On the Mirror P = W Conjecture

- **Weight** filtr.

Grothendieck's Dream (Motif)

$$X \to M(X) \to H^{\cdot}(X)$$

sm. proj. var. $\mapsto$ pure motive $\mapsto$ pure Hodge str.

general var. $\mapsto$ mixed motive $\mapsto$ mixed Hodge str.

[Pure wg = k]

$$H^{k} \overset{\exists}{=} \bigoplus H^{i,k-i} \Leftrightarrow \exists\ F^{p} = \bigoplus_{i\geq p} H^{i,n-i}$$

$$H^{i,k-i} = F^{i} \cap \bar{F}^{k-i}$$

ana. : $\mathcal{L}^2$-analysis of elliptic $\Delta \curvearrowright H_{harm}$

alg. (Deligne - Illusie)

E_1-dege. of Hodge-deRham S.S.

[Mixed] $\phi \overset{\text{Frob.}}{\curvearrowright} H^{*}_{ét}(X/_{\mathbb{F}_q} \otimes \bar{\mathbb{F}}_q ; \mathbb{Q}_\ell)$

Weil Conj. $\exists$ Functorial filtr. from geo.

$$W_p := \bigoplus_{i\leq p} \text{Eigspace}(|\lambda| = q^{\frac{i}{2}})$$

Thm. (Deligne Hodge II, III)

$\forall$ alg. var. X $(\xrightarrow{f} Y)$

$\Rightarrow \exists$ "Functorial" **Mixed Hodge Structure**

$(H^n(X;\mathbb{Q}), F^\bullet_{\mathbb{C}}, W_\bullet)$

i.e. $\begin{cases} \forall\, k \in [0, 2n],\ (Gr^k_W \otimes \mathbb{C}, F^\bullet) \text{ pure wg} = k \\ \text{"strictness"} : \operatorname{Im}(f) \cap Gr^i_{F/W} H^n(X) = f(Gr^i_{F/W} H^n) \end{cases}$

Ex. 1 (Deligne Hodge II : Open Sm.)

"good compactification"

$$\underset{\substack{\uparrow \\ \text{sm. cpt.}}}{U = X} - \underset{\nwarrow \text{simple NCD}}{D} \overset{j}{\hookrightarrow} X$$

$\Omega^\bullet_X(\log D) := \{\text{logarithmic de Rham complex}\} \subseteq j_*\Omega^\bullet_U$

$\overset{\text{q.is.}}{\simeq} Rj_*\underline{\mathbb{C}}_U$

Thm.

$$H^k(U;\mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(\log D))$$

given "mixed Hodge struct."

by $\begin{cases} W_m = \operatorname{Im}(\mathbb{H}^k(X, W_{m-k}\Omega^\bullet_X(\log D)) \to H^k(U)) \\ F^p = \operatorname{Im}(\mathbb{H}^k(X, F^p\Omega^\bullet_X(\log D)) \to H^k(U)) \end{cases}$

where

$$W_m \Omega_X^p(\log D) := \begin{cases} 0 & m < 0 \\ \Omega_X^p(\log D) & m \geq p \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq p \end{cases}$$

Ex. 2 (Schmid: Limiting from family)

$\mathcal{X} \to \Delta^*$ sm. proj. family

$\exists$ Quasi-unip. Monodromy $T \circlearrowright H^*(X_t : \text{fixed})$

$$T = T_{ss} T_u \quad \text{s.t.} \quad \begin{cases} (T_u - I)^\ell = 0 \\ T_{ss}^m = I \end{cases}$$

Set

$$N := \log T_u = \sum_{k \geq 1} \frac{(-1)^{k-1}(T_u - I)^k}{k} \quad \text{s.t.} \quad N^\ell = 0$$

Fact

$\forall$ Nilpotent $N \in \mathrm{End}(V)$ s.t. $N^{\ell+1} = 0$

$$\exists! \quad 0 \subset \underset{\mathrm{Im}\, N^\ell}{\overset{\shortparallel}{W_0}} \subset \cdots \subset \underset{\ker N^\ell}{\overset{\shortparallel}{W_{2\ell-1}}} \subset W_{2\ell} = V$$

s.t.

$$\begin{cases} N(W_i) \subset W_{i-2} \\ N^i : Gr^W_{k+i} \xrightarrow{\sim} Gr^W_{k-i} \end{cases}$$

Moreover "Lefschetz-type"

$$Gr_i^W = \bigoplus_{j \geq \ell - i} N^j \ker(N^{i-\ell+2j+1})$$

Thm.

