Enumerative geometry and Gromov-Witten invariants

§0 Some questions in enumerative geometry.

Question 1: Given four general $L_1, \dots, L_4 \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ will meet all four?

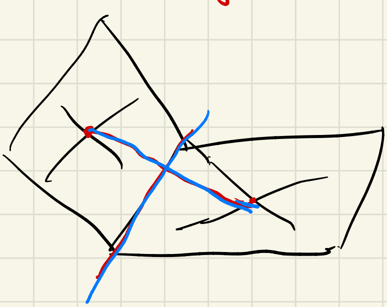
$$\overset{\text{Line}}{\Lambda} \subset \mathbb{P}^3$$

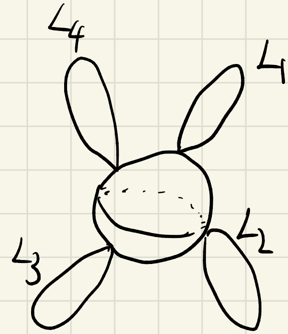
$$\Updownarrow$$

$$P(L_i) := \{ \Lambda \in Gr(2,4) \mid \dim(\Lambda \cap L_i) \geq 1 \}$$

$$\# \bigcap_{i=1}^{4} P(L_i) = \int_{Gr(2,4)} [P(L_1)] \cup \cdots \cup [P(L_4)]$$

if tranversely meet





Question 2: Given general m lines $L_1, \cdots, L_m \subset \mathbb{P}^2$, how many (rational) curves of degree d in $\mathbb{P}^2$ will meet $L_1, L_2, \cdots, L_m$?

genus 0

(Answer in Example 4.2)

Question 3: Given general m lines $L_1, \cdots, L_m \subset X$ (X: smooth proj variety), how many genus g curves of homology class β in X will meet $L_1, L_2, \cdots, L_m$?

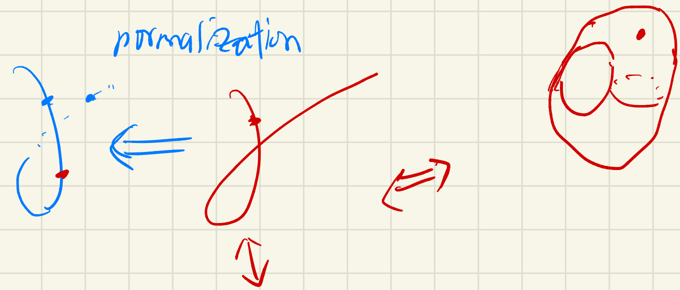
$\beta \in H_2(X;\mathbb{Z})$

In fact, we should consider the "moduli space":

$$\{ f: \Sigma_g \longrightarrow X \text{ morphism} \mid f_*[\Sigma_g] = \beta,\ \beta \in H_2(X;\mathbb{Z}) \}$$

(or holomorphic) Σ_g: smooth proj curve of genus g

§1 Stable curve.



Def 1.1 A connected, nodal complex proj curve

C is called stable curve if

$$\mathrm{Aut}(C) = \{\varphi : C \to C \mid \varphi \text{ is isomorphism}\}$$

is finite.

$$\mathrm{Aut}(E) = E \times \text{finite group}.$$

$$\mathrm{Aut}(\mathbb{P}^1) = PSL_2(\mathbb{C})$$

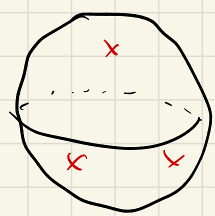
Given $p_1, \cdots, p_n$ distinct smooth points of C, we say that $(C, p_1, \cdots, p_n)$ is stable if the group $\mathrm{Aut}(C, p_1, \cdots, p_n) = \{ \varphi : C \to C \text{ iso} \mid \varphi(p_i) = p_i \}$ is finite

