"Nonabelian analogue" : Hodge Thm.

X : cpt. Kä. / sm. proj.

$$(H^k_{sing}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \overset{\text{DeRham}}{\cong} H^k_{DR}(X,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$$
$$\cong \mathcal{H}^k_{harm}(X,g)_{\mathbb{C}}$$
$$\overset{\text{Hodge}}{\cong} H^k_{Dol}(X,\mathbb{C})$$

Particularly $k=1$

$$H^1(X,\mathbb{C}) \overset{\text{Hurewicz}}{\cong} \mathrm{Hom}(\pi_1^{ab}(X),\mathbb{C})$$
$$\text{“}=\text{”}\ \mathrm{Hom}(\pi_1(X),GL(1,\mathbb{C}))$$
$$\cong H^1_{Dol} = H^1(\mathcal{O}_X)\oplus H^0(\Omega^1_X)$$

flat $\mathcal{C}^\infty$ (V,∇) $\leftrightarrow$ $(E$ holo , $\theta)$

General Picture

"Topo."		"smooth"	$\mathcal{H}^1_{Harm}$	"holo."
$H^1_B(X,G)$	$\cong$	$H^1_{dR}(X,G)$	$\cong$	$H^1_{Dol}(X,G)$
$\mathrm{Hom}(\pi_1(X),G)$ ↑		↑		↑
G - rep.		flat G-conn.		G-Higgs bd.

$$\begin{array}{ccc} & & \text{Harmonic } (\exists \text{ metric}) \end{array}$$

Harmonic ($\exists$ metric) $\xleftrightarrow{\text{Donaldson Corlette}}$ $\mathcal{C}^\infty$ FlatVb.

Harmonic ($\exists$ metric) $\xleftrightarrow{\text{D-U-Y}}$ Stab $\mathcal{C}^\infty$ Higgs

Harmonic ($\exists$ metric) $\longleftrightarrow$ Flat λ-Vb.

LocSys. $\xleftrightarrow{\text{RH}}$ $\mathcal{C}^\infty$ FlatVb.

LocSys. $\xleftrightarrow{\text{RH}}$ Rep(π_1)

$\mathcal{C}^\infty$ FlatVb. $\xleftrightarrow{\text{Koszul Malgrange}}$ holo. FlatVb.

Stab $\mathcal{C}^\infty$ Higgs $\longleftrightarrow$ Stab Higgs

holo. FlatVb. $\xleftrightarrow{\text{NAHC}}$ Stab Higgs

Flat λ-Vb. $\xrightarrow{\lambda=1}$ holo. FlatVb.

Flat λ-Vb. $\xrightarrow{\lambda=0}$ Stab Higgs

Cpt. case $G = U(n)$

Thm. (M. S. Narasimhan and C. S. Seshadri, Stable and unitary bundles on a compact Riemann surface, *Ann. of Math.*, **82** (1965), 540-564.

Seshadri, C.S.: Space of unitary vector bundles on a compact Riemann surface. Ann. Math. (2) **85**, 303–336 (1967))

holo. $E \to \Sigma_{g \geq 2} =: X$

$$\mathrm{Vect}^s_X(r, \deg = 0) \cong \mathrm{Hom}^{irr}(\pi_1(X), U(r))$$

Mumford : GIT "stable"

Higher dim. via "Diff. Geo."

Thm. (cpt. Kä. / proj. var. K. Uhlenbeck & S. T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **34** (1986) 257–293.

S. K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geometry **18** (1983) 269–277.

S. K. Donaldson, Anti self dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.* (3), **50** (1985), 1-26.

S. K. Donaldson, Infinite determinants, stable bundles, and curvature, *Duke Math. J.*, **54** (1987), 231-247.)

indecomposable E "stable" iff

"$\exists$" HE. metric h : Chern D_h

s.t. $\Lambda_h F_{D_h} = 2\pi i \, \lambda \cdot \mathrm{Id}_E$, $\lambda \in \mathbb{R}$

i.e.

$$\mathrm{Vect}^s_X(r, c_{i \geq 1} = 0) \cong U\mathrm{Flat}^{irr}_X(r) \cong \mathrm{Hom}^{irr}(\pi_1(X), U(r))$$

General G ? Ans : D. C., Hitchin, Simpson ...

$$\mathrm{Higgs}^s_X(G, c_i = 0) \cong \mathrm{Harm}(G) \cong \mathrm{Hom}^{irr}(\pi_1(X), G)$$

If X : nonsing. conn. curve $g \geq 2$

Construction from GIT :

alg. var. $\mathcal{N}_{r,d}^{g(ss)} := \{$ semistable $rk = r,\ \deg = d \}$

Fact "coprime" $= \mathcal{N}_{r,d}^{g(s)} \overset{C^\infty}{\cong} \mathcal{N}_{YM}$

i.e. $\gcd(r,d) = 1$ $\quad$ "smooth" proj. $\dim_{\mathbb{C}} = r^2(g-1)+1$

[Atiyah & Bott (1981) Yang-Mills over Rie.]

$$\mathcal{N}_{YM} = \mathcal{N}_{r,d} \xrightarrow[\sim]{C^\infty} \{A, B_{1\to g} \mid [A_i, B_i] = e^{\frac{2\pi\sqrt{-1}d}{r}} \mathrm{id}\} / U(r) \text{ conj.}$$

$\Downarrow$

$\mathbb{R}$-mfd. $\quad$ "complexification"

hyperKä

$$\mathcal{M}_{Hit} \overset{C^\infty(I)}{\cong} \mathcal{M}_{Dol} := \{S\text{-equi. ss. Higgs}\} \overset{\theta(J)}{\cong} \cdots /\!/ GL(r)$$

Zar. open $\cup$ (r,d) cop. $\qquad$ qisproj. sch.

