$+ \quad \times$

$- \quad \div$

§ Def of associative alg.

· Eg: $\boxed{R_t := k[x] / (x^2 - t)} = (k \oplus k\langle x \rangle, *_t)$

$\langle 1, x \rangle$

· Eg: $R_t := k[x,y] / (y^2 - x^3 - t)$

· Goal: Def := one-para family of product on the same vector space $(V, *_t)$

· Setup: Given an associative alg A/k.

The underlying v.s. is denoted by V.

$R := k[[t]]$ $\quad K := k((t))$. $\quad V_K := V \otimes_k K$

$$A := (V, f: V \times V \longrightarrow V)$$

$\uparrow$ multiplication $\Rightarrow$ bilinear $/k$.

$\forall t$, want to define $f_t : V_K \times V_K \longrightarrow V_K$ bilinear map over K.

· Def: A one parameter family of $A = (V, f: V\times V \to V)$ is the same vector space together with

$f_t: V_K \times V_K \longrightarrow V_K$ s.t.

① f_t is bilinear over K

② $f_t(a,b) = f(a,b) + t F_1(a,b) + t^2 F_2(a,b) + t^3 F_3(a,b) + \cdots$

$= a\cdot b + t F_1(a,b) + \cdots$

$\boxed{F_i: V\times V \longrightarrow V}$ bilinear map over k. , $\forall a,b \in V$.

· Eg: $k[x]/(x^2-t)$ $\langle V = k \oplus k\langle x\rangle ,\ f_t = ?\ \rangle$

$f_t(x,x) = \cancel{f(x,x)}^{\to 0} + t\cdot F_1(x,x)$

$= t$

Here $F_1(x,x) = 1. \in \langle 1, \cdot\rangle \in k \oplus k\langle x\rangle$.

$k[x]/(x^2) \underset{k}{\otimes} k((t)) = V_K$

We hope f_t be an associative prod.

· Associative law

$$\boxed{f_t(a, f_t(b,c)) = f_t(f_t(a,b), c)}$$

$$LHS = f_t(a, bc + tF_1(b,c) + t^2F_2(b,c) + \cdots)$$

$$= a(bc) + t\left(\boxed{F_1(a,bc) + aF_1(b,c)}\right) + \cdots$$

$$RHS = f_t(ab + tF_1(a,b) + \cdots, c)$$

$$= (ab)c + t(\boxed{F_1(ab,c) + F_1(a,b)\cdot c}) + \cdots$$

$\Rightarrow$ $F_1 : V \times V \longrightarrow V$

satisfies $\boxed{F_1(a,bc) + aF_1(b,c) - F_1(ab,c) - F_1(a,b)\cdot c} = 0$

$$\underbrace{}_{= \, d_{Hoch}F_1(a,b,c)}$$

More generally

$(\ast)(I_\nu)$ $$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu\geq 0}} F_\lambda(F_\mu(a,b), c) = \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu\geq 0}} F_\lambda(a, F_\mu(b,c))$$

$F_i : V \times V \longrightarrow V$ bilinear $/ \mathbb{R}$.

· Fact: If f_t is associative, then $F_1 : V \times V \longrightarrow V$ is a Hochschild 2-cocycle.

§ Hochschild cohomology of an associative alg

① cochains

② Hochschild cohomology can be defined as $\mathrm{Ext}_{A\otimes A}(A, A)$.

· Def: Given an associative alg A/k & M be an A-bimod.

Cochain $C^n(A, M) := \mathrm{Hom}_k(A^{\otimes n}, M)$

together with the following differential

$\forall f \in C^n(A, M) \qquad f: A^{\otimes n} \longrightarrow M$

$$
\begin{aligned}
d_H f(a_1, \cdots, a_{n+1}) := \; & a_1 \cdot f(a_2, \cdots, a_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\
& + (-1)^{n+1} f(a_1, \cdots, a_n) \cdot a_{n+1}
\end{aligned}
$$

· Lemma: $(d_H)^2 = 0$.

$$HH^n(A, M) := H^n(C^\bullet(A, M), d_H)$$

$n=0$, $C^\circ(A, M) = \mathrm{Hom}_k(A^{\otimes 0}, M) = \mathrm{Hom}_k(k, M) \simeq M$.

$\quad\quad\quad\quad\ni f$

$$df(a) := a \cdot f(1) - f(1) \cdot a$$

$$= a \cdot m - m \cdot a$$

$\Rightarrow HH^\circ(A, M) = Z(M) := \{ m \in M \mid a \cdot m = m \cdot a \quad \forall a \in A \}$.

$n=1$, $d_H f(a_1, a_2) = a_1 f(a_2) - f(a_1 a_2) + f(a_1) \cdot a_2$

$\Rightarrow d_H f = 0 \Rightarrow f \in \mathrm{Der}_k(A, M)$.

