TOWARDS SMALL QUANTUM CHERN CHARACTER

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Abstract. We show a quantum version of Chern character homomorphism from the small quantum K-theory to the small quantum cohomology in the cases of projective spaces and incidence varieties, whose classical limit gives the classical Chern character homomorphism. We also provide a ring presentation of the small quantum K-theory of Milnor hypersurfaces.

1. Introduction

Let X be a smooth complex projective variety. Let $K(X) = K^0(X;\mathbb{Q})$ be the Grothendieck ring of topological complex vector bundles on X with rational coefficients. Consider the even part $H^{\mathrm{ev}}(X)$ of the rational cohomology $H^*(X) = H^*(X;\mathbb{Q})$. It is well-known that there is the Chern character ring homormorphism [Gr58, AH61]

$$\mathrm{ch} : K(X) \longrightarrow H^{\mathrm{ev}}(X).$$

There are quantum deformations of both sides. It is natural to ask the following.

Question 1.1. *Is there a quantum Chern character homomorhism in the quantum world?*

More precisely, the deformation $QH_{\mathrm{big}}(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q},\mathbf{s}]\!],*_{\mathbf{s}})$ of $(H^{\mathrm{ev}}(X),\cup)$, called the (even) big quantum cohomology ring, encodes genus-zero Gromov-Witten invariants of X [RT95,BeFa97,LT98]. Its K-analogue is the big quantum K-theory $QK_{\mathrm{big}}(X) = (K(X)\otimes\mathbb{Q}[\![\mathbf{Q},\mathbf{t}]\!],\star_{\mathbf{t}})$, encoding genus-zero K-theoretic Gromov-Witten invariants [Gi00,Le04]. Their restrictions to the fiber at the origin give the small quantum cohomology/K-theory

$$QH(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q}]\!],*), \qquad QK(X) = (K(X)\otimes\mathbb{Q}[\![\mathbf{Q}]\!],\star)$$

respectively, which are relatively more accessible than the big ones. Whenever the small quantum product is finite, we can consider the corresponding polynomial version

$$QH_{\mathrm{poly}}(X) = (H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\mathbf{q}],*), \qquad QK_{\mathrm{poly}}(X) = (K(X)\otimes\mathbb{Q}[\mathbf{Q}],\star)$$

For instance, $QH_{\mathrm{poly}}(X)$ always makes sense for any Fano manifold X, while $QK_{\mathrm{poly}}(X)$ are known to make sense for very few examples. There is a so-called fake quantum K-theory, which may be seen as building block of the big quantum K-theory. There has been the (big) quantum Chern(-Dold) character [Co03, Gi04, CG06], defined for the fake quantum K-theory. On the one hand, it gives a comparison of the genus-0 fake K-theoretic Gromov-Witten invariants with the cohomological ones in terms of symplectic geometry of loop spaces; moreover, by [IMT15, Remark 5.13], the big quantum K-theory is isomorphic to the big quantum cohomology as F-maniolds through the Hirzebruch-Riemann-Roch theorem for fake quantum K-theory [GT14]. On the other hand, in general such isomorphism does not preserve the small loci, and it looks unclear how the (big) quantum Chern character is viewed as a quantum version of the classical Chern character above.

The main aim of this paper is to explore a (small) quantum Chern character ring homomorphism qch for the small quantum K-theory, such that its classical limit gives ch, i.e. the following diagram of ring homomorphisms is commutative, where vertical maps are natural projections.

$$\begin{tikzcd} QK(X) \arrow[rr, "\mathrm{qch}"] \arrow[d, "\text{mod } \mathbf{Q}"'] & & QH^*(X) \arrow[d, "\text{mod } \mathbf{q}"] \\ QK(X)/(\mathbf{Q}) = K(X) \arrow[r, "\mathrm{ch}"] & H^*(X) = QH^*(X)/(\mathbf{q}) & \end{tikzcd} \tag{1.1}$$

As we will see below, we succeed to find the small quantum Chern character morphism qch in the cases of projective spaces $\mathbb{P}^n$ and incidence varieties $\mathbb{F}\ell_{1,n-1;n}$.

Both $\mathbb{P}^n$ and $\mathbb{F}\ell_{1,n-1;n}$ are examples of flag varieties G/P, which are Fano manifolds and are central objects of study in enumerative geometry. There have been extensive studies of $QH_{\text{poly}}(G/P)$ (see e.g. the survey [LL17] and the references therein). There have also been studies of $QK(G/P)$ in individual cases [BM11, BCMP18, BCP23, LNS24, KLNS24, Xu24, LLSY25] from the lens of Schubert calculus, giving the quantum multiplication of certain Schubert classes. Ring presentations have recently been provided for the small quantum K-theory of type A partial flag varieities [KPSZ21, GMSZ22, GM$^+$23, GM$^+$24, HK24b, MNS25a, MNS25b, AHK$^+$25] and of type C complete flag varieties [KN24], as well as for general G/P in terms of K-theoretic version of Peterson's 'quantum = affine' statement [LLMS18, IIM20] proved in the groundbreaking works [Ka18, Ka19] (see also [ChLe20]). Moreover, the small quantum K product is in fact finite for general G/P [ACTI22] (see also [BCMP13, BCMP16] for certain Grassmannians), so that $QK_{\text{poly}}(G/P)$ is meaningful.

Let us start with the simplest case $\mathbb{P}^n$. The line bundle class $L := [\mathcal{O}_{\mathbb{P}^n}(1)]$ generates $K(\mathbb{P}^n)$ with inverse $L^{-1} = [\mathcal{O}_{\mathbb{P}^n}(-1)]$, and the hyperplane class $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ (or simply $c_1(L)$) generates $H^*(\mathbb{P}^n) = H^{\text{ev}}(\mathbb{P}^n)$. We have

$$QK_{\text{poly}}(\mathbb{P}^n) = \mathbb{Q}[L^{-1}, Q]/((1 - L^{-1})^{n+1} - Q), \qquad QH_{\text{poly}}(\mathbb{P}^n) = \mathbb{Q}[h, q]/(h^{n+1} - q),$$

both of which are isomorphic to $\mathbb{Q}[x]$ as $\mathbb{Q}$-algebras. However, it is less obvious whether there is an (iso)morphism compatible with ch in the sense of making diagram (1.1) commutative. To achieve our aim, we need the completion $QH(\mathbb{P}^n) = QH_{\text{poly}}(\mathbb{P}^n) \otimes_{\mathbb{Q}[q]} \mathbb{Q}[\![q]\!]$ and introduce the *quantum Todd class* of the tangent bundle $T_{\mathbb{P}^n}$,

$$\mathrm{Td}_q(T_{\mathbb{P}^n}) := \left(\frac{h}{1 - e^{-h}}\right)^{n+1} \in QH(\mathbb{P}^n), \tag{1.2}$$

in whose series expansion each term h^k is read off as the quantum product h^{*k}. In particular, we have $\mathrm{Td}_q(T_{\mathbb{P}^n}) \equiv \mathrm{Td}(T_{\mathbb{P}^n}) \mod \mathbf{q}$. As the warm-up theorem, we have

Theorem 1.2 (Theorem 3.1). *The map* $\mathrm{qch} : QK(\mathbb{P}^n) \longrightarrow QH(\mathbb{P}^n)$, *given by*

$$L^{-1} \mapsto e^{-h} = \sum_{m=0}^{+\infty} \frac{(-h)^{*m}}{m!}, \qquad Q \mapsto q\ (\mathrm{Td}_q(T_{\mathbb{P}^n}))^{-1},$$

is a well-defined ring homomorphism, and its classical limit gives ch.

Remark 1.3. *As pointed out in [IMT15, Remark 5.13], Givental-Tonita's work [GT14] implies that the big quantum K-theory is isomorphic to the big quantum cohomology as F-maniolds. From the discussion with Hiroshi Iritani, such isomorphism does not restrict to the small locus in the following sense: when $n \geq 2$, there is no continuous ring homomorphism $\mathrm{ch}^q : QK(\mathbb{P}^n) \to QH(\mathbb{P}^n)$ (with respect to the natural Q-adic/q-adic topology) such that $\mathrm{ch}^q(Q) \in \mathbb{Q}[\![q]\!] \setminus \{0\}$ and (1.1) is a commutative diagram of ring homomorphisms. In Theorem 1.2, the image of Q does lie in $QH(\mathbb{P}^n) \setminus \mathbb{Q}[\![q]\!]$.*

The second simplest case may be the incidence varieties $\mathbb{F}\ell_{1,n-1;n} = \{V_1 \leq V_{n-1} \leq \mathbb{C}^n \mid \dim V_1 = 1, \dim V_{n-1} = n-1\}$. Let $\underline{\mathbb{C}^n}$ denote the trivial $\mathbb{C}^n$-bundle over $\mathbb{F}\ell_{1,n-1;n}$, and $\mathcal{S}_1$ (resp. $\mathcal{S}_{n-1}$) denote the tautological vector subbundle whose fiber at a point $V_\bullet$ is just the vector space V_1 (resp. V_{n-1}). Let L_1 (resp. L_2) denote the class of the dual line bundle $\mathcal{S}_1^\vee$ (resp. quotient line bundle $\underline{\mathbb{C}^n}/\mathcal{S}_{n-1}$) in $K(\mathbb{F}\ell_{1,n-1;n})$. We have the inverse $L_1^{-1} = [\mathcal{S}_1]$ and $L_2^{-1} = [(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee]$ in $K(\mathbb{F}\ell_{1,n-1;n})$, and denote $h_i := c_1(L_i) \in H^2(\mathbb{F}\ell_{1,n-1;n})$ for $i = 1, 2$. We use the ring presentation for $QH_{\mathrm{poly}}(\mathbb{F}\ell_{1,n-1;n})$ with generators $\{h_1, h_2\}$ in [CP11, Proposition 7.2] (see Proposition 4.1 below). For $QK(\mathbb{F}\ell_{1,n-1;n})$, the ring presentation could be obtained by simplifying the Whitney presentation in [GM$^+$23, Theorem 1.3] (with $2n-1$ generators) or derived from the quantum Littlewood-Richardson rule in [Xu24]. Here we consider the ring presentation with generators $\{L_1^{-1}, L_2^{-1}\}$ read off from Theorem 1.5 as a special case of Milnor hypersurfaces. They are of the following form

$$QK(\mathbb{F}\ell_{1,n-1;n}) = \frac{\mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1, Q_2]\!]}{(F_1^Q, F_2^Q)}, \qquad QH(\mathbb{F}\ell_{1,n-1;n}) = \frac{\mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]}{(f_1^q, f_2^q)}$$

with $F_i^Q = F_i^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2)$ and $f_i^q = f_i^q(h_1, h_2, q_1, q_2)$ being polynomials. Similar to $\mathbb{P}^n$, we introduce the quantum Todd class

$$(\mathrm{Td}_q(L_i^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2) := (\frac{1 - e^{-h_i}}{h_i})^n / (\frac{1 - e^{-h_1 - h_2}}{h_1 + h_2}). \tag{1.3}$$

In particular, $(\mathrm{Td}_q(L_1^{\oplus n})) * \mathrm{Td}_q(L_1 \otimes L_2)^{-1} \mod \mathbf{q}$ is the classical Todd class of the difference between the pullback of $T_{\mathbb{F}\ell_{1;n}}$ and the line bundle $L_1 \otimes L_2$ (see Section 4 for more details). We have the following quantum Chern character homomorphism for $\mathbb{F}\ell_{1,n-1;n}$.

