Lecture 1 PART I Introduction to nonabelian Hodge

§1. 0th Motivation (Example: mflds with 2 diff. alg. structures)

X smooth proj. curve $/\mathbb{C}$ genus g

cpt. Riemann surface

$g=1$ $E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$ elliptic curve

$$\begin{array}{ccccc} T^*E \simeq E\times\mathbb{C} & \not\simeq & (\mathbb{C}^*)^2 \simeq \mathrm{Hom}(\pi_1(E),\mathbb{C}^*) =: M_B(E,1) \\ \| \quad\quad \wr\!\| \quad \| & & \\ M_{Dol}(X,1) \quad \mathbb{C}/\mathbb{Z}\oplus\mathbb{C}/\mathbb{Z} \quad \mathbb{R}^2 & & \\ \wr\!\| & & \\ (S^1)^2 & & \end{array}$$

affine variety

$\downarrow$ fibers proj

$\mathbb{C}$

$g \geq 2$ $K_X := \Omega^1_X$ canonical line bundle

$$\underbrace{\mathrm{Pic}^0(X)\times \underbrace{H^0(X,K_X)}_{\cong\ \mathbb{C}^g \simeq \mathbb{R}^{2g}}}_{=:\ M_{Dol}(X,1)} \not\simeq (\mathbb{C}^*)^{2g} \simeq \mathrm{Hom}(\pi_1(X),\mathbb{C}^*) = M_B(X,1)$$

quasi-proj

affine

$\mathrm{Pic}^0(X)$ iso classes of line bundles over X of deg 0

$\simeq H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$

$\simeq H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$

$\simeq (S^1)^{2g}$

"Higgs bundles" $\longleftrightarrow$ "local systems"

§2. 1st Motivation: nonabelian analogue of Hodge theory

X smooth cplx mfd. $\dim_{\mathbb{C}} X = n$

$\rightsquigarrow$ 3 information:

(1) topological world:

X^{top}: underlying top. space by forgetting all extra str.

(2) differential world:

$X_{\mathbb{R}}$: forgetting complex str. looking as a real mfld of dim $2n$

(3) holo./analytic world:

X, picking up all str.

$\Rightarrow$ 3 kinds of coho. gps.

(1) Betti coho.

$$H^k_B(X,\mathbb{Z}) := H^k(X^{top},\mathbb{Z})$$

$\rightsquigarrow$ k-th singular coho. gp.

(2) de Rham coho.

$$H^k_{dR}(X,\mathbb{R}) := H^k(X_{\mathbb{R}},\mathbb{R})$$
$$:= H^k(C^{\bullet}_{dR})$$

$$C^{\bullet}_{dR}: 0 \to A^0(X,\mathbb{R}) \xrightarrow{d} A^1(X,\mathbb{R}) \xrightarrow{d} A^2(X,\mathbb{R}) \to \cdots$$

(3) Dolbeault coho.

$$H^k_{Dol}(X,\mathbb{C}) := \bigoplus_{p+q=k} H^{p,q}(X)$$
$$:= \bigoplus_{p+q=k} H^q(C^{p,\bullet}_{Dol})$$

$$C^{p,\bullet}_{Dol}: 0 \to A^{p,0}(X) \xrightarrow{\bar\partial} A^{p,1} \to \cdots$$

<u>Thm</u> (Main thms in Hodge theory)

(1) $\forall k$
$$H^k_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^k_{dR}(X,\mathbb{R})$$

(2) $\forall p,q$
$$H^{p,q}(X) \simeq H^q(X, \underbrace{\Omega^p_X}_{\text{sheaf of holo. } p\text{-forms on } X})$$

(3) X cpt. kähler $\forall k$
$$H^k_{dR}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^k_{Dol}(X,\mathbb{C})$$
$$= \bigoplus_{p+q=k} H^{p,q}(X)$$

Taking $k=1$:
$$H^1_{Dol}(X,\mathbb{C}) = H^1(X,\mathcal{O}_X) \oplus H^0(X,\Omega^1_X)$$
$$\wr\wr$$
$$H^1_{dR}(X,\mathbb{C}) \simeq H^1_B(X,\mathbb{C}) \simeq \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C})$$
$$\simeq \mathrm{Hom}(\pi_1(X)^{ab},\mathbb{C})$$
$$= \mathrm{Hom}(\pi_1(X),\mathbb{C})$$

$\Rightarrow$

$$\begin{tikzcd}
H^1(X,\mathcal{O}_X) \oplus H^0(X,\overset{\omega_X}{K_X}) \arrow[r, phantom, "\sim"] & \underline{\mathrm{Hom}(\pi_1(X),\mathbb{C})} \\
(e,f) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \rho \arrow[d, leftrightarrow] \\
\mathcal{H}^1(X,\mathbb{C}) \arrow[ur, phantom, ""] \arrow[r] & \underline{H^1_{dR}(X,\mathbb{C})}
\end{tikzcd}$$

$\mathcal{H}^1(X,\mathbb{C}) = \mathcal{H}^{0,1}(X) \oplus \mathcal{H}^{1,0}(X)$

"space of harmonic 1-forms"

nonabelian <u>analogue</u>:

$\mathbb{C} \to G$ general reductive gp. ($G = GL(n,\mathbb{C})$
$SL(n,\mathbb{C})$)

$$\rightarrow \quad H^1_B(X.G) \simeq H^1_{dR}(X.G) \simeq H^1_{Dol}(X.G)$$

$H^1_B(X.G) = \mathrm{Hom}(\pi_1(X), G)$

$H^1_{dR}(X.G)$: $\uparrow$ "space of diff. object" $\uparrow$ flat G-connections

$H^1_{Dol}(X.G) = H^1(X. Gl(\mathcal{O}_X)) \oplus H^0(X. ad(\mathcal{E}) \otimes \Omega_X^1)$ $\uparrow$ G-Higgs bundles

$\underbrace{\hspace{6em}}$ Riemann-Hilbert correspondence $\qquad \underbrace{\hspace{4em}}$ "harmonic objects"

§3. 2nd motivation: generalization of Narasimhan-Seshadri correspondence

Thm 1 (Narasimhan-Seshadri 1965)

X cpt. R. S. $g \geq 2$

holomorphic vector bundle over X is <u>stable</u> $\iff$ comes from irreducible proj. unitary representation of $\pi_1(X)$

In particular holo. v.b. of deg 0 is stable $\iff$ comes from irreducible unitary representation of $\pi_1(X)$

- $\mathcal{C}^s_{Vect}(X, n)$ (resp $\mathcal{C}^s_{Vect}(X, n, 0)$) : category of stable v.b. of rk n (resp. rk n + deg 0)
- $\mathcal{C}^{irr}_{Rep}(X, PU(n))$ (resp. $\mathcal{C}^{irr}_{Rep}(X, U(n))$) : category of irreducible reps $\pi_1(X) \to PU(n)$ (resp. $\pi_1(X) \to U(n)$)

$$\mathcal{C}^s_{Vect}(X, n) \simeq \mathcal{C}^{irr}_{Rep}(X. PU(n))$$

$$\mathcal{C}^s_{Vect}(X, n, 0) \simeq \mathcal{C}^{irr}_{Rep}(X. U(n))$$

higher dim. generalization:

Thm 2 (Donaldson, Uhlenbeck-Yau)

(X,ω) cpt. Kähler mfd.

$$\mathcal{C}^{s}_{Vect}(X,n) \simeq \mathcal{C}^{irr}_{Rep}(X, PU(n))$$
$$\mathcal{C}^{s}_{Vect}(X,n,c_1=\cdots=0) \simeq \mathcal{C}^{irr}_{Rep}(X, U(n))$$

Q: what objects correspond to $\mathcal{C}^{irr}_{Rep}(X, GL(n,\mathbb{C}))$?

Thm 3 (Donaldson, Corlette, Hitchin, Simpson)

$$\begin{array}{ccc} \mathcal{C}^{s}_{Higgs}(X,n,c_1=\cdots=0) & \simeq & \mathcal{C}^{irr}_{Rep}(X, GL(n,\mathbb{C})) \\ \shortparallel & & \shortparallel \\ \mathcal{C}^{s}_{Dol}(X,n) & & \mathcal{C}^{irr}_{B}(X,n) \end{array}$$

§4. key point: Donaldson's diff-geom reproof of NS correspondence.

Thm 1' (Donaldson 1983) X cpt R.S $g\geq 2$ $\mathcal{E}$ indecomposable v.b.

$\mathcal{E}$ is stable $\Leftrightarrow$ $\mathcal{E}$ is projectively unitary flat bundle

In particular, $\mathcal{E}$ is stable of deg $0 \Leftrightarrow \mathcal{E}$ is unitary flat bundle.

where • proj. unitary flat bundle $:= \exists$ unitary connection ∇ s.t.

$$\underbrace{\Lambda_\omega F_\nabla}_{\text{curvature}} = -2\pi\sqrt{-1}\,\underbrace{\mu(\mathcal{E})}\,\mathrm{Id}$$

$$\mu(\mathcal{E}) := \frac{\int_X c_1(\mathcal{E})}{rk(\mathcal{E})}$$

$\Leftrightarrow$ $\exists$ hermitian metric h s.t.

$$\Lambda_\omega F_{\nabla_h} = -2\pi\sqrt{-1}\,\mu(\mathcal{E})\,\mathrm{Id}$$

∇_h : Chern connection.

$\mathcal{E}$ is unitary flat $\triangleq$ $\sqrt{-1}\Lambda_\omega F_{\nabla} = 0$

$\vdots$

$$\boxed{\begin{aligned} \mathcal{C}^{s}_{\mathrm{Vect}}(X,n) &\simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{PUF}}(X,n) \\ \mathcal{C}^{s}_{\mathrm{Vect}}(X,n,0) &\simeq \mathcal{C}^{\mathrm{irr}}_{\mathrm{UF}}(X,n) \end{aligned}}$$

"stability" ~ "special metric" $\leftarrow$ called Kobayashi-Hitchin correspondence

Goal of this ~~short~~ course:

explain explicitly the idea on show Thm 3 :

$$\mathcal{C}^{s}_{\mathrm{Ad}}(X,n) \simeq \mathcal{C}^{\mathrm{irr}}_{B}(X,n)$$

KHC $\rightarrow$ $\wr\wr$ $\wr\wr$ $\uparrow$ Riemann-Hilbert corresp.

<u>key!</u>

$$\mathcal{C}^{\mathrm{irr}}_{\mathrm{dR}}(X,n)$$

$\uparrow$

category of irreducible flat bundle of rk n

PART I A crash course course on Hodge theory

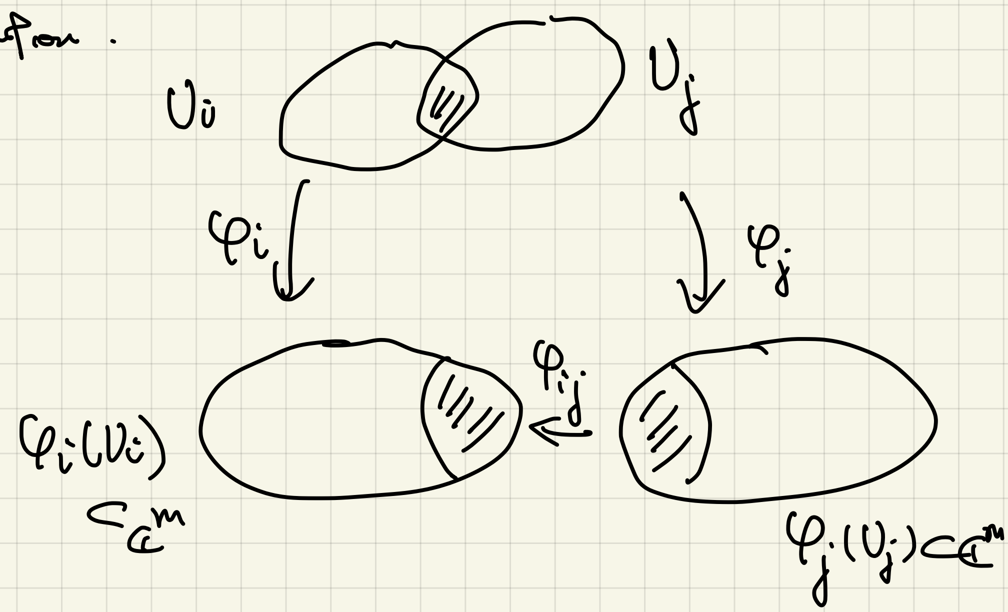
§1. Complex mflds ($\mathbb{Z}$-equiv. descriptions)

<u>Def 1.1</u> A cplex mfld of dim m is top. space X together with a complex structure (complex atlas) $\{(U_i, \varphi_i)\}_{i\in I}$ where:

- $\mathcal{U} = \{U_i\}_{i\in I}$ open covering of X
- $\forall i \quad \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^m$ homeomorphism
- whenever $U_i \cap U_j \neq \emptyset$, the transition function

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \underset{\subset \mathbb{C}^m}{\varphi_j(U_i \cap U_j)} \longrightarrow \underset{\subset \mathbb{C}^m}{\varphi_i(U_i \cap U_j)}$$

is holo. function.



