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ON THE EXISTENCE OF DEFORMATIONS OF COMPLEX ANALYTIC STRUCTURES

BY K. KODAIRA, L. NIRENBERG AND D. C. SPENCER

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1. Introduction

Let $M = \{t \,|\, |t| < 1\}$ be a spherical domain on the space of m complex variables $t = (t_1, \cdots, t_\lambda, \cdots, t_m)$, where $|t|^2 = \sum |t_\lambda|^2$, and let $\mathscr{V} = \{V_t \,|\, t \in M\}$ be a complex analytic family of compact complex manifolds V_t over M (see Kodaira and Spencer [4], § 1). $\mathscr{V}$ may be defined as a family of complex structures V_t defined on one and the same differentiable manifold X in the following manner:

Let $\{U_j\}$ be a finite covering of X by sufficiently small neighborhoods U_j.

(i) *There exists on each U_j a set of n complex-valued C^∞-differentiable functions $\zeta_j^1(x, t), \cdots, \zeta_j^\beta(x, t), \cdots, \zeta_j^n(x, t)$ of x and t, $x \in U_j$, $t \in M$, such that, for each t, $(\zeta_j^1(x, t), \cdots, \zeta_j^\beta(x, t), \cdots, \zeta_j^n(x, t))$ forms a system of local holomorphic coordinates defining the complex structure V_t.*

(ii) *For each pair U_j, U_i with non-empty intersection $U_j \cap U_i$, there exists a set of n holomorphic functions $h_{ij}^1(\zeta_j, t), \cdots, h_{ij}^n(\zeta_j, t)$ in $\zeta_j = (\zeta_j^1, \cdots, \zeta_j^n)$ and t (defined on a subdomain of $\mathbf{C}^n \times M$, $\mathbf{C}^n$ being the space of n complex variables) such that*

$$\zeta_i^\beta(x, t) = h_{ij}^\beta(\zeta_j(x, t), t)\,, \qquad \text{for } x \in U_j \cap U_i. \tag{1}$$

In what follows we identify X with V_0, denote a point on $X = V_0$ by z instead of x and consider $\zeta_j^\beta(z, t) = \zeta_j^\beta(x, t)$ as C^∞-*differentiable* functions of z and t.

We denote by $(z^1, \cdots, z^\beta, \cdots, z^n)$ local holomorphic coordinates (not specified) of a point z on the *complex manifold* V_0 and let

$$\partial_\beta = \frac{\partial}{\partial z^\beta}\,,\ \ \bar{\partial}_\beta = \frac{\partial}{\partial \bar{z}^\beta}\,.$$

Moreover let ∂, $\bar{\partial}$ be the exterior derivatives on the complex manifold V_0. Now we define (0, 1)-forms

$$\varphi_j^\beta(z, t) = \sum\nolimits_{\nu=1}^n \varphi_{j\bar{\nu}}^\beta(z, t) d\bar{z}^\nu, \qquad \beta = 1, 2, \cdots, n,$$

on $U_j \subset V_0$ by the simultaneous linear equations

$$\bar{\partial}\zeta_j^\beta(z, t) = \sum\nolimits_{\gamma=1}^n \varphi_j^\gamma(z, t)\partial_\gamma \zeta_j^\beta(z, t)\,, \qquad |t| < \varepsilon. \tag{2}$$

It is clear that $\varphi_j^\gamma(z, t)$ are uniquely determined and C^∞-differentiable for

$|t| < \varepsilon$, provided that $\varepsilon > 0$ is sufficiently small. From (1) we obtain

$$\bar{\partial}\zeta_i^\beta(z, t) = \sum_{\mu=1}^n \frac{\partial h_{ij}^\beta}{\partial \zeta_j^\mu} \cdot \bar{\partial}\zeta_j^\mu(z, t) ,$$

$$\partial_\gamma \zeta_i^\beta(z, t) = \sum_{\mu=1}^n \frac{\partial h_{ij}^\beta}{\partial \zeta_j^\mu} \cdot \partial_\gamma \zeta_j^\mu(z, t) .$$

Comparing these equalities with (2) we infer immediately that

$$\varphi_i^\beta(z, t) = \varphi_j^\beta(z, t) , \qquad \text{for} \quad z \in U_i \cap U_j.$$

