Mirror Symmetry

CY manifolds $:=$ $K_X \simeq \mathcal{O}_X$ or nowhere vanishing holomorphic volume form

$\wedge$ cpt or noncpt

$s \in \Gamma(X, \Omega_X^n) = H^{n,0}(X)$

· Adjunction formula : $Y \underset{\text{codim } 1}{\hookrightarrow} X$, X nonsing proj.

$$K_Y \sim K_X(Y)|_Y$$

· Eg 1: $X = \mathbb{P}^n$ $\quad K_X \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$

$\Rightarrow$ $Y =$ deg $n+1$ hypersurface in $\mathbb{P}^n$

$\Rightarrow$ $K_Y \simeq K_X(n+1)|_Y = \mathcal{O}_{\mathbb{P}^n}|_Y(-n-1+n+1) = \mathcal{O}_Y$

$Y =$ nonsing proj. CY.

· Eg 2: $X =$ Fano $\quad Y \in |-K_X| \xrightarrow{\text{Adjunction formula}} K_Y = \mathcal{O}_Y$

$\Rightarrow Y$ CY. proj.

$H^1(\mathbb{P}^n, \mathcal{O}^*_{\mathbb{P}^n}) \xrightarrow{c_1} H^2(\mathbb{P}^n) = \mathbb{Z}$ $\qquad Fl(1, n-1, n)$

· Rmk: Eg 1 gives simply connected proj. nonsing CY.

for $n \geq 3$.

· Thm (Lefschetz hyperplane thm)

Let X be a smooth cpx proj. var of $\dim_{\mathbb{C}} n$, and let D be an effective ample divisor on X.

Then $r_i : H^i(X, \mathbb{Z}) \longrightarrow H^i(D, \mathbb{Z})$

is iso for $i \leq n-2$ & inj. for $i = n-1$

· Eg: $\mathbb{P}^n$ $\quad H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$ $\quad |x| = 2$.

$\Rightarrow H^1(\mathbb{P}^n, \mathbb{Z}) = 0$

for $n \geq 3$. $\quad \underset{\deg n+1}{Y \hookrightarrow \mathbb{P}^n}$, $\quad H^1(\mathbb{P}^n, \mathbb{Z}) \xrightarrow{\sim} H^1(Y, \mathbb{Z}) = 0$

· Eg: $n = 3$. $\quad Y = K3$ surface $\quad \binom{3+4}{3} - 1 - \mathrm{Aut}(PGL_4)$

$= \frac{7 \times \cancel{6} \times 5}{\cancel{6}} - 1 - 4 \times 4 + 1 = 18$

$n = 4$ $\quad Y =$ quintic 3-fold $\quad H^2(\mathbb{P}^4, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z}) = \mathbb{Z}$

$\dim H^1(Y, T_Y) = 101$ $\qquad$ Moduli of deg 5 $\subseteq \mathbb{P}^4$

$\quad \| \; H^1(Y, \Omega^2_Y)$ $\qquad \binom{4+5}{5} - 1 - \mathrm{Aut}(PGL_5)$

$H^1(\check{Y}, \Omega^1_{\check{Y}}) = H^{1,1}(\check{Y}) = 101$

$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} - 1 - 24$

$\Rightarrow \check{Y} \not\hookrightarrow \mathbb{P}^4$

$= 126 - 1 - 24$

$= 101$

· Construct $\check{Y}$ crepant resolution

Crepant resolution Need not exist in general.

· Eg 3: Abelian var has trivial canonical bundles.

· Eg 4: $\mathbb{C}^n$.

log CY mfd

Fano X, anticanonical divisor D

$X \setminus D$ admits a nowhere vanishing holomorphic volume form

noncpt.

$\Gamma(X, -K_X) \ni s$

· $\dim_{\mathbb{C}} X = 2$ D normal crossing. [Gross-Hacking-Keel]

$(X, D) \sim$ toric model [Gross-Hacking-Keel-Kontsevich]

blow up, blow down $\sim (X_{\text{toric}}\ D_{\text{toric}})$

$\uparrow$ Fano $\uparrow$ toric divisor

· Eg 5: toric CY var

Fact: Fano toric var X. $\rightsquigarrow$ total space of canonical bundle is still toric & it's CY.