- (U_i, φ_i) holo. chart
- $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^m$ coordinate function
  $\quad x \mapsto (z_1, \dots, z_m)$

<u>Ex</u> (complex proj. space)

$$\mathbb{CP}^m := \mathbb{C}^{m+1} - \{0\} / \sim$$

$(z_0, \ldots, z_m) \sim (z_0', \ldots, z_m') \Leftrightarrow (z_0, \ldots, z_m) = \lambda (z_0', \ldots, z_m')$

for some $\lambda \in \mathbb{C}^*$.

$[z_0 : \cdots : z_m]$

$\{(U_i, \varphi_i)\}_{0 \le i \le m}$

- $U_i := \{[z_0 : \cdots : z_m] : z_i \neq 0\}$
- $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{C}^m$

$$[z_0 : \cdots : z_m] \longmapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}\right)$$

- $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \quad (i < j)$

$$(w_1, \ldots, w_m) \mapsto \left(\frac{w_1}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \ldots, \frac{w_m}{w_i}\right)$$

Def 1.2 An almost cplex mfld is a diff. mfld X together with an almost cplex str. J, i.e. $J : TX \to TX$ s.t. $J^2 = -\mathrm{Id}$.

Prop 1.3 (1) Any almost cplex mfld has even real dim.

(2) complex mfld $\Rightarrow$ almost cplex mfld.

pf

(1) $\forall x \in X \quad \dim_{\mathbb{R}} T_x X = \dim_{\mathbb{R}} X$

$\forall v \in T_x X$. v & Jv $\mathbb{R}$-linear independent

$V := \mathrm{Span}_{\mathbb{R}}\{v, Jv\}$ J-inv. $V^\perp \subset T_x X$ is J-inv.

induction on dim gives $2 \mid \dim_{\mathbb{R}} T_x X$

(2) X cplex ~~of~~ $\dim_{\mathbb{C}} X = m \Rightarrow$ real mfld. of dim $2m$.

$\forall x \in X \quad (U, \varphi ; z_1, \ldots, z_m)$

$z_j = x_j + \sqrt{-1}\, y_j$. $1 \le j \le m$.

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ local basis of $T_x U$

define $J_x : T_x U \to T_x U$

$\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j}$ $\rightsquigarrow$ $J_x^2 = -\mathrm{Id}$

$\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}$

transition functs are holo.

$\rightsquigarrow$ globally defined $J : TX \to TX$ s.t. $J^2 = -\mathrm{Id}$.

$\blacksquare$

<u>Remk</u>. W. $\mathbb{R}$-v.s. $J : W \to W$ $J^2 = -\mathrm{Id}$

$\Rightarrow W$ is $\mathbb{C}$-v.s. via

$(a + \sqrt{-1}b) \cdot w := a \cdot w + b J(w)$

<u>But</u>. almost cplex mflds are not necessarily cplex mflds.

(X, J) almost cplex mfld. $\dim_{\mathbb{R}} X = 2n$

$J : TX \to TX$ $J^2 = -\mathrm{Id}$ $\rightsquigarrow$ $J : T^*X \longrightarrow T^*X$ via

$T^*X = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{R})$

$(J\theta)(v) := \theta(J(v))$ $\forall \theta \in T^*X$. $v \in TX$

let $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$ complexified bundle.

$T^*_{\mathbb{C}}X := T^*X \otimes_{\mathbb{R}} \mathbb{C}$

$J : TX \to TX$ & $J : T^*X \to T^*X$ extend $\mathbb{C}$-linearly

- $J : T_{\mathbb{C}}X \to T_{\mathbb{C}}X$ $v \in TX$. $\lambda \in \mathbb{C}$

  $v \otimes \lambda \mapsto J(v) \otimes \lambda$

- $J : T^*_{\mathbb{C}}X \longrightarrow T^*_{\mathbb{C}}X$ $\qquad J^2 = -Id$

  $\theta \otimes \lambda \longmapsto J(\theta) \otimes \lambda$

$\rightsquigarrow$ $\pm i$-eigenbundles

- $T_{\mathbb{C}}X = T_{1,0}X \oplus T_{0,1}X$

$$\begin{cases} T_{1,0}X = \{v - iJv : v \in TX\} \\ T_{0,1}X = \{v + iJv : v \in TX\} \end{cases}$$

- $T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X$

$$\begin{cases} T^*_{1,0}X = \{\theta - iJ\theta : \theta \in T^*X\} \\ T^*_{0,1}X = \{\theta + iJ\theta : \theta \in T^*X\} \end{cases}$$

define conjugation:

$$\begin{aligned} \bar{\ } : T_{\mathbb{C}}X &\longrightarrow T_{\mathbb{C}}X \\ v \otimes \lambda &\longmapsto v \otimes \bar{\lambda} \end{aligned}$$

$$\begin{aligned} \bar{\ } : T^*_{\mathbb{C}}X &\longrightarrow T^*_{\mathbb{C}}X \\ \theta \otimes \lambda &\longmapsto \theta \otimes \bar{\lambda} \end{aligned}$$

$\rightsquigarrow$ $\overline{T_{1,0}X} = T_{0,1}X$

$\overline{T^*_{1,0}X} = T^*_{0,1}X$

Thm 1.4 (Newlander-Nirenberg) (X, J) almost cplex str. TFAE:

(1) X cplex mfld

(2) The Nijenhuis tensor $N^J \equiv 0$

where $N^J(v,w) := J\big([Jv, w] + [v, Jw]\big) + [v,w] - [Jv, Jw]$

(3) $[\cdot,\cdot]$ is closed under $T_{0,1}X$ i.e. $[v,w] \in T_{0,1}X$ for $\forall v,w \in T_{0,1}X$

§2. Vector bundles and sheaves

$k = \mathbb{C}$ or. $\mathbb{R}$

<u>Def 2.1</u> A diff. surj map of diff. mflds $\pi : E \to X$ is <u>k-vector bundle of rk n</u> if $\forall x \in X$. $\exists$ open nbhd $x \in U \subseteq X$

& homeomorphism $\psi_U : \pi^{-1}(U) \to U \times k^n$ ← local trivialization

s.t.

(1) $pr_1 \circ \psi_U = \pi$ $\longleftrightarrow$

$$\begin{tikzcd} & U \times k^n \arrow[d, "pr_1"] \\ \pi^{-1}(U) \arrow[ur, "\psi_U"] \arrow[r, "\pi"'] \arrow[ur, phantom, shift left=3, "\cong"] & U \end{tikzcd}$$

(2) each fiber $E_x := \pi^{-1}(x)$ has a str. of k-v.s. s.t.

$E_x \to k^n$ k-linear iso.

A <u>section</u> of $\pi : E \to X$ is a C^∞ map $s : X \to E$ s.t. $\pi \circ s = id_X$

- $A^0(X, E)$: space of sections of E

$\Updownarrow$ transition functions:

$\forall$ 2 local trivializations $\psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times k^n$

$\psi_j : \pi^{-1}(U_j) \xrightarrow{\sim} U_j \times k^n$

with $U_i \cap U_j \neq \phi$

$$\rightsquigarrow \psi_{ij} := \psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times k^n \longrightarrow (U_i \cap U_j) \times k^n$$
$$(x, v) \longmapsto (x, g_{ij}(x) v)$$

determines $g_{ij} : U_i \cap U_j \to GL(n, k)$ ← transition functions

$$x \longmapsto g_{ij}(x)$$

s.t. • $g_{ii} = Id$ on U_i

• $g_{ij} g_{jk} g_{ki} = Id$ on $U_i \cap U_j \cap U_k \neq \phi$

Def 2.2 X cplex mfld. A <u>holomorphic vector bundle</u> over X is a $\mathbb{C}$-v.b. over X s.t. the transitions functions $\{g_{ij}\}_{i,j}$ are holomorphic.

Def 2.3 X top. space. a <u>presheaf</u> $\mathcal{F}$ of abelian gps (resp. k-v.s., rings, modules, ...) over X is

← space of sections on U

(1) $\forall U \subset X$ open. $\mathcal{F}(U)$ is abelian gp (resp. k-v.s., ...)

(2) $\forall U, V \subseteq X$ open. with $U \subset V$. the <u>restriction map</u>

$$r_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$
$$s \longmapsto s|_U$$

is gp. homomorphism. (resp. linear map of k-v.s. ..)

s.t.

(a) $r_{U,U} = id_{\mathcal{F}(U)}$

(b) $\forall U, V, W \subset X$ open. $U \subset V \subset W$

$$r_{V,U} \circ r_{W,V} = r_{W,U}$$

$s \in \mathcal{F}(W)$

$$(s|_V)|_U = s|_U$$

It is a <u>sheaf</u> if moreover,

(c) $\forall U = \bigcup_i U_i \subset X$ open. $\forall s_1, s_2 \in \mathcal{F}(U)$ with $s_1|_{U_i} = s_2|_{U_i}$ $\forall i$

$\Rightarrow s_1 = s_2$ (uniqueness)

(d) $s_i \in \mathcal{F}(U_i)$ s.t. whenever $U_i \cap U_j \neq \phi$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j$$

$\Rightarrow \exists_! s \in \mathcal{F}(U)$ s.t.

$s|_{U_i} = s_i \quad \forall i$ gluing

Ex X cplex mfd.

(1) C^∞_X : sheaf of C^∞ $\mathbb{C}$-valued functions on X

(2) $\mathcal{O}_X$: sheaf of holo. functions on X

(3) $E \to X$ $\mathbb{C}$-v.b. $\rightsquigarrow$ $\mathcal{E}$ sheaf of sections

i.e. $\forall\, U \subset X$ open.

$\mathcal{E}(U) := \{ s : U \to E \text{ section} \}$

(4) Ω^p_X : sheaf of holo. p-forms on X

$\Omega^p_X(U) := \{ \alpha \in A^{p,0}(U) : \bar\partial \alpha = 0 \}$

- $\mathcal{F}$: sheaf of $\mathcal{O}_X$-modules over X

$\mathcal{O}_X(U)$: ring

(1) $\mathcal{F}$ is coherent if $\forall x \in X$ $\exists$ $x \in U \subseteq X$ open s.t.

$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{F}_U \to 0$ exact for some $p, q \in \mathbb{Z}_{>0}$

$\Updownarrow$ $\mathcal{F}$ locally of finite type

& $\ker(\mathcal{O}_U^p \to \mathcal{F}_U)$ finite.

(2) $\mathcal{F}$ is torsion free if $\forall x \in X$ $\mathcal{F}_x$ is torsion free $\mathcal{O}_{X,x}$-module

$\Uparrow$

(3) $\mathcal{F}$ is locally free if $\forall$ $x \in X$. $\exists_{x \in} U \subset X$ s.t.

$\mathcal{F}_U \underset{\uparrow}{\simeq} \mathcal{O}_U^p$ for some $p \in \mathbb{Z}_{>0}$ 秩 of $\mathcal{F}$

iso. of sheaves of $\mathcal{O}_U$-modules.

for us. X cplex mfld

$\mathcal{O}_X$: sheaf of holo. functions on X

$(X, \mathcal{O}_X)$

<u>Prop 2.5</u> We have the following equiv. of categories (analytic version)

$$\left\{ \begin{array}{c} \text{holo. v.b. of rk } n \\ \text{over } X \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Locally free sheaves of } \mathcal{O}_X\text{-modules} \\ \text{of rk } n \end{array} \right\}$$

$$E \longmapsto \mathcal{E}$$

$\uparrow$ associated sheaf of holo. sections

<u>pf.</u>
- fully faithful: $\mathrm{Hom}(E, F)$ $\quad$ $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$

$$\exists\, \tilde{f} \in \mathrm{Hom}(E,F) \longleftrightarrow f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

- essentially surj. $\mathcal{E}$ on right side $\longrightarrow$ find $E = \coprod_{x \in X} \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$

$\blacksquare$

<u>Prop 2.6</u>

(1) (X, J) almost cplex $\Rightarrow$ TX (and T^*X) is $\mathbb{C}$-v.b.

