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ON DEFORMATIONS OF COMPLEX ANALYTIC STRUCTURES, I

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Owing to its length, this paper is published in two parts : Part I consisting of Chapters I—V and Part II consisting of Chapters VI—VIII. Part II is to be found at the beginning of the next issue of this journal.

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GLOSSARY OF SYMBOLS

For the convenience of the reader, some symbols whose meanings are *usually* the same wherever they occur are listed here.

$\boldsymbol{C}$: field of complex numbers
$\boldsymbol{C}^*$: multiplicative group of complex numbers
c_i : i-th Chern class of a complex manifold
$\mathfrak{E}$: tangent bundle of a differentiable family of structures
$\mathfrak{E}_P$: tangent bundle of a family $\mathscr{P} \to M$ of complex principal bundles
$\mathscr{E}$: family of complex line bundles (complex line bundle over a family of complex structures)
E_t : restriction of $\mathscr{E}$ to a fibre V_t (complex analytic line bundle over V_t)
$\mathfrak{F}$: bundle along the fibres of a differentiable family of complex structures
$\mathfrak{F}_P$: bundle along the fibres of $\mathscr{P} \to \mathscr{V}$ where $\mathscr{P} \to \mathscr{V} \to M$ is a family of complex principal bundles
$\mathscr{F}_P$: bundle along the fibres of $\mathscr{P} \to M$ where $\mathscr{P} \to \mathscr{V} \to M$ is a family of complex principal bundles
G_t : Green's operator on a fibre V_t of a differentiable family
$G_t(r, s)$: Green's operator on a fibre V_t for forms of type (r, s)
$\mathscr{H}^q(\Theta)$: q-dimensional cohomology sheaf over the base space of a differentiable family with coefficients in Θ
$\boldsymbol{H}(r, s)$: space of harmonic $\mathscr{B}$-forms on $\mathscr{V}$ of type (r, s)
$\boldsymbol{H}_t(r, s)$: space of harmonic B_t-forms on V_t of type (r, s)
$H_t(r, s)$: orthogonal projection onto $\boldsymbol{H}_t(r, s)$
$H_{\bar\partial}^{r,s}(\mathscr{B})$: $\bar\partial$-cohomology of $\mathscr{B}$-forms on $\mathscr{V}$ of type (r, s)
$H_{\bar\partial_t}^{r,s}(B_t)$: $\bar\partial_t$-cohomology of B_t-forms on V_t of type (r, s)

$\mathfrak{J}$: imaginary part

J : complex structure tensor (along the fibres) of a family $\mathscr{V} \to M$ of complex structures

$\mathfrak{L}$: the bundle $\mathfrak{F}_P/G$ where G is a complex Lie group

$\mathfrak{M}$: the bundle $\mathscr{F}_P/G$

M : the base space of a differentiable family $\mathscr{V} \to M$ of complex structures

$m(V_t)$: number of moduli of a compact complex manifold V_t

Ω : sheaf of germs of holomorphic functions on a complex manifold

$\Omega(B_t)$: sheaf of germs of holomorphic sections of a vector bundle $B_t \to V_t$

$\mathfrak{D}$: sheaf of germs of differentiable functions on $\mathscr{V}$ which are holomorphic along the fibres of $\mathscr{V}$

$\mathfrak{D}(\mathscr{B})$: sheaf of germs of differentiable sections of $\mathscr{B} \to \mathscr{V}$ which are holomorphic along the fibres of $\mathscr{V} \to M$

$\mathfrak{D}^*$: multiplicative sheaf of germs of differentiable functions on $\mathscr{V}$ which are holomorphic along the fibres of $\mathscr{V}$

O : subsheaf of $\mathfrak{D}$ composed of real-valued differentiable functions which are constant along the fibres of $\mathscr{V}$

Π : subsheaf of the sheaf of sections of $\mathfrak{E}$

ϖ : projection map for a differentiable family $\mathscr{V} \to M$ of complex structures

$\mathscr{P}$: differentiable family of complex principal bundles

$P_n(\boldsymbol{C})$: complex projective n-space

ρ : the map $T_M \to \mathscr{H}^1(\Theta)$

ρ_t : the map $(T_M)_t \to H^1(V_t, \Theta_t)$

ρ_d : the map $T_M \to \mathscr{H}^0(\Psi)$

$\rho_{d,t}$: the map $(T_M)_t \to H^0(V_t, \Psi_t)$

$\boldsymbol{R}$: field of real numbers

Θ : sheaf of germs of differentiable sections of $\mathfrak{F} \to \mathscr{V}$ which are holomorphic along the fibres of $\mathscr{V} \to M$

Θ_t : sheaf of germs of holomorphic sections of the tangent bundle of the complex manifold V_t

T : the quotient sheaf Π/Θ

T_M : sheaf of germs of differentiable sections of the tangent bundle of the base space M of a differentiable family $\mathscr{V} \to M$

$(T_M)_t$: the tangent space of the base space M at the point $t \in M$

$\mathscr{U}$: neighborhood on $\mathscr{V}$

$\mathscr{V}$: family of complex structures
V_t: fibre of $\mathscr{V} \to M$ over the point $t \in M$
$\boldsymbol{Z}$: ring of integers

INTRODUCTION

Deformation of the complex structure of a Riemann surface is an idea which goes back to Riemann who, in his famous memoir on abelian functions published in 1857, calculated the number of independent parameters on which the deformation depends and called these parameters "moduli". Riemann's well known formula states that the number of moduli of a compact Riemann surface of genus p is equal to the dimension of the complex Lie group of complex analytic automorphisms of the surface plus the number $3p-3$. This formula has been generalized by Klein to the case of a Riemann surface with boundary. During the hundred years which have elapsed since the publication of Riemann's memoir, the questions centering around the deformation of the complex structure of a Riemann surface have never lost their interest and, during the last twenty years, have received fresh stimuli from two sources at least, namely: from the theory of extremal quasi-conformal mappings which was initiated by O. Teichmüller (see reference [38] listed at the end of this paper) and which has been developed by L. Ahlfors [1] and others; from the variational calculus for Riemann surfaces which began with a paper by M. Schiffer in 1938 and which has been developed and applied by several authors (see, e. g., [34]). In both of these developments the quadratic differentials of the surface play a fundamental role.

The deformation of higher-dimensional complex manifolds, or of algebraic surfaces at least, seems to have been considered first by Max Noether in 1888 ([33]). However, in sharp constrast with the case of complex dimension 1 (Riemann surfaces), the deformation of higher-dimensional complex manifolds has been curiously neglected. Last year Frölicher and Nijenhuis, as an outgrowth of their earlier work on vector-valued differential forms, obtained an important theorem (see [16]) which is the starting point of this study.

The purpose of the present paper is to develop a more or less systematic theory of deformations of complex structures of higher dimensional manifolds. The concept of deformations of complex structures, or of a family of complex structures depending differentiably on a parameter, can be defined in terms of structure tensors determining the complex structures (see Kodaira and Spencer [28], Frölicher and Nijenhuis [16]). Given a family $\{V_t\}$ of complex structures V_t defined on a compact differentiable manifold X which depend differentiably on a parameter t moving on a connected differentiable manifold M, the union $\mathscr{V} = \cup_{t \in M} V_t$ of the complex manifolds V_t may be regarded as a kind of fibre space over M whose structure is a mixture of differentiable structure and complex structure along the fibres. Reversing this process, we *define* a differentiable family of compact complex structures (manifolds) as a fibre space $\mathscr{V}$ over a connected differentiable manifold M whose structure is a mixture of differentiable and complex structures (Definition 1.1). In particular, if $\mathscr{V}$ and M are both complex manifolds and if $\mathscr{V}$ is a complex-analytic fibre space over M, we call $\mathscr{V}$ a complex analytic family of compact complex structures (manifolds).

Our first task is to define an object which measures the magnitude of dependence of the complex structure of V_t on the parameter t. We introduce a sheaf Θ on $\mathscr{V}$ (Section 4), the corresponding sheaf of cohomology $\mathscr{H}^1(\Theta)$ on M, and construct a homomorphism $\rho : T_M \to \mathscr{H}^1(\Theta)$ of the sheaf T_M of germs of differentiable vector fields on M into $\mathscr{H}^1(\Theta)$(Section 5). This homomorphism ρ may be considered to be the object which represents the magnitude of dependence of the complex structure of V_t on t. In fact, $\mathscr{V}$ is locally trivial in the sense that $\mathscr{V}$ has a local product structure (and therefore the complex structure of V_t is independent of t) if and only if ρ vanishes (Theorem 5.1). The restriction Θ_t of Θ to a fibre V_t of $\mathscr{V}$ (in the sense of Section 4) coincides with the sheaf of germs of holomorphic vector fields on V_t and therefore $H^1(V_t, \Theta_t)$ depends only on V_t. By restricting ρ to V_t we obtain a homomorphism $\rho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ which was first introduced by Frölicher and Nijenhuis [16], where $(T_M)_t$ denotes the tangent space of M at t. For any tangent vector $v_t \in (T_M)_t$, the image $\rho_t(V_t) \in H^1(V_t, \Theta_t)$ represents the "infinitesimal deformation" of the complex structure of V_t along the vector v_t (Section 6). Clearly $\rho = 0$ implies $\rho_t = 0$ for all $t \in M$, but the converse is not necessarily true, as an example shows (Section 15). This is connected with the fact that the complex structure can "jump" from one structure to another by an arbitrarily small deformation, a phenomenon which does not occur in the case of Riemann surfaces. Because of this phenomenon, it is impossible to generalize the concept of distance between two compact Riemann surfaces in the sense of Teichmüller [38] to higher dimensional manifolds (Section 15). With the help of the theory of harmonic forms (Section 2) we prove that, if dim $H^1(V_t, \Theta_t)$ is independent of t, $\rho_t = 0$ for all $t \in M$ implies $\rho = 0$, and derive from this the theorem of Frölicher and Nijenhuis [16] concerning the *rigidity* of compact complex manifolds (Section 6).

In case $\mathscr{V}$ is a complex analytic family, i. e., a complex analytic fibre space, we may ask whether $\mathscr{V}$ is a complex analytic fibre bundle if $\mathscr{V}$ is locally trivial as a differentiable family. We give an affirmative answer to this question (Section 18).

Each imbedding of a compact complex manifold V_o as fibre over the point $o \in M$ in a differentiable family $\mathscr{V} \to M$ of complex structures determines a space of infinitesimal deformations in $H^1(V_o, \Theta_o)$, namely the image of the map $\rho_o : (T_M)_o \to H^1(V_o, \Theta_o)$. A space of infinitesimal deformations in $H^1(V_o, \Theta_o)$ determined in this way will be called *maximal* if it is not a proper subspace of a space of infinitesimal deformations determined by some other imbedding of V_o as fibre in a differentiable family. A maximal space of infinitesimal deformations will be called a *deformation space*. An example (Section 16) shows that there exist manifolds V_o with more than one deformation space in $H^1(V_o, \Theta_o)$, at least for complex dimension exceeding 2.

We generalize the above results to differentiable families of complex manifolds with the additional structure of complex fibre bundle (Sections 7 and 17).

Next, we extend Riemann's concept of number of moduli to higher dimensional complex manifolds (Section 11). The main point here is to avoid the use of the concept of the space of moduli of complex manifolds which cannot be defined in general for higher dimensional manifolds (Section 14, (γ)). Moreover, a necessary condition for the existence of a number $m(V_o)$ of moduli of a complex manifold V_o is that $H^1(V_o, \Theta_o)$ contain only one deformation space ; hence $m(V_o)$ is not defined for all compact complex manifolds. We compute the number $m(V_o)$ of moduli of some simple types of complex manifolds V_o and find that $m(V_o) = \dim H^1(V_o, \Theta_o)$ (Section 14). We remark that, in the case of a Riemann surface V_o, the duality theorem implies that $H^1(V_o, \Theta_o)$ is isomorphic to the space of quadratic differentials which are everywhere regular on V_o ; therefore $m(V_o) = \dim H^1(V_o, \Theta_o)$ coincides with the number of moduli of V_o in the sense of Riemann. On the other hand, the classical formula for the number of moduli of algebraic surfaces due to Noether [33] may be regarded as the "postulation formula" for $\dim H^1(V_o, \Theta_o)$. In view of the above statements, we would like to propose, as the main problem in the theory of deformations of compact complex manifolds, that of understanding the reasons why the equality $m(V_o) = \dim H^1(V_o, \Theta_o)$ holds for many examples of complex manifolds.

