Classical Hodge theory

Hodge theorem

+ Hodge decomposition (dolbeault cohomology & Hodge symmetry)

+ Hard Lefschetz decomposition.

$\Downarrow$ Hodge diamond.

LG h_1 Perverse sheave

Mirr h_2 weighted Hodge

h_3

(Piece of history: Mirror symmetry gets it name for "symmetry" in Hodge diamond of certains (pairs of) CY 3-folds,

namely $h_X^{1,1} = h_Y^{2,1}$

Hodge theorem on compact Riemannian mfd M.

Notation: $*$: Hodge star: $A^p(M) \longrightarrow A^{n-p}(M)$

$$d: A^p(M) \to A^{p+1}(M)$$

$$\delta := (-1)^{n(p+1)+1} * d * \; : \; A^p \to A^{p-1}$$

Laplacian-Hodge operator: $d^* := *d*$

$$\Delta := \delta d + d\delta$$

$$= (-1)^{n(p+1)+1}(dd^* + d^* d)$$

Fact: $[\Delta, *] = 0$, $[\Delta, d] = 0$

$[\Delta, d^*] = 0$.

Notation: $\langle \cdot , \cdot \rangle : A^p \times A^p \to \mathbb{R}$

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge * \beta$$

Poincare pairing

$\alpha \in A^p$ $\alpha^\perp \in A^{n-p}$

$\langle \alpha, \alpha^\perp \rangle^M = \int_M \alpha \wedge \alpha^\perp$

Harmonic forms:

$$\mathcal{H}^p_\Delta(M) := \{\alpha \in A^p(M) \mid \Delta\alpha = 0\}$$

Notice Laplace-de Rham act on forms (antisymmetric tensors).

$$\Delta_{dR} := (-1)^{(p+1)n+1}(dd^* + d^* d)$$

Laplace - Beltrami operator act on tensors :

different literature have different names

$\forall$ T a tensor (symmetric or antisymmetric)

$$\Delta_{LB} T := tr(\nabla^2 T)$$

∇^2 is the Hessian.

in local coord

$$\Delta_{LB} T (X_1, X_2) := g^{ij} \left(\nabla_{X_1} \nabla_{X_2} T - \nabla_{\nabla_{X_1} X_2} T \right)$$

X_1, X_2 vector fields

$$\nabla^2 T (X_1, X_2) := \left(\nabla_{X_1} \nabla_{X_2} T - \nabla_{\nabla_{X_1} X_2} T \right)$$

$$- \nabla^2 T (X_2, X_1) \qquad + \nabla_{\nabla_{X_2} X_1} T$$

$$= \nabla_{[X_2, X_1]} T$$

$$\nabla^2 T (X_1, X_2) - \nabla^2 T (X_2, X_1)$$

$$= \nabla_{X_1}\nabla_{X_2}T - \nabla_{X_2}\nabla_{X_1}T + \nabla_{[X_2,X_1]}T$$

T is a vector field

T is symmetric k-tensor ;

∇T is symmtric $(k,1)$ tensor

$\nabla(\nabla T)$ $(k,2)$ tensor

$\left(\nabla(\nabla T)\right)(X_1, X_2) \in$ k-tensor

$$\underline{\nabla\left((\nabla T)(X_1)\right)(X_2)}$$

exercise: for f function & w 1-form, compute:

① $\nabla \Delta_{LB} f - \Delta_{LB} \nabla f$

② $\nabla \Delta_{LB} W - \Delta_{LB} \nabla W$

Hodge Theorem:

$\exists$ a canonical mapping:

$$\varphi : \mathcal{H}^p_\Delta(M) \to H^p_{dR}(M) \quad \frac{\text{closed}}{\text{Im } d}$$

such that this φ is an isomorphism as vector space.

