Group action and Quotient space

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1 group action

Definition 1.1. *An (left) action of a group G on a set X is a map*

$$\rho : G \times X \longrightarrow X,$$

$$(g, x) \longmapsto g.x$$

which satisfies the following conditions:
(1) $e.x = x$, for all $x \in X$,
(2) $g_1.(g_2.x) = (g_1g_2).x$, for all $x \in X, g_1, g_2 \in G$.

For $x \in X$, define the orbit of x as

$$O(x) := Orb(x) = \{g.x | x \in X\} =: G.x,$$

and the stablizer (or isotropy group) of x is defined by

$$G_x := Stab(x) = \{g \in G | g.x = x\}$$

The orbit space is defined by

$$X/G := \{O(x) | x \in X\}.$$

Remark: the definition of group action is equivalent to the statment:

$$\bar{\rho} : G \longrightarrow Aut(X)$$

is a group homomorphism.

Moreover,if we add some other topological (or smooth, algebraic) conditions on G and X, and the group action ρ adds additional continuous (or smooth, algebraic) conditions. Then the homomorphism $\bar{\rho}$ of equivalent statement also needs to add additional continuous (or smooth, algebraic) conditions. Typical examples are:

(1) G is a topological group, X is a topological space, and the group action ρ is continuous. Then $\bar{\rho}$ is a continuous group homomorphism, i.e. continuous map + group homomorphism.

(2) G is a Lie group, X is a smooth manifold, and the group action ρ is smooth. Then $\bar{\rho}$ is a Lie group homomorphism, i.e. smooth map + group homomorphism.

(3) G is a complex Lie group (or discrete group), X is a complex manifold (e.g. Riemann surface), and the group action ρ is holomorphic. Then $\bar{\rho}$ is a complex Lie group homomorphism, i.e. holomorphic map + group homomorphism.

(4) G is an algebraic group over an algebraically closed field k, X is an algebraic variety over k, and the group action ρ is regular. Then $\bar{\rho}$ is an 'algebraic group homomorphism' , i.e. regular map (morphism) + group homomorphism.

Remark:

1. If X is a topological space, then $Aut(X)$ means the homoemorphism group $Homeo(X)$, in general, we can't expect that it is a topological group, but when X is compact, then $Homeo(X)$ is a topological group with the compact-open topology [Are46].

2. If X is a smooth manifold, then $Aut(X)$ means the diffeorphism group $Diff(X)$, in general, we can't expect that it is a Lie group, but when X is a compact smooth manifold, then $Diff(X)$ is a Lie group [Les67].

3. If X is a complex manifold, then $Aut(X)$ means the biholomorphism group $BiHol(X)$, in general, we can't expect that it is a complex Lie group, but when X is a compact complex manifold, then $BiHol(X)$ is a complex Lie group [Dea47].

4. If X is an algebraic variety over a field k, then $Aut(X)$ means the biregular automorphism group , in general, we can't expect that $Aut(X)$ is an algebraic group over k, but when X is a complete variety, then $Aut(X)$ is a algebraic group scheme [Fra67](i.e. corresponding functor is locally representable), if the characteristic of k is 0, then the connected component of the unit of this scheme is a variety.

In particularly, if X is a complete variety over $\mathbb{C}$, Then $Aut(X) \cong Bihol(X)$.

In lower dimension, if X is a smooth, complete algebraic curve of genus $g > 1$, then the group $Aut(X)$ is finite. And the automorphism group $Aut(S)$ of a complete algebraic surface S over k is the group of $k-$points of a certain group scheme, the connected component $Aut^0(S)$ of which is an algebraic group.

Besides, the automorphism group of algebraic varieties (over k)

with an ample canonical or anti-canonical invertible sheaf is an algebraic subgroup of the group $PGL(N, k)$ for some N. For a more general case, refer to [Mat58].

Question 1.1. *Whether can we endow X/G with a topological (resp. smooth manifold or Riemann surface, variety) structure such that the orbit map $\pi : X \longrightarrow X/G$ is a surjective continous (resp. smooth, holomorphic, or regular) map ?*

2 Quotient space

2.1 Step 1 :quotient topology

X/G can be endowed with the quotient topology: U is open in X/G if and only if $\pi^{-1}(U)$ is open in X. Then the orbit map $\pi : X \longrightarrow X/G$ is a surjective continous map. In this case, π is an open map.

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a continuous surjective open map. Then Y is a Hausdorff space if and only if*

$$R := \{(x_1, x_2)|f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

Proof. By the knowledge of basic topology, we know Y is a Hausdorff space if and only if the diagonal $\triangle_Y := \{(y, y) \in Y \times Y|y \in Y\}$ is closed in $Y \times Y$, it suffices to show the statement:

$\triangle_Y := \{(y, y) \in Y \times Y|y \in Y\}$ is closed in $Y \times Y$ if and only if

$$R := \{(x_1, x_2)|f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

Denote the map

$$\phi : X \times X \longrightarrow Y \times Y,$$

$$(x_1, x_2) \longmapsto (f(x_1), f(x_2)).$$

$\Longleftarrow$ Since ϕ is continuous and $\phi^{-1}(\triangle_Y) = R$.

$\Longrightarrow$ For point $(z_1, z_2) \in Y \times Y - \triangle_Y$, since f is surjective, then there exist $x_1, x_2 \in X$, such that $f(x_1) = z_1, f(x_2) = z_2$. But $(x_1, x_2) \notin R$, then there exist open neighborhoods V_{x_1}, V_{x_2} of x_1, x_2 respectively such that $V_{x_1} \times V_{x_2} \notin R$, due to the closedness of R .

Thus, $f(V_{x_1}) \times f(V_{x_2})$ is open in $Y \times Y - \triangle_Y$ and $f(V_{x_1}), f(V_{x_2})$ are open subsets of Y because f is an open map.

$\square$

Proposition 2.2. *If G is a compact group and X a Hausdorff space, then X/G is a Hausdorff space.*

Proof. consider action $t : (g, (a, b)) \mapsto (a, gb)$ of G on $X \times X$, since the diagonal $D \subset X \times X$ is closed, then

Claim: $GD := \{g.z | g \in G, z \in D\} = R \subset X \times X$ is closed.

It suffices to show that the complement of GD on $X \times X$ is open, choose $y \notin GD$, take $g \in G$, since the action t is continuous and $X \times X - gD$ is open , then there exist open nbhds V_g of $e \in G$ and W_g of y such that $t(V_g \times W_g) = V_g W_g \subset X \times X - gD$. And $W_g \cap V_g^{-1} gD = \varnothing$. But G is compact, then we can choose a finite number of elements $g_1, g_2, \cdots, g_n \in G$ such that

$$G \subset \bigcup_{i=1}^{n} V_{g_i}^{-1} g_i$$

Set $W = \bigcap_{i=1}^{n} W_{g_i}$, then W is an open nbhd of y. Moreover, $W \cup GD = \varnothing$.

$\square$

Proposition 2.3. *Let G be a topological group, and H be its subgroup, then G/H is a Hausdorff space if and only if H is closed in G.*

2.2 Step 2: proper action

Definition 2.1. *Let X, Y be two topological spaces, let $f : X \longrightarrow Y$ be a continuous map, then f is called proper if f is a continuous closed map and the preimage of every point in Y is compact.*

Remark:

1. If X is Hausdorff, Y is locally compact and Hausdorff, then the following are equivalent [Nic66, I.10]:

(I) $f : X \longrightarrow Y$ is proper.

(II) The preimage of every compact set Y is compact in X.

(III) (universally closed) For each topological space Z, the map $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is closed .

(IV) For any continuous map $g : Z \longrightarrow Y$ the pullback $X \times_Y Z \longrightarrow Z$ be closed, as follows from the fact that $X \times_Y Z$ is a closed subspace of $X \times Z$, where $X \times_Y Z := \{(x, y)|f(x) = g(y)\}$.

2. Every continuous map from a compact space to a Hausdorff space is both proper and closed.

3. A topological space is compact if and only if the map from that space to a single point is proper.

4. In algebraic geometry, the analogous concept is called a proper morphism. (seperated + of finite type +univerally closed), and a seperated scheme of finite type (such as a variety) over $\mathbb{C}$ is proper over $\mathbb{C}$ if and only if the space $X(\mathbb{C})$ of complex points with the classical (Euclean) topology is compact and Hausdorff. And $f : X \longrightarrow Y$ is a proper morphism of varieties (over $\mathbb{C}$) if and only if the continuous map $f : X(\mathbb{C}) \longrightarrow Y(\mathbb{C})$ is a proper map [eeae71, SGA 1, XII Proposition 3.2].

· It is not easy to endow X/G with a smooth manifold structure, we need to add some other condition to the group action [Die87]:

Definition 2.2. *An action*

$$\rho : G \times X \longrightarrow X,$$

$$(g, x) \longmapsto g.x$$

of the topological group G on the topological space X is called proper if the associated map

$$\theta = \theta_\rho : G \times X \longrightarrow X \times X,$$

$$(g, x) \longmapsto (x, g.x)$$

is proper.

Example 2.4. *1. $G = \mathbb{R}, X = \mathbb{S}^1$.*

$$\rho : G \times X \longrightarrow X,$$

$$(n, z) \longmapsto e^{2\pi i n\lambda} z,$$

where λ is an irrational number. Consider

$$\theta = \theta_\rho : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1,$$

since we choose a closed subset $\mathbb{Z}$ of $\mathbb{R}$ and a point $1 \in \mathbb{S}^1$, then $\theta(\mathbb{Z} \times \{1\})$ is dense in $\{1\} \times \mathbb{S}^1$, which is not closed in $\mathbb{S}^1 \times \mathbb{S}^1$. So This action is not a proper action.