$T^* \mathcal{N}_{r,d}^g$ $\qquad$ "smooth" $\dim_{\mathbb{C}} = 2r^2(g-1)+2$

Fact (N. Nitsure, *Moduli spaces of semistable pairs on a curve*, Proc. Lon. Math. Sot. **62** (1991), 275–300)

"since"

$$T^*_{[\mathcal{E}]} \mathcal{N}_{r,d}^g = H^1(\Sigma_g, \mathcal{E}nd\ \mathcal{E}) \overset{\text{Serre}}{\cong} H^0(\Sigma_g, K \otimes (\mathcal{E}nd\ \mathcal{E})^*)$$

Generally

$$\mathcal{M}_B(GL_n) \overset{\text{biholo.}}{\cong} \mathcal{M}_{DR}(GL_n) \overset{\text{homeo.}}{\cong} \mathcal{M}_{Dol}(GL_n)$$

"Abelian Prototype"

Classical Abel - Jacobi

$$H_1(X,\mathbb{Z}) \hookrightarrow H^0(X,\Omega^1_X)^* \cong \mathbb{C}^g$$
$$[\gamma] \mapsto (\omega \mapsto \int_{\gamma_i} \omega)$$

$$\Rightarrow \mathrm{Jac}(X) := \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})} = \mathbb{C}^g/\mathbb{Z}^{2g}$$

$$\mathrm{Vect}_X(1, \deg = 0) \cong \mathrm{Hom}(\pi_1(X), \underline{U(1)})$$
$$\overset{\shortparallel}{\mathrm{Pic}^0(X)} \overset{A\cdot J}{\cong} \mathrm{Jac}(X) = \overset{\shortparallel}{(U(1))^{2g}}$$

Eg. (rk = 1)

$$M_B(\Sigma_g, 1) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g} \quad \text{affine}$$
$$= (S^1)^{2g} \times (\mathbb{R}^*_+)^{2g}$$
$$\text{S}\uparrow \text{ "homeo."}$$

$$M_{Dol}(\Sigma_g, 1) = \mathrm{Jac}(\Sigma_g) \times H^0(\Sigma_g, K) \cong T^*\mathrm{Jac}(\Sigma_g)$$
$$\downarrow$$
$$H^0(\Sigma_g, K) = \mathbb{C}^g$$

by $\theta \mapsto (\ldots, \exp(-\int_{\gamma_i}(\theta+\bar\theta)), \ldots)$

Remark.

$$M_B(\Sigma_g, 1) \underset{T}{\cong} M_{Dol}(\Sigma_g, 1)$$

not biholo. / alg. since M_B affine

Remark 1. Why "stable"

"jumping phenomenon"

$\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \mathbb{C}$

$$\Rightarrow \quad 0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to V_t \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

$$" \begin{cases} \mathcal{O} \oplus \mathcal{O} & , t \neq 0 \\ \mathcal{O}(-1) \oplus \mathcal{O}(1) & , t = 0 \end{cases}$$

Also "unbounded" ...

Def. E (semi)stable

$$\forall\ E' \subsetneq E \qquad \mu(E') := \frac{\deg E'}{\mathrm{rk}\ E'} \le \mu(E)$$

[(Via D.G.) $E \to X$ with $\forall$ 1-d-conn. ∇

$$\deg E = \int_X \Big[\underbrace{\tfrac{i}{2\pi} \mathrm{Tr}\ \nabla^2}_{\text{"}ch_1(E) = c_1(E)\text{"}}\Big] \wedge \omega^{\dim X - 1}$$
]

Remark 2. Why $g \geq 2$

($g = 0$) Grothendieck

$$V = \oplus\, \mathcal{O}(a_i), \quad a_1 \geq a_2 \geq \cdots$$

$\exists\ \mathcal{O}(a_1) \le V$

$$\text{s.t.} \quad \frac{\deg \mathcal{O}(a_1)}{\mathrm{rk} = 1} = a_1 \geq \frac{\deg V}{\mathrm{rk}\ V} = \frac{\sum a_i}{n} \qquad \times$$

($g = 1$) Atiyah

indecompo. $V \rightsquigarrow$ semistable

$\Rightarrow$ stable iff $(r, d) = 1$

$$\text{Loring W. Tu} \rightsquigarrow \begin{cases} \mathcal{N}^s_{n,d} = \begin{cases} E & , (n,d) = 1 \\ \emptyset & , (n,d) \neq 1 \end{cases} \\ \mathcal{N}^{ss}_{n,d} = S^h E, \quad h := \gcd(n,d) \end{cases}$$

Step 1 Riemann - Hilbert

Thm. (Classical RH)

$$\{\text{Rep. } \pi_1\} \xleftrightarrow{\sim} \{\text{Loc}\} \xleftrightarrow{\sim} \{\text{Flat bd.}\}$$

irreducible

no "ρ-inv." $W \subset \mathbb{C}^n$ other than $\{0\}$ & $\mathbb{C}^n$

where $\rho(\gamma)(W) \subset W$, $\forall \gamma \in \pi_1(X)$

$$\Updownarrow s$$

no proper holo. flat subbd. $(\tilde{\mathcal{E}}, D|_{\tilde{\mathcal{E}}}) \subset (\mathcal{E}, D)$

Remark.