(2) X cplex mfld. $\Rightarrow$ $T_{1,0}X$ (and $T^*_{1,0}X$) is a holo. v.b.

$\uparrow$ holo. tangent $\quad$ $\uparrow$ holo. cotangent.

2. Hodge theory (left)

3. affine GIT

4. Betti spaces

5. de Rham spaces

6. Dolbeault spaces

7. Proof of KHC

8. Other topics

X cplex mfld $\quad$ X analytic space (allow singularities)

$\mathcal{O}_X$ $\quad$ sheaf of $\mathcal{O}_X$-modules

E v.b. $\quad$ $F \subseteq E$ torsion-free sheaf

E/F may have torsion

$$0 \to F \to E \to E/F \to 0$$

Lecture 2

A crash course on Hodge theory ②

§3. Differential forms on cplex mflds

X cplex mfld, $\dim_{\mathbb{C}} X = m$

$\forall\ 0 \le k \le 2m$, introduce

(1) k-th exterior cotangent bundle

$$\Lambda^k X := \Lambda^k (T^* X)$$

(2) k-th exterior complexified cotangent bundle

$$\Lambda^k_{\mathbb{C}} X := \Lambda^k (T^*_{\mathbb{C}} X)$$

$$= \Lambda^k (T^*_{1,0} X \oplus T^*_{0,1} X)$$

$$\Lambda^k (E \oplus F) = \bigoplus_{p+q=k} \Lambda^p E \otimes \Lambda^q F$$

$$\to = \bigoplus_{p+q=k} \left(\Lambda^p (T^*_{1,0} X) \otimes \Lambda^q (T^*_{0,1} X) \right)$$

$$=: \bigoplus_{p+q=k} \Lambda^{p,q} X$$

- $A^k(X, \mathbb{R}) := \Gamma(X, \Lambda^k X)$ space of C^∞ real k-forms on X
- $A^k(X, \mathbb{C}) := \Gamma(X, \Lambda^k_{\mathbb{C}} X)$ ... .. complex ..
- $A^{p,q}(X) := \Gamma(X, \Lambda^{p,q} X)$ . .. (p,q)-forms ...

$$\Rightarrow \Lambda^k_{\mathbb{C}} X = \Lambda^k X \otimes_{\mathbb{R}} \mathbb{C}$$

$$\Lambda^k_{\mathbb{C}} X = \Lambda^k (T^*_{\mathbb{C}} X) = \Lambda^k (T^* X \otimes_{\mathbb{R}} \mathbb{C}) = \Lambda^k (T^* X) \otimes_{\mathbb{R}} \mathbb{C}$$

$$A^k(X, \mathbb{C}) = A^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

$$A^k(X, \mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$$

Taking local coordinate chart $(U, \varphi : z_1, \dots, z_m)$, then $\forall\, \alpha \in A^k(X, \mathbb{C})$, locally it looks like

$$\alpha = \sum_{p+q=k} \underbrace{\alpha_{i_1 \cdots i_p j_1 \cdots j_q}}_{\in C^\infty(U, \mathbb{C})} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \qquad (*)$$

Def 3.1 The exterior derivative $d : A^k(X, \mathbb{R}) \to A^{k+1}(X, \mathbb{R})$ extends $\mathbb{C}$-linearly to

$$d : A^k(X, \mathbb{C}) \to A^{k+1}(X, \mathbb{C})$$

define

$$\partial := \pi_{p+1,q} \circ d : A^{p,q}(X) \to A^{p+1,q}(X)$$

$$\bar\partial := \pi_{p,q+1} \circ d : A^{p,q}(X) \to A^{p,q+1}(X)$$

where $\pi_{p,q} : A^{p+q}(X, \mathbb{C}) \to A^{p,q}(X)$

Prop 3.2

X cplex mfld $\overset{(*)}{\Rightarrow}$ $d = \partial + \bar\partial$

i.e. $d : A^{p,q}(X) \to A^{p+1,q}(X) \oplus A^{p,q+1}(X)$

In particular, $d^2 = 0 \iff \partial^2 = 0 = \bar\partial^2$

$\partial\bar\partial = -\bar\partial\partial$

Rmk If X is merely an almost cplex mfld, then $d = \partial + \bar\partial$ not true!

$$d : A^{p,q}(X) \to A^{p+1,q}(X) + A^{p,q+1}(X) + \underline{A^{p-1,q+2}(X) + A^{p+2,q-1}(X)}$$

§4. Kähler mflds

(X, J) cplex mfld, $\dim_{\mathbb{C}} X = m$. g a Riemannian metric

Def 4.1 g is Hermitian if J is an isometry w.r.t. g.

i.e.

$$g(J\cdot, J\cdot) = g(\cdot, \cdot)$$

$\rightsquigarrow$ (X, J, g) is called a <u>Hermitian mfld</u>.

Rmk Hermitian metrics always exist on cplex mflds!

g a Riemannian metric

$$\tilde g(\cdot,\cdot) := g(J\cdot, J\cdot) + g(\cdot, \cdot)$$

$\rightsquigarrow$ $\tilde g$ is Hermitian.

(X, J, g) Hermitian mfd.. define the following 2-tensor

$$\omega(\cdot,\cdot) := g(J\cdot,\cdot) \qquad \left(\in \Gamma(X, T^*X \otimes T^*X) \right)$$

called the fundamental form.

Rmk (1) $g(JV, JW) = g(V,W) \Rightarrow \omega(V,W) = -\omega(W,V)$

$\rightarrow \omega$ is 2-form. $(\in A^2(X,\mathbb{R}))$

(2) J is an isometry w.r.t. ω.

i.e. $\omega(JV, JW) = \omega(V,W)$

(3) $g \,\&\, J \Rightarrow \omega$; $\omega \,\&\, J \Rightarrow g$ $\left(g(\cdot,\cdot) = \omega(\cdot, J\cdot) \right)$

Def 4.2 (X, J, g) is called Kähler if ω is closed. i.e. $d\omega = 0$.

Hermitian metric

• $g : TX \times TX \to \mathbb{R}$ extends $\mathbb{C}$-linearly to $T_{\mathbb{C}}X$

$$g : T_{\mathbb{C}}X \times T_{\mathbb{C}}X \to \mathbb{C}$$

$$(V \otimes \lambda, W \otimes \mu) \mapsto g(V,W)\lambda\mu \qquad \lambda, \mu \in \mathbb{C}$$

$\Rightarrow$ (a) $g(\bar{V}, \bar{W}) = \overline{g(V,W)}$

(b) $g(V, \bar{V}) \geq 0$ & "$=0$" iff $V = 0$

(c) $g(V,W) = 0 \quad \forall V, W \in T_{1,0}X$

(d) $g(V,W) = 0 \quad \forall V, W \in T_{0,1}X$

Rmk (1) we say g Hermitian metric, because

$$g_{\mathbb{C}}(V,W) := g(V, \bar{W})$$

defines a Hermitian metric on the $\mathbb{C}$-v.b. $T_{\mathbb{C}}X$.

$T_{\mathbb{C}}X = T_{1,0}X \oplus T_{0,1}X$ is orthogonal w.r.t. $g_{\mathbb{C}}$

(2) $\omega \in A^2(X,\mathbb{R}) \cap A^{1,1}(X)$ from (c) & (d)

(3) local coord. char $(U, \varphi; z_1, \dots, z_m)$ $z_j = x_j + i y_j$

$$\mathrm{Vol}_g = \sqrt{\det(g)}\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n$$

$$= \frac{\omega^m}{m!}$$

§5. Hodge theory.

(X, J, g) cplex mfld. for g a Riemannian metric. $\dim_{\mathbb{C}} X = m$

g inner product on TX

$\rightsquigarrow$ induces g on $T^*X = \mathrm{Hom}_{\mathbb{R}}(TX, \mathbb{R})$

$\rightsquigarrow$ induces g on $\Lambda^k X = \Lambda^k(T^*X)$ $1 \le k \le 2m$

Locally. choose orthogonal normal basis $\{e^1, \dots, e^{2m}\}$ for T^*X

then $\{e^{j_1} \wedge \dots \wedge e^{j_k},\ 1 \le j_1 < j_2 < \dots < j_k \le 2m\}$ is an orthogonal normal basis for $\Lambda^k X$.

Def 5.1 $\forall\, 0 \le k \le 2m$, on $A^k(X, \mathbb{R})$ we define

(1) global L^2-norm:

$\forall\, \theta, \eta \in A^k(X, \mathbb{R})$

$$(\theta, \eta)_{L^2} := \int_X g(\theta, \eta)\, \mathrm{vol}_g$$

In particular, $\eta = \theta$, $(\theta, \theta)_{L^2} =: \|\theta\|_{L^2}^2$

(2) Hodge $*$:

$$* : A^k(X,\mathbb{R}) \longrightarrow A^{2m-k}(X,\mathbb{R})$$

via

$$\theta \wedge *\eta = g(\theta,\eta)\,\mathrm{Vol}_g.$$

Well-defined since $\wedge : A^k(X,\mathbb{R}) \times A^{2m-k}(X,\mathbb{R}) \to A^{2m}(X,\mathbb{R})$
$(\theta\ ,\ \eta) \mapsto \theta\wedge\eta$

is non-degenerate.

(3) adjoint exterior derivative

$$d^* : A^k(X,\mathbb{R}) \to A^{k-1}(X,\mathbb{R})$$

via

$$d^* = - * d *$$

(4) Hodge Laplacian

$$\Delta := dd^* + d^*d : A^k(X,\mathbb{R}) \to A^k(X,\mathbb{R})$$

(5) harmonic forms:

$$\mathcal{H}^k_g(X,\mathbb{R}) := \{\theta \in A^k(X,\mathbb{R}) : \Delta(\theta) = 0\}$$

space of "harmonic k-forms".

From now on. g is Hermitian metric. $\rightsquigarrow$ ω fundamental 2-form

(6) Lefschetz operator:

$$L_\omega : A^k(X,\mathbb{R}) \longrightarrow A^{k+2}(X,\mathbb{R})$$
$$\theta \longmapsto \theta\wedge\omega$$

(7) dual Lefschetz operator:

$$\Lambda_\omega : A^k(X,\mathbb{R}) \to A^{k-2}(X,\mathbb{R})$$

via

$$g(\Lambda_\omega(\theta),\eta) = g(\theta, L_\omega(\eta)) \qquad \forall\, \theta \in A^k(X,\mathbb{R}),\ \eta \in A^{k-2}(X,\mathbb{R})$$

$\mathbb{R} \rightsquigarrow \mathbb{C}$

$*$, L_ω, Λ_ω extend $\mathbb{C}$-linearly to $A^k(X,\mathbb{C}) = A^k(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. $\rightsquigarrow d^*$, Δ

real in the following sense:

$$\overline{*(\theta)} = *(\bar\theta). \quad \overline{L_\omega(\theta)} = L_\omega(\bar\theta). \quad \overline{\Lambda_\omega(\theta)} = \Lambda_\omega(\bar\theta)$$

g on TX $\rightsquigarrow$ $g_{\mathbb{C}}$ Hermitian metric on $T_{\mathbb{C}}X$

on $\Lambda^k X = \Lambda^k(T^*X)$ $\rightsquigarrow$ $g_{\mathbb{C}}$ Hermitian metric $\Lambda^k_{\mathbb{C}}X$

$$g_{\mathbb{C}}(\theta,\eta) := g(\theta,\bar\eta) \qquad \forall\, \theta,\eta \in A^k(X,\mathbb{C})$$

$\rightsquigarrow$ global L^2-norm on $A^k(X,\mathbb{C})$:

$$(\theta,\eta)_{L^2} := \int_X g_{\mathbb{C}}(\theta,\bar\eta)\, \mathrm{vol}_g$$

Rmk.:

(1) $* : A^{p,q}(X) \longrightarrow A^{m-q,m-p}(X)$

(2) $*^2 = (-1)^{k(2m-k)}\, \mathrm{Id}$ on $A^k(X,\mathbb{C})$

(3) $g(*\theta,*\eta) = g(\theta,\eta)$ $\rightsquigarrow$ $g_{\mathbb{C}}(*\theta,*\eta) = g(*\theta,\overline{*\eta})$

$$= g(*\theta,*\bar\eta)$$
$$= g(\theta,\bar\eta)$$
$$= g_{\mathbb{C}}(\theta,\eta)$$

