_o = \boldsymbol{C}^n/\mathscr{G}$, and we know from Section 14, (γ), that, if U is small enough, there exist coordinates covering $\bigcup_{t \in U} D_t$ of the type described in Proposition 21.1; hence Proposition 21.2 follows from Proposition 21.1.

Propositions 21.1 and 21.2 are false if $\dim_C V_t = 1$. In fact, for any compact Riemann surface V_o, any sufficiently small deformation V_t of the complex structure of V_o is obtainable in the manner described in these propositions, i.e. by a deformation of the structure inside a single coordinate domain (see [34]).

22. Problems

In the Introduction we proposed the problem of investigating the reasons why the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ holds for many examples of complex manifolds V_o. This problem may be regarded as a special case of the following problem concerning the *existence of deformations*:

PROBLEM 1. Given a compact complex manifold V_0 and an element $\theta_0 \in H^1(V_0, \Theta_0)$, find a useful sufficient condition for the existence of a 1-parameter family $\mathscr{V} = \{V_t \mid -\varepsilon < t < \varepsilon\}$ of deformations V_t of V_0 whose infinitesimal deformation $\rho_o(d/dt)$ at $t = 0$ coincides with the given θ_0: $\rho_0(d/dt) = \theta_0$.

We have pointed out in Section 6 that $[\theta_o, \theta_o] = 0$ is a necessary condition for the existence of $\mathscr{V}$ with $\rho_o(d/dt) = \theta_o$, but we do not know whether $[\theta_o, \theta_o] = 0$ is a sufficient condition. In this connection we propose

PROBLEM 2. Is any deformation space of a compact complex manifold V_o a maximal abelian Lie algebra in $H^1(V_t, \Theta_i)$? (Compare Section 6).

PROBLEM 3. Is the vector space $H^1(V_o, \Theta_o)$ spanned by the deformation spaces of V_o?

In case the answer to Problem 2 or 3 is negative, it would not be difficult to solve the problem by exhibiting a counter example.

PROBLEM 4. Is the number of moduli $m(V_o)$ of a compact complex manifold V_o equal to dim $H^1(V_o, \Theta_o)$?

This is a formulation in stronger form of the problem proposed in the Introduction. The corresponding problem for the relative theory is as follows:

PROBLEM 5. Is the number of relative moduli $m_W(V_o)$ of a submanifold V_o of an algebraic manifold W equal to dim $H^1_W(V_o, \Theta_o)$?

We have shown in Section 12 that, in case V_o is a submanifold of W of codimension 1, the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ holds, provided that $H^1(W, \Omega(V_o)) = 0$. The assumption $H^1(W, \Omega(V_o)) = 0$ is necessary for the *proof* of the theorem of completeness of the characteristic linear systems of complete continuous systems from which we derive the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$. In fact, Zappa [40] has exhibited an example of a complete continuous system of curves on an algebraic surface whose characteristic linear system is not complete. We remark, however, that the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ still holds for Zappa's example. In Section 14 we have shown that the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ is valid for several simple types of compact complex manifolds V_o. We have examined further examples and found that the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ always occurs. Since it is difficult to believe that this equality occurs by accident in so many cases, we state the following *conjecture*: The equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ holds, possibly under some additional restricitions on V_o. By additional restrictions we mean here some condition corresponding to the condition $H^1(V_o, \Omega(V_o)) = 0$ in the relative theory mentioned above. We might point out that the above conjecture has actually been useful as a "working hypothesis" in our study of deformations.

PROBLEM 6. Let $\mathscr{V} \to M$ be a differentiable family of compact complex structures such that the map ρ_t: $(T_M)_t \to H^1(V_t, \Theta_t)$ is surjective for each $t \in M$. Then is $\mathscr{V} \to M$ complete?

We list again, because of its importance, the following problem, which was stated in Section 3:

PROBLEM 7. Is an arbitrary deformation of a compact Kähler manifold again a Kähler manifold?

We remark that, if the answer to Problem 7 is affirmative, then the answer to the following problem (compare Section 20) will also be affirmative.

PROBLEM 8. Is an arbitrary deformation of complex projective n-space again complex projective n-space? Is an arbitrary deformation of a complex n-torus again a complex n-torus?

We have shown, in case $n = 2$, that the answer to the problem is affirmative.