"smooth" $A^k(E) := \Gamma(X, \Lambda^k(T_{\mathbb{C}}^* X) \otimes_{\mathbb{C}} E) \cong \bigoplus_{p+q=k} A^{p,q}(E)$

∇ : flat 1-d-conn. $= \nabla' + \nabla''$

$\Updownarrow$ $\exists$ "Dolbeault operator" $\bar{\partial}_E$: flat 1-$\bar{\partial}$-conn.

[$\xrightarrow{\text{extension}}$ $A^{p,q}(E) \xrightarrow{\bar{\partial}_E} A^{p,q+1}(E)$ by $\bar{\partial}\alpha \otimes s + (-1)^{p+q} \alpha \wedge \bar{\partial}_E(s)$]

"holo." $\Omega^k(\mathcal{E}) = \ker(\bar{\partial}_E : A^{k,0}(E) \to A^{k,1}(E))$

D : flat 1-d_h-conn. ($\mathcal{O}$-lin.)

where "holo. de Rham" $0 \to \mathcal{O}_X \xrightarrow{d_h} \Omega^1 \xrightarrow{d_h} \cdots$

Hence

$$\{\mathcal{C}^\infty \text{ flat}\} \xleftrightarrow{\sim} \{\text{holo. flat}\}$$

$$(E, \bar{\partial}_E, \nabla := D + \bar{\partial}_E) \longleftarrow (\mathcal{E}, D)$$

$$(E, \bar{\partial}_E = \nabla'', \nabla = \nabla' + \nabla'') \longmapsto (\mathcal{E}, D = \nabla')$$

Step 2 Harmonic World $\left[\mathcal{O}_X \xrightarrow{d_h} \Omega^1_X \to \cdots \right]$ "holo. diff."

Higgs bundle

"field" $\in H^0(X, \mathrm{End}\, \mathcal{E} \otimes \Omega^1_X)$

"holo." description $(\mathcal{E}, \phi : \text{flat } 0\text{-}d_h\text{-conn.})$

$$\underbrace{\mathcal{E} \xrightarrow{\phi} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\phi \otimes \mathrm{id}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\mathrm{id} \otimes (- \wedge -)} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X}_{\phi \wedge \phi = 0 \ \in H^0(X, \mathrm{End}\, \mathcal{E} \otimes \Omega^2_X)}$$

"smooth" description $(E, \bar\partial_E, \theta : \text{flat } 0\text{-}\partial\text{-conn.})$

s.t. $(D'')^2 = (\bar\partial_E + \theta)^2 = 0$

$$\text{i.e.} \begin{cases} (\bar\partial_E)^2 = 0 \\ \bar\partial_E \theta = -\theta \bar\partial_E \underline{\equiv 0} \ (\text{on } \mathcal{E}) \\ \theta \wedge \theta = 0 \end{cases} \Rightarrow (\theta, E) \to (\phi, \mathcal{E})$$

Eg. (dim = 1)

$$K_X := T^*_{1,0} X = \Omega^1_X$$

Hence

$$\mathcal{E} := K_X^{1/2} \oplus K_X^{-1/2}, \quad \theta := \begin{pmatrix} & q_2 \\ 1 & \end{pmatrix}$$

$$\mathcal{E} := K_X^1 \oplus \mathcal{O}_X \oplus K_X^{-1}, \quad \theta := \begin{pmatrix} & & q_3 \\ 1 & & \\ & 1 & \end{pmatrix}$$

where

$$1 : K_X^{1/2} \to K_X^{-1/2} \otimes K_X = K_X^{1/2}$$

$$q_2 : K_X^{-1/2} \to K_X^{1/2} \otimes K_X \in H^0(X, K_X^2)$$

$$q_3 \in H^0(X, K_X^3)$$

In general

$$\mathcal{E} = \mathrm{Sym}^{n-1}(K_X^{1/2} \oplus K_X^{-1/2}) = K_X^{n-1/2} \oplus K_X^{n-3/2} \oplus \cdots \oplus K_X^{-n+1/2}$$

$$\theta = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ 1 & 0 & q_2 & \cdots & q_{n-1} \\ & 1 & 0 & \ddots & \vdots \\ & & 1 & \ddots & \end{pmatrix} \qquad q_i : K_X^{n-2i+1/2} \to K_X^{n-1/2} \otimes K_X$$

$\forall\ \mathcal{C}^\infty : E \to X$, $\exists$ "hermitian metric" h

$$A^0(X, E) \times A^0(X, E) \to A^0_{\mathbb{C}}(X)$$

$$\xrightarrow{\text{extend}} A^p(X, E) \times A^q(X, E) \longrightarrow A^{p+q}_{\mathbb{C}}(X)$$

$$(\alpha \otimes u, \beta \otimes v) \mapsto \alpha \wedge \bar{\beta} \otimes h(u, v)$$

"Chern connection" s.t. $\exists !$ $D_h := \underline{\partial_h} + \bar{\partial}_E$

$$d(h(u, v)) = h(\underline{D_h}(u), v) + h(u, D_h(v))$$

with $\exists !$ ∂_h : 1 - ∂ - conn. of type $(1, 0)$

2.1 For $\boxed{\text{Higgs}}$ $(E, \bar{\partial}_E, \theta, h : \text{Herm.})$

Defining "adjoint" of type $(0, 1)$

i.e. $h(\theta(u), v) = h(u, \theta^*(v))$

pluri - harmonic metric "h"