<u>Example 1.2</u>

not stable

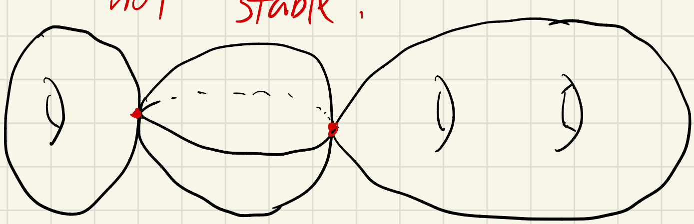
$\mathbb{P}^1$

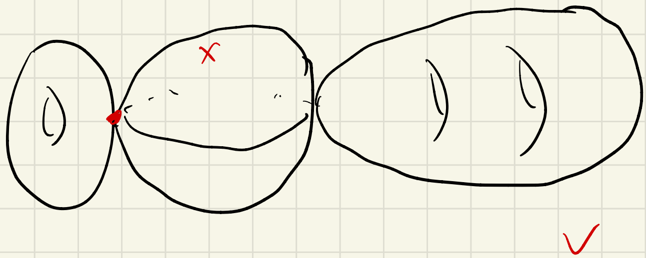
$(\mathbb{P}^1, x_1, x_2, x_3)$



stable

not stable.





§2 Stable map and moduli space

Def 2.1 An n-pointed stable map consists of a proj connected marked nodal curve $(C, \underbrace{p_1, \cdots, p_n}_{\text{distinct smooth pts}})$ and a morphism $f: C \to X$

s.t.

$$|\mathrm{Aut}(C, p_1, \cdots, p_n, f)| < +\infty.$$

Here $\varphi \in \mathrm{Aut}(C, p_1, \cdots, p_n, f)$, i.e.

$$\begin{tikzcd}
(C, p_1, \ldots, p_n) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & (C, p_1, \ldots, p_n) \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

$p \qquad \varphi(p) \qquad \varphi(p_i) = p_i$

$f'(\varphi(p)) = f(p)$

$f \simeq f' \iff \exists\, \varphi : C \to C$ s.t.

$$f' \circ \varphi = f$$

Remark 2.2

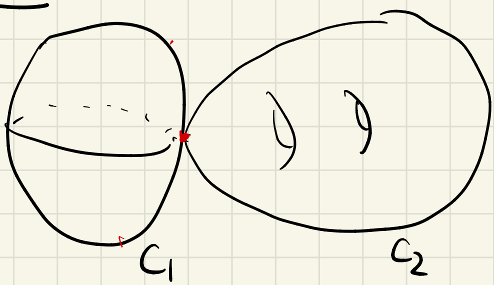
(1) If C_i is a component of C such that $C_i \simeq \mathbb{P}^1$ and f is constant on C_i, then C_i contains at least 3 special (i.e. nodal or marked) points.

(2) If C has arithmetic genus 1 and $n=0$, then f is not constant.

Example 2.3

not stable map ✗

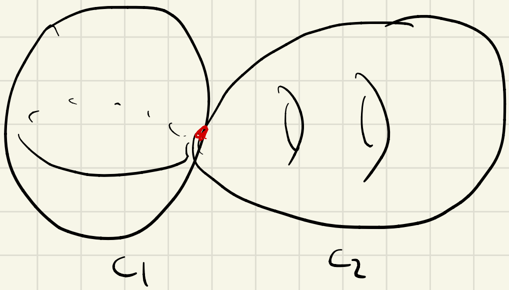
(1)



$\xrightarrow{\text{constant}} \cdot pt$

since $C_1 \simeq \mathbb{P}^1$, it has only 1 special points.

(2)



