Motivation

- M cpx. $TM \otimes_{\mathbb{R}} \mathbb{C}$
- Hodge star
- R.S. Hodge decompositions
- Riemannian mfd (X, g). Intuition of the decomposition. $\Omega^k(X)$.

§

- Riemann Surface: $2\text{-}\dim_{\mathbb{R}}$ mfd

$$p \in U \subseteq M \xrightarrow{\varphi_U} \mathbb{C} \supseteq W$$

$\varphi_V \circ \varphi_U^{-1}$ is holomorphic

- Rmk: Can be noncpt. But it doesn't have bdry.

  $\mathbb{C}$, $\mathbb{C}^*$, open subsets $\subseteq \mathbb{C}$

  Complex 1-dim mfd.

- Differential forms

  M: Riemann Surface

As a smooth $\mathbb{R}$-mfd,

$\Omega^k(M;\mathbb{R})$ $\quad \Omega^0(M;\mathbb{R}) = C^\infty(M)$ $\quad \Omega^2(M;\mathbb{R}) = C^\infty(M)\cdot \mathrm{vol}_g$

- Ex: Riemannian Surface is orientable. vol_g

$\boxed{\Omega^1(M;\mathbb{R})}$

As a cpx mfd, $\boxed{\text{we should consider } \mathbb{C}\text{-value functions}}$

$\Rightarrow \Omega^k(M;\mathbb{C})$

$\Omega^0(M;\mathbb{C})$ = cpx valued functions $\quad f: M \to \mathbb{C} \cong \mathbb{R}^2$

$\Omega^2(M;\mathbb{C})$ = cpx-valued func $\cdot$ vol_g

$\boxed{\Omega^1(M;\mathbb{C})\ ?}$

· Rmk: $\Omega^k(M;\mathbb{C}) \cong \Gamma(M, \Lambda^k(T^*M \otimes_{\mathbb{R}} \mathbb{C}))$.

· Lemma: If $T: V \to W$ is $\mathbb{R}$-linear map btw cpx v.s.

then there is a unique decomposition $T = T_1 + T_2$ s.t.

T_1 is $\mathbb{C}$-linear & T_2 is anti-$\mathbb{C}$-linear. i.e.

$$\boxed{T_1 i = i T_1, \quad T_2 i = -i T_2.}$$

pf: $T_1 := \frac{1}{2}(T - iTi)$ $\qquad T_1 i = \frac{1}{2}(Ti + iT)$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad = iT_1$

$T_2 = \frac{1}{2}(T + iTi)$

$\rightsquigarrow \omega \in \Omega^1(M;\mathbb{C})$

$$\omega| = \underset{\substack{\uparrow \\ \mathbb{C}\text{-linear}}}{\omega_1} + \underset{\substack{\uparrow \\ \mathbb{C}\text{-anti linear}}}{\omega_2} \quad \text{locally}$$

$\rightsquigarrow \Omega^1(M;\mathbb{C}) = A^{1,0}(M) \oplus A^{0,1}(M)$ $\qquad T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0^*}M \oplus T^{0,1^*}M$

$A^{1,0} = \Gamma(M, T^{1,0*}M)$ $\quad$ $A^{0,1} = \Gamma(M, T^{0,1*}M)$.

- In particular, $w = df$ with $f \in \Omega^0(M; \mathbb{C})$

$$w = \partial_z f \, dz + \partial_{\bar z} f \, d\bar z$$

If $f \in \mathcal{H}(M) =$ holomorphic, df is $\mathbb{C}$-linear.

$\Rightarrow \partial_{\bar z} f = 0$ $\mathbb{C}$-anti linear vanishes.

- Goal:

Hodge decomposition:

$$\Omega^1_2(M; \mathbb{R}) = E \oplus *E \oplus \mathfrak{h}_2(M; \mathbb{R})$$

exact $\quad$ coclosed $\quad$ Harmonic

Hodge star operator

- Motivation: consider square integrable 1-forms. (L^2-function)

- Pointwise:

V^n, $\langle \cdot, \cdot \rangle$ $\quad$ V is $\mathbb{R}$-v.s.