THE INSTITUTE FOR ADVANCED STUDY AND PRINCETON UNIVERSITY

BIBLIOGRAPHY

1. L. AHLFORS, *On quasiconformal mappings*, J. d'Analyse Math., III (1953/54), 1–58.
2. Y. AKIZUKI and S. NAKANO, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. of the Jap. Acad., 30 (1954), 266–272.
3. A. ANDREOTTI, *Recherches sur les surfaces irrégulières*, Mem. Acad. royale de Belgique, 27 (1952), fasc. 4, 1–56.
4. M. F. ATIYAH, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc., 85 (1957), 181–207.
5. W. L. BAILY, *The decomposition theorem for V-manifolds*, Amer. J. Math., 78 (1956), 862–888.
6. A. BLANCHARD, Sur les variétés analytiques complexes, Thèse, Paris, 1956.
7. R. BOTT, *Homogeneous vector bundles*, Annals of Math., 66 (1957), 203–248.
8. E. CALABI, *Fibre bundle techniques in the modification of complex analytic structure*, to appear.
9. H. CARTAN, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. de France, 85 (1957), 77–99.
10. ——— and S. EILENBERG, Homological algebra, Princeton University Press, 1956.
11. G. CASTELNUOVO and F. ENRIQUES, *Sopra alcune questioni fondamentali nella teoria delle superficie algebriche*, Ann. Mat. pura appl. III, 6 (1901) 165–225.
12. P. DOLBEAULT, *Sur la cohomologie des variétés analytiques complexes*, C. R. Paris, 236 (1953), 175–177.
13. F. ENRIQUES, *Sopra le superficie che possegono un fascio di curve razionali*, Atti della Reale Accad. dei Lincei, Rendiconti, 7 (1898), 2° Semestre, 344–347.
14. A. FRÖLICHER and A. NIJENHUIS, *Theory of vector-valued differential forms, Part I: Derivations in the graded ring of differential forms*, Proc. Kon. Ned. Akad. Wet. Amsterdam, 59 (1956), 338–359.
15. ———, *Some new cohomology invariants for complex manifolds*. I, Proc. Kon. Ned. Akad. Wet. Amsterdam, 59 (1956), 540–564.
16. ———, *A theorem on stability of complex structures*, Proc. Nat. Acad. Sci., U.S.A., 43 (1957), 239–241.
17. F. HIRZEBRUCH, Neue topologische Methoden in der algebraischen Geometrie, Ergebnisse der Math., Neue Folge, Heft 9, 1956.
18. ——— and K. KODAIRA, *On the complex projective spaces*, Jour. de math. pures et appliquées, 36 (1957), 201–216.
19. H. HOPF, *Zur Topologie der komplexen Mannigfaltigkeiten*, in Studies and essays presented to R. Courant, New York, 1948.
20. K. KODAIRA, *Harmonic fields in Riemannian manifolds (generalized potential theory)*, Annals of Math., 50 (1949), 587–665.
21. ———, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci., U. S. A., 39 (1953), 1268–1273.
22. ———, *Some results in the transcendental theory of algebraic varieties*, Annals of Math., 59 (1954), 86–134.

23. K. KODAIRA, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Annals of Math., 60 (1954), 28–48.
24. ———, *Characteristic linear systems of complete continuous systems*, Amer. J. Math., 78 (1956), 716–744.
25. ———, *On compact complex analytic surfaces* I, polycopied note, Princeton University (1955); to appear in Annals of Math.
26. ———, *On compact complex analytic surfaces* II, to appear in Annals of Math.
27. ——— and D. C. SPENCER, *On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski*, Proc. Nat. Acad. Sci., U. S. A., 39 (1953), 1273–1278.
28. ———, *On the variation of almost-complex structure*, in Algebraic geometry and topology, Princeton University Press, 1957.
29. ———, *Stability theorems for complex structures*, to appear.
30. A. KRAZER, Lehrbuch der Thetafunktionen, Leipzig, 1903.
31. S. NAKANO, *On complex analytic vector bundles*, J. Math. Soc. Japan, 7 (1955), 1–12.
32. M. NOETHER, *Ueber Flächen, welche Schaaren rationaler Curven besitzen*, Math. Annalen, 3 (1871), 161–227.
33. ———, *Anzahl der Moduln einer Classe algebraischer Flächen*, Sitz. Königlich Preuss. Akad. der Wiss. zu Berlin, erster Halbband 1888, 123–127.
34. M. SCHIFFER and D. C. SPENCER, Functionals of finite Riemann surfaces, Princeton University Press, 1954.
35. C. L. SIEGEL, *Discontinuous groups*, Annals of Math., 44 (1946), 674–689.
36. ———, Analytic functions of several complex variables, Lectures delivered at the Institute for Advanced Study, 1948–1949 (mimeographed).
37. N. STEENROD, The topology of fibre bundles, Princeton University Press, 1951.
38. O. TEICHMÜLLER, *Extremale quasikonforme Abbildungen und quadratiche Differentiale*, Abh. Preuss. Akad. der Wiss., Math.-naturw. Klasse. Nr. 22(1940), 1–197.
39. A. WEIL, *On Picard varieties*, Amer. J. Math., 74 (1952), 865–894.
40. G. ZAPPA, *Sull'esistenza, sopra le superficie algebriche, di sistemi continui completi infiniti, la cui curva generica e a serie caratteristica incompleta*, Pont. Acad. Sci. Acta, 9 (1945), 91–93.
41. O. ZARISKI, Algebraic surfaces, Ergebnisse der Math., Bd. 5.