$\xrightarrow[\text{nonconstant}]{f}$

s.t. $f|_{C_1} : C_1 \to X$ nonconstant

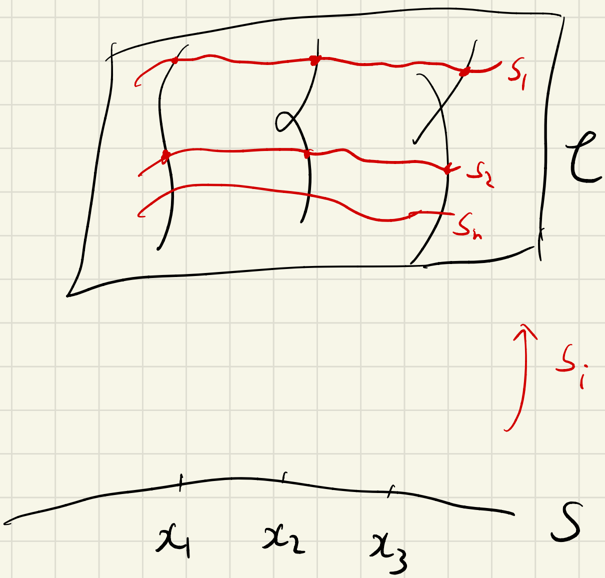
stable map ✓

family of stable map

Def 2.4 X: smooth proj variety, S: scheme$/\mathbb{C}$

An n-pointed stable map over S is a flat proper morphism $\mathcal{C} \to S$ together with n sections $s_1, \cdots, s_n$ and a map

$f: \mathcal{C} \to X$ such that for geometric point x of S, the restriction $f_x : \mathcal{C}_x \to X$ of f to the geometric fibers of $\mathcal{C}$ over x, together with images of the sections s_i, defines a stable map.



$(\mathcal{C}_x, s_1(x), \dots, s_n(x)) \xrightarrow{f_x} X$

is a stable map

$$\begin{tikzcd} \mathcal{C}_x \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \{x\} \arrow[r] \arrow[u, bend right, "s_i"'] \arrow[ur, phantom, "\mathcal{C}"] & S \arrow[u, bend right, "s_i"'] \end{tikzcd}$$

Furthermore (i) We say that $f: \mathcal{C} \to X$ has genus g if for each geometric point s of S, the curve $\mathcal{C}_s$ has arithmetic genus g.

"family invariant"

(ii) Given a homology class $\beta \in H_2(X;\mathbb{Z})$ if for each geometric point s of S, $(f_s)_*[\mathcal{C}_s] = \beta$.

Given $\beta \in H_2(X;\mathbb{Z})$, we can define the contravariant functor

$$\overline{\mathcal{M}}_{g,n}(X,\beta) : (\mathbb{C}\text{-schemes})^{op} \longrightarrow (\text{Sets})$$

$$S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of } n\text{-pointed stable} \\ \text{maps over } S \text{ of} \\ \text{genus } g \text{ and} \\ \text{class } \beta \end{array} \right\}$$

Given $S \xrightarrow{\varphi} T$ morphism

$$\overline{\mathcal{M}}_{g,n}(X,\beta)(T) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)(S)$$

$$\begin{tikzcd} & \mathcal{C} \arrow[d] \arrow[dr] & \\ S \arrow[r, "\varphi"] & T \arrow[u, bend right, "s_i"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times_T S \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \arrow[r, bend left] & X \\ S \arrow[r, "\varphi"] \arrow[u, bend right] & T \arrow[u, bend right, "s_i"'] & \end{tikzcd}$$

Question : Is $\overline{\mathcal{M}}_{g,n}(X,\beta)$ representable ?

i.e. $\exists\, Z \in \text{schemes}/\mathbb{C}$, s.t. $\overline{\mathcal{M}}_{g,n}(X,\beta) \simeq h_Z = \mathrm{Hom}_{Sch/\mathbb{C}}(\ ,Z)$

(Z is called fine moduli space)

Remark : In general, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not representable.

Thm 2.5 ([Alexeev], [FP] - Theorem 1)

X : proj $/\mathbb{C}$, $0 \neq \beta \in H_2(X;\mathbb{Z})$, there exists a projective coarse moduli space $\overline{M}_{g,n}(X,\beta)$ respresenting the moduli functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$

i.e. $\exists$ a map (set morphism) given by a natural transformation between two functors

$$\phi : \overline{\mathcal{M}}_{g,n}(X,\beta)(-) \Rightarrow \mathrm{Hom}_{Sch/\mathbb{C}}(-, \overline{M}_{g,n}(X,\beta))$$

$$\mathrm{Hom}_{Sch/\mathbb{C}}(-, \overline{M}_{g,n}(X,\beta)) : (Sch/\mathbb{C}) \longrightarrow (Sets)$$

s.t.