H: projection: $A^p \to \mathcal{H}^p_\Delta$

$\Uparrow$

Theorem (decomposition)

$$A^p(M) = \mathcal{H}^p_\Delta \oplus \text{Im } d \oplus \text{Im } d^*.$$

to imply isomorphism with H^p_{dR}, we need the fact:

Fact: $\forall \alpha \in \text{Im } d^*$, $d\alpha = 0 \Rightarrow \alpha = 0$.

pf: Let $\alpha = d^*\beta$, assume $d\alpha = 0$,

then $\langle d\alpha, \beta \rangle \overset{(\text{def})}{=} \langle \alpha, d^*\beta \rangle$

$= \langle d^*\beta, d^*\beta \rangle$

$\overset{\text{assumption}}{=} 0$

$\Rightarrow \alpha = d^*\beta = 0.$

Definition: A weak solution of $\Delta(\cdot) = \alpha$

is a $\boxed{\text{bounded}}$ linear functional $l : A^p(M) \to \mathbb{R}$ s.t.

$\|l(\alpha)\| \le c\|\alpha\|$ $\quad l(\Delta \cdot) = l(\Delta^* \cdot) := \langle \alpha, \cdot \rangle$ $\quad C^\infty$

Remark: $\langle \Delta a, b \rangle = \langle a, \Delta b \rangle$

Δ is self-adjoint, i.e. $\Delta = \Delta^*$.

$\ell(\beta) = ?$ we don't know,

but as long as $\ell(\Delta\varphi) = \langle \alpha, \varphi \rangle$,

then this functional ℓ is a weak solution.

$\boxed{\Delta w = \alpha}$

$$\ell(\beta) = \langle \beta, w \rangle \overset{\beta = \Delta\varphi}{=\!=} \langle \Delta\varphi, w \rangle = \langle \varphi, \Delta w \rangle$$
$$= \langle \varphi, \alpha \rangle$$

Reize representation theorem.

why we cannot use it to show?

Theorem (Regularity thm: 6.5)

Assume $\ell : A^p \to \mathbb{R}$ is a weak solution of $\Delta(\ \cdot\ ) = \alpha$ for C^∞ p-form α, i.e.

$$\ell(\Delta^* \cdot) = \langle \alpha, \cdot \rangle.$$

$\exists$ u C^∞ p-form s.t. $\ell(\ ) = \langle u, \ \rangle$

$\boxed{\Delta u = \alpha}$ Δ elliptic "weak solution" is real solution:

for f is C^∞ p-form

$\Delta u = f$ alway have a C^∞ solution.

Note $\Delta K = 0$ iff K is constant function on cpt M.

$H^0(M) \cong \mathbb{R}$

$\boxed{\text{Thm}}$ 6.6 (Δ^{-1} being compact on L^2 p-form). depend on M being cpt.

Let $\{\alpha_n\}$ be a sequence of smth p-forms on M s.t. $\langle \alpha_n, \alpha_n \rangle \leq C$ and $\langle \Delta\alpha_n, \Delta\alpha_n \rangle \leq C$. $\forall n$, and for some $C > 0$.

Then a subsequence of $\{\alpha_n\}$ is $\boxed{\text{Cauchy}}$.

(Cauchy can be change to that $\{\alpha_n\}$ converges to a L^2 p-form β, that's not nece C^∞)

(cpt operator: it takes bounded sequence into a sequence that has subsequence converge).

Show decomposition theorem:

$${}_{C^\infty}A^p = \Delta(A^p) \oplus \mathcal{H}^p_{\Delta\, C^\infty}$$

Weyl's Law

Weyl formula

asymptotic

50's

① $\boxed{\mathcal{H}^p_\Delta \text{ is finite dim}}$

(use thm 6.6).

Assume $\mathcal{H}^p_\Delta$ is of infinite dimensional. Let $\{\alpha_n\}$ be an orthonormal sequence in $\mathcal{H}^p_\Delta$.

$\lambda_i \varphi_i = \Delta \varphi_i$

$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$

$\lambda_i \sim \boxed{\frac{(vol(M))^{-\frac{2}{n}}}{(Vol(B))^{-\frac{2}{n}}} \cdot (2\pi)^{\frac{1}{2}}} \cdot \boxed{i^{\frac{2}{n}}}$

By theorem 6.6, $\{\alpha_n\}$ has a Cauchy subsequence

$$\|\alpha_i' - \alpha_j'\|^2 < \varepsilon$$

$$= \langle \alpha_i' - \alpha_j', \alpha_i' - \alpha_j' \rangle$$

$$= \|\alpha_i'\|^2 + \|\alpha_j'\|^2 = 2$$

hence it's impossible to be Cauchy.