By restricting our considerations to submanifolds of a *fixed* compact complex manifold W, we obtain the corresponding theory of deformations of complex structures *relative* to W (Section 12). Thus we introduce the number $m_W(V_o)$ of relative moduli and the "relative cohomology" $H^1_W(V_o, \Theta_o)$ for a submanifold V_o of W. In the particular case in which W is an algebraic manifold and V_o is a submanifold of W of codimension 1, we derive the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ from the theorem of completeness of the characteristic linear systems of complete continuous systems (Section 12).

Finally, we discuss several other topics connected with the above problems, but for these we refer the reader to the table of contents.

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Chapter I. Definitions; Harmonic forms along the fibres of a differentiable family

1. Fundamental definitions

There are several ways of defining deformation of complex analytic structures. In our earlier paper [28] we employed a differential-geometric definition. We may also define deformation of complex structures as a special case of deformation of structures in general. In this paper we employ an elementary definition of deformation which is convenient for our purpose.

By "differentiable" we shall always mean "differentiable of class C^∞", and we shall assume that all manifolds are paracompact. Let M be a *connected* differentiable manifold and let $\mathscr{V}$ be a differentiable fibre bundle over M such that each fibre V_t of $\mathscr{V}$ over $t \in M$ is a connected complex analytic manifold whose complex structure is compatible with the differentiable structure of V_t induced from the differentiable structure of $\mathscr{V}$. Obviously $\{V_t \mid t \in M\}$ can be regarded as a family of complex structures whose underlying differentiable structures are equivalent to a fixed differentiable structure on one and the same manifold X. We denote by ϖ the canonical projection of $\mathscr{V}$ onto M.

Definition 1.1. We say that the complex structures V_t on X depend differentiably on t and we call the fibre space $\mathscr{V} = \{V_t \mid t \in M\}$ a differentiable family of complex structures (or complex manifolds) if each point of $\mathscr{V}$ has a neighborhood $\mathscr{U}$ satisfying the following conditions: There exists a differentiable homeomorphism h of $\mathscr{U}$ into $\boldsymbol{C}^n \times \varpi(\mathscr{U})$ such that, for each point $t \in \varpi(\mathscr{U})$, the restriction h_t of h to $\mathscr{U} \cap V_t$ is a biregular map of $\mathscr{U} \cap V_t$ into $C^n \times t$ in the complex-analytic sense, where C^n is the space of n complex variables $(z^1, z^2, \cdots, z^n)$, n being the complex dimension of V_t. Moreover, with reference to a complex structure V_o, $o \in M$, we call any V_t, $t \in M$, a deformation of V_o.

We remark that " biregular " will mean " holomorphic biregular " unless otherwise specified. We call M the base space or the parameter variety of $\mathscr{V}$ and we write $\mathscr{V} \to M$ or $\mathscr{V} \overset{\varpi}{\to} M$ for $\mathscr{V}$ when we want to indicate the base space M.

A differentiable family of complex structures is an important example of " mixed structure ". Another example of mixed structure will be given at the end of this section.

We are concerned in this paper with deformations of complex structure, and we begin by stating definitions of several basic concepts relating to differentiable families of complex structures.

DEFINITION 1.2. Two differentiable families $\mathscr{V} \to M$, $\mathscr{W} \to N$ of complex structures will be called *equivalent* if and only if there is a bundle map $\varphi : \mathscr{V} \to \mathscr{W}$ which is biregular, in the sense of the category of differentiable fibre bundles, and which, restricted to a fibre of $\mathscr{V}$, is a biregular map of that fibre onto the corresponding fibre of $\mathscr{W}$ in the sense of complex analytic structure. In the particular case where $\mathscr{V}$ coincides with $\mathscr{W}$, we say that φ is an *automorphism* of the family ; if φ passed to the base space is the identity, we say that it is a *fibre-preserving automorphism*.

DEFINITION 1.3. A differentiable family of complex structures is trivial if there exists a differentiable map $h: \mathscr{V} \to V_o = \varpi^{-1}(o)$, $o \in M$, which maps each fibre $V_t = \varpi^{-1}(t)$, $t \in M$, biregularly onto V_o in the complex analytic sense.

DEFINITION 1.4. In the particular case in which $\mathscr{V}$ and M are both complex analytic manifolds and ϖ is a holomorphic map (regular map in the category of complex analytic structures), we call $\mathscr{V} \overset{\varpi}{\to} M$ a complex analytic family of complex structures.

By a complex analytic fibre space we mean a triple $(\mathscr{W}, \varpi, N)$ of connected complex manifolds $\mathscr{W}$, N and a holomorphic map ϖ of $\mathscr{W}$ onto N. A fibre $\varpi^{-1}(s)$, $s \in N$, of the complex analytic fibre space $(\mathscr{W}, \varpi, N)$ will be called *singular* if there exists a point $p \in \varpi^{-1}(s)$ such that the rank of the jacobian of the map ϖ at p is less than the dimension of N. A complex analytic family of complex structures is a complex-analytic fibre space without singular fibres whose fibres are connected

A complex analytic family of complex structures is called complex analytically trivial if there exists a regular map $h: \mathscr{V} \to V_o = \varpi^{-1}(o)$ in the sense of complex analytic structure which is a biregular complex-analytic map of each fibre V_t onto V_o.

Unless the contrary is explicitly stated, we shall assume that the family is differentiable and that the base manifold M has only a real differentiable structure.

Suppose given a differentiable family $\mathscr{V} \xrightarrow{\varpi} M$ of complex structures and a differentiable map $f: N \to M$ of the differentiable manifold N into M.

DEFINITION 1.5. A differentiable family $\mathscr{W} \xrightarrow{\pi} N$ will be called the family of complex structures over N induced from $\mathscr{V}$ by the map f if there exists a differentiable map h of $\mathscr{W}$ into $\mathscr{V}$ which maps each fibre $W_s = \pi^{-1}(s)$, $s \in N$, biregularly onto $V_{f(s)} = \varpi^{-1}(f(s))$ in the complex analytic sense.

The induced family $\mathscr{W} \xrightarrow{\pi} N$ always exists and is unique up to an equivalence. In fact, let N' be another examplar of N and let $s' \in N'$ correspond to $s \in N$. By identifying $s \in N$ with $(f(s), s') \in M \times N'$, we may consider N as a submanifold of $M \times N'$. Consider $\mathscr{V} \times N'$ as a differentiable family over $M \times N'$ in an obvious manner. Then the induced family is given, up to an equivalence, by the restriction $\mathscr{W} = \mathscr{V} \times N' | N$ of $\mathscr{V} \times N'$ to $N \subset M \times N'$.

DEFINITION 1.6. Two complex analytic manifolds V_0 and V_1 will be said to be complex homotopic, shortly c-homotopic, if and only if both occur as fibres in the same differentiable family of complex structures.

So far as c-homotopy is concerned, it is plainly sufficient to consider only 1-parameter differentiable families $\mathscr{V} \to M$ where M is an interval: $M = \{t | a \leqq t \leqq b\}$ (the family over the closed interval $M = [a, b]$ is understood to be the restriction to $[a, b]$ of a family defined over a slightly larger open interval $(a - \varepsilon, b + \varepsilon)$). In order to prove that c-homotopy is an equivalence relation, it is sufficient to show that it is transitive. Suppose then that V_0, V_1 are the fibres over the points $t = 0, 1$ respectively in the family $\mathscr{V} = \{V_t | 0 \leqq t \leqq 1\}$ and that V_1, V_2 are the fibres over $t' = 1, 2$ respectively in the family $\mathscr{V}' = \{V_{t'} | 1 \leqq t' \leqq 2\}$. To prove transitivity we construct a family $\mathscr{V}'' = \{V''_s | 0 \leqq s \leqq 2\}$ whose fibres over $s = 0, 1, 2$ coincide respectively with V_0, V_1, V_2. In fact, let $I = \{t | 0 \leqq t \leqq 1\}$, $I' = \{t' | 1 \leqq t' \leqq 2\}$ and induce from $\mathscr{V}$, $\mathscr{V}'$ new families $\mathscr{W} \to I$, $\mathscr{W}' \to I'$ by means of differentiable maps $f: I \to I\, f': I' \to I'$ where $f(0) = 0$, $f(1) = f'(1) = 1$, $f'(2) = 2$ and where all derivatives of f, f' vanish at 1. Then $\mathscr{W} = \{W_s | 0 \leqq s \leqq 1\}$, $\mathscr{W}' = \{W'_s | 1 \leqq s \leqq 2\}$

where $W_s = V_{f(s)}$, $W'_s = V'_{f'(s)}$, and the desired family $\mathscr{V}''$ is obtained by the composition of $\mathscr{W}$ and $\mathscr{W}'$. Namely, let $\mathscr{V}'' = \{V''_s \mid 0 \leqq s \leqq 2\}$ where $V''_s = W_s$ for $0 \leqq s \leqq 1$, $V''_s = W'_s$ for $1 \leqq s \leqq 2$; then $\mathscr{V}''$ is clearly a differentiable family whose fibres over $s = 0, 1, 2$ are V_0, V_1, V_2 respectively. Thus c-homotopy is an equivalence relation.

DEFINITION 1.7. We say that the family $\mathscr{V} \xrightarrow{\varpi} M$ is complete at a point $t \in M$ if, for every differentiable family $\{V_{t,s} \mid s \in N\}$ of deformations $V_{t,s}$ of $V_{t,o} = V_t = \varpi^{-1}(t)$, there exists a differentiable map $s \to t(s)$, $t(o) = t$, of a neighborhood U of o on N into M such that $\{V_{t,s} \mid s \in U\}$ coincides with the differentiable family $\{V_{t(s)} \mid s \in U\}$ induced from $\mathscr{V}$ by the map $s \to t(s)$. The family $\mathscr{V}$ will be called complete if $\mathscr{V}$ is complete at each point t on M.

Next, we state a more intrinsic definition of differentiable family of complex structures. Let $\mathscr{V} \xrightarrow{\varpi} M$ be a differentiable fibre bundle with fibre X, where X is a (connected) differentiable manifold of real dimension $2n$, and let $\mathfrak{F}_+$ be the bundle over $\mathscr{V}$ of tangent vectors along the fibres of $\mathscr{V}$, $\mathfrak{S}_+$ the tangent bundle of $\mathscr{V}$. Denote by m the real dimension of M. We have the exact sequence

$$0 \longrightarrow \mathfrak{F}_+ \longrightarrow \mathfrak{S}_+ \longrightarrow \mathfrak{S}_+/\mathfrak{F}_+ \longrightarrow 0$$

of vector bundles over $\mathscr{V}$ where the structure group of $\mathfrak{S}_+$ is $GL(2n, m; \boldsymbol{R})$, the subgroup of those matrices of $GL(2n+m, \boldsymbol{R})$ which map the linear subspace $x_{2n+1} = x_{2n+2} = \cdots = x_{2n+m} = 0$ of $\boldsymbol{R}^{2n+m}$ (with coordinates $x_1, \cdots, x_{2n+m}$) onto itself. The group of $\mathfrak{F}_+$ is a subgroup of $GL(2n, m; \boldsymbol{R})$.

Denote by $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ the group of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in GL(n, \boldsymbol{C})$, $C \in GL(m, \boldsymbol{R})$, and where B is an arbitrary complex matrix of n rows and m columns. Then $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ can be imbedded in $GL(2n, m; \boldsymbol{R})$ as a real subgroup by the map

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \longrightarrow \begin{pmatrix} \Re A & -\Im A & \Re B \\ \Im A & \Re A & \Im B \\ 0 & 0 & C \end{pmatrix}.$$

Definition 1.1 is equivalent to the following:

DEFINITION 1.1′. A differentiable family of complex structures is a differentiable fibre bundle $\mathscr{V} \to M$ with fibre X, $\dim_R X = 2n$, together with a differentiable reduction of the structure group $GL(2n, m; \boldsymbol{R})$ of $\mathfrak{S}_+$ to $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ (regarded as a real subgroup) which imparts to each fibre a complex analytic structure.