$\mathcal{C}^\infty$ 1 - d - conn. $\mathbb{D}_h := D_h + \theta + \theta^*$

$$= \partial_h + \bar{\partial}_E + \theta + \theta^* \quad \text{flat}$$

Hitchin's SDE $\begin{cases} D_h^2 = -[\theta, \theta^*] \\ \bar{\partial}_h \theta = 0 \end{cases}$ $(F_h = 0)$

$\rightsquigarrow$ harmonic Higgs $(E, \bar{\partial}_E, \theta, h)$

Harmonic mertic "h"

$$F_h := (\mathbb{D}_h)^2$$

$$\underline{\Lambda_\omega}(F_h) = 0 \quad \text{w.r.t. Kä } \omega$$

where "contraction" is dual of Lefschetz

i.e. $h(\Lambda_\omega \alpha, \beta) := h(\alpha, \beta \wedge \omega)$

Herm-Yang-Mills (Herm-Einstein) metric

$$\Lambda_\omega(F_h) = \lambda \cdot \mathrm{Id}_E$$

$\hookrightarrow$ "topo. const." $= -2\pi i \dfrac{\deg E}{\mathrm{rk}\, E \cdot \mathrm{Vol}_\omega(E)}$

Since $\det(\mathrm{id} + \frac{\sqrt{-1}}{2\pi} F) = 1 + c_1(E) + c_2(E) + \cdots$

Hence

pluri-harmonic $\Leftrightarrow$ harmonic $\Leftrightarrow$ Hermitian-Einstein

$(ch_2(E) = 0)$ $\qquad$ $(c_1(E) = ch_1(E) = 0)$

"sketch" $\Downarrow$

$$\int_X ch_2(E) \wedge \frac{\omega^{\dim X =: m-2}}{(m-2)!} = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(F_h \wedge F_h) \wedge \frac{\omega^{m-2}}{(m-2)!}$$
$$= \frac{1}{8\pi^2} (\|F_h\|^2 - \|\Lambda_\omega(F_h)\|^2)$$

Particularly

$\dim = 1 \Rightarrow$ pluri-harmonic = harmonic

2.2 For $\boxed{\text{Flat}}$ $(E, D \overset{\exists!}{=} D_h + \Phi_h, h)$

unitary conn. $\nearrow$ $\qquad$ $\in$ self-adj. $A^2(X, \mathrm{End}(E))$

with

$$h(\Phi_h(u), v) := \frac{1}{2}(h(Du, v) + h(u, Dv) - dh(u, v))$$

Decompose by types :

$$D_h = \partial_h + \bar\partial_h, \quad \Phi_h = \varphi_h + \varphi_h^*$$

Rearrange

$$D_h' := \partial_h + \varphi_h^*, \quad D_h'' := \bar\partial_h + \varphi_h : \tfrac{1}{2}-\bar\partial\text{-conn.}$$

$\rightsquigarrow$ pseudo-curvature $\quad G_h := (D_h'')^2$

Similarly def.

pluri-harmonic metric $G_h = 0$

$\Rightarrow$ Higgs $(E, \bar\partial_h, \varphi_h)$ harmonic flat bd.

Moreover

harmonic metric $\Lambda_\omega(G_h) = 0$

Claim X : cpt. Kähler

h_{DR} "pluri-harmonic" iff h_{DR} "harmonic"

pf.

$D = d' + d'' \overset{\exists}{\Rightarrow} d' + \delta_h''$ & $d'' + \delta_h'$ unitary

put
$$\partial_h := \tfrac{1}{2}(d' + \delta_h') \qquad \bar\partial_h := \tfrac{1}{2}(d'' + \delta_h'')$$
$$\varphi_h := \tfrac{1}{2}(d' - \delta_h') \qquad \varphi_h^* := \tfrac{1}{2}(d'' - \delta_h'')$$

Then
$$\left.\begin{aligned} (D_h')^2 &= -(D_h'')^2 \\ D^2 &= (D_h' + D_h'')^2 = 0 \end{aligned}\right\} \Rightarrow D_h'' D_h' + D_h' D_h'' = 0$$

$$\Rightarrow (D_h^c := D_h'' - D_h')^2 = 0$$

by
$$G_h = (D_h'')^2 = \frac{1}{4}(D_h^c D + D D_h^c)$$
$$= \frac{1}{2} D(\varphi_h - \varphi_h^*)$$

$\Rightarrow$ Bianchi identities
$$\begin{cases} D_h^c G_h = [D_h^c, G_h] = 0 \\ D G_h = [D, G_h] = 0 \end{cases}$$

Apply to Kä id. $D^* G_h = \sqrt{-1}[\Lambda_\omega, D_h^c] G_h$

$= 0$ (by $\Lambda_\omega G_h = 0$)

Consequently
$$\|G_h\|_E^2 := \int_X \langle G_h, G_h \rangle \frac{\omega^m}{m!}$$
$$= \frac{1}{2}\int_X \langle \varphi_h - \varphi_h^*, D^* G_h \rangle \cdots = 0$$

i.e. $G_h = 0$

$\square$

Remark.

flatness $\nabla^2 = 0$ more "rigid" than

Higgs $(D_h'' = \bar\partial_E + \theta)^2 = 0$

Thm 0. (Dol $\overset{\text{Harm.}}{=}$ De Rham)

$\{$ Harmonic Higgs $\} \xrightarrow{\sim} \{$ Harmonic flat $\}$

$(E, \bar\partial_E, \theta, h) \mapsto (E, \bar\partial_E, D_h := (\bar\partial_E + \partial_h) + \theta + \theta^*, h)$

$(E, \bar\partial_h, \varphi_h, h) \leftarrow (E, \nabla = D_h + \Phi_h, h), (D_h'')^2 = 0$

Moreover preserve operation

$\oplus, (\ )^\vee, \otimes$ and funtorial f^*

pf.