$\exists\ F^{\cdot}_{\lim}$ on $H^{\cdot}(X_t)$

s.t. $(H^{\cdot}(X_t), W_{\cdot}, F^{\cdot}_{\lim})$ MHS

Particularly N : $(-1,-1)$ type

- **Perverse filtr.**

Thm. (Deligne decomposition)

$\forall\ X \xrightarrow[f]{\text{sm. proj.}} Y$

$$Rf_*\underline{\mathbb{Q}}_X \cong \bigoplus_{i\geq 0} \mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X)[-i] \in D^b_c(Y)$$

$\Rightarrow$ E_2 degeneration of Leray S.S.

$$E_2^{p,q} = H^p(Y; R^q f_*\underline{\mathbb{Q}}_X) \Rightarrow H^{p+q}(X;\mathbb{Q})$$

Define "Leray filtration"

$$L_q H^{\cdot}(X;\mathbb{Q}) := \mathrm{Im}(H^{\cdot}(Y;\tau_{\leq q} Rf_*\underline{\mathbb{Q}}_X) \to H^{\cdot}(X;\mathbb{Q}))$$

$\Downarrow$ Generalize

Thm. (Beilinson - Berstein - Deligne - Gabber)

$\forall$ sm. $X \xrightarrow{\text{"proper"}} Y$

$$Rf_*\mathbb{Q}_X[n] \cong \bigoplus_{i\geq 0} {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n])[-i] \in D^b_c(Y)$$

$\Rightarrow$ E_2 - dege. of "perverse" Leray S.S.

$${}^pE_2^{p,q} = H^p(Y; {}^pR^qf_*\mathbb{Q}_X) \Rightarrow H^{p+q}(X;\mathbb{Q})$$

Similarly "perverse" Leray filtration

$$P_q H^\bullet(X;\mathbb{Q}) := \mathrm{Im}(H^\bullet(Y; {}^p\tau_{\leq q}Rf_*\mathbb{Q}_X) \to H^\bullet(X;\mathbb{Q}))$$

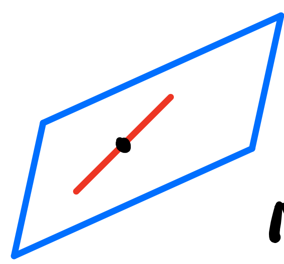
Geometric characterization

Λ^k : codim $= k$ linear subspace

$$f : X \to \mathbb{A}^n$$

$\Updownarrow$ "generic n-flag"

$$\underline{\phi = \Lambda_{n+1} \subset \Lambda_n \subset \cdots \subset \Lambda_0 = \mathbb{A}^n}$$



$\Lambda^3 \subset \Lambda^2 \subset \Lambda^1$

Flag filtration

$$F_k H^\bullet(X) := \mathrm{Ker}(H^\bullet(X) \to H^\bullet(f^{-1}(\Lambda^{k-1})))$$

Thm. (Cataldo - Migliorini 10')

$X \xrightarrow{f} Y$: affine

$$P_i H^\bullet(X) \cong F_i H^\bullet(X)$$

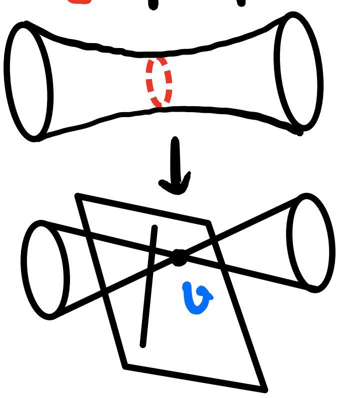
Ex. 1

trivial $\quad X \times \mathbb{C}^n \xrightarrow{pj} \mathbb{C}^n$

Künneth $\quad P_{k+n} H^k(X) = F_{k+n} H^k(X) = H^k(X)$

Ex. 2

$E = \mathbb{P}^1 \times \mathbb{P}^1$



$$\begin{array}{l} X \\ \downarrow f = Bl_v \\ Y = \mathrm{cone}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{A}^4 \end{array}$$

$$Rf_* \mathbb{Q}_X \overset{BBDG}{\cong} \underbrace{\underbrace{\underbrace{\overset{H^2(E)}{\overset{\updownarrow}{\mathbb{Q}_v}}[-2]}_{P_2} \oplus IC_Y}_{P_3} \oplus \overset{H^4(E)}{\overset{\updownarrow}{\mathbb{Q}_v}}[-4]}_{P_4}$$

Generic 4-flag

$(Y_0 = Y) \supset (Y_1 = Y \cap \mathbb{A}^3) \supset (Y_2 = Y \cap \mathbb{A}^2) \supset (Y_3 = Y \cap \mathbb{A}^1) \supset \emptyset$