Thus, letting

$$\varphi(t) = (\varphi^1(t), \cdots, \varphi^\beta(t), \cdots, \varphi^n(t)), \quad \varphi^\beta(t) = \varphi_j^\beta(z, t) ,$$

we obtain a *vector* (0, 1)-*form* $\varphi(t)$ on V_0 depending differentiably on t. Since $\bar{\partial}\bar{\partial}\zeta_j^\beta(z, t) = 0$, it follows from (2) that

$$\bar{\partial}\varphi^\beta(t) - \sum_{\gamma=1}^n \varphi^\gamma(t) \wedge \partial_\gamma \varphi^\beta(t) = 0, \qquad \beta = 1, 2, \cdots, n, \tag{3}$$

where $\wedge$ denotes exterior multiplication of forms and

$$\partial_\gamma \varphi^\beta(t) = \sum_\nu \partial_\gamma \varphi_{\bar{\nu}}^\beta (z, t)\, d\bar{z}^\nu .$$

This formula (3) represents the *integrability condition* for the system of linear partial differential equations (2).

It follows from $\bar{\partial}\zeta_j^\beta(z, 0) = 0$ that $\varphi(0) = 0$. Hence we infer from (3) that, for any tangent vector $v = \sum_{\nu=1}^m v^\lambda(\partial/\partial t_\lambda)$ of M at 0 ,

$$v\varphi(0) = \sum_{\lambda=1}^m v^\lambda \left(\frac{\partial \varphi(t)}{\partial t_\lambda}\right)_{t=0}$$

satisfies

$$\bar{\partial}(v\varphi(0)) = 0 .$$

We consider the $\bar{\partial}$-cohomology $H_{\bar{\partial}}^{0,1}(\Phi)$ of vector forms on V_0 (see § 3 below) and denote by $\rho_0(v) \in H_{\bar{\partial}}^{0,1}(\Phi)$ the $\bar{\partial}$-cohomology class of $v\varphi(0)$. Let Θ_0 be the sheaf over V_0 of germs of holomorphic vector fields. In view of the Dolbeault isomorphism (see § 3), $H_{\bar{\partial}}^{0,1}(\Phi)$ may be canonically identified with $H^1(V_0, \Theta_0)$ and thus $\rho_0(v)$ may be considered as an element of $H^1(V_0, \Theta_0)$. $\rho_0(v)$ is called the *infinitesimal deformation* of V_0 along v (see Frölicher and Nijenhuis [3]; Kodaira and Spencer [4], § 6). We note that $\rho_0 : v \to \rho_0(v)$ is a linear map of the tangent space $(T_M)_0$ of M at 0 into $H^1(V_0, \Theta_0)$.

Suppose given a compact complex manifold V. If there exists a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ such that $V_0 = V$, we call $\mathscr{V}$ a complex analytic family of *deformations* V_t of V. An important problem concerning deformations of complex manifolds is to find useful sufficient conditions for the existence of a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ of deformations of a given compact complex manifold $V_0 = V$ such that $\rho_0((T_M)_0) = H^1(V_0, \Theta_0)$ (see Kodaira and Spencer [4], § 22). The purpose of the present note is to prove the following theorem which gives *an* answer to this problem:

THEOREM. *Let V_0 be a compact complex manifold and let Θ_0 be the sheaf over V_0 of germs of holomorphic vector fields. If $H^2(V_0, \Theta_0) = 0$, then there exists a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ of deformations V_t of V_0 such that ρ_0 maps the tangent space $(T_M)_0$ of M at 0 isomorphically onto $H^1(V_0, \Theta_0)$.*

Our proof of the theorem is based on the theory of elliptic partial differential equations. Simultaneously, an entirely different approach for proving this theorem was outlined to us by H. Grauert.