Theorem 1.4 (Theorem 4.4). *The map* $\mathrm{qch} : QK(\mathbb{F}\ell_{1,n-1;n}) \longrightarrow QH(\mathbb{F}\ell_{1,n-1;n})$*, given by*

$$L_i^{-1} \mapsto e^{-c_1(L_i)} := \sum_{m=0}^{+\infty} \frac{(-h_i)^{*m}}{m!},$$
$$Q_i \mapsto q_i(\mathrm{Td}_q(L_i^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2), \qquad i = 1, 2.$$

is a well-defined ring homomorphism, and its classical limit gives ch.

We notice that the inverse of (classical) Todd class appeared in the Grothendieck-Riemann–Roch formula (see e.g. [EH16, Theorem 14.5]). The inverse of Todd function also appeared in the study of BCOV theory [CoLi20, Formula (38)].

The degree-$(1,1)$ hypersurfaces in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, known as Milnor hypersurfaces, are an important class of algebraic varieties in topology and algebraic geometry. Amongest them the smooth one is usually denoted as $H_{n-1,m-1}$. In particular, we have $H_{n-1,n-1} =$

$\mathbb{F}\ell_{1,n-1;n}$. We can view $H_{n-1,m-1}$ as a smooth Schubert variety of $\mathbb{F}\ell_{1,n-1;n}$ via a natural embedding ι where $n \geq m \geq 2$ (see Section 5.1 for more details). Then we have the line bundle classes $[\iota^*(\mathcal{S}_1^\vee)]$ and $[\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})]$ in $K(H_{n-1,m-1})$, denoted again as L_1 and L_2 respectively by abuse of notation. By Proposition 5.4, we have $K(H_{n-1,m-1}) = \mathbb{Q}[L_1^{-1}, L_2^{-1}]/(F_1(L_1^{-1}, L_2^{-1}), F_2(L_1^{-1}, L_2^{-1}))$, where

$$F_1(x,y) := (1-y)^m, \qquad F_2(x,y) := (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{m-1}(-1)^{n-1-t}x^{t-1}(1-x)^{n-t-1}(1-y)^t.$$

As we will show in **Theorem 5.7**, we have the following ring presentation.

Theorem 1.5. *Let $n \geq m \geq 3$. The small quantum K-theory of $H_{n-1,m-1}$ is presented by*

$$QK(H_{n-1,m-1}) \cong \mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1, Q_2]\!]/(F_1^Q, F_2^Q),$$

where $F_1^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = F_1(L_1^{-1}, L_2^{-1}) - Q_2 + Q_2 L_1^{-1} L_2^{-1}$ and

$$F_2^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = F_2(L_1^{-1}, L_2^{-1}) - (-1)^{n-m}Q_2(L_1^{-1})^{m-1}(1-L_1^{-1})^{n-m} - (-1)^{n-1}Q_1 L_2^{-1}.$$

On the one hand, there is an approach, known to experts, to derive a ring presentation of the quantum K-theory of a Fano manifold X. Namely, firstly one may compute the K-theoretic J-function J_X of X; secondly, one may find the symbols of finite difference operators annihilating J_X, which would give relations in $QK(X)$ by [IMT15, Proposition 2.10] and [HK24a, Theorem 4.12]; thirdly, one may show those relations are sufficient by a Nakayama-type result in [GM$^+$23, Appendix A]. We notice that when $K(X)$ is generated by line bundle classes, there has also been a heuristic argument on obtaining the ring presentation of $QK(X)$ by using finite difference equations in [Lee99]. On the other hand, there are very few examples that could really be carried out in this way, including the type A complete flag variety [AHK$^+$25, Appendix A]. Our Theorem 1.5 provides one new class of examples with this approach. We directly write down the J-function of $H_{n-1,m-1}$ by the quantum Lefschetz principle [Gi15, Gi20]. We notice that the K-theoretic J-function has been calculated for type A flag varieties [GL03, Ta13] (see [BrFi11] for the quasi-map version for complete flag varieties).

Remark 1.6. *Theorem 1.4 can be generalized to Milnor hypersurfaces $H_{n-1,m-1}$ as follows. We first derive the following ring presentation*

$$QH(H_{n-1,m-1}) \cong \mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]/(h_2^m - q_2(h_1+h_2), \sum_{k=0}^{m-1}(-1)^k h_1^{n-1-k}h_2^k - q_1 - (-1)^{m-1}q_2 h_1^{n-m})$$

by investigating the quantum differential equations for Givental's (cohomological) J-function of $H_{n-1,m-1}$. Then we can define the map qch : $QK(H_{n-1,m-1}) \to QH(H_{n-1,m-1})$ *by sending L_i^{-1} to e^{-h_i} and sending Q_1 (resp. Q_2) to* $q_1(\mathrm{Td}_q(L_1^{\oplus n}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2)$ *(resp.* $q_2(\mathrm{Td}_q(L_2^{\oplus m}))^{-1} * \mathrm{Td}_q(L_1 \otimes L_2)$*). The strategy to show that such* qch *gives a ring homomorphism* qch *is similar to the case for $\mathbb{F}\ell_{1,n-1;n} = H_{n-1,n-1}$. The full details will be slightly more involved, and we plan to address the cohomological part in the study of D-module mirror symmetry for $H_{n-1,m-1}$ elsewhere.*

The paper is organized as follows. In Section 2, we review the background of quantum cohomology and quantum K-theory. In Sections 3 and 4, we show the small quantum Chern character homomorphism for the cases $\mathbb{P}^n$ and $\mathbb{F}\ell_{1,n-1;n}$ respectively. In Section 5, we derive a ring presentation of the small quantum K-theory of Milnor hypersurface $H_{n-1,m-1}$. Finally, we prove Lemma 4.5 by studying mirror symmetry for $\mathbb{F}\ell_{1,n-1;n}$.

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2. Preliminaries

In the section, we review some basic facts on quantum cohomology and quantum K-theory, mainly following [FP97,Gi00,Le04,IMT15]. For simplicity, we assume the topological K-theory of X coincides with the algebraic K-theory of X, which is sufficient in the rest of this paper. We refer to [Le04] for more details on the set-up of the quantum K-theory in general cases.

2.1. **Cohomological/K-theoretic Gromov-Witten invariants.** Let $\overline{\mathcal{M}}_{0,k}(X,\beta)$ denote the moduli (Degline-Mumford) stack of genus 0, k-pointed stable maps to X of degree $\beta \in H_2(X;\mathbb{Z})$. Let $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,k}(X,\beta) \longrightarrow X$ be the i-th evaluation map, and $\mathcal{L}_i$ denote the universal cotangent line bundle on $\overline{\mathcal{M}}_{0,k}(X,\beta)$, where $1 \le i \le k$. Given nonnegative integers d_i and classes $\xi_i \in H^{\mathrm{ev}}(X)/\gamma_i \in K(X)$, the cohomological/$K$-theoretic Gromov-Witten invariants are respectively defined by

$$\langle \tau_{d_1}\xi_1, \tau_{d_2}\xi_2, \cdots, \tau_{d_k}\xi_k \rangle^H_{0,k,\beta} = \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{vir}} \prod_{i=1}^{k} (c_1(\mathcal{L}_i)^{d_i} \cup \mathrm{ev}_i^*(\xi_i)),$$

$$\langle \tau_{d_1}\gamma_1, \tau_{d_2}\gamma_2, \cdots, \tau_{d_k}\gamma_k \rangle_{0,k,\beta} = \chi\Big(\overline{\mathcal{M}}_{0,k}(X,\beta), \mathcal{O}^{vir} \otimes (\otimes_{i=1}^{k} \mathcal{L}_i^{\otimes d_i} \mathrm{ev}_i^*(\gamma_i))\Big).$$

Here $[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{vir}$ denotes the virtual fundamental class [RT95, BeFa97, LT98, FO99], which is of expected dimension $\dim X + k - 3 + \int_\beta c_1(X)$; $\mathcal{O}^{vir}$ denotes the virtual structure sheaf [Le04, BeFa97], and χ is the pushforward to a point $\mathrm{Spec}(\mathbb{C})$.