Eg: $\mathbb{P}^1$ $\mathcal{O}_{\mathbb{P}^1}(-2)$ $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1)$

$\cong T^*\mathbb{P}^1$ is CY.

$\mathbb{P}^2$ $\mathcal{O}_{\mathbb{P}^2}(-3)$ $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2)$

$= K_{\mathbb{P}^2}$

· Eg 6: <u>Hypertoric var</u>

· Eg 7: Nakajima quiver var is HK.

$T^*\mathrm{Flag}$ HK

· Eg 8: Sometimes CY variety come from crepant resolution.

A resolution $f: X \to Y$ (f is birational, onto, X nonsing, Y is sing)

is crepant if $f^* K_Y \simeq K_X$.

- Suppose Y is normal ($\Rightarrow$ regular in codim 1)

($\Leftarrow$ singular locus has codim ≥ 2)

Let $Y_{ns} = Y \setminus Y_{sing}$, codim $Y_{sing} \geq 2$. $i: Y_{ns} \to Y$

$i_* K_{Y_{ns}} =: \omega_Y$. ω_Y need not be a line bundle.

· Def: Y is Gorenstein if ω_Y is a line bundle.

$\omega_Y = K_Y$ canonical bundle.

· Fact: Any hypersurface in a nonsingular var is Gorenstein.

$\underset{\text{sing}}{Y} \hookrightarrow X \implies K_Y \simeq K_X(Y)|_Y$ is a line bundle.

· Eg: $\qquad \Gamma = \mathbb{Z}/n\mathbb{Z} \qquad \mathbb{Z}/2\mathbb{Z}$

Thm: $\Gamma \underset{\text{finite}}{\subseteq} SL_2(\mathbb{C})$, $\Gamma \curvearrowright \mathbb{C}^2$ $\quad \mathbb{C}^2/\Gamma$ is a hypersurface in $\mathbb{C}^3$

$$\Gamma = \mathbb{Z}/2\mathbb{Z} \quad \Gamma \curvearrowright \mathbb{C}^2 \implies \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[z_1, z_2]^\Gamma$$

$$= \left\langle \begin{bmatrix} \varepsilon_2 & \\ & \varepsilon_2^{-1} \end{bmatrix} \right\rangle \qquad = \operatorname{Spec} \mathbb{C}[z_1^2, z_2^2, z_1 z_2]$$

$$= \operatorname{Spec} \mathbb{C}[X, Y, Z]/(XY - Z^2)$$

$$\Gamma = \mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{C}^2 \implies \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[z_1, z_2]^\Gamma$$

$$= \left\langle \begin{bmatrix} \varepsilon_n & \\ & \varepsilon_n^{-1} \end{bmatrix} \right\rangle \qquad = \operatorname{Spec} \mathbb{C}[z_1^n, z_2^n, z_1 z_2]$$

$$= \operatorname{Spec} \mathbb{C}[X, Y, Z]/(XY - Z^n)$$

· Thm: $\mathbb{C}^2/\Gamma$ admits a crepant resolution

$$\widetilde{\mathbb{C}^2/\Gamma} \xrightarrow{f} \mathbb{C}^2/\Gamma \qquad f^* K_{\mathbb{C}^2/\Gamma} \simeq K_{\widetilde{\mathbb{C}^2/\Gamma}}$$

$$\simeq f^* \mathcal{O}_{\mathbb{C}^2/\Gamma} \simeq \mathcal{O}_{\widetilde{\mathbb{C}^2/\Gamma}}$$

