Grothendieck's Dream (Motif)

$X \to M(X) \to H^{\cdot}(X)$: lin.

sm. proj. var. $\mapsto$ pure motive $\mapsto$ pure Hodge str.

general var. $\mapsto$ mixed motive $\mapsto$ mixed Hodge str.

[Pure]

Riemann $\supseteq$ Kä. $\supseteq$ sm. proj.

$$H^{\cdot}_{Harm} \text{ with } (\ ,\ )_{L^2} \xRightarrow[\text{analysis}]{\Delta\,:\,\text{elliptic}} \overset{\text{ortho.}}{\bigoplus} H^{i,j}$$

$$\text{Hodge-deRham S.S.} \xRightarrow[\text{alg.}\,:\,\text{char } p \text{ reduction}]{E_1\text{-degen.}} \bigoplus H^{i,j}$$

(Deligne-Illusie)

sm. proper sch. X/k : char $= 0$

$$H^n_{sing}(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} \cong H^n_{DR}(X;\mathbb{C})$$

$$= \bigoplus H^{p,q}_{Harm} \cong \bigoplus H^{p,q}_{\bar\partial,\,Dol}$$

$\Leftrightarrow$

$\exists\ F^i := \bigoplus_{j\geq i} H^{j,n-j}$ "pure" wg $= n$ H.S. over $\mathbb{C}$

[Mixed]

Frob. auto. $\phi \curvearrowright H^*_{ét}(X_{/\mathbb{F}_q} \otimes \overline{\mathbb{F}}_q ; \mathbb{Q}_\ell)$

$\Rightarrow$ Cor. from Weil Conj.

- Eigenval(ϕ) : $|\text{alg. num.}| = q^{\frac{i}{2}}$ w.r.t. $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$
- X sm. proj. $\rightsquigarrow$ $|\cdot| = q^{\frac{n}{2}}$
- $\exists$ Functorial from geo.

  $W_p := \bigoplus_{i \le p} V_i := \text{Eigspace}(q^{\frac{i}{2}})$

9.1. D	Spec (V)
D^*	Spec (K)
un revêtement universel $\tilde{D}^*$ de D^*	Spec $(\bar{K})$
groupe fondamental $\pi_1(D^*)$	groupe d'inertie I
(avec $\pi_1(D^*) = \mathbb{Z} \simeq \mathbb{Z}(1)_{\mathbb{Z}}$)	(avec $I_l = \mathbb{Z}_l(1)$)
X	schéma projectif X sur Spec (V)
$X^* = f^{-1}(D^*)$	X_K
$\tilde{X} = X \times_D \tilde{D}^*$	$X_{\bar{K}}$
système local $R^i f_* \mathbb{Z} \mid D^*$	module galoisien $H^i(X_{\bar{K}}, \mathbb{Z}_l)$
$H^i(\tilde{X}, \mathbb{Z})$	$H^i(X_{\bar{K}}, \mathbb{Z}_l)$

$\text{Gal}(\bar{k}/k) \cong \text{Gal}(\bar{K}/K)/I$ - mod. $H \overset{\text{unip.}}{\curvearrowleft} I$

$\overset{\text{natural}}{\rightsquigarrow}$ $M : \mathbb{Z}_\ell(1) \otimes W_n(H) \subset W_{n-2}(H)$

Hope

Limiting MHS $\rightsquigarrow$ asymptotic $H^\cdot(X_t)$

Formal definition

Set $\begin{cases} \text{Noe. } R \subset \mathbb{C} \text{ s.t. } R \otimes_{\mathbb{Z}} \mathbb{Q} : \text{field} \\ V_R : \text{finite type } R\text{-mod.} \end{cases}$

(R-) mixed hodge structure (on V_R)

weight fil. $\nearrow$ $W_\bullet$ on $V_R \otimes_R (R \otimes \mathbb{Q})$

Hodge fil. $\searrow$ $F^\bullet$ on $V_{\mathbb{C}} := V_R \otimes_R \mathbb{C}$

inducing pure $(R \otimes \mathbb{Q})$-HS of wg. $= k$

on each $Gr_k^W (V_R \otimes_{\mathbb{Z}} \mathbb{Q}) := W_k / W_{k-1}$

graded-polarizable if polarizable

"Hodge num." $h^{p,q} := \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W V_{\mathbb{C}}$

recorded in Hodge-Euler poly.

$$e_{Hdg}(V) := \sum h^{p,q}(V) u^p v^q$$

morphism $V_R \xrightarrow{f} V_R'$

s.t. $Gr_m^W(f) : Gr_m^W V \hookrightarrow Gr_m^W V'$

Thm. (Strictness : De II 1.3.2.)