2. Vector forms

Let V_0 be the given compact complex manifold. We denote by $\Phi^{0,q}$ the linear space of C^∞-differentiable vector $(0, q)$-forms

$$\psi = (\psi^1, \cdots, \psi^\beta, \cdots, \psi^n), \quad \psi^\beta = 1/q\,! \sum \psi^\beta_{\bar\mu_1 \cdots \bar\mu_q} d\bar z^{\mu_1} \wedge \cdots \wedge d\bar z^{\mu_q} \,.$$

The exterior derivative $\bar\partial\psi$ of ψ is simply defined by

$$\bar\partial\psi = (\bar\partial\psi^1, \cdots, \bar\partial\psi^\beta, \cdots, \bar\partial\psi^n) \,.$$

We define the Poisson bracket

$$[\varphi, \psi] = ([\varphi, \psi]^1, \cdots, [\varphi, \psi]^\beta, \cdots, [\varphi, \psi]^n)$$

of $\varphi \in \Phi^{0,p}$ and $\psi \in \Phi^{0,q}$ by

$$[\varphi, \psi]^\beta = 1/2 \sum_{\mu=1}^{n} (\varphi^\mu \wedge \partial_\mu \psi^\beta + (-1)^{pq+1} \psi^\mu \wedge \partial_\mu \varphi^\beta) \,,$$

where

$$\partial_\mu \psi^\beta = 1/q\,! \sum \partial_\mu \psi^\beta_{\bar\mu_1 \cdots \bar\mu_q} d\bar z^{\mu_1} \wedge \cdots \wedge d\bar z^{\mu_q} \,.$$

$[\varphi, \psi]$ is a vector form $\in \Phi^{0,p+q}$. (For an intrinsic definition of the Poisson bracket, see e. g., Kodaira and Spencer [4], § 4. This bracket was first studied systematically by Frölicher and Nijenhuis; see references given in [3]). In terms of the Poisson bracket, the integrability condition (3) is written in the form

$$(3)' \qquad \bar\partial\varphi(t) - [\varphi(t), \varphi(t)] = 0.$$

We have, for $\varphi \in \Phi^{0,p}$, $\psi \in \Phi^{0,q}$, $\tau \in \Phi^{0,r}$,

$$[\varphi, \psi] = (-1)^{pq+1}[\psi, \varphi] \,, \tag{4}$$

$$\bar{\partial}[\varphi, \psi] = [\bar{\partial}\varphi, \psi] + (-1)^{p}[\varphi, \bar{\partial}\psi] \,, \tag{5}$$

$$(-1)^{pr}[\varphi, [\psi, \tau]] + (-1)^{qp}[\psi, [\tau, \varphi]] + (-1)^{rq}[\tau, [\varphi, \psi]] = 0 \,. \tag{6}$$

3. Formal solutions

Let $Z(\Phi^{0,q})$ be the subspace of $\Phi^{0,q}$ of all $\bar{\partial}$-closed vector $(0, q)$-forms. The $\bar{\partial}$-cohomology $H^{0,q}_{\bar{\partial}}(\Phi)$ is, by definition, the factor space :

$$H^{0,q}_{\bar{\partial}}(\Phi) = Z(\Phi^{0,q})/\bar{\partial}\Phi^{0,q-1} \,.$$

We have the Dolbeault isomorphism

$$H^{q}(V_0, \Theta_0) \cong H^{0,q}_{\bar{\partial}}(\Phi)$$

(see Dolbeault [1] ; cf. also Kodaira and Spencer [4], § 2). By hypothesis $H^{0,2}_{\bar{\partial}}(\Phi) \cong H^{2}(V_0, \Theta_0) = 0$, or

$$Z(\Phi^{0,2}) = \bar{\partial}\Phi^{0,1} \,. \tag{7}$$

Let m be the dimension of the linear space $H^{0,1}_{\bar{\partial}}(\Phi)$ and let $\{\eta_1, \cdots, \eta_\lambda, \cdots, \eta_m\}$ be a set of m elements of $Z(\Phi^{0,1})$ which represents a *base* of the linear space $H^{0,1}_{\bar{\partial}}(\Phi) = Z(\Phi^{0,1})/\bar{\partial}\Phi^{0,0}$. Moreover let

$$\varphi_1(t) = \eta_1 t_1 + \cdots + \eta_m t_m$$

where $t_1, \cdots, t_m$ are complex variables. Now we construct homogeneous polynomials $\varphi_r(t)$ of $t_1, \cdots, t_m$ with coefficients in $\Phi^{0,1}$ of degrees $r=2, 3, 4, \cdots$, such that the formal power series