2. Let G be a Lie group (or locally compact Hausdorff topological group), let H be a closed subgroup of G, then the action

$$\theta = \theta_\rho : H \times G \longrightarrow G \times G,$$
$$(h, g) \longmapsto (g, h.g := gh^{-1})$$

is proper. Moreover, this action is free.

Proposition 2.5. *Given a proper action of G on X. Then X/G is a Hausdorff space.*

Proof. Let C be the image of θ, let $f : X \longrightarrow X/G$ be the quotient map, which is a surjective open map, and $C = \{(x_1, x_2)|f(x_1) = f(x_2)\}$ is closed in $X \times X$, by lemma 2.1, X/G is Hausdorff. □

Proposition 2.6. *Let G act properly on X. For each $x \in X$ the following holds:*

(1) $\omega : G \longrightarrow X$, $g \longmapsto g.x$ is proper.

(2) The isotropy group G_x is compact.

(3) The map $\bar{\omega} : G/G_x \longrightarrow G.x$ induced by ω is a homeomorphism.

(4) The orbit $G.x$ is closed in X.

Proof. (1) Since $G \cong G \times \{x\}$, every closed subset $E \subset G$, $\omega(E) \cong \{x\} \times \omega(E) \cong \theta(\{x\} \times E)$ is closed in $\{x\} \times X \cong X$, and for every point $y \in X$, $\omega^{-1}(y) \cong \theta^{-1}((x, y)) \subset G \times \{x\}$ is compact on $G \times X$, but $G \times \{x\}$ is closed on $G \times X$, then $\omega^{-1}(y)$ is compact on G.

(2) $G_x = \omega^{-1}(x)$ is compact by (1).

(3) $G \longrightarrow G/G_x \longrightarrow G.x$,since $\pi : G \longrightarrow G/G_x$ is surjective , and $\bar{\omega} \circ \pi = \omega$ is proper, it suffices to show ω is proper (actually closed enough). For any closed subset $E \subset G/G_x$, then $\omega(\pi^{-1}(E)) = \bar{\omega}(E)$ is closed in $G.x$.

closed+continuous+bijective $\Rightarrow$ homoemorphism.

(4) by(1).

□

Proposition 2.7 (Equivalent Formulations of compactness)**.** *Suppose X is a metrizable space. The following are equivalent [Mun14, page 179]:*

(a) X is compact.

(b) Every infinite subset of X has a limit point in X.

(c) Every sequence in X has a convergent subsequence in X.

Remark: If X is just a Hausdorff space, then X is compact if and only if every net in X has a convergent subnet [J.L55, page 136].

The next result is an important characterization of proper actions for locally compact groups.

Proposition 2.8. *[Die87, page 28] Let the locally compact Hausdorff group G act on the Hausdorff space X. Then G acts properly if and only if the following holds:*

For each pair x, y of points in X, there exist neighborhoods V_x of x and V_y of y in X such that

$$S_{x,y} := \{g \in G | gV_x \cap V_y \neq \varnothing\}$$

is relatively compact in G, i.e. the closure of $S_{x,y}$ is compact.

Proof. Suppose the latter condition is satisfied. We show that

$$\theta = \theta_\rho : G \times X \longrightarrow X \times X$$

is closed. Let $A \subset G \times X$ be closed. Let $((x_j, y_j)|j \in J)$ be a set of points in $\theta(A)$ which converges to $(x, y) \in X \times X$. We have to show that $(x, y) \in \theta(A)$.

Write $y_j = g_j x_j$ with $(g_j, x_j) \in A$. Choose V_x and V_y such that $\{g \in G | gV_x \cap V_y \neq \varnothing\}$ is contained in a compact set K. We may assume that $x_j \in V_x, y_j \in V_y$ for all j. Then $g_j \in K$ and, by compactness, there exists a subnet (g_α) of (g_j) which converges to $g \in K$. Since A is closed, we have $(g, x) \in A$ and since θ is continuous, $\theta(g, x) = (x, g.x) = (x, y)$. And θ has compact preimages of points, since $\theta^{-1}((x, y)) \subset \{g \in G | gV_x \cap V_y \neq \varnothing\} \times \{x\}$ and $\theta^{-1}((x, y))$ is closed. Thus θ is proper.

Conversely, assume that θ is proper. Then

$$G \times X \longrightarrow G \times X \times X, (g, x) \mapsto (g, x, g.x)$$

is a homeomorphism onto its images D and it transforms the proper map θ to the proper map $p : D \longrightarrow X \times X, (g, x, g.x) \longmapsto (x, g.x)$. Let $F = G \cup \{\infty\}$ be the one-point compactification of G.

Claim: D is closed in $F \times X \times X$.

The set $E := \{(g, g)|g \in G\} \subset F \times G$ is closed, being the graph of the inclusion $G \hookrightarrow F$. Therefore,

$$(E \times X \times X) \cap (F \times D) =: H$$

is closed in $F \times D$. Since p is proper, then

$$u : F \times D \longrightarrow F \times X \times X,$$
$$(h, g, x, y) \longmapsto (h, x, y)$$

is closed. And $u(H) = D$, we conclude that D is closed in $F \times X \times X$, as claimed.

We have $(\{\infty\} \times X \times X) \cap D = \varnothing$. Therefore, there exist neighborhoods V of $\{\infty\}$ in F and W of (x, y) in $X \times X$ such that $(V \times W) \cap D = \varnothing$. By definition of F, we can take V to be of the form $(G - K) \cup \{\infty\}$, $K \subset G$ compact. If $V_x \times V_y \subset W$, then

$$(G - K) \times (V_x \times V_y)) \cap D = \varnothing,$$

which is equivalent to ($g \notin K$ *implies* $gV_x \cap V_y = \varnothing$). □

Remark: In other book [M.L03, page 543], if X is a metrizable space (such as smooth manifold, by Urysohn metrization theorem, every second-countable regular (e.g.locally compact Hausdorff) space is metrizable), the equivalent condition can be also written as following:

For every compact subset $K \subset X$, the set $G_K := \{g \in G | g.K \cap K \neq \varnothing\}$ is compact.

To show that G_K is compact, suppose (g_i) is any sequence of points in G_K, this means that for each i, there exist $p_i \in g_i.K \cap K$, which is to say that $p_i \in K$ and $g_i^{-1}.p_i \in K$, after passing to a subsequence of that , we may assume that (p_i) converge amd $(g_i^{-1}.p_i)$ converge, since G acts properly on X, then there is a subsequence (g_{i_k}) converges. Since each subsequence of G_K has a convergent subsequence, G_K is compact.

Conversely, suppose $L \subset X \times X$ is compact, and take $K = \pi_1(L) \cup \pi_2(L) \subset X$, where $\pi_1, \pi_2 : X \times X \longrightarrow X$ are the projections onto the first and second factors, respectively. Then

$$\theta^{-1}(L) \subset \theta^{-1}(K \times K) = \{(g, p) \in G \times X | g.p \in K \textit{ and } p \in K\} \subset G_K \times K.$$

since $\theta^{-1}(L)$ is closed and $G_K \times K$ is compact, then $\theta^{-1}(L)$ is compact.

Corollary 2.9 (properly discontinuous)**.** *A discrete group G acts properly on the Hausdorff space X if and only if for each pair of points (x,y) in X there exist neighborhoods V_x of x and V_y of y such that $\{g \in G | gV_x \cap V_y \neq \varnothing\}$ is finite.*

Example 2.10. *$G = SL(2,\mathbb{C}), X = \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$.*

$$\rho : G \times X \longrightarrow X,$$

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \longmapsto \frac{az+b}{cz+d}$$

is not a proper action.

choose a point $x = [1,0]$ of $\mathbb{C}_\infty \cong \mathbb{CP}^1$, then the isotropy group of x is the upper triangle subgroup

$$\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a, b, d \in \mathbb{C}, ad = 1\},$$

which is not compact. Thus the action is not proper.

Example 2.11. *Let $\Lambda := \{mw_1 + nw_2 | m, n \in \mathbb{Z}\}$, where $\frac{w_1}{w_2} \notin \mathbb{R}$. consider the group action:*

$$\rho : \Lambda \times \mathbb{C} \longrightarrow \mathbb{C},$$

$$(s, z) \longmapsto z + s,$$

is free and properly discontinuous .

· proper: Consider a pair of points (z_1, z_2), if $z_1 \neq z_2$, let

$$\eta_1 := \inf_{w \in \Lambda} |z_2 - (z_1 + w)| > 0,$$

let V_1 and V_2 be open disc of radius $\frac{\eta_1}{2}$, centered at z_1, z_2, respectively. Then we have

$$(V_1 + w) \cap V_2 = \varnothing, \ for\ all\ w \in \Lambda.$$

If $z_1 = z_2$, let

$$\eta_2 := \inf_{w \in \Lambda - (0,0)} |z_2 - (z_1 + w)| > 0,$$

let V_1 and V_2 be open disc of radius $\frac{\eta_2}{2}$, centered at z_1, z_2, respectively. then we have

$$\{g \in \Lambda | gV_1 \cap V_2 \neq \varnothing\} = \{e\}.$$

· free: for all $z \in \mathbb{C}$, the stablizer of z is trivial.

Example 2.12. *$G = SL_2(\mathbb{Z}), X := \mathcal{H} = \{z \in \mathbb{C} | Im(z) > 0\}$. Consider the group action*

$$\rho : G \times X \longrightarrow X,$$
$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \longmapsto \frac{az+b}{cz+d}$$

is properly discontinuous.