TAFE

- $\{E_r^{p,q}\}$ degenerate at $E_1 = E_\infty$
- $\forall p$, $H^i(F^p(K)) \xrightarrow{\sim} F^p H^i(K)$
- $d_0 : K^i \to K^{i+1}$ strict w.r.t. $F^\bullet$

i.e. $d_0(F^n A) = d_0(A) \cap F^n A$, $\forall n$

Deligne's Splitting

$$I^{p,q} := F^p \cap W_{p+q} \cap (\overline{F}^q \cap W_{p+q} + \sum_{j\geq 2} \overline{F}^{q-j+1} \cap W_{p+q-j})$$

s.t.

$$\begin{cases} Gr_k^W = \bigoplus_{p+q=k} I^{p,q} \quad \text{wg} = k \quad \text{pure } \mathbb{R}\text{-HS} \\ W_k = \bigoplus_{p+q\leq k} I^{p,q} \\ F^p = \bigoplus_{r\geq p} I^{r,s} \end{cases}$$

$$I^{p,q} \neq \overline{I^{q,p}} \quad \text{but} \quad I^{p,q} \equiv \overline{I^{q,p}} \ (\mathrm{mod}\ W_{p+q-2})$$

Yoga of weight

Morphism bewteen MHS $f(I^{p,q}) \subset I'^{p,q}$

$\Downarrow$

.... are all

"strict" $\rightsquigarrow \begin{cases} \mathrm{Im} f \cap Gr_m^W V' = f(Gr_m^W V) \\ \mathrm{Im} f \cap Gr_F^n V' \overset{(?)}{=} f(Gr_F^n V) \end{cases}$

i.e. $Gr_m^W V \hookrightarrow Gr_m^W V'$

prop. (Strictness)

$V' \xrightarrow{f} V \xrightarrow{g} V''$ e.q. of MHS

$$\Rightarrow \begin{cases} Gr_k^W V'_{\mathbb{Q}} \to Gr_k^W V_{\mathbb{Q}} \to Gr_k^W V''_{\mathbb{Q}} \\ Gr_F^p V'_{\mathbb{C}} \to Gr_F^p V_{\mathbb{C}} \to Gr_F^p V''_{\mathbb{C}} \\ Gr_F^p Gr_k^W V'_{\mathbb{C}} \to Gr_F^p Gr_k^W V_{\mathbb{C}} \to Gr_F^p Gr_k^W V''_{\mathbb{C}} \end{cases}$$

Moreover Category (MHS) : Abelian

cor.

$0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$ s.e.q. of $\mathbb{Q}$-Vect.

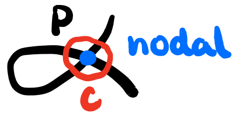
($\exists$ MHS)

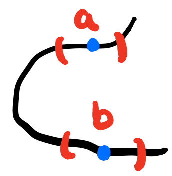
Compatible with F, W

Then (V, F, W) MHS iff f, g strict

Motivating Eg.

(i) Singular but cpt.

X 

$\xleftarrow{\lambda\,:\,\text{normalization}}$ 

$\tilde{X}$

Consider $0 \to \mathbb{Q}_X \to \lambda^* \mathbb{Q}_{\tilde{X}} \to \mathbb{Q}_p \to 0$

$$c \mapsto (c, c)$$
$$(a, b) \mapsto a - b$$

$$\Rightarrow \quad 0 \to \underset{\substack{\| \\ \mathbb{Q}}}{H^0(p, \mathbb{Q})} \to H^1(X, \mathbb{Q}) \to \underset{\substack{\uparrow \\ \text{weight} = 1}}{H^1(\tilde{X}, \mathbb{Q})} \to 0$$