Compute

$X_\bullet := f^{-1}(Y_\bullet)$	X_0	X_1	X_2	X_3	X_4
$H^2(X_\bullet)$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$	0	0	0

$$P_i H^2(X) = F_i H^2(X) = H^2(X) = \mathbb{Q}, \quad i \geq 3$$

While

$$P_2 H^2(X) = H^2(Y, \mathbb{Q}_v[-2]) = \mathrm{Ker}(\mathbb{Q} \to H^2(X_1)) = 0$$

- Classical $P = W$

Setting

- C_g : proj. sm. alg. curve genus ≥ 2
- $d \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$ s.t. (d, n) coprime

Defined "twisted" moduli

$$\mathcal{M}_B(GL_n(\mathbb{C})) := \left\{ (A_i, B_i) \in (GL_n(\mathbb{C}))^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi\sqrt{-1}\, d}{n}} I_n \right\} /\!/ GL_n(\mathbb{C})$$

diffeo. $\cong$ ← "NAHC"

$$\mathcal{M}_{Dol}(GL_n(\mathbb{C})) := \left\{ (\mathcal{E}, \theta) \;\middle|\; \begin{array}{c} \deg = d,\ rk = n \\ S\text{-equi. stable Higgs bd.} \end{array} \right\}$$

$$\Rightarrow \quad H^*(\mathcal{M}_{Dol}; \mathbb{Q}) = H^*(\mathcal{M}_B; \mathbb{Q})$$

Thm. ($P = W$: type A_{n-1} (GL_n, PGL_n, SL_n))

$$P_k H^*(\mathcal{M}_{Dol}; \mathbb{Q}) = W_{2k = 2k+1} H^*(\mathcal{M}_B; \mathbb{Q})$$

"topo. of integrable sys." (Lagr. fibration) "Hodge of char. var."

[Maulik - Shen, Hausel - Mellit - Minets - Schiffmann 22'
Maulik - Shen - Yin 23']

Toy Ex. (Line bd. $n = 1$)

$$\begin{array}{ccc} \mathcal{M}_B = (\mathbb{C}^*)^{2g} & \cong & \mathcal{M}_{Dol} = Jac(C_g) \times H^0(K_{C_g}) \\ & & \downarrow h \\ & & \mathbb{C}^g = H^0(K_{C_g}) \end{array}$$

$$\Rightarrow \quad W_2 H^1(\mathcal{M}_B) = P_1 H^1(\mathcal{M}_{Dol}) = \mathbb{Q}^{2g}$$

- Mirror P = W Conjecture
  [Katzarkov - Przyjalkowski - Harder 19']

$$\begin{tikzcd}
\mathcal{M}_B(G) \arrow[d, squiggly, "\text{Simposon}"', "\text{Geo. } P=W \text{ conj.}"] & \cong & \mathcal{M}_{Dol}(G) \arrow[dr, "h"] \arrow[rr, leftrightarrow, "T\text{-dual}"] & & \mathcal{M}_{Dol}(G^L) \arrow[dl, "h^\vee"'] \arrow[llll, dashed, bend left, "\text{HMS}"] \\
\text{log CY} & & & B &
\end{tikzcd}$$

log CY (X : proj. sm., snc $D := \sum D_i \in |-K_X|$)

$$\begin{tikzcd}
 & & X \arrow[d, squiggly, "\text{cptification}"] & & & & \operatorname{Spec}(\mathbb{C}[U^\vee]) \\
\text{sm. qisproj.} & U := & X - D \arrow[rr, Rightarrow, "\text{“mirror”}", "(\text{LG})"'] & & U^\vee \arrow[rr, "w := (w_1, \dots, w_k)", "\text{“affinization”}"'] & & \mathbb{C}^k \arrow[u, equal]
\end{tikzcd}$$

U "Fano" $U^\vee$ "CY fib."

perverse mixed Hodge poly.

$$PW_M(u,t,w,p) := \sum_{a,b,r,s} \dim Gr_F^a Gr^W_{s+b} Gr^P_r (H^s(U)) u^a t^s w^b p^r$$

Conj. (Numerical Mirror P = W)

$$PW_U(u^{-1}t^{-2}, t, p, w) u^d t^d \overset{?}{=} PW_{U^\vee}(u,t,w,p)$$

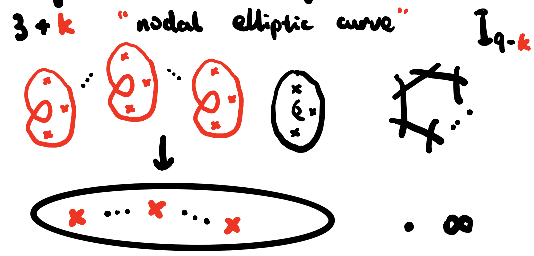
$w=1$

$$\rightsquigarrow \dim Gr_F^q Gr^W_{p+q+r} H^{p+q}(U) = \dim Gr_F^{d-q} Gr^P_{d+p-q+r} H^{d+p-q}(U^\vee)$$

