$F : D^b(\mathcal{A}) \longrightarrow \mathcal{A}$: Abelian "cohomological"

$X \to Y \to Z \to X[1]$ distinguished

i.e.

$$\Downarrow$$

$$FX \to FY \to FZ \to F^1 X := F(X[1]) \quad \text{exact}$$

Thm. (Deligne : Lefschetz prop. 1.2.)

$X \in D^b(\mathcal{A})$

$\forall$ Cohomological $F : D^b(\mathcal{A}) \longrightarrow \mathcal{A}$

$\Rightarrow$ s.s. $E_2^{p,q} = F^p(\mathcal{H}^q(X)) \Rightarrow F^{p+q}(X)$

degenerate at E_2 iff

$\exists$ decomposition in $D^b(\mathcal{A})$: $X \cong \bigoplus_i \mathcal{H}^i(X)[-i]$

$\Big\}$ Application

<u>prop</u>. (Hard Lefschetz Type)

$X \in D^b(\mathcal{A})$, if $L : X \to X[2]$

s.t. $\forall\ i \geq 0$, $L^i : \mathcal{H}^{n-i}(X) \xrightarrow{\cong} \mathcal{H}^{n+i}(X)$

$\Rightarrow X \cong \bigoplus_i \mathcal{H}^i(X)[-i]$

pf.

Put $P^{-i} := \ker(L^{i+1} : \mathcal{H}^{-i}(X) \to \mathcal{H}^{i+2}(X))$

$\forall\ k \geq 0$, $L^k P^{-i-2k} \hookrightarrow \mathcal{H}^{-i}(X)$

$$\rightsquigarrow \begin{cases} \mathcal{H}^{-i}(X) = \bigoplus_{k \geq 0} L^k P^{-i-2k} \\ \mathcal{H}^{i}(X) = \bigoplus_{k \geq 0} L^{k+i} P^{-i-2k} \end{cases}$$

For $L : E_2^{p,q} \to E_2^{p,q+2}$

$$PE_2^{p,q} := \operatorname{Ker}(L^{i+1} : E_2^{p,-i} \to E_2^{p,-i+2})$$

$$\rightsquigarrow \begin{cases} E_2^{p,-i} = \bigoplus_{k\ge 0} L^k \, PE_2^{p,-i-2k} \\ E_2^{p,i} = \bigoplus_{k\ge 0} L^{k+i} \, PE_2^{p,-i-2k} \end{cases}$$

$$\begin{tikzcd} PE_2^{p,-i} \arrow[r, "d_2 = 0"] \arrow[d, "0 = L^{i+1}"'] & E_2^{p+2,-i-1} \arrow[d, "L^{i+1}", "\wr"'] \\ E_2^{p,i+2} \arrow[r, "d_2"] & E_2^{p+2,i+1} \end{tikzcd} \quad \rightsquigarrow \quad d_{\ge 2} = 0$$

$\square$

Thm.

$\forall\ X \xrightarrow[\text{sm. proj.}]{f} Y$, $L \overset{f\text{-ample}}{\in} \mathrm{Pic}(X)$

- (Smooth Relative Hard Lefschetz) $(n = \dim f)$

$$\eta := c_1(L) \Rightarrow \eta^{i\ge 0} : R^{n-i} f_* \underline{\mathbb{Q}}_X \xrightarrow{\sim} R^{n+i} f_* \underline{\mathbb{Q}}_X$$

- (Deligne decomposition)

$$Rf_* \underline{\mathbb{Q}}_X \cong \bigoplus_{i\ge 0} \underline{\mathcal{H}^i(Rf_* \underline{\mathbb{Q}}_X)}_{\,= R^i f_* \underline{\mathbb{Q}}_X} [-i] \in D^b_c(Y)$$

"constructible"

- (Deligne Semisimplicity)

Local systems [polarizable VHS] $\quad R^i f_* \underline{\mathbb{Q}}_X[-i]$: semisimple