Let $\overline{\mathrm{NE}}(X)$ denote the Mori cone of effective curve classes. It follows from the definition that cohomological/K-theoretic Gromov-Witten invariants of degree β are vanishing unless $\beta \in \overline{\mathrm{NE}}(X)$. We may take nef line bundles L_i's, such that $\{h_i := c_1(L_i)\}_{i=1}^r$ form a nef integral basis of $H^2(X;\mathbb{Z})$/torison. We then introduce the Novikov variables $\{q_i\}_{i=1}^r$ (resp. $\{Q_i\}_{i=1}^r$) for quantum cohomology (resp. K-theory). For $\beta \in \overline{\mathrm{NE}}(X)$, we have $d_i := d_i(\beta) = \int_\beta h_i \ge 0$, and denote

$$q^\beta := \prod_{i=1}^{r} q_i^{d_i}, \qquad Q^\beta := \prod_{i=1}^{r} Q_i^{d_i}.$$

2.2. **Quantum cohomology.** Let $\{\xi_i\}_{i=1}^N$ be a basis of $H^{\mathrm{ev}}(X)$, and $\{\xi^j\}_j$ be the dual basis of $H^{\mathrm{ev}}(X)$, where $\int_{[X]}\xi_b\cup\xi^a=\delta_{a,b}$. Write $\mathbf{s}=\sum_{i=1}^N s_i\xi_i\in H^{\mathrm{ev}}(X)$. The (even) big quantum cohomology $QH_{\mathrm{big}}(X)=(H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q},\mathbf{s}]\!],*_{\mathbf{s}})$ is a $\mathbb{Q}[\![\mathbf{s},\mathbf{q}]\!]$-algebra with the big quantum product defined by

$$\xi_i *_{\mathbf{s}} \xi_j := \sum_{k=0}^{\infty}\sum_{\beta\in\overline{\mathrm{NE}}(X)}\sum_{a=1}^{N}\frac{q^\beta}{k!}\langle\xi_i,\xi_j,\xi^a,\mathbf{s},\cdots,\mathbf{s}\rangle^H_{0,k+3,\beta}\xi_a. \tag{2.1}$$

The restriction of $QH_{\mathrm{big}}(X)$ to $\mathbf{s}=0$, i.e. $\xi_i*\xi_j=\sum_{\beta,a}\langle\xi_i,\xi_j,\xi^a\rangle^H_{0,3,\beta}\xi_a q^\beta$, gives the small quantum cohomology $QH(X)=(H^{\mathrm{ev}}(X)\otimes\mathbb{Q}[\![\mathbf{q}]\!],*)$.

For $\alpha\in H^{\mathrm{ev}}(X)$ and $f(x)=x^m$ where $m\geq 0$, we take the notation conventions

$$f_{\mathrm{cl}}(\alpha)=\alpha^{\cup m}:=\underbrace{\alpha\cup\cdots\cup\alpha}_{m\text{ copies}},\qquad f(\alpha)=\alpha^{*m}:=\underbrace{\alpha*\cdots*\alpha}_{m\text{ copies}}.$$

We can take similar conventions for formal power series in multi-variables by observing the following. For any $f(x_1,\dots,x_k)\in\mathbb{Q}[\![x_1,\dots,x_k]\!]$ and any homogeneous $\alpha_i\in H^{\mathrm{ev}}(X)\setminus H^0(X)$, $1\leq i\leq k$, we have

(1) $f_{\mathrm{cl}}(\alpha_1,..,\alpha_k)\in H^{\mathrm{ev}}(X)$, by the nilpotency of α_i's in $H^{\mathrm{ev}}(X)$;
(2) $f(\alpha_1,..,\alpha_k)\in QH(X)$ for the reason: for each q^β, by the dimension axiom, there are only finitely many k-tuples $(m_1,...,m_k)$ such that q^β appears in $\alpha_1^{*m_1}*\cdots*\alpha_k^{*m_k}$.

Moreover, under the natural surjective ring homomorphism $QH(X)\xrightarrow{\bmod q}H^{\mathrm{ev}}(X)$, we have $f(\alpha_1,...,\alpha_k)\mapsto f_{\mathrm{cl}}(\alpha_1,...,\alpha_k)$.

More generally, for $\alpha\in H^2(X)$ and function $f(x)$ holomorphic at $x=0$, we can define $f_{\mathrm{cl}}(\alpha)\in H^{\mathrm{ev}}(X)$ and $f(\alpha)\in QH(X)$ by the Taylor expansion of $f(x)$ at $x=0$. In this article, we consider the cases $f(x)=e^x$, $\frac{1-e^{-x}}{x}$ or $\frac{x}{1-e^{-x}}$, giving $f(\alpha)\in QH(X)$.

2.3. **Quantum K-theory.** The definition of quantum K-theory is more involved. Let $\mathbf{t}=\sum_{i=1}^N t_i\gamma_i\in K(X)$. We first introduce the quantum K-potential, which is a generating series of genus-zero K-theoretic Gromov-Witten invariants given by

$$\mathcal{G}(\mathbf{t},\mathbf{Q}):=\frac{1}{2}\sum_{i,j}t_it_jg_{ij}+\sum_{k=0}^{\infty}\sum_{\beta\in\overline{\mathrm{NE}}(X)}\frac{Q^\beta}{k!}\langle\mathbf{t},\cdots,\mathbf{t}\rangle_{0,k,\beta}$$

with $g_{ij}=g(\gamma_i,\gamma_j):=\chi(X,\gamma_i\otimes\gamma_j)$ being the classical metric on $K(X)$ given by the Euler characteristic map.

Denote $\partial_i:=\frac{\partial}{\partial t_i}$. The quantum K-metric is defined by

$$G_{ij}=G(\gamma_i,\gamma_j):=\partial_i\partial_j\mathcal{G}.$$

We have $G_{ij}\equiv g_{ij}\mod\mathbf{Q}$, and $\big(G_{ij}\big)\in GL_N(\mathbb{Q}[\![\mathbf{Q},\mathbf{t}]\!])$.

Definition 2.1. *The big quantum K-theory $QK_{big}(X):=(K(X)\otimes[\![\mathbf{Q},\mathbf{t}]\!],\star_{\mathbf{t}})$ is a $[\![\mathbf{Q},\mathbf{t}]\!]$-algebra with the big quantum K-product $\gamma_i\star_{\mathbf{t}}\gamma_j$ defined by using the quantum K-metric,*

$$G(\gamma_i\star_{\mathbf{t}}\gamma_j,\gamma_k):=\partial_i\partial_j\partial_k\mathcal{G}.$$

The big quantum K-theory $QK_{\mathrm{big}}(X)$ is a commutative and associative algebra with identity $\mathbf{1}$ (lying in $K(X)$). Its restriction to $\mathbf{t}=0$ gives the small quantum K-theory $QK(X)=(K(X)\otimes\mathbb{Q}[\![\mathbf{Q}]\!],\star)$.

2.4. **Quantum connection and K-theoretic J-function.** Let $\hbar$ be a formal variable. The quantum connection is given by the operators

$$\nabla_i^{\hbar} := (1-\hbar)\frac{\partial}{\partial t_i} - \gamma_i\star_{\mathbf{t}},\ 1\leq i\leq N,$$

acting on $K(X)\otimes\mathbb{Q}[\hbar,\hbar^{-1}][\![\mathbf{Q},\mathbf{t}]\!]$. It could also be seen as a connection on the tangent bundle $T_{K(X)}$ by identifying γ_i with $\frac{\partial}{\partial t_i}$. This quantum connection is flat, with the fundamental solution $\{S_{ab}\}_{1\leq a,b\leq N}$ to the K-theoretic quantum differential equation $\nabla_i^{\hbar}(\sum_{b=1}^{N} S_{ab}\gamma_b)=0$ given by

$$S_{ab}(\mathbf{t},\mathbf{Q}) = g_{ab} + \sum_{l=0}^{\infty}\sum_{d\in\overline{\mathrm{NE}}(\mathrm{X})}\frac{Q^d}{l!}\langle\gamma_a,\mathbf{t},\cdots,\mathbf{t},\frac{\gamma_b}{1-\hbar\mathcal{L}}\rangle_{0,l+2,d}.$$

Here $\frac{\gamma_b}{1-\hbar\mathcal{L}}=\sum_{n=0}^{\infty}\gamma_b\hbar^n\mathcal{L}^{\otimes n}$, where $\mathcal{L}$ is the universal cotangent line bundle at the $(l+2)$th-marked point on $\overline{\mathcal{M}}_{0,l+2}(X,d)$. Following [IMT15], we introduce an endomorphism-valued function $T\in End(K(X))\otimes\mathbb{Q}(\hbar)[\![Q,t]\!]$ by the formula $g(T\gamma_a,\gamma_b)=S_{ab}$.

Definition 2.2. *The big K-theoretic J-function* [1] *of X is defined by*

$$J_X(\hbar,\mathbf{Q},\mathbf{t}) := T(\mathbf{1}) = \mathbf{1} + \sum_{l=0}^{\infty}\sum_{d\in\overline{\mathrm{NE}}(\mathrm{X})}\sum_{i,j}\frac{Q^d}{l!}\langle\mathbf{1},\mathbf{t},\cdots,\mathbf{t},\frac{\gamma_i}{1-\hbar\mathcal{L}}\rangle_{0,l+2,d}\ g^{ij}\gamma_j.$$

The restriction $J_X(\hbar,\mathbf{Q})=J_X(\hbar,\mathbf{Q},\mathbf{t})|_{\mathbf{t}=0}$ is called the small K-theoretic J-function.

Givental and Tonita [GT14] showed that the space $T(K(X)\otimes\mathbb{Q}[\hbar,\hbar^{-1}][\![\mathbf{Q},\mathbf{t}]\!])$ is preserved by the operator $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}$ for all $1\leq i\leq r$, where L_i^{-1} acts by tensor product, and $\hbar^{Q_i\partial_{Q_i}}$ is the $\hbar$-shift operator acting on functions in $Q_1,Q_2,\cdots,Q_r$ by

$$f(Q_1,Q_2,\cdots,Q_r)\mapsto f(Q_1,\cdots,Q_{i-1},\hbar Q_i,Q_{i+1},\cdots,Q_r).$$

Set $A_i := \left(T^{-1}\circ L_i^{-1}\hbar^{Q_i\partial_{Q_i}}\circ T\right)|_{\mathrm{t}=0,\hbar=1}$, which is an element in $\mathrm{End}(K(X))\otimes\mathbb{Q}[\![\mathbf{Q}]\!]$. Suppose that a $\hbar$-difference operator $D\in\mathbb{Q}[\hbar,\mathbf{Q}]\left\langle L_i^{-1}\hbar^{Q_i\partial_{Q_i}}\right\rangle$ satisfies

$$D(L_1^{-1}\hbar^{Q_1\partial_{Q_1}},\cdots,L_r^{-1}\hbar^{Q_r\partial_{Q_r}},\hbar,Q_1,\cdots,Q_r)J_X(\hbar,\mathbf{Q})=0. \tag{2.2}$$

Then $D(A_1,\cdots,A_r,1,Q_1,\cdots,Q_r)(\mathbf{1})=0$ by the definition of A_i's. To simplify A_i, we note that if the coefficient of Q^d in $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_X(\hbar,\mathbf{Q}))$ vanish when $\hbar=\infty$ for all $d>0$, then $A_i=(L_i^{-1}\star)$ by [IMT15, Proposition 2.10] and [HK24a, Theorem 4.12]. In summary, we have

Proposition 2.3. *Suppose that: (i) $D\in\mathbb{Q}[\hbar,\mathbf{Q}]\left\langle L_i^{-1}\hbar^{Q_i\partial_{Q_i}}\right\rangle$ satisfies Equation (2.2); (ii) for $1\leq i\leq r$, $\mathrm{Coeff}_{Q^d}\left(L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_X(\hbar,\mathbf{Q}))\right)|_{\hbar=\infty}=0$ for all $d\in\overline{\mathrm{NE}}(X)\setminus\{0\}$. Then*

$$D(L_1^{-1},\cdots,L_r^{-1},1,Q_1,\cdots,Q_r)(\mathbf{1})=0\quad in\ QK(X). \tag{2.3}$$

[1] The K-theoretic J-function in [IMT15, Definition 2.4] is equal to our $(1-\hbar)J$ with $\hbar=q$.