Orthonormal basis $\quad e_1, e_2, \ldots, e_n$

$\rightsquigarrow \Lambda^j V^n$, $\langle \cdot, \cdot \rangle$ $\quad e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ $\quad i_1 < i_2 < \cdots < i_j$

$\Lambda^n V^n \cong \mathbb{R}$ $\quad e_1 \wedge e_2 \wedge \cdots \wedge e_n = \mathrm{vol}$

$* : \Lambda^j V^n \longrightarrow \Lambda^{n-j} V^n$ $\mathbb{R}$ - linear

s.t. $\forall\ \alpha, \beta \in \Lambda^j V^n$, $\alpha \wedge * \beta = \underbrace{\langle \alpha, \beta \rangle}_{\in\ \mathbb{R}}\ \underbrace{\mathrm{vol}}_{\text{top form}}$

Eg: $V^4, \langle \cdot, \cdot \rangle$ $\quad e_1, e_2, e_3, e_4$

Q: $* e_1 \in \Lambda^{4-1} V = \Lambda^3 V$? $\quad e_1 \wedge e_2 \wedge e_3 \cdots$

$$* e_1 = a_1\, e_1 \wedge e_2 \wedge e_3 + a_2\, e_1 \wedge e_2 \wedge e_4 + a_3\, e_1 \wedge e_3 \wedge e_4$$
$$+ a_4\, e_2 \wedge e_3 \wedge e_4 .$$

$e_1 \wedge * e_1 = \langle e_1, e_1 \rangle \mathrm{vol} = \mathrm{vol} \quad \Rightarrow \quad a_4 = 1.$

$e_4 \wedge * e_1 = \langle e_4, e_1 \rangle \mathrm{vol} = 0 \quad \Rightarrow \quad a_1 = 0.$

$= e_4 \wedge (a_1 e_1 \wedge e_2 \wedge e_3)$ $\quad \vdots$

$\Rightarrow \quad * e_1 = e_2 \wedge e_3 \wedge e_4 .$ $\qquad e_I := e_{i_1} \wedge \cdots \wedge e_{i_j}$

$* e_I = (-1)^{\sigma_{IJ}} e_J$ $\qquad \{ I, J \} = \{ 1, \cdots, n \}$

· Rmk: $\alpha \wedge * \alpha = \underbrace{\boxed{\langle \alpha, \alpha \rangle}}_{|\alpha|^2} \mathrm{vol}$

Globally, (M, g) $\quad g$: Riemannian metric $\quad$ oriented $\quad$ cpt.

$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ s.t.

$\forall\ \alpha, \beta \in \Omega^k(M) \qquad \alpha \wedge * \beta = \langle \alpha, \beta \rangle_g \mathrm{vol}_g .$

$\Downarrow$ Exercise

Back to Riemann Surface

Def: To every $w \in \Omega^1(M;\mathbb{C})$ we associate a 1 form $*w$ defined as follow:

$dz = dx + i dy$

$d\bar{z} = dx - i dy$

if $w = u\,dz + v\,d\bar{z} = f\,dx + g\,dy$,

then $*w = f\,dy - g\,dx = -iu\,dz + iv\,d\bar{z}$. ($*$ is $\mathbb{C}$-linear).

Moreover, if $w, \eta \in \Omega^1_{comp}(M;\mathbb{C})$, we define an inner product (Hermitian)

$$\langle w, \eta\rangle := \int_M w \wedge \overline{*\eta}.$$

If $w = u\,dz + v\,d\bar{z}$, $\eta = r\,dz + s\,d\bar{z}$,

$$w \wedge \overline{*\eta} = (u\,dz + v\,d\bar{z}) \wedge \overline{(-ir\,dz + is\,d\bar{z})}$$

$$= (u\,dz + v\,d\bar{z}) \wedge (i\bar{r}\,d\bar{z} - i\bar{s}\,dz)$$

$$= i(u\bar{r} + v\bar{s})\,dz \wedge d\bar{z} = 2(u\bar{r} + v\bar{s})\,dx \wedge dy$$

In particular, $\eta = w$.

$$\langle w, w\rangle = \int_M w \wedge \overline{*w} = 2\int_M (|u|^2 + |v|^2)\,dx \wedge dy$$

square integrate.