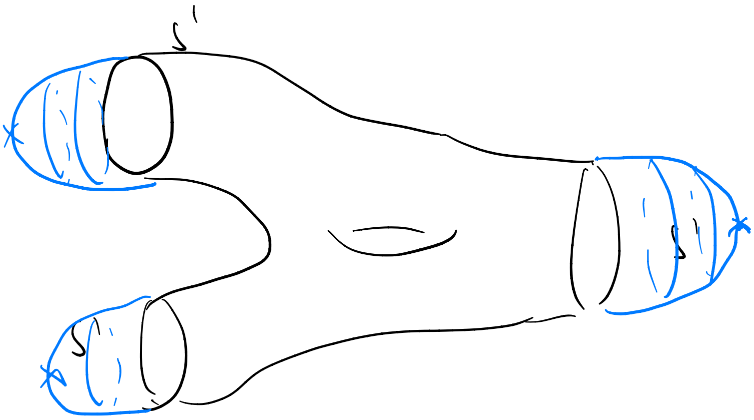
In particular, $\widetilde{\mathbb{C}^2/\Gamma}$ is a CY (HK & Nak quiver var).

$$K_{\mathbb{C}^2/\Gamma} \simeq \mathcal{O}_{\mathbb{C}^3}(\mathbb{C}^2/\Gamma)\big|_{\mathbb{C}^2/\Gamma} \simeq \mathcal{O}_{\mathbb{C}^3}\big|_{\mathbb{C}^2/\Gamma} \simeq \mathcal{O}_{\mathbb{C}^2/\Gamma}$$

$$\Gamma = \mathbb{Z}/2\mathbb{Z} \qquad \widetilde{\mathbb{C}^2/\mathbb{Z}/2\mathbb{Z}} = T^*\mathbb{P}^1 = \begin{array}{c} \mathcal{O}_{\mathbb{P}^1}(-2) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

$$\mathbb{P}\left(\begin{array}{c} \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \\ \downarrow \\ \mathbb{P}^1 \end{array} \right) = F_2$$

closed string



X mfd

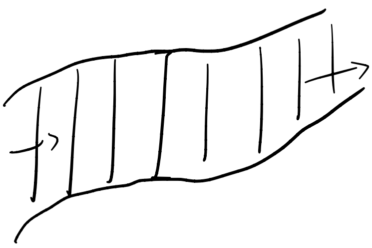
$\mathcal{L}X = \mathrm{Map}(S^1, X)$

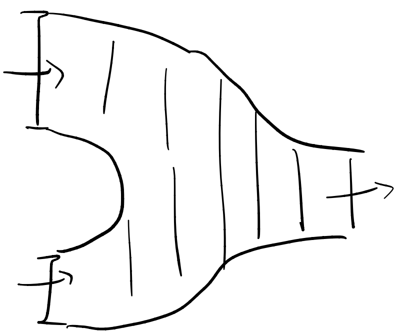
Open string I closed interval

$I \longrightarrow X$ mfd

consider how I changes.

But open string has bdry conditions





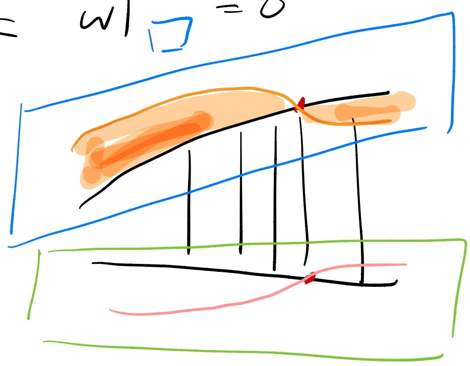
For A-model

Hope bdry $\longrightarrow$ V.S.

$I \qquad \int \omega = 0$

$\Rightarrow \square$ isotropy submfd

$\Leftarrow \omega|_{\square} = 0$



$\square$ isotropy submfd

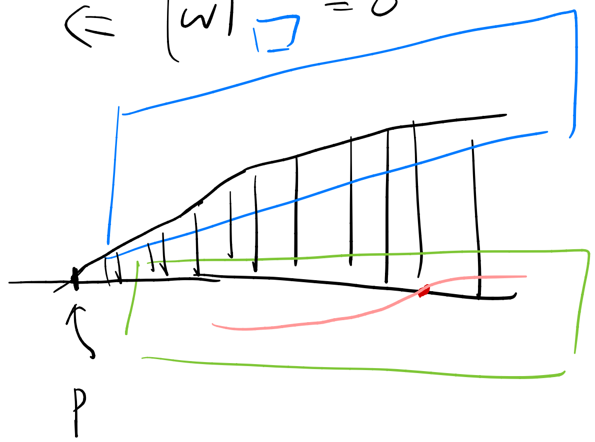
For A-model

Hope body $\longrightarrow$ V.S.