weight $= 0$

Fact

$$\mu : \tilde{X} \to X, \quad E = \mu^{-1} Z_{sing}$$

$$\to H^{n-1}(E) \xrightarrow{\delta} \underline{H^n(X)} \xrightarrow{(\mu^*, i^*)} H^n(\tilde{X}) \oplus H^n(Z) \xrightarrow{\tilde{i}^* - \mu^*} H^n(E)$$

$$\Rightarrow \begin{cases} W_{n-2} = 0 \\ W_{n-1} = \operatorname{Im} \delta = \ker(H^n(X) \to H^n(\tilde{X})) \\ W_n = H^n(X) \end{cases}$$

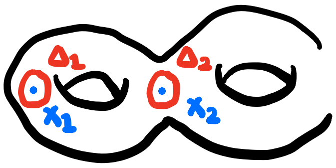
Since

$$\mu^*|_E : H^n(Z) \hookrightarrow H^n(E)$$

$$\begin{array}{ccc} E & \to & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \to & X \end{array}$$

Eg 2.

Compute $X^0 := \underset{\cup\{x_i\}}{X - D} \overset{j}{\hookrightarrow} X = X^0 \cup (\dot{\bigcup} \Delta_i)$



MV seq.

$$0 \to \underset{\mathbb{Q}}{\underset{\parallel}{H^0(X)}} \to \underset{\mathbb{Q}^{r+1}}{\underset{\parallel}{H^0(X^0) \oplus \bigoplus^r H^0(\Delta_i)}} \to \underset{\mathbb{Q}^r}{\underset{\parallel}{\bigoplus^r H^0(\Delta_i^*)}} \; (\to 0)$$

$$0 \to H^1(X) \to H^1(X^0) \to \underset{\mathbb{Q}^r}{\underset{\parallel}{\bigoplus^r}} H^1(\underset{S^1}{\underset{\cong}{\Delta_i^*}}) \to H^2(X) \to H^2(X^0) \to 0$$

wg = 1 ? "wg = 2"

Important observation

(deRham) $\mathbb{C}_X \simeq_{qis.} \Omega^\bullet_X$

(Deligne) $Rj_*\mathbb{C}_{X^0} \simeq_{qis.} \Omega^\bullet_X(\log D)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "id"] \arrow[d] & \mathcal{O}_X \arrow[rr] \arrow[d] & & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_X \arrow[r, hook] & \Omega_X(\log D) \arrow[rr, "Res"] & & \mathcal{O}_D = \mathbb{C}^{\oplus r} \arrow[r] & 0
\end{tikzcd}$$

$$f(z_i)\frac{dz_i}{z_i} \mapsto f|_{z_i=0} = (f(x_1), \ldots, f(x_r))$$

$W_0 = \Omega_X$

$W_1 = \Omega_X(\log D)$

$W_1 / W_0 = \mathcal{O}_D[-1] = \Omega^\bullet_D[-1]$

"s.e.q. of complex"

$$0 \to W_0 \to W_1 \to W_1/W_0 \to 0$$

$$\Downarrow \mathbb{H}^{\cdot}(\cdots)$$

$$\begin{aligned}
0 \to \underset{\overset{\parallel}{H^i(X)}}{}\ \mathbb{H}^0(\Omega_X^{\cdot}) &\to \mathbb{H}^0(\Omega_X^{\cdot}(\log D)) \to \mathbb{H}^0(\Omega_D^{\cdot}[-1]) = 0 \\
\to \mathbb{H}^1(\Omega_X^{\cdot}) &\to \mathbb{H}^1(\Omega_X^{\cdot}(\log D)) \to \mathbb{H}^1(\Omega_D^{\cdot}[-1]) = \mathbb{Q}^{\oplus r} \\
\to \mathbb{H}^2(\Omega_X^{\cdot}) &\to \mathbb{H}^2(\Omega_X^{\cdot}(\log D)) \to \mathbb{H}^2(\Omega_D^{\cdot}[-1]) = 0
\end{aligned}$$

with $\mathbb{H}^i(\Omega_X^{\cdot}) = H^i(X)$ and $\mathbb{H}^i(\Omega_X^{\cdot}(\log D)) = H^i(X^\circ)$.