3. Quantum Chern character for $\mathbb{P}^n$

In this section, as a warm-up, we construct a quantum version of Chern character for $\mathbb{P}^n$, i.e. a ring homomorphism $QK(\mathbb{P}^n) \xrightarrow{\mathrm{qch}} QH(\mathbb{P}^n)$ that is a lift of $K(\mathbb{P}^n) \xrightarrow{\mathrm{ch}} H^*(\mathbb{P}^n) = H^{ev}(\mathbb{P}^n)$.

Let us consider line bundle class $L := [\mathcal{O}(1)] \in K(\mathbb{P}^n)$ and hyperplane class $h := c_1(L) \in H^2(\mathbb{P}^n)$. We have the following ring presentations (see e.g. [BM11]):

$$QK(\mathbb{P}^n) \cong \mathbb{Q}[L^{-1}][\![Q]\!]/((1-L^{-1})^{n+1} - Q), \qquad QH(\mathbb{P}^n) \cong \mathbb{Q}[h][\![q]\!]/(h^{n+1} - q),$$

where $L^{-1} = [\mathcal{O}(-1)]$ is the classical inverse of L in $K(\mathbb{P}^n)$.

We define a map $\mathrm{qch} : QK(\mathbb{P}^n) \longrightarrow QH(\mathbb{P}^n)$ as follows. We firstly set

$$\mathrm{qch}(L^{-1}) = e^{-h} \text{ and } \mathrm{qch}(Q) = q\left(\frac{1-e^{-h}}{h}\right)^{n+1},$$

which both lie in $QH(\mathbb{P}^n)$ by our convention in Section 2.2. Secondly, for any $\alpha \in QK(\mathbb{P}^n)$, the above ring presentation implies that there exists $\Phi_\alpha(x, y) \in \mathbb{Q}[x][\![y]\!]$ such that $\alpha = \Phi_\alpha(L^{-1}, Q)$. We then set

$$\mathrm{qch}(\alpha) = \Phi_\alpha\left(\mathrm{qch}(L^{-1}), \mathrm{qch}(Q)\right).$$

Theorem 3.1. *The map* qch *is independent of the choice of* Φ_α*'s and is a ring homomorphism. Moreover, its classical limit gives* ch*.*

Proof. To prove the first statement, by the ring presentation for $QK(\mathbb{P}^n)$, it suffices to show that $\left(1 - \mathrm{qch}(L^{-1})\right)^{n+1} = \mathrm{qch}(Q)$, or equivalently, $\left(1-e^{-h}\right)^{n+1} = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}$.

The equality $1 - e^{-x} = x \cdot \frac{1-e^{-x}}{x}$ in $\mathbb{Q}[\![x]\!]$ implies that $1 - e^{-h} = h \star \frac{1-e^{-h}}{h}$. Thus we have

$$\left(1-e^{-h}\right)^{n+1} = h^{n+1} * \left(\frac{1-e^{-h}}{h}\right)^{n+1} = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}.$$

In the last equality, we use the relation $h^{n+1} = q$ in $QH(\mathbb{P}^n)$. This proves the first statement.

The second statement follows directly from the definition of qch. This finishes the proof of the proposition. □

Remark 3.2. *The image* $\mathrm{qch}(Q)$ *is actually uniquely determined by* $\mathrm{qch}(L^{-1}) = e^{-h}$ *and the requirement that* qch *is a ring homomorphism. Indeed, that* qch *is a ring homomorphism implies that* $\left(1 - \mathrm{qch}(L^{-1})\right)^{n+1} = \mathrm{qch}(Q)$*. Together with the relation* $h^{n+1} = q$ *in* $QH(\mathbb{P}^n)$*, we obtain*

$$\mathrm{qch}(Q) * h^{n+1} = q\left(1-e^{-h}\right)^{n+1}.$$

This equation for $\mathrm{qch}(Q)$ *has a solution* $\mathrm{qch}(Q) = q\left(\frac{1-e^{-h}}{h}\right)^{n+1}$*. Moreover, noting* $QH(\mathbb{P}^n)$ *is a* $\mathbb{Q}[\![q]\!]$*-algebra, the relation* $h^{n+1} = q$ *implies that* h *is NOT a zero divisor in* $QH(\mathbb{P}^n)$*. As a consequence, the equation for* $\mathrm{qch}(Q)$ *has a unique solution.*

Remark 3.3. *The factor* $\left(\frac{1-e^{-h}}{h}\right)^{n+1}$ *in* $\mathrm{qch}(Q)$ *can be interpreted as a quantum version of the inverse of Todd class of* $T_{\mathbb{P}^n}$*. On the one hand, the factor is equal to* $f(h)$ *with*

$f(x) = \left(\frac{1-e^{-x}}{x}\right)^{n+1}$. *On the other hand, the Euler exact sequence*

$$0 \to \mathcal{O} \to L^{\oplus(n+1)} \to T\mathbb{P}^n \to 0$$

gives

$$\mathrm{Td}(T_{\mathbb{P}^n})^{-1} = \left(\mathrm{Td}(L)^{n+1}\right)^{-1} = f_{\mathrm{cl}}(h).$$

So $\left(\frac{1-e^{-h}}{h}\right)^{n+1}$ *is mapped to* $\mathrm{Td}(T_{\mathbb{P}^n})^{-1}$ *via* $QH(\mathbb{P}^n) \xrightarrow{mod\ q} H^*(\mathbb{P}^n)$.

4. Quantum Chern character for $\mathbb{F}\ell_{1,n-1;n}$

In this section, we construct a quantum version of Chern character for the incidence variety $\mathbb{F}\ell_{1,n-1;n} = \{V_1 \le V_{n-1} \le \mathbb{C}^n \mid \dim V_1 = 1, \dim V_{n-1} = n-1\}$, i.e. a ring homomorphism $QK(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{qch}} QH(\mathbb{F}\ell_{1,n-1;n})$ that gives a lift of $K(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{ch}} H^*(\mathbb{F}\ell_{1,n-1;n}) = H^{\mathrm{ev}}(\mathbb{F}\ell_{1,n-1;n})$.

Let $L_1 := [(\mathcal{S}_1)^\vee], L_2 := [\underline{\mathbb{C}^n}/\mathcal{S}_{n-1}] \in K(\mathbb{F}\ell_{1,n-1;n})$, where $\mathcal{S}_1$ and $\mathcal{S}_{n-1}$ are the tautological vector subbundles of $\mathbb{F}\ell_{1,n-1;n}$. Note that $\{h_a := c_1(L_a)\}_{1\le a\le 2}$ form a nef basis of $H^2(\mathbb{F}\ell_{1,n-1;n}, \mathbb{Z})$. Recall that q_a (resp. Q_a), $1 \le a \le 2$, denote the corresponding Novikov variables in the quantum cohomology (resp. K-theory) of $\mathbb{F}\ell_{1,n-1;n}$. We have

Proposition 4.1 ([CP11, Proposition 7.2]). *The small quantum cohomology ring of* $\mathbb{F}\ell_{1,n-1;n}$ *is canonically given by* $QH(\mathbb{F}\ell_{1,n-1;n}) \cong \mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]/(f_1^q, f_2^q)$, *where*

$$f_1^q(h_1, h_2, q_1, q_2) = h_2^n - q_2(h_1 + h_2),$$
$$f_2^q(h_1, h_2, q_1, q_2) = \sum_{l=0}^{n-1} (-1)^{n-1-l} h_1^l h_2^{n-1-l} - q_1 - (-1)^{n-1} q_2.$$

The isomorphism in the above proposition is canonical in the sense that the hyperplane class h_a in $QH(\mathbb{F}\ell_{1,n-1;n})$ is maps to the generator h_a on the right, for $a = 1, 2$. Moreover, there is a canonical selfduality of $\mathbb{F}\ell_{1,n-1;n}$, which induces an automorphism of the small quantum cohomology by interchanging (h_1, q_1) and (h_2, q_2). As an immediate consequence, we have

Corollary 4.2. *We have* $h_a^n = q_a(h_1 + h_2)$ *in* $QH(\mathbb{F}\ell_{1,n-1;n})$ *for* $a = 1, 2$.

Ring presentations for $QK(\mathbb{F}\ell_{1,n-1;n})$ could be obtained from [GM+23, Xu24]. For our purpose of constructing qch, we treat $\mathbb{F}\ell_{1,n-1;n}$ as the special Milnor hypersurface $H_{n-1,n-1}$, and read off one from the general case $QK(H_{n-1,m-1})$ in Theorem 5.7 directly; namely

Proposition 4.3. *The small quantum K-theory of* $\mathbb{F}\ell_{1,n-1;n}$ *is canonically given by*

$$QK(\mathbb{F}\ell_{1,n-1;n}) \cong \mathbb{Q}[L_1^{-1}, L_2^{-1}][\![Q_1, Q_2]\!]/(F_1^Q, F_2^Q),$$

$$\text{where}\quad F_1^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = (1 - L_2^{-1})^n - Q_2 + Q_2(L_1^{-1}L_2^{-1}),$$
$$F_2^Q(L_1^{-1}, L_2^{-1}, Q_1, Q_2) = F_2(L_1^{-1}, L_2^{-1}) - Q_2(L_1^{-1})^{n-1} - (-1)^{n-1}Q_1 L_2^{-1},$$

with $F_2(x, y) = (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{n-1} (-1)^{n-1-t} x^{t-1}(1-x)^{n-t-1}(1-y)^t$.