$$\varphi(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) + \cdots$$

satisfies

$$\bar{\partial}\varphi(t) - [\varphi(t), \varphi(t)] = 0 \,. \tag{3'}$$

Writing

$$\sigma_r(t) = \varphi_1(t) + \cdots + \varphi_r(t) \,,$$

we infer that (3)′ is equivalent to

$$\bar{\partial}\sigma_r(t) - [\sigma_r(t), \sigma_r(t)] \equiv 0 \bmod (t^{r+1}), \qquad r = 1, 2, \cdots, \tag{8}$$

where we indicate, for any polynomial $\psi(t)$ of $t_1, \cdots, t_m$, by "$\psi(t) \equiv 0 \bmod (t^{r+1})$" that $\psi(t)$ contains no terms of degree $\leqq r$. It is obvious that $\sigma_1(t) = \varphi_1(t)$ satisfies (8). Suppose therefore that $\varphi_2(t), \cdots, \varphi_r(t)$ are already determined in such a way that (8) holds for $\sigma_r(t)$. Let

$$\bar{\partial}\sigma_r(t) - [\sigma_r(t), \sigma_r(t)] \equiv \psi_{r+1}(t) \bmod (t^{r+2}) \tag{9}$$

where $\psi_{r+1}(t)$ is a homogenous polynomial in $t_1, \cdots, t_m$ of degree $r + 1$ with coefficients in $\Phi^{0,2}$. Then, using (4) and (5), we get

$$\bar{\partial}\psi_{r+1}(t) \equiv 2[\sigma_r(t), \bar{\partial}\sigma_r(t)] \bmod (t^{r+2}) .$$

Hence, by (8),

$$\bar{\partial}\psi_{r+1}(t) \equiv 2[\sigma_r(t), [\sigma_r(t), \sigma_r(t)]] \bmod (t^{r+2}) ,$$

while (6) implies that $[\psi, [\psi, \psi]] = 0$ for all ψ. Consequently we obtain

$$\bar{\partial}\psi_{r+1}(t) = 0 .$$

Thus each coefficient of $\psi_{r+1}(t)$ belongs to $Z(\Phi^{0,2})$ and therefore, by (7), we can find a homogeneous polynomial $\varphi_{r+1}(t)$ of degree $r+1$ with coefficients in $\Phi^{0,1}$ such that

$$\bar{\partial}\varphi_{r+1}(t) = -\psi_{r+1}(t) . \tag{10}$$

Now it is easy to verify that $\sigma_{r+1}(t) = \sigma_r(t) + \varphi_{r+1}(t)$ satisfies

$$\bar{\partial}\sigma_{r+1}(t) - [\sigma_{r+1}(t), \sigma_{r+1}(t)] \equiv 0 \bmod (t^{r+2}) .$$

This completes our inductive construction of $\varphi(t)$.

4. Potential-theoretic lemma

We define the norm $\|\psi\|_{k+\alpha}$ of $\psi \in \Phi^{0,q}$, $k = 1, 2, \cdots, 0 < \alpha < 1$, as follows: Let $\{U_j\}$ be a finite covering of V_0 by coordinate neighborhoods U_j and let $(z_j^1, \cdots, z_j^\beta, \cdots, z_j^n)$ be the system of holomorphic coordinates on U_j *fixed once and for all*. We write ψ explicitly in the form

$$\psi = (\psi^1, \cdots, \psi^\beta, \cdots, \psi^n), \quad \psi^\beta = 1/q! \sum \psi^\beta_{j\bar{\mu}_1\cdots\bar{\mu}_q}(z_j) d\bar{z}_j^{\mu_1} \wedge \cdots \wedge d\bar{z}_j^{\mu_q}$$

in terms of the local coordinates $(z_j^1, \cdots, z_j^n)$ and let

$$\|\psi\|_{k+\alpha} = \max_j \|\psi\|^{U_j}_{k+\alpha},$$
$$\|\psi\|^{U_j}_{k+\alpha} = \sum_{h=0}^{k} \sup |D_j^h \psi^\beta_{j\bar{\mu}_1\cdots\bar{\mu}_q}(z_j)| + \sup \frac{|D_j^k\psi^\beta_{j\bar{\mu}_1\cdots\bar{\mu}_q}(z_j) - D_j^k\psi^\beta_{j\bar{\mu}_1\cdots\bar{\mu}_q}(y_j)|}{|z_j - y_j|^\alpha},$$

where the "sup" is extended over all points $z_j, y_j \in U_j$, all indices β, $\bar{\mu}_1, \cdots, \bar{\mu}_q$ and all partial derivatives D_j^h, D_j^k of order h, k with respect to $z_j^1, \cdots, z_j^n, \bar{z}_j^1, \cdots, \bar{z}_j^n$.