Hence

$$0 \to H^0(X) \to H^0(X^\circ) \to 0$$

$$0 \to \underbrace{H^1(X)}_{\text{"wg = 1"}} \to \underbrace{H^1(X^\circ)}_{\text{open sm.}} \to \mathbb{Q}^{\oplus r} \xrightarrow{\text{"strict"}} \underbrace{H^2(X)}_{=\,\mathbb{Q}} \to \underbrace{H^2(X^\circ)}_{=\,0} \to 0$$

Ker

"wg = 2"

Then $0 \to H^1(X) \to H^1(X^\circ) \to K \to 0$

$\Downarrow$

"mixed"

"spectral sequence"

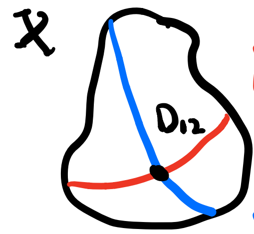
E_1 - page

$$H^q(W_1/W_0) \to H^{q+1}(W_0/W_{-1}) \quad \overset{\text{cov.}}{\Longrightarrow} \quad H^{\cdot}(X^*)$$

$$\begin{array}{ccc} \wr\!\parallel & & \wr\!\parallel \\ H^{q-1}(\Omega_D^{\cdot}) \text{ "}H^{q-1}(D)\text{"} & \to & H^{q+1}(\Omega_X^{\cdot}) \end{array} \quad \text{"wg = q+1"}$$

More general

$$U = X - D_1 - D_2 + D_{12}$$



$D_2 : \{z_2 = 0\}$

$D_1 : \{z_1 = 0\}$

$$\begin{tikzcd}[column sep=small]
 & W_0 & & W_1 \ (\text{order} \le 1) & & W_1/W_0 & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "="] \arrow[d] & & \mathcal{O}_X \arrow[rr] \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_X \arrow[rr, hook] \arrow[d] & & \Omega_X(\log D) \arrow[rr, "\mathrm{Res}_D"] \arrow[d] & & \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_X^2 \arrow[rr, hook] \arrow[d] & & \Omega_X \wedge \Omega_X(\log D) \arrow[rr, "\mathrm{Res}_D"] \arrow[d] & & \Omega_{D_1} \oplus \Omega_{D_2} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & \Omega_X^p \arrow[rr, hook] & & \Omega_X^{p-1} \wedge \Omega_X(\log D) \arrow[rr, "\mathrm{Res}_D"] & & \Omega^{p-1}_{D_1} \oplus \Omega^{p-1}_{D_2} \arrow[r] & 0
\end{tikzcd}$$

$$\alpha \wedge \frac{dz_1}{z_1} + \beta \wedge \frac{dz_2}{z_2} \mapsto (\alpha|_{D_1}, \beta|_{D_2})$$

$$\begin{tikzcd}[column sep=small]
 & W_1 & W_2 & W_2/W_1 & \\
0 \arrow[r] & \Omega_X^{p-2} \wedge \Omega_X(\log D) \arrow[r, hook] & \Omega_X^p(\log D) \arrow[r, "\mathrm{Res}_{D_{12}}"] & \Omega^{p-2}_{D_{12}} \arrow[r] & 0
\end{tikzcd}$$

$$\alpha \wedge \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \mapsto \alpha|_{D_{12}}$$

$\Rightarrow$ " cut pure wg. out of $\Omega^\bullet_X(\log D)$ "

$$W_0 \subseteq W_1 \subseteq W_2 \quad \left\{ \begin{array}{l} W_1/W_0 \cong \Omega^\bullet_{D_1}[-1] \oplus \Omega^\bullet_{D_2}[-1] \\ W_2/W_1 \cong \Omega^\bullet_{D_{12}}[-2] \end{array} \right.$$

$\rightsquigarrow$ E_1 - page

$$H^q(W_2/W_1) \to H^{q+1}(W_0/W_1) \to H^{q+2}(W_0)$$

$$H^{q-1}(W_2/W_1) \to H^q(W_0/W_1) \to H^{q+1}(W_0)$$

$$\|$$

$$H^{q-2}(D_{12}) \to H^q(D_1) \oplus H^q(D_2) \to H^{q+2}(X)$$

$$H^{q-3}(D_{12}) \xrightarrow{(j_1, -j_2)} H^{q-1}(D_1) \oplus H^{q-1}(D_2) \xrightarrow{j_1 - j_2} H^{q+1}(X)$$

open sm.

conv. $\Rightarrow H^\bullet(U)$

E_2 - degen.