$\forall \alpha \in (C^\infty) A^p$, let $\{w_i\}$ be orthonormal basis of $\mathcal{H}^p$. then α can be uniquely written as

$\alpha = \beta + \sum_i^{n} \langle \alpha, w_i \rangle w_i$ for $\beta \in (\mathcal{H}^p_\Delta)^\perp$

Hence $\mathcal{H}^p_\Delta$ is finite dim.

$(C^\infty) A^p \cap \{\alpha \in L^2 \mid \langle \alpha, h \rangle = 0 \ \forall h \in \boxed{C^\infty} \mathcal{H}^p_\Delta\}$

Since $\mathcal{H}^p_\Delta \subset A^p$, then $\underline{A^p = \underbrace{(\mathcal{H}^p_\Delta)^\perp}_{C^\infty} \oplus \mathcal{H}^p_\Delta}$ C^∞

C^∞ p-form

Want to show $\Delta(A^p) = (\mathcal{H}^p_\Delta)^\perp$.

First $\forall \alpha \in A^p$, $h \in \mathcal{H}^p_\Delta$,

$$\langle \Delta \alpha, h \rangle = \langle \alpha, \Delta h \rangle = 0$$

Thus $\Delta \alpha \perp h$, $\Delta(A^p) \subset (\mathcal{H}^p_\Delta)^\perp$

Next, need to show $(\mathcal{H}^p_\Delta)^\perp \subset \Delta(A^p)$.

② $\|\beta\| \leq c\|\Delta\beta\| \quad \forall \beta \in (\mathcal{H}^p_\Delta)^\perp$ smooth

(this is saying Δ^{-1} is bounded in L^2)

(Δ^{-1} being compact implies Δ^{-1} being bounded in L^2)

pf of ②: omit

Let $\alpha \in (\mathcal{H}^p_\Delta)^\perp$, l be defined as a functional on $\Delta(A^p)$:

$\forall \Delta\varphi \in \Delta(A^p) \qquad l(\Delta\varphi) = \langle \alpha, \varphi \rangle$.

l is well-defined on $\overline{\Delta(A^p)}$.

$\Delta(\ ) = \alpha$

③ Want to show l is bounded on $\Delta(A^p)$

$$l(\Delta\varphi) = \langle \alpha, \varphi \rangle \leq c\|\Delta\varphi\| \quad \text{for } \varphi \in A^p.$$

$$A^p = (\mathcal{H}^p_\Delta)^\perp \oplus \mathcal{H}^p_\Delta$$

$$\varphi = \psi + H(\varphi)$$

$H(\varphi) :=$ projection onto $\mathcal{H}^p_\Delta$
$= \sum_i^n \langle \varphi, w_i \rangle w_i$

for w_i orthonormal basis of $\mathcal{H}^p_\Delta$.

$$\underline{|l(\Delta\varphi)|} = |l(\Delta\psi)|$$

$$= |\langle \alpha, \psi \rangle|$$

$\overset{\text{bilinear}}{\leq} \|\alpha\| \cdot \|\psi\|$

$\overset{②}{\leq} \|\alpha\| \cdot C \cdot \|\Delta\psi\| = C \cdot \|\alpha\| \cdot \|\Delta\psi\|$.

Hence the aforedefined l is bounded on $\Delta(A^p)$.

l is a weak solution on $\Delta(A^p)$.

④ By Hahn-Banach thm:
a bounded linear functional on $(V, \|\cdot\|)$ can be extended to a bounded linear functional on $(W, \|\cdot\|)$ where $V \subset W$.

we can extend l bounded on $\Delta(A^p)$ to bounded on A^p.

Then l is a weak solution of $\Delta(\cdot) = \alpha$, $\boxed{\alpha \in (\mathcal{H}^p_\Delta)^\perp}$.

Apply Thm 6.5, $\exists\ w \in A^p$, s.t. $\Delta w = \alpha$.

Hence $(\mathcal{H}^p_\Delta)^\perp \subset \Delta(A^p)$.

Remark: The above proof didn't ensure that weak solution always exists of $\Delta(\cdot)=\alpha$ for any $\alpha \in (C^\infty)A^p$.

only ensures for $\alpha \in (\mathcal{H}^p_\Delta)^\perp$.