$\Omega^1_{\boxed{comp}}(M;\mathbb{C})$ Abstract completion denoted by $\Omega^1_2(M)$.

smooth

It contains all 1-forms that locally have L^2-coefficient,

$*$ $\boxed{\text{need not smooth}}$

and globally $\|w\|^2 = \langle w, w\rangle < +\infty$.

· Lemma: For any $\omega, \eta \in \Omega^1_2(M)$, we have

(1) $* \bar{\omega} = \overline{*\omega}$ (2) $\underline{** \omega = -\omega}$ (3) $\underline{\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle}$

Automorphism on $\Omega^1_2(M)$ Isometry. ☐

· Def: We say that $f \in \Omega^0(M;\mathbb{C})$ is harmonic if f is harmonic in every chart i.e. $\Delta f = \partial_z \partial_{\bar{z}} f = 0$.

$\omega \in \Omega^1(M;\mathbb{C})$ is $\underline{\text{harmonic}}$ if $d\omega = 0 = d*\omega$.

$\updownarrow$ ($d^* = \pm * d *$)

(locally, $\omega = dg$, g is Harmonic)

We denote the set of all harmonic forms by $\mathfrak{h}(M;\mathbb{C})$ or $\mathfrak{h}(M;\mathbb{R})$

$$\Delta = d^*d + dd^*$$

· Thm: Let $\mathfrak{h}_2(M;\mathbb{R}) := \mathfrak{h}(M;\mathbb{R}) \cap \Omega^1_2(M;\mathbb{R})$.

$$E = \overline{\{df \mid f \in \Omega^0_{comp}(M;\mathbb{R})\}} \quad (dA^0)$$

$$*E := \overline{\{*df \mid f \in \Omega^0_{comp}(M;\mathbb{R})\}} \quad (d^* A^2)$$

then $\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus \mathfrak{h}_2(M;\mathbb{R})$.

$E \cap \Omega^1_{comp}(M;\mathbb{R})$ is exact

$(\mathfrak{h}_2(M;\mathbb{R}) \oplus E) \cap \Omega^1_{comp}(M;\mathbb{R})$ is closed.

$$A^{1,0}(M) = \partial A^{0,0} \oplus \partial^* A^{2,0} \oplus \mathfrak{h}^{1,0}_{\partial} \qquad \Delta_\partial = \partial\partial^* + \partial^*\partial.$$

· Intuition

$\Rightarrow$ Δ is discrete spectrum

(X, g) Riemannian mfd, $\boxed{\text{cpt}}$, oriented.

T^*X $\quad dx_1, \dots, dx_n$ orthonormal frame

$\Lambda^h(X)$ $\quad dx_{i_1} \wedge \dots \wedge dx_{i_h}$ inner product.

d^* adjoint to d.

$\alpha \in \Lambda^h(X)$ $\quad \beta \in \Lambda^{h+1}(X)$

$$g(d\alpha, \beta) \overset{\text{formal}}{=} g(\alpha, d^*\beta)$$

· Lemma: $d^* = \pm * d *$. $\square$.

$\Delta = dd^* + d^*d$ Laplace operator. $\circlearrowright \Lambda^h(X)$.