(4) $A^k(X,\mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$ orthogonal decom. w.r.t. $g_{\mathbb{C}}$

Def 5.2

$\partial^* := -*\bar\partial* : A^{p,q}(X) \longrightarrow A^{p-1,q}(X)$ adjoint

$\bar\partial^* := -*\partial* : A^{p,q}(X) \longrightarrow A^{p,q-1}(X)$ adjoint

$\Delta_\partial := \partial\partial^* + \partial^*\partial : A^{p,q}(X) \longrightarrow A^{p,q}(X)$

$\Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{p,q}(X) \longrightarrow A^{p,q}(X)$

Prop 5.3 (1) $\Lambda_\omega(\theta) = (-1)^k * L_\omega * \theta \quad \forall\, \theta \in A^k$

(2) X cpt (i.e. without boundary). then

$$(\partial\theta, \eta)_{L^2} = (\theta, \partial^* \eta)_{L^2} \qquad \theta \in A^{p,q} \quad \eta \in A^{p+1,q}$$

$$(\bar\partial\theta, \eta)_{L^2} = (\theta, \bar\partial^* \eta)_{L^2} \qquad \theta \in A^{p,q} \quad \eta \in A^{p,q+1}$$

$$(d\theta, \eta)_{L^2} = (\theta, d^* \eta)_{L^2} \qquad \theta \in A^{k} \quad \eta \in A^{k+1}$$

pf (sketch)

(1) $g(\Lambda_\omega(\theta), \eta) = g(\theta, L_\omega(\eta)) \qquad \forall\, \theta \in A^k,\ \eta \in A^{k-2}$

$$\Rightarrow \eta \wedge * \Lambda_\omega(\theta) = L_\omega(\eta) \wedge * \theta$$
$$= \eta \wedge \omega \wedge * \theta$$
$$= \eta \wedge * \theta \wedge \omega$$
$$= \eta \wedge L_\omega(*\theta)$$

$$\Rightarrow * \Lambda_\omega(\theta) = L_\omega * \theta$$

(2) show $(d\theta, \eta)_{L^2} = (\theta, d^*\eta)_{L^2} \qquad \forall\, \theta \in A^k \quad \eta \in A^{k+1}$

$$g(d\theta, \eta)\, \mathrm{vol}_g = d\theta \wedge * \eta = d(\theta \wedge * \eta) - (-1)^k \theta \wedge \underbrace{d * \eta}$$
$$= d(\theta \wedge * \eta) - (-1)^k (-1)^{k(2m-k)} \theta \wedge * * d * \eta$$
$$= d(\theta \wedge * \eta) - g(\theta, * d * \eta)\, \mathrm{vol}_g$$
$$= d(\theta \wedge * \eta) + g(\theta, d^* \eta)\, \mathrm{vol}_g$$

$\boxtimes$

Prop 5.4 (Kähler identities) (X, J, g, ω) Kähler mfld

(1) $[\bar\partial, L_\omega] = [\partial, L_\omega] = [\bar\partial^*, \Lambda_\omega] = [\partial^*, \Lambda_\omega] = 0$

(2) $[\bar\partial^*, L_\omega] = i\partial. \quad [\partial^*, L_\omega] = -i\bar\partial$

$$[\Lambda_\omega, \bar\partial] = -i\partial^*, \quad [\Lambda_\omega, \partial] = i\bar\partial^*$$

(3) $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$

Δ commutes with $*$, ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, L_ω, and Λ_ω.

Pf (sketch)

(1) $[\bar\partial, L_\omega](\theta) = \bar\partial(L_\omega(\theta)) - L_\omega(\bar\partial\theta)$

$$= \bar\partial(\theta\wedge\omega) - \bar\partial\theta\wedge\omega$$

$$= \bar\partial\theta\wedge\omega + (-1)^k\theta\wedge\bar\partial\omega - \bar\partial\theta\wedge\omega$$

$$= 0$$

(2) define $H : A^k(X, \mathbb{C}) \longrightarrow A^k(X, \mathbb{C})$ $\qquad H = (k-m)\,\mathrm{Id}.$

$$\theta \longmapsto (k-m)\theta$$

check: $[L_\omega, \Lambda_\omega] = H$

$$-i\bar\partial = [\partial^*, L_\omega]$$

$$[\Lambda_\omega, \bar\partial] = i[\Lambda_\omega, [\partial^*, L_\omega]]$$

$$= -i\left([\partial^*, [L_\omega, \Lambda_\omega]] + [L_\omega, [\Lambda_\omega, \partial^*]]\right)$$

$$= -i\left([\partial^*, H]\right)$$

$$= -i\partial^*$$

(3) $\underline{\partial\bar\partial^* + \bar\partial^*\partial = 0}$ because

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial[\Lambda_\omega, \partial] + [\Lambda_\omega, \partial]\partial$$

$$= \partial\Lambda_\omega\partial - \partial^2\Lambda_\omega + \Lambda_\omega\partial^2 - \partial\Lambda_\omega\partial$$

$$= 0$$

$\underline{\bar\partial\partial^* + \partial^*\bar\partial = 0}$

$\Rightarrow \Delta_\partial = \Delta_{\bar\partial}$

- $\Delta = \Delta_\partial + \Delta_{\bar\partial}$

"spaces of harmonic forms": (X, J, g) Hermitian mfd.

(1) $\mathcal{H}^k(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta(\theta) = 0\}$

(2) $\mathcal{H}^k_\partial(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta_\partial(\theta) = 0\}$

(3) $\mathcal{H}^k_{\bar\partial}(X, g) := \{\theta \in A^k(X, \mathbb{C}) : \Delta_{\bar\partial}(\theta) = 0\}$

(4) $\mathcal{H}^{p,q}(X, g) := \{\theta \in A^{p,q}(X) : \Delta(\theta) = 0\}$

(5) $\mathcal{H}^{p,q}_\partial(X, g) := \{\theta \in A^{p,q}(X) : \Delta_\partial(\theta) = 0\}$

(6) $\mathcal{H}^{p,q}_{\bar\partial}(X, g) := \{\theta \in A^{p,q}(X) : \Delta_{\bar\partial}(\theta) = 0\}$

$\rightsquigarrow$

$$\mathcal{H}^k(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$$

orthogonal decomp. w.r.t. $g_{\mathbb{C}}$

$$\mathcal{H}^k_\partial(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\partial(X, g)$$

$$\mathcal{H}^k_{\bar\partial}(X, g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial}(X, g)$$

In particular, X Kähler $(\Rightarrow \Delta = 2\Delta_\partial = 2\Delta_{\bar\partial})$

$\Rightarrow$

$$\mathcal{H}^k(X, g) = \mathcal{H}^k_\partial(X, g) = \mathcal{H}^k_{\bar\partial}(X, g)$$

$$\mathcal{H}^{p,q}(X, g) = \mathcal{H}^{p,q}_\partial(X, g) = \mathcal{H}^{p,q}_{\bar\partial}(X, g)$$

Prop 5.5

(1) (Poincaré duality)

$$* : \mathcal{H}^k(X,g) \xrightarrow{\simeq} \mathcal{H}^{2m-k}(X,g) \qquad \left| [\Delta, *] = 0 \right.$$
$$\theta \longmapsto *\theta$$

(2) (Hodge duality)

$$* : \mathcal{H}^{p,q}_{\partial}(X,g) \xrightarrow{\simeq} \mathcal{H}^{m-q,m-p}_{\partial}(X,g)$$
$$\theta \longmapsto *\theta \qquad \left| [\Delta_\partial, *] = 0 \right.$$

(3) (Serre duality)

$$\bar{*} : \mathcal{H}^{p,q}_{\bar\partial}(X,g) \xrightarrow{\simeq} \mathcal{H}^{m-p,m-q}_{\bar\partial}(X,g)$$
$$\theta \longmapsto *\bar\theta$$

Def 5.6 (X,J) cplex mfld. $\dim_{\mathbb{C}} X = m$. $\forall\, 0 \le k \le 2m$

(1) Betti cohomology:

$$H^k_B(X,\mathbb{Z}) := H^k(C^\bullet_{sing}, d)$$

where $(C^\bullet_{sing}, d)$: singular cochain complex. Betti complex

(2) De Rham cohomology:

$$H^k_{dR}(X,\mathbb{C}) := H^k(C^\bullet_{dR}, d)$$
$$= \frac{\ker(d : A^k(X,\mathbb{C}) \to A^{k+1}(X,\mathbb{C}))}{\operatorname{Im}(d : A^{k-1}(X,\mathbb{C}) \to A^k(X,\mathbb{C}))}$$

$$(C^\bullet_{dR}, d): \quad 0 \to A^0(X,\mathbb{C}) \xrightarrow{d} A^1(X,\mathbb{C}) \xrightarrow{d} A^2(X,\mathbb{C}) \to \cdots$$

de Rham complex

(3) Dolbeault cohomology:

$$H^k_{Dol}(X,\mathbb{C}) := \bigoplus_{p+q=k} H^{p,q}(X)$$

$$:= \bigoplus_{p+q=k} H^q(C^{p,\bullet}_{Dol}, \bar\partial)$$

$$= \bigoplus_{p+q=k} \frac{\operatorname{Ker}(\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{Im}(\bar\partial : A^{p,q-1}(X) \to A^{p,q}(X))}$$

$(C^{p,\bullet}_{Dol}, \bar\partial)$:

$$0 \to A^{p,0}(X) \xrightarrow{\bar\partial} A^{p,1}(X) \xrightarrow{\bar\partial} \cdots \to A^{p,n}(X) \to 0$$

Dolbeault complex

Rmk (1) Betti coho is enough for X top space

(2) De Rham coho. is enough for X diff. mfd. ($d^2=0$)

(3) Dolbeault coho. need X cplex mfld ($d^2=0 \overset{d=\partial+\bar\partial}{\Rightarrow} \bar\partial^2=0$)

Thm 5.7 (Main thms in classical Hodge theory)

(1) De Rham thm:

$$H^k_B(X,\mathbb{C}) := H^k_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k_{dR}(X,\mathbb{C})$$

(2) Dolbeault thm:

$$H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$$

where Ω^p_X : sheaf of holo. p-forms.

$\forall\, U \subset X$ open $\Omega^p_X(U) := \{\alpha \in A^{p,0}(U) : \bar\partial\alpha = 0\}$

(3) Hodge decomposition thm:

(X, J, g, ω) compact Kähler mfld.

$$H^k_{dR}(X,\mathbb{C}) \simeq H^k_{Dol}(X,\mathbb{C})$$

$$= \bigoplus_{p+q=k} H^{p,q}(X)$$

$$\overline{H^{q,p}(X)} = H^{p,q}(X) \quad \leadsto \text{ conjugation.}$$

$$H^{p,q}(X) \simeq H^{m-p,m-q}(X) \quad \leadsto \text{ Serre duality}$$

pf (sketch)

(1) $H_B^k(X,\mathbb{C}) \simeq H^k(X,\underline{\mathbb{C}})$

local Poincaré lemma implies

$\mathcal{A}_{\mathbb{C}}^k$: sheaf of C^∞ $\mathbb{C}$-valued k-forms

In particular, $\mathcal{A}_{\mathbb{C}}^k(X) = A^k(X,\mathbb{C})$

$$0 \to \underline{\mathbb{C}} \hookrightarrow \mathcal{A}_{\mathbb{C}}^0 \to \mathcal{A}_{\mathbb{C}}^1 \to \cdots \quad \text{exact}$$

$$\Rightarrow H^k(X,\underline{\mathbb{C}}) \simeq H^k(\Gamma(X,\mathcal{A}_{\mathbb{C}}^\bullet)) = H_{dR}^k(X,\mathbb{C})$$

$$\rotatebox{90}{$\circlearrowleft$} \quad 0 \to A^0(X,\mathbb{C}) \xrightarrow{d} A^1(X,\mathbb{C}) \to \cdots$$

(2) $\mathcal{A}_X^{p,q}$: sheaf of C^∞ (p,q)-forms

In part. $\mathcal{A}_X^{p,q}(X) = A^{p,q}(X)$

$$0 \to \Omega_X^p \hookrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots \quad \text{exact}$$

$$\leadsto H^q(X,\Omega_X^p) \simeq H^{p,q}(X)$$

(3) to show $H_{dR}^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \quad \ni [\theta^{p,q}]$

$[\theta]_k \in$ $\quad \wr\wr \qquad\qquad \Downarrow$

$$\theta \in \quad \mathcal{H}^k(X,g) = \bigoplus_{p+q=k} \mathcal{H}^k(X,g) = \bigoplus_{p+q=k} \mathcal{H}^k_{\bar\partial}(X,g) \qquad \theta^{p,q}$$

$\uparrow$ Kähler

check independ of g

$\blacksquare$

Rmk $b_k := \dim_{\mathbb{C}} H^k_{dR}(X, \mathbb{C})$ Betti number

$h_{p,q} := \dim_{\mathbb{C}} H^{p,q}(X)$ Hodge number

(X, J, g, ω) compact Kähler $\dim_{\mathbb{C}} X = m$

$\Rightarrow$

- $b_k = b_{2m-k}$ (Poincaré duality)
- $h_{p,q} = h_{m-q,m-p}$ (Hodge duality)
- $h_{p,q} = h_{q,p}$ (conjugation)

$\Big\}\Rightarrow h_{p,q} = h_{m-p,m-q}$ (Serre duality)

- $b_k = \bigoplus_{p+q=k} h_{p,q}$. (Hodge decomposition)

$\rightsquigarrow$ b_k is even if k odd.