We introduce a positive-definite Hermitian metric of class C^∞ on V_0 and denote by (φ, ψ) the scalar product of vector forms φ, ψ defined with the help of the Hermitian metric in an obvious manner. Moreover we denote by $\mathfrak{b}$ the *adjoint* of the operator $\bar{\partial}$ with respect to the scalar product, i. e., we have $(\mathfrak{b}\varphi, \psi) = (\varphi, \bar{\partial}\psi)$ for all $\varphi \in \Phi^{0,q}$, $\psi \in \Phi^{0,q-1}$, $q = 1, 2, \cdots, n$ (see Kodaira and Spencer [4], § 2). One verifies easily that $\mathfrak{b}\bar{\partial} + \bar{\partial}\mathfrak{b}$ is an *elliptic* partial differential operator of second order acting on the spaces $\Phi^{0,q}$, $q = 0, 1, \cdots, n$.

LEMMA. *For any $\psi \in \bar{\partial}\Phi^{0,q}$, the differential equation*

$$\bar{\partial}\varphi = \psi$$

has a unique solution $\varphi \in \Phi^{0,q}$ such that

$$(\varphi, \eta) = 0 , \qquad \textit{for all } \eta \in Z(\Phi^{0,q}) . \tag{11}$$

The solution φ satisfies the inequality

$$\| \varphi \|_{k+\alpha} < c_{k,\alpha} \cdot \| \psi \|_{k-1+\alpha} , \qquad k \geqq 1, \tag{12}$$

where $c_{k,\alpha}$ is a constant which is independent of ψ.

PROOF. Let $\mathfrak{L}$ denote the Hilbert space of all Lebesgue measurable vector forms φ of type $(0, q)$ with $(\varphi, \varphi) < +\infty$, and let $\mathfrak{S}$ denote the subspace of $\mathfrak{L}$ of vector forms φ satisfying

$$(\varphi, \mathfrak{d}\tau) = 0 \qquad \text{for all } \tau \in \Phi^{0,q+1}.$$

Thus $\mathfrak{S}$ consists of vector forms which are $\bar{\partial}$-closed in some generalized ("weak") sense. Suppose now that $\psi = \bar{\partial}\varphi_0$, $\varphi_0 \in \Phi^{0,q}$. We wish to show that there is a vector form $\varphi \in \Phi^{0,q}$ with $\bar{\partial}\varphi = \psi$ which satisfies (11). Choose the vector form $\varphi \in \mathfrak{L}$ with smallest norm (φ, φ) such that $\varphi - \varphi_0 \in \mathfrak{S}$. Clearly φ exists, is unique, and furthermore satisfies $(\varphi, \omega) = 0$ for all $\omega \in \mathfrak{S}$. For any $\sigma \in \Phi^{0,q}$ we therefore have

$$(\varphi, (\mathfrak{d}\bar{\partial} + \bar{\partial}\mathfrak{d})\sigma) = (\varphi, \mathfrak{d}\bar{\partial}\sigma) = (\varphi_0, \mathfrak{d}\bar{\partial}\sigma) = (\psi, \bar{\partial}\sigma) = (\mathfrak{d}\psi, \sigma) ,$$

and it follows that φ is a generalized ("weak") solution of the elliptic partial differential equation

$$(\mathfrak{d}\bar{\partial} + \bar{\partial}\mathfrak{d})\varphi = \mathfrak{d}\psi . \tag{13}$$

However, it is well known that such solutions are of class C^∞, so φ is of class C^∞ and satisfies (13) in the usual sense. Moreover it follows from $\varphi - \varphi_0 \in \mathfrak{S}$ that $\bar{\partial}\varphi = \bar{\partial}\varphi_0 = \psi$, and, since $Z(\Phi^{0,q}) \subset \mathfrak{S}$, φ satisfies (11).