$$\int_{\cancel{\quad}} w = 0$$

$\Rightarrow$ $\square$ isotropy submfd

$\Leftarrow$ $|w|_{\square} = 0$



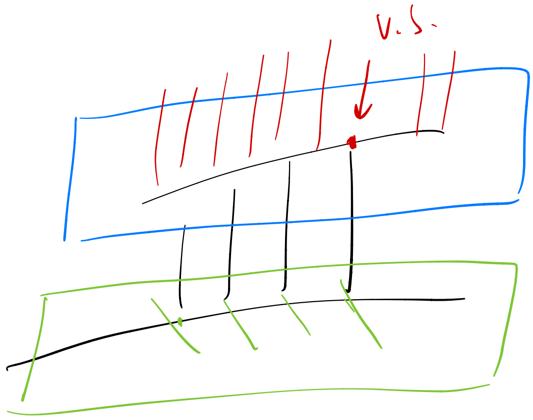
$\square$ isotropy submfd.

$p \in \square \cap \square$ $\Longleftrightarrow$ blue & green generically intersect

$\Longleftrightarrow$ $\boxed{\text{Lags !}}$

B-model

$\cdot \longrightarrow$ V.S.



$\Longrightarrow$ VB. E over cpx submfd. Y

$i : Y \hookrightarrow X$

$i_* E$ coh$/X$.

- HMS conj:

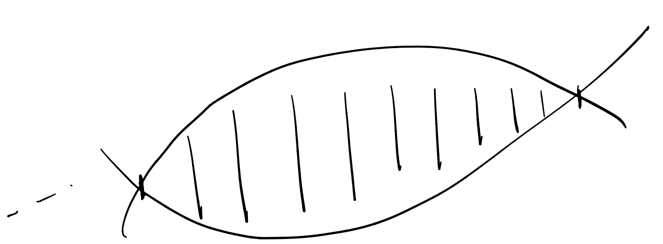
If X & $\check{X}$ are mirror CY mfds,

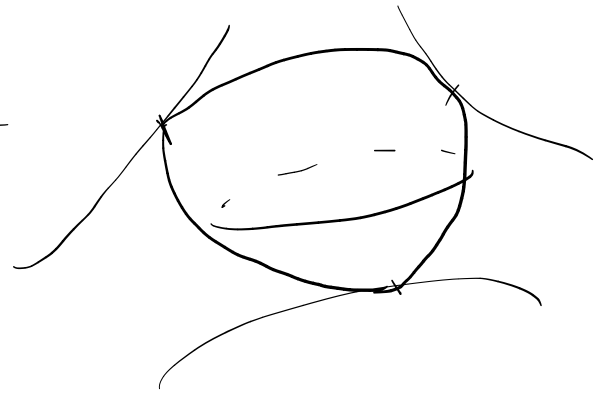
then $D\,\mathrm{Fuk}(X) \simeq D^b\mathrm{coh}(\check{X})$ equivalent as triangulated cat.

$CF^{\cdot}(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} k\langle p\rangle$ $\qquad$ $R\mathrm{Hom}(E_1, E_2) \cong \mathrm{Ext}^k(E_1, E_2)$

(intersection)

(objs are bdy of open strings
Mor ... intersections)





- Rmk:

(1) Seidel, Sheridan

HMS: $\quad D\,\mathrm{Fuk}(X_k, \omega_k) \xrightarrow{\sim} D^b\mathrm{coh}(\check{X}_{\psi(k)}, J_{\psi(k)})$

$\psi: \mathcal{M}_{K\ddot{a}h}(X_k) \longrightarrow \mathcal{M}_{cpx}(\check{X})$ $\quad$ mirror map

(2) $GW(X_k, \omega_k)$ $\qquad$ $GW(\check{X}_{\psi(k)}, \omega)$

Period (X_k, J) $\qquad$ Period $(\check{X}_{\psi(k)}, J_{\psi(k)})$

(3) Triangulated cat. $\Rightarrow$ closed subsch $\longleftrightarrow$ Ideal sheaf

$\updownarrow$ HMS

'Lags'

(4) Bridgeland stability condition coincides

$\Rightarrow$ moduli of stable objs is isomorphic

A - side X B - side $\check{X}$

GV invariant $=$ 'Counting of stable Lags'

why torus ?