Cor. $\Rightarrow$ $\underline{E_2 \text{ degeneration}}$!! of Leray S.S.

$$E_2^{p,q} = H^p(Y; R^q f_* \underline{\mathbb{Q}}_X) \Rightarrow H^{p+q}(X; \mathbb{Q})$$

Hence cohomological decomposition

$$H^{p+q}(X;\mathbb{Q}) \cong \bigoplus H^p(Y; R^q f_* \underline{\mathbb{Q}}_X)$$

Y : simply conn. $\rightsquigarrow$ Künneth $= \bigoplus (H^p(Y;\mathbb{Q}) \otimes H^q(F;\mathbb{Q}))$

Define "Leray filtration"

$$L_q H^\bullet(X;\mathbb{Q}) := \mathrm{Im}(H^\bullet(Y; \tau_{\le q} Rf_* \underline{\mathbb{Q}}_X) \to H^\bullet(X;\mathbb{Q}))$$

Remark. (Failure Ex.)

- "holo. but not alg."

$$\underbrace{\mathbb{C}^*/q^{\mathbb{Z}}}_{g=1} \to \mathbb{C}^2 - \{0\}/q^{\mathbb{Z}} \xrightarrow[\text{proper}]{\text{sm.}} \mathbb{C}^2 - \{0\}/\mathbb{C}^*$$

$\left(\begin{matrix} q \in \mathbb{C} \\ |q| < 1 \end{matrix}\right)$

Then $\pi_1(X) = q^{\mathbb{Z}} \overset{\text{Hurewicz}}{\rightsquigarrow} h^1 = 1$

If Leray degen.

$$\Rightarrow h^1(X) = h^1(\mathbb{CP}^1) + h^0(R^1 f_* \underline{\mathbb{Q}}_X = \mathbb{Q}\oplus\mathbb{Q}) = 2$$

- "$\mathbb{R}$-alg."

Hopf fib. $S^1 \to S^3 \overset{\text{proper}}{\underset{\text{subm.}}{\twoheadrightarrow}} S^2$

If Leray degen. $\rightsquigarrow H^\bullet(S^3) = H^\bullet(S^1) \otimes H^\bullet(S^2)$ ✗

- "singular" $X \overset{\text{reso.}}{\twoheadrightarrow} Y$

If Leray degen. $\Rightarrow H^\bullet(Y) \overset{f^*}{\hookrightarrow} H^\bullet(X)$ ✗

Cohomology Coeff.

Abelian. Noe.
Artin.

$$\mathrm{Loc}(X) \subset \mathrm{Sh}(X)$$
$$\cap \qquad \subset_{\text{full}}$$
$$\text{"}\mathrm{Sh}_c(X)\text{"}$$

- "Noe." ×

$j_n : \mathbb{A}^1 - \bigcup_{1\to n}\{z_i\} \hookrightarrow \mathbb{A}^1 \Rightarrow \{j_{n!}\mathbb{C}\}_1^\infty \downarrow$

- "Stable under duality" ×

$j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \Rightarrow \mathbb{D}(j_!\mathbb{C}_{\mathbb{G}_m}) = j_*\mathbb{C}_{\mathbb{G}_m}[2]$

Ans.

- $j_{n!}\mathbb{C} \to \mathbb{C}_{\mathbb{A}^1} \to \tilde{F}_n \xrightarrow{+1}$

$$\bigoplus i_*\mathbb{C}_{\{z_i\}} \to j_{n!}\mathbb{C}[1] \to \mathbb{C}_{\mathbb{A}^1}[1]$$

- $\mathbb{D}(j_!\mathbb{C}_{\mathbb{G}_m} \to \mathbb{C}_{\mathbb{A}^1} \to \mathbb{C}_{\{0\}} \xrightarrow{+1})$

$$\|$$

$$\mathbb{C}_{\{0\}} \to \underline{\mathbb{C}_{\mathbb{A}^1}[2]} \;?\to \underline{j_*\mathbb{C}_{\mathbb{G}_m}[2]} \;?\xrightarrow{+1}$$