Let $\bar{U_1}, \bar{U_2}$ be any neighborhood of x, y with compact closure in $\mathcal{H}$. Then it suffices to show

$$\{\gamma \in SL_2(\mathbb{Z}) | \gamma(\bar{U_1}) \cap \bar{U_2} \neq \varnothing\}$$

is finite.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, If one of c, d is zero , then the other will be 1 or -1; If both of c, d are not zero, then $gcd(c, d) = 1$, define $a_1 := \inf\{Im(x) | x \in \bar{U_1}\}$, $a_2 := \inf\{Im(x) | x \in \bar{U_2}\}$, $b_1 := \sup\{Im(x) | x \in \bar{U_1}\}$, then

$$Im(\gamma(x)) = \frac{Im(x)}{|cx+d|^2} \leq min\{\frac{1}{c^2 a_1}, \frac{a_2}{(cRe(x)+d)^2}\}.$$

since $\frac{1}{c^2 a_1} < b_1$ for all but fintely many c, denote $\{c_1, c_2, \cdots, c_n\}$ and for each $c_i (1 \leq i \leq n)$, $\frac{a_2}{(cRe(x)+d)^2} < b_1$ for all fintely many d, denote $d_i j (1 \leq j \leq m_i)$, then

$$\{(c, d) | \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \gamma(\bar{U_1}) \cap \bar{U_2} = \varnothing\}$$

is finite. By $gcd(c_i, d_{ij}) = 1$, then there exist $s_0, t_0 \in \mathbb{Z}$ such that $cs_0 - dt_0 = 1$. However, the matrices $\gamma \in SL_2(\mathbb{Z})$ with bottom row (c, d) are

$$\left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_0 & t_0 \\ c_i & d_{ij} \end{pmatrix} \middle| k \in \mathbb{Z} \right\}$$

Therefore,

$$\gamma(\bar{U_1}) \cap \bar{U_2} = (\begin{pmatrix} s_0 & t_0 \\ c_i & d_{ij} \end{pmatrix} (\bar{U_1}) + k) \cap \bar{U_2}$$

is empty for all but finitely many γ with bottom row (c, d), by the above argument, $\gamma(\bar{U_1})$ intersects $\bar{U_2}$ for only finitely many γ.

Remark:In fact, we can have a stronger result: for $x, y \in \mathcal{H}$, then there exist neighhoods V_x of x and V_y in $\mathcal{H}$ of y with the property:

for all $\gamma \in SL_2(\mathbb{Z})$, if $\gamma(V_x) \cap V_y \neq \varnothing$, then $\gamma(x) = y$ [FD05, page 46].

Corollary 2.13. *Every continuous action by a compact group on the Hausdorff space X is proper .*

A proper action of a discrete group has locally the orbit space of a finite group action:

Proposition 2.14. *Let the discrete group G act properly on the Hausdorff space X. Then the isotropy group G_x of $x \in X$ is finite. Moreover:*

(1) There exists an open neighborhood U of x which is a G_x-subspace and satisfies $U \cap gU = \varnothing$ for $g \notin G_x$.

(2) U can be chosen in such a way that the canonical map $\alpha : U/G_x \longrightarrow X/G$ is a homeomorphism onto an open set.

(3) Moreover, if G act holomorphically and effectively on a Riemann surface X (effectively means $\bigcap_{x \in X} G_x = \{e\}$), then no point of U except x is fixed by any element of G_x. In this case, G_x is always a finite cyclic group.

Proof. (1), (2) G_x is finite by proposition 2.6, and by proposition 2.8, there exists an open neighborhood V_x of x such that

$$K_x := \{g \in G | gV_x \cap V_x \neq \varnothing\}$$

is finite. We have $G_x \subset K_x$, let $g_1, g_2, \cdots, g_n$ be the elements of $K_x - G_x$, the points $x_i = g_i.x$ are different from x. Since X is a Hausdorff space, there exist open neighborhood V_i of x and W_i of $g_i.x$ such that $V_i \cap W_i = \varnothing$. Let $U_i = V_i \cap g_i^{-1} W_i$, this is an open neighborhood of x satisfying $U_i \cap g_i U_i \subset V_i \cap W_i = \varnothing$. Let $\hat{U} := U_0 \cap U_1 \cap \cdots \cap U_n$, this open neighborhood of x satisfies $\hat{U} \cap g\hat{U} = \varnothing$ for $g \notin G_x$. The neighborhood $U = \bigcap_{g \in G_x} g\hat{U}$ is a G_x space and satisfies $U \cap gU = \varnothing$ for $g \notin G_x$.

The canonical map $U/G_x \longrightarrow X/G$ is injective by construction. Moreover, it is continuous an open, thus a homeomorphism onto its image.

(3) The last statement is followed by the following claim (simply shrink U if necessary):

The points of X with nontrivial stablizers are discrete.

suppose not, there is a sequence $\{p_n\}$ converging to p such that each p_i has a nontrivial element g_i fixing it. choose a compact neighborhood K_p of p, then by the equivalent condition of proper map, $\{g \in G | gK_p \cap K_p \neq \varnothing\}$ is finite, so we can choose a subsequence of $\{p_{i_k}\}$ such that each p_{i_k} is fixed by the same nontrivial element g. Since g is continuous, then $g.p = p$, but g is a holomorphic automorphism of X, the identity theorem and $\bigcap_{x \in X} G_x = \{e\}$ implies that g is identity.

Next, we will show that G_x is always a finite cyclic group [Mir95, page 76].

□

Corollary 2.15. *Let the discrete group G act freely and properly on the Hausdorff space X. Then the orbit map $X \longrightarrow X/G$ is a covering, i.e. a locally trivial map with typical fibre G.*

2.3 Step 3: smooth structure

Theorem 2.16 (Quotient manifold theorem)**.** *[M.L03, page 544] Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to $dimM - dimG$, and has a unique smooth structure with the property that the quotient map $\pi : M \longrightarrow M/G$ is a smooth submersion.*

Remark:Let G be a Lie group , let H be a closed subgroup of G, then the action

$$\theta = \theta_\rho : H \times G \longrightarrow G \times G,$$
$$(h, g) \longmapsto (g, h.g := gh^{-1})$$

is proper, free and smooth. Then the orbit space G/H can be endowed with a smooth manifold structure. This space is often called homogeneous space.

Theorem 2.17. *Let $\eta : G \times M \longrightarrow M$ be a transitive smooth action of the Lie group G on the smooth manifold M on the left. Let $m \in M$, and $H := \{g \in G | g.m = m\}$. Define a mapping*

$$\beta : G/H \longrightarrow M,$$

$$gH \longmapsto \eta(g, m) := g.m.$$

Then β is a diffeomorphism.

Example 2.18.

$$GL(n,\mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Consider the orthogonal group

Theorem 2.19 (Homomorphism with disrete kernels)**.** *Let G and H be connected Lie groups. For any Lie group homomorphism $F : G \longrightarrow H$, the following are equivalent [M.L03, page 557]:*

(a) F is surjective and has discrete kernel.

(b) F is a smooth covering map.

(c) F is a local diffeomorphism.

(d) The induced homomorphism $F_* : Lie(G) \longrightarrow Lie(H)$ is an isomorphism.

2.4 Step 4: Riemann Surface case

If X is a Riemann surface and discrete group G acts properly discontinuously (holomorphically) on X, by proposition 2.14, G_x is finite for all $x \in X$, and we will define charts on U/G_x and transport these to X/G via the map α.

Case 1: $|G_x| = 1$, by proposition 2.14 (2), there is a neighborhood U of x such that $\pi|_U : U \longrightarrow W \subset X/G$ is a homeomorphism onto a neighborhood W of $\pi(x)$. By shrinking U if necessary, we may assume that U is the domain of a chart $\varphi : U \longrightarrow V$ on X. We take a chart on X/G the composition $\phi = \varphi \circ \pi|_U^{-1} : W \longrightarrow V$. This is a chart on X/G.

Case 2: $|G_x| := m \geq 2$, using proposition 2.14, choose a G_x-invariant neighborhood U of x such that the natural map $\alpha : U/G_x \longrightarrow W \subset X/G$ is a homeomorphism onto a neighborhood W of $\pi(x)$. Moreover, the map $U \longrightarrow U/G_x$ is exactly $m - to - 1$ away from the point x.

We seek a mapping $\phi : W \longrightarrow \mathbb{C}$ to serve as a chart near $\pi(x)$. The composition of such a map with α and the quotient map from U to U/G_x would be a G_x-invariant function

$$h : U \longrightarrow U/G_x \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{C}$$

on a neighborhood of x. We will find ϕ by first finding this function h.

Let z be a coordinate centered at x, for each $g \in G_x$, we have the function $g(z)$, which has multiplicity one at x, define

$$h(z) := \prod_{g \in G_x} g(z).$$

Note that h has multiplicity $m = |G_x|$ at x, and is defined in some G_x-invariant neighborhood of x , we may shrink U to this neighborhood if necessary, and assume that h is defined on U. Clearly h is holomorphic and G_x-invariant. Therefore h descends to a continuous function $\bar{h} : U/G_x \longrightarrow \mathbb{C}$. Moreovr, since h is open, so is $\bar{h}$.

Finally we claim that $\bar{h}$ is $1-1$, this is simply because the holomorphic map h has multiplicity m, and hence is $m - to - 1$ near x, so is the map from U to U/G_x away from x. There $\bar{h}$ is $1-1$.

Since $\bar{h}$ is $1-1$, continuous and open, it is a homeomorphism to its image; composing it with the inverse of $\alpha : U/G_x \longrightarrow W$ gives a chart map ϕ on W:

$$\phi : W \xrightarrow{\alpha^{-1}} U/G_x \xrightarrow{\bar{h}} V \subset \mathbb{C}.$$

Note that the first case of multiplicity one is really a special case of the second case: if $m = 1$, then $h(z) = z$.