Abstraction : Setting "Good Compactification"

sm. open

$$U \overset{\text{Nagata}}{\hookrightarrow} (Z,\ Y := Z - U) \overset{\text{Hironaka}}{\rightsquigarrow} (X,\ \underset{\text{snc}}{D} = \bigcup^{\text{irr}} D_i^{\text{sm}}) \quad \overset{\text{locally}}{=} \{z_1 \cdots z_k = 0\}$$

Define

$$\Omega_X^\bullet(\log D) := \{\text{logarithmic de Rham complex}\} \subseteq j_*\Omega_U^\bullet$$

locally

$$\Omega_X^1(\log D)_p = \mathcal{O}_{X,p}\frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{X,p}\frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p}\,dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{X,p}\,dz_n$$

$$\Omega_X^n(\log D)_p = \bigwedge^n \Omega_X^1(\log D)_p$$

<u>Fact</u>. $\Omega_X^\bullet(\log D) \underset{\sim}{\xrightarrow{\text{q.is.}}} j_*\Omega_U^\bullet$

Remark. $\begin{cases} j_*\Omega^q_{U_{alg}} = \Omega^q_{X_{alg}}(\log D) \\ j_*\Omega^q_{U_{an}} \neq \Omega^q_{X_{an}}(\log D) \end{cases}$

pf sketch.

Check stalk-iso. $\mathcal{H}^\bullet(\ \ )_{p\in D}$

locally "polycylinder"

$$H^\bullet(\Gamma(\Delta^n, j_*\Omega_U^\bullet)) = H^\bullet(\Gamma((\Delta^*)^k \times \Delta^{n-k}, \Omega_U^\bullet))$$

$$\cong H^\bullet((S^1)^k, \mathbb{C})$$
$$= \Lambda^\bullet\left(\left[\frac{dz_1}{z_1}\right], \cdots, \left[\frac{dz_k}{z_k}\right]\right)$$

Consider

$$\Gamma(\Delta, \Omega^q(\log D)) \xrightarrow{\mathrm{res}_1} \Gamma(\Delta, \Omega^{q-1}(\log D))$$
$$\varphi \mapsto \alpha_1$$

s.t. $[\varphi] = \underbrace{[\varphi_0]} + \left[\frac{dz_1}{z_1}\right] \wedge \underbrace{[\alpha_1]}$

"no poles at z_1"

$$\Rightarrow \mathcal{H}^k(\Omega^q(\log D))_p \cong \mathcal{H}^k(j_*\Omega^q_U)_p$$

Moreover

$$H^q(U; \mathbb{C}) = H^q_{DR}(\Omega^\bullet_U) = H^q(\Gamma(U, \Omega^\bullet_U))$$
$$= \mathbb{H}^q(X, Rj_*\mathbb{C}_U)$$
$$= \mathbb{H}^q(X, \Omega^\bullet_X(\log D))$$

Thm. (Deligne II)

$$H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(\log D)) \ (\exists!) \ \text{MHS}$$

$$\begin{cases} W_m \\ F^p \end{cases} H^k(U) = \begin{cases} \operatorname{Im}(\mathbb{H}^k(X, W_{m-k}\Omega^\bullet_X(\log D)) \to H^k(U)) \\ \operatorname{Im}(\mathbb{H}^k(X, F^p\Omega^\bullet_X(\log D)) \to H^k(U)) \end{cases}$$

where

$$W_m\Omega^p_X(\log D) := \begin{cases} 0 & m < 0 \\ \Omega^p_X(\log D) & m \geq p \\ \Omega^{p-m}_X \wedge \Omega^m_X(\log D) & 0 \leq m \leq p \end{cases}$$