$\Rightarrow \mathrm{Inn}(A, M) := \{ D \in \mathrm{Der}_k(A, M) \mid D$ is of the following form $a \longmapsto a \cdot m - m \cdot a$ for some $m \}$

$\Rightarrow HH^1(A, M) := \mathrm{Der}_k(A, M) / \mathrm{Inn}(A, M)$.

$n=2$,

$$d_H F_1(a, b, c) = a \cdot F_1(b, c) - F_1(ab, c) + F_1(a, bc) - F_1(a, b) \cdot c$$

§ Back to deformation.

$F_1 \in Z^2(A,A)$

Q1: Whether the 1st order def only depends on the cohomology class in $HH^2(A)$?

Q2: Given $[F] \in HH^2(A)$. Does there exist an 1-parameter family of def of associative algs s.t. $[F_1]=[F]$?

Obstruction

For Q1, need to define <u>equivalent deformations</u>.

$$g_t : V_K \times V_K \longrightarrow V_K$$

$$g_t(a,b) = a\cdot b + t\, G_1(a,b) + t^2 G_2(a,b) + \cdots$$

Def: A 1-para family of def of an associative alg with multiplication

$$g_t(a,b) = a\cdot b + t\, G_1(a,b) + t^2 G_2(a,b) + \cdots$$

is called <u>trivial</u> if $\exists$ an auto $\Phi_t : V_K \xrightarrow{\sim} V_K$ linear over K &

$$\Phi_t(a) := a + t\,\varphi_1(a) + t^2 \varphi_2(a) + \cdots$$

$\forall a \in V$, where $\varphi_i : V \longrightarrow V$ linear over k.

s.t. $\Phi_t(g_t(a,b)) = \Phi_t(a)\cdot\Phi_t(b)$. $\left(g_t(a,b) = \Phi_t^{-1}(\Phi_t(a)\cdot\Phi_t(b)) \right.$

$$\Phi_t : (V_K, g_t) \longrightarrow (V_K, \cdot)$$

More generally,

$f_t \sim g_t$ if $\exists\ \Phi_t$ as above

$$\boxed{\Phi_t(g_t(a,b)) = f_t(\Phi_t(a), \Phi_t(b))}.$$

$$\text{LHS} := \Phi_t(ab + t\,G_1(a,b) + t^2 G_2(a,b) + \cdots)$$

$$= ab + t\left(\boxed{G_1(a,b) + \varphi_1(ab)}\right) + \cdots$$

$$\text{RHS} = f_t(a + t\varphi_1(a) + \cdots,\ b + t\varphi_1(b) + \cdots)$$

$$= ab + t\left(\boxed{\varphi_1(a)b + a\varphi_1(b) + F_1(a,b)}\right)$$

$$\Rightarrow (F_1 - G_1)(a,b) = \varphi_1(a)b + a\varphi_1(b) - \varphi_1(ab)$$

$$= d_H\varphi_1(a,b)$$

- equiv $\Rightarrow$ $[F_1] = [G_1] \in HH^2(A,A)$.

$\boxed{\text{Thm:}}$ $HH^2(A,A)$ is the first order def of A.

For Q2, we need to go back to (1_ν)

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F_\lambda(F_\mu(a,b),c) = \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F_\lambda(a, F_\mu(b,c))$$

$$(2_\nu) \Rightarrow \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0\\ \lambda,\mu<\nu}} \Big(F_\lambda(F_\mu(a,b),c) - F_\lambda(a,F_\mu(b,c)) \Big) = a\cdot F_\nu(b,c) + F_\nu(a,bc) - F_\nu(ab,c) - F_\nu(a,b)\cdot c$$

$$= d_H F_\nu(a,b,c)$$

For $\nu = 2$, $RHS = d_H F_2(a,b,c)$

$$LHS := F_1(F_1(a,b),c) - F_1(a,F_1(b,c)) \in Z^3(A,A)$$

- The LHS is the first order obstruction to extend to def to second order (F_2)

$\xRightarrow{(2_\nu)}$

- Thm: LHS defines the $(\nu-1)$-th order obstruction, which is a class in $HH^3(A,A)$.

  If the class vanishes i.e. exact, then the deformation can be extended to F_ν.

In summary,

$$HH^2(A,A) = \text{1 st order def of } A$$

$$HH^3(A,A) = \text{obstruction of } \underline{\text{integrability}}.$$

Given $F_1 \longrightarrow F_2$?

$F_2 \longrightarrow F_3$? ...

· Cor: $HH^2(A,A) = 0$, then A is rigid.

§ Back to HH^n

M is an A-bimod

$$HH^n(A,M) := \mathrm{Ext}^n_{A\text{-bimod}}(A, M)$$

· Fact: $\forall$ associative alg A, there exists a free A-bimo resolution of A.

$\boxed{\text{Bar cpx or bar construction.}}$ B.A

$\Downarrow$

$$\mathrm{Ext}^n_{A\text{-bimod}}(A, M) := H^n(\mathrm{Hom}_{A\text{-bimod}}(B.A, M))$$

$$\cong (C^n(A, M), d)$$