We define a map $QK(\mathbb{F}\ell_{1,n-1;n}) \xrightarrow{\mathrm{qch}} QH(\mathbb{F}\ell_{1,n-1;n})$ as follows. Firstly, for $a=1,2$, we set

$$\mathrm{qch}(L_a^{-1}) = e^{-h_a} \quad \text{and} \quad \mathrm{qch}(Q_a) = q_a \left(\frac{1-e^{-h_a}}{h_a}\right)^n * \left(\frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}}\right).$$

Secondly, for any $\alpha \in QK(\mathbb{F}\ell_{1,n-1;n})$, the above ring presentation implies that there exists $\Phi_\alpha(x_1,x_2,y_1,y_2) \in \mathbb{Q}[x_1,x_2][\![y_1,y_2]\!]$ such that $\alpha = \Phi_\alpha\left(L_1^{-1}, L_2^{-1}, Q_1, Q_2\right)$, and we set

$$\mathrm{qch}(\alpha) = \Phi_\alpha\left(\mathrm{qch}(L_1^{-1}), \mathrm{qch}(L_2^{-1}), \mathrm{qch}(Q_1), \mathrm{qch}(Q_2)\right).$$

Theorem 4.4. *The map* qch *is independent of the choice of Φ_α's and is a ring homomorphism. Moreover, its classical limit gives* ch.

To prove the theorem, we prepare several lemmas first. Recall that in a commutative ring R, an element $x \in R$ is not a zero-divisor if and only if for any $y \in R$, we have $y=0$ whenever $xy=0$. We first assume the next lemma, and leave its proof at the end of the appendix, which will involve mirror symmetry for $\mathbb{F}\ell_{1,n-1;n}$.

Lemma 4.5. *The element h_1+h_2 is not a zero-divisor in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Lemma 4.6. *The element $1-e^{-(h_1+h_2)}$ is not a zero-divisor in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Proof. The equality $(1-e^{-x})\cdot\frac{x}{1-e^{-x}} = x$ in $\mathbb{Q}[\![x]\!]$ implies that $\left(1-e^{-(h_1+h_2)}\right) * \frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}} = h_1+h_2$. So it follows from Lemma 4.5 that $1-e^{-(h_1+h_2)}$ is not a zero-divisor. □

Lemma 4.7. *For $a \in \{1,2\}$, we have $(1-e^{-(h_1+h_2)})*\mathrm{qch}(Q_a) = (1-e^{-h_a})^n$ in $QH(\mathbb{F}\ell_{1,n-1;n})$.*

Proof. By the equalities $x \cdot \frac{1-e^{-x}}{x} = 1-e^{-x}$ and $(1-e^{-x})\cdot\frac{x}{1-e^{-x}} = x$ in $\mathbb{Q}[\![x]\!]$, we have

$$h_a * \frac{1-e^{-h_a}}{h_a} = 1-e^{-h_a} \text{ and } (1-e^{-(h_1+h_2)}) * \frac{h_1+h_2}{1-e^{-(h_1+h_2)}} = h_1+h_2.$$

So we see that

$$\begin{aligned}
(1-e^{-(h_1+h_2)}) * \mathrm{qch}(Q_a) &= q_a \left(\frac{1-e^{-h_a}}{h_a}\right)^n * \left((1-e^{-(h_1+h_2)}) * \frac{h_1+h_2}{1-e^{-(h_1+h_2)}}\right) \\
&= q_a \left(\frac{1-e^{-h_a}}{h_a}\right)^n * (h_1+h_2) \\
&= h_a^n * \left(\frac{1-e^{-h_a}}{h_a}\right)^n \qquad (\text{from } Corollary\ 4.2) \\
&= (1-e^{-h_a})^n.
\end{aligned}$$

□

Proof of Theorem 4.4. To prove the first statement, by the ring presentation in Proposition 4.1, it suffices to show that the two elements

$$F_a^Q\left(\mathrm{qch}(L_1^{-1}), \mathrm{qch}(L_2^{-1}), \mathrm{qch}(Q_1), \mathrm{qch}(Q_2)\right) \in QH(\mathbb{F}\ell_{1,n-1;n}), \quad a=1,2,$$

are vanishing.

For F_1^Q, by Lemma 4.6, we only need to prove that

$$(1-e^{-(h_1+h_2)})*\Big((1-e^{-h_2})^n-\mathrm{qch}(Q_2)+\mathrm{qch}(Q_2)*e^{-h_1}*e^{-h_2}\Big)=0.$$

It follows from Lemma 4.7 that

$$\begin{aligned}&(1-e^{-(h_1+h_2)})*\Big((1-e^{-h_2})^n-\mathrm{qch}(Q_2)+\mathrm{qch}(Q_2)*e^{-h_1}*e^{-h_2}\Big)\\ =&\,(1-e^{-h_2})^n*\Big((1-e^{-(h_1+h_2)})-1+e^{-h_1}*e^{-h_2}\Big).\end{aligned}$$

So the required vanishing result follows from $e^{-(h_1+h_2)}=e^{-h_1}*e^{-h_2}$.

For F_2^Q, by Lemma 4.6, we only need to prove that LHS = RHS, where

$$\begin{aligned}\mathrm{LHS}&:=(1-e^{-(h_1+h_2)})*F_2(e^{-h_1},e^{-h_2}),\\ \mathrm{RHS}&:=(1-e^{-(h_1+h_2)})*\Big(\mathrm{qch}(Q_2)*(e^{-h_1})^{*(n-1)}+(-1)^{n-1}\mathrm{qch}(Q_1)*e^{-h_2}\Big).\end{aligned}$$

We use Lemma 4.7 to find that

$$\mathrm{RHS}=(1-e^{-h_2})^n*(e^{-h_1})^{n-1}+(-1)^{n-1}(1-e^{-h_1})^n*e^{-h_2}.$$

For LHS, observe that $F_2(e^{-h_1},e^{-h_2})$ is equal to

$$(-1)^{n-1}(1-e^{-h_1})^{n-1}+(1-e^{-h_2})*\sum_{l=0}^{n-2}(e^{-h_1}-1)^{n-2-l}*\Big(e^{-h_1}*(1-e^{-h_2})\Big)^l.$$

Note that $(e^{-h_1}-1)-\big(e^{-h_1}*(1-e^{-h_2})\big)=-(1-e^{-(h_1+h_2)})$, and we find that

$$\begin{aligned}\mathrm{LHS}=&\,(-1)^{n-1}(1-e^{-h_1})^{n-1}*(1-e^{-(h_1+h_2)})\\ &-(1-e^{-h_2})*\Big((e^{-h_1}-1)^{*(n-1)}-(e^{-h_1}*(1-e^{-h_2}))^{n-1}\Big).\end{aligned}$$

Now one can check that LHS = RHS, and this proves the first statement.

The second statement directly follows from the definition of qch. This finishes the proof. $\square$

Remark 4.8. *If we assume a prior that* qch *is a ring homomorphism, then the images* $\mathrm{qch}(Q_a)$*'s are uniquely determined by* $\mathrm{qch}(L_a^{-1})=e^{-h_a}$*, similar to Remark 3.2.*

Remark 4.9. *Let* π_1,π_2 *be the natural projection from* $\mathbb{F}\ell_{1,n-1;n}$ *to* $\mathbb{F}\ell_{1,n},\mathbb{F}\ell_{n-1;n}$*, respectively. We have the exact sequences*

$$0\to\mathcal{O}_{\mathbb{F}\ell_{1,n-1;n}}\to L_1^{\oplus n}\to\pi_1^*T_{\mathbb{F}\ell_{1,n}}\to 0,\qquad 0\to\mathcal{O}_{\mathbb{F}\ell_{1,n-1;n}}\to L_2^{\oplus n}\to\pi_2^*T_{\mathbb{F}\ell_{n-1;n}}\to 0.$$

Similar to Remark 3.3, the factor $\left(\frac{1-e^{-h_a}}{h_a}\right)^n$ *in* $\mathrm{qch}(Q_a)$*,* $1\le a\le 2$*, may be interpreted as a quantum version of the inverse of the Todd classes of* $\pi_1^*T_{\mathbb{F}\ell_{1;n}}$ *and* $\pi_2^*T_{\mathbb{F}\ell_{n-1;n}}$ *respectively. The factor* $\frac{(h_1+h_2)}{1-e^{-(h_1+h_2)}}$ *may be interpreted as a quantum version of the Todd class of* $L_1\otimes L_2$*.*

5. Quantum K-theory of Milnor hypersurfaces

In the section, we provide a ring presentation of the small quantum K-theory of smooth Milnor hypersurfaces.

5.1. **Milnor hypersurfaces.** Let $n \geq m \geq 3$ [2]. The (smooth) Milnor hypersurface $H_{n-1,m-1}$ is a degree-$(1,1)$ hypersurface in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, defined by the equation

$$x_1y_1 + x_2y_2 + \cdots + x_my_m = 0,$$

where $[x_1 : \cdots : x_n]$ and $[y_1 : \cdots : y_m]$ are homogeneous coordinates of $\mathbb{P}^{n-1}, \mathbb{P}^{m-1}$, respectively. Note that the incidence variety $\mathbb{F}\ell_{1,n-1;n}$ is a degree $(1,1)$-hypersurface of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ defined by $\sum_{j=1}^{n}(-1)^j x_jy_j = 0$ via the Plücker embedding. We can identify $H_{n-1,n-1}$ with $\mathbb{F}\ell_{1,n-1;n}$ by a simple coordinate change ($y_j \mapsto (-1)^j y_j$).