Obj. : ✓

Mor. : Claim $H^0_{Dol}(E, D_h'') \cong H^0_{DR}(E, D_h)$

i.e. $D_h'' s = 0$ iff $D_h s = 0$

$\Rightarrow$ $\|D_h'(s)\| = \int_X \langle (D_h')^* D_h' s, s\rangle \frac{\omega^n}{n!}$

$\overset{\text{Kä}}{=} \int \langle -i\Lambda_\omega D_h' D_h'' s, s\rangle \frac{\omega^n}{n!} = 0$

hence $\nabla s = (D_h' + D_h'') s = 0$

$\Leftarrow$ $\|D_h^c\|^2 \overset{\text{Kä}}{=} \int_X \langle -i\Lambda_\omega D_h D_h^c s, s\rangle \frac{\omega^n}{n!} = 0$

since $D_h^c D_h = - D_h D_h^c$

then $D_h'' s = \frac{1}{2}(D_h + D_h^c) s = 0$

$\square$

Step 3 DG : $\exists$ Metric = AG : Stability

Higgs slope-(semi)stable

$\forall$ (ϕ-inv.) $F \subsetneq E$ i.e. $\phi(F) \subset F \otimes \Omega_X^1$

torsion free coh. with $0 < \operatorname{rk} F < \operatorname{rk} E$

s.t. $\mu(F) \lneq \mu(E)$

slope-polystable

$F = \oplus F_i$: stable s.t. $\mu(F_i) = \mu(F)$

Eg.

- $(g=0)$ $V = \oplus \mathcal{O}(a_i)$, $a_1 \geq a_2 \geq \cdots$

$\Phi = \begin{pmatrix} 0 & & \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{pmatrix}$ since $a_{i,j} = H^0(\mathcal{O}(a_i - a_j - 2)) = 0$

$\exists$ Φ-inv. : $(\mathcal{O}(a_1), 0)$

$\mu(\mathcal{O}(a_1)) = a_1 \geq \mu(V)$ ×

- "Trivial"

previous $(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, (\ ,\ ^q))$ ...

- $(g=1 \Rightarrow K_X = \mathcal{O}_X : \operatorname{rk} = 2)$

"decomposable" $V = L_1 \oplus L_2$, $\Phi = \begin{pmatrix} a & b \in H^0(L_2^* L_1) \\ c & -a \end{pmatrix}$

assume $\deg(L_2^* L_1 : \text{nontrivial}) \geq 0 \rightarrow c = 0$

but $\deg L_1 \geq \frac{1}{2} \deg(L_1 L_2)$ ×

Else $L_1 \cong L_2$ a . b . c const.

with $\deg = 0$ eigenspace ×

"indecomposable" Fix $\deg = 1$: $A \in \mathrm{Pic}(X)$

$\Rightarrow \mathcal{E}(2, d) \xrightarrow{\sim} \mathcal{E}(2, d+2)$

$V \mapsto V \otimes A$

i. $\mathcal{E}(2, \underset{\sim}{0})$

extension $0 \to \mathcal{O} \hookrightarrow V \to \mathcal{O} \to 0$

Φ inv. s.t. $\deg \mathcal{O} = 0 = \frac{1}{2}\deg(\Lambda^2 V)$ ✗

ii. $\mathcal{E}(2, \underset{\sim}{1})$

extension $0 \to \mathcal{O} \hookrightarrow V \to \mathcal{O}(1) \to 0$

<u>Only</u> stable pair $(V, 0)$ ✓

- $rk = 2$, $\Sigma_{g \geq 2}$, (V, Φ) stable

(Hitchin 86 prop. 3.3)

(i) *V is stable;*

(ii) *V is semi-stable and $g > 2$;*

(iii) *if V is semi-stable and $g = 2$ then $V \cong U \otimes L$ where U is either decomposable or an extension of the trivial bundle by itself;*

(iv) *V is not semi-stable and $\dim H^0(M; L_V^{-2}K \otimes \Lambda^2 V)$ is greater than 1, where L_V is the canonical subbundle;*

(v) *V is decomposable as*

$$V = L_V \oplus (L_V^* \otimes \Lambda^2 V) \quad \textit{and} \quad \dim H^0(M; L_V^{-2}K \otimes \Lambda^2 V) = 1.$$

Remark

"Betti" side

$\pi_1(E) \cong \mathbb{Z} \times \mathbb{Z} \to GL_n$

$ab = ba \mapsto \begin{cases} \deg = 0 & AB = BA \\ \deg = 1 & AB = -BA \end{cases}$

$AB = BA \rightsquigarrow$ shared eigenvect.

no "irr. rep."

Thm 1. (DUY / KH corr.)

Higgs $(E, \bar\partial_E, \theta)$ $\exists$ "pluri-harm." h

iff polystable $+$ $c_1 = ch_2 = 0$

(such "h" unique up to sclar mult.)