Theorem 2.20. *[Mir95, page 78] Let G be a discrete group acting holomorphically, effectively, and properly discontinuously on a Riemann surface X. Then the above construction of complex charts on X/G makes X/G into a Riemann surface. Moreover if G is finite, then the quotient map $\pi : X \longrightarrow X/G$ is holomorphic of degree $|G|$, and $mult_x(\pi) = |G_x|$ for any point $x \in X$.*

Proof. According to above discussion, we get the compplex charts of X/G, it suffices to check that they are all compatible. Since the points with nontrivial stabilizers are discrete, we may assume that no two chart domians, constructed in the $m \geq 2$ case, meet; hence there is nothing to check there.

Suppose next that the two charts are both constructed in the $m = 1$ case. Then they are compatible, since the original charts on X are compatible.

Finally, suppose that we have one chart

$$\phi : W_1 \xrightarrow{\alpha_1^{-1}} U_1 \xrightarrow{\varphi_1} V_1 \subset \mathbb{C}.$$

constructed in the $m = 1$ case, and one

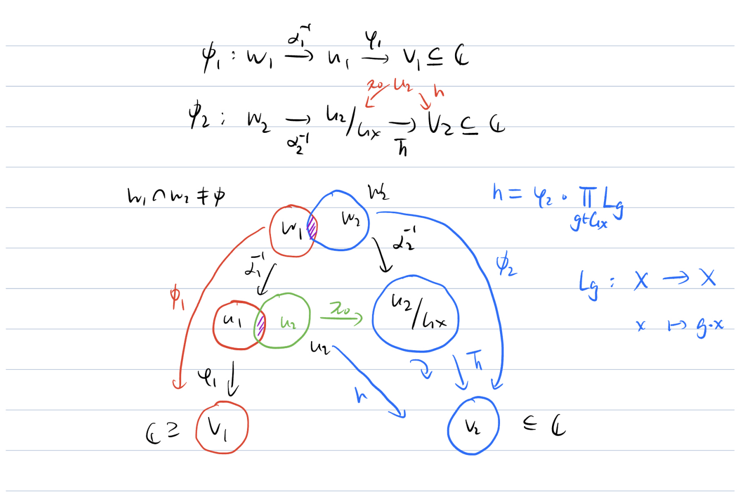
$$\phi_2 : W_2 \xrightarrow{\alpha_2^{-1}} U_2/G_x \xrightarrow{\bar{h}} V_2 \subset \mathbb{C}.$$

constructed in the $m \geq 2$ case. Let U_1 and U_2 be the open sets in X used to construct these If $W_1 \cap W_2 \neq \varnothing$, then

$$\bar{h} \circ \pi_0 \circ \varphi_1^{-1} : \phi_1(W_1 \cap W_2) \longrightarrow \phi_2(W_1 \cap W_2)$$

is holomorphic, since

$$\bar{h} \circ \pi_0 \circ \varphi_1^{-1} = h \circ \varphi_1^{-1} = (\varphi_2 \circ \prod_{g \in G_x} L_g \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1})$$



□

Example 2.21. *1. From example 2.11, we know that the discrete group Λ acts holomorphically, effectively (free implies effective), and properly discontinuously on $\mathbb{C}$, by choosing suitable complex chart , $\mathbb{C}/\Lambda$ has a Riemann surface structure, which is called complex tori.*

2. From example 2.12, we know that the discrete group $SL_2(\mathbb{Z})$ acts holomorphically, and properly on $\mathcal{H}$, actually this action is effective, so $\mathcal{H}/SL_2(\mathbb{Z})$ has a Riemann surface structure, which is called a fundamental domain for $SL_2(\mathbb{Z})$.

Remark: this action is not free, because there exists a point τ such that G_τ is not trivial, for example, *let $\tau = i$,*

$$G_i = \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

such points with nontrivial isotropy group are called elliptic points for $SL_2(\mathbb{Z})$.

Proposition 2.22. *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X, with quotient map $\pi : X \longrightarrow Y = X/G$. Suppose that there are k branch points $y_1, y_2, \cdots, y_k$ in Y, with π having multiplicity r_i at the $\frac{|G|}{r_i}$ points above y_i. Then*

$$\begin{aligned} 2g(X) - 2 &= |G|(2g(X/G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|[2g(X/G) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{r_i})]. \end{aligned}$$

3 Automophism group of Riemann surface

Lemma 3.1. *Suppose that k integers $r_1, r_2, \cdots, r_k$ with $r_i \geq 2$ for each i are given. Let $R = \sum_{i=1}^{k}(1 - \frac{1}{r_i})$.*

$$(1) R < 2 \iff k, \{r_i\} = \begin{cases} k = 1, any\ r_1; \\ k = 2, any\ r_1, r_2; \\ k = 3, \{r_i\} = \{2, 2, any\ r_3\}; or \\ k = 3, \{r_i\} = \{2,3,3\}, \{2,3,4\}, \{2,3,5\}. \end{cases}$$

$$(2) R = 2 \iff k, \{r_i\} = \begin{cases} k = 3, \{r_i\} = \{2,3,6\}, \{2,4,4\}, \{3,3,3\}; or \\ k = 4, \{r_i\} = \{2,2,2,2\}. \end{cases}$$

(3) *If $R > 2$, then in fact $R \geq 2 + \frac{1}{42}$.*

Theorem 3.2 (Hurwitz's theorem)**.** *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X of genus $g \geq 2$. Then*

$$|G| \leq 84(g - 1).$$

Proof. According to proposition 2.22, we have

$$2g(X) - 2 = |G|[2g(X/G) - 2 + R].$$

where $R = \sum_{i=1}^{k}(1 - \frac{1}{r_i})$.

Case 1: $g(X/G) = 0$, then $R \geq 2$, by lemma 3.1, $R - 2 \geq \frac{1}{42}$, then $|G| \leq 84(g(X) - 2)$.

Case 2: $g(X/G) \geq 1$, if $R = 0$, then there is no ramification to the quotient map, so we have $g(X/G) \geq 2$, which implies $|G| = 1$; if $R \neq 0$, then $2g(X) - 2 \geq |G| \cdot R \geq |G|\frac{1}{2}$, i.e. $|G| \leq 4(g(X) - 1)$. □

Remark: In fact, the automorphism group $Aut(X)$ of a compact Riemann surface X of genus ≥ 2 is a finite group, then we have

$$|Aut(X)| \leq 84(g(X) - 1).$$

4 GIT quotient

Definition 4.1. *Let G be an (not necessarily irreducible) algebraic variety (over an algebraically closed field k) endowed with the structure of a group, we call G an algebraic group (over k) if*

$$\begin{aligned} m : G \times G &\longrightarrow G, \\ (x, y) &\longmapsto xy, \end{aligned}$$

$$\begin{aligned} i : G &\longrightarrow G, \\ x &\longmapsto x^{-1}, \end{aligned}$$

are morphisms of varieties. We call G an affine algebraic group if the underlying variety is affine, and we call G an abelian variety if the underlying variety is complete (or projective).

Remark: If we use the language of schemes, sometimes we define an algebraic group over k as a group scheme of finite type over k. In this note, we just consider affine algebraic group over k.

Example 4.1. *(1)*

$$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{A}^4 \middle| ad - bc - 1 = 0 \right\}.$$

(2) If we endow $\mathbb{A}^1$ with the addicitve group structure, and

$$m : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1,$$

$$(x, y) \longmapsto x + y.$$

$$i : \mathbb{A}^1 \longrightarrow \mathbb{A}^1,$$

$$x \longmapsto -x.$$

then we denote this algebraic group as $\mathbb{G}_a$.
(3)

$$\begin{aligned} GL_n &= \{g \in \mathbb{A}^{n^2} | det\ g \neq 0\} \\ &= \{(g, \delta) \in \mathbb{A}^{n^2+1} | \delta \cdot det\ g - 1 = 0\}. \end{aligned}$$

Sometimes we denote GL_1 as $\mathbb{G}_a$.

Suppose G is an algebraic group over an algebraically closed field k, X is an algebraic variety over k, and the group action $\rho : G \times X \longrightarrow X$ is regular, i.e. ρ is a morphism of varieties.

Question: Whether can we endow X/G with a variety structure such that the orbit map $\pi : X \longrightarrow X/G$ is a surjective morphism ?

Example 4.2.

$$\mathbb{G}_m \times \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k,$$

$$(\lambda, x) \longmapsto \lambda x.$$

There are two kinds of orbits: the origin O and lines excluding the origin O.

Example 4.3.

$$\mathbb{G}_m \times \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k,$$

$$(\lambda, (x, y)) \longmapsto (\lambda x, \lambda^{-1} y).$$

There are three kinds of orbits: the origin O, $O_1 := \{(x, 0) \in \mathbb{A}^2_k | x \neq 0\}$, $O_2 := \{(0, y) \in \mathbb{A}^2_k | y \neq 0\}$, and $O(t) := \{(x, y) \in \mathbb{A}^2_k | xy = t \neq 0\}$.

Example 4.4. *Let G be an affine algebraic group, and let H be a closed algebraic subgroup of G.*

$$H \times G \longrightarrow G,$$

$$(h, g) \longmapsto gh.$$

The orbits are $\{gH\}$.

In order to answer the question of variety structure of orbit space X/G, we introduce the definition of geometric quotient.