Question: Can we solve

$$\Delta w = \alpha \quad \text{for } \alpha \in \mathcal{H}^p_\Delta$$

in weak sense?

$$A^P \cap \{\alpha \in L^2 \mid \langle \alpha, h\rangle = 0, \ \forall h \in \mathcal{H}^P_\Delta\}$$

$$\left(\{y-x=0\} \cap \{x=0\}\right)^{\perp} = \mathbb{R}^2 \qquad A^P \cap \mathcal{H}$$

$$A^P \cap \{\alpha \in L^2 \mid \langle \alpha, h\rangle = 0, \ \forall h \in \overline{\mathcal{H}}\}$$

$$\{y+x=0\} = \{y-x=0\}^{\perp}$$

$$\{ \quad \} \supset \{ \quad \}$$

Yin : 2023 Summer.

10-30 Hard Lefschetz theorem.

(X, ω) cpt Kähler mfd,

$$L := \omega \wedge : H^{p,q} \to H^{p+1,q+1}$$

$$\Lambda := *^{-1} \omega \wedge * : H^{p,q} \to H^{p-1,q-1}$$

$$\left(** = (-1)^{\cdots} \qquad *^{-1} = * \cdot (-1)^{\cdots} \right)$$

Thm (Hard Lefschetz theorem).

The map $L^k : H^{n-k}(X) \to H^{n+k}(X)$ is an isomorphism. And if we define the primitive cohomology

$$P^{n-k}(X) := \ker L^{k+1} \quad \left(L^{k+1} : H^{n-k} \to H^{n+k+2} \right)$$

then we have decomposition:

$$H^m(X) = \bigoplus_k L^k P^{m-2k}(X). \quad \forall m.$$

In fact, since L^k is actually : $H^{p,q} \to H^{p+k,q+k}$.

we have isomorphism: for $p+q \leq n$

$$H^{p,q}(X) \underset{\simeq}{\xrightarrow{L^{n-p-q}}} H^{n-q,n-p}$$

Example: $X = \mathbb{P}^2 bl_{pt}$. $H^*(X) = \{[X], H, E, [pt]\}$

$$H^2 = [pt], \quad E^2 = -[pt]$$

$$H \cdot E = 0$$

$$H^*_{dR}(X) = \{\underset{\deg 0}{1}, \underset{\deg 2}{\alpha_H}, \underset{\deg 2}{\alpha_E}, \underset{\deg 4}{\alpha_{vol}}\}$$

$$\alpha_H \wedge \alpha_H = \alpha_{vol}, \quad \alpha_E \wedge \alpha_E = -\alpha_{vol}$$

$$\alpha_H \wedge \alpha_E = 0.$$

$$w = c_1 = 3\alpha_H - \alpha_E.$$

$$\begin{array}{llll}
H^*_{dR} & & & \\
H^4 & 8\alpha_{vol} & & 0 \\
 & \uparrow w\wedge & & \uparrow w\wedge \\
H^2 & 3\alpha_H - \alpha_E & & \beta = \alpha_H - 3\alpha_E \\
 & \uparrow w\wedge & & \\
H^0 & 1 & &
\end{array}$$

$$P^0 := \operatorname{Ker}(w\wedge)^3 = \langle 1 \rangle$$

$$P^2 := \operatorname{Ker}(w\wedge) = \langle \beta \rangle = \langle \alpha_H - 3\alpha_E \rangle$$

$$\beta \wedge w = 0 \Rightarrow \beta = \alpha_H - 3\alpha_E$$

$$H^2 = L^1 \cdot P^{2-2\cdot 1} \oplus L^0 \cdot P^{2-2\cdot 0}$$

$$= L P^0 \oplus P^2$$

Remark: this thm has a simple result:

Since $h^0 = 1$, we have for cpt kähler X,

$$\boxed{h^{2k} \geq 1, \ \forall n \geq k \geq 0}$$

Hence Hopf manifold $S^1 \times S^{2j-1}$, $j \geq 2$, although can be a compact complex mfd it is not kähler manifold.

H^{2j}	1
H^{2j-1}	1
	0
	0
H^1	1
H^0	1

Representation of sl_2:

sl_2 can be realized a v.s. of 2×2 complex matrices with trace zero.