We now establish (12) with the aid of known estimates for elliptic partial differential equations (see Douglis and Nirenberg [2]). Let $\{W_j\}$ be another finite covering of V_0 by coordinate neighborhoods where the closure of each W_j is contained as a compact subset in U_j. According to the results stated in [2], we may assert that

$$\| \varphi \|^{W_j}_{k+\alpha} \leqq \text{constant} \cdot (\| \psi \|^{U_j}_{k-1+\alpha} + \sup_{U_j} | \varphi^{\beta}{}_{j\bar{\mu}_1\cdots\bar{\mu}_q}(z_j)|)$$

where the constant is independent of ψ. Here the norm on the left is evaluated with the aid of the coordinates z_j covering U_j. (Theorem 4 of [2] yields the inequality for $k \geqq 2$ while Theorem 4′ of [2] implies it for $k = 1$). Hence we find

$$\| \varphi \|_{k+\alpha} \leqq \text{constant} \cdot (\| \psi \|_{k-1+\alpha} + \| \varphi \|_0) \tag{14}$$

where

$$\| \varphi \|_0 = \max \sup_{U_j} | \varphi^{\beta}_{j \bar{\mu}_1 \cdots \bar{\mu}_q}(z_j) | \; .$$

Thus, to complete the proof of (12), it suffices finally to show that φ satisfies the inequality

$$\| \varphi \|_0 \leqq \text{constant} \cdot \| \psi \|_{k-1+\alpha}$$

for some constant independent of ψ. Assume the contrary ; then there is a sequence $\{\psi_n\}$ with $\| \psi_n \|_{k-1+\alpha} \to 0$ and for which the corresponding vector forms φ_n satisfy $\| \varphi_n \|_0 = 1$. From (14) it follows that the coefficients of φ_n have uniformly bounded first derivative in the U_j which are also equi-continuous. We may therefore select a subsequence converging together with its first derivatives to a form $\tilde{\varphi}$ with $\|\tilde{\varphi}\|_0 = 1$ which satisfies, as one sees immediately from (13),

$$\bar{\partial} \tilde{\varphi} = 0, \qquad (\tilde{\varphi}, \omega) = 0 \qquad\qquad \text{for all } \omega \in \mathfrak{S}.$$

It follows that $\tilde{\varphi} = 0$ which contradicts the assertion $\| \tilde{\varphi} \|_0 = 1$.

5. Proof of convergence

Consider a formal power series

$$\psi(t) = \sum \psi_{h_1 h_2 \cdots h_m} t_1^{h_1} t_2^{h_2} \cdots t_m^{h_m}$$

with coefficients $\psi_{h_1 h_2 \cdots h_m} \in \Phi^{0,q}$ and a power series

$$a(t) = \sum a_{h_1 h_2 \cdots h_m} t_1^{h_1} t_2^{h_2} \cdots t_m^{h_m}, \qquad\qquad a_{h_1 h_2 \cdots h_m} \geqq 0 \; .$$

We indicate by $\| \psi \|_{k+\alpha}(t) << a(t)$ that

$$\| \psi_{h_1 h_2 \cdots h_m} \|_{k+\alpha} \leqq a_{h_1 h_2 \cdots h_m}.$$

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{s^n}{n^2} \; .$$

Since

$$\sum_{l=1}^{n} \frac{1}{l^2 (n+1-l)^2} < \frac{4\pi^2}{3} \frac{1}{n^2} < \frac{16}{n^2} \, ,$$

we have

$$(15) \qquad f(s)^2 = s\sum_{n=1}^{\infty} s^n \sum_{l=1}^{n} \frac{1}{l^2(n+1-l)^2} << 16\, sf(s)\ .$$

Now we show that the construction of $\varphi(t)$ in § 3 can be carried out in such a way that

$$(16) \qquad \|\varphi\|_{k+\alpha}(t) << (A/B)f(B(t_1+t_2+\cdots+t_m))\ ,$$

where A and B are constants (depending on k, α). There exists a constant c' (depending on k, α) such that

$$(17) \qquad \|[\varphi, \psi]\|_{k-1+\alpha} < c'\|\varphi\|_{k+\alpha}\cdot\|\psi\|_{k+\alpha}$$

for all $\varphi, \psi \in \Phi^{0,1}$. It is clear that $\sigma_1(t) = \varphi_1(t)$ satisfies