If the last phrase is omitted, we obtain a differentiable family of almost-complex structures, namely a differentiable fibre bundle $\mathscr{V} \to M$ with fibre X, $\dim_R X = 2n$, together with a differentiable reduction of the structure group $GL(2n, m; \boldsymbol{R})$ of $\mathfrak{E}_+$ to $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$.

It is easy to see that Definition 1.1 implies Definition 1.1′, but the converse implication requires a non-trivial theorem (see [29]).

By Definition 1.1 there exists a locally finite covering $\mathfrak{U} = \{\mathscr{U}_i\}$ of $\mathscr{V}$ by neighborhoods $\mathscr{U}_i$ with associated homeomorphisms $h_i : \mathscr{U}_i \to \boldsymbol{C}^n \times \varpi(\mathscr{U}_i)$ satisfying the conditions stated in that definition. Let

$$h_i(p) = (z_i^1(p), \cdots, z_i^n(p), t), \qquad t = \varpi(p), \qquad \text{for } p \in \mathscr{U}_i.$$

Then $(z_i^1, \cdots, z_i^n, t) = (z_i^1(p), \cdots, z_i^n(p), \varpi(p))$ can be used as the local coordinates of p in $\mathscr{U}_i$ and for each fixed t, $(z_i^1, \cdots, z_i^n)$ form a system of local holomorphic coordinates on $V_t \cap \mathscr{U}_i$, $V_t = \varpi^{-1}(t)$. Further we may suppose that $t = \varpi(p) = (t_i^1, \cdots, t_i^m)$, where (t_i^λ) is a system of local coordinates for M covering $U_i = \varpi(\mathscr{U}_i)$. Let

$$z_i^\alpha(p) = h_{ik}^\alpha(z_k(p), t), \qquad p \in \mathscr{U}_i \cap \mathscr{U}_k,$$

where $z_k(p) = (z_k^1(p), \cdots, z_k^n(p))$. Clearly the $h_{ik}^\alpha(z_k, t)$ are differentiable functions defined in $\mathscr{U}_i \cap \mathscr{U}_k$ which are holomorphic in z_k. Now let $g_{ik}(p)$ be the jacobians:

$$g_{ik}(p) = \left(\frac{\partial z_i^\alpha(p)}{\partial z_k^\beta(p)}\right)_{\alpha, \beta = 1, \cdots, n}.$$

Denote by $\mathfrak{F}$ the complex vector bundle over $\mathscr{V}$ determined by the system $\{g_{ik}(p)\}$ of transition functions with respect to the covering $\{\mathscr{U}_i\}$. Moreover, denote by $\mathfrak{E}$ the vector bundle over $\mathscr{V}$ determined by the system of transition functions

$$\begin{pmatrix} \dfrac{\partial z_i^\alpha(p)}{\partial z_k^\beta(p)} & \dfrac{\partial z_i^\alpha(p)}{\partial t_k^\nu} \\ 0 & \dfrac{\partial t_i^\lambda}{\partial t_k^\nu} \end{pmatrix}_{\substack{\alpha, \beta = 1, 2, \cdots, n \\ \lambda, \nu = 1, 2, \cdots, m}}$$

We remark that the transition functions will always be assumed to operate on the left. Clearly $\mathfrak{F}$ is a sub-bundle of $\mathfrak{E}$ in an obvious manner, and we have the exact sequence

$$0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{E}/\mathfrak{F} \to 0. \tag{1.1}$$

We call this the *fundamental sequence of (vector) bundles for the family* $\mathscr{V} \to M$.

In connection with Definition 1.1′, the bundles $\mathfrak{E}$ and $\mathfrak{F}$ can be defined intrinsically as follows. The structure group $GL(2n, m; \boldsymbol{R})$ of $\mathfrak{E}_+$ is reduced, by hypothesis, to the subgroup $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ of matrices of the form

$$\begin{pmatrix} \Re A & -\Im A & \Re B \\ \Im A & \Re A & \Im B \\ 0 & 0 & C \end{pmatrix}$$

where C are real matrices. Hence we can complexify the first $2n$ components of the vectors of $\mathfrak{E}_+$, and we obtain a bundle $C\mathfrak{E}_+$ whose fibre is $\boldsymbol{C}^{2n} \oplus \boldsymbol{R}^m$ and whose structure group is the group $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ of matrices of the form indicated. Now, letting

$$E = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$E \begin{pmatrix} \Re A & -\Im A & \Re B \\ \Im A & \Re A & \Im B \\ 0 & 0 & C \end{pmatrix} E^{-1} = \begin{pmatrix} A & 0 & B \\ 0 & \bar{A} & \bar{B} \\ 0 & 0 & C \end{pmatrix}.$$

Hence $C\mathfrak{E}_+$ contains two sub-bundles $\mathfrak{E}$ and $\bar{\mathfrak{E}}$ where $\mathfrak{E}$ is the sub-bundle with fibre $\boldsymbol{C}^n \oplus \boldsymbol{R}^m$ and structure group the group $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ of matrices of the form

$$\begin{pmatrix} A & \mathrm{B} \\ 0 & C \end{pmatrix},$$

and where $\bar{\mathfrak{E}}$ is the conjugate bundle. Finally, the bundle $\mathfrak{F}$ is obtained from $\mathfrak{F}_+$ by splitting its complexification into two parts

$$C\mathfrak{F}_+ = \mathfrak{F}_+ \oplus i\mathfrak{F}_+ = \mathfrak{F} \oplus \bar{\mathfrak{F}}$$

where $\mathfrak{F}$ has fibre $\boldsymbol{C}^n$, structure group $GL(n, \boldsymbol{C})$, and where $\bar{\mathfrak{F}}$ is the conjugate bundle.

The above considerations obviously apply to a differentiable family of almost-complex structures. Let $P, \bar{P}$ denote the projections of $C\mathfrak{F}_+$ onto $\mathfrak{F}, \bar{\mathfrak{F}}$ respectively, and define

$$J = \sqrt{-1}\,(P - \bar{P}).$$

We have $J^2 = -I$ where I is the identity map of $C\mathfrak{F}_+$. We call J the "complex structure tensor" of the family $\mathscr{V} \to M$. The "torsion tensor" T of the almost-complex structure along the fibres of $\mathscr{V}$ is defined by

$$4T(u, v) = J[Ju, v] + J[u, Jv] + [u, v] - [Ju, Jv]$$

where u, v are differentiable cross-sections of $C\mathfrak{F}_+$ and where $[u, v]$ denotes the Poisson bracket of vector fields u, v. A differentiable family of almost-complex structures is a family of complex structures if and only if T vanishes.

We turn now to the deformations of complex structures which possess an additional structure of complex fibre bundle. Namely, let $\mathscr{V} \xrightarrow{\varpi} M$ be an arbitrary differentiable family of complex structures ; then :

DEFINITION 1.8. A differentiable family $\mathscr{B} \to \mathscr{V} \to M$ of complex fibre bundles over M is a differentiable fibre bundle $\mathscr{B} \to \mathscr{V}$, with structure group a complex Lie group G, whose restriction to each fibre of $\mathscr{V}$ is a complex-analytic fibre bundle over that fibre (in the sense that the restriction of the associated principal bundle is holomorphic along that fibre). A family of complex fibre bundles is a complex-analytic family if and only if $\mathscr{V} \to M$ is a complex analytic family of complex structures and $\mathscr{B} \to \mathscr{V}$ is a complex-analytic fibre bundle.

We abbreviate by writing $\mathscr{B} \to \mathscr{V}$ or $\mathscr{B} \to M$ for $\mathscr{B} \to \mathscr{V} \to M$ according to the emphasis desired.

The particular case where $\mathscr{V} = V \times M$ (V a complex analytic manifold) corresponds to a differentiable or a complex analytic family of complex fibre bundles over a *fixed* complex analytic manifold V.

We shall be concerned mainly with families of complex vector bundles. The bundle $\mathfrak{F}$ along the fibres of $\mathscr{V}$ which was introduced above is an example of a differentiable family of complex fibre bundles, but the bundle $\mathfrak{E}$ is *not* a differentiable family of complex fibre bundles according to our definition because the structure group is not a complex Lie group.

Let $\mathfrak{O}(G)$ denote the sheaf of germs of differentiable maps $\mathscr{V} \to G$ whose restrictions to each fibre of $\mathscr{V}$ are holomorphic. Thus $\mathfrak{O}(G)$ is a sheaf of groups, generally non-abelian. Although cohomology groups $H^q(\mathscr{V}, \mathfrak{O}(G))$ cannot be defined in the non-abelian case, nevertheless for $q = 1$ it is possible, as is well known, to define a cohomology set $H^1(\mathscr{V}, \mathfrak{O}(G))$ with a distinguished element which, in the case of an abelian group G, agrees with the first cohomology group of $\mathscr{V}$ with coefficients in $\mathfrak{O}(G)$, the distinguished element in this case corresponding to the zero element of the cohomology group.

DIEFINITION 1.9. Let $\mathscr{B}$, $\mathscr{B}'$ be two differentiable families of complex fibre bundles over $\mathscr{V}$ with F as fibre and G as structure group. We say that $\mathscr{B}$, $\mathscr{B}'$ are *equivalent* if and only if $\mathscr{B}$ and $\mathscr{B}'$ are isomorphic in the category of differentiable fibre bundles where the isomorphism is established by a homeomorphism whose restriction to each fibre $B_t \to V_t$ of $\mathscr{B} \to M$ is an isomorphism in the category of complex-analytic fibre bundles.

We obtain, by the standard argument, the following proposition :

PROPOSITION 1.1. *The equivalence classes of differentiable families of complex fibre bundles over $\mathscr{V}$ with F as fibre and G as structure group, where G operates effectively on F, correspond one-one and in a natural way with the elements of the cohomology set $H^1(\mathscr{V}, \mathfrak{D}(G))$. The equivalence class of the trivial bundle $\mathscr{B} = F \times \mathscr{V}$ corresponds to the distinguished element of $H^1(\mathscr{V}, \mathfrak{D}(G))$.*

In the cases $G = \boldsymbol{C}$ (additive group of complex numbers), $G = \mathbf{C}^* = GL(1, \boldsymbol{C})$ (multiplicative group of complex numbers), we write $\mathfrak{D} = \mathfrak{D}(\boldsymbol{C})$, $\mathfrak{D}^* = \mathfrak{D}(\boldsymbol{C}^*)$. Thus $\mathfrak{D}$ is the sheaf of germs of differentiable functions on $\mathscr{V}$ whose restrictions to each fibre are holomorphic. Let $\boldsymbol{Z}$ be the (sheaf on $\mathscr{V}$ of) integers ; then we have the exact sequence

$$0 \longrightarrow \boldsymbol{Z} \xrightarrow{\iota} \mathfrak{D} \xrightarrow{\varepsilon} \mathfrak{D}^* \longrightarrow 0$$

where i is the canonical inclusion map and ε is the exponential map : $f \longrightarrow \exp(2\pi i f), f \in \mathfrak{D}$. The corresponding exact cohomology sequence is

$$\cdots \to H^1(\mathscr{V}, \mathfrak{D}) \xrightarrow{\varepsilon^*} H^1(\mathscr{V}, \mathfrak{D}^*) \xrightarrow{\delta^*} H^2(\mathscr{V}, \boldsymbol{Z}) \xrightarrow{\iota^*} H^2(\mathscr{V}, \mathfrak{D}) \to \cdots$$

where $H^1(\mathscr{V}, \mathfrak{D}^*)$ is in one-one correspondence with the equivalence classes of differentiable families of complex line bundles over $\mathscr{V}$ with $\boldsymbol{C}$ as fibre and $\boldsymbol{C}^*$ as structure group. The image under δ^* of an element of $H^1(\mathscr{V}, \mathfrak{D}^*)$ is the characteristic class (Chern class) of the given element.