Ex. 1 (dim = 2) Auroux – Katzarkov – Orlov 05')

Del Pezzo — Rational Elliptic Surface

$$X_k = Bl_{k\text{-pts}}\,\mathbb{P}^2 \quad \underset{\text{"HMS"}}{\overset{\text{"SYZ"}}{\longleftrightarrow}} \quad \begin{array}{ccc} M_k & \subset & \bar{M}_k \\ \downarrow w_k & & \downarrow \bar{w}_k \\ \mathbb{C} & \subset & \mathbb{P}^1 \end{array}$$



$$PW_{U_d} = p^2 + k\, v^2 u + v^2 w u^2 + v^2 w u$$

$$PW_{Y_d} = p + p v^2 u + k\, v^2 u + v^2 w^2 u^2$$

Ex. 2 (dim = 3) Przyjalkowski)

"Fanography" 105 families sm. Fano 3-fold

Iskovskikh - Prokhorov Table

$\rho(X)$	1	2	3	4	5	6	7	8	9	10
#	17	36	31	13	3	1	1	1	1	1

Thm.

"$\exists$" Toric LG model for Fano 3-fold

i.e. Laurent $f : (\mathbb{C}^\times)^n \to \mathbb{C}$

- Period : $I_f = \tilde{I}_0^{X,D}$: I-series from GW
- CY - cptfi.
- Toric : $\exists$ degen. $X \rightsquigarrow$ toric $X_{\Delta = N(f)}$

$$\begin{tikzcd}
(\mathbb{C}^*)^3 \arrow[r, hook] \arrow[d, "f"'] & Y \arrow[r, hook, "(K_Y \sim 0)"] \arrow[d, "w"] & Z \arrow[d, "\bar{w}"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, hook, "\text{log CY}", "\text{cptify}"'] & \mathbb{P}^1
\end{tikzcd}$$

Z : sm. cpt. s.t. $-K_Z \sim \bar{w}^{-1}(\infty)$

PW_U :

p^3 $\quad (p-1)pt^2u \quad h^{1,2}t^3u \quad wt^3u$

$h^{1,2}t^3u^2 \quad (20-p)wt^3u^2$

wt^3u^3

PW_Y [Type III by Harder]

$w^3t^3u^3 \quad (ph-3)wt^3u^2 \quad k\, t^4u^2 \quad pt^4u^2$

$k\, t^2u \quad (22-ph)pt^2u$

$h^{1,2}(Z)t^3u^2 \quad p$

$h^{1,2}(Z)t^3u$

where

$$k(Y, w) := \sum_{\text{crit.}} \#(\text{irr. comp. of } w^{-1}(p)) - 1$$

$$ph(Y, w) := \dim \operatorname{coker}(H^2(Y) \to H^2(V))$$

Thm.

$$h^{1,2}(X) = k(Y, w)$$

$$p(X) = ph(Y, w) - 2$$

$$h^{1,2}(Z) = 0$$

Ex 3.

Toric wk Fano dim ≤ 3

- Application & Generalization

Conj. (Relative HMS : D sm.)

$$
\begin{tikzcd}
D^b coh(D) \arrow[r, shift left, "i_*"] \arrow[d, leftrightarrow, dashed, "\cong"] & D^b coh(X_0) \arrow[l, shift left, "i^*"] \arrow[r, "j^*"] \arrow[d, leftrightarrow, dashed, "\cong"] & D^b coh(U) \arrow[d, leftrightarrow, dashed, "\cong"] \\
Fuk(Y_{sm}) \arrow[r, shift left, "\cup"] & FS(Y, w) \arrow[l, shift left, "\cap"] \arrow[r, "\iota"] \arrow[d, squiggly, "HH."] & Fuk^{wr}(Y) \\
 & {} &
\end{tikzcd}
$$

$$HH_i(D^b(X_0)) \cong \bigoplus_{q-p=i} H^q(X, \Omega_X^p)$$
$$\wr\wr$$
$$HH_i(FS(Y, w)) \cong H^{n+i}(Y, Y_{sm})$$