· Observation: Δ is $\boxed{\text{self-adjoint}}$ & $\boxed{\text{semi-positive definite}}$.

pf: $g(\Delta\alpha, \beta) = g(\alpha, \Delta\beta)$

$$= g(dd^*\alpha, \beta) + g(d^*d\alpha, \beta) = g(\alpha, dd^*\beta) + g(\alpha, d^*d\beta)$$

$$g(\Delta\alpha, \alpha) = g(dd^*\alpha, \alpha) + g(d^*d\alpha, \alpha)$$

$$= g(d^*\alpha, d^*\alpha) + g(d\alpha, d\alpha) \geq 0. \quad \text{semipositive def.}$$

Furthermore, $\Delta\alpha = 0 \iff d^*\alpha = 0$ & $d\alpha = 0$

$\Rightarrow$ $\operatorname{Ker}(\Delta) = \operatorname{Ker}(d) \cap \operatorname{Ker}(d^*)$

Harmonic = closed & coclosed.

If ignore the technical issue from inf dim space,

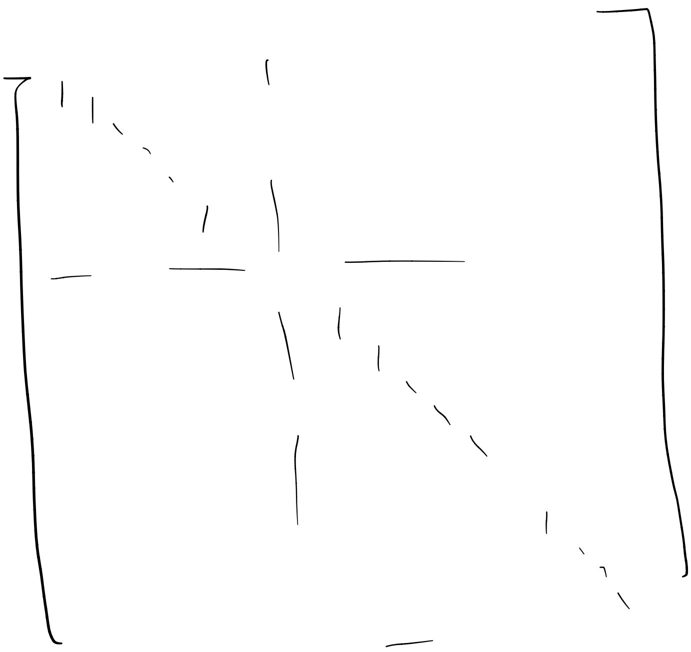
Eigenvalue decomposition

$$\Delta = \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & \lambda_1 & & & \\ & & & & & \lambda_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \\ & & & & & & & & \ddots \end{bmatrix}$$

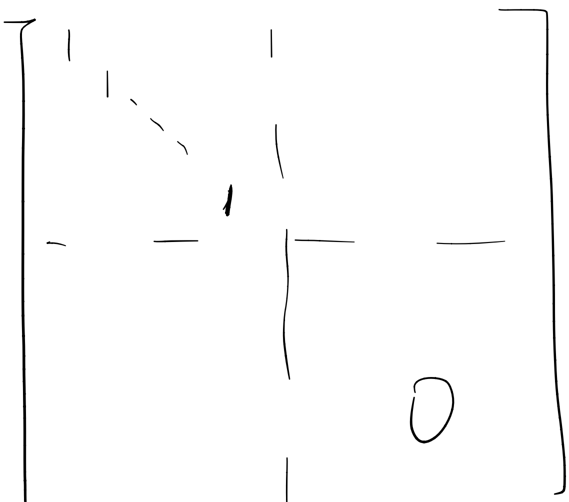
$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots$

Id on $\Omega^k(X)$?