Definition 4.2 (Geometric quotient)**.** *Let G be an affine algebraic group acting regularly on an variety X (over k), a geometric quotient for the action is a pair (Y,π), where Y is a variety and $\pi : X \longrightarrow Y$ is a morphism such that*

(1) π is open,

(2) π is surjective, and the fibers of π are the G-orbits of X, (i.e. for each $y \in Y$, the fiber $\pi^{-1}(y)$ is a single closed orbit),

(3) For every open subsets U of Y, the map $\pi^\sharp_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism of k-algebras, where $\mathcal{O}_X(\pi^{-1}(U))^G := \{f \in \mathcal{O}_X(\pi^{-1}(U)) | g.f = f \ for \ all \ g \in G\}$.

It is not hard to observe that the geometric quotient of algebraic group action does not exist in example 4.2 and example 4.3, because some orbits are not closed. Fortunately, the geometric quotient in example 4.4 exists [Bor91, chapter II, thm 6.8], the quotient G/H is also called homogeneous space. In fact, we can modify the action in example 4.2:

Example 4.5.

$$\mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \longrightarrow (\mathbb{A}^n - \{0\}),$$

$$(\lambda, x) \longmapsto \lambda x.$$

There is only one kind of orbits: lines excluding the origin O. The geometric quotient of this action exists, i.e. $(\mathbb{A}^n - \{0\})/\mathbb{G}_m \cong \mathbb{P}^{n-1}$.

Unfortunately, the geometric quotient doesn't exist in most cases, so we introduce a 'weaker' quotient.

Definition 4.3 (Good quotient)**.** *Let G be an affine algebraic group acting regularly on an variety X (over k), a good quotient for the action is a pair (Y,π), where Y is a variety and $\pi : X \longrightarrow Y$ is a morphism such that*

(1) π is surjective, and constant along the G-orbits,

(2) For every open subsets U of Y, the map $\pi^\sharp_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism of k-algebras,

(3) If $W \subset X$ is closed and G-stable, then $\pi(W) \subset Y$ is closed,

(4) If W_1, W_2 are two closed and G-stable subsets of X, and $W_1 \cap W_2 = \emptyset$, then $\pi(W_1) \cap \pi(W_2) = \emptyset$,

(5) π is affine (i.e. the preimage of every affine open subsets of Y is affine).

Remark: good quotient and geometric quotient are local with respect to the base.

Proposition 4.6. *Let G be an affine algebraic group acting regularly on an variety X (over k),*

(1) Suppose $\pi : X \longrightarrow Y$ is a good (resp. geometric) quotient of the action, then for any open subsets $U \subset X$, $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \longrightarrow U$ is a good (resp. geometric) quotient for G acting regularly on $\pi^{-1}(U)$.

(2) Suppose $\pi : X \longrightarrow Y$ is a G-invariant morphism, if there is an open covering $Y = \cup_i U_i$ of Y such that for each i, $\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \longrightarrow U_i$ is a good (resp. geometric) quotient, then $\pi : X \longrightarrow Y$ is a good (resp. geometric) quotient.

Example 4.7. *In example 4.3, if we identify the origin O and orbit O_1, O_2 as the same point, then we can define another quotient: $\mathbb{A}_k^2//\mathbb{G}_m := Spec(k[x,y]^{\mathbb{G}_m})$, where $k[x,y]^{\mathbb{G}_m} := \{f \in k[x,y] | t.f = f \text{ for all } t \in \mathbb{G}_m\} = k[xy]$, according to $k[xy] \hookrightarrow k[x,y]$, then we can obtain the surjective morphism*

$$\pi : \mathbb{A}_k^2 \longrightarrow Spec\ k[xy] \cong \mathbb{A}_k^1,$$
$$(x,y) \longmapsto xy.$$

$(\mathbb{A}_k^2//\mathbb{G}_m, \pi)$ is a good quotient, although it is not a geometric quotient. Moreover, if we consider the pricipal open subset

$$\mathbb{A}_{xy}^2 = \{(x,y) \in \mathbb{A}_k^2 | xy \neq 0\} = \bigcup_{t \in k^*} O(t),$$

which is G-stable, and also is the union of all the closed G-orbits of maximal dimension. Then $(k^*, \pi|_{\mathbb{A}_{xy}^2})$ is the geometric quotient.

Remark:

1. geometric quotient $\Rightarrow$ good quotient, but geometric quotient $\not\Leftarrow$ good quotient.
2. Given a good quotient $(Y, \pi : X \longrightarrow Y)$, and $x_1, x_2 \in X$, then

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \Leftrightarrow \pi(x_1) = \pi(x_2).$$

Definition 4.4 (Affine GIT quotient)**.** *Let G be a reductive algebraic group acting regularly on an affine variety (scheme) X (over k), then $(X//G, \pi : X \longrightarrow X//G)$ is called the affine GIT quotient of the action, where $X//G := Spec(k[X]^G)$ and π is induced by $k[X]^G \hookrightarrow k[X]$.*

Theorem 4.8. *Let G be a reductive algebraic group acting regularly on an affine variety X (over k), then $(X//G, \pi : X \longrightarrow X//G)$ is also a good quotient.*

Question 4.9. *What do the k-points $X//G(k)$ represent ?*

Definition 4.5. *Let G be a reductive algebraic group acting regularly on an affine variety X (over k),*
(1) $x \in X$ is polystable if $G.x \subset X$ is closed,
(2) $x \in X$ is stable if $G.x \subset X$ is closed and $\dim G_x = 0$.
(3) $x_1, x_2 \in X$ are called S-equivalent if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$, denoted as $x_1 \sim_S x_2$.

Denote X^{ps}(resp.X^s) as the set of polystable (resp. stable) points.

Example 4.10. *take $G = \mathbb{G}_m, X = \mathbb{A}^2_k$,*

$$\mathbb{G}_m \times \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k,$$
$$(\lambda, (x, y)) \longmapsto (\lambda x, \lambda^{-1} y).$$

$X^s = \{(x, y) \in \mathbb{A}^2_k | xy \neq 0\}$.

Example 4.11. *Take $G = GL_n(k)$, denote $X = \mathfrak{g} = \mathfrak{gl}_n(k) \cong \mathbb{A}^{n^2}_k$, where k is algebraically closed and $char(k) = 0$. We consider the conjugation action:*

$$GL_n(k) \times \mathfrak{gl}_n(k) \longrightarrow \mathfrak{gl}_n(k),$$
$$(g, A) \longmapsto g.A := gAg^{-1}.$$

The GIT quotient $\mathfrak{g}//G := Spec(k[\mathfrak{g}^*]^G)$, according to chevalley theorem, we have $k[\mathfrak{g}^*]^G \cong k[\mathfrak{t}^*]^{\mathbb{S}_n}$, where $\mathfrak{t}$ is the Cartan subalgebra consisting of diagonal matrices, and $\mathbb{S}_n$ is the permutaion group (Wely group of A type). Therefore

$$\mathfrak{g}//G = Spec(k[c_{n-1}, c_{n-2}, \cdots, c_0]),$$

where c_i is the coeffient of the characteristic polynomial $det(\lambda(Id_n) - T) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda^1 + c_0$, $c_0 = det, c_{n-1} = tr$.

Question: Can we generalize this result to general adjoint action of (reductive or semisimple) algebraic groups G on its Lie algebras $\mathfrak{g}$?

Question: What is the orbit (conjugacy class of matrices) ? How to classify the different orbit? First consider dimension, what is the orbit of maximal dimension ? What is the orbit of maximal dimension ? [Ste74]

$x \in \mathfrak{g}, O(x) = \{g.x | g \in G\}$ has maximal dimension if and only if the stablizer of x, $G_x = \{g \in G | g.x = gxg^{-1} = x\}$ has minimal dimension. By direct computation, if x is a semisimple element (i.e. diagonable) with maximal dimension orbit , then $O(x) = O(y)$, with

$$y = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_{n-1} & \\ & & & & a_n \end{pmatrix},$$

where $a_1, a_2, \cdots a_n$ are all distinct. If x is a nilpotent element with maximal dimension orbit, then $O(x) = O(z)$, with

$$z = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix},$$

In fact the element with maximal dimension orbit is called regular element:

Definition 4.6. *x is called the regular element of $\mathfrak{gl}_n(k)$ if dim $G_x =$ rank $\mathfrak{gl}_n(k)$. x is called the subregular element of $\mathfrak{gl}_n(k)$ if dim $G_x =$ rank $\mathfrak{gl}_n(k) + 2$.*

If x is a semisimple element, then the orbit $O(x)$ of x is closed, i.e. $\overline{O(x)} = O(x)$, but the closure $\overline{O(y)}$ of orbit of nilpotent element y has singularity. For example, choose y as a regular nilpotent element of $\mathfrak{g} = \mathfrak{gl}_n(k)$, then $\overline{O(y)} = \mathcal{N} = \{A \in \mathfrak{g} | A \text{ is nilpotent}\}$, called nilpotent cone , which is also the preimage of $0 \in \mathfrak{t}//\mathbb{S}_n$ of the quotient map $\chi : \mathfrak{g} \longrightarrow \mathfrak{g}//G \cong \mathfrak{t}//\mathbb{S}_n$.

4.1 Flag geometry and Springer resolution

Springer studies resolution of nilpotent cone, which nowdays is called Springer resolution. Next we consider $\mathfrak{g}$ as a simple Lie algebra (e.g. $\mathfrak{g} = \mathfrak{sl}_n(k)$). Let $\mathcal{B}$ be the space of all the Borel subalgebras (maximal solvable subalgebra) of $\mathfrak{g}$.