(Poincaré) Residue map

Index $I := \{i_1, \dots, i_m\}$ $\qquad \sum_{j \notin I} D_I \cap D_j$

$$\mathrm{res}_I : \Omega^\bullet_X(\log D) \twoheadrightarrow \Omega^\bullet_{D_I}(\log \overset{!!}{\underbrace{D(I)}})[-m]$$

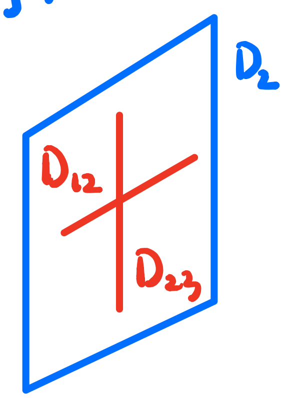
$$\frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_m}}{x_{i_m}} \wedge \eta + \eta' \mapsto \eta \big|_{D(I)}$$

$\Downarrow$ Descend to

$$W_m \Omega^\bullet_X(\log D) \twoheadrightarrow \Omega^\bullet_{D_I}[-m]$$

$$\rightsquigarrow \bigoplus_{|I|=m} \mathrm{res}_I : \mathrm{Gr}^W_m \Omega^\bullet_X(\log D) \xrightarrow{\simeq} a_{m*} \Omega^\bullet_{\underset{\underset{\coprod_{|I|=m} D_I}{\shortparallel}}{D(m)}}[-m]$$

Eg. $(m = 1)$



$I = \{2\} \subset \{1, 2, 3\}$

$D(\{2\}) = D_{21} + D_{23}$

$D(1) = D_1 + D_2 + D_3$

Geometric description

$${}_W\tilde{E}_1^{-m, k+m} = \mathbb{H}^m(X, \mathrm{Gr}^W_k \Omega^\bullet_X(\log D))$$

$$\cong H^{k-m}(D(m); \mathbb{C})$$

Map Level

Thm.

$$\begin{array}{ccc} {}_w\tilde{E}_1^{-m,\,k+m} & \xrightarrow[\cong]{\mathrm{res}_m} & H^{k-m}(\mathbb{D})(m)\,;\,\mathbb{C}) \\ \downarrow d_1 & & \downarrow \; j_m := \oplus (-1)^s \underbrace{(j_s^m)_!} \\ {}_w\tilde{E}_1^{-m+1,\,k+m} & \xrightarrow[\cong]{\mathrm{res}_{m-1}} & H^{k-m+2}(\mathbb{D})(m-1)\,;\,\mathbb{C}) \end{array}$$

Gysin map

$$X^n \xrightarrow{f} Y^m \rightsquigarrow H_k(X) \xrightarrow{f_*} H_k(Y)$$

$$f_! := \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X \qquad k \in [0, 2n]$$

$$H^{-k+2n}(X, \underline{(n)}) \xrightarrow[\text{H.S.}]{} H^{-k+2m}(Y, \underline{(m)})$$

Tate twist

Deep Fact $\quad \mathbb{C} \overset{\text{reduce}}{\rightsquigarrow} \mathbb{Q}$

Goal

${}_wE_2$ - degeneration &

compatibility

Linear Alg.

$f : (K, F) \to (K', F')$

compatible : $f(F^i) \subset F'^i$

strict : $f(F^i) \overset{\text{"no jump"}}{=} \operatorname{Im} f \cap F'^i$

iff

$$\begin{tikzcd}
(K, F) \arrow[r] \arrow[d] & (K', F') \\
(\operatorname{CoIm}(f), F) \arrow[r, "\text{"}\sim\text{"}"] & (\operatorname{Im}(f), F') \arrow[u, hook]
\end{tikzcd}$$

Eg.

$B \hookrightarrow A$ strict $F^\bullet(B) := F^\bullet A \cap B$

Sub - quo. filtr.