Complx better than cohomology $\{\mathcal{F}^\cdot \mid \mathcal{H}^k(\mathcal{F}^\cdot) \in \mathrm{Sh}_c(X)\}$

$$\mathrm{Sh}_c(X) \subset D^b(\mathrm{Sh}_c(X)) \xrightarrow{\simeq} D^b_c(X) \quad \text{!!}$$
$$\cap$$
$$\mathrm{Perv}(X) \subset D^b(\mathrm{Perv}(X)) \quad \text{"} \quad \text{(Beilinson)}$$

strictly full $D^{\leq,\geq 0} \subseteq D$: tri. category

$\begin{cases} \cdot\ \mathrm{Mor}_{\leq,\geq 0}(\ ,\ ) = \mathrm{Mor}(\ ,\ ) \\ \cdot\ \forall\, \mathcal{U} \in D \text{ s.t. } \mathcal{U} \cong \mathcal{F} \in D^{\leq,\geq 0} \Rightarrow \mathcal{U} \in D^{\leq,\geq 0} \end{cases}$

Set $D^{\leq n} := D^{\leq 0}[-n]$, $D^{\geq n} := D^{\geq 0}[-n]$, $n \in \mathbb{Z}$

$(D^{\leq 0}, D^{\geq 0})$ t-structure on D :

(i) $D^{\leq -1} \subset D^{\leq 0}$, $D^{\geq 1} \subset D^{\geq 0}$

(ii) $\mathrm{Hom}_D(D^{\leq 0}, D^{\geq 1}) = 0$

(iii) $\forall\ P \in D$, $\exists$ distinguished triag.

$$\underset{D^{\leq 0}}{\underset{\cap}{P'}} \to P \to \underset{D^{\geq 1}}{\underset{\cap}{P''}} \xrightarrow{+1}$$

Prop.

• (truncation)

$\iota : D^{\leq n} \hookrightarrow D \Rightarrow \exists$ radj. $\tau_{\leq n} : D \to D^{\leq n}$

i.e. $\mathrm{Hom}_{D^{\leq n}}(Y, \tau_{\leq n} X) \xrightarrow{\sim} \mathrm{Hom}_D(\iota(Y), X)$

similarly $\mathrm{Hom}_{D^{\geq n}}(\tau_{\geq n} X, Y) \xrightarrow{\sim} \mathrm{Hom}_D(X, \iota(Y))$

s.t. $\tau_{\leq n}(P[m]) \cong \tau_{\leq n+m}(P)$

• ("Heart" $\Rightarrow$ Abelian)

$$\mathcal{A} := D^{\leq 0} \cap D^{\geq 0} \subseteq D$$

cohomology functor

$${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0} = \tau_{\leq 0}\tau_{\geq 0} : D \to \mathcal{A}$$

$${}^tH^i := {}^tH^0 \circ [i]$$

Eg 1. (Standard $\sim$)

$$\mathcal{D}_c^{\le 0} := \{ \tilde{\mathcal{F}}^\bullet \mid \mathcal{H}^j(\tilde{\mathcal{F}}^\bullet) = 0, \ \forall j > 0 \}$$

$$\mathcal{D}_c^{\ge 0} := \{ \tilde{\mathcal{F}}^\bullet \mid \mathcal{H}^j(\tilde{\mathcal{F}}^\bullet) = 0, \ \forall j < 0 \}$$

$$\tau_{\le 0}(\tilde{\mathcal{F}}^\bullet) = \{ \cdots \to \tilde{\mathcal{F}}^{-2} \to \tilde{\mathcal{F}}^{-1} \to \ker d^0 \to 0 \cdots \}$$

$$\tau_{\ge 0}(\tilde{\mathcal{F}}^\bullet) = \{ \cdots \to 0 \to \operatorname{coker} d^{-1} \to \tilde{\mathcal{F}}^1 \to \tilde{\mathcal{F}}^2 \cdots \}$$