Take standard generators: $sl_2 = \langle e, H, f \rangle$

$$\underset{e}{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underset{f}{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the relations:

$[X,Y]=H$, $[H,X]=2X$, $[H,Y]=-2Y$.

ρ is a representation of sl_2 action on V:

$$\rho: sl_2 \longrightarrow gl(V) \text{ s.t.}$$

$$\rho([A,B]) = [\rho(A), \rho(B)], \quad \forall A, B \in sl_2.$$

V is called sl_2-module.

V (or ρ) is called irreducible if V has no notrivial submodule.

Classical theory on representation of sl_2:

① Every sl_2-module is the direct sum of irreducible sl_2-modules.

② On irreducible sl_2-module, $\rho(H)$ has eigenspaces V. For eigenspace V, the eigenvalue of $\rho(H)$ on $\underline{V}$, say, is $\underline{\lambda}$,

then $\underline{\rho(X)V}$ & $\rho(Y)V$ are of eigenvalues $\lambda+2$ & $\lambda-2$.

Note: $\rho(H)=H$, $\rho(X)=X$, $\rho(Y)=Y$.

check: $H(XV) = XHV + [H,X]V$

$$= XHV + 2XV$$

$$= X\cdot\lambda V + 2XV$$

$$= (\lambda+2)XV.$$

same for Y.

Since H has finite eigenvalues (hence X has to reach 0 vector after finite times), X and Y are nilpotents.

③ Under assumption of V irreducible, V can be decomposed into 1-dim eigenspaces:

$$V = V_\lambda \oplus V_{\lambda-2} \oplus V_{\lambda-4} \oplus \cdots$$

$H(V_{\tilde\lambda}) = V_{\tilde\lambda}$, $X(V_{\tilde\lambda}) = V_{\tilde\lambda+2}$, $Y(V_{\tilde\lambda}) = V_{\tilde\lambda-2}$,

④ (Using the primitive 1-dim space, one can argue) all eigenvalues are actually integers, i.e. for irreducible V of dim n,

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n}$$

Kähler mfd. $h(\alpha) = (k-n)\alpha$

Define: $h(\alpha) = (n-k)\alpha \quad \forall \alpha \in H^k(X)$

$$h(\alpha) = \sum_{k=0}^{2n} (n-k)\pi^k \alpha \quad \forall \alpha \in H^*(X)$$

$$\pi^k : H^* \to H^k$$

Remark: h has eigenvalues on H^k with $(n-k)$.
the degree of α higher,
the eigenvalue of h correspond to α lower.

Naturally: as actions on H^*,

$$[h, L] = -2L$$

$$[h, \Lambda] = 2\Lambda$$

Check: for $\alpha \in H^k$,

$$hL\alpha - Lh\alpha$$

$$= h(\omega \wedge \alpha) - \omega \wedge ($$

Define: $\underline{h(\alpha) = (k-n)\alpha}$ $\quad\quad \alpha \in H^k.$

$$h(1) = -n\cdot 1 \quad\quad 1 \in H^0.$$

for $\alpha \in H^k(X)$

$$\begin{aligned}
[h, L]\alpha &= h(\omega\wedge\alpha) - \omega\wedge h(\alpha) \\
&= (k+2-n)(\omega\wedge\alpha) - \omega\wedge(k-n)\alpha \\
&= 2\,\omega\wedge\alpha \\
&= 2L(\alpha).
\end{aligned}$$

$$[h, \Lambda]\alpha = -2\Lambda(\alpha).$$

Hodge identi(ties):

$$[L, \Delta] = 0, \quad\quad [\Lambda, \Delta] = 0,$$

$$\boxed{[\Lambda, L] = -h.} \iff [\underset{+2}{L}, \underset{-2}{\Lambda}] = h.$$

check on $\mathbb{C}^n$:

$$L = \omega\wedge = \Big(\sum_i dx_i \wedge dy_i\Big)\wedge$$

Experiment: $\Lambda L(\alpha) = \underline{*}^{-1} w \wedge (\underline{*}\, w \wedge \alpha)$