· Thm (Arnold - Liouville)

If $f: X \to B$ is a proper Lag fib with connected fiber (without singular fibers), then $f^{-1}(b)$ is a cpt torus.

$\forall b \in B$

· HMS

If HMS is true for X & $\check{X}$ i.e.

$$D^b Fuk(X) \simeq D^b coh(\check{X})$$

then

$$\begin{array}{ccc} \cup & & \cup \\ L & \longmapsto & \mathcal{O}_p \end{array}$$

& $HF^{\cdot}(L) \cong Ext^*(\mathcal{O}_p, \mathcal{O}_p) \cong \Lambda^* V$, $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} \check{X}$

$\simeq H^*(L)$

$\Rightarrow$ L is a topologically torus

· Eg: $V = \mathbb{C}^2$

$$\Lambda^0 \mathbb{C}^2 \cong \mathbb{C}$$

$$\Lambda^1 \mathbb{C}^2 \cong \mathbb{C}^2$$

$$\Lambda^2 \mathbb{C}^2 \cong \mathbb{C}$$

· SYZ conj:

$\check{X} = Hilb^1(\check{X}) =$ moduli space of Lags with flat $U(1)$-connection in X

SYZ conj : (ideal case)

Given X : CY n-fold. (1) $\exists$ $f : X \longrightarrow B$ s.t.

f is a (special) Lag torus fib $\left(\Rightarrow B = \frac{1}{2} \dim_{\mathbb{R}} X \right)$

(2) $\check{X}$ can be constructed as T-duality (torus)

$$\check{X} := \coprod_{b \in B} (f^{-1}(b))^{\vee}$$

conj:

X & $\check{X}$ are mirror CY n-folds. Then

$\exists$

$$\begin{tikzcd} X \arrow[dr, "f"'] & & \check{X} \arrow[dl, "\check{f}"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} \check{f}^{-1}(b) = (f^{-1}(b))^{\vee} \arrow[d] \\ \ni b \end{tikzcd}$$

s.t. f & $\check{f}$ are (special) Lag torus fib.

furthermore, for generic $b \in B$, (smooth)

$$(f^{-1}(b))^{\vee} \simeq \check{f}^{-1}(b).$$

· Rmk: $f^{-1}(b) \simeq T^n \simeq V/\Lambda \simeq H_1(T^n, \mathbb{R}) / H_1(T^n, \mathbb{Z})$ $\quad \Lambda \underset{\text{lattice}}{\subseteq} V$

$$(f^{-1}(b))^{\vee} := V^*/\Lambda^* \qquad \Lambda^* := \underset{\text{group}}{\mathrm{Hom}}(\Lambda, \mathbb{Z})$$

$$\simeq H^1(T^n, \mathbb{R}) / H^1(T^n, \mathbb{Z})$$

$$\simeq \underset{\text{group}}{\mathrm{Hom}}(H_1(T;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \simeq \mathrm{Hom}(H_1(T;\mathbb{Z}), U(1))$$

$$\simeq (U(1))^n \simeq \check{T}^n$$

· Fact: $(\check{T}^n)$ is the moduli space of

The flat $U(1)$-connections over T^n

$\Big($ $p \in (\check{T}^n) \longleftrightarrow \nabla$ over T^n

$\text{holo}: H_1(T;\mathbb{Z}) \to U(1) \longleftrightarrow \nabla$

$\cap$

$\text{Hom}(H_1(T;\mathbb{Z}), U(1)) \simeq (\check{T}^n)$

· Rmk: (1) It's hard to find Lag torus fib.

(Ω. L is special if $\text{Im}\,\Omega|_L = 0$)

(2) Existence of singular fibers. (dual $\check{X}$).

· Idea:

Cor of Conj:

Given a CY n-fold X that has a special Lag torus

fib $f: X \to B$.