① $\varphi(\mathrm{Spec}\,\mathbb{C}) : \overline{\mathcal{M}}_{g,n}(X,\beta)(\mathrm{Spec}\,\mathbb{C}) \longrightarrow$

$$\mathrm{Hom}_{Sch/\mathbb{C}}(\mathrm{Spec}\,\mathbb{C}, \overline{M}_{g,n}(X,\beta))$$

is a set bijection

② If $T \in Sch_{\mathbb{C}}$ and $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta)(-)$

$\Rightarrow \mathrm{Hom}_{Sch_{\mathbb{C}}}(-, T)$ is a natural transformation

of functors, then there exists unique

$\nu : \overline{M}_{g,n}(X,\beta) \longrightarrow T$ s.t. $\psi = \tilde{\nu} \circ \varphi$

$\tilde{\gamma} : \mathrm{Hom}(-, \overline{M}_{g,n}(X,\beta)) \longrightarrow \mathrm{Hom}(-, T)$ is

the natural transformation associated to γ.

$$\begin{tikzcd}
\overline{M}_{g,n}(X,\beta)(-) \arrow[r, Rightarrow, "\phi"] \arrow[dr, Rightarrow, "\psi"'] & h_{\overline{M}_{g,n}(X,\beta)} \arrow[d, Rightarrow, "\tilde{\gamma}"] \\
 & h_T
\end{tikzcd}$$

Remark 2.6

By definition,

$$\overline{M}_{g,n}(X,\beta)(S) \longrightarrow \mathrm{Hom}_{Sch/\mathbb{C}}(S, X)$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, shift right] \arrow[drr, "f"] & & \\
S \arrow[u, shift right, "s_i"'] & & X
\end{tikzcd} \longmapsto \left(f \circ s_i : S \to X \right)$$

$\exists !$ evaluation map (morphism) $ev_i : \overline{M}_{g,n}(X,\beta) \to X$

$$(\mathcal{C}, x_1, \dots, x_n, f) \longmapsto f(x_i)$$

Example 2.7

① $X = pt$, $\beta = 0$, and $n+2g \geqslant 3$.

$$\overline{M}_{g,n}(\text{point}, 0) = \overline{M}_{g,n}$$

② $X = \mathbb{P}^r$, $\beta = d\square$, $\square \in H_2(\mathbb{P}^r, \mathbb{Z})$

$$\overline{M}_{0,0}(\mathbb{P}^r, 1\cdot\square) = \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^r)$$

$$= Gr(2, r+1)$$

<u>Example</u> (Special case) $X = G/P$, $g=0$

<u>Thm 2.8</u> ([FP] - Theorem 2, Theorem 3)

(i) $\overline{M}_{0,n}(X,\beta)$ is a normal projective variety of pure dim

$$\dim_{\mathbb{C}}(X) + \int_\beta c_1(T_X) + n - 3$$

(ii) $\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a nonsingular variety by a finite group (orbifold) (The boundary of $\overline{M}_{0,n}(X,\beta)$ is a divisor with normal crossing)

(iii) $\overline{M}^*_{0,n}(X,\beta)$ is a nonsingular, fine moduli space (for automorphism-free stable maps) equipped with a universal family.

Remark: We explain the reason why the moduli space $\overline{M}_{0,n}(X,\beta)$ of $X = G/P$ behaves well from symplectic geometry

Def 2.9 A (compact Kähler mfd) nonsingular variety M is called convex if for any morphism $f : \mathbb{P}^1 \to M$ (or holomorphic map), s.t.

$$H^1(\mathbb{P}^1, f^* TM) = 0.$$

Example: $X = G/P$. since $T_{G/P}$ is globally generated.