$$\Downarrow$$

$\rightsquigarrow$ Heart $\quad Sh_c(X) = \mathcal{D}_c^{\ge 0} \wedge \mathcal{D}_c^{\le 0}$

$$= \{ \cdots 0 \to \mathcal{H}^0(\tilde{\mathcal{F}}^\bullet) \to 0 \cdots \}$$

Generally $\quad \mathcal{H}^j = \tau_{\le j} \circ \tau_{\ge j}$

Eg 2. (Perverse $\sim$)

$$\underline{{}^p\mathcal{D}_c^{\le 0}} := \{ \tilde{\mathcal{F}}^\bullet \mid \dim \operatorname{supp} H^k(\tilde{\mathcal{F}}^\bullet) \le -k \}$$

$$= \{ H^j(i_{X_\alpha}^{-1} \tilde{\mathcal{F}}^\bullet) = 0, \ \forall \alpha, \ \forall j > -d_{X_\alpha} \}$$

$$\underline{{}^p\mathcal{D}_c^{\ge 0}} := \{ \tilde{\mathcal{F}}^\bullet \mid \dim \operatorname{supp} H^k(\mathbb{D}_X \tilde{\mathcal{F}}^\bullet) \le -k \}$$

$$= \{ H^j(i_{X_\alpha}^! \tilde{\mathcal{F}}^\bullet) = 0, \ \forall \alpha, \ \forall j > -d_{X_\alpha} \}$$

$\rightsquigarrow$ Heart $\underline{\mathrm{Perv}(X)} = {}^p\mathcal{D}_c^{\leq 0} \cap {}^p\mathcal{D}_c^{\geq 0}$

"perverse cohomology"

$${}^p\mathcal{H}^j = {}^p\tau_{\leq j} \circ {}^p\tau_{\geq j} : D_c^b(X) \to \mathrm{Perv}(X)$$

Remark.

Perv are not "shf."!

while $\underline{U \mapsto \mathrm{Perv}(U)}$ can be glued

"stack"

Ex.

(i) $\mathcal{L} \in \mathrm{Loc}(X)$

$X \begin{cases} \text{sm.} \\ \text{local complete intersection} \end{cases} \Rightarrow \mathcal{L}[n] \in \mathrm{Perv}(X)$

(ii) • Skyscraper

• $Z^d \xrightarrow{\text{cl.}} X$, $i_*\mathcal{L}[d] \in \mathrm{Perv}(X)$

• $\mathrm{pt} \subset \mathbb{C}^1 \subset \mathbb{C}^2$

$$\mathbb{C}_{\mathbb{C}^2}[2] \oplus \mathbb{C}_{\mathbb{C}^1}[1] \oplus \mathbb{C}_{\{pt\}} \in \mathrm{Perv}(\mathbb{C}^2)$$

(iii) "Intersection cohomology complex"

General setting

Whitney $\rightsquigarrow$ $\exists$ $\mathfrak{X}$-stratification $X = X_n \geq X_{n-2} \geq \dots \geq X_0 \geq \emptyset$

set $U_k := X - X_{n-k}$, $k \geq 2$

with $U_k \xrightarrow{j_k} U_{k+1} \xleftarrow{i_k} X_{n-k} - X_{n-k-1} = U_{k+1} - U_k$