$A \twoheadrightarrow A/B$ strict $F^\bullet(A/B) := \dfrac{B + F^\bullet A}{B} \cong \dfrac{F^\bullet A}{B \cap F^\bullet A}$

Lemma. (Deligne) For $C \subset B \subset (A, F)$

Strict comm. diag.

$$\begin{tikzcd}
A/B & & A/C \arrow[ll, two heads, "f"'] & \\
& A \arrow[ul, two heads] \arrow[ur, two heads] & & B/C \arrow[ul, hook] \\
C \arrow[ur, hook] \arrow[rr, hook] & & B \arrow[ul, hook] \arrow[ur, two heads] &
\end{tikzcd}$$

$B/C = \ker f$ $\begin{cases} \text{sub of quo. } \ker \subset A/C \\ \text{quo. of sub. } B \end{cases}$

double filtered Cplx. (K, W, F)

1st direct filtration

$$F_d^p := \mathrm{Im}\left(E_r(F^pK, W) \to E_r(K, W)\right)$$

$E_r(K,W)$ = Spectral seq. w.r.t. (K, W)

2nd direct filtration

$$F_{d*}^p := \mathrm{Ker}\left(\underline{E_r(K, W)} \to E_r(K/F^pK, W)\right)$$

Fact $F_d^p = F_{d*}^p$ on $\begin{cases} E_0^{p,q} = Gr_W^p(K^{p+q}) \\ E_1^{p,q} = H^{p+q}(Gr_p^W K) \end{cases}$

recurrent filtration

$$F_{rec}^p := F_d^p = F_{d*}^p \quad \text{on} \quad E_0^{p,q}$$

Inductively

$$E_{k+1}^{p,q} \twoheadleftarrow \ker d_k \hookrightarrow E_k^{p,q}$$

$$\Rightarrow F_{rec}^p(\underline{\ker d_r} / \dots) \quad \text{where} \quad \underline{F^p E_k \cap \ker}$$

prop.

- $F_d \subset F_{rec} \subset F_{d*}$ on $E_r^{p,q}$
- $F_d \subset F \subset F_{d*}$ on $E_\infty^{p,q}$
- d_r compatible F_d, F_d^*

Thm. (Two filtrations lemma 1.3.16.)

[Cond. $r_0 \in \mathbb{N}_+$]

$\forall\ 1 \le k < r_0$, d_k strict with F_r

$\Rightarrow$

- $r \le r_0$

$0 \to \cdots \to 0$ exact

- $r = r_0 + 1$

$\underline{E_r(F^pK, W) \to E_r(K, W) \to E_r(K/F^pK, W)}$ exact

- $F_r = F_d = F_{d^*}$ on $E_{r \le r_0+1}$

Particularly d_{r_0+1} compatible with F_r

If Cond. $\forall\ r_0$ ✓

$\Rightarrow$

- $F_r = F_d = F_{d^*}$ on E_r

Particularly $\cdots = F$ on $E_\infty^{p,q} = Gr_W^{-p}(H^{p+q}(K))$

$\rightsquigarrow\ Gr_F^p E_r(K, W) \cong E_r(Gr_F^p K, W)$

- $E(K, F)$ degen. at E_1-page.

Main pf. sketch (Induction)

- $d_0 = d$ strict with $F_r = F_d$ at E_0

  (Hodge-deRham Sptr)

- d_1 strict with $F^{\bullet}$ at E_1 (Gysin)

  is Hodge-morphism

- $2 \le k \le r-1$, $d_k = 0$ & $E_k^{p,q} = E_2^{p,q}$

lemma. $\rightsquigarrow$ $F_d = F_{d^*} = F_r$ on $E_{k \le r}^{p,q}$

Particularly d_r compatible with F

$$d_r(E_r^{p,q}) \subset \sum_{\dots} [F^a(E_r^{-p+r,\, q-r+1}) \cap \bar{F}^b(\dots)] = 0$$

lemma. $\Rightarrow$ $\forall\, r \geq 2$, $d_r = 0$ strict with F_r

Particularly

$$\exists\ (F,\ {}_wE_\infty^{p,q} = Gr_W^p H^{p+q}(X;\mathbb{C}))$$

$\square$

Eg.