In symplectic geometry, $(\Sigma = \mathbb{P}^1, j_\Sigma)$, (M, J)

$\beta \in H_2(M)$

holomorphic map : $(\mathbb{P}^1, j_\Sigma) \xrightarrow{f} (M, J)$

$$\Longleftrightarrow \quad J \circ df = df \circ j_{\mathbb{P}^1} \quad \text{i.e.} \quad \begin{tikzcd} T\mathbb{P}^1 \arrow[r, "df"] \arrow[d, "j_{\mathbb{P}^1}"'] \arrow[dr, phantom, "\circlearrowright"] & TM \arrow[d, "J"] \\ T\mathbb{P}^1 \arrow[r, "df"'] & TM \end{tikzcd} \text{ commutes.}$$

Remark 2.10 $g : \mathbb{C}^n \to \mathbb{C}$ holomorphic $\Longleftrightarrow J_{\mathbb{R}^{2n}} \circ dg = dg \circ J_{\mathbb{R}^2}$

$\mathbb{C}^n \cong \mathbb{R}^{2n}$, $\mathbb{C} \cong \mathbb{R}^2$

$\Longleftrightarrow$ Cauchy-Riemann eqn

$$J_{\mathbb{R}^{2n}} : \frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$$

$J \circ J = -1$

Denote $\bar{\partial}_J(f) := \frac{1}{2}\left(df + J \circ df \circ j_{\mathbb{P}^1}\right)$

$\mathrm{Hol}^*(\mathbb{P}^1, M, \beta, J) := \{ f \in C^\infty(\mathbb{P}^1, M) \mid \bar{\partial}_J(f) = 0,\ f_*[\mathbb{P}^1] = \beta \}$

smooth map — holomorphic

f is simple.

$\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1 \to M$

$\mathrm{Aut}(\varphi) = 1$

$\Updownarrow$

$\deg(\varphi) > 1$

$\gamma : (-\varepsilon, \varepsilon) \to C^\infty(\mathbb{P}^1, M)$

$t \mapsto \gamma_t : \mathbb{P}^1 \to M.$

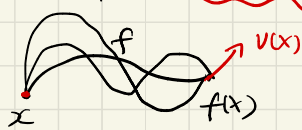
s.t. $\gamma_0 = f : \mathbb{P}^1 \to M.$

$T_f\mathcal{B} \ni \boxed{\frac{d}{dt}\Big|_{t=0} \gamma_t} : \mathbb{P}^1 \to TM$

$x \mapsto \frac{d}{dt}\Big|_{t=0} \gamma_t(x) \in T_{f(x)}M$

$\uparrow \bar{\partial}_J$

$f \in \mathcal{B} = \{ f \in C^\infty(\mathbb{P}^1, M) \mid f_*[\mathbb{P}^1] = \beta \}.$



smooth section

$T_f\mathcal{B} \simeq \Omega^0(\mathbb{P}^1, f^*TM)$

$x \in \mathbb{P}^1 \longrightarrow v(x) \in T_{f(x)}M$

$$\begin{array}{ccc} f^*TM & \longrightarrow & TM \\ (x, v) & & \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{f} & M \\ x & \mapsto & f(x) \end{array}$$

$f^*TM = \{ (x, v) \in \mathbb{P}^1 \times TM \mid v \in T_{f(x)}M \}$

$\bar{\partial}_J : \mathcal{B} \longrightarrow \mathrm{Map}(T\mathbb{P}^1, TM)$

$\forall v \in T\mathbb{P}^1 \simeq T^{1,0}\mathbb{P}^1$, $\bar{\partial}_J(f)(v) \in T^{0,1}M$.

$T\mathbb{P}^1 \otimes_{\mathbb{R}} \mathbb{C} \simeq T^{1,0}\mathbb{P}^1 \oplus T^{0,1}\mathbb{P}^1$

$\underline{v \in T^{0,1}\mathbb{P}^1 \ , \ \bar{\partial}_J(f)(v) \in T^{1,0}M.}$