Returning to the general case of a differentiable family $\mathscr{B} \to \mathscr{V}$ of complex bundles, we denote by $\mathscr{P} \to \mathscr{V}$ the associated principal bundle of $\mathscr{B} \to \mathscr{V}$; $\mathscr{P} \to \mathscr{V}$ is a differentiable family of complex principal bundles with fibre and structure group G. The equivalence theorem for ordinary fibre bundles ([37], p. 36) follows from Proposition 1.1 for families of complex bundles (case of mixed structure) and it asserts that two families $\mathscr{B}$, $\mathscr{B}'$ of complex fibre bundles over the same $\mathscr{V}$ which have the same fibre and group are equivalent if and only if their associated principal bundles $\mathscr{P}$, $\mathscr{P}'$ are equivalent. Thus the investigation of a family of complex bundles is reduced to the consideration of the associated family of principal bundles, and a family of complex bundles has a certain property, such as triviality, if and only if the associated family of principal bundles has it.

We remark that a family $\mathscr{P} \longrightarrow \mathscr{V} \longrightarrow M$ of complex principal bundles may be regarded as a family of complex structures in two different ways, namely: (1) $\mathscr{P} \longrightarrow M$; (2) $\mathscr{P} \longrightarrow \mathscr{V}$. In (1) the fibres are complex-analytic principal bundles $P_t \longrightarrow V_t$ where $V_t = \varpi^{-1}(t)$, $P_t = \mathscr{P} \mid V_t$, $t \in M$. Let

$$0 \longrightarrow \mathscr{F}_P \longrightarrow \mathfrak{E}_P \longrightarrow \mathfrak{E}_P/\mathscr{F}_P \longrightarrow 0 \tag{1.2}$$

be the sequence (1.1) for the family $\mathscr{P} \longrightarrow M$, and let $\mathfrak{F}_P$ be the sub-bundle of $\mathscr{F}_P$ composed of those vectors which are tangent to the fibres of $\mathscr{P} \longrightarrow \mathscr{V}$. Then we also have the exact sequence of bundles

$$0 \longrightarrow \mathfrak{F}_P \longrightarrow \mathfrak{E}_P \longrightarrow \mathfrak{E}_P/\mathfrak{F}_P \longrightarrow 0\ . \tag{1.3}$$

We call $\mathfrak{F}_P$ the bundle along the fibres of the family $\mathscr{P} \longrightarrow \mathscr{V}$. The group G operates by right translation on $\mathscr{P}$ (where the transition functions operate on the left) and hence it operates on the sequences (1.2) and (1.3). We are therefore able to replace (1.2) and (1.3) by new sequences of bundles over $\mathscr{V}$ which are obtained by factoring these sequences with respect to the action of G. In fact, let

$$\mathfrak{L} = \mathfrak{F}_P/G, \qquad \mathfrak{M} = \mathscr{F}_P/G, \qquad \mathfrak{N} = \mathfrak{E}_P/G$$

where the quotient spaces are obtained by identifying two points of a fibre one of which is the right translation of the other by an element of G. Then $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ are (isomorphic to) bundles over $\mathscr{V}$ whose fibres are equivalence classes of vectors tangent to $\mathscr{P}$ which arise from one another by right translations under G. For example, consider the bundle $\mathfrak{L}$. The group G operates on its tangent space $\mathfrak{g}$ at the identity (Lie algebra of G) by Ad (G) and $\mathfrak{L}$ is the bundle over $\mathscr{V}$, with fibre $\mathfrak{g}$, which is associated to $\mathscr{P} \longrightarrow \mathscr{V}$, i. e., $\mathfrak{L}$ is the bundle $\mathscr{P} \times_G \mathfrak{g}$ which is obtained from $\mathscr{P} \times \mathfrak{g}$ by means of the identification $(pg, \lambda) = (p, \mathrm{Ad}(g)\lambda)$, $g \in G$.

We obtain, as is readily seen, the following exact commutative diagram of bundles:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{F} & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{E}/\mathfrak{F} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{E}/\mathfrak{F} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{L} & \longrightarrow & \mathfrak{L} & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{1.1$_P$}$$

We call $(1.1)_P$ the *fundamental bundle diagram* for the family $\mathscr{P} \longrightarrow \mathscr{V} \longrightarrow M$. In the special case where $\mathscr{P}$ is the associated principal bundle of $\mathfrak{F}$, the bundle along the fibres of $\mathscr{V} \longrightarrow M$, we call $(1.1)_P$ the *associated bundle diagram* of $\mathscr{V} \longrightarrow M$.

Finally let $\mathscr{B} \longrightarrow \mathscr{V}$ be a differentiable family of complex vector bundles over $\mathscr{V}$ with fibre $\boldsymbol{C}^q$ and structure group $GL(q, \boldsymbol{C})$. Define $\mathfrak{D}(\mathscr{B})$ to be the sheaf of germs of differentiable sections of $\mathscr{B}$ whose restrictions to each fibre of $\mathscr{V} \rightarrow M$ are holomorphic. Let $V_t = \varpi^{-1}(t)$ be the fibre over a particular point $t \in M$ and write $B_t = \mathscr{B} \mid V_t$ for the restriction of the bundle $\mathscr{B}$ to the fibre V_t. Denote by $\Omega(B_t)$ the sheaf of germs of holomorphic sections of B_t; then we have the exact sequence of sheaves

$$0 \longrightarrow K_t \longrightarrow \mathfrak{D}(\mathscr{B}) \mid V_t \longrightarrow \Omega(B_t) \longrightarrow 0$$

where K_t is the subsheaf of the restriction $\mathfrak{D}(\mathscr{B}) \mid V_t$ of $\mathfrak{D}(\mathscr{B})$ to the fibre V_t which is composed of those germs which vanish on V_t. We denote by r_t the map defined by the commutative triangle

$$\begin{tikzcd} \mathfrak{D}(\mathscr{B}) \arrow[rr, "r_t"] \arrow[dr] & & \Omega(B_t) \\ & \mathfrak{D}(\mathscr{B}) \mid V_t \arrow[ur] & \end{tikzcd}$$

We call r_t the *restriction map* to the fibre V_t. We note that r_t is quite different from the usual restriction of sheaves in topology. The map r_t induces a restriction map in the cohomology with coefficients in $\mathfrak{D}(\mathscr{B})$, namely

$$r_t^* : H^*(\mathscr{V}, \mathfrak{D}(\mathscr{B})) \longrightarrow H^*(V_t, \Omega(B_t))$$

where, as usual, $H^*(\cdots)$ denotes the direct sum $\sum_q H^q(\cdots)$.

For purposes of comparison, we conclude this section by giving another example of mixed structure (not required in the sequel), a continuous family of differentiable structures defined as follows. Let M be a connected topological manifold and let $\mathscr{V}$ be a continuous fibre bundle over M whose fibre V_t over $t \in M$ is a (connected) differentiable manifold the differentiable structure of which is compatible with the continuous structure of V_t induced from that of $\mathscr{V}$. We denote by ϖ the canonical projection of $\mathscr{V}$ onto M. Each point of $\mathscr{V}$ has a neighborhood $\mathscr{U}$ satisfying the following conditions: there exists a continuous homeomorphism h of $\mathscr{U}$ into $\boldsymbol{R}^n \times \varpi(\mathscr{U})$ whose restriction h_t to $\mathscr{U} \cap V_t$ is a biregular map of $\mathscr{U} \cap V_t$ into $\boldsymbol{R}^n \times t$ in the sense of the category of differentiable structures, where $\boldsymbol{R}^n$ is the space of n real variables $x^1, \cdots, x^n$. The continuous dependence of the differentiable structure of the fibre V_t on the point t is secured by imposing the following condition: Let $\mathfrak{U} = \{\mathscr{U}_i\}$ be a locally finite covering of $\mathscr{V}$ by neighborhoods $\mathscr{U}_i$ with associated homeomorphisms $h_i: \mathscr{U}_i \to \boldsymbol{R}^n \times \varpi(\mathscr{U}_i)$ satisfying the above requirement, and let $h_i(p) = (x_i^1(p), \cdots, x_i^n(p), t)$, $t = \varpi(p)$, $p \in \mathscr{U}_i \cap \mathscr{U}_k$; then $x_i^\alpha(p) = h_{ik}^\alpha(x_k(p), t)$, $p \in \mathscr{U}_i \cap \mathscr{U}_k$, where the partial derivatives of the $h_{ik}^\alpha(x_k, t)$ with respect to the coordinates $x_k^1, \cdots, x_k^n$(of arbitrary orders) are continuous functions defined in $\mathscr{U}_i \cap \mathscr{U}_k$. With reference to a particular V_o, $o \in M$, any V_t, $t \in M$, is a deformation of the differentiable structure of V_o. At least if the fibres V_t of $\mathscr{V} \to M$ are compact, it can be shown that an arbitrary point $o \in M$ has a neighborhood U such that $\varpi^{-1}(U)$ is trivial in the sense that there exists a continuous map $h: \varpi^{-1}(U) \to V_o = \varpi^{-1}(o)$ whose restriction to each fibre $V_t = \varpi^{-1}(t)$, $t \in U$, is a biregular differentiable homeomorphim of V_t onto V_o. Thus every continuous family of compact differentiable manifolds is locally trivial, a circumstance which obviously does not generally occur for differentiable families of compact complex-analytic manifolds.

2. Harmonic forms along the fibres of a differentiable family

We summarize here certain basic theorems, relating to harmonic forms, which will be used in the sequel.

Let $\mathscr{V} \to M$ be a differentiable family of complex structures, $\mathfrak{F}$ the bundle along the fibres of $\mathscr{V}$ as defined in Section 1, and denote by $\mathfrak{F}^*$ the dual vector bundle of $\mathfrak{F}$. Let $\mathfrak{F}^*(r, s) = (\wedge^r\mathfrak{F}^*) \wedge (\wedge^s\overline{\mathfrak{F}}^*)$ where $\wedge^r\mathfrak{F}^*$ denotes the r-tupel exterior product of $\mathfrak{F}^*$, $\wedge^s\overline{\mathfrak{F}}^*$ the s-tupel exterior product of $\overline{\mathfrak{F}}^*$, the conjugate bundle of $\mathfrak{F}^*$. The differentiable sections of $\mathfrak{F}^*(r, s)$ will be called differential forms of type (r, s) (along the fibres) on $\mathscr{V}$. Let $A(r, s)$ be the space of differential forms of type (r, s) on $\mathscr{V}$ and write $A = \sum_{r,s} A(r, s)$ where A is the space of differential forms of $\mathscr{V}$ of all degrees and types. Denote by d the exterior derivation along the fibres of $\mathscr{V}$ which is the anti-derivation of degree 1 on A characterized by the following two conditions : (1) if f is a function on $\mathscr{V}$, df is the gradient of f along the fibres of $\mathscr{V}$; (2) $d^2 = d \circ d = 0$. We say that an operator on A is of type (μ, ν) if, for each (r, s), it maps $A(r, s)$ into $A(r + \mu, s + \nu)$. Then the exterior differential operator d on A clearly splits : $d = \partial + \bar{\partial}$ where ∂ is of type $(1, 0)$, $\bar{\partial}$ (the conjugate operator) of type $(0, 1)$.

Now let $\mathscr{B}$ be an arbitrary differentiable family of complex vector bundles over $\mathscr{V}$ (with fibre C^q and structure group $GL(q, C)$) and let $B_t = \mathscr{B} \mid V^t$ (restriction of $\mathscr{B}$ to the fibre $V_t = \varpi^{-1}(t)$ of $\mathscr{V}$, $t \in M$).

DEFINITION 2.1. The differentiable sections of $\mathscr{B} \otimes \mathfrak{F}^*(r, s)$ over $\mathscr{V}$ will be called $\mathscr{B}$-forms of type (r, s) on $\mathscr{V}$; the differentiable sections of $B_t \otimes F_t^*(r, s)$ will be called B_t-forms of type (r,s) on V_t.

A differentiable section of $\mathscr{B} \otimes \sum_{r,s} \mathfrak{F}^*(r, s)$ will be called a $\mathscr{B}$-form; the space of $\mathscr{B}$-forms on $\mathscr{V}$ thus decomposes into a direct sum of subspaces of $\mathscr{B}$-forms of the various types. A similar remark applies to B_t-forms (differentiable sections of $B_t \otimes \sum_{r,s} F_t^*(r, s)$ over V_t). It is obvious that the restriction to a fibre V_t of a $\mathscr{B}$-form of type (r, s) is a B_t-form of type (r, s) on V_t. An arbitrary family $\{\varphi_t \mid t \in M\}$ of differentiable B_t-forms φ_t on V_t determines uniquely a (not necessarily differentiable) $\mathscr{B}$-form φ on $\mathscr{V}$ whose restriction to each fibre V_t coincides with $\varphi^t : \varphi \mid V_t = \varphi_t$. We say that φ_t *depends differentiably on* t if and only if φ is a differentiable $\mathscr{B}$-form on $\mathscr{V}$.

Let $d_t = d \mid V_t$, $\partial_t = \partial \mid V_t$, $\bar{\partial}_t = \bar{\partial} \mid V_t$ be the restrictions of the operators d, ∂, $\bar{\partial}$ respectively to the fibre V_t. Then d_t is the usual exterior differential operator on V_t and $d_t = \partial_t + \bar{\partial}_t$ where ∂_t is an operator of type $(1, 0)$ and $\bar{\partial}_t$ is the conjugate operator of type $(0, 1)$. It is clear that $\bar{\partial}$ operates on the space of $\mathscr{B}$-forms and that it maps a $\mathscr{B}$-form of type (r, s) into one of type $(r, s+1)$. Similarly, $\bar{\partial}_t$ operates on the space of B_t-forms on V_t and it maps a B_t-form of the type (r, s) into one of type $(r, s+1)$. These operations commute with restriction, namely $\bar{\partial}_t \circ r_t = r_t \circ \bar{\partial}$. We denote by $H_{\bar{\partial}}^{r,s}(\mathscr{B})$ the $\bar{\partial}$-cohomology of $\mathscr{B}$-forms of type (r, s) on $\mathscr{V}$ and by $H_{\bar{\partial}_t}^{r,s}(B_t)$ the $\bar{\partial}_t$-cohomology of B_t-forms of type (r, s) on V_t.

PROPOSITION 2.1. *The module $H^q(\mathscr{V}, \mathfrak{D}(\mathscr{B}))$ is isomorphic to $H_{\bar{\partial}}^{0,q}(\mathscr{B})$; $H^q(V_t, \Omega(B_t))$ is isomorphic to $H_{\bar{\partial}_t}^{0,q}(B_t)$. Moreover the diagram*

$$\begin{array}{ccc} H^q(\mathscr{V}, \mathfrak{D}(\mathscr{B})) & \cong & H_{\bar{\partial}}^{0,q}(\mathscr{B}) \\ \downarrow r_t & & \downarrow r_t \\ H^q(V_t, \Omega(B_t)) & \cong & H_{\bar{\partial}_t}^{0,q}(B_t) \end{array}$$

is commutative.

The isomorphism $H^q(V_t, \Omega(B_t)) \cong H_{\bar{\partial}_t}^{0,q}(B_t)$ is well known (Dolbeault lemma); its proof is based on the so-called "Poincaré lemma" for the operator $\bar{\partial}_t$, namely: if φ_t is a B_t-form of type (r, s), $s > 0$, which is defined in a neighborhood of a point of V_t and satisfies $\bar{\partial}_t \varphi_t = 0$, then there exists a B_t-form ψ_t of type $(r, s-1)$, which is defined in some (possibly smaller) neighborhood of the same point of V_t, such that $\varphi_t = \bar{\partial}_t \psi_t$ there. The proof of the isomorphism $H^q(\mathscr{V}, \mathfrak{D}(\mathscr{B})) \cong H^{0,q}_{\bar{\partial}}(\mathscr{B})$ is entirely similar and is based on the Poincaré lemma for the operator $\bar{\partial}$. The proof of the Poincaré lemma for $\bar{\partial}_t$ (due to A. Grothendieck—see e. g., [12]) is a reduction to the case where φ_t is real analytic; the lemma is then established by the classical method of power series. The proof of the Poincaré lemma for $\bar{\partial}$ may by carried out along similar lines. The reduction to the case where φ is real analytic along the fibres presents no difficulty; the lemma is then established by means of power series whose coefficients depend differentiably on the point $t \in M$.

DEFINITION 2.2. An hermitian metric along the fibres of $\mathscr{B}$ is a differentiable reduction of the structure group $GL(q, \boldsymbol{C})$ of $\mathscr{B}$ to the unitary group $U(q)$.

We note here that an hermitian metric on a complex manifold is equivalent to a differentiable reduction of the structure group of the tangent bundle to the unitary group.

Since $GL(q, \boldsymbol{C})$ is the complexification of the compact group $U(q)$, a differentiable reduction of the structure group to $U(q)$ is always possible. In the particular case of the vector bundle $\mathfrak{F}$ with fibre $\boldsymbol{C}^n$ and group $GL(n, \boldsymbol{C})$, where n is the complex dimension of the fibres V_t of $\mathscr{V}$, we say that a differentiable reduction of $GL(n, \boldsymbol{C})$ to $U(n)$ defines an hermitian metric along the fibres of $\mathscr{V}$.

DEFINITION 2.3. A quasi-hermitian metric on $\mathscr{V}$ is a differentiable reduction of the structure group $GL(n, \boldsymbol{C}; m, \boldsymbol{R})$ of the bundle $\mathfrak{E}$ (with fibre $\boldsymbol{C}^n \oplus \boldsymbol{R}^m$) to $U(n) \times O(m)$.

Thus a quasi-hermitian metric on $\mathscr{V}$ is a riemannian metric on $\mathscr{V}$ whose restriction to each fibre is an hermitian metric. In particular, a quasi-hermitian metric on $\mathscr{V}$ defines a positive hermitian form over each fibre $\mathfrak{F}_p$, $p \in \mathscr{V}$, depending differentiably on p. For $u, v \in \mathfrak{F}_p$ let $(u, v)_p$ be the corresponding hermitian bilinear form. Then the hermitian property is characterized by the statement that $(u, v)_p = (Ju, Jv)_p$ where J is the complex structure tensor. Now let ω be the differential form of type $(1, 1)$ on $\mathscr{V}$ which, at the point $p \in \mathscr{V}$, is defined by $\omega(u, v) = (Ju, v)_p$, $u, v \in \mathfrak{F}_p$.

DEFINITION 2.4. A quasi-hermitian metric will be called quasi-kählerian if and only if its restriction to each fibre is kählerian.

It is clear that a quasi-hermitian metric is quasi-kählerian if and only if $d\omega = 0$ where d is the exterior differential operator along the fibres of $\mathscr{V}$. If a family $\mathscr{V}$ possesses a quasi-hermitian (quasi-kählerian) metric, we shall say that $\mathscr{V}$ is quasi-hermitian (quasi-kählerian).

PROPOSITION 2.2. *A differentiable family* $\mathscr{V} \xrightarrow{\varpi} M$ *of compact complex structures is quasi-kählerian if and only if each fibre is kählerian.*

PROOF. If $\mathscr{V}$ is quasi-kählerian, then (as remarked above) each fibre is kählerian. Assume that each fibre is kählerian. By Theorem 3.1 below, there exists a locally finite covering $\{U_\iota\}$ of M by open sets U_ι such that $\varpi^{-1}(U_\iota)$ is quasi-kählerian. A partition of unity subordinate to the covering $\{U_\iota\}$ can then be lifted up to $\mathscr{V}$ and used to define a (differentiable) quasi-kählerian metric on $\mathscr{V}$.

Since quasi-hermitian metrics on $\mathscr{V}$ always exist, assume that one is given and assume further that an hermitian metric along the fibres of $\mathscr{B}$ is given. We define an hermitian scalar product (φ, ψ) for any two $\mathscr{B}$-forms φ, ψ on $\mathscr{V}$ by

$$\int_M (\varphi_t, \psi_t)_t dM$$

where $(\varphi_t, \psi_t)_t$ is the hermitian scalar product on the fibre V_t of the restrictions φ_t, ψ_t of φ, ψ and where dM is the volume element on the base space M. Denote by ϑ the formal adjoint of $\bar\partial$ satisfying $(\bar\partial\varphi, \psi) = (\varphi, \vartheta\psi)$ for $\mathscr{B}$-forms φ, ψ one of which, at least, has compact support, and introduce the laplacian $\square = \bar\partial\vartheta + \vartheta\bar\partial$ which is an operator of type (0,0) on the space of $\mathscr{B}$-forms and hence operates on the subspace of $\mathscr{B}$-forms composed of forms of fixed type (r, s). The operators $\square$, ϑ depend on $\mathscr{B}$ but, for simplicity, we do not indicate explicitly this dependence. We remark that the restriction $\vartheta_t = \vartheta \mid V_t$ coincides with the formal adjoint of $\bar\partial_t$ on the fibre V_t and $\square_t = \square \mid V_t = \bar\partial_t\vartheta_t + \vartheta_t\bar\partial_t$.

We remark that, if $\mathscr{B}$ is the trivial product complex line bundle $\boldsymbol{C} \times \mathscr{V}$, then the space of $\mathscr{B}$-forms coincides with the space A of (scalar) forms along the fibres of $\mathscr{B}$ and, in this case, the conjugate laplacian $\bar\square = \bar\vartheta\partial + \partial\bar\vartheta$ also operates on A and, like $\square$, is an operator of type (0, 0). Moreover, since d operates on A, we can define the formal adjoint δ of d, $(d\varphi, \psi) = (\varphi, \delta\psi)$ for forms $\varphi, \psi \in A$ one of which, at least, has compact support, and introduce the real laplacian $\Delta = d\delta + \delta d$ as operator on A. We note that we have $\delta = \vartheta + \bar\vartheta$.

PROPOSITION 2.3. *If the fibres of* $\mathscr{V} \to M$ *are compact, then the usual Green's formulas without boundary terms are valid, in particular* $(d\varphi, \psi) = (\varphi, \delta\psi)$ *for* $\varphi, \psi \in A$, $(\bar\partial\varphi, \psi) = (\varphi, \vartheta\psi)$ *for* $\mathscr{B}$*-forms* φ, ψ.

The proof is straightforward and will be omitted.

PROPOSITION 2.4. *If* $\mathscr{V}$ *possesses a quasi-Kähler metric, then* $\frac{1}{2}\Delta = \square = \bar\square$ *for scalar forms.*

The proof does not differ from the standard proof for a single Kähler manifold.

We return now to the general case of a non-trivial vector bundle $\mathscr{B}$ over $\mathscr{V}$ and we suppose that the fibres V_t of $\mathscr{V}$ are compact (closed). Since the restriction $\Box_t = \bar{\partial}_t\vartheta_t + \vartheta_t\bar{\partial}_t$ of $\Box$ to the fibre V_t of $\mathscr{V}$ is an elliptic operator on the space of B_t-forms on V_t, it is well known that there exists a unique Green's operator G_t satisfying $\varphi_t = \Box_t G_t\varphi_t + H_t\varphi_t$ for any B_t-form φ_t and $G_tH_t = H_tG_t = 0$, where H_t signifies orthogonal projection onto the subspace $\boldsymbol{H}_t$ of harmonic B_t-forms ψ_t on V_t which are characterized by $\bar{\partial}_t\psi_t = \vartheta_t\psi_t = 0$. Let $\Pi_t(r, s)$ denote projection from the space of B_t-forms on V_t onto the subspace of B_t-forms of type (r, s). Then G_t commutes with $\bar{\partial}_t$, H_t and $\Pi_t(r, s)$ (since $\Box_t$ commutes with these operators); H_t commutes with $\Pi_t(r, s)$ and with $\bar{\partial}_t : \bar{\partial}_tH_t = H_t\bar{\partial}_t = 0$. We set $H_t(r, s) = H_t\circ\Pi_t(r, s)$ and $G_t(r, s) = G_t\circ\Pi_t(r, s)$. It follows immediately that the $\bar{\partial}_t$-cohomology $H^{r,s}_{\bar{\partial}_t}(B_t)$ of B_t-forms of type (r, s) on V_t is isomorphic to $\boldsymbol{H}_t(r, s)$, the subspace of $\boldsymbol{H}_t$ composed of forms of type (r, s). By Proposition 2.1 we have

$$\boldsymbol{H}_t(0, q) \cong H^p(V_t, \Omega(B_t)) \, . \tag{2.1}$$

The question arises concerning the existence of a Green's operator $G(r, s)$ and a harmonic projection operator $H(r, s)$ acting on the space of differentiable $\mathscr{B}$-forms of type (r, s) on $\mathscr{V}$. Let $\mathfrak{L}$ be the Hilbert space of all Lebesgue measurable $\mathscr{B}$-forms φ of type (r, s) with $(\varphi, \varphi) < +\infty$ and let $\mathfrak{L}^\infty$ be the subspace of $\mathfrak{L}$ composed of differentiable $\mathscr{B}$-forms of type (r, s). Moreover let $\boldsymbol{H}(r, s)$ be the space of differentiable $\mathscr{B}$-forms ψ of type (r, s) satisfying $\bar{\partial}\psi = \vartheta\psi = 0$, let $\mathfrak{H} = [\boldsymbol{H}(r, s) \cap \mathfrak{L}]$ be the *closure* of $\boldsymbol{H}(r, s) \cap \mathfrak{L}$ in $\mathfrak{L}$, and let $H(r, s)$ denote the orthogonal projection of $\mathfrak{L}$ onto $\mathfrak{H}$. Now, if $H(r, s)\cdot\mathfrak{L}^\infty \subseteqq \boldsymbol{H}(r, s)$, we call $H(r, s)$ a harmonic (projection) operator: otherwise we say that the harmonic operator does *not* exist. Assume that the harmonic operator exists. Then, by the Green's operator $G(r, s)$, we mean a linear operator $G(r, s)$: $\mathfrak{L}^\infty \to \mathfrak{L}^\infty$ satisfying

$$\varphi = \Box G(r, s)\varphi + H(r, s)\varphi \, , \qquad \text{for all } \varphi \in \mathfrak{L}^\infty;$$
$$G(r, s)\, H(r, s) = H(r, s)\, G(r, s) = 0 \, .$$

For any $\varphi \in \mathfrak{L}^\infty$ we denote by φ_t the restriction $\varphi \mid V_t$ of φ to V_t. Suppose that the Green's operator $G(r, s)$ and the harmonic operator $H(r, s)$ exist. Then, comparing

$$\varphi_t = \Box_t(G(r, s)\varphi)_t + (H(r, s)\varphi)_t$$

with

$$\varphi_t = \Box_t G_t(r, s)\varphi_t + H_t(r, s)\varphi_t ,$$

we get

$$\begin{cases} G_t(r, s)\varphi_t = (G(r, s)\varphi)_t , \\ H_t(r, s)\varphi_t = (H(r, s)\varphi)_t . \end{cases} \tag{2.2}$$

Thus $G_t(r, s)\varphi_t$ and $H_t(r, s)\varphi_t$ depend differentiably on t. Conversely, if $G_t(r, s)\varphi_t$ and $H_t(r, s)\varphi_t$ depend differentiably on t for all $\varphi \in \mathfrak{L}^\infty$, then the Green's operator $G(r, s)$ and the harmonic operator $H(r, s)$ are defined by (2.2).

FUNDAMENTAL THEOREM. *$G_t(r, s)\varphi_t$ and $H_t(r, s)\varphi_t$, $\varphi_t = \varphi | V_t$, depend differentiably on t for all differentiable $\mathscr{B}$-forms φ on $\mathscr{V}$ if and only if* $\dim \boldsymbol{H}_t(r, s)$ *is independent of the point $t \in M$.*

This "fundamental theorem" is proved by a method implicitly contained in our paper [28]. For a complete proof, see [29]. Combining the fundamental theorem with the above remarks, we obtain.

PROPOSITION 2.5. *The Green's operator $G(r, s)$ and the harmonic operator $H(r, s)$ exist if and only if* $\dim \boldsymbol{H}_t(r, s)$ *is independent of the point $t \in M$.*

Assume that $\dim \boldsymbol{H}_t(r, s)$ is independent of t. In view of (2.2), the operators $G(r, s)$ and $H(r, s)$ defined on the space $\mathfrak{L}^\infty$ of *norm finite* differentiable $\mathscr{B}$-forms of type (r, s) can be extended uniquely to operators on the space of *all* differentiable $\mathscr{B}$-forms of type (r, s). In fact, let $1 = \sum \rho_i$ be a partition of unity on the base space M such that the supports of ρ_i are compact. We lift up the ρ_i to $\mathscr{V}$ and denote them by the same symbols ρ_i and, for any differentiable $\mathscr{B}$-forms φ of type (r, s) on $\mathscr{V}$, we define

$$G(r, s)\varphi = \textstyle\sum_i G(r, s)\rho_i\varphi ,$$

$$H(r, s)\varphi = \textstyle\sum_i H(r, s)\rho_i\varphi .$$

Obviously $H(r, s)\varphi$ belongs to $\boldsymbol{H}(r, s)$ and

$$\varphi = \Box G(r, s)\varphi + H(r, s)\varphi ; \tag{2.3}$$

in particular, if $\bar\partial\varphi = 0$,

$$\varphi = \bar\partial\vartheta G(r, s)\varphi + H(r, s)\varphi . \tag{2.4}$$

From this we obtain the following

PROPOSITION 2.6. *If* $\dim \boldsymbol{H}_t(r, s)$ *is independent of $t \in M$, then $H^{r,s}_{\bar\partial}(\mathscr{B})$ is isomorphic to $\boldsymbol{H}(r, s)$ and, moreover, we have the commutative diagram*

$$\begin{array}{ccc} H_{\bar\partial}^{r,s}(\mathscr{B}) & \cong & \boldsymbol{H}(r,s) \\ \downarrow r_t & & \downarrow r_t \\ H_{\bar\partial_t}^{r,s}(B_t) & \cong & \boldsymbol{H}_t(r,s) \end{array}$$

PROPOSITION 2.7. *If* dim $\boldsymbol{H}_t(r, s)$ *is independent of* $t \in M$, *then the homomorphism* $r_t : \boldsymbol{H}(r, s) \to \boldsymbol{H}_t(r, s)$ *is surjective; moreover,* $\bigcup_{t \in M} \boldsymbol{H}_t(r, s)$ *forms a differentiable complex vector bundle over* M *in a canonical manner and* $\boldsymbol{H}(r, s)$ *coincides with the space of differentiable sections of* $\bigcup_{t \in M} \boldsymbol{H}_t(r, s)$.

PROOF. It suffices to show that $r_t : \boldsymbol{H}(r, s) \to H_t(r, s)$ is surjective. Let $\psi_t \in \boldsymbol{H}_t(r, s)$. Then there exists a differentiable $\mathscr{B}$-form ψ of type (r, s) on $\mathscr{V}$ such that $\psi \mid V_t = \psi_t$. We have

$$r_t(H(r, s)\psi) = H_t(r, s)\psi_t = \psi_t ,$$

q. e. d.

We now state the basic theorems of potential theory which are required in the sequel, namely:

THEOREM 2.1. (*upper semi-continuity*). *For each point* $o \in M$ *there exists a neighborhood* U *of* o *such that*

$$\dim H^q(V_t, \Omega(B_t)) \leqq \dim H^q(V_o, \Omega(B_o))$$

for $t \in U$.

THEOREM 2.2. *If* $\dim H^q(V_t, \Omega(B_t))$ *is independent of* $t \in M$, *then*:

(i) *the restriction map* $r_t : H^k(\mathscr{V}, \mathfrak{D}(\mathscr{B})) \to H^k(V_t, \Omega(B_t))$ *is surjective for* $k = q - 1, q$;

(ii) $r_t(\varphi) = 0$ *for all* $t \in M$ *implies* $\varphi = 0$, *where* $\varphi \in H^q(\mathscr{V}, \mathfrak{D}(\mathscr{B}))$.

THEOREM 2.3. *If* $\dim H^1(V_t, \Omega(B_t))$ *is independent of* $t \in M$, *then* $\dim H^0(V_t, \Omega(B_t))$ *is independent of* t.

Theorem 2.1, which may be called the "principle of upper semi-continuity", has been proved by the authors in their paper [28] (see also [29]); it has many applications for deformations of complex structures.

PROOF OF THEOREM 2.2. First we consider the case $k = q-1$ of (i). In view of Proposition 2.1, it suffices to show that $r_t : H_{\bar\partial}^{0,q-1}(\mathscr{B}) \to H_{\bar\partial_t}^{0,q-1}(B_t)$ is surjective. By Proposition 2.5 the Green's operator $G(0, q)$ and the harmonic operator $H(0, q)$ exist. Given a $\bar\partial_t$-closed B_t-form φ_t of type $(0, q-1)_t$ on V_t, let ψ be a $\mathscr{B}$-form of type $(0, q-1)$ on $\mathscr{V}$ such that $r_t(\psi) = \varphi_t$, and let $\eta = \vartheta G(0, q)\bar\partial\psi$. Then we have

$$\bar\partial\eta = \bar\partial\vartheta G(0, q)\bar\partial\psi = \square\, G(0, q)\bar\partial\psi = \bar\partial\psi$$

since $H(0, q)\bar\partial\psi = 0$, and hence, writing $\varphi = \psi - \eta$, we have $\bar\partial\varphi = 0$. On the other hand, $r_t(\varphi) = r_t(\psi) = \varphi_t$, since $r_t(\eta) = \vartheta_t G_t(0, q)\bar\partial_t\psi_t$ where $\bar\partial_t\psi_t = \bar\partial_t\varphi_t = 0$. Thus φ_t is in the image of $r_t : H_{\bar\partial}^{0,q-1}(\mathscr{B}) \to H_{\bar\partial_t}^{0,q-1}(B_t)$.

Part (ii) and the case $k = q$ of (i) are immediate consequences of Propositions 2.6 and 2.7, q. e. d.

PROOF OF THEOREM 2.3. By Theorem 2.2, (i), the restriction map $r_t : H^0(\mathscr{V}, \mathfrak{O}(\mathscr{B})) \to H^0(V_t, \Omega(B_t))$ is surjective. Take a base $\{\varphi_{t0}, \varphi_{t1}, \cdots, \varphi_{tk}\}$ of $H^0(V_t, \Omega(B_t))$; then there exist $\varphi_\nu \in H^0(\mathscr{V}, \mathfrak{O}(\mathscr{B}))$ such that $\varphi_{t\nu} = r_t(\varphi_\nu)$ and, if s belongs to a sufficiently small neighborhood of the point t, $r_s(\varphi_0), \cdots, r_s(\varphi_k)$ are complex linearly independent elements of $H^0(V_s, \Omega(B_s))$. Hence $\dim H^0(V_s, \Omega(B_s)) \geqq \dim H^0(V_t, \Omega(B_t))$ while, by Theorem 2.1, the opposite inequality is valid for s near t, q. e. d.

3. Stability theorems

A stability theorem asserts that a certain property of a complex analytic manifold is stable under " sufficiently small deformations " of the complex analytic structure—more precisely that, if a certain fibre V_o of a differentiable family $\mathscr{V} \xrightarrow{\varpi} M$ possesses the property, then all neighboring fibres also possess the property. We assume throughout this section that $\mathscr{V} \xrightarrow{\varpi} M$ is a differentiable family of complex structures whose fibre $V_t = \varpi^{-1}(t)$, $t \in M$, is *compact*.

As first example of a stability theorem we have :

THEOREM 3.1. *If V_0 is a Kähler manifold, then any small deformation V_t of V_0 is also a Kähler manifold ; more precisely, there exists a neighborhood U of the point $o \in M$ such that V_t is a Kähler manifold for any $t \in U$. Moreover we can choose a Kähler metric on each fibre V_t which depends differentiably on t.*

For a proof of this theorem see Kodaira and Spencer [29].

The question remains open whether *any* deformation V_t of a Kähler manifold V_o is also a Kähler manifold ; more precisely : if V_o is kählerian, then is every manifold which is c-homotopic to V_o also kählerian? (Compare Section 1.)

Now let φ be a real-valued differential form of type (1, 1) on a compact complex-analytic manifold V of complex dimension n; then φ, referred to local holomorphic coordinates $(z^1, \cdots, z^\alpha, \cdots, z^n)$, is of the form $i \sum \varphi_{\alpha\bar\beta}(z, \bar z) dz^\alpha \wedge d\bar z^\beta$. We say that φ is positive on V and we write $\varphi > 0$ if the hermitian form $\sum \varphi_{\alpha\bar\beta}(z, \bar z) u^\alpha \bar u^\beta$ in n variables $u^1, \cdots, u^n$ is positive definite at each point z of V. Suppose that V has an hermitian metric $\sum g_{\alpha\bar\beta}(z, \bar z)\, dz^\alpha \cdot d\bar z^\beta$, where $dz^\alpha \cdot d\bar z^\beta$ denotes the symmetric product of dz^α and $d\bar z^\beta$, and let g denote the determinant of the $g_{\alpha\bar\beta}$. The Ricci curvature of this metric is the tensor $R_{\alpha\bar\beta} = -\partial^2 \log g / \partial z^\alpha \partial \bar z^\beta$ and $(i/2\pi) \sum R_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$, a closed real form of type (1, 1), is the so-called Ricci form.

We recall that an algebraic manifold is a compact complex-analytic manifold which can be imbedded complex analytically and without singularities in a complex projective space of suitably chosen dimension. It has been proved by Kodaira [23] that, if the Ricci form of a compact complex-analytic manifold is positive or negative definite, then the manifold is algebraic. Therefore we have:

THEOREM 3.2. *If V_o is an algebraic manifold with positive or negative definite Ricci form, there exists a neighborhood U of $o \in M$ such that V_t is an algebraic manifold for $t \in U$.*

It is also known (Kodaira [25], Theorem 3.5) that a Kähler manifold V_o for which $\dim H^2(V_o, \Omega_o) = 0$ is an algebraic manifold, where Ω_o denotes the sheaf over V_o of germs of holomorphic functions. By Theorems 2.1 and 3.1 we therefore have:

THEOREM 3.3. *If V_o is an algebraic manifold with* $\dim H^2(V_o, \Omega_o) = 0$, *there is a neighborhood U of $o \in M$ such that V_t is an algebraic manifold for $t \in U$.*

In the remainder of this section we consider small deformations of algebraic surfaces. Let $\{V_t \mid t \in M\}$ be a differentiable family of deformations V_t of the complex structure of an algebraic surface V_o. Let K_t be the canonical bundle over V_t and let K_t^m denote the tensor product m times of K_t with itself. Write $P_m(V_t) = \dim H^0(V_t, \Omega(K_t^m))(m = 1, 2, 3, \cdots)$, $q(V_t) = \dim H^1(V_t, \Omega_t)$. We note that $P_{mn}(V_t) = 0$ implies $P_m(V_t) = 0$. It follows from Theorem 3.1 that any small deformation V_t of V_o is a Kähler surface. Since $P_1(V_t) = H^2(V_t, \Omega_t)$, we infer from Theorems 2.1 and 3.3 that, if $P_1(V_o) = 0$, any small deformation V_t of V_o is an algebraic surface. Now assume that V_o is a rational surface. Since rational surfaces are characterized by the condition $q = P_2 = 0$ (Castelnuovo and Enriques [11]) and since $P_2 = 0$ implies $P_1 = 0$, we obtain the following

THEOREM 3.4. *Any small deformation of a rational surface is rational.*

Similarly, since algebraic surfaces which are birationally equivalent to ruled surfaces are characterized by the condition $P_{12} = 0$, we have

THEOREM 3.5. *If V_o is birationally equivalent to a ruled surface, then any small deformation V_t of V_o is also an algebraic surface which is birationally equivalent to a ruled surface.*

Let c_1 be the first Chern class of V_t and denote by $c^2(V_t)$ the value of c^2 on the 4-cycle V_t (with the natural orientation). It is obvious that $c^2(V_t) = c^2(V_o)$.

THEOREM 3.5. *Let V_o be an algebraic surface. If $c_1^2(V_o) > 0$, then any small deformation V_t of V_o is also an algebraic surface.*

PROOF. A detailed analysis of the structures of compact analytic surfaces shows that, if a compact Kähler surface V is not algebraic, the inequality $c_1^2(V) \leqq 0$ holds (Kodaira [26]). Hence we obtain our theorem.

THEOREM 3.7. *Let V_0 be an algebraic surface containing no exceptional curve of the first kind. If $c_1^2(V_0) \neq 0$ then any small deformation V_t of V_0 is an algebraic surface.*

PROOF. If $c_1^2(V_0) < 0$, V_0 is birationally equivalent to a ruled surface. Our theorem follows therefore from Theorems 3.5 and 3.6.

We note that, in case $c_1^2(V_0) = 0$, V_0 can be deformed into a non-algebraic surface by an arbitrarily small deformation, except for some special cases (Kodaira [26]).

4. Fundamental sheaves

Let $\mathscr{V} \xrightarrow{\varpi} M$ be a differentiable family of complex structures. We have introduced, in Section 1, the sheaf $\mathfrak{O}$ of germs of complex-valued differentiable functions on $\mathscr{V}$ whose restrictions to each fibre of $\mathscr{V}$ are holomorphic. In particular, $\mathfrak{O}$ may be regarded as a sheaf of $\boldsymbol{R}$-modules and we denote by O the subsheaf of $\mathfrak{O}$ composed of germs of real-valued differentiable functions on $\mathscr{V}$ which are constant along the fibres of $\mathscr{V}$. Thus O is the sheaf induced over $\mathscr{V}$, by the map $\varpi : \mathscr{V} \to M$, from the sheaf of germs of differentiable functions on M.

Now let Θ, Ψ respectively be the sheaves of germs of differentiable sections of $\mathfrak{F}$, $\mathfrak{G}$ whose restrictions to each fibre of $\mathscr{V}$ are holomorphic. We have the following exact sequence of sheaves

$$0 \longrightarrow \Theta \longrightarrow \Psi \longrightarrow \Lambda \longrightarrow 0 \tag{4.1}$$

where Λ denote the quotient sheaf Ψ/Θ.

Let T_M be the sheaf of germs of differentiable sections of the tangent bundle of M, and let $\tilde{T}$ be the sheaf induced over $\mathscr{V}$ from T_M by the map $\varpi : \mathscr{V} \to M$. There is a natural injection $\tilde{T} \to \Lambda$ and the image of $\tilde{T}$ in Λ will be denoted by T. We define $\Pi = j^{-1}(T)$ to be the inverse image of T in Ψ under the map j of (4.1). (For explicit representations of germs of Θ and Π see Section 5.) Thus we obtain the exact sequence of sheaves

$$0 \longrightarrow \Theta \longrightarrow \Pi \xrightarrow{j} T \longrightarrow 0 \, . \tag{4.2}$$

We call (4.2) the *fundamental sequence* of sheaves for the differentiable family $\mathscr{V} \to M$.

An alternative way of defining Π, T is the following. Since Ψ is a sheaf of germs of vector fields tangent to $\mathscr{V}$, Ψ operates by differentiation on O. Let Π be the largest subsheaf of Ψ for which O is stable under the differentiation operations of Π. Clearly Θ is a subsheaf of Π, namely the subsheaf which annihilates O. Hence we have the exact sequence (4.2) where T is the quotient sheaf.

Suppose that $\mathscr{B} \to \mathscr{V}$ is a family of complex fibre bundles over $\mathscr{V}$ (see Definition 1.8) and let $\mathscr{P} \to \mathscr{V}$ be the associated family of complex principal bundles. Then we have the fundamental bundle diagram $(1.1)_P$ and we can define a corresponding sheaf diagram. In fact, let Ξ, Σ respectively be the sheaves of germs of differentiable sections of the bundles $\mathfrak{L}$, $\mathfrak{M}$, whose restrictions to each fibre of $\mathscr{V}$ are holomorphic. Furthermore, let Φ be the sheaf of germs of differentiable sections of the bundle $\mathfrak{R}$ whose restrictions to each fibre of $\mathscr{V}$ are holomorphic; then we have the exact sequence of sheaves

$$0 \longrightarrow \Xi \longrightarrow \Phi \xrightarrow{k} \Psi \longrightarrow 0 \tag{4.3}$$

where Ψ is the middle term of the sequence (4.1). Since we have the injection $\Pi \to \Psi$, we define $\Gamma = k^{-1}(\Pi)$ to be the inverse image of Π under the map k of (4.3) and we obtain the exact sequence

$$0 \longrightarrow \Xi \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow 0. \tag{4.4}$$

As is readily seen, the various sheaves fit together to form the following exact commutative diagram:

$$(4.2)_P \qquad \begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \Theta \arrow[r] \arrow[u] & \Pi \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma \arrow[r] \arrow[u] & \Gamma \arrow[r] \arrow[u, "\alpha"'] & T \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Xi \arrow[r] \arrow[u] & \Xi \arrow[r] \arrow[u] & 0 & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

We call $(4.2)_P$ the *fundamental sheaf diagram* for the family $\mathscr{P} \to \mathscr{V} \to M$. In the special case where $\mathscr{P}$ is the associated principal bundle of $\mathfrak{F}$ along the fibres of $\mathscr{V} \to M$, we call $(4.2)_P$ the *associated sheaf diagram* of $\mathscr{V} \to M$.

We recall (see [10]) that a graded Lie algebra $\mathfrak{g}$ over a commutative ring $\boldsymbol{K}$ is a graded $\boldsymbol{K}$-module $\mathfrak{g} = \Sigma_q \mathfrak{g}^q$ together with a $\boldsymbol{K}$-homomorphism $x \otimes y \to [x, y]$ of $\mathfrak{g}^p \otimes \mathfrak{g}^q \to \mathfrak{g}^{p+q}$ such that, for $x \in \mathfrak{g}^p$, $y \in \mathfrak{g}^q$, $z \in \mathfrak{g}^r$, we have

(i) $$[x, y] = (-1)^{pq+1}[y, x];$$

(ii) $$(-1)^{pr}[[x, y], z] + (-1)^{qp}[[y, z], x] + (-1)^{rq}[[z, x], y] = 0 .$$

Now let B denote a sheaf of germs of vector fields over $\mathscr{V}$, e. g. $B = \Theta$ or $B = \Psi$ (compare (4.1)). Then the usual Poisson bracket of vector fields induces a multiplication in B but this bracket operation is bilinear in general only over the constants ($\boldsymbol{R}$ or $\boldsymbol{C}$). In the general case B is therefore a sheaf of Lie algebras over $\boldsymbol{K}$ where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$. In the special case $B = \Theta$, however, the bracket is bilinear over O and hence Θ is a sheaf of germs of Lie algebras over $\boldsymbol{K} = O$. As a consequence of general results in the theory of cohomology with coefficients in sheaves we have:

PROPOSITION 4.1. *Let B be a sheaf of Lie algebra sover $\boldsymbol{K}$; then $H^*(\mathscr{V}, B) = \Sigma_q H^q(\mathscr{V}, B)$ is a graded Lie algebra over $\boldsymbol{K}$.*

The most important case for us is that in which $B = \Theta = \mathfrak{O}(\mathfrak{F})$. By Proposition 2.1:

$$H^q(\mathscr{V}, \Theta) = H^{0,q}_{\bar\partial}(\mathfrak{F}) \,. \tag{4.5}$$

An element of $H^{0,q}_{\bar\partial}(\mathfrak{F})$ is represented by a $\bar\partial$-closed $\mathfrak{F}$-form of type $(0, q)$, that is by a section of $\mathfrak{F}\otimes\mathfrak{F}^*(0, q)$ over $\mathscr{V}$ (compare Definition 2.1) which is closed under $\bar\partial$. A structure of graded Lie algebra can be defined on the space of all $\mathfrak{F}$-forms which induces the structure of graded Lie algebra on $H^*_{\bar\partial}(\mathfrak{F})$ asserted by Proposition 4.1. In the case where the base space M of the differentiable family $\mathscr{V}\to M$ is a single point, this graded Lie algebra of $\mathfrak{F}$-forms coincides with that introduced by Frölicher and Nijenhuis in their papers [14, 15]. In order to define the bracket of an arbitrary pair of $\mathfrak{F}$-forms, it will be sufficient to define the bracket for a pair of $\mathfrak{F}$-forms of the special type $\theta\otimes\varphi$, $\lambda\otimes\psi$ where θ, λ are sections of $\mathfrak{F}$ and where φ, ψ are arbitrary elements of A (the space of forms along the fibres of $\mathscr{V}$) of total degrees p, q respectively.

Before defining $[\theta\otimes\varphi, \lambda\otimes\psi]$, we recall two classical operations from the theory of exterior algebras of vector spaces. Namely, let $\boldsymbol{V}$ be a vector space, $\boldsymbol{V}^*$ the dual vector space, and denote by $\wedge \boldsymbol{V}$, $\wedge \boldsymbol{V}^*$ respectively the exterior algebras of $\boldsymbol{V}, \boldsymbol{V}^*$. Let $e(u)$, $u \in \wedge \boldsymbol{V}$, be the linear endomorphism of $\wedge \boldsymbol{V}$ defined by $e(u)\colon v \to u \wedge v$ for all $v \in \wedge \boldsymbol{V}$. Clearly $e(u + v) = e(u) + e(v)$ and $e(u \wedge v) = e(u) \circ e(v)$ for $u, v \in \wedge \boldsymbol{V}$, by the linearity and associativity of exterior multiplication. The transposed endomorphism $e^*(u)$ will be denoted by $i(u)\colon \wedge \boldsymbol{V}^* \to \wedge \boldsymbol{V}^*$ (contraction by u); it is easily verified that, if u is of degree 1, then $i(u)$ is an antiderivation of degree $+1$ on $\wedge \boldsymbol{V}^*$.

Denote by $[\theta, \lambda]$ the Poisson bracket of θ, λ, and by $i(\theta)$ the (above) operation $i(\theta)\colon A \to A$ (contraction by θ). We denote by $l(\theta)$ the "Lie derivative along the fibres of $\mathscr{V}$", namely $l(\theta) = i(\theta) \circ d + d \circ i(\theta)$ where d is the exterior differential along the fibres of $\mathscr{V}$, and we define

$$
\begin{aligned}
[\theta \otimes \varphi, \lambda \otimes \Psi] = \lambda \otimes (\varphi \wedge l(\theta)\psi) &+ (-1)^p \lambda \otimes (d\varphi \wedge i(\theta)\psi) \\
&+ (-1)^{pq+1} \theta \otimes (\psi \wedge l(\lambda)\varphi) \\
&+ (-1)^{pq+q+1} \theta \otimes (d\psi \wedge i(\lambda)\varphi) \\
&+ [\theta, \lambda] \otimes (\varphi \wedge \psi).
\end{aligned} \tag{4.6}
$$

This definition also applies to $\mathfrak{F}$-forms defined in a neighborhood of a point of $\mathscr{V}$ and, if f is a function on this neighborhood which represents a germ of $\mathfrak{O}$, it may be verified that $[\theta \otimes f\varphi, \lambda \otimes \psi] = [f\theta \otimes \varphi, \lambda \otimes \psi]$. Finally it may be verified that the bracket satisfies conditions (i) and (ii) for a graded Lie algebra where the degree of $\theta \otimes \varphi$ is the total degree of φ. We remark that, while the bracket is defined for tensor products $\theta \otimes \varphi$ over $\mathfrak{O}$, it is bilinear only over O.

We consider next the fibre $V_t = \varpi^{-1}(t)$ of $\mathscr{V}$ over a particular point $t \in M$ which is assumed fixed once for all. Denote by F_t, E_t the restrictions to V_t of the bundles $\mathfrak{F}, \mathfrak{E}$; then we have the sequence of bundles over V_t:

$$0 \longrightarrow F_t \longrightarrow E_t \longrightarrow E_t/F_t \longrightarrow 0 .$$

These restricted bundles are holomorphic and we let Θ_t, Ψ_t be the sheaves of germs of holomorphic sections of F_t, E_t respectively. We have the exact sequence of sheaves

$$0 \longrightarrow \Theta_t \longrightarrow \Psi_t \xrightarrow{j_t} \Lambda_t \longrightarrow 0 \tag{4.1$_t$}$$

where Λ_t denotes the quotient sheaf. Let $(T_M)_t$ be the tangent space of M at the point t (sheaf of sections of the restriction to t of the tangent bundle of M), and let T_t be the subsheaf of Λ_t induced from $(T_M)_t$ by the map $\varpi_t : V_t \to t$. Then T_t is isomorphic to the trivial sheaf $V_t \times (T_M)_t$. Define $\Pi_t = j_t^{-1}(T_t)$; then

$$0 \longrightarrow \Theta_t \longrightarrow \Pi_t \longrightarrow T_t \longrightarrow 0 . \tag{4.2$_t$}$$

We have the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_t & \longrightarrow & I_t & \longrightarrow & N_t & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta|V_t & \longrightarrow & \Pi|V_t & \longrightarrow & T|V_t & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta_t & \longrightarrow & \Pi_t & \longrightarrow & T_t & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where K_t is the subsheaf of the restriction $\Theta \mid V_t$ of Θ to the fibre V_t which is composed of those germs which vanish on V_t and where I_t, N_t have similar meanings relative to $\Pi \mid V_t$, $T \mid V_t$ respectively. As in Section 1, we denote by r_t the map defined by the commutative diagram

$$\begin{tikzcd} \Theta \arrow[rr, "r_t"] \arrow[dr] & & \Theta_t \\ & \Theta \mid V_t \arrow[ur] & \end{tikzcd}$$

and we call r_t the *restriction map* to the fibre V_t. Similarly we define the restriction maps $\Pi \to \Pi_t$, $T \to T_t$, and we denote them by the same symbol r_t. We have the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & \Pi & \longrightarrow & T & \longrightarrow & 0 \\ & & r_t\downarrow & & r_t\downarrow & & r_t\downarrow & & \\ 0 & \longrightarrow & \Theta_t & \longrightarrow & \Pi_t & \longrightarrow & T_t & \longrightarrow & 0 \end{array}$$

Suppose that $\mathscr{B} \to \mathscr{V}$ is a family of complex fibre bundles over M, $\mathscr{P} \to \mathscr{V}$ the associated family of complex principal bundles. Similar considerations then apply to all the sheaves of the fundamental sheaf diagram $(4.2)_P$. Namely, denote by L_t, M_t, R_t the restrictions to V_t of the bundles $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{R}$ respectively, and let Ξ_t, Φ_t be respectively the sheaves of germs of holomorphic sections of L_t, R_t; then

$$(4.3)_t \qquad 0 \longrightarrow \Xi_t \longrightarrow \Phi_t \xrightarrow{k_t} \Psi_t \longrightarrow 0 .$$

Since Π_t is a subsheaf of Ψ_t, we may define $\Gamma_t = k_t^{-1}(\Pi_t) \subset \Phi_t$ and we obtain

$$(4.4)_t \qquad 0 \longrightarrow \Xi_t \longrightarrow \Gamma_t \longrightarrow \Pi_t \longrightarrow 0 .$$

Denoting by Σ_t the sheaf of germs of holomorphic sections of M_t, we obtain the following exact commutative diagram:

$$(4.2)_{P,t} \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Theta_t & \longrightarrow & \Pi_t & \longrightarrow & T_t & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma_t & \longrightarrow & \Gamma_t & \longrightarrow & T_t & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Xi_t & \longrightarrow & \Xi_t & \longrightarrow & 0 & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

We call $(4.2)_{P,t}$ the fundamental sheaf diagram for the fibre $P_t \to V_t$ of the family $\mathscr{P} \to M$.

We remark that the sequence

$$(4.5)_t \qquad 0 \longrightarrow \Xi_t \longrightarrow \Sigma_t \longrightarrow \Theta_t \longrightarrow 0$$

is the sequence (4.4) for the trivial family $V_t \xrightarrow{\varpi_t} t$ in which the base space consists of a single point.

CHAPTER II. GENERAL THEORY OF DEFORMATIONS OF COMPLEX STRUCTURES

5. Local structure of a differentiable family; map ρ

Given a differentiable family $\mathscr{V} \xrightarrow{\varpi} M$ of complex structures, it will be convenient to introduce for each sheaf $\mathscr{S}$ over $\mathscr{V}$ the corresponding sheaf of cohomology $\mathscr{H}^q(\mathscr{S})$ over M which is defined as follows: we assign to each open set $U \subset M$ the cohomology module $H^q(\varpi^{-1}(U), \mathscr{S})$, and $\mathscr{H}^q(\mathscr{S})$ is the sheaf over M induced from the presheaf defined by the assignment $U \to H^q(\varpi^{-1}(U), \mathscr{S})$.

We consider the fundamental sequence

$$0 \longrightarrow \Theta \longrightarrow \Pi \xrightarrow{j} T \longrightarrow 0$$

for the differentiable family $\mathscr{V} \xrightarrow{\varpi} M$ (see (4.2)) and the corresponding exact cohomology sequence

$$(5.1) \qquad \begin{aligned} 0 \longrightarrow H^0(\mathscr{V}|U, \Theta) \longrightarrow H^0(\mathscr{V}|U, \Pi) \xrightarrow{j} \\ H^0(\mathscr{V}|U, T) \xrightarrow{\delta^*} H^1(\mathscr{V}|U, \Theta) \longrightarrow \cdots, \end{aligned}$$

where $\mathscr{V}|U = \varpi^{-1}(U)$, U being a sufficiently small neighborhood of a point on M. The map $\varpi: \mathscr{V} \to M$ induces an isomorphism:

$$H^0(\mathscr{V}|U, T) \cong T_M(U),$$

where $T_M(U) = H^0(U, T_M)$ is the space of differentiable vector fields on U. In view of this we replace $H^0(\mathscr{V}|U, T)$ by $T_M(U)$ and consider δ^* in (5.1) as a homomorphism $T_M(U) \to H^1(\mathscr{V}|U, \Theta)$ which we denote by ρ_U:

$$(5.2) \qquad \rho_U : T_M(U) \longrightarrow H^1(\mathscr{V}|U, \Theta).$$

This map ρ_U, of fundamental importance, is described in terms of local coordinates as follows: take a locally finite covering $\{\mathscr{U}_i\}$ of $\mathscr{V}|U$ by neighborhoods $\mathscr{U}_i$ with associated homeomorphisms $h_i: \mathscr{U}_i \to \boldsymbol{C}^n \times \varpi(\mathscr{U}_i)$ as described in Definition 1.1 and let

$$h_i(p) = (z_i^1(p), \cdots, z_i^\alpha(p), \cdots, z_i^n(p), t), \ t = \varpi(p), \qquad \text{for } p \in \mathscr{U}_i,$$

$$z_i^\alpha(p) = h_{ik}^\alpha(z_k(p), t), \quad g_{ik\beta}^{\ \alpha}(p) = \frac{\partial h_{ik}^\alpha(z_k(p), t)}{\partial z_k^\beta(p)}, \qquad \text{for } p \in \mathscr{U}_i \cap \mathscr{U}_k,$$

as in Section 1. Moreover let $(t^1, \cdots, t^\mu, \cdots, t^m)$ be a system of local coordinates which covers U and let

$$\theta_{ik\mu}^{\ \alpha}(p) = \frac{\partial h_{ik}^\alpha(z_k(p), t)}{\partial t^\mu}.$$

Let $\pi: p \to \pi(p)$, be a section of Π over an open subset $\mathscr{U}$ of $\mathscr{V}$. In terms of the system of local coordinates $(z_i^1, \cdots, z_i^\alpha, \cdots, t^1, \cdots t^\mu, \cdots)$ on $\mathscr{U}_i$, $\pi(p)$ is written in the form

$$\pi(p) = (\theta_i^1(p), \cdots, \theta_i^\alpha(p), \cdots, v_i^1(t), \cdots, v_i^\mu(t), \cdots), \quad \text{for } p \in \mathscr{U} \cap \mathscr{U}_i,$$

where $t = \varpi(p)$, $\theta_i^\alpha(p) = \theta_i^\alpha(z_i, t)$ are differentiable functions which are holomorphic in $z_i = z_i(p)$ and $v_i^\mu(t)$ are differentiable functions in t. The law of transition of the fibre coordinates is given by

$$(5.3) \qquad \begin{cases} \theta_i^\alpha(p) = \sum_\beta g_{ik\beta}^{\ \alpha}(p)\theta_k^\beta(p) + \sum_\mu \theta_{ik\mu}^{\ \alpha}