Deligne complex

- Start with $\mathcal{G}^\bullet_{\bar{p}}|_{U_2} := A_{U_2}[n]$
- Inductively $\mathcal{G}^\bullet_{\bar{p}}|_{U_{k+1}} := \tau_{\leq \bar{p}(k)-n} Rj_{k*}(\mathcal{G}^\bullet_{\bar{p}}|_{U_k})$

where

perversity func. $\bar{p} : \mathbb{Z}_{\geq 2} \to \mathbb{N}$

s.t. $\begin{cases} \bar{p}(2) = 0 \\ \bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k)+1 \end{cases}$

eg. (most common)

zero-peru. $\bar{0}(k) \equiv 0$

top-peru. $\bar{t}(k) = k-2 = (0, 1, 2, 3, \dots)$

lower-middle $\bar{m}(k) = \lfloor \frac{k-2}{2} \rfloor = (0, 0, 1, 1, 2, 2, 3 \dots)$

upper-middle $\bar{n}(k) = \lceil \frac{k-2}{2} \rceil = (0, 1, 1, 2, 2, 3, 3 \dots)$

complementary

$\{\bar{p}, \bar{q}\}$ s.t. $\bar{p} + \bar{q} = \bar{t}$

Geometrically $\mathcal{G}^\bullet_{\bar{p}} \overset{\text{qis.}}{\simeq} IC^\bullet_{\bar{p}}$

$$X \supseteq U \longmapsto IC^{\bar{p}}_{-i}(\!(U)\!)$$

where

$$\xi \in \underbrace{IC^{\bar{p}}_{i}(\!(U)\!)}_{\text{"locally finite PL } i\text{-chain"}} \quad \forall\, k \geq 2 \quad \begin{cases} \dim(|\xi| \cap |X_{n-k}|) \leq i - k + \bar{p}(k) \\ \dim(|\partial\xi| \cap |X_{n-k}|) \leq i - k - 1 + \bar{p}(k) \end{cases}$$

Remark. "perversity cond."

$\begin{cases} \text{1st} & i\text{-cycle lies mostly in } X_{reg} \\ \text{2rd} & \text{inv. under restratification} \end{cases}$

Particularly

$$I^{BM}H^{\bar{p}}_{i}(X;\mathcal{L}) := H_i(IC^{\bar{p}}_{\bullet}(\!(X;\mathcal{L})\!)) = \mathbb{H}^{-i}(\mathcal{G}^{\bullet}_{\bar{p}}(\mathcal{L}))$$

$$IH^{\bar{p}}_{i}(X;\mathcal{L}) := H_i(IC^{\bar{p}}_{\bullet}(X;\mathcal{L})) = \mathbb{H}^{-i}_{c}(\mathcal{G}^{\bullet}_{\bar{p}}(\mathcal{L}))$$

Put $IC_X := IC^{\bullet}_{\bar{m}}[-n] \in Perv(X)$

$$\underbrace{IH^k(X)}_{\substack{\text{topological} \\ \text{stratified}}} := \mathbb{H}^k(IC_X) \quad \text{satifies} \begin{cases} \text{Poincáre Duality} \\ \text{Lefschetz thm.} \\ \text{MHS} \\ \text{Decomposition package} \end{cases}$$

Thm. (Deligne's construction)

$A^\bullet \in D^b(X)$ satifies axioms $[AX_{\bar{p}}]$

(AX0) *Normalization*: $\mathcal{A}^\bullet|_{U_2} \simeq \underline{A}_{U_2}[n]$.
(AX1) *Lower bound*: $\mathcal{H}^j(\mathcal{A}^\bullet) = 0$ if $j < -n$.
(AX2) *Vanishing condition*: $\mathcal{H}^j(\mathcal{A}^\bullet|_{U_{k+1}}) = 0$ if $j > \overline{p}(k) - n$.
(AX3) *Attaching condition*: $\mathcal{H}^j(i_k^*\mathcal{A}^\bullet|_{U_{k+1}}) \to \mathcal{H}^j(i_k^* R j_{k*} j_k^* \mathcal{A}^\bullet|_{U_{k+1}})$ is an isomorphism if $j \leq \overline{p}(k) - n$.