Theorem 4.12. *In the case of $G = SL_n(\mathbb{C}), \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the space $\mathcal{B}$ is identified naturally with the complete flag variety $\left\{F_\cdot = (\{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n) | dim F_i = i\right\}$.*

Define $Y = \left\{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} | x \in \mathfrak{b}\right\}$,

Theorem 4.13. *Let G be a reductive algebraic group acting regularly on an affine variety X (over k), and let $(X//G, \pi : X \longrightarrow X//G)$ be the affine GIT quotient of the action, then*

(1) $X^s \subset X$ is open and G-invariant,

(2) $\pi(X^s) \subset X//G$ is open and $\pi^{-1}(\pi(X^s)) = X^s$,

(3) $(X^s, \pi|_{X^s})$ is a geometric quotient of the restriction action.

Corollary 4.14. *The following set of k-points are 1-1 correspondence*

$$X//G(k) \leftrightarrow X^{ps}(k)/G(k) \leftrightarrow X(k)/\sim_S .$$

Question: How about projective case of GIT quotient ?

In projective case, we can't use the corrodinate ring of projective variety, since the global section is less (For example, the global section of projective space is constant function), therefore we need to consider Line bundle.

Definition 4.7 (Linearisation)**.** *Let X be a variety (or scheme) and G be an affine algebraic group acting on X via a morphism $\sigma : G \times X \longrightarrow X$. Then a linearisation of the G-action on X is a line bundle $\pi : L \longrightarrow X$ with an isomorphism of line bundles*

$$\phi : pr_2^* L = G \times L \cong \sigma^* L,$$

$$
\begin{tikzcd}
G\times L \arrow[dr, "\phi"] \arrow[drr, bend left, "\Phi"] \arrow[ddr, bend right, "id_G\times\pi"'] & & \\
& \sigma^*L \arrow[r, "\hat{\sigma}"] \arrow[d] & L \arrow[d, "\pi"'] \\
& G\times X \arrow[r, "\sigma"] & X
\end{tikzcd}
$$

where $pr_2 : G \times X \longrightarrow X$ is the second projection, such that $\Phi = \hat{\sigma} \circ \phi : G \times L \longrightarrow L$ is an action of G on L, where $\hat{\sigma} : \sigma^* L \longrightarrow L$.

$$
\begin{tikzcd}
& G\times L \arrow[rr] \arrow[dd] & & L \arrow[dd] \\
G\times L\times L \arrow[ur] \arrow[rr] \arrow[dd] & & G\times L \arrow[ur] \arrow[dd] & \\
& G\times X \arrow[rr] & & X \\
G\times G\times X \arrow[ur] \arrow[rr] & & G\times X \arrow[ur] &
\end{tikzcd}
$$

L is sometimes called total space of G-linearied line bundle, denote $Pic^G(X)$ as the set of all G-linearied line bundles.

Remark: If we have a linearisation of the G-action on X, then we have a $G-$action on $H^0(X, L)$:

$$H^0(X, L) \longrightarrow H^0(G\times X, \sigma^* L) = H^0(G\times X, G\times L) \cong H^0(G, O_G)\otimes H^0(X, L)$$

Assume X is projective over k , G is a reductive algebraic group acting on X, and L is ample linearisation of the G-action on X, since

$$\bigoplus_{r\geq 0} H^0(X, L^{\otimes r})^G \hookrightarrow \bigoplus_{r\geq 0} H^0(X, L^{\otimes r}),$$

then we have

$$X = Proj(\bigoplus_{r\geq 0} H^0(X, L^{\otimes r})) \dashrightarrow Proj(\bigoplus_{r\geq 0} H^0(X, L^{\otimes r})^G) =: X//_L G.$$

If we want to get a morphism between X and $X//_L G$, it needs to introduce the concept of semistable points and stable points.

Definition 4.8. *(1) A point $x \in X$ is semistable with respect to (σ, L, ϕ) if there is an invariant section $\sigma \in H^0(X, L^{\otimes r})$ for some $r > 0$ such that $\sigma(x) \neq 0$.*

(2) A point $x \in X$ is semistable with respect to (σ, L, ϕ) if $dim G \cdot x = dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes r})$ for some $r > 0$ such that $\sigma(x) \neq 0$, and the action of G on $X_\sigma := \{x \in X | \sigma(x) \neq 0\}$ is closed.

Denote X^{ss}, X^s as all semistable points and stable points, repsectively.

Remark:

1. Semistable and stable points depends on the linearization of the G-action on X, which means semistable and stable points not only depends on (σ, L), but also depends on ϕ.

2. Actually, we can define semistable and stable points of general case, which add the condition '$X_\sigma := \{x \in X | \sigma(x) \neq 0\}$ is affine' on definition 4.8, it is not hard to see this condition is redundant if we consider projective variety X and ample linearisation L.

3. If we consider G-action on affine variety X, trivial line bundle $X \times \mathbb{A}^1_k$, and trivial G-action on $X \times \mathbb{A}^1_k$, then all points on X is semistable and the stable points conincides the definition of 4.5.

Example 4.15. *Let $\chi : \mathbb{G}_m \longrightarrow \mathbb{G}_m, t \longmapsto t^l$ be the character of $\mathbb{G}_m$, $X = \mathbb{A}^1_k$, $L = \mathbb{A}^n_k \times \mathbb{A}^1_k$,*

$$\mathbb{G}_m \times \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k,$$

$$(\lambda, v) \longmapsto \lambda v.$$

$$\mathbb{G}_m \times L \longrightarrow L,$$

$$(\lambda, v, x) \longmapsto (v, \chi(\lambda)(x)).$$

this action induces a $\mathbb{G}_m$-action on $H^0(\mathbb{A}^n_k, L)$, then the $\mathbb{G}_m$-invariant section is $H^0(\mathbb{A}^n_k, L)^{\mathbb{G}_m} := \{f | \lambda.f = \chi(\lambda^{-1})f\} = \{$ homogeneous polynomials of degree l $\}$.

If $l > 0$, then $X^{ss} = \mathbb{A}^n_k - \{0\}, X^s = \mathbb{A}^n_k - \{0\}$, $X//_\chi \mathbb{G}_m := Proj(\bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^{\mathbb{G}_m}) = \mathbb{P}^{n-1}_k$ with line bundle $O(l)$.

If $l = 0$, then $X^{ss} = \mathbb{A}^n_k, X^s = \emptyset$.

If $l < 0$, then $X^{ss} = X^s = \emptyset$.

Next we introduce a criterion to find the semistable points and stable points.

Definition 4.9. *Suppose X is a proper variety (or scheme) over k, and reductive group G acts on X. λ is a one-parameter subgroup of G (i.e. $\lambda : \mathbb{G}_m \longrightarrow G$ is an algebraic group homomorphism), and $L \in Pic^G(X), x \in X$ (or x is a closed point of X), then $x_0 := \lim_{\alpha\to 0} \lambda(\alpha).x$ exists and is unique, and the induced $\mathbb{G}_m$-action on fiber L_{x_0}, which is a one-dimensional subspace. Therefore we obtain a charater*

$$\chi_{x_0} : \mathbb{G}_m \longrightarrow Aut(L_{x_0}) \cong \mathbb{G}_m$$
$$t \longmapsto t^p,$$

define $\mu^L(x, \lambda) := -p$, which is called Hilbert-Mumford weight.

Theorem 4.16 (Hilbert-Mumford criterion)**.** *Suppose X is a proper variety (or scheme) over k, and reductive group G acts on X. Let $L \in Pic^G(X)$, and assume L is ample, $x \in X$ (or x is a closed point of X). Then*

$$x \in X^{ss} \Leftrightarrow \mu^L(x, \lambda) \geq 0 \ for \ all \ one-parameter \ subgroups \ \lambda.$$

$$x \in X^{ss} \Leftrightarrow \mu^L(x, \lambda) > 0 \ for \ all \ one-parameter \ subgroups \ \lambda.$$

Example 4.17.

$$\mathbb{G}_m \times \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k,$$
$$(\lambda, [x_0 : x_1 : \cdots : x_n]) \longmapsto [\lambda x_0 : \lambda^{-1}x_1 : \cdots : \lambda^{-1}x_n].$$

This group action can natually induce an action on tautological line bundle $O(-1) := \{(l, x) \in \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k | x \in l\}$, then induce an action on $L = O(1)$.

$$\Phi : \mathbb{G}_m \times O(-1) \longrightarrow O(-1),$$

$$z_0 := \lim_{\lambda \longrightarrow 0} \lambda.[x_0 : x_1 : \cdots : x_n] = \begin{cases} [0 : x_1 : \cdots : x_n], x_0 \neq 0; \\ [0 : x_1 : \cdots : x_n], x_0 = 0. \end{cases}$$

$$z_\infty := \lim_{\lambda \longrightarrow 0} \lambda^{-1}.[x_0 : x_1 : \cdots : x_n] = \begin{cases} [x_0 : 0 : 0 : \cdots : 0], \ x_0 \neq 0; \\ [x_0 : x_1 : \cdots : x_n], \ x_0 = 0. \end{cases}$$

Let $x = [x_0 : x_1 : \cdots : x_n]$,

Case 1: $x_0 \neq 0$,

$$O(-1)_{z_0} \longrightarrow O(-1)_{z_0},$$
$$x = (0, x_1, \cdots, x_n) \longmapsto \lambda^{-1}(0, x_1, \cdots, x_n)$$
$$\mu^L(x, \lambda) = -(-1) > 0,$$

$$O(-1)_{z_\infty} \longrightarrow O(-1)_{z_\infty},$$
$$x = (x_0, 0 \cdots, 0) \longmapsto \lambda^{-1}(x_0, 0, \cdots, 0)$$
$$\mu^L(x, \lambda^{-1}) = -(-1) > 0,$$