Sketch

$\Rightarrow$ Only check "saturated" subshf. $\mathcal{F} \leq \mathcal{E}$

i.e. $0 \to \mathcal{F} \to \mathcal{E} \to \underbrace{\mathcal{E}/\mathcal{F}}_{} \to 0$ "tor.-free"

$\mu(\mathcal{E}) \overset{?}{<} \mu(\mathcal{E}/\mathcal{F})$

Canonical $\mathscr{C}^\infty$-vectbd. splitting w.r.t. h

outside codim $\mathrm{Sing}(\mathcal{E}/\mathcal{F}) \geq 2$

$$0 \to \mathcal{F} \underset{\pi}{\rightleftarrows} \mathcal{E} \cong \mathcal{F} \oplus Q \xrightarrow{Id-\pi} Q := \mathcal{E}/\mathcal{F} \to 0$$

$\forall\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in A^0(X, \mathcal{E})$

write $D_h \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi \circ D_h & \beta \in A^1(Hom(Q,\mathcal{F})) \\ \alpha \in A^1(Hom(\mathcal{F}, Q)) & (Id-\pi)\circ D_h \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

compute that $\alpha = D_h'(\pi)$, $\beta = -D_h''(\pi)$

put $D_{\mathcal{F}} := \pi \circ D_h$

$$\Rightarrow \quad \pi \circ F_h \circ \pi = F_{D_{\mathcal{F}}} - D''(\pi) \circ D_h'(\pi)$$

Hence

$$\deg \mathcal{F} = \frac{i}{2\pi} \int_X \mathrm{Tr}(F_{D_{\mathcal{F}}}) \wedge \underline{\frac{\omega^{m-1}}{(m-1)!}}$$

$$= \frac{i}{2\pi} \left(\int_X \mathrm{Tr}(\pi \circ F_h) \wedge \cdot - \int_X \mathrm{Tr}(D''(\pi) \circ D_h'(\pi)) \wedge \cdot \right)$$

$$= -\| D''(\pi) \|^2 \leq 0$$

"$=$" iff $D''(\pi) = (\bar\partial_E + \theta)(\pi) = 0$

i.e. θ-inv. holo. decomp. $\mathcal{E} = \tilde{\mathcal{E}} \oplus Q$

$\rightsquigarrow$ $\mathcal{E} = \bigoplus (E_i, \bar{\partial}_{E_i}, \theta_i)$ polystable

$\Leftarrow$ [Hard:]

fixed $H(0) := K \in \mathrm{Herm}(E)$

$H(t) := K \cdot h(t)$, $h(t) \in \mathrm{Aut}(E)$ s.t. $h(t) = (h(t))^{*K}$

"Donaldson func."

$M : \mathrm{Herm}(E) \times \mathrm{Herm}(E) \to \mathbb{R}$

$$(H, K) \mapsto \int_0^1 \int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_H \cdot H^{-1}\frac{\partial H}{\partial t}) \frac{\omega^m}{m!} dt$$

Fact (path idp.)

$$\begin{cases} H_0(0) = H_1(0) = K \\ H_0(1) = H_1(1) \\ \det H_0 = \det H_1 = \det K \end{cases} \Rightarrow M(H_0, K) = M(H_1, K)$$

$\rightsquigarrow$ Choose explicit path $H(t) = Ke^{ts}$

s.t. $\mathrm{Tr}(S = H^{-1}\frac{\partial H}{\partial t}) = 0$

Along "Donaldson heat flow"

$$H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega F_H$$

$$\frac{d}{dt} M(H, K) = -\int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_H \cdot \sqrt{-1}\Lambda_\omega F_H) \frac{\omega^m}{m!}$$
$$= -\|\Lambda_\omega F_H\|^2_{L^2} < 0$$

Particularly

$$M(K, K) = 0$$

Prop. (Simpson estimate)

(1988) Constructing VHS using Y-M theory and app. to uniformalization prop. 5.3.)

For $H = Ke^{s}$ s.t. $Tr(S) = 0$, $\sup_x |S| < \infty$

stable $(E, \bar{\partial}_E, \theta)$, $\sup_x |\Lambda_\omega F_H| \le C$

$\Rightarrow \sup_x |S| \le C_1 + C_2 M(H, K)$

Sketch

$\| S_i \| \geq C_i M(K, H_i)$, $C_i \to \infty$

subseq. $S_{i_j} \to \lambda := S_\infty \in \underset{\{1^{st}\text{ deri } L^2\text{-integrable}\}}{\underset{\shortparallel}{W^{1,2}}}(X, End(E))$

s.t. $\begin{cases} \lambda^2 = \lambda^* = \lambda & \text{ortho. proj.} \\ (id - \lambda) \circ \bar{\partial}_E(\lambda) = 0 & \text{holo.} \end{cases}$

$$\Big\Downarrow \text{U-Y}$$

$\exists$ coh. $F \subset E$ with $\mathrm{codim}(S : \mathrm{sing}) \geq 2$

s.t. $\begin{cases} \lambda \in C^\infty(X \setminus S, End(E)) \\ \lambda^2 = \lambda^* = \lambda \ \& \ (id - \lambda) \circ \bar{\partial}_E(\lambda) = 0 \text{ on } X \setminus S \\ F|_{X \setminus S} = \lambda_{X \setminus S}(E|_{X \setminus S}) \overset{\text{holo.}}{\le} E|_{X \setminus S} \end{cases}$

Apply in Higgs $\bar{\partial}_E \rightsquigarrow D'' = \bar{\partial}_E + \theta$

Check

$\mu(F) \geq \mu(E)$!