$$\|\sigma_1\|_{k+\alpha}(t) << (A/B)f(B(t_1+\cdots+t_m))\ ,$$

provided that $A > 0$ is sufficiently large. We choose B such that

$$(18) \qquad B > 64\, c_{k,\alpha}c'A\ ,$$

where $c_{k,\alpha}$ is the number occurring in (12). Suppose that $\varphi_2(t), \cdots, \varphi_r(t)$ are already determined and that

$$(19) \qquad \|\sigma_r\|_{k+\alpha}(t) << (A/B)f(B(t_1+\cdots+t_m))\ .$$

Then, by (17), we get

$$\|[\sigma_r, \sigma_r]\|_{k-1+\alpha}(t) << c'(A^2/B^2)f^2(B(t_1+\cdots+t_m))\ ,$$

and therefore, by (15),

$$\|[\sigma_r, \sigma_r]\|_{k-1+\alpha}(t) << 16\, c'(A^2/B)(t_1+\cdots+t_m)f(B(t_1+\cdots+t_m)).$$

$\psi_{r+1}(t)$ is composed of all terms of $[\sigma_r(t), \sigma_r(t)]$ of degree $r+1$, as (9) shows. Hence we obtain

$$\|\psi_{r+1}\|_{k-1+\alpha}(t) << \frac{16\, c'A^2B^{r-1}}{r^2}\cdot(t_1+\cdots+t_m)^{r+1}\ .$$

In view of the above lemma (with $q = 1$) the differential equation $\bar{\partial}\varphi_{r+1}(t) = -\psi_{r+1}(t)$ (see (10)) has a unique solution $\varphi_{r+1}(t)$ such that

$$(20) \qquad (\varphi_{r+1}(t), \eta) = 0 \qquad \text{for all } \eta \in Z(\Phi^{0,q})$$

and the solution $\varphi_{r+1}(t)$ satisfies

$$\|\varphi_{r+1}\|_{k+\alpha}(t) << c_{k,\alpha}\cdot\|\psi_{r+1}\|_{k-1+\alpha}(t)\ .$$

Hence, by (18),

$$\| \varphi_{r+1} \|_{k+\alpha}(t) << \frac{AB^r}{4r^2} \cdot (t_1 + \cdots + t_m)^{r+1} . \tag{21}$$

On the other hand, $\sigma_r(t)$ is a polynomial of degree r and the term of degree $r+1$ of $(A/B)f(B(t_1 + \cdots + t_m))$ has the form

$$\frac{AB^r}{(r+1)^2} \cdot (t_1 + \cdots + t_m)^{r+1} .$$

Consequently we infer from (19) and (21) that $\sigma_{r+1}(t) = \sigma_r(t) + \varphi_{r+1}(t)$ satisfies

$$\| \sigma_{r+1} \|_{k+\alpha}(t) >> (A/B)f(B(t_1 + \cdots + t_m)) .$$

This completes our inductive proof of (16).

6. Proof of the theorem

We fix $k \geqq 2$ and α, $0 < \alpha < 1$. It follows from (16), that, if $\varepsilon_0 > 0$ is sufficiently small, the series

$$\varphi(t) = \varphi_1(t) + \varphi_2(t) + \cdots + \varphi_r(t) + \cdots$$

converges in $\| \quad \|_{k+\alpha}$ for $|t| < \varepsilon_0$ and therefore each component $\varphi^{\beta}_{\bar{\mu}}(z, t)$ of $\varphi(t)$ is a differentiable function in z and t of class C^k which is *holomorphic* in t. Since (20) implies $\mathfrak{d}\varphi_{r+1}(t) = 0$, we infer from (3)$'$ that $\varphi(t)$ satisfies the quasi-linear differential equation

$$\sum_{\lambda=1}^{m} \frac{\partial^2}{\partial t_\lambda \partial \bar{t}_\lambda} \varphi(t) + (\mathfrak{d}\bar{\partial} + \bar{\partial}\mathfrak{d})\varphi(t) - \mathfrak{d}[\varphi(t), \varphi(t)] = \bar{\partial}\mathfrak{d}\varphi_1(t) .$$

This equation is *elliptic* for $|t| < \varepsilon^*$, provided that $\varepsilon^* > 0$ is sufficiently small. It follows that *$\varphi^{\beta}_{\bar{\mu}}(z, t)$ is of class C^∞ in z and t, $|t| < \varepsilon^*$,* (see Douglis and Nirenberg [2], Theorem 5).