Case 2: $x_0 = 0$ *and* $(x_1 : \cdots : x_n) \neq (0, 0, \cdots, 0)$

$$O(-1)_{z_0} \longrightarrow O(-1)_{z_0},$$
$$x = (0, x_1, \cdots, x_n) \longmapsto \lambda^{-1}(0, x_1, \cdots, x_n)$$
$$\mu^L(x, \lambda) = -(-1) > 0,$$

$$O(-1)_{z_\infty} \longrightarrow O(-1)_{z_\infty},$$
$$x = (0, x_1, \cdots, x_n) \longmapsto \lambda(0, x_1, \cdots, x_n)$$
$$\mu^L(x, \lambda^{-1}) = -1 < 0,$$

Thus

$$X^s = X^{ss} = \{[x_0 : x_1 : \cdots : x_n] | x_0 \neq 0,\ and\ (x_1 : \cdots : x_n) \neq (0, 0, \cdots, 0)\}$$

$$\mathbb{P}_k^n // _\Phi \mathbb{G}_m = X^s / \mathbb{G}_m = \mathbb{P}_k^{n-1}$$

In general, if $\mathbb{G}_m$ acts on $\mathbb{P}^n$ linearly, $z = [x_0 : x_1 : \cdots : x_n]$,

$$\mathbb{G}_m \times \mathbb{P}_k^n \longrightarrow \mathbb{P}_k^n,$$
$$(\lambda, [x_0 : x_1 : \cdots : x_n]) \longmapsto [\lambda^{r_0} x_0 : \lambda^{r_1} x_1 : \cdots : \lambda^{r_n} x_n].$$

Then

$$\mu^{O(1)}(z, \lambda^{-1}) := -min\{r_i | x_i \neq 0\},$$
$$\mu^{O(1)}(z, \lambda^{-1}) := -min\{-r_i | x_i \neq 0\}.$$

5 Symplectic quotient

5.1 Hamiltonian group action

Let (M, ω) be a symplectic manifold, and G be a connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{X}(M)$ be the vector fields on M. Suppose the group action G on (M, ω) is symplectic, i.e. there is a group homomorphism between G and symplectomorphism group $Symp(M, \omega) := \{\phi \in Diff(M) | \phi^* \omega = \omega\}$.

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & Symp(M, \omega) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow[\phi]{} & \mathfrak{X}(M, \omega) \end{array}$$

$$\xi \longmapsto X_\xi,$$

Here $(X_\xi)_p := \frac{d}{dt}\Big|_{t=0} \Phi_{exp(t\xi)}(p)$, $\mathcal{L}$ is the Lie derivative and

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) | \mathcal{L}_X w = 0\}.$$

Due to the Cartan's (homotopy) formula, $\mathcal{L}_X w = (d\iota_X + \iota_X d)w = d\iota_X w$, then $\mathcal{L}_X w = 0$ is equivalent to $\iota_X w$ is a closed 1-form, where ι_X is the contration with vector field X. Denote

$$\mathfrak{X}_{Ham}(M, \omega) := \{X \in \mathfrak{X}(M) | \iota_X w \textit{ is exact } \text{1-form}\}.$$

Definition 5.1. *The symplectic group action G on (M, ω) is called weakly Hamiltonian if $X_\xi \in \mathfrak{X}_{Ham}(M, \omega)$, i.e. $\exists\, H_\xi \in C^\infty(M)$ s.t. $\iota_{X_\xi} w = dH_\xi$, where $C^\infty(M)$ is the ring of smooth function on M.*

Define the Poisson bracket on $C^\infty(M)$ via $\{f, g\} := w(X_f, X_g)$, where X_f, X_g are the Hamiltonian vector field associated to f, g, respectively, which means $\iota_{X_f} w = df, \iota_{X_g} w = dg$. It is not hard to see there is an Lie algebra (anti)-isomorphism between $(\mathfrak{X}_{Ham}(M, \omega), [,])$ and $(C^\infty(M)/\mathbb{R}, \{,\})$, which is denoted by $\overline{H}$.

Proposition 5.1. $[\mathfrak{X}(M, \omega), \mathfrak{X}(M, \omega)] \subset \mathfrak{X}_{Ham}(M, \omega)$.

Proof. Suppose X, Y are symplectic vector fields with corresponding flows Φ_X^t, Φ_Y^t, respectively. $[X,Y] := \mathcal{L}_X Y = -\frac{d}{dt}\Big|_{t=0} (\phi_X^t)^* Y$.

$$\begin{aligned}
\iota_{[X,Y]} w &= \frac{d}{dt}\bigg|_{t=0} \iota_{(\phi_X^t)^* Y} w \\
&= \frac{d}{dt}\bigg|_{t=0} (\phi_X^t)^* \iota_Y w \quad (since (\phi_X^t)^* w = w) \\
&= \mathcal{L}_X(\iota_Y w) \\
&= (d\iota_X + \iota_X d)\iota_Y w \\
&= dw(Y, X).
\end{aligned}$$

□

Proposition 5.2. *ϕ is Lie algebra homomorphism.*

Proof. Let $\xi, \eta \in \mathfrak{g}$,

$$[\xi, \eta] = \frac{d}{ds}\bigg|_{s=0} exp(-s\xi)\eta exp(s\xi),$$

$$\begin{aligned}
X_{[\xi,\eta]} &= \frac{d}{ds}\bigg|_{s=0} X_{exp(-s\xi)\eta exp(s\xi)} \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \Phi_{exp(t(exp(-s\xi)\eta exp(s\xi)))} \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \Phi_{exp(-s\xi)} \Phi_{exp(t\eta)} \Phi_{exp(s\xi)} \\
&= \frac{d}{ds}\bigg|_{s=0} \Phi_{exp(-s\xi)} X_\eta \Phi_{exp(s\xi)} \\
&= \mathcal{L}_{X_\xi} X_\eta = [X_\xi, X_\eta].
\end{aligned}$$

□

Remark:

1. $H^1_{dR}(M;\mathbb{R})$ characterizes the difference between $\mathfrak{X}(M,\omega)$ and $\mathfrak{X}_{Ham}(M,\omega)$, if $H^1_{dR}(M;\mathbb{R}) = \{0\}$, then $\mathfrak{X}(M,\omega) = \mathfrak{X}_{Ham}(M,\omega)$.

2. $\varphi := \overline{H} \circ \phi$ is Lie algebra anti-homomorphism. sometimes people want to obtain a Lie algebra homomorphism between $(\mathfrak{g}, [,])$ and $(C^\infty(M)/\mathbb{R}, \{,\})$ via using another notation of Lie derivative

$[X, Y] := -\mathcal{L}_X Y$. [MS17, Remark 3.1.6]. Later we just consider φ as a Lie algebra homomorphism.

$$\begin{tikzcd} & (\mathfrak{X}_{Ham}(M,\omega), [,]) \arrow[dr, "\overline{H}"] & \\ (\mathfrak{g}, [,]) \arrow[ur, "\phi"] \arrow[rr, "\varphi"] & & (C^\infty(M)/\mathbb{R}, \{,\}) \end{tikzcd}$$

3. If the Lie algebra $\mathfrak{g} = Lie(G)$ is semisimple (imply $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), then any symplectic group action of G on (M, w) is always weakly Hamiltonian.

Question 5.3. *Suppose the action of G on (M, w) is weakly Hamiltonian. Is it possible to obtain a compatible lifting of Lie algebra homomorphism from $(\mathfrak{g}, [,])$ to $(C^\infty(M), \{,\})$?*

Proposition 5.4. *Suppose the action of connected Lie group G on (M, w) is weakly Hamiltonian. Let $\hat{\varphi} := H \circ \phi$*

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{X}_{Ham}(M,\omega) & \xrightarrow{H} & C^\infty(M) \\ \xi & \longmapsto & X_\xi & \longmapsto & H_\xi \end{array}$$

be a linear map such that $\iota_{X_\xi} w = dH_\xi$ for any $\xi \in \mathfrak{g}$. The following are equivalent:

(1) $\quad (\mathfrak{g}\ , [\ ,\ ]) \xrightarrow{\hat{\varphi}} (C^\infty(M), \{,\})$ *is a Lie algebra homomorphism such that*

$$\begin{tikzcd} (\mathfrak{g}\ , [\ ,\ ]) \arrow[r, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & (C^\infty(M), \{,\}) \arrow[d, "q"] \\ & (C^\infty(M)/\mathbb{R}, \{,\}) \end{tikzcd}$$

commutes.

(2) $\quad \mathfrak{g} \xrightarrow{\hat{\varphi}} C^\infty(M)$ *is G-equivariant.*

(3) $\exists$ a smooth map $\mu : M \longrightarrow \mathfrak{g}^*$ such that $\langle \mu(x), \xi \rangle = H_\xi(x)$ for $\xi \in \mathfrak{g}$, where $\langle , \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. And μ is G-equivariant.

If one of the above three conditions satisfies, then the group action of G on (M, w) is called Hamiltonian group aciton. μ of (3) is called moment map.