"destabilize"

$\square$

$$\frac{d}{dt}|\Lambda_\omega F_H|^2 = 2\sqrt{-1}\,\mathrm{Tr}(\Lambda_\omega(F_H)\cdot\Lambda_\omega D'' D'_H(\Lambda_\omega F_H))$$

$$\Delta'_\omega |\Lambda_\omega F_H|^2 = -\partial^*\partial\,\mathrm{Tr}(\Lambda_\omega F_H \cdot \Lambda_\omega F_H)$$

$$\Downarrow$$

$$\left(\frac{d}{dt} + \Delta'_\omega\right)|\Lambda_\omega F_H|^2 = -2|D''(\Lambda_\omega F_H)|^2 \le 0$$

$\Big\}$ parabolic PDE

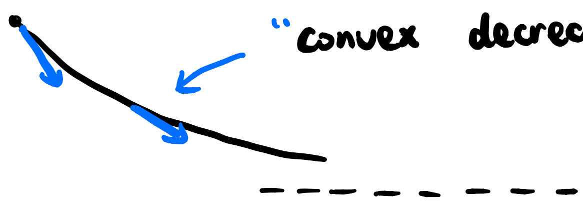
$$\sup_X |\underline{\Lambda_\omega F_H}| \le M$$

"uniformly bd."

Moreover

$$\frac{d^2}{dt^2}M(H,K) = \frac{d}{dt}\int_X \mathrm{Tr}(\Lambda_\omega F_H\cdot\Lambda_\omega F_H)\frac{\omega^m}{m!}$$
$$= 2\int_X |D''(\Lambda_\omega(F_H))|^2\frac{\omega^m}{m!} \ge 0$$

$f(0) = M(K,K) = 0$



"convex decrease" $\begin{cases} -M \le f' \le 0 \\ f'' \ge 0 \end{cases}$

Simpson lower bd.

$$t\to\infty, \quad \frac{d}{dt}M(H,K) = -\|\Lambda_\omega F_H\|^2_{L^2} \to 0$$

$$\exists\ s_i \xrightarrow{\text{wk cov.}} s_\infty \in W^{2,p} \underset{\text{regularity}}{\overset{\text{elliptic}}{\rightsquigarrow}} C^\infty \Rightarrow H_\infty = K\cdot e^{s_\infty}$$

Supply Fact (Existence)

$K \in \mathrm{Herm}(E)$ s.t. $|\Lambda_\omega(F_K)| \le C$

$\Rightarrow$ $\exists!$ sol. $H(t)$: Donaldson heat flow

s.t. $\det H = \det K$, $\forall\ t\in\mathbb{R}$

$\square$

Similarly

Thm 2.

flat (E, ∇) "pluri-harm." iff semisimple

sketch $(\Leftarrow)$

$$M(H, K) := \int_0^1 \int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega G_K \cdot H^{-1}\frac{\partial H}{\partial t})\frac{\omega^m}{m!}dt$$

almost same

- $M(H(0) = K, K) = 0$
- $\frac{d}{dt}M(H(t), K) = -2\|\Lambda_\omega(G_H)\|_{L^2}^2 < 0$
- $\sup_X |\Lambda_\omega(G_H)| \leq C$

  $\rightsquigarrow \sup_X |S| \leq C_1 + C_2 M(H, K)$
- Along flow "convex"

  i.e. $\frac{d^2}{dt^2}M(H(t), K) \geq 0$

$\Downarrow$ $t \to \infty$, $\frac{d}{dt}M(H(t), K) \to 0$ up to subseq.

$S(t_n)$ weakly cov. to $S_\infty \in W^{1,p}$

$\xrightarrow{\text{elliptic regularity}} h_\infty := e^{S_\infty} \in \mathcal{C}^\infty$

$H_\infty := K \cdot h_\infty$ pluriharmonic

$(\Rightarrow)$

$(E, \nabla = \bar\partial_E + \partial_h + \theta + \theta^*) \xrightarrow{\text{Thm 1.}} \bigoplus(E_i, \bar\partial_{E_i}, \theta_i)$

$\rightsquigarrow (E, \nabla) = \bigoplus(E_i, \nabla_i := \bar\partial_{E_i} + \partial_{h_i} + \theta_i + \theta_i^*)$

Further Generalization

λ-connection ($\lambda \in \mathbb{C}$)

- holo. $(\mathcal{E}, D^\lambda : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$ s.t.

"λ-twisted Leibniz":

$$D^\lambda(fs) = f D^\lambda s + \lambda s \otimes df$$

"flat" $(D^\lambda + \bar{\partial})^2 = 0$

- $\mathcal{C}^\infty$ $(E, \bar{\partial}_E, D^\lambda : \mathcal{A}^0(X, E) \to \mathcal{A}^1(X, E))$

s.t. $D^\lambda(fs) = f D^\lambda s + \lambda s \otimes \partial f + s\, \bar{\partial}_E f$

"flat" $(D^\lambda)^2 = 0$

Particularly

$$\lambda = \begin{cases} 0 & \rightsquigarrow \text{Higgs} \\ 1 & \rightsquigarrow \text{usual flat} \end{cases}$$

Mochizuki (2009)

K-H correspondence for tame harm. bd. II

Thm 3.

λ-flat (E, D_λ) $(\lambda \neq 0)$ pluri-harm. iff polystable + $c_i = 0$

Another Approach

($m = 1$, $rk = 2$) Donaldson (1987)

Twisted harmonic maps and self-duality eq.