Consider the system of linear partial differential equations

$$\begin{cases} \bar{\partial}_\mu w - \sum_{\gamma=1}^{n} \varphi^{\gamma}_{\bar{\mu}}(z, t)\partial_\gamma w = 0 , & \mu = 1, 2, \cdots, n, \\ \dfrac{\partial w}{\partial \bar{t}_\lambda} = 0, & \lambda = 1, 2, \cdots, m. \end{cases} \tag{22}$$

We infer from (3)$'$ (or (3)) and

$$\frac{\partial}{\partial \bar{t}_\lambda} \varphi(t) = 0$$

that *the system* (22) *satisfies the required integrability conditions*. Hence, by a theorem of Newlander and Nirenberg [5], for each point $z_j \in V_0$, there exists a neighborhood U_j of z_j on V_0 and $\varepsilon_j > 0$ such that, in the domain: $z \in U_j$, $|t| < \varepsilon_j$, the system of equations (22) has n solutions

$w = \zeta_j^\beta(z, t)$, $\beta = 1, 2, \cdots, n$, of class C^∞ which are independent in the sense that the Jacobian of $\zeta_j^1(z, t), \cdots, \zeta_j^n(z, t), \overline{\zeta_j^1(z, t)}, \cdots, \overline{\zeta_j^n(z, t)}$ with respect to $z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n$ is different from 0. Moreover (22) is equivalent to the system of equations

$$\text{(23)} \qquad \frac{\partial w}{\partial \bar{\zeta}_j^\beta} = 0 \, , \quad \frac{\partial w}{\partial \bar{t}_\lambda} = 0 \, , \; \beta = 1, 2, \cdots, n \, ; \; \lambda = 1, 2, \cdots, m$$

where $\zeta_j^\beta = \zeta_j^\beta(z, t)$. Clearly V_0 is covered by a finite number of U_j, say $U_1, U_2, \cdots, U_N$. Let $\varepsilon = \min_{1 \leq j \leq N} \varepsilon_j$ and let

$$\zeta_i^\beta(z, t) = h_{ij}^\beta(\zeta_j^1(z, t), \cdots, \zeta_j^n(z, t), t) \, , \quad \text{for } z \in U_i \cap U_j, \, |t| < \varepsilon.$$

Then, since (22) is equivalent to (23), $h_{ij}^\beta(\zeta_j^1, \cdots, \zeta_j^n, t)$ are holomorphic functions in $\zeta_j^1, \cdots, \zeta_j^n, t$. Thus, for each t, $|t| < \varepsilon$, the system of local coordinates $(\zeta_j^1(z, t), \cdots, \zeta_j^n(z, t))$, $j = 1, 2, \cdots, N$, defines a complex structure V_t on V_0 and $\{V_t \mid t \in M\}$, $M = \{t \mid |t| < \varepsilon\}$, forms a complex analytic family of deformations of V_0. It is clear that

$$v\varphi(0) = \sum_{\lambda=1}^m v^\lambda \eta_\lambda$$

for $v = \sum v^\lambda (\partial/\partial t_\lambda) \in (\mathrm{T}_M)_0$. This proves that ρ_0 gives the isomorphism: $(T_M)_0 \cong H^1(V_0, \Theta_0)$.

We observe that, although our choice of the successive terms φ_r was made in a special way, the proof would be unaffected by an arbitrary choice of σ_r, for some finite r, satisfying (8).

Institute for Advanced Study and Princeton University
Institute for Advanced Study
Princeton University

References

1. P. Dolbeault, *Sur la cohomologie des variétés analytiques complexes*, C. R. Paris, 236 (1953), 175–177.
2. A. Douglis and L. Nirenberg, "Interior estimates for elliptic systems of partial differential equations," Comm. on Pure and Applied Mathematics, VIII (1955), 503–538.
3. A. Frölicher and A. Nijenhuis, *A theorem on stability of complex structures*, Proc. Nat. Acad. Sci., U. S. A., 43 (1957), 239–241.
4. K. Kodaira, and D. C. Spencer, *On deformations of complex analytic structures*, I, II, Ann. of Math., 67 (1958), 328–466.
5. A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math., 65 (1957), 391–404.