Remark 5.1. *Any smooth Schubert variety in $\mathbb{F}\ell_{1,n-1;n}$ is either a Milnor hypersurface or a product of projective spaces.*

Consider the following commutative diagram:

$$\begin{tikzcd} H_{n-1,m-1} \arrow[r, hook, "\iota"] \arrow[d, "\pi_2^m"] & H_{n-1,n-1} \cong \mathbb{F}\ell_{1,n-1;n} \arrow[d, "\pi_2"] \\ \mathbb{P}^{m-1} \arrow[r, hook, "\iota_m"] & \mathbb{P}^{n-1} \cong \mathbb{F}\ell_{n-1;n} \end{tikzcd} \tag{5.1}$$

Here ι is the natural embedding, π_2 (resp. π_2^m) is the natural projection to the second factor of the product of projective spaces, and ι_m denotes the natural inclusion. Let $\mathcal{U}_{n-1}$ be the tautological bundle of $\mathbb{F}\ell_{n-1;n}$. Then $\mathbb{F}\ell_{1,n-1;n} = \mathbb{P}(\mathcal{U}_{n-1})$ is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^{n-1}$, and $H_{n-1,m-1} = \mathbb{P}^{m-1} \times_{\mathbb{F}\ell_{n-1;n}} \mathbb{F}\ell_{1,n-1;n} = \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$ is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^{m-1}$.

5.2. **Ring presentation of $K(H_{n-1,m-1})$.** There is a standard procedure to obtain a ring presentation of the classical K-theory of the projective bundle $H_{n-1,m-1} = \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$. Here we provide a precise presentation of $K(H_{n-1,m-1})$ that is helpful for us to derive a ring presentation of $QK(H_{n-1,m-1})$.

Lemma 5.2. *For any nonnegative integers b, n, t with $0 \leq b \leq n-t-1$, we have*

$$\sum_{c=0}^{b} \binom{n}{n-b+c}\binom{t+c}{c}(-1)^c = \binom{n-t-1}{b}.$$

Proof. It follows from differentiating $\sum_{c=0}^{\infty} z^c = \frac{1}{1-z}$, where ($|z| < 1$), that

$$A(z) := \sum_{c=0}^{\infty} \binom{t+c}{c} z^c = \frac{1}{(1-z)^{t+1}}.$$

The coefficient of z^b-term in $(1+z)^n \cdot A(-z)$ is given by $\sum_{c=0}^{b} \binom{n}{n-b+c}\binom{t+c}{c}(-1)^c$. Hence, it is equal to $\binom{n-t-1}{b}$, by noting $(1+z)^{n-t-1} = (1+z)^n \cdot A(-z)$. $\square$

Recall the following polynomials defined in the introduction.

$$F_1(x,y) := (1-y)^m, \tag{5.2}$$

$$F_2(x,y) := (-1)^{n-1}(1-x)^{n-1} + \sum_{t=1}^{m-1}(-1)^{n-1-t}x^{t-1}(1-x)^{n-t-1}(1-y)^t. \tag{5.3}$$

[2] We exclude the case $m = 2$, for which there is a nontrivial mirror map in the K-theoretic J-function by quantum Lefschetz principle and Proposition 2.3 is not applicable directly.

Lemma 5.3. *There exists $a(x,y) \in \mathbb{Q}[x,y]$ such that*

$$F_2(x,y) - a(x,y)F_1(x,y) = (-1)^n \sum_{l=0}^{n-1}(-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s}(-y)^s.$$

Proof. Let RHS denote the right-hand side and $a_1(x,y) := -x^{n-1}(1-y)^{n-m}$. By direct calculation, we have RHS $= a_1(x,y)F_1(x,y) + x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s}(-y)^s$. Note $(-y)^s = \sum_{t=1}^{s}(1-y)^t \binom{s}{t}(-1)^{s-t} + (-1)^s$. By changing the order of summation over t and s, l, we have RHS $= a_1(x,y)F_1(x,y) + g_0(x) + \sum_{t=1}^{n-1} g_t(x)(1-y)^t$, where

$$g_0(x) = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l \sum_{s=1}^{l+1} \binom{n}{n-1-l+s}(-1)^s,$$

$$g_t(x) = (-1)^n \sum_{l=t-1}^{n-2}(-x)^l \sum_{s=t}^{l+1} \binom{n}{n-1-l+s}\binom{s}{t}(-1)^{s-t}.$$

By Lemma 5.2 with $t = 0$, we have

$$g_0(x) = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l(-1)\sum_{c=0}^{l}\binom{n}{n-l+c}(-1)^c = x^{n-1} + (-1)^n \sum_{l=0}^{n-2}(-x)^l(-1)\binom{n-1}{l}.$$

Hence, $g_0(x) = (-1)^{n-1}(1-x)^{n-1}$. For $1 \le t \le m-1$, by substituting l with $b = l+1-t$ and then applying Lemma 5.2 with $c = s-t$, we have

$$g_t(x) = (-1)^n \sum_{b=0}^{n-t-1}(-x)^{b+t-1}\sum_{s=t}^{b+t}\binom{n}{n-b-t+s}\binom{s}{t}(-1)^{s-t} = (-1)^n \sum_{b=0}^{n-t-1}\binom{n-t-1}{b}(-x)^{b+t-1}.$$

Hence, $g_t(x) = (-1)^{n-1-t}x^{t-1}(1-x)^{n-t-1}$.

Set $a(x,y) = a_1(x,y) + \sum_{t=m}^{n-1} g_t(x)(1-y)^{t-m}$. We are done. □

Recall that $\mathcal{S}_1$ (resp. $\mathcal{S}_{n-1}$) denotes the tautological vector subbundle of $\mathbb{F}\ell_{1,n-1;n}$ whose fiber at $V_\bullet$ is given by the vector space V_1 (resp. V_{n-1}). Let P_1 (resp. P_2) be the class of line bundle $\iota^*\mathcal{S}_1$ (resp. $\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee$) in $K(H_{n-1,m-1})$, namely $P_i = L_i^{-1}$ in Theorem 1.4.

Proposition 5.4. *The classical K-theory of $H_{n-1,m-1}$ is generated by P_1, P_2 with*

$$K(H_{n-1,m-1}) = \mathbb{Q}[P_1,P_2]/(F_1(P_1,P_2), F_2(P_1,P_2)).$$

Proof. Note $H_{n-1,m-1} \cong \mathbb{P}(\iota_m^*\mathcal{U}_{n-1})$. By the classical K-theory for projective bundles (see e.g. [Ka78, Theorem 2.16]), we have

$$K(H_{n-1,m-1}) = K(\mathbb{P}^{m-1})[P_1]/(P_1^{n-1} - [\lambda^1(\iota_m^*\mathcal{U}_{n-1})]P_1^{n-2} + \cdots + (-1)^{n-1}[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1})]).$$

Here $\lambda^k(\iota_m^*\mathcal{U}_{n-1})$ is the k-th exterior power of $\iota_m^*\mathcal{U}_{n-1}$, and $K(H_{n-1,m-1})$ is a free $K(\mathbb{P}^{m-1})$-module via the ring homomorphism $(\pi_2^m)^* : K(\mathbb{P}^{m-1}) \longrightarrow K(H_{n-1,m-1})$.

Let $\underline{\mathbb{C}^n}$ also denote the trivial bundle over $\mathbb{F}\ell_{n-1;n}$ by abuse of notation, and denote $\mathcal{Q} := \underline{\mathbb{C}^n}/\mathcal{U}_{n-1}$, which is a line bundle. So $(1-(\iota_m^*\mathcal{Q})^\vee)^m = 0$ in $K(\mathbb{P}^{m-1})$. Notice $P_2 = [\iota^*(\underline{\mathbb{C}^n}/\mathcal{S}_{n-1})^\vee] = [(\pi_2^m)^*(\iota_m^*\mathcal{Q})^\vee] \in K(H_{n-1,m-1})$. Hence, $(1-P_2)^m = 0$.

Let $\mathbf{1} \in K(\mathbb{P}^{m-1})$ be the class of the trivial line bundle. For $1 \leq k \leq n$, it follows from the exact sequence $0 \to \mathcal{U}_{n-1} \to \underline{\mathbb{C}^n} \to \mathcal{Q} \to 0$, together with $\mathcal{Q}$ being a line bundle, that

$$\binom{n}{k}\mathbf{1} = \sum_{t+b=k} [\iota_m^*(\lambda^t(\mathcal{U}_{n-1}))] \cdot [\iota_m^*(\lambda^b \mathcal{Q})] = [\lambda^{k-1}(\iota_m^* \mathcal{U}_{n-1}) \otimes \iota_m^* \mathcal{Q})] + [\lambda^k(\iota_m^* \mathcal{U}_{n-1})]. \tag{5.4}$$

In particular, we have $[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1}) \otimes \iota_m^* \mathcal{Q})] = \mathbf{1}$, implying $[\lambda^{n-1}(\iota_m^*\mathcal{U}_{n-1})] = [\iota_m^* \mathcal{Q})^\vee]$. By (reverse) induction on k, we have $[\lambda^k(\iota_m^*\mathcal{U}_{n-1})] = \sum_{s=1}^{n-k}(-1)^{s-1}\binom{n}{k+s}[(\iota_m^*\mathcal{Q})^\vee]^s$ in $K(\mathbb{P}^{m-1})$ for $1 \leq k \leq n-1$. Hence, $K(H_{n-1,m-1})$ is the quotient of $\mathbb{Q}[P_1, P_2]$ by the ideal generated by $(1-P_2)^m$ and

$$\sum_{l=0}^{n-1}(-1)^{n-1-l}(P_1)^l\bigg(\sum_{s=1}^{l+1}(-1)^{s-1}\binom{n}{n-1-l+s}P_2^s\bigg). \tag{5.5}$$