Assume $\alpha \in H^0(X)$

$$= \alpha \cdot \underline{*}^{-1}\, w \wedge \left(\sum_{i=1}^{n} (-1)^? \, dx_1 \wedge dy_1 \wedge \cdots \widehat{dx_i} \wedge \widehat{dy_i} \wedge \cdots dx_n \wedge dy_n \right)$$

$$= \alpha \cdot (-1)^? \, *^{-1} \sum_{i=1}^{n} dx_1 \wedge dy_1 \wedge \cdots dx_n \wedge dy_n$$

$$= \alpha \cdot (-1)^? \, n \cdot \underline{*(vol)} \quad \cancel{\alpha} (-1)^?$$

$$= (-1)^? \alpha \cdot n$$

$$L \Lambda (\alpha) = w \wedge *^{-1} \underline{w \wedge (} \underline{* \alpha}) \quad \in H^{2n}$$

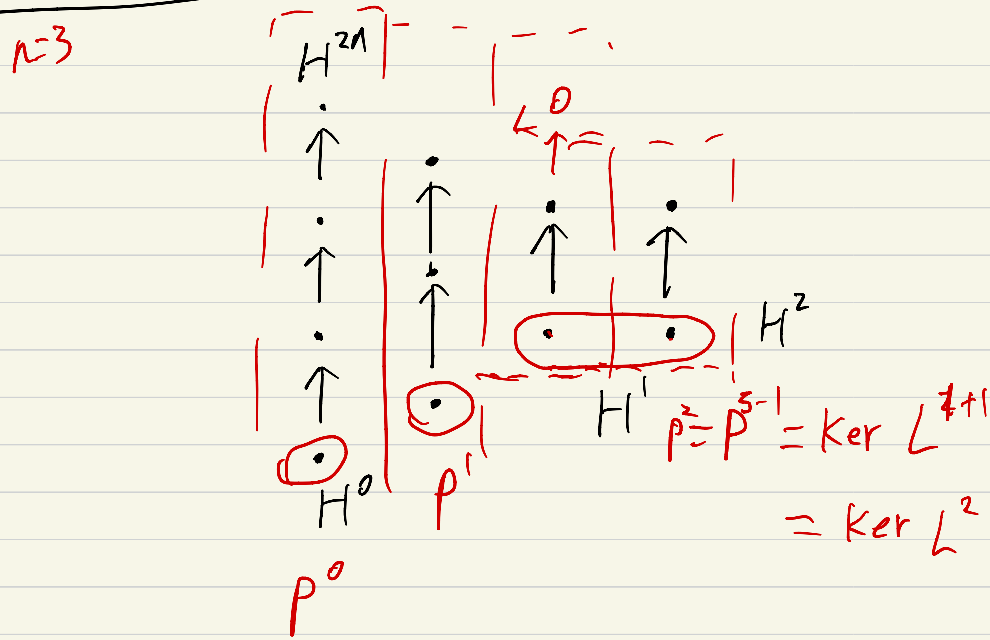
$$= w \wedge *^{-1}(0) \,.\; = 0 \,.$$

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to L$$

$$Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \to \Lambda$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to h \,.$$

The theorm on primitive decomposition is natural consequence under this representation. $\square$



Looijenga – Luts – Verbitsky

Int. Math. 96′   94′–95′

Let

$V = \bigoplus_{k \in \mathbb{Z}} V_k$ be a finite dim graded vector space.

<u>Def</u>: Let $e: V \to V$ be a deg 2 endomorphism.

We say e has <u>Lefschetz property</u> if

$$e^k : V_{-k} \longrightarrow V_k$$

is an isomorphism.

Lemma: (Jacobson-Morozov [Jac 51])

An operator e has above Lefschetz property if and only if $\exists$ a unique deg -2 endomorphism $f: V \to V$ s.t.

$$[e, f] = h,$$

here $h(\alpha) = k \cdot \alpha \quad \forall \alpha \in V_k$.

Such (e, h, f) as a triple acting on V is called sl_2-triple.

- - - - - - - - - - - - - - -

Well known that for hyperkähler X,

$\exists$ I, J, K acting on TX, s.t. $I^2 = J^2 = K^2 = -Id$,

$IJ = -JI = K$.

And $\omega_I := g(I, \ )$, $\omega_J := g(J, \ )$, $\omega_K := g(K, \ )$

are all kähler forms on X.