Then $\check{X}$ = moduli of special Lags (in the fiber of f w.r.t. f)

together with a flat $U(1)$-connection.

$= \{ (L, \nabla) \mid L \underset{\substack{\text{special}\\ \text{Lag}}}{\subseteq} X,\ \nabla: U(1)\text{-connection} \}$

special $\overset{\text{Auroux}}{\Rightarrow}$ $\check{X}$ has a symplectic structure.

$\overset{\text{TY}}{\Longrightarrow}$ Ham det class of a Lag contains at most 1

Special Lag.

(Joyce) $\Longrightarrow$ 'Stable' obj in Fukaya cat.

$$T_{(L,\nabla)}\check{X} \overset{\text{McLean}}{\simeq} H^1(L;\mathbb{R}) \oplus V \subsetneq C^\infty(NL) \oplus \Omega^1(L;\mathbb{R})$$

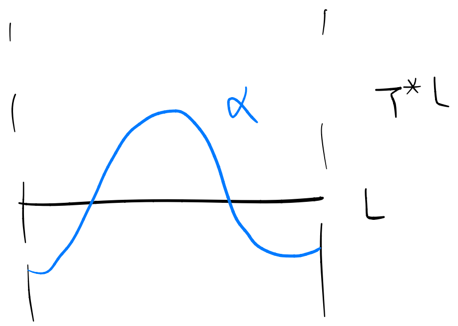
$$\uparrow A$$

L cpt smooth $\xrightarrow[\text{thm}]{\text{Weinstein nbd}}$ $\exists\ L \subseteq U \underset{\text{open}}{\subseteq} X$ s.t.

$$U \simeq T^*L$$

closed 1-form of L, $\alpha \in \Gamma(T^*L)$

submfd in T^*L. $\overset{\text{closed}}{\Longrightarrow}$ α is Lag.



exact 1-form of L, $\alpha = dg \in \Gamma(T^*L)$

$\overset{\text{exact}}{\Longrightarrow}$ $\alpha = dg$ Hamilton def of L.

$$dg = \iota_{X_g}\omega \qquad X_g \text{ v.F.}$$

$\Rightarrow$ Fact: $H^1(L;\mathbb{R})$ is def space of a smooth cpt Lag.

$\Rightarrow$ Idea:

$$\begin{array}{ccc} \check{X} & & X \\ \Downarrow & & \cup\!\mid \\ P & \longrightarrow & L \\ \cap\!\mid & & \\ U_P & \longleftarrow & \mathrm{Def}(L) \end{array}$$

$\Rightarrow$ $\check{X}$ = gluing of Maurer-Cartan def spaces of Lag

· Eg:

V v.s. of $\dim_{\mathbb{R}} n$.

A)

T^*V $\quad V = (y_1, \cdots, y_n)$

$\Lambda^*|_p := \mathbb{Z}\langle dy_1, \cdots, dy_n\rangle|_p \in T^*_pV$

$\Lambda^* \curvearrowright T^*V$

$\Rightarrow T^*V/\Lambda^* \leftarrow T^n (\simeq T^*_pV/\Lambda^*_p)$

$\downarrow$ Lag torus fib

V

$\boxed{V = \mathbb{R}}$

B)

TV

$\Lambda|_p = \mathbb{Z}\langle \partial y_1, \cdots, \partial y_n\rangle|_p \in T_pV$

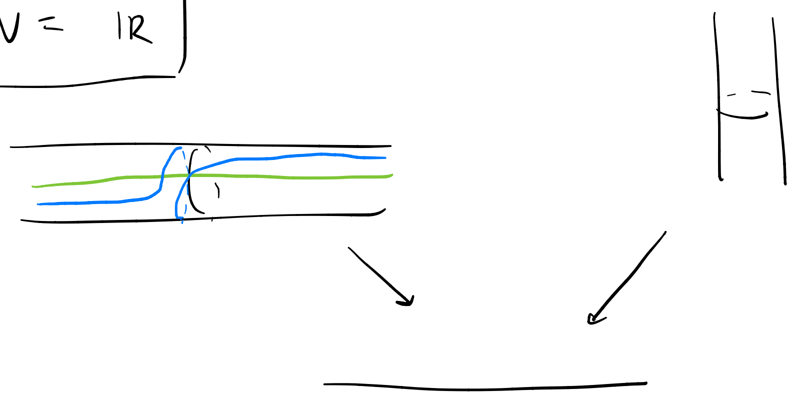
$\Lambda \curvearrowright TV$

$\Rightarrow TV/\Lambda \leftarrow \check{T}^n$

$\downarrow$ torus fib

V

(Fact: cpx structure)