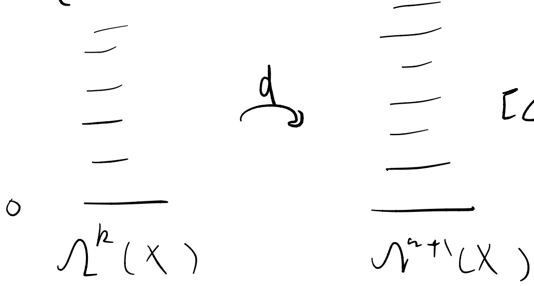
$\mathcal{H}^k$ Harmonic-forms



=



+



$[\Delta, d] = 0$

$\Delta\alpha = \lambda\alpha$

$d(\Delta\alpha) = d\lambda\alpha \qquad d(dd^*\alpha) + \boxed{d(d^*d\alpha)}$

Technical issue (existence of G)

$$\left[\begin{array}{ccc|cccc} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ \hline & & & \lambda_1^{-1} & & & \\ & & & & \lambda_2^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n^{-1} \\ & & & & & & & \ddots \end{array}\right] \rightsquigarrow \text{operator } G$$

Green operator

$$\Rightarrow \left[\begin{array}{c|cccc} 0 & & & & \\ \hline & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{array}\right] = \Delta G$$

$$\left[\begin{array}{c|cccc} 0 & & & & \\ \hline & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_n \\ & & & & & \ddots \end{array}\right] \left[\begin{array}{c|cccc} 0 & & & & \\ \hline & \lambda_1^{-1} & & & \\ & & \lambda_2^{-1} & & \\ & & & \ddots & \\ & & & & \lambda_n^{-1} \\ & & & & & \ddots \end{array}\right]$$

$$\xRightarrow{\text{Hodge thm}} \quad \Omega^k(X) = \underbrace{\mathcal{H}^k(X)}_{\text{Harmonic}} \oplus \Delta G(\Omega^k(X))$$

$$= \mathcal{H}^k(X) \oplus \underbrace{d(d^* G)}_{\operatorname{Im} d} \oplus \underbrace{d^*(d G)}_{\operatorname{Im} d^*} . \qquad \square .$$

Rmk: $A^{p,q}(X)$ $\quad \Delta_\partial = \partial\partial^* + \partial^*\partial$ self-adjoint

semi-positive def.

· Review

$$\Omega^1_{cpt}(M;\mathbb{R}) \xrightarrow{comp} \Omega^1_2(M;\mathbb{R}) \qquad L^2\text{- coefficient differential forms.}$$

· Hope: $\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus \mathcal{H}_2(M;\mathbb{R})$

$$E := \overline{\{ df \mid f \in \Omega^0_{comp}(M;\mathbb{R})\}} \qquad *E := \overline{\{ *df \mid f \in \Omega^0_{comp}(M;\mathbb{R})\}}$$

$$\mathcal{H}_2(M;\mathbb{R}) := \mathcal{H}(M;\mathbb{R}) \cap \Omega^1_2(M;\mathbb{R}) \qquad \text{smooth.}$$

Lemma: Let $\alpha \in \Omega^1_2(M;\mathbb{R})$ be a smooth 1-form.

Then $\alpha \in E^\perp$ iff $d*\alpha = 0$ $\quad [E^\perp \cap \Omega^1(M;\mathbb{R}) \subseteq \operatorname{Ker}(d^*)]$

$\alpha \in (*E)^\perp$ iff $d\alpha = 0$ $\quad [(*E)^\perp \cap \Omega^1(M;\mathbb{R}) \subseteq \operatorname{Ker}(d)]$

Furthermore, $E \subset (*E)^\perp$, & $(*E) \subset E^\perp$.

$$\Rightarrow \quad \Omega^1_2(M;\mathbb{R}) = E \oplus (*E) \oplus (E^\perp) \cap (*E)^\perp$$

It remains to show $(E^\perp) \cap (*E)^\perp = \mathcal{H}_2(M;\mathbb{R})$.

① smooth ② Harmonic.