__Def__ (L-L-V algebra). Let $\mathfrak{g}_g \subset End(H^*(X, \mathbb{R}))$ be the Lie algebra generated by sl_2-triples

(e_a, h, f_a) where a runs over $\langle [w_I], [w_J], [w_K] \rangle \subset H^2(X, \mathbb{R})$.

where the e_a is defined as the action about $a \in H^2(X, \mathbb{R})$,

s.t. $e_a : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R})$

$\alpha \mapsto a \cdot \alpha$.

Remark: for any a being a kähler form,

e_a satisfies Lefschetz property,

hence by J-M Lemma, (e_a, h, f_a) is an sl_2-triple.

Definition ("total Lie algebra associted with kähler mfd (M, w)")

$\mathfrak{g}_{tot}$ is defined as the Lie algebra generated by sl_2-trip (e_a, h, f_a) where a runs over any 2-form that e_a satisfies Lefschetz property.

Key observation: such $a \in H^2(M, \mathbb{R})$ that makes e_a satisfies Lef. prop. is Zarisky dense in $H^2(M, \mathbb{R})$.

Example: $\mathbb{P}^2 \, bl_{pt}$.

Thm (L-L-V) $\mathfrak{g}(X) = \mathfrak{so}(4,1)$ (dim 10)

(L-L-V algebra for hyperkähler X)

$= \langle e_I, e_J, e_K,$
$f_I, f_J, f_K,$
$h,$
$[e_I, f_J]^?, [e_J, f_K]^?,$
$[e_K, f_I]^? \rangle$

(Li-Leung 08' simple way to compute)

$[f_I, f_J] = [e_I, e_J] = 0$ and so on.

Example $\mathbb{P}^2 bl_{pt}$ dim 2. $\mathfrak{g}_{tot} = ?$

e_α satisfies Lef prop

$\Leftrightarrow e_\alpha^2 : H^0 \to H^4$ is an isomorphism.

$\Leftrightarrow \alpha \wedge \alpha$ is nonzero. $(h^0 = h^4 = 1)$

$H^1 = H^3 = 0$
$e_\alpha : H^{n-1} \to H^{n+1}$ is iso is true automatically.

$\alpha \in H^2(M, \mathbb{R})$

α_H, α_E

$\alpha_H \wedge \alpha_H = 1$, $\alpha_E \wedge \alpha_E = -1$

$\alpha = k_1 \alpha_H + k_2 \alpha_E$

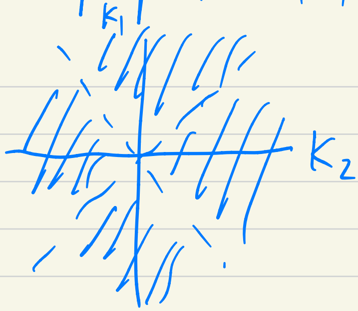
$k_1 \alpha_H + k_2 \alpha_E$ is in kähler cone.

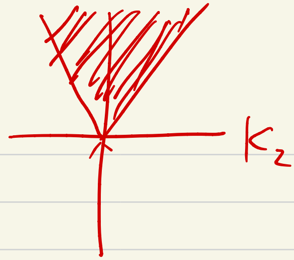
$\Rightarrow (k_1 \alpha_H + k_2 \alpha_E)^{\wedge 2} > 0$

$\Leftrightarrow k_1^2 > k_2^2$

k_1

Lef prop $\Leftrightarrow k_1^2 \neq k_2^2$





Question: $g_{tot}(M)$ for Kähler M

is in general hard,

say $M = \mathbb{P}^3 bl_{pt}$.

$e_a : \underline{H^2} \underline{\to} \underline{H^4}$

$e_a^3 : \overline{H^0} \to \underbrace{H^2} \to \underbrace{H^4 \to H^6}$

$e_a^2 : H^1 \to H^5$ trivial ✓

T. C. Dinh etl. asked:

Can one characterize $\alpha_i \in H^2$, $i \in \{1, \cdots, r\}$, s.t.

$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r$ gives isomorphism

$H^{n-r} \longrightarrow H^{n+r}$

check:

Zhangchi Chen '22 PAMS

example $\subset$ Lef mfd $\subset$ example?