$\rightsquigarrow$ Hodge diamond

$$
\begin{array}{ccccccccc|c}
 & & & & h_{0,0} & & & & & b_0 \\
 & & & h_{1,0} & & h_{0,1} & & & & b_1 \\
 & & h_{2,0} & & h_{1,1} & & h_{0,2} & & & b_2 \\
 & \iddots & & & & & & & & \vdots \\
h_{m,0} & & \cdots & & \cdots & & \cdot & & h_{0,m} & b_m \\
 & \ddots & & & & & & \iddots & & \\
 & & h_{m,m-2} & & h_{m-1,m-1} & & h_{m-2,m} & & & b_{2m-2} \\
 & & & h_{m,m-1} & & h_{m-1,m} & & & & b_{2m-1} \\
 & & & & h_{m,m} & & & & & b_{2m}
\end{array}
$$

Hodge $\updownarrow$ *

Serre

Poincaré $\updownarrow$

$\longleftrightarrow$ conjugation

Lecture 3

Algebraic groups and affine GIT

Setting: $k = \bar{k}$, char $k = 0$. e.g. $k = \mathbb{C}$

algebraic varieties: defined as $Z(I)$. + Zariski top.

zero locus — I: (homogeneous) ideal generated by (homog). polynomials in $k[x_1, \dots, x_n]$

e.g. affine varieties

proj varieties

quasi-proj. varieties

integral separated scheme of finite type.

Q: Classification problems in algebraic geometry:

$(\mathcal{A}, \sim)$ for · $\mathcal{A}$: collection of objects e.g. top. spaces, varieties, bundles, ...

· $\sim$: equivalence relation e.g. homeomorphisms, iso, ...

find an algebraic variety M to describe $\mathcal{A}/\sim$ so that
M / k

$$\begin{array}{ccc} M(k) & \sim & \mathcal{A}/\sim \\ \cup\!\!\mid & & \cup\!\!\mid \\ x & \longleftrightarrow & [a] \end{array}$$

via GIT (geometric invariant theory).

§1. Linear algebraic groups

Def 1.1 An algebraic group G/k is a group that admits a structure of algebraic variety $/k$ s.t.

(1) multiplication $m: G \times G \to G$

$(g, h) \mapsto gh$

(2) inverse $i: G \to G$

$g \mapsto g^{-1}$

are morphisms of alg. varieties.

Def 1.2 A <u>homomorphism</u> of alg. gps $f: G \to G'$ is both a morphism of alg. varieties and a homomorphism of gps.

Prop 1.3 Every algebraic group is non-singular as an alg. variety.

- non-singular at some $g \in G$.
- $\forall h = hg^{-1} \cdot g \rightarrow$ non-singular at h.

E.g. $GL(n, k) := \{X \in M_{n \times n}(k) : \det X \neq 0\}$

$$\simeq \{(g, \lambda) \in M_{n \times n}(k) \times k : \det(g) \cdot \lambda = 1\} \subseteq \mathbb{A}_k^{n^2+1}$$

closed subvariety. so affine

Def 1.4 A <u>linear algebraic group</u> is an algebraic group that can be identified with a closed subgroup of some $GL(n, k)$.

An <u>affine algebraic group</u> is an algebraic group that is also an affine variety.

Rmk over $k = \bar{k}$, char $k = 0$

linear alg. gps = affine alg. groups.

Ex. (1) $G_m := (k^*, \times) \simeq GL(1, k)$

(2) $G_a := (k, +)$

as $\mathbb{G}_a \simeq \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \bar{k} \right\} \subset GL(2, \bar{k})$

(3) $SL(n, \bar{k}) \cong GL(n, \bar{k}) \cap Z(\det - 1)$

(4) $O(n, \bar{k})$

$SO(n, \bar{k})$

$PGL(n, \bar{k})$

...

(5) (Non-linear alg. group) · elliptic curves.

· abelian varieties

<u>Now</u>. G is linear alg. group. $/\bar{k}$.

V. finite dimensional vector space

$\rho : G \to GL(V)$ representation

Def 1.5 (1) ρ is irreducible if there is no <u>non-trivial</u> ρ-invariant subspace of V.

· non-trivial: except $\{0\}$. and V.

· ρ-invariant subspace: $W \subseteq V$ subspace s.t. $\rho(g)(W) \subseteq W$ $\forall g \in G$

(2) ρ is semisimple / completely reducible if $\forall$ non-trivial ρ-invariant subspace has ρ-invariant complement.

or equivalently. $\rho = \bigoplus_{i=1}^{\ell} \rho_i$ for each ρ_i irreducible

<u>Check!</u>

i.e., · $V = \bigoplus_{i=1}^{\ell} V_i$ $V_i \subset V$ subspace, ρ-inv.

· $\rho_i := \rho|_{V_i} : G \to GL(V_i)$

(3) G is <u>reductive</u> if $\forall$. finite dimensional representation of G

is completely reducible.

$\Updownarrow$ Weyl, Nagata, Mumford

G is reductive if unipotent radical is trivial. i.e. the maximal connected unipotent closed subgroup is trivial.

e.g. $GL(n,\mathbb{k})$. the maximal normal connected closed subgp is $T \subset GL(n,\mathbb{k})$ maximal torus. i.e.

$$\mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

unipotent : $\mathrm{diag}(1, \dots, 1)$

$SL(n,\mathbb{k})$. $PGL(n,\mathbb{k})$

$O(n,\mathbb{k})$. $SO(n,\mathbb{k})$

<u>But</u>. G_a is <u>not</u> reductive!

$$G_a \simeq \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{k} \right\} \subset GL(2,\mathbb{k})$$

but the unipotent radical is itself. non-trivial.

or equiv. $\rho : G_a \to GL(\mathbb{k}^2)$

$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle < \mathbb{k}^2$ G_a-inv.

but it has no G_a-inv. complement!

§2. Algebraic group action.

<u>Def 2.1</u> <u>G-variety</u> is an alg. variety $X/\mathbb{k}$ s.t. $G \curvearrowright X$

i.e. $\exists$ a group action $G \times X \to X$ also morphism of varieties
$(g,x) \mapsto gx$

- $g.(h.x) = (gh).x$ $\quad \forall g,h \in G$
- $e.x = x$ $\quad \forall x \in X$

Prop 2.2 X is a G-variety. then for $\forall x \in X$

(1) $G \cdot x \subset X$ is smooth locally closed. i.e. $G \cdot x \subset \overline{G \cdot x}$ open

(2) $G \cdot x$ is equi-dimensional. of $\dim(G \cdot x) = \dim(G) - \dim(G_x)$

for $G_x := \{ g \in G : gx = x \}$

(3) $\overline{G \cdot x} \setminus G \cdot x$ is the union of some closed orbits of $< \dim(G \cdot x)$

$\Rightarrow$ any orbit of minimal dimension is closed.

$\forall$ G-inv. $Y \subseteq X$ contains a closed orbit. (In particular, $\overline{G \cdot x}$ contains a closed orbit.)

pf. **Lem 2.3 (Chevalley)** $\varphi : X \to Y$ regular map of varieties $/k$.

$\Rightarrow \forall\, U \subset X$ constructible, tes image $\varphi(U)$ is constructible.

$\uparrow$ finite union of locally closed subsets.

In particular, $\varphi(X)$ is constructible

(1) $\sigma_x : G \to X$ $\quad \Rightarrow \quad G \cdot x = \operatorname{Im}(\sigma_x) \Rightarrow$ constructible
$\quad g \mapsto g \cdot x$

$\Rightarrow U \subset G \cdot x \subset \overline{G \cdot x}$ for $U \subset \overline{G x}$ open.

$G \curvearrowright \overline{G x}$ transitively $\forall\, g \in G \cdot x$ can be covered by moving U.

$\Rightarrow G \cdot x$ locally closed

(2) $\sigma_x : G \to G \cdot x$ flat with the fiber at x is $\sigma_x^{-1}(x) = G_x$

$\Rightarrow \dim G = \dim(G \cdot x) + \dim(G_x)$

(3) $\overline{G \cdot x} \setminus G \cdot x \subset \overline{G \cdot x}$ closed by (1)

$$G \curvearrowright \overline{G\cdot x}\setminus G\cdot x \quad \Rightarrow \quad \overline{G\cdot x}\setminus G\cdot x = \bigcup_{y\in U} G\cdot y$$
some $U \subset G$

☒

§3. Quotients and affine GIT.

X G-variety. G linear alg. group. $/k$

affine

$$G\times X \to X$$
$$(g, x) \mapsto g\cdot x$$

⇝ want "quotient".

(1) "stronger sense": $X/G := \{G\cdot x : x\in X\}$ has the str. of alg. var.

at least $\forall\, G\cdot x \subset G$ closed

(2) "weaker sense": allow $\exists$ non-closed orbits.. $\Rightarrow$ we already know closed orbits alway exist! Need a method to identify non-closed orbits. with non-empty intersection of their closures.

Def 3.1 A <u>categorical quotient</u> of $G \curvearrowright X$ is (Y, φ) s.t.

$\varphi : X \to Y$ <u>G-invariant</u> morphism of varieties. which is universal

i.e. $\forall$ G-invariant morphism $f : X \to Z$ uniquely factors through φ.

$G \curvearrowright X$

$\varphi(g\cdot x) = \varphi(x)$ $\forall g\in G$

$$\begin{tikzcd} & Y \arrow[d, dashed, "\exists ! g"] \\ X \arrow[ur, "\varphi"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\circlearrowright"] & Z \end{tikzcd}$$

Rmk 1) Not necessarily surjective (for general alg. var. see. arxiv: 9806049)

2) categorical quotient. if exist. is unique up to isomorphism (check!)

3). $\overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \phi \Rightarrow$

$$\begin{array}{ccc} \varphi(G\cdot x_1) & = & \varphi(G\cdot x_2) \\ \| & & \| \\ \varphi(x_1) & & \varphi(x_2) \\ \| & & \| \\ \varphi(\overline{G\cdot x_1}) & = & \varphi(\overline{G\cdot x_2}) \end{array}$$

• $G\cdot x_1$, $G\cdot x_2$ closed $\quad G\cdot x_1 \cap G\cdot x_2 = \phi \Rightarrow \varphi(G\cdot x_1) \neq \varphi(G\cdot x_2)$ (check!)

$k[X]$ coordinate ring $= k[x_1,\dots,x_n]/I(X) \simeq \mathcal{O}_X(X)$

finitely generated k-alg.

$G \curvearrowright X \rightsquigarrow G \curvearrowright k[X]$ via

$f \in k[X]$, define

$$g\cdot f := f(g^{-1}\cdot)$$
$$\overset{\text{!!}}{f^g}$$

define $k[X]^G := \{ f \in k[X] : f^g = f \ \forall g \in G \}$

Def 3.2 A good quotient of $G \curvearrowright X$ is G-invariant morphism of varieties $\varphi : X \to Y$ s.t.