· Expectation: $\quad \mathrm{Fuk}(X) \xrightarrow{\Phi} D^b(\check{X})$

• (L, ∇) Lag torus fiber $\longmapsto \mathcal{O}_p$

($HF^*((L,\nabla),(L,\nabla)) \longmapsto \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) = H^*(T)$)

(LYZ Chan) • Lag section $\longmapsto$ line bundle

· Beyond CY - case

Fano $\xleftrightarrow{SYZ}$ LG model

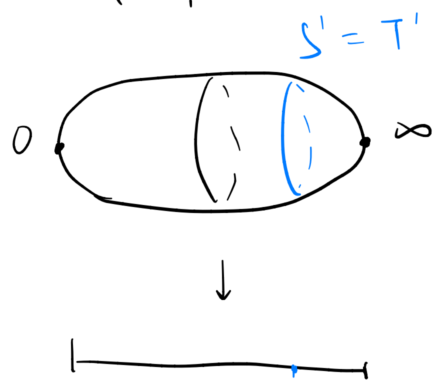
· Conj: (Auroux & SYZ)

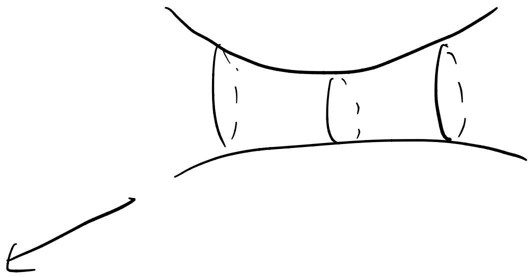
Given a Fano variety X. $D \in |-K_X|$ s.n.c.

$\Rightarrow$ $\check{X}$ is conj. to be moduli space of (special) Lags with flat $U(1)$-connection in $X \setminus D$ together with a function W called superpotential.

· Eg: $X = \mathbb{P}^1$, $D = \{0, \infty\}$ $\xleftrightarrow{\text{Hori-Vafa}}$ $(\mathbb{C}^*, W)$

$S^1 = T^1$





$X \setminus D \simeq \mathbb{C}^*$

(1) Toric Fano

$X \setminus D \xleftrightarrow{SYZ} (\mathbb{C}^*)^n$

(2) body $\longleftrightarrow$ W

$$(X, D) \rightsquigarrow \text{LG model}$$

Fano

$$X \setminus D \xrightarrow{SYZ} \check{X}$$

$(\mathbb{C}^*)^n$

$$D \longrightarrow W$$

superpotential

$\Rightarrow$ X Fano $\longrightarrow$ $(\check{X}, W)$ LG - model.

· Eg: $X = \mathbb{P}^1$, $D = \{0, \infty\}$

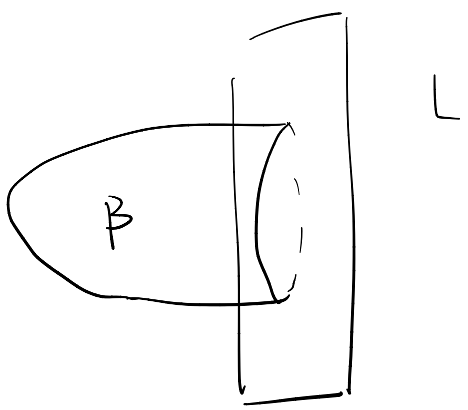
$$\mathbb{P}^1 \setminus D \simeq \mathbb{C}^* \longrightarrow \mathbb{C}^*$$

$$D \longrightarrow W = z + \frac{q}{z}$$

· Def: Superpotential

$$W_L = \sum_{\substack{\beta \in \pi_2(X, L) \\ \mu(\beta) = 2}} n(\beta) \cdot \exp\left(-\int_\beta \omega\right) \cdot \mathrm{hol}_\nabla(\partial\beta)$$