$TM \otimes_{\mathbb{R}} \mathbb{C} \simeq T^{1,0}M \oplus T^{0,1}M.$

$$J\left(\bar{\partial}_J(f)(v)\right)$$

$$= \frac{1}{2} J \left(df + J \circ df \circ \underline{j_{\mathbb{P}^1}\right)(v)}_{\sqrt{-1}\,v}$$

$$= \frac{1}{2}\left(J \circ df \circ j_{\mathbb{P}^1}\underbrace{(-j_{\mathbb{P}^1}(v))}_{-\sqrt{-1}\,v} + (-1) \circ df(\sqrt{-1}\,v)\right)$$

$$= (-\sqrt{-1})\left(\bar{\partial}_J(f)(v)\right)$$

$$T\mathbb{P}^1 \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{df} TM \otimes_{\mathbb{R}} \mathbb{C}.$$

$$\bar{\partial}_J(f) : T^{0,1}\mathbb{P}^1 \longrightarrow T^{1,0}M$$

$$\bar{\partial}_J(f) \in \Gamma\left(f^* T^{1,0}M \otimes (T^{0,1}\mathbb{P}^1)^*\right)$$

$$0 \to f^* T^{1,0}M \xrightarrow{D\bar{\partial}_J} (T^{0,1}\mathbb{P}^1)^* \otimes f^* T^{1,0}M \to 0.$$

$\mathcal{E}$ $\qquad \mathcal{E}_f = \Omega^{0,1}(\mathbb{P}^1, f^* T^{1,0}M)$ $\leftrightarrow$ smooth section

$$\| $$

$$\Gamma(\mathbb{P}^1, f^* T^{1,0}M \otimes (T^{0,1}\mathbb{P}^1)^*)$$

$\downarrow \uparrow \ S(f) := (f, \bar{\partial}_J(f))$

$$\mathcal{B} = \{ f \in C^\infty(\mathbb{P}^1, M) \mid f_*[\mathbb{P}^1] = \beta \}$$

$Hol(\mathbb{P}^1, M, \beta, J)$ is the zero set of this section

$$\begin{array}{c}\mathcal{E}\\ \downarrow\uparrow s\\ \mathcal{B}\end{array}$$

i.e. $s^{-1}(0) = Hol(\mathbb{P}^1, M, \beta, J)$

differential $\quad D_f \overset{\text{denote}}{:=} DS(f) = D_f\bar{\partial}_J : T_f\mathcal{B} \longrightarrow \mathcal{E}_f$.

$$0 \longrightarrow \Omega^0(\mathbb{P}^1, f^*T^{1,0}M) \xrightarrow{D_f\bar{\partial}_J} \Omega^{0,1}(\mathbb{P}^1, f^*T^{1,0}M)$$

(0,1)-Dolbeault cohomology $\quad$ sheaf cohomology

$$\mathrm{Coker}\, D_f = H^{0,1}(\mathbb{P}^1, f^*T^{1,0}M) \simeq H^1(\mathbb{P}^1, f^*TM \otimes \Omega^0)$$

$$\ker D_f = H^{0,0}(\mathbb{P}^1, f^*T^{1,0}M) \simeq H^0(\mathbb{P}^1, f^*TM)$$

Since M is complex mfd, then $T^{1,0}_M \simeq$ holomorphic tangent bundle TM

· If $\boxed{H^1(\mathbb{P}^1, f^*TM) = 0}$ $\Longrightarrow$ $D_f\bar{\partial}_J$ is surjective.

implicit function thm.

$\Longrightarrow$

$s^{-1}(0) = Hol^*(\mathbb{P}^1, M, \beta, J)$ is a "smooth" mfd.

with "Sobolev completions"

In fact,

$$\text{Index } D_f = \ker D_f - \operatorname{coker} D_f$$

$$= H^0(\mathbb{P}^1, f^*TM) - H^1(\mathbb{P}^1, f^*TM)$$

$$= \chi(\mathbb{P}^1, f^*TM)$$

complex dim Riemann-Roch.

$$= (\dim_{\mathbb{C}} M)(1-g) + \langle \beta, c_1(T_X) \rangle$$

§3 Gromov–Witten invariants of $X = G/P$.