(1) φ surjective & affine

(2) $\forall\, U \subset Y$ open, the pull-back

$$\varphi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U))$$
$$f \longmapsto f\circ\varphi$$

φ is G-inv. $\Rightarrow f\circ\varphi$ is G-inv. i.e. $\operatorname{Im}(\varphi^*) \subset \mathcal{O}_X(\varphi^{-1}(U))^G$

induces an isomorphism $\mathcal{O}_Y(U) \xrightarrow{\simeq} \mathcal{O}_X(\varphi^{-1}(U))^G$.

(3) $\forall\, W \subset X$ G-invariant closed $\Rightarrow$ $\varphi(W) \subset Y$ closed.

(4) $\forall\, W_1, W_2 \subset X$. G-inv. & closed s.t. $W_1 \cap W_2 = \phi$

$$\Rightarrow \varphi(W_1) \cap \varphi(W_2) = \phi.$$

Def 3.3 A geometric quotient of $G \curvearrowright X$ is a good quotient $\varphi: X \to Y$ which is also an $\underbrace{\text{orbit space}}$.

$\forall\, y \in Y$ $\varphi^{-1}(y)$ is a single closed orbit.

The following property says good / geom. quotients are local w.r.t. the base.

Prop 3.4 (1) $\varphi: X \to Y$ good (resp. geometric) quotient of $G \curvearrowright X$

$\Rightarrow \forall\, U \subset Y$ open. $\varphi|_{\varphi^{-1}(U)}: \varphi^{-1}(U) \to U$ is good (resp. geometric) quotient for $G \curvearrowright \varphi^{-1}(U)$.

(2) $\varphi: X \to Y$ G-inv. morphism. if $\exists\, Y = \bigcup_i U_i$ U_i open covering of Y s.t. $\varphi|_{\varphi^{-1}(U_i)}: \varphi^{-1}(U_i) \to U_i$ is good (resp. geom.) quotient $\forall i$

$\Rightarrow \varphi: X \to Y$ is a good (resp. geom.) quotient.

Prop 3.5 Geometric quotient $\Rightarrow$ good quotient $\Rightarrow$ categorical quotient

$\not\Leftarrow$ $\qquad\qquad\qquad$ $\not\Leftarrow$

$\uparrow$ very hard to construct.

pf (sketch) good $\Rightarrow$ categorical:

$\forall$ G-invariant morphism $f: X \to Z$ uniquely factors through $\varphi: X \to Y$

define set-theoretic map $\quad g: Y \to Z$

$\qquad y \mapsto f(x)$

$$\begin{tikzcd} & Y \arrow[d, dashed, "\exists ! g"] \\ X \arrow[ur, "\varphi"] \arrow[r, "f"'] & Z \end{tikzcd}$$

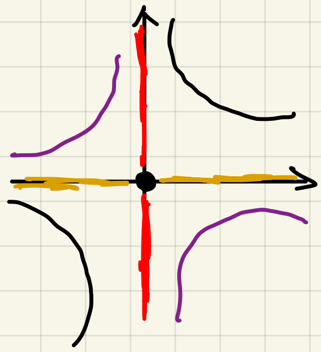
x is any element in $\varphi^{-1}(y)$

Show: • well-defined.

• g is a morphism of varieties.:

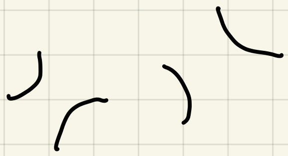
- continuous under Zariski top.
- $\forall\, V \subset Z$ affine open. $g|_{g^{-1}(V)} : g^{-1}(V) \to V$ is a morphism.

$\blacksquare$



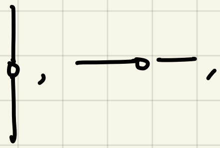
Ex 1) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$

$t.(x,y) := (t^{-1}x, ty)$



closed • closed

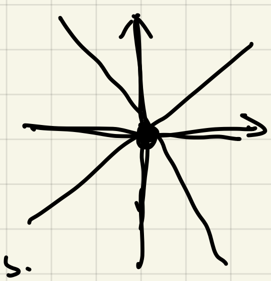
• $\subseteq$  $\subseteq$

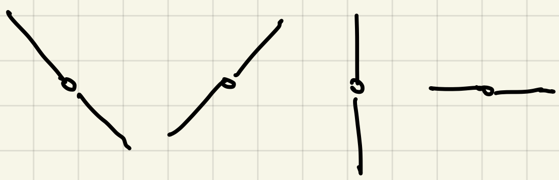
$\rightsquigarrow$ orbits  , • will be identified.

$\Rightarrow \varphi : \mathbb{A}^2_k \to \mathbb{A}^1_k \ (= \mathrm{Spec}\, k[xy])$ good quotient

$(x,y) \mapsto xy$ not geometric quotient

2) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k$ via $t.(x,y) = (tx,ty)$





non-closed orbits.

• closed $\subseteq$ closure of all orbits $\Rightarrow$ all orbits will be identified.

$\Rightarrow \varphi : \mathbb{A}^2_k \to pt$

3) $\mathbb{G}_m \curvearrowright \mathbb{A}^2_k \setminus \{0\}$. via $t.(x,y) := (tx,ty)$

$\rightsquigarrow$ all orbits are closed

$\rightsquigarrow$ geom. quotient $\varphi : \mathbb{A}^2_k \setminus \{0\} \to \mathbb{P}^1_k$.

$G \curvearrowright X \rightsquigarrow G \curvearrowright k[X] \rightsquigarrow k[X]^G$

<u>Q</u>: Is $k[X]^G$ finitely generated?

<u>Thm 3.6</u> (Nagata) G reductive $\Rightarrow$ $k[X]^G$ is finitely generated k-alg.

<u>Def 3.7</u> The affine GIT quotient is $\varphi: X \to X/\!/G := \operatorname{Spec}(k[X]^G)$ induced from $k[X]^G \hookrightarrow k[X]$

<u>Thm 3.8</u> $\varphi: X \to X/\!/G$ is good quotient. Hence categorical quotient.

Pf. Denote $Y := X/\!/G$

(1) $\varphi: X \to Y$ is G-invariant:

if not. $\exists\, g \in G$ s.t. $\varphi(g\cdot x) \neq \varphi(x)$

$\Rightarrow \exists\, f \in \mathcal{O}_Y(Y)$ s.t. $f(\varphi(g\cdot x)) \neq f(\varphi(x))$

i.e. $(\varphi^* f)(g\cdot x) \neq (\varphi^* f)(x)$.

contradicts to $\varphi^*(\mathcal{O}_Y(Y)) \subset \mathcal{O}_X(X)^G$.

(2) $\forall\, W_1, W_2 \subset X$ closed disjoint G-inv. subsets.

then $\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi$

[closed disjoint G-invariant subsets are distinguished by G-invariant functions.
i.e. $\exists\, f \in k[X]^G$. s.t. $f(W_1) = 1$. $f(W_2) = 0$.

$k[X]^G \cong k[Y]$ view. $f \in k[Y] \Rightarrow f(\varphi(W_1)) = 1$

$f(\varphi(W_2)) = 0$

$\Rightarrow \varphi(W_1) \cap \varphi(W_2) = \phi$

(3) $\forall\, W \subset X$ closed G-invariant. $\Rightarrow \varphi(W) \subset Y$ closed.

if not. $\exists\, y \in \overline{\varphi(W)} \setminus \varphi(W)$. $W_1 := W$ $\quad W_2 := \varphi^{-1}(y)$

$\Rightarrow W_1, W_2$ are closed, disjoint, G-inv.

$\overset{(2)}{\Rightarrow}$ $\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi$ i.e. $\overline{\varphi(W_1)} \cap \{y\} = \phi$ $\lightning$

$\square$

$G \curvearrowright X \rightsquigarrow \varphi : X \to X/\!/G = \mathrm{Spec}(\mathbb{k}[X]^G)$.

Q: What do the $\mathbb{k}$-pts $X/\!/G(\mathbb{k})$ represent?

Def 3.9 (1) $x \in X$ is polystable if $G \cdot x \subset X$ is closed.

(2) $x \in X$ is stable if • $G \cdot x \subset X$ closed

• $\dim(G_x) = 0$

(3) $x_1, x_2 \in X$ are called S-equivalent if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$

denoted as $x_1 \sim_S x_2$.

Seshadri

X^{ps} (resp. X^s) $\subset X$: sets of polystable (resp. stable) points.

Thm 3.10

(1) $X^s \subset X$ open & G-invariant.

(2) $\varphi(X^s) \subset X/\!/G$ open & $\varphi^{-1}(\varphi(X^s)) = X^s$

(3) $\varphi|_{X^s} : X^s \to \varphi(X^s)$ geometric quotient.

Thm 3.11 (Hilbert-Mumford criterion)

$x \in X$. let $O_x \subset \varphi^{-1}(\varphi(x))$ the unique closed orbit.

$\Rightarrow \exists$ 1-PS $\lambda : \mathbb{G}_m \to G$ s.t.

1-PS: 1-parameter subgroup is a group homomorphism $\lambda : \mathbb{G}_m \to G$

$\lim_{t \to 0} \lambda(t) \cdot x$ exists $\in O_x$

In particular.

$x \in X^s \Leftrightarrow \lim_{t\to 0} \lambda(t)\cdot x$ does not exist for $\forall$ non-trivial 1-ps $\lambda: G_m \to G$

pf:

"$\Rightarrow$": if $\exists$ non-trivial $\lambda: G_m \to G$ s.t. $\lim_{t\to 0} \lambda(t)\cdot x =: y$ exists.

$\Rightarrow y \in \overline{G\cdot x} = G\cdot x \Rightarrow y \in X^s$ since $x \in X^s$

but G_y contains $\lambda(G_m)$ $\rightsquigarrow$ $\dim(G_y) > 0$

contradicts to $y \in X^s$.

"$\Leftarrow$" if $\lim_{t\to 0} \lambda(t)\cdot x$ does not exist for $\forall$ non-trivial 1-ps.

$\Rightarrow G\cdot x$ closed.

($\because$ if not. $O_x \subset \overline{G\cdot x} \setminus G\cdot x$ by $\lim_{t\to 0} \lambda(t)\cdot x' \in O_x$ for trivial λ

$\Rightarrow g\cdot x \in O_x$ for some $g \in G$.)

G_x finite: otherwise it contains a non-trivial 1-ps. impossible.

$\boxtimes$

Cor 3.12 The following sets of k-pts are 1:1 correspondence (bijective)

$$X /\!/ G(k) \simeq X^{ps}(k)/G(k) \simeq X(k)/\sim_s$$

pf:

(1) $\forall$ $\overline{G\cdot x}$ contains a unique closed orbit.

(2) $\forall x_1, x_2$. $\varphi(x_1) = \varphi(x_2) \Leftrightarrow \overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \phi$

$\Leftrightarrow x_1 \sim_s x_2$

$\boxtimes$

Lecture 4

Betti spaces

Recall.

$k = \bar{k}$, char $k = 0$

G reductive alg. group $/k$

X: affine G-variety

$\rightsquigarrow$ affine GIT quotient $\varphi : X \to X/\!\!/G := \mathrm{Spec}(k[X]^G)$

Prop $\varphi : X \to X/\!\!/G$ is good quotient. in particular, φ is surjective

Lem: $\forall$ ideal $I \subset k[X]^G$

$$k[X]I \cap k[X]^G = I$$

Pf of lemma.

A $\forall$ k-algebra. $G \curvearrowright A$

$\Rightarrow \exists$ a linear map of k-algs $R : A \to A^G$ s.t.

$$R(ff') = R(f)f' \qquad \forall f \in A.\ f' \in A^G$$

In particular, $R(f) = f$ whenever $f \in A^G$.

This operator is called the Reynolds operator.

Applying R to $A := k[X]$, $A^G := k[X]^G$.

It suffices to show $k[X]I \cap k[X]^G \subset I$:

$\forall\ f = \sum_i f_i h_i \in k[X]^G \qquad f_i \in k[X],\ h_i \in I \subset k[X]^G$

$$\Rightarrow f = R(f) = \sum_i R(f_i h_i) = \sum_i \underbrace{R(f_i)}_{\in k[X]^G} \underbrace{h_i}_{\in I} \in k[X]^G I \subset I$$

$\blacksquare$

proof of proposition:

$\forall\ y \in X/\!\!/G \Longleftrightarrow$ maximal ideal $I_y \subset k[X]^G$

Lemma $\Rightarrow$ $k[X] I_y \subset k[X]$ proper ideal

$\Rightarrow$ $k[X] I_y \subset I_x$, for some $I_x \subset k[X]$ maximal ideal

by definition $\varphi(x) = y$.