Then $IC^\bullet_{\bar{p}} \overset{qis.}{\cong} A^\bullet \in D^b_{c.*}(X)$

For $U \underset{open}{\overset{j}{\hookrightarrow}} \overset{dense}{X} \underset{cl.}{\overset{i}{\hookleftarrow}} Z = X - U$

intermediate extension

$\mathcal{G}^\bullet \in Perv(U) \overset{\exists!}{\rightsquigarrow} \underline{j_{!*}\mathcal{G}^\bullet} \in Perv(X)$ s.t. TAFE

(i) $Im({}^p j_! \mathcal{G}^\bullet \to {}^p j_* \mathcal{G}^\bullet)$

(ii) $\exists!$ extension as $\mathcal{G}^\bullet \to \tilde{\mathcal{F}}^\bullet \in D^b_c(X)$

s.t. $\begin{cases} i^* \tilde{\mathcal{F}}^\bullet \in {}^pD^{\leq -1}(Z) \\ i^! \tilde{\mathcal{F}}^\bullet \in {}^pD^{\geq 1}(Z) \end{cases}$

$\Updownarrow$

$\forall$ stratum $V \subseteq Z \overset{i_V}{\hookrightarrow} X$

$\begin{cases} \mathcal{H}^k(i_V^* \tilde{\mathcal{F}}^\bullet) = 0, & k \geq -\dim_{\mathbb{C}} V \\ \mathcal{H}^k(i_V^! \tilde{\mathcal{F}}^\bullet) = 0, & k \leq -\dim_{\mathbb{C}} V \end{cases}$

Eg. $U_{\ell+2} \overset{v=v_\ell}{\hookrightarrow} U_\ell := \coprod_{\dim \geq \ell} X_\alpha$

Then $\mathcal{G}^\bullet \in Perv(U_{\ell+1})$

$\Rightarrow v_{!*}\mathcal{G}^\bullet = \tau_{\leq -\ell-2} v_* \mathcal{G}^\bullet \in Perv(U_\ell)$

Particularly

$U = X_{reg} \overset{j}{\hookrightarrow} X \qquad IC_X(L) \cong j_{!*}(L[n])$

Thm.

$\mathrm{Perv}(X)$ $\begin{cases} \text{Artinian} \\ \text{Noetherian} \end{cases}$

with simple object of the form :

$$j_{!*}(\mathcal{L}^{irr}[d]) \quad \text{where} \quad j : Z^d_{sm} \xrightarrow[\text{cl.}]{\text{locally}} X$$

Thm. (Beilinson - Berstein - Deligne - Gabber)

$\forall$ pure - dimensional $X \xrightarrow{\text{"proper"}} Y$

$$Rf_* IC_X \cong \bigoplus_{i \geq 0} {}^p\mathcal{H}^i(Rf_* IC_X)[-i] \in D^b_c(Y)$$

$$= \bigoplus \left(\bigoplus_S IC_{\bar{S}}(\mathcal{L}_{i,S} : ss.) \right)[-i]$$

Particularly X : sm.

$$Rf_* \underline{\mathbb{Q}}_X[n] \cong \bigoplus_{i \geq 0} {}^p\mathcal{H}^i(Rf_* \underline{\mathbb{Q}}_X[n])[-i]$$

$\Rightarrow$ E_2 - dege. of "perverse" Leray S.S.

$${}^pE_2^{p,q} = H^p(Y ; {}^pR^q f_* \underline{\mathbb{Q}}_X) \Rightarrow H^{p+q}(X ; \mathbb{Q})$$

Proof

1st	BBDG	reduction to finite field
2rd	M. Saito	MHM
3rd	Cataldo . Migliorini	Classical Hodge

Similarly "perverse" Leray filtration

$$P_q H^\bullet(X;\mathbb{Q}) := \mathrm{Im}\left(H^\bullet(Y; {}^p\tau_{\le q} Rf_*\underline{\mathbb{Q}}_X) \to H^\bullet(X;\mathbb{Q})\right)$$

$$\overset{\text{If } \exists\, \text{BBDG}}{=} \bigoplus_{i\le q} H^{\bullet - q}(Y, {}^p\mathcal{H}^i(Rf_*\underline{\mathbb{Q}}_X))$$