([FP] - Section 7)

For the moduli space $\overline{M}_{0,n}(X,\beta)$, $\beta \in H_2(X;\mathbb{Z})$ we get the fillowing natural maps:

$$\pi_1 : \overline{M}_{0,n}(X,\beta) \longrightarrow X \times \cdots \times X$$

$$(C, p_1, \ldots, p_n, f) \longmapsto (f(p_1), \ldots, f(p_n))$$

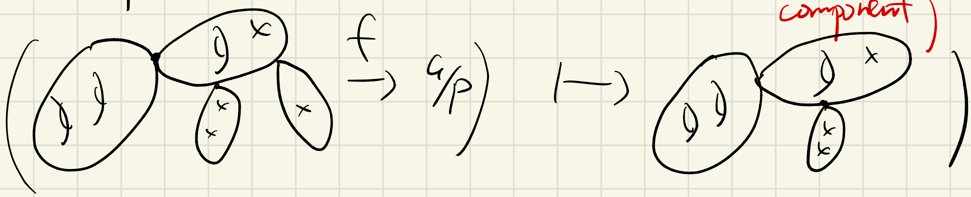
$$\pi_2 : \overline{M}_{0,n}(X,\beta) \longrightarrow \overline{M}_{0,n} \qquad (n \geq 3)$$

$(C, p_1, \ldots, p_n, f) \longmapsto$ stable curve $\tilde{C}$

by contracting the non-stable components of C gives a stable curve $\tilde{C}$

(contracting the unstable component)

Example



Remark 3.1

① π_1 is a morphism with the help of Remark 2.6.

② $n \geq 3$ $(n+2g \geq 3)$, π_2 can be seen to be a morphism using the techniques of Knudsen [Knudsen]

Since $\overline{M}_{0,n}(X,\beta)$ is an orbifold (due to Thm 2.8) of dimension $\dim_{\mathbb{C}} X - 3 + \int_\beta c_1(T_X) + n$, and π_1, π_2 induces the natural maps

$$\pi_1^* : H^*(X,\mathbb{Q})^{\otimes n} \longrightarrow H^*(\overline{M}_{0,n}(X,\beta),\mathbb{Q})$$

$$(\pi_2)_* : H_*(\overline{M}_{0,n}(X,\beta),\mathbb{Q}) \longrightarrow H_*(\overline{M}_{0,n},\mathbb{Q})$$

Bacause the cohomology of orbifold admits poincaré duality and π_2 is proper, then we have

$$(\pi_2)_! : H^*(\overline{M}_{0,n}(X,\beta),\mathbb{Q}) \longrightarrow H^{2m+*}(\overline{M}_{0,n},\mathbb{Q})$$

where $m = -\dim_{\mathbb{C}} X - \int_\beta c_1(T_X) = \dim \overline{M}_{0,n} - \dim \overline{M}_{0,n}(X,\beta)$

<u>Def 3.2</u> For $n \geq 3$, we define the Gromov–Witten class $I_{0,n,\beta}(\alpha_1, \cdots, \alpha_n)$ by the formula

$$I_{0,n,\beta}(\alpha_1, \cdots, \alpha_n) := (\pi_2)_! \, \pi_1^*(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n)$$

where $\alpha_1, \cdots, \alpha_n \in H^*(X)$

<u>Remark 3.3</u>

The Gromov–Witten class is proposed by Kontsevich–Manin [KM]. In fact, the Gromov–Witten class need to satisfy some axioms (such as splitting axiom, divisor axiom, fundamental class axiom, ...)

<u>Def 3.3</u>

The Gromov–Witten invariant (genus 0) for $X = G/P$ is defined by

$$\langle I_{0,n,\beta} \rangle (\alpha_1, \cdots, \alpha_n) := \int_{\overline{M}_{0,n}} I_{0,n,\beta}(\alpha_1, \cdots, \alpha_n) \quad (n \geq 3)$$

Remark 3.4

(1) By Def 3.2, $\langle I_{0,n,\beta}\rangle(\alpha_1,\dots,\alpha_n) = \int_{\overline{M}_{0,n}(X,\beta)} \pi_1^*(\alpha_1\otimes\cdots\otimes\alpha_n)$

$$= \int_{\overline{M}_{0,n}(X,\beta)} ev_1^*(\alpha_1)\cup\cdots ev_n^*(\alpha_n)$$

where ev_i $(1\le i\le n)$ are the i-th evaluation map.