$\blacksquare$

$x \in X$ is

- polystable if $G \cdot x \subset X$ is closed
- stable if
  - $G \cdot x \subset X$ is closed
  - G_x finite i.e. $\dim(G_x) = 0$

- $x_1, x_2 \in X$ are S-equivalent if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$

denote $x_1 \sim_S x_2$

$X^{ps} \subset X$ subset of polystable points

$X^s \subset X$ subset of stable points

Thm (1) $X^s \subset X$ open & G-invariant

(2) $\varphi(X^s) \subset X /\!/ G$ open & $\varphi^{-1}(\varphi(X^s)) = X^s$

(3) $\varphi|_{X^s} : X^s \to \varphi(X^s)$ geometric quotient

proof

(1) $X^s \subset X$ open: $\forall x \in X^s$, $\exists$ open nbhd of x in X^s

- define $X_+ := \{x \in X : \dim(G_x) > 0\}$

then $X_+ \subset X$ closed

- $G \cdot x \subset X$ closed

$\Rightarrow$ $X_+ \cap G \cdot x \neq \phi$. so they can be distinguished by some G-invariant function $f \in k[X]^G$.

i.e. $f(X_+) = 0$. $f(G \cdot x) = 1$

define
$$X_f := \{x \in X : f(x) \neq 0\} \subset X \quad \text{open.}$$

Claim: X_f is the desired ngbh. i.e. $x \in X_f \subset X^s$

1^{st}: $\forall x' \in X_f.$ $f(x') \neq 0 \Rightarrow x' \notin X_+ \Rightarrow \dim(G_{x'}) = 0$

2^{nd}: suppose $\exists x' \in X_f$ s.t. $G \cdot x'$ not closed.

$\Rightarrow \overline{G \cdot x'} \setminus G \cdot x'$ contains a closed orbit. say $G \cdot x''$ with

$$\dim(G \cdot x'') < \dim(G \cdot x') \leq \dim(G) \Rightarrow \dim(G_{x''}) > 0$$

but $x'' \in \overline{G \cdot x'} \setminus G \cdot x' \Rightarrow f(x'') \neq 0 \Rightarrow x'' \notin X_+ \Rightarrow \dim(G_{x''}) = 0$ ↯

$\Rightarrow X_f \subset X^s$

(2) $\varphi(X^s) \subset X /\!\!/ G$ open.

$f \in k[X]^G \simeq k[X /\!\!/ G] \rightsquigarrow \varphi(X_f) = \{y \in X /\!\!/ G : f(y) \neq 0\} \subset X /\!\!/ G$ open

$\Rightarrow \varphi(X^s) \subset X /\!\!/ G$ open

$X_f = \varphi^{-1}(\varphi(X_f))$:

if not. let $x \in \varphi^{-1}(\varphi(X_f)) \setminus X_f \Rightarrow f(x) = 0$

$f \in k[X /\!\!/ G] \Rightarrow f(\varphi(x)) = 0$ ↯

$\Rightarrow X^s = \varphi^{-1}(\varphi(X^s))$

∎

§2. Local systems = fundamental group representations

X connected. locally simply connected top. space.

Def 2.1 A k-local system of rk n on X is a rk n locally constant sheaf of k-vector spaces. $\mathcal{F}$. i.e. $\forall x \in X$ $\exists$ open $x \in U \subset X$ s.t.

$$\mathcal{F}|_U \simeq \underline{k}_U^n \leftarrow \text{constant sheaf on } U$$

$\mathcal{C}_{Loc}(X,n)$: category of k-local systems of rk n on X

objects : ...

morphisms : morphism of sheaves.

$\phi : \mathcal{F} \to \mathcal{F}'$ is morphism of sheaves is collection of morphism of k-vector spaces . $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "r^{\mathcal{F}}_{UV}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}'(U) \arrow[d, "r^{\mathcal{F}'}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\phi_V"'] & \mathcal{F}'(U) \end{tikzcd}$$

$\mathcal{C}_{Rep}(X,n)$: category of fundamental group representations $\rho : \pi_1(X,x) \to GL(n,k)$

fix $x \in X$ base point.

objects : ...

morphisms : a morphism of 2 representations $\rho_1, \rho_2 : \pi_1(X,x) \to GL(n,k)$

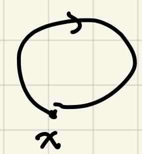
is a linear map $g : k^n \to k^n$ s.t.

$$\rho_1(\gamma) \circ g = g \circ \rho_2(\gamma) \quad \forall \gamma \in \pi_1(X,x)$$

i.e. $\rho_1(\gamma) = g \circ \rho_2(\gamma) \circ g^{-1}$

Thm 2.2 The above 2 categories are equivalent :

$$\mathcal{C}_{Loc}(X,n) \simeq \mathcal{C}_{Rep}(X,n)$$



pf (sketch)

"$\Rightarrow$" $\mathcal{F} \in \mathcal{C}_{Loc}(X,n)$. $\forall$ loop $\gamma : [0,1] \to X$ based at $x \in X$

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ X \end{array}$$

$\gamma^*\mathcal{F}$ is a k-local system on $[0,1]$

$[0,1]$ simply-connected $\Rightarrow$ $\gamma^*\mathcal{F}$ constant sheaf

$$\Rightarrow (\gamma^*\mathcal{F})_0 \xleftarrow[\sim]{r_{A0}} (\gamma^*\mathcal{F})_{[0,1]} \xrightarrow[\sim]{r_{A1}} (\gamma^*\mathcal{F})_1$$

$$\Rightarrow \gamma_* : \mathcal{F}_{\gamma(0)} \xrightarrow{\simeq} \mathcal{F}_{\gamma(1)}$$

$$\begin{tikzcd} \mathcal{F}_{\gamma(0)} \arrow[r, "\simeq"] \arrow[d, phantom, "\wr\wr"] & \mathcal{F}_{\gamma(1)} \arrow[d, phantom, "\wr\wr"] \\ \mathbb{R}^n & \mathbb{R}^n \end{tikzcd}$$

it suffices to show:

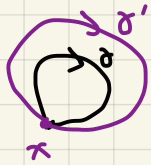
(check)! • γ_* is $\mathbb{R}$-linear: $\gamma_*(v+\lambda w) = \gamma_*(v) + \lambda \gamma_*(w)$ $\quad \forall\, v,w \in \mathcal{F}_{\gamma(0)}$, $\lambda \in \mathbb{R}$

• homotopy invariant: $\gamma \sim \gamma' \Rightarrow \gamma_* = \gamma'_*$.

homotopy inv.:

$\gamma \sim \gamma' \Rightarrow H : [0,1] \times [0,1] \to X$ s.t. $H(0,t) = \gamma(t)$

$H(1,t) = \gamma'(t)$



$$\Rightarrow \begin{tikzcd} \mathcal{F}_{H(0,0)} \arrow[r, "\gamma_*", "\simeq"'] \arrow[d, phantom, "\parallel"] & \mathcal{F}_{H(0,1)} \arrow[d, phantom, "\parallel"] \\ \mathcal{F}_{H(1,0)} \arrow[r, "\gamma'_*"] & \mathcal{F}_{H(1,1)} \end{tikzcd}$$

$\Rightarrow \gamma_* = \gamma'_*$

$\rightsquigarrow$ well-defined map $\rho : \pi_1(X,x) \longrightarrow GL(n,\mathbb{R})$. called the
$[\gamma] \longmapsto \gamma_*$
monodromy representation of the $\mathbb{R}$-local system $\mathcal{F}$.

"$\Leftarrow$" $\rho : \pi_1(X,x) \to GL(n,\mathbb{R})$. let $\pi : \tilde{X} \to X$ universal covering.

$E_\rho := \tilde{X} \times_\rho \mathbb{R}^n := \tilde{X} \times \mathbb{R}^n / \pi_1$

$\gamma \cdot (\tilde{x}, v) := (\gamma\tilde{x}, \rho(\gamma)v)$

$\mathcal{E}_\rho$ the associated sheaf of sections of E_ρ.

$\forall\, U \subset X$ open

$$\mathcal{E}_\rho(U) := \Gamma(U, E_\rho)$$

$$\simeq \{ s : \pi^{-1}(U) \to k^n : s(\gamma\tilde{x}) = \rho(\gamma) s(\tilde{x}) \ \forall \tilde{x} \in \tilde{X},\ \gamma \in \pi_1(X,x) \}$$

check $\mathcal{E}_\rho$ locally constant sheaf.. k-local system of rk n

Finally, show the functor $\mathcal{F} \mapsto (\gamma \mapsto \gamma_*)$ is an equivalence of categories.

$$\Rightarrow \quad \mathcal{C}_{Loc}(X,n) \simeq \mathcal{C}_{Rep}(X,n)$$

$\boxtimes$

§3. Betti spaces (abstract theory)

X smooth irreducible projective variety $/k$. fix $x \in X$

$\Rightarrow \pi_1(X,x)$ is finitely generated:

$$\pi_1(X,x) = \langle \gamma_1, \dots, \gamma_\ell : r_1(\gamma_1, \dots, \gamma_\ell) = \cdots = r_m(\gamma_1, \dots, \gamma_\ell) = Id \rangle,$$

where
- $\gamma_1, \dots, \gamma_\ell$ generators
- $r_1, \dots, r_m$ relations on generators.

Ex $\dim_k X = 1 \qquad g \geq 2$

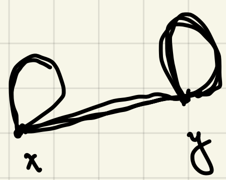
$$\pi_1(X,x) = \langle \gamma_1, \dots \gamma_g, \gamma_1', \dots \gamma_g' : \prod_{i=1}^{g} \underbrace{[\gamma_i, \gamma_i']}_{\gamma_i \gamma_i' \gamma_i^{-1} \gamma_i'^{-1}} = Id \rangle$$

Define

$$\mathcal{U}(X,x,n) := \mathrm{Hom}(\pi_1(X,x), GL(n,k)) = \{ \rho : \pi_1(X,x) \to GL(n,k) \}$$

space of fundamental group representations.

Choose $y \in X$.



$\Rightarrow \bullet \ \pi_1(X,x) \simeq \pi_1(X,y)$

$\Rightarrow \mathcal{U}(X, x, n) \simeq \mathcal{U}(X, y, n)$

we omit the base point, and denote it as $\mathcal{U}(X, n)$

- $\rho \in \mathcal{U}(X, n)$ $\quad \rho : \pi_1(X, x) \to GL(n, \mathbb{k})$ is fully determined by $\rho(\gamma_1), \dots, \rho(\gamma_\ell) \in GL(n, \mathbb{k})$

$\Rightarrow$ embedding
$$i : \mathcal{U}(X, n) \hookrightarrow GL(n, \mathbb{k})^\ell$$
$$\rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_\ell))$$

with image
$$\mathcal{U}(X, n) \simeq i(\mathcal{U}(X, n)) = \mathrm{Rel}^{-1}(Id, \dots, Id)$$
$$\mathrm{Rel} : GL(n, \mathbb{k})^\ell \to GL(n, \mathbb{k})^m$$
$$(A_1, \dots, A_\ell) \mapsto (r_1(A_1, \dots, A_\ell), \dots, r_m(A_1, \dots, A_\ell))$$

$\Rightarrow \mathcal{U}(X, n)$ can be thought as a closed subvariety of $GL(n, \mathbb{k})^\ell$.

so it is an affine variety.

Ex $\dim_{\mathbb{k}} X = 1$.
$$\mathcal{U}(X, n) \simeq \left\{ (A_1, \dots, A_g, A_1', \dots, A_g') \in GL(n, \mathbb{k})^{2g} : \prod_{i=1}^{g} [A_i, A_i'] = Id \right\}$$
$$\subset GL(n, \mathbb{k})^{2g}$$

$GL(n, \mathbb{k}) \curvearrowright \mathcal{U}(X, n)$ by conjugation:

$$\sigma : GL(n, \mathbb{k}) \times \mathcal{U}(X, n) \to \mathcal{U}(X, n)$$
$$(g, \rho) \mapsto g \cdot \rho \cdot g^{-1}$$
$$(g \rho g^{-1})(\gamma) := g \cdot \rho(\gamma) \cdot g^{-1}$$

this is compatible with the identification
$$\mathcal{U}(X, x, n) \simeq \mathcal{U}(X, y, n)$$

$\rightsquigarrow$ well-defined.!