Definition 5.2. *(1) $\Longrightarrow$ (2)*

(2) $\Longrightarrow$ (1) For $g \in G$, we have a natural G-action on $C^\infty(M)$, $g.f(m) := f \circ \Phi_{g^{-1}}(m)$ for all $m \in M$. $\hat{\varphi}$ is G-equivariant means $H_\xi \circ \Phi_g = H_{g^{-1}\xi g}$. Then we obtain

$$\begin{aligned} H_{[\xi,\eta]} &= -\frac{d}{ds}\bigg|_{s=0} H_{exp(-s\xi)\eta exp(s\xi)} \\ &= -\frac{d}{ds}\bigg|_{s=0} H_{eta} \circ \Phi_{exp(s\xi)} \text{ (since } G\text{-equivariant)} \\ &= -X_\xi(H_\eta) = -\{X_\eta, X_\xi\} = \{H_\xi, H_\eta\}. \end{aligned}$$

(2)$\Longleftrightarrow$ (3) Define $\mu : M \longrightarrow \mathfrak{g}^*$ such that $\langle \mu(m), \xi \rangle = H_\xi(m)$ for $\xi \in \mathfrak{g}, m \in M$. Then μ is G-equivariant if and only if $Ad_g^* \circ \mu = \mu \circ \Phi_g$, i.e.

$$H_{Ad_{g^{-1}}\xi}(m) = \langle \mu(m), Ad_{g^{-1}}\xi \rangle = \langle Ad_g^*(\mu(m)), \xi \rangle = \langle \mu \circ \Phi_g(m), \xi \rangle = H_\xi(\Phi_g(m)).$$

if and only if $\hat{\varphi}$ is G-equivariant.

Remark 5.5. *In general, there are obstructions to lift the morphism φ to the morphism $\hat{\varphi}$, which can be characterized by second Lie algebra cohomology $H^2(\mathfrak{g}; \mathbb{R})$.*

$$d : \wedge^k \mathfrak{g}^* \longrightarrow \wedge^{k+1} \mathfrak{g}^*$$

$$\alpha \longmapsto d\alpha,$$

where $d\alpha(X_0, X_1, \cdots, X_k) := \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k)$ for $X_0, X_1, \cdots, X_k \in \mathfrak{g}$. Denote $Z^p(\mathfrak{g}; \mathbb{R}) := \{\alpha \in \wedge^p \mathfrak{g}^* | d\alpha = 0\}$, $B^p(\mathfrak{g}; \mathbb{R}) := \{d\beta | \beta \in \wedge^{p-1} \mathfrak{g}^*\}$, therefore we define

$$H^p(\mathfrak{g}; \mathbb{R}) := \frac{Z^p(\mathfrak{g}; \mathbb{R})}{B^p(\mathfrak{g}; \mathbb{R})}.$$

Let $\eta, \xi \in \mathfrak{g}$, we define $\tau(\xi, \eta) := \{H_\xi, H_\eta\} - H_{[\xi,\eta]}$. It is not hard to check $d\tau(\xi, \eta, \gamma) = \frac{1}{2} d_M w(X_\xi, X_\eta, X_\gamma)$. Due to closedness of w, so $\tau \in Z^2(\mathfrak{g}; \mathbb{R})$. In fact weakly Hamiltonian action of connected Lie group G on (M, w) is Hamiltonian is equivalent to $0 = [\tau] \in H^2(\mathfrak{g}; \mathbb{R})$.

Example 5.6. *Consider the 2-dim sphere* $\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$*, choose the cylindrical polar coordinates* (φ, x_3) *on* $\mathbb{S}^2 - \{(0, 0, \pm 1)\}$ *with the symplectic form* $w = d\varphi \wedge dx_3$*. Define a* $\mathbb{S}^1$*-action on* $\mathbb{S}^2$ *as follows:*

$$\Phi : \mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2,$$
$$(e^{i\theta}, (e^{i\varphi}, x_3)) \longmapsto (e^{i(\theta+\varphi)}, x_3),$$

and fix the points $\{(0, 0, \pm 1)\}$*. Actually this is a Hamiltonian group action.*

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "\Phi"] \arrow[d] & Symp(\mathbb{S}^2, \omega) \arrow[d] \\
i\theta \arrow[r, "\phi"'] & \mathfrak{X}(\mathbb{S}^2, \omega)
\end{tikzcd}$$

Let $i\theta \in Lie(\mathbb{S}^1)$*,*

$$\begin{aligned}
(X_{i\theta})_{(\varphi, x_3)} &= \frac{d}{ds}\bigg|_{s=0} \Phi_{exp(si\theta)}(\varphi, x_3) \\
&= \frac{d}{ds}\bigg|_{s=0} (s\theta + \varphi, x_3) = \theta \frac{\partial}{\partial \varphi}.
\end{aligned}$$

since $H^1(\mathbb{S}^2; \mathbb{R}) = \{0\}$*, then this group action is weakly Hamiltonian, suppose* $w(X_{i\theta}, \cdot) = dH_{i\theta}$*, then* $H_{i\theta} = \theta x_3$*. so we can define*

$$\mu : \mathbb{S}^2 \longrightarrow Lie(\mathbb{S}^1)^* \cong i\mathbb{R},$$

$$(\varphi, x_3) \longmapsto -ix_3 \longmapsto x_3,$$

via $\langle \mu(\varphi, x_3), i\theta \rangle = H_{i\theta}(\varphi, x_3) = \theta x_3$*. It is not hard to check that* μ *is* $\mathbb{S}^1$*-equivariant.*

Example 5.7. *Consider the group action*

$$\Phi : \mathbb{S}^1 \times \mathbb{C} \longrightarrow \mathbb{C},$$

$$(e^{i\theta}, z) \longmapsto e^{i\theta} z,$$

where $\mathbb{C} \cong \mathbb{R}^2$, $z = x + iy$ *is identified via* (x, y)*, and the standard symplectic form is* $w = dx \wedge dy$.

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "\Phi"] \arrow[d] & Symp(\mathbb{C}, \omega) \arrow[d] \\ i\theta \arrow[r, "\phi"'] & \mathfrak{X}(\mathbb{C}, \omega) \end{tikzcd}$$

Let $i\theta \in Lie(\mathbb{S}^1)$*,*

$$\begin{aligned} (X_{i\theta})_{(x+iy)} &= \frac{d}{ds}\bigg|_{s=0} \Phi_{exp(si\theta)}(x+iy) \\ &= \frac{d}{ds}\bigg|_{s=0} \begin{pmatrix} \cos(s\theta) & \sin(s\theta) \\ \sin(s\theta) & \cos(s\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\theta y \frac{\partial}{\partial x} + \theta x \frac{\partial}{\partial y}. \end{aligned}$$

$w(X_{i\theta}, \cdot) = d(-\frac{\theta(x^2+y^2)}{2})$*, take* $H_{i\theta} = -\frac{\theta(x^2+y^2)}{2}$*, similarly, we can compute the moment map*

$$\mu : \mathbb{C} \longrightarrow Lie(\mathbb{S}^1)^* \cong i\mathbb{R},$$
$$(x+iy) \longmapsto \frac{i(x^2+y^2)}{2} \longmapsto -\frac{(x^2+y^2)}{2}.$$

It is also not hard to check that μ *is* $\mathbb{S}^1$*-equivariant.*

Moreover,

$$\Phi : \mathbb{S}^1 \times \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$
$$(e^{i\theta}, (z_1, z_2, \cdots, z_n) \longmapsto (e^{i\theta} z_1, e^{i\theta} z_2, \cdots, e^{i\theta} z_n).$$

We have the moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathbb{R},$$
$$(z_1, z_2, \cdots, z_n) \longmapsto -\frac{\sum_{i=1}^n |z_i|^2}{2}.$$

Example 5.8.

$$\Phi : \mathbb{S}^1 \times \mathbb{P}^n \longrightarrow \mathbb{P}^n,$$
$$(e^{i\theta}, [z_0 : z_1 : \cdots : z_n] \longmapsto [e^{i\theta} z_0 : e^{-i\theta} z_1 : \cdots , e^{-i\theta} z_n].$$

We have the moment map

$$\mu : \mathbb{P}^n \longrightarrow \mathbb{R},$$
$$[z_0 : z_1 : \cdots : z_n] \longmapsto -\frac{|z_0|^2 - \sum_{i=1}^n |z_i|^2}{\sum_{i=0}^n |z_i|^2}.$$

5.2 Hamiltonian Lifting

be a Lie group, acting on a smooth manifold M smoothly, i.e. $\Psi : G\times M \longrightarrow M, (g,m) \longmapsto g.m$ is a smooth group action, and denote $\Psi_g : M \longrightarrow M, m \longmapsto g.m$. And this group action can induce a natrual G-action on cotangent bundle T^*M of M as follows, which is defined by

$$\Phi : G \times T^*M \longrightarrow T^*M,$$

$$(g,(m,v^*)) \longmapsto (\Psi_g(m), \Psi^*_{g^{-1}}(v^*)) := \Phi_g(m,v^*).$$

Recall that the cotangent bundle $(T^*M, \pi : T^*M \longrightarrow M)$ of a smooth manifold M has a canonical 1-form λ_{can}:

$$\langle \lambda_{can}, \nu \rangle := \langle v^*, \pi_*(\nu) \rangle,$$

where $\nu \in T_{(m,v^*)}(T^*M)$, $\pi_* : T(T^*M) \longrightarrow TM$ is the tangent map of π.

$$\begin{tikzcd}
T_{(m,v^*)}(T^*M) \arrow[d, "\pi_*"] \arrow[dr, "\lambda_{can}"] & \\
T_mM \arrow[r, "v^*"] & \mathbb{R}
\end{tikzcd}$$

If we choose a coordinate chart $(U, x_1, x_2, \cdots, x_n)$ of M, $m \in U$ and a basis $\{dx_1, dx_2, \cdots, dx_n\}$ of T^*U, take $v^* = \sum_i y_i dx_i$, define the coordinate function

$$y^*_j : T^*U \longrightarrow \mathbb{R}, \ \sum_i y_i dx_i \longmapsto y_j$$

then $(T^*U, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ is a coordinate chart of T^*M. $\lambda_{can} = \sum_i y^*_i dx_i$.

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