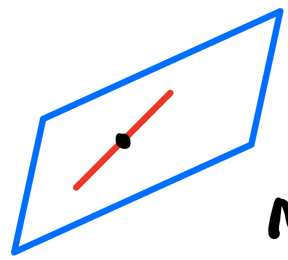
Geometric characterization

Λ^k : codim $= k$ linear subspace

$$f : X \to \mathbb{A}^n$$

$\updownarrow$ "generic n-flag"

$$\underline{\emptyset = \Lambda_{n+1} \subset \Lambda_n \subset \cdots \subset \Lambda_0 = \mathbb{A}^n}$$



$\Lambda^3 \subset \Lambda^2 \subset \Lambda^1$

Flag filtration

$$F_k H^\bullet(X) := \mathrm{Ker}\left(H^\bullet(X) \to H^\bullet(f^{-1}(\Lambda^{k+1-\bullet}))\right)$$

Thm. (Cataldo – Migliorini '10)

$$X \xrightarrow{f} Y : \text{affine} \Rightarrow P_i H^\bullet(X) \cong F_i H^\bullet(X)$$

Pf. (sketch for proper)

BBDG $\rightsquigarrow$ $Rf_*\mathbb{Q}_X[n] \cong \bigoplus \mathcal{Q}_i[-i]$

$$H^\bullet(X;\mathbb{Q}) = \cdots \oplus \overbrace{H^{\bullet-i}(Y, \mathcal{Q}_i) \oplus H^{\bullet-i+1}(Y, \mathcal{Q}_{i-1}) \oplus \cdots}^{P_i}$$

"perverse Lefschetz Hyperplane" $\updownarrow$; $\downarrow$ "res" ; $\downarrow 0$: Artin vanishing

$$\cdots \oplus \underline{H^{\bullet-i}(Y_{i-\bullet+1}, \mathcal{Q}_i|_{\cdots})} \oplus \cdots$$

$\square$

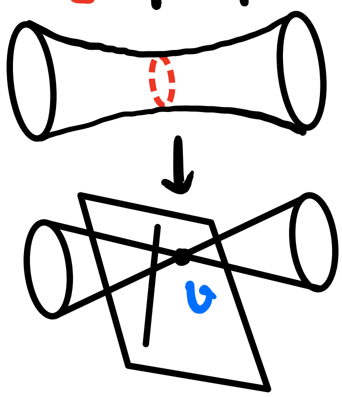
Ex. 1

trivial $\quad X \times \mathbb{C}^n \xrightarrow{pj} \mathbb{C}^n$

Künneth $\quad P_{k+n} H^k(X) = F_{k+n} H^k(X) = H^k(X)$

Ex. 2

$E = \mathbb{P}^1 \times \mathbb{P}^1$



$$\begin{array}{c} X \\ \downarrow f = Bl_v \\ Y = \mathrm{cone}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{A}^4 \end{array}$$

$$Rf_* \mathbb{Q}_X \overset{BBDG}{\cong} \overset{H^2(E)}{\mathbb{Q}_v[-2]} \oplus IC_Y \oplus \overset{H^4(E)}{\mathbb{Q}_v[-4]}$$

P_2

P_3

P_4

Generic 4-flag

$(Y_0 = Y) \supset (Y_1 = Y \cap \mathbb{A}^3) \supset (Y_2 = Y \cap \mathbb{A}^2) \supset (Y_3 = Y \cap \mathbb{A}^1) \supset \phi$

Compute

$X_\cdot := f^{-1}(Y_\cdot)$	X_0	X_1	X_2	X_3	X_4
$H^2(X_\cdot)$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$	0	0	0

$$P_i H^2(X) = F_i H^2(X) = H^2(X) = \mathbb{Q} \,. \quad i \geq 3$$

While

$$P_2 H^2(X) = H^2(Y, \mathbb{Q}_v[-2]) = \mathrm{Ker}(\mathbb{Q} \to H^2(X_1)) = 0$$