Last time:

$\forall \ w = u\,dz + v\,d\bar{z} \in E^\perp \cap (*E)^\perp$

$$\langle w, df \rangle = 0 \quad \Rightarrow \quad \cdot \ 2\int (u f_{\bar z} + v f_{\bar z})\, dx\,dy = 0$$

$$\langle w, *df \rangle = 0 \qquad \cdot \ -2i\int (u f_{\bar z} - v f_{\bar z})\, dx\,dy = 0$$

$$\Rightarrow \quad \int u f_{\bar z}\, dx\,dy = 0 \quad \& \quad \int v f_{\bar z}\, dx\,dy = 0$$

$$\Downarrow f := g_{\bar z} \qquad\qquad \Downarrow f := g_z$$

$$\int u \,\Delta g\, dx\,dy = 0 \qquad\qquad \int v\, \Delta g\, dx\,dy = 0$$

Def: u is weakly harmonic i.e.

$$\int u\, \Delta\phi\, dx\,dy = 0 \qquad \forall \phi \in C^\infty_{comp}(M).$$

[

$$\int u\cdot(-)\, dx\,dy : L^2(M) \longrightarrow \mathbb{R}$$

satisfies $\int u(\Delta\phi)\, dx\,dy = 0$]

$$\int u\cdot(-)\, dx\,dy = \langle \alpha, - \rangle \qquad \alpha \in L^2(M)$$

$$\langle \alpha, \Delta\phi \rangle = 0 \qquad \forall \phi \in C^\infty_{comp}(M)$$

$$\overset{\text{if smooth}}{\Longleftrightarrow} \langle \Delta\alpha, \phi \rangle = 0 \quad \overset{\text{if smooth}}{\Longleftrightarrow} \quad \Delta\alpha = 0$$

· Weyl lemma: Let $V \subset \mathbb{C}$ be open & $u \in L^1_{comp}(V)$.

Weakly Harmonic = Harmonic

Suppose u is weakly Harmonic i.e. $\int_V u\,\Delta\phi\, dx\,dy = 0$

$$\forall \phi \in \Lambda^\infty_{comp}(V).$$

Then u is smooth & $\Delta u = 0$.

Step 1:

Suppose $\{u_n\}_{n=1}^{\infty} \subset C^{\infty}(V)$ $\Delta u_n = 0$ that converges to u_∞, in the sense of L^1. Then $u_\infty \in C^{\infty}(V)$ & u_∞ is Harmonic.

· Key: Mean-value property characterizes the Harmonic functions.

· Recall that:

Mean value: $u_n(z) = \frac{1}{2\pi r} \int_{|z-\xi|=r} u_n(\xi)\, d\xi$

as $n \to \infty$ $\downarrow$ $\downarrow$

$u_\infty(z) = \frac{1}{2\pi r} \int_{|z-\xi|=r} u_\infty(\xi)\, d\xi$

Thus, u_∞ has mean-value property.

PDE: $\begin{cases} \Delta \tilde{u} = 0 & \text{in } \mathbb{D} := \{z \mid |z-\xi| = r\} \subset V. \\ \tilde{u}|_{\partial \mathbb{D}} = u_\infty|_{\partial D} \end{cases}$

$\Longrightarrow$ This PDE has a solution $\tilde{u}$.

Claim: $\tilde{u} = u_\infty$.

pf: Mean-value property $\Rightarrow$ Maximum principle.

$\tilde{u} - u_\infty$ satisfies mean-value property.

& $\tilde{u} - u_\infty|_{\partial D} = 0$ $\xRightarrow{\text{Maximum}}$ $\tilde{u} = u_\infty$

$\Rightarrow$ u_∞ is Harmonic.

Step 2: $u \in L^1_{comp}(V)$ weakly Harmonic.

Hope to find $u_n \longrightarrow u$ & $\Delta u_n = 0$.