Then we are done by Lemma 5.3. $\square$

5.3. **Ring presentation of $QK(H_{n-1,m-1})$.** Recall $L_i = P_i^{-1} \in K(H_{n-1,m-1})$, and $\{h_i = c_1(L_i)\}_{1\leq i\leq 2}$ form a nef basis of $H^2(H_{n-1,m-1}, \mathbb{Z})$. We have the Mori cone $\overline{\mathrm{NE}}(H_{n-1,m-1}) = \mathbb{Z}_{\geq 0}\beta_1 + \mathbb{Z}_{\geq 0}\beta_2 \subset H_2(H_{n-1,m-1}, \mathbb{Z})$ with with $\int_{\beta_i} h_j = \delta_{i,j}$. By [Ta13, Proposition 1] and [Lee99, Theorem 10], the small K-theoretic J-function of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is given by

$$J_{\mathbb{P}^{n-1}\times\mathbb{P}^{m-1}}(\hbar, \mathbf{Q}) = \sum_{d_1\geq 0,\, d_2\geq 0}^{\infty} \frac{Q_1^{d_1}Q_2^{d_2}}{\prod_{l=1}^{d_1}(1-\mathcal{L}_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-\mathcal{L}_2^{-1}\hbar^l)^m}. \tag{5.6}$$

Here $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is the class of the line bundle obtained by the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ (resp. $\mathcal{O}_{\mathbb{P}^m}(1)$), and we still denote by Q_1, Q_2 the corresponding Novikov variables for $QK(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})$ by abuse of notation. As a degree-$(1,1)$ hypersurface of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, by the quantum Lefschetz principle [Gi15, the last Theorem] (see also [Gi20, Theorem 3] for general cases), we obtain the small K-theoretic J-function of $H_{n-1,m-1}$, [3]

$$J_{H_{n-1,m-1}}(\hbar, \mathbf{Q}) = \sum_{d_1\geq 0, d_2\geq 0}^{\infty} \frac{Q_1^{d_1}Q_2^{d_2}\prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m}. \tag{5.7}$$

Theorem 5.5. *Denote $\vartheta_k := 1 - L_k^{-1}\hbar^{Q_k\partial_{Q_k}}, k = 1, 2$. For the $\hbar$-difference operators*

$$D_1 := \vartheta_2^m, \qquad D_2 := (-1)^{n-1}(\vartheta_1)^{n-1} + \sum_{t=1}^{m-1}(-1)^{n-1-t}(1-\vartheta_1)^{t-1}\vartheta_1^{n-t-1}\vartheta_2^t,$$

we have

$$\big(D_1 - Q_2 + Q_2\hbar(1-\vartheta_1)(1-\vartheta_2)\big)J_{H_{n-1,m-1}} = 0, \tag{5.8}$$

$$\big(D_2 - (-1)^{n-m}Q_2(1-\vartheta_1)^{m-1}\vartheta_1^{n-m} - (-1)^{n-1}Q_1(1-\vartheta_2)\big)J_{H_{n-1,m-1}} = 0. \tag{5.9}$$

Proof. We show the statement by direct computations as follows.

[3] The J-function of $H_{2,2} = F\ell_{1,2;3}$ was computed in [GL03, Example 2.5] up to a scalar by $(1-\hbar)$.

$$\begin{aligned}
D_1(J_{H_{n-1,m-1}}) &= \Bigg(\sum_{d_2=0} \frac{(1-L_2^{-1})^m Q_1^{d_1} \prod_{l=1}^{d_1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n} \\
&\quad + \sum_{d_2\geq 1}^{\infty} \frac{(1-L_2^{-1}\hbar^{d_2})^m Q_1^{d_1} Q_2^{d_2} \prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= 0 + \sum_{d_2\geq 1}^{\infty} \frac{Q_1^{d_1} Q_2^{d_2} \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)\cdot 1}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m} \\
&\quad + Q_2\hbar \sum_{d_2\geq 1}^{\infty} \frac{Q_1^{d_1} Q_2^{d_2-1} \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)(-L_1^{-1}\hbar^{d_1}\cdot L_2^{-1}\hbar^{d_2-1})}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m} \\
&= Q_2 J_{H_{n-1,m-1}} - Q_2\hbar(L_1^{-1}\hbar^{Q_1\partial_{Q_1}})(L_2^{-1}\hbar^{Q_2\partial_{Q_2}})(J_{H_{n-1,m-1}}).
\end{aligned}$$

Here the term 0 in the second equality follows from the relation $(1-L_2^{-1})^m = 0$ in $K(H_{n-1,m-1})$. The third equality holds because $d_2 \geq 1$ is replaced by $d_2 - 1 \geq 0$, so that the J-function $J_{H_{n-1,m-1}}$ appears again.

To simplify the expressions, we denote $S_i := (1 - L_i^{-1}\hbar^{d_i}), i = 1, 2$. We have

$$\begin{aligned}
&D_2(J_{H_{n-1,m-1}}) \\
&= \Bigg(F_2(L_1^{-1}, L_2^{-1}) + \sum_{0\neq(d_1,d_2)}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{F_2(L_1^{-1}\hbar^{d_1}, L_2^{-1}\hbar^{d_2}) \prod_{l=1}^{d_1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= 0 + \sum_{0\neq(d_1,d_2)}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{F_2(1-S_1, 1-S_2)(-S_1S_2+S_1+S_2) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \\
&= \Bigg(\sum_{d_1=0,d_2\geq 1}^{\infty} Q_2^{d_2} \frac{((-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \\
&\quad + \sum_{d_1\geq 1,d_2=0}^{\infty} Q_1^{d_1} \frac{((-1)^{n-1}S_1^n(1-S_2)) \prod_{l=1}^{d_1-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n} \\
&\quad + \sum_{d_1\geq 1,d_2\geq 1}^{\infty} Q_1^{d_1}Q_2^{d_2} \frac{((-1)^{n-1}S_1^n(1-S_2)+(-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= \Bigg(Q_2 \sum_{d_2\geq 1}^{\infty} Q_1^{d_1}Q_2^{d_2-1} \frac{((-1)^{n-m}(1-S_1)^{m-1}S_1^{n-m}S_2^m) \prod_{l=1}^{d_1+d_2-1}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{\prod_{l=1}^{d_1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2-1}(1-L_2^{-1}\hbar^l)^m S_2^m} \\
&\quad + Q_1 \sum_{d_1\geq 1}^{\infty} Q_1^{d_1-1}Q_2^{d_2} \frac{((-1)^{n-1}S_1^n(1-S_2)) \prod_{l=1}^{d_1-1+d_2}(1-L_1^{-1}\cdot L_2^{-1}\hbar^l)}{S_1^n \prod_{l=1}^{d_1-1}(1-L_1^{-1}\hbar^l)^n \prod_{l=1}^{d_2}(1-L_2^{-1}\hbar^l)^m} \Bigg) \\
&= (-1)^{n-m}Q_2(L_1^{-1}\hbar^{Q_1\partial_{Q_1}})^{m-1}(1-L_1^{-1}\hbar^{Q_1\partial_{Q_1}})^{n-m}(J_{H_{n-1,m-1}}) + (-1)^{n-1}Q_1(L_2^{-1}\hbar^{Q_2\partial_{Q_2}})(J_{H_{n-1,m-1}}).
\end{aligned}$$

Here $F_2(x,y)$ is the polynomial defined in Equation (5.3). The term 0 in the second equality follows from the relation $F_2(L_1^{-1},L_2^{-1})=0$ in $K(H_{n-1,m-1})$ by Proposition 5.4. The term $(-S_1S_2+S_1+S_2)$ in the second equality comes from $1-L_1^{-1}L_2^{-1}\hbar^{d_1+d_2}$. The last equality holds because $d_2\geq 1$ (resp. $d_1\geq 1$) could be replaced by $d_2-1\geq 0$ (resp. $d_1-1\geq 0$) after canceling out the same term S_2^m (resp. S_1^n), so that the J-function appears again. □

To achieve a ring presentation of $QK(H_{n-1,m-1})$, it remains to apply the following Nakayama-type result proved in [GMSZ22, Proposition A.3].

Proposition 5.6 (Gu-Mihalcea-Sharpe-Zou). *Let A be a Noetherian integral domain, and let $\mathfrak{a}\subset A$ be an ideal. Assume that A is complete in the $\mathfrak{a}$-adic topology. Let M,N be finitely generated A-modules. Assume that the A-module N, and the $A/\mathfrak{a}$-module $N/\mathfrak{a}N$, are both free modules of the same rank $r\leq\infty$, and that we are given an A-module homomorphism $f:M\to N$ such that the induced $A/\mathfrak{a}$-module map $\overline{f}:M/\mathfrak{a}M\to N/\mathfrak{a}N$ is an isomorphism of $A/\mathfrak{a}$-modules. Then f is an isomorphism.*

Now we restate Theorem 1.5 as follows.

Theorem 5.7. *Let $n\geq m\geq 3$. The small quantum K-theory of $H_{n-1,m-1}$ is presented by*

$$QK(H_{n-1,m-1})\cong\mathbb{Q}[L_1^{-1},L_2^{-1}][\![Q_1,Q_2]\!]/(F_1^Q,F_2^Q),$$

where $F_1^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=F_1(L_1^{-1},L_2^{-1})-Q_2+Q_2L_1^{-1}L_2^{-1}$ and

$$F_2^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=F_2(L_1^{-1},L_2^{-1})-(-1)^{n-m}Q_2(L_1^{-1})^{m-1}(1-L_1^{-1})^{n-m}-(-1)^{n-1}Q_1L_2^{-1}.$$