(2) Although the Gromov-Witten class is defined with the necessary condition $n\geq 3$ $(n+2g\geq 3)$, the Gromov-Witten invariant in (1) is defined for all $n\geq 0$, $g=0$ $(g\geq 0)$

(3) In general case for smooth proj variety X, $\beta\in H_2(X)$ the moduli space $\overline{M}_{g,n}(X,\beta)$ is far from orbifold and the boundary of $\overline{M}_{g,n}(X,\beta)$ is complicated. So the Gromov-Witten invariant is defined by virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{vir}$. In other words,

$$[\overline{M}_{0,n}(G/P,\beta)]^{vir} = [\overline{M}_{0,n}(G/P,\beta)].$$

§4 Enumerative meaning of Gromov-Witten invariant for G/P

__Lemma 4.1__ : (Fulton - Pandharipande [FP, Lemma 14])

Let $X = G/P$, Let $g_1, \ldots, g_n \in G$

be general elements, let $\Gamma_1, \ldots, \Gamma_n$ be the

pure dimensional subvarieties of X. Let

$[Y_i] := P.D.\,[\Gamma_i] \in H^*(X)$ be the

corresponding cohomology class. Assume

$$\sum_i \operatorname{codim}(\Gamma_i) = \dim_{\mathbb{C}} \overline{M}_{0,n}(G/P, \beta)$$

$$= \dim_{\mathbb{C}}(G/P) + \int_\beta c_1(T_X) + n - 3.$$

Then the scheme theoretic intersection

$$ev_1^{-1}(g_1\Gamma_1) \cap ev_2^{-1}(g_2\Gamma_2) \cap \cdots \cap ev_n^{-1}(g_n\Gamma_n)$$

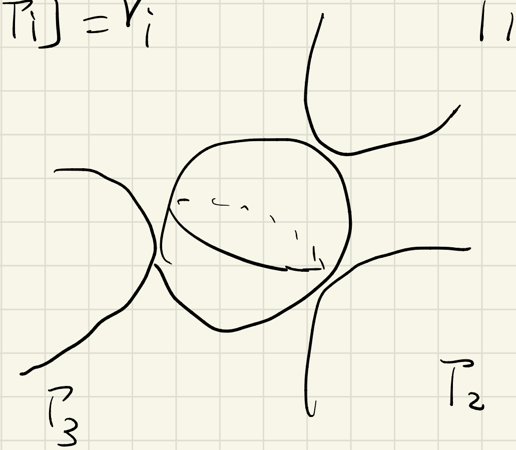
is a finite number of reduced points

supported on $M_{0,n}(X,\beta)$ and.

$$\langle I \rangle_{0,n,\beta}(\gamma_1, \cdots, \gamma_n) = \# \bigcap_i ev_i^{-1}(s_i \Gamma_i)$$

GW invariant $\longleftrightarrow$ enumerative geometry.

P.D $[\Gamma_i] = \gamma_i$



Example 4.2

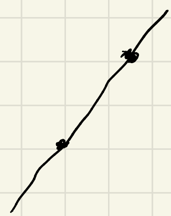
$X = \mathbb{P}^2$, $d = 1$ degree,

$[pt] \in H^4(\mathbb{P}^2)$, $1 \in H_2(\mathbb{P}^2, \mathbb{Z})$

$$\langle I_{0,2,1}\rangle([pt],[pt]) = \# \, ev_1^{-1}(S_1 [pt]) \cap ev_2^{-1}(S_2 [pt]) = 1$$

from enumerative meaning

$$4 = 2\,\mathrm{codim}_{\mathbb{C}}([pt]) = \dim_{\mathbb{C}} \overline{M}_{0,2}(\mathbb{P}^2, \beta) = 2 + 3 + 2 - 3$$



$\mathbb{P}^2$

In fact, we can compute the number N_d of rational curves of degree d passing through $3d-1$ points by WDVV equation ($\Leftrightarrow$ splitting axiom) and $N_1 = 1$.

$$N_d = \langle I_{0,3d-1,d}\rangle(\underbrace{[pt], \cdots, [pt]}_{d})$$

$$N_d \overset{\text{Kontsevich}}{=} \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

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