(i) (NCD) $X = X_1 \cup X_2$, X_i : sm. proj.

$$\xrightarrow{\beta_{k-1}} \underbrace{H^{k-1}(X_1\cap X_2)}_{wt = k-1} \xrightarrow{\delta} \underbrace{H^k(X)}_{} \to \underbrace{H^k(X_1)\oplus H^k(X_2)}_{wt=k} \xrightarrow{\beta_k} \underbrace{H^k(X_1\cap X_2)}_{wt=k} \to$$

$$\begin{cases} gr^W_k = \ker \beta_k \\ gr^W_{k-1} = \operatorname{Im} \delta = \operatorname{coker} \beta_{k-1} \end{cases}$$

(ii) (Open)

sm. Div$(X) \ni D \overset{\text{closed}}{\subset} X$: sm. proj. $\overset{\text{open}}{\underset{i}{\hookleftarrow}} U := X - D$

"Gysin" seq.

$$\cdots \to H^{k-2}(D) \xrightarrow{\gamma_k} H^k(X) \xrightarrow{i^*} H^k(U) \xrightarrow{\text{res}} H^{k-1}(D) \xrightarrow{\gamma_{k+1}} H^{k+1}(X) \to \cdots$$

$$\begin{cases} gr^W_k = \operatorname{Im} i^* = \operatorname{coker} \gamma_k \\ gr^W_{k+1} = \ker \gamma_{k+1} \end{cases}$$

Similarly

$\Rightarrow \cdots \to H^\cdot_c(U) \to H^\cdot(X) \to H^\cdot(D) \to H^{\cdot+1}_c(U) \to \cdots$

Long exact seq. comp. with MHS

Particularly

$$H^i_c(U) = \underbrace{\operatorname{Ker}[H^i(X) \to H^i(D)]}_{wq = i} \oplus \operatorname{Im}[\underbrace{H^{i-1}(X) \to H^{i-1}(D)}_{wq = i-1}]$$

Prop.

i. Functoriality

ii. Indp. of cptification

iii. Künneth : MHS iso.

$$\bigoplus_{p+q=n} H^p(U) \otimes H^q(V) \xrightarrow{\sim} H^{p+q}(U \times V)$$

iv. Cup product compactible with MHS

$$H^p(U) \times H^q(U) \xrightarrow{\cup} H^{p+q}(U)$$

pf. (i)

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, hook] & V \arrow[d, hook] \\ (X, D) \arrow[r, "\tilde{f}"] & (Y, E) \end{tikzcd} \quad \Rightarrow \exists \quad \begin{tikzcd} (\tilde{f}^* \Omega^\bullet_Y(\log E), W, F) \arrow[d] \\ (\Omega^\bullet_X(\log D), W, F) \end{tikzcd}$$

(ii)

$$\begin{tikzcd} & (X_1, D_1) & \\ U \arrow[ur] \arrow[rr, "\text{Resolution } U \xrightarrow{\Delta} U \times U \subset"', "\text{cptify in } X_1 \times X_2"] \arrow[dr] & & (Z, E) \arrow[ul, dashed, "pr_1"'] \arrow[dl, dashed, "pr_2"] \\ & (X_2, D_2) & \end{tikzcd}$$

$$\Rightarrow \mathbb{H}^\bullet(Z ; \Omega^\bullet_Z(\log E)) \xrightarrow{\sim} \mathbb{H}^\bullet(X_i ; \Omega^\bullet_{X_i}(\log D_i))$$

Sum up

Thm. (Deligne II . III)

$H^i(X,\mathbb{Q})$: functorial mixed Hodge with "restrictions on the <u>weight</u>"

(i) weights lies in $[0,2i]$

i.e. $$W_a H^i(X) = \begin{cases} 0 & , \quad a < 0 \\ H^i(X) & , \quad a \geq 2i \end{cases}$$

(ii) smooth not cpt. $\Rightarrow$ lies in $[i,2i]$

$$W_i H^i(X) = \operatorname{Im}(H^i(\bar{X}) \to H^i(X))$$

$\bar{X}$: smooth compactification

(iii) cpt. (complete) $\Rightarrow$ lies in $[0,i]$

$$W_{i-1} H^i(X) = \operatorname{Ker}(H^i(X) \to H^i(\tilde{X}))$$

$\tilde{X}$: resolution of sing.