Proof. Let M be the right hand side of the expected isomorphism, set $N:=QK(H_{n-1,m-1})$, and take the ideal $\mathfrak{a}=(Q_1,Q_2)$ of $A:=\mathbb{Q}[\![Q_1,Q_2]\!]$. Clearly, we have $A/\mathfrak{a}=\mathbb{Q}$. Moreover, N (resp. $N/\mathfrak{a}N$) is a free A-module (resp. $A/\mathfrak{a}$-module) of rank $\dim_{\mathbb{Q}}K(H_{n-1,m-1})=m(n-1)<\infty$. Note $n\geq m\geq 3$. It follows directly from the expression of $J_{H_{n-1,m-1}}$ that each (d_1,d_2)-term of $L_i^{-1}\hbar^{Q_i\partial_{Q_i}}(J_{H_{n-1,m-1}})$ vanishes at $\hbar=\infty$ for $1\leq i\leq 2$, whenever $d_1>0$ or $d_2>0$. By Proposition 2.3 and Theorem 5.5, the relations $F_1^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=0$ and $F_2^Q(L_1^{-1},L_2^{-1},Q_1,Q_2)=0$ hold in $QK(H_{n-1,m-1})$ (with respect to the quantum K-product). Therefore we have a canonical ring homomorphism $f:M\to N$ defined by $L_i^{-1}\mapsto L_i^{-1}$ and $Q_i\mapsto Q_i, i=1,2$. Since $f(\mathfrak{a}M)\subset\mathfrak{a}N$, it induces an $A/\mathfrak{a}$-module homomorphism $\overline{f}:M/\mathfrak{a}M\to N/\mathfrak{a}N$. Note $M/\mathfrak{a}M\cong(A/\mathfrak{a})\otimes_A M\cong\mathbb{Q}[L_1^{-1},L_2^{-1}]/(F_1(L_1^{-1},L_2^{-1}),F_2(L_1^{-1},L_2^{-1})$ and $N/\mathfrak{a}N\cong K(H_{n-1,m-1})$. Thus $\overline{f}$ is an isomorphism of $A/\mathfrak{a}$-modules by Proposition 5.4. Therefore f is an isomorphism of A-modules by Proposition 5.6. □

Remark 5.8. *The presentation for the special case $QK(H_{n-1,n-1})$ can also be derived from the quantum Littlewood-Richardson rule for $QK(\mathbb{F}\ell_{1,n-1;n})$ [Xu24, Section 5]. Therein, the Schubert class $\mathcal{O}^{[1,n-1]}$ (resp. $\mathcal{O}^{[2,n]}$) is given by $1-L_1^{-1}$, (resp. $1-L_2^{-1}$).*

6. Appendix: proof of Lemma 4.5

By [BCFKS00], the toric superpotential mirror to $\mathbb{F}\ell_{1,n-1;n}$ is the Laurent polynomial

$$f_{\rm tor}:=\sum_{k=1}^{2n-3}\frac{x_k}{x_{k-1}}+\frac{q_2}{x_{n-2}}+\frac{x_n}{q_2}+\frac{q_1q_2}{x_{2n-3}}\in\mathbb{Q}[x_1^{\pm 1},\cdots,x_{2n-3}^{\pm 1},q_1^{\pm 1},q_2^{\pm 1}],$$

where $x_0 := 1$. The Jacobi ring of $f_{\rm tor}$ is defined by

$$\mathrm{Jac}(f_{\rm tor}) := \mathbb{Q}[x_1^{\pm 1}, \cdots, x_{2n-3}^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}] \Big/ \Big(x_1 \frac{\partial f_{\rm tor}}{\partial x_1}, \cdots, x_{2n-3} \frac{\partial f_{\rm tor}}{\partial x_{2n-3}} \Big). \tag{6.1}$$

Proposition 6.1. *There is an injective homomorphism of $\mathbb{Q}[\![q_1, q_2]\!]$-algebras*

$$\Phi : QH(\mathbb{F}\ell_{1,n-1;n}) \hookrightarrow \mathrm{Jac}(f_{\rm tor}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!],$$

such that $\Phi(h_1) = [\frac{q_1 q_2}{x_{2n-3}}], \quad \Phi(h_2) = [x_1]$.

Proof. Assume $n \geq 4$ first. Let R_a be the relation given by $x_a \frac{\partial f_{\rm tor}}{\partial x_a}$. From $R_1, ..., R_{n-3}$ and $R_{n+1}, \ldots, R_{2n-3}$, we get

$$x_k = \begin{cases} x_1^k, & \text{if } 2 \leq k \leq n-2, \\ (\frac{x_{2n-3}}{q_1 q_2})^{2n-3-k} x_{2n-3}, & \text{if } n \leq k \leq 2n-4. \end{cases} \tag{6.2}$$

It follows immediately that

$$\mathrm{Jac}(f_{\rm tor}) \cong \mathbb{Q}[x_1^{\pm 1}, x_{2n-3}^{\pm 1}, x_{n-1}^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}] \big/ \big(R_{n-2}, R_{n-1}, R_n \big). \tag{6.3}$$

It follows from R_{n-2} that

$$x_{n-1} = x_1^{n-1} - q_2. \tag{6.4}$$

It follows from R_{n-1} and (6.2) that in $\mathrm{Jac}(f_{\rm tor})$,

$$x_n = \frac{(x_1^{n-1} - q_2)^2}{x_1^{n-2}} = (\frac{x_{2n-3}}{q_1 q_2})^{n-3} x_{2n-3}. \tag{6.5}$$

By R_n we have $\frac{x_n}{x_{n-1}} = \frac{x_{n+1}}{x_n} - \frac{x_n}{q_2}$. By (6.4) and (6.5), we have $\frac{x_n}{x_{n-1}} = \frac{x_1^{n-1} - q_2}{x_1^{n-2}}$. By (6.2), we have $\frac{x_{n+1}}{x_n} = \frac{q_1 q_2}{x_{2n-3}}$. By (6.5), we have $\frac{x_n}{q_2} = \frac{1}{q_2} \frac{(x_1^{n-1} - q_2)^2}{x_1^{n-2}}$. All these relations together give the relation in $\mathrm{Jac}(f_{\rm tor})$

$$\frac{x_1^n}{q_2} - x_1 - \frac{q_1 q_2}{x_{2n-3}} = 0. \tag{6.6}$$

One can check that the ideal (R_{n-2}, R_{n-1}, R_n) in (6.3) is generated by the three relations (6.4), (6.5), (6.6). Recall $QH(\mathbb{F}\ell_{1,n-1;n}) = \mathbb{Q}[h_1, h_2][\![q_1, q_2]\!]/(f_1^q(h_1, h_2, q_1, q_2), f_2^q(h_1, h_2, q_1, q_2))$ by Proposition 4.1. It remains to show the relations (6.5) and (6.6) are equivalent to $f_1^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$ and $f_2^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$ in (6.3).

Since q_2 is a unit in (6.3), (6.6) is equivalent to $x_1^n - q_2(x_1 + \frac{q_1 q_2}{x_{2n-3}}) = f_1^q(\frac{q_1 q_2}{x_{2n-3}}, x_1, q_1, q_2)$. By direct calculations using (6.5) and (6.6), we have

$$q_1 q_2 - (\frac{q_1 q_2}{x_{2n-3}})^{n-2} (-\frac{x_1^{n-1} - q_2}{x_1^{n-2}} + \frac{q_1 q_2}{x_{2n-3}}) q_2 = 0 \tag{6.7}$$

Therefore, in (6.3), we note q_2 is a unit again, and then by (6.7) we have

$$\begin{aligned}
0 =& q_1 - (\frac{q_1q_2}{x_{2n-3}})^{n-2}(-x_1 + \frac{q_2}{x_1^{n-2}} + \frac{q_1q_2}{x_{2n-3}}) \\
=& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + \frac{q_1q_2}{x_{2n-3}}\frac{q_2}{x_1^{n-2}}(\frac{q_1q_2}{x_{2n-3}})^{n-3}\big) \\
=& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + (\frac{q_1q_2}{x_{2n-3}})^{n-3}x_1^2 - \frac{q_2}{x_1^{n-3}}(\frac{q_1q_2}{x_{2n-3}})^{n-3}\big) \\
=& q_1 - \big((\frac{q_1q_2}{x_{2n-3}})^{n-1} - (\frac{q_1q_2}{x_{2n-3}})^{n-2}x_1 + (\frac{q_1q_2}{x_{2n-3}})^{n-3}x_1^2 \\
&\qquad - (\frac{q_1q_2}{x_{2n-3}})^{n-4}x_1^3 + \cdots + (-1)^{n-1}x_1^{n-1} - (-1)^{n-1}q_2\big).
\end{aligned}$$

Here the third equality follows from (6.6), and the last equality follows from (6.6) together with the induction on the power of x_1. Therefore we obtain the following relation

$$0 = \sum_{k=0}^{n-1}(-1)^k(\frac{q_1q_2}{x_{2n-3}})^{n-1-k}x_1^k - (-1)^{n-1}q_2 - q_1 = f_2^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2). \tag{6.8}$$

It follows directly from the process of obtaining f_1^q, f_2^q that

$$\big(R_{n-2}, R_{n-1}, R_n\big) = \big(x_{n-1} - (x_1^{n-1} - q_2), f_1^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2), f_2^q(\frac{q_1q_2}{x_{2n-3}}, x_1, q_1, q_2)\big)$$

as ideals in $\mathbb{Q}[x_1^{\pm1}, x_{2n-3}^{\pm1}, x_{n-1}^{\pm1}, q_1^{\pm1}, q_2^{\pm1}]$. Hence, $h_1 \mapsto \frac{q_1q_2}{x_{2n-3}}, h_2 \mapsto x_1$, $q_1 \mapsto q_1$ and $q_2 \mapsto q_2$ cannonically defines an injective ring homomorphism Φ.

The case $n = 3$ can be verified by similar but easier calculations (in which case $\mathbb{F}\ell_{1,2;3}$ is a complete flag variety and the expected statement is known earlier to the experts). $\square$

Remark 6.2. *The injective homomorphism Φ can naturally be extended to an isomorphism from a suitable localization of $QH(\mathbb{F}\ell_{1,n-1;n})$ to $\mathrm{Jac}(f_{\mathrm{tor}}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!]$.*

Proof of Lemma 4.5. By Proposition 6.1, $QH(\mathbb{F}\ell_{1,n-1;n})$ is a subring of $\mathrm{Jac}(f_{\mathrm{tor}}) \otimes_{\mathbb{Q}[q_1,q_2]} \mathbb{Q}[\![q_1, q_2]\!]$ via the embeding Φ. Note that both $\Phi(h_1) = \frac{q_1q_2}{x_{2n-3}}$ and q_1 are invertible in the Jacobi ring. Since $h_1^n = q_1(h_1 + h_2)$ holds in $QH(\mathbb{F}\ell_{1,n-1;n})$, $\Phi(h_1 + h_2) = q_1^{-1}\Phi(h_1)^n$ is invertible in the Jacobi ring. Hence, $h_1 + h_2$ is not a zero-divisor in (the subring) $QH(\mathbb{F}\ell_{1,n-1;n})$. $\square$

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