$$V_n := \{ z \in V \mid \operatorname{dist}(z, \partial V) > 1/n \}$$

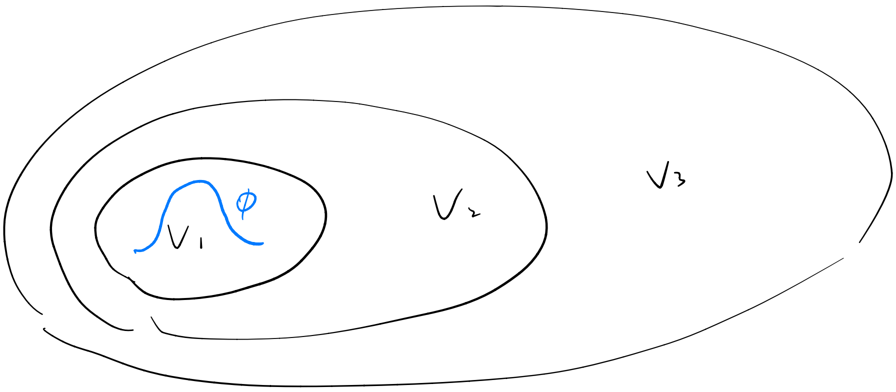
$$V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n \subset \cdots \subset V$$

Then set $\boxed{u_n(z) := (u * \phi_n)(z)}$ $\forall\, z \in V_n$

where $\phi_n(\zeta) := n^2 \phi(n\zeta)$ for $n \geq 1$ with $\phi \geq 0$ a C^∞.

$\operatorname{supp}(\varphi) \subseteq \mathbb{D} \subset V_1$ & $\int \phi = 1$.

$\operatorname{Supp}(\phi_n) \subseteq \mathbb{D}_n \subset V_n$



$$u_n(z) = \int_{\mathbb{R}^2} u(z-\zeta)\,\phi_n(\zeta)\, d\zeta$$
$$= -\int_{\mathbb{R}^2} u(\eta)\,\phi_n(z-\eta)\, d\eta$$

$\Delta u_n(z) = -\int_{\mathbb{R}^2} u(\eta)\, \Delta\phi_n(z-\eta)\, d\eta = 0$ since u is weakly Harmonic

$\xRightarrow{\text{Lemma 3.3}}$ $u_n(z) \longrightarrow u(z)$ L^1_{comp} converge

$\square$.

So weakly Harmonic is Harmonic $\Rightarrow$ $(*E)^\perp \cap E^\perp = \mathcal{H}_2(M;\mathbb{R})$.

· Thm: $\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus \mathcal{H}_2(M;\mathbb{R})$!

Q: $(E \cap \Omega^1(M;\mathbb{R}))$. And want to describe exact forms in $\Omega^1(M;\mathbb{R})$.

Is this exact? ✓

$\mathcal{H}_2(M;\mathbb{R})$ is finite dim $\Leftarrow$ Δ^{-1} is compact operator.

· Cor: The following criteria for exactness hold

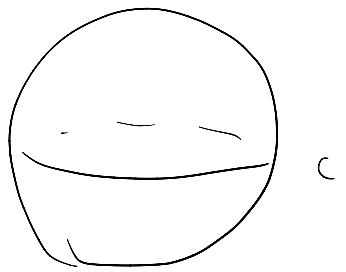
(1) Let $\alpha \in \Omega^1(M;\mathbb{R})$. Then α is exact iff $\langle \alpha, \beta \rangle = 0$ for all coclosed $\beta \in \Omega^1_{comp}(M;\mathbb{R})$.

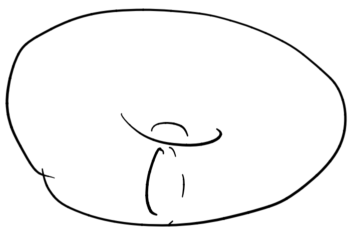
(2) Let $\alpha \in E$ be smooth. Then α is exact.

(i.e. $E \cap \Omega^1(M;\mathbb{R}) = \{ df \mid f \in C^\infty_{comp}(M) \}$.)

Recall: $E := \overline{\{ df \mid f \in C^\infty_{comp}(M) \}}$

(3) Suppose the closed loop c doesn't separate M i.e. $M \setminus c([0,1])$ is connected. Then there exists a closed form $\alpha \in \Omega^1(M;\mathbb{R})$ which is not exact. In particular, $\mathcal{H}_2(M;\mathbb{R}) \neq 0$.





Q: Can we find α which is Harmonic on $M \setminus \{p\}$?

$\Downarrow$