(General) Corlette (1988)

Flat G-bd. with canonical metrics

sketch

Step 1.

monodromy $\rho : \pi_1(X) \to GL_n(\mathbb{C})$

$$\Downarrow \text{RH}$$

flat vb. $(E := \tilde{X} \times_\rho \mathbb{C}^n, \nabla := d)$

$h \in \mathrm{Herm}(E, \nabla)$ iff ρ-equi. $h_\rho : \tilde{X} \to GL/U(n)$

i.e. $h_\rho(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot h_\rho(\tilde{x})$

$$:= (\rho(\gamma)^{-1})^* h_\rho(\tilde{x}) \rho(\gamma)^{-1}$$

since

$\Gamma(X, E) = \{ C^\infty$ ρ-equi. $\tilde{X} \xrightarrow{u} \mathbb{C}^n \}$

Def.

$$h(u_1, u_2)(\tilde{x}) := \langle u_1(\tilde{x}), h_\rho(\tilde{x}) u_2(\tilde{x}) \rangle_{\mathbb{C}^n}$$

Recall $(E, h, \nabla \overset{\exists!}{=} D_h + \Phi_h)$

unitary conn. $\quad A^1(X, \mathrm{End}(E))$ self-adj.

Direct compute $\Phi_h = -\frac{1}{2} h_\rho \, dh_\rho$

Hence $\exists!$ h_ρ dp. on h only

Step 2.

$h_p : \tilde{X} \to GL/U$ harmonic

ift h harmonic i.e. $\Lambda_\omega G_h = 0$

sketch

h_p harmonic $\rightsquigarrow$ $E(h_p) = \int_X |dh_p|^2 \, dvol = E_{min}$

$\xrightarrow{\text{Siu. Sampson}}$ E-L eq. $D^{0,1}(dh_p) = 0$, $D := h_p^* \nabla^{LC}_{T(GL/U)}$

perturb $\tilde{h} := h \cdot e^{S(t)}$, $e^{S(t)} \in Aut(E)$, $S(0) = 0$

denote $\begin{cases} \delta'_{\tilde{h}} = \delta'_{h} + \delta'_{h}(S) \\ \delta''_{\tilde{h}} = \delta''_{h} + \delta''_{h}(S) \end{cases}$

$$\Rightarrow \frac{1}{4} E_{\tilde{h}_p} = \int_X \langle \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^*, \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^* \rangle \frac{\omega^m}{m!}$$

$$= \int_X |\Phi_h| \, dvol - \int_X \langle (\delta'_h)^*(\varphi_h) + (\delta''_h)^*(\varphi_h^*), S \rangle + \frac{1}{4} \int_X |\delta'_{\tilde{h}}(S) + \delta''_{\tilde{h}}(S)|^2 \, dvol$$

Hence E-L eq. $\Leftrightarrow$ $(\delta'_h)^*(\varphi_h) + (\delta''_h)^*(\varphi_h^*) = 0$

$\Downarrow$ Kä id.

$$0 = \Lambda_\omega (d'' \varphi_h - d' \varphi_h^*) = 2 \Lambda_\omega G_h$$

$\square$

Step 3.

$h_\rho : \tilde{X} \to GL/U$ harmonic

iff $\rho : \pi_1(X) \to GL(n;\mathbb{C})$ semisimple

Remark. (General)

$\rho : \pi_1(X : \text{cpt. Rie.}) \to G : \mathbb{C}\text{-reductive}$ "semisimple"

i.e. $\forall$ dim $< \infty$ rep. $G \xrightarrow{\tau} GL(V)$

$\tau \circ \rho : \pi_1(X) \to GL(V)$ ss.

iff ρ-equi. $h_\rho : \tilde{X} \to G/K : \text{max cpt.}$ "harmonic"

Sketch

Consider ρ-equi. family $h_{\rho,t} : \tilde{X} \to GL/U$

satisfy heat flow $\begin{cases} \dfrac{\partial h_{\rho,t}}{\partial t} = -D^{0,1} dh_{\rho,t} \\ h_{\rho,0} = h_\rho \end{cases}$

"Eells - Sampson tech."

Fact

- $\exists$ long time sol. $h_{\rho,t}$
- energy density uni. bd.

  i.e. $\sup_{\tilde{x},t} |dh_{\rho,t}|^2(\tilde{x}) \le C$
- $\sup_{\tilde{x}} |h_{\rho,t}(\tilde{x})| \le C_1$
- $\lim_{t\to\infty} \|D^{0,1} dh_{\rho,t}\|^2_{L^2} = 0$

Applying Arzela - Ascoli thm.

$h_{\rho,t_n} \xrightarrow{\text{cov.}} h_{\rho,\infty}$ s.t. $D^{0,1} dh_{\rho,\infty} = 0$

$\xrightarrow[\text{reg.}]{\text{elliptic}} h_{\rho,\infty} \; \mathcal{C}^\infty$

$\square$

To sum up

Cor. (NAHC)

$\{\text{semisimple } \pi_1(X) \xrightarrow{\rho} GL(n;\mathbb{C})\}/\sim \quad \longleftrightarrow \quad \left\{\begin{array}{c} \rho\text{-equi. } \tilde{X} \xrightarrow{h_\rho} GL/U \\ \text{harmonic} \end{array}\right\}$

$\updownarrow$ RH

$\{\text{semisimple flat } c_i = 0\}/\sim$

$\updownarrow$ Thm. 2 $\qquad\qquad \updownarrow$ D-C

$\{\text{harmonic bundle} : \exists \text{ pluri-harmonic}\}$

$\updownarrow$ Thm. 1 $\qquad\qquad \updownarrow$ Thm. 3

$\{\text{polystable Higgs } c_i = 0\}/\sim$

$\{\text{polystable } \lambda\text{-flat } c_i = 0\}/\sim$

Remark.

Simpson (proj.)

Nie Yanci - Zhang Xi (Kä)

$\downarrow$ extend

$\{\text{semistable Higgs } c_i = 0\} \cong \{\text{flat vb.}\}$