$\Rightarrow$ Affine GIT quotient

$$\varphi : \mathcal{U}(X,n) \longrightarrow \mathcal{U}(X,n) /\!/ GL(n,\Bbbk) := \operatorname{Spec}\left(\Bbbk[\mathcal{U}(X,n)]^{GL(n,\Bbbk)} \right)$$
$$=: \mathcal{M}_B(X,n)$$

called the moduli space of fundamental group representations.

"Betti moduli space".

Eg. $\dim_{\Bbbk} X = 1$

(1) $g=0$ $\qquad \mathcal{M}_B(X,n) = \begin{cases} \emptyset & n \geq 2 \\ \{pt\} & n=1 \end{cases}$

(2) $g \geq 2$ $\qquad \mathcal{M}_B(X,n) \simeq (\mathbb{C}^*)^{2g}$

For $\rho \in \mathcal{U}(X,n)$, recall

Def 3.1 (1) ρ is called irreducible/simple if $\nexists$ non-trivial ρ-invariant subspace $W < \Bbbk^n$ $\neq \{0\}, \Bbbk^n$. $\rho(\gamma)(W) \subseteq W$ $\forall \gamma \in \pi_1(X,x)$

(2) ρ is called completely reducible/semisimple if $\forall$ ρ-invariant subspace of $\Bbbk^n$ has a ρ-invariant complement.

or. direct sum of irreducible reps.

Also recall:

Def 3.2

(1) ρ is polystable if the orbit $GL(n,\Bbbk)\cdot\rho \subset \mathcal{U}(X,n)$ is closed.

(2) ρ is stable if
- $GL(n,\Bbbk)\cdot\rho \subset \mathcal{U}(X,n)$ closed
- $\dim(PGL(n,\Bbbk)_\rho) = 0$

(3) $\rho_1 \sim_S \rho_2$ if $\overline{GL(n,\mathbb{k})\cdot\rho_1} \cap \overline{GL(n,\mathbb{k})\cdot\rho_2} \neq \phi$

Lem 3.3 : (1) ρ is stable $\Leftrightarrow$ ρ is irreducible

(2) ρ is polystable $\Leftrightarrow$ ρ is completely reducible.

pf

Recall Hilbert-Mumford criterion of stability:

For $\rho \in \mathcal{U}(X,n)$. $\exists$ 1-ps $\lambda : \mathbb{G}_m \to GL(n,\mathbb{k})$ s.t.

$$\lim_{t\to 0} \lambda(t)\cdot\rho := \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1} \text{ exists } \& \in O_\rho$$

the unique closed orbit inside $\overline{GL(n,\mathbb{k})\cdot\rho}$

In particular, ρ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ doesn't exist for $\forall$ non-trivial 1-ps $\lambda : \mathbb{G}_m \to GL(n,\mathbb{k})$

(1) $\rho : \pi_1(X,x) \to GL(n,\mathbb{k})$ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ does not exist. $\forall$ nontrivial 1-ps λ

By Borel thm. up to conjugation.

$$\lambda(t) = \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} \in GL(n,\mathbb{k})$$

$$a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbb{Z}$$

w.r.t. $\{e_1, \dots, e_n\}$

write $\underbrace{a_1 = a_2 = \cdots = a_{n_1}}_{n_1} > \underbrace{a_{n_1+1} = \cdots = a_{n_1+n_2}}_{n_2} > \cdots \underbrace{= a_{n_1+\cdots+n_m} = a_n}_{n_m}$

write

$$\rho(\gamma) = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

$\Rightarrow$

$$\lambda(t)\cdot\rho(\gamma) := \lambda(t)\rho(\gamma)\lambda(t)^{-1}$$

$$= \begin{pmatrix} \rho_{11} & t^{a_1-a_2}\rho_{12} & \cdots & t^{a_1-a_n}\rho_{1n} \\ t^{a_2-a_1}\rho_{21} & \rho_{22} & \cdots & t^{a_2-a_n}\rho_{2n} \\ \vdots & \vdots & & \vdots \\ t^{a_n-a_1}\rho_{n1} & t^{a_n-a_2}\rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

$\Rightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ exists $\iff$ $\rho_{ij} = 0$ whenever $a_i < a_j$

$$\iff \rho(\gamma) = \begin{pmatrix} \rho_{11} & \rho_{12} \cdots & \rho_{1n} \\ \vdots & \vdots & \vdots \\ \rho_{n_1 1} & \vdots & \vdots \\ 0 & \rho_{n_1+n_2\, n_1+1} & \vdots \\ \vdots & 0 & \\ 0 & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{0} & & & \\ & \boxed{0} & & \\ 0 & & 0 \ \ddots & \\ & & & \boxed{0} \end{pmatrix}$$

$\iff$ ρ preserves the flag

$$0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m = k^n$$

for $W_i = \mathrm{span}\langle e_1, \ldots, e_{n_1+\cdots+n_i}\rangle$

$\Rightarrow$

ρ stable $\iff$ ρ irreducible

(2) ρ polystable i.e. $GL(n,k)\cdot\rho \subset \mathcal{U}(X,n)$ closed $\Rightarrow$ ρ completely reducible

if not. $\Rightarrow \exists\ 0 \subsetneq W \subsetneq k^n$ s.t. $\rho(\gamma)(W) \subset W \quad \forall \gamma \in \pi_1(X,x)$

but W has no ρ-invariant complement.

Take 1-ps $\lambda : \mathbb{G}_m \to GL(n,k)$ s.t. $\lambda(t)w = tw \quad \forall w \in W$

& $\exists\ W'$ complement W' s.t. $\lambda(t)w' = w' \quad \forall w' \in W'$

define $\rho' := \lim_{t\to 0} \lambda(t).\rho = \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1}$

$\Rightarrow \forall w \in W,\ \gamma \in \pi_1(X,x) \quad \rho'(\gamma)w = \rho(\gamma)w \in W \quad \forall w \in W$

$\forall w' \in W' \quad \gamma \in \pi_1(X,x) \quad \rho'(\gamma)w' = 0$ or $\in W'$

$\Rightarrow$ both W & W' are ρ'-invariant

$\Rightarrow \rho' \not\sim \rho$

$\Rightarrow GL(n,\mathbb{k}).\rho$ not closed!

- ρ completely reducible $\Rightarrow$ ρ polystable, i.e. $GL(n,\mathbb{k}).\rho \subset \mathcal{U}$ closed:

ρ c.r. $\Rightarrow \exists$ flag $0 \subsetneq W_1 \subsetneq \dots \subsetneq W_m = \mathbb{k}^n$ preserved by ρ

$$\rho = \begin{pmatrix} \rho_1 & & & * \\ & \rho_2 & & \\ & & \rho_3 & \\ 0 & & & \ddots \ \rho_m \end{pmatrix}$$

$$\rho' = \lim_{t\to 0} \lambda(t).\rho = \begin{pmatrix} \rho_1 & & & 0 \\ & \rho_2 & & \\ & & \rho_3 & \\ 0 & & & \ddots \ \rho_m \end{pmatrix}$$

For simplicity, assume $m = 1$

$\Rightarrow \exists$ complement $W_1^\perp$ of W_1 in $\mathbb{k}^n$ which is ρ-inv.

$$\mathbb{k}^n = W_1 \oplus W_1^\perp$$

ρ & ρ' act on W_1 & $W_1^\perp$ ~~the~~ in same way.

$\Rightarrow \rho \sim \rho'$

$\square$

Cor 3.4 $\mathcal{U}^{ss}(X,n) \subset \mathcal{U}(X,n)$ semisimple reps. (cr above ss)

$\mathcal{U}^{irr}(X,n) \subset \mathcal{U}(X,n)$ irreducible reps.

$$\Rightarrow \quad \varphi : \mathcal{U}^{irr}(X,n) \longrightarrow \varphi(\mathcal{U}^{irr}(X,n)) =: \mathcal{M}_B^s(X,n)$$

geometric quotient.

In particular, $\mathcal{M}_B^s(X,n)$ is non-singular.

need deformation

Cor 3.5 Bijections of sets:

$$\mathcal{M}_B(X,n)(k) \simeq \mathcal{U}^{ss}(X,n)(k) / GL(n,k) \simeq \mathcal{U}(X,n)(k) / \sim_S$$

Known. each $\overline{GL(n,k)\cdot\rho} \supset \underbrace{GL(n,k)\cdot\rho'}$

- closed $\Rightarrow$ ρ' polystable
- unique (up to iso.)

ρ' is the semisimple representative of ρ, called the semisimplification

Lem 3.6 $\forall\, \rho \in \mathcal{U}(X,n)$ admits a Jordan-Hölder filtration:

$$0 \subsetneq \rho_1 \subsetneq \cdots \subsetneq \rho_n = \rho$$

s.t. each quotient $gr_i = \rho_i / \rho_{i-1}$ is irreducible.

such filtration is unique up to iso. of $gr(\rho) := \bigoplus_{i=1}^{n} gr_i$

- subrep: $\rho' \subset \rho : \pi_1(X,x) \to GL(n,k)$ if $\exists\, 0 \subsetneq W \subsetneq k^n$
  ρ-inv. s.t. $\rho' = \rho|_W : \pi_1(X,x) \to GL(W)$.

- quotient rep.: ρ/ρ' for ρ' subrep
  $: \pi_1(X,x) \to GL(k^n/W)$

$$\rho/_{\rho'}(\gamma)(w+v) := \rho(\gamma)(w) + v.$$

$\gamma \in \pi_1$
$w \in W$
$v \in k^n$.

Lem 3.7 $\rho_1 \sim_s \rho_2 \iff gr(\rho_1) \simeq gr(\rho_2)$

$\rightsquigarrow \quad \rho' \simeq gr(\rho)$

In conclusion, we can write
$$\varphi : \mathcal{U}(X,n) \to \mathcal{M}_B(X,n)$$
$$\rho \longmapsto [gr(\rho)]$$

§4. Twisted version

In this part, $\dim_{\mathbb{R}} X = 1$

Though $\mathcal{M}_B(X,n)$ contains an open smooth subvar. $\mathcal{M}_B^s(X,n)$

Problem: $\mathcal{M}_B(X,n)$ is not smooth $<$ non-closed orbits / infinite stabilizers.

How to solve this?

Taking a primitive n-th root of unity ξ. is $\xi = e^{\frac{2\pi i d}{n}}$
$\gcd(n,d) = 1$.

Defined the space of twisted representations.

$$\mathcal{U}(X,n,d) := \mathrm{Hom}(\pi_1^{tw}(X,x), GL(n,k))$$

$$\uparrow$$

$$\langle \gamma_1, \dots, \gamma_g, \gamma_1', \dots, \gamma_g' : \prod_{i=1}^{g} [\gamma_i, \gamma_i'] = \xi\, \mathrm{Id} \rangle$$

$$\simeq \{ (A_1, \dots, A_g, A_1', \dots, A_g') \in GL(n,k)^{2g} : \prod_{i=1}^{g} [A_i, A_i'] = \xi\, \mathrm{Id} \}.$$

$\subset GL(n,\mathbb{k})^{2g}$ closed subvariety

$\rightsquigarrow \mathcal{U}(X,n,d)$ affine variety.

similarly, $GL(n,\mathbb{k}) \curvearrowright \mathcal{U}(X,n,d)$ by conjugation :

$\rightsquigarrow$ affine GIT quotient

$$\varphi : \mathcal{U}(X,n,d) \to \mathcal{U}(X,n,d) /\!\!/ GL(n,\mathbb{k}) =: \mathcal{M}_B(X,n,d)$$

"twisted Betti moduli space"

$$\mathcal{M}_B(X,n) \simeq \mathcal{M}_{Dol}(X,n) \simeq \mathcal{M}_{dR}(X,n)$$

$$\mathcal{M}_B(X,n,d) \simeq \mathcal{M}_{Dol}(X,n,d) \simeq \mathcal{M}_{dR}(X,n,d)$$

Thm 4.1

(1) $\mathcal{U}(X,n,d)$ and $\mathcal{M}_B(X,n,d)$ are connected.

(2) $\mathcal{M}_B(X,n,d)$ is non-singular

(3) $\dim \mathcal{U}(X,n,d) = n^2(2g-1)+1$

$\dim \mathcal{M}_B(X,n,d) = n^2(2g-2)+2$