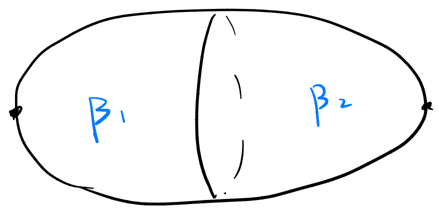
$n(\beta) = \#$ of discs



· Prop (Auroux): For toric Fano,

Disc β. $\mu(\beta) = 2\cdot[\beta\cdot D]$

· Eg: $\mathbb{P}^1$ $\qquad$ L



$$W_L = \sum_{\substack{\mu(\beta)=2 \\ \beta\in\pi_2(X,L)}} n(\beta)\exp\left(-\int_\beta \omega\right) \mathrm{hol}_\nabla(\partial\beta)$$

$$= \exp\left(-\int_{\beta_1}\omega\right)\mathrm{hol}_\nabla(\partial\beta_1) + \exp\left(-\int_{\beta_2}\omega\right)\mathrm{hol}_\nabla(\partial\beta_2)$$

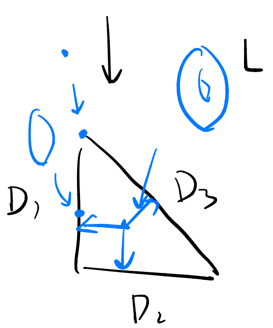
$z' := \mathrm{hol}_\nabla(\partial\beta_1)$

$$= \exp\left(-\int_{\beta_1}\omega\right) z' + \exp\left(-\int_{\beta_2}\omega\right)\cdot\frac{1}{z'}$$

$$= z + \exp\left(-\int_{\beta_1+\beta_2}\omega\right)\frac{1}{z} = z + \frac{q}{z}$$

$$n(\beta_1) = n(\beta_2) = 1$$

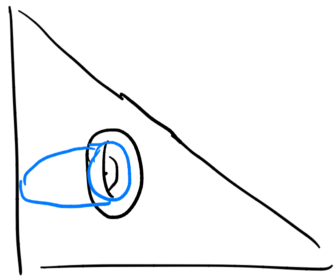
· Eg: $\mathbb{P}^2$, $D = D_1 + D_2 + D_3$ $\qquad$ $\mathbb{P}^2 \setminus D \cong (\mathbb{C}^*)^2$



$$\check{X} \simeq (\mathbb{C}^*)^2$$

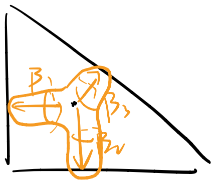
$$W = \sum_{\substack{\mu(\beta)=2 \\ \beta \in \pi_2(X, L)}} n(\beta) \cdot \exp\left(-\int_\beta \omega\right) \cdot \mathrm{hol}_\nabla(\partial\beta)$$

$$W = \exp\left(-\int_{\beta_1} \omega\right) \underbrace{\mathrm{hol}_\nabla(\partial\beta_1)}_{z_1'} + \exp\left(-\int_{\beta_2} \omega\right) \underbrace{\mathrm{hol}_\nabla(\partial\beta_2)}_{z_2'}$$



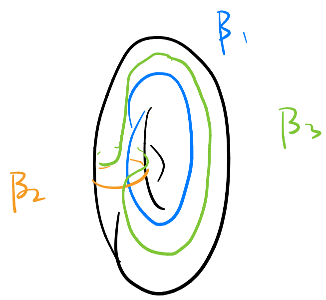
$$+ \exp\left(-\int_{\beta_3} \omega\right) \underbrace{\mathrm{hol}_\nabla(\partial\beta_3)}_{\frac{1}{z_1' z_2'}}$$

$$= z_1 + z_2 + \frac{q}{z_1 z_2}$$



$$q = \exp\left(-\int_{S^2} \omega\right)$$

$\beta_3 = \beta_1 + \beta_2$



- For toric Fano, see Cho–Oh, Auroux.