Conjecturally

(E_1^W - page)

$$
\begin{tikzcd}
\boxed{\bigoplus H^q(X, \Omega_D^p) \arrow[r] \arrow[d, leftrightarrow, dashed, "\cong"] & \bigoplus H^q(X, \Omega_X^p) \arrow[r] \arrow[d, leftrightarrow, dashed, "\cong"] & \bigoplus H^q(X, \Omega_U^p)} \arrow[d, leftrightarrow, dashed, "\cong"] \\
\boxed{H^{i+n-1}(Y_{sm}) \arrow[r] & H^{i+n}(Y, Y_{sm}) \arrow[r] & H^{i+n}(Y)}
\end{tikzcd}
$$

(E_1^P - page)

$$\overset{Gr}{\Rightarrow} Gr_F^q \, Gr^W_{p+q+i} H^{p+q}(U) \cong Gr^{W_{Mon}}_{2(n-q)} Gr^P_{n+p-q+i} H^{n+p-q}(Y)$$

[Kontsevich - Katzarkov - Pantev Conj. 17']

$$\Omega_Z^i(\log D_Z, f) := \{\alpha \in \Omega_Z^i(\log D_Z) \mid df \wedge \alpha \in \Omega_Z^{i+1}(\dots)\}$$

LG Hodge num.

$$h^{p,q}(Y, w) := \dim_{\mathbb{C}} gr_p^{W(N,p+q)} H^{p+q}(Y, Y_{\infty}; \mathbb{C})$$

$$f^{p,q}(Y, w) := \dim_{\mathbb{C}} H^p(Z, \Omega_Z^q(\log D_Z, f))$$

$$\overset{[\text{Harder}]}{=} h^{p,q}(Y, V \underset{\text{val.}}{\overset{\text{reg.}}{:=}} w^{-1}(t); \mathbb{C})$$

Thm.

$$\dim H^n(Y, V; \mathbb{C}) = \sum_{p+q=n} h^{p,q} = \sum f^{p,q}$$

Conj.

$$h^{p,q}(Y, w) \overset{(1)}{=} f^{p,q}(Y, w)$$

$$\overset{(2)}{=} h^{n-p,q}(X)$$

dim = 2 Lunts - Przyjalkowski ✓

dim = 3 Przyjalkowski ✓

Prop. (Mirror $P = W$ $\Rightarrow$ KKP 1)

$$PW_V(u, t, 1, p) = PW_Y(u^{-1}t^{-2}, t, w, 1)u^n t^n$$

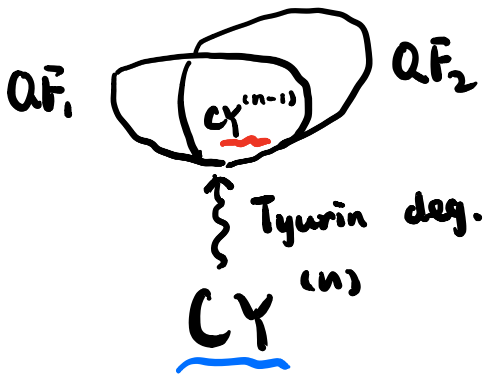
$$\Rightarrow \quad h^{p,q}(Y, w) = f^{p,q}(Y, w)$$

When $D = \sum D_i$

"Hybrid LG model" ... (Sukjoo Lee, Harder)

"DHT Mirror Philo."

Degeneration $\rightsquigarrow$ fibration



$$\overset{\text{Gluing}}{\rightsquigarrow} \qquad LG_{1,2} \to \mathbb{A}^1$$

$$\check{CY}^{(n-1)} \to \check{CY}^{(n)} \downarrow \mathbb{P}^1$$

Setting

$\mathcal{X} \xrightarrow{\pi} \Delta$: Type N ss. degen. of CY X^n

i.e. $\Gamma(X_0 = \cup X_i)$ = standard N-cplx

If $\exists\ (X_i, X_{ij}) \overset{\text{mirror}}{\leftrightsquigarrow} (Y_i, w_i) \overset{\text{Glue}}{\Rightarrow} Y \xrightarrow{w} \mathbb{P}^N$

Conj.

$$\dim Gr_F^q Gr^{W_{lim}}_{p+q} H^{p+q+i}(X) = \dim Gr_F^{n-q} Gr^{P}_{p+n-q} H^{p+n-q+i}(Y)$$

Moreover $H \overset{\text{generic}}{\subset} \mathbb{P}^N$, $U := w^{-1}(\mathbb{P}^N - H)$

$$\dim Gr_F^q Gr^{W}_{p+q} H^{p+q+i}(X_0)$$
$$= \dim Gr_F^{n-q} Gr^{P}_{p+n-q} H_c^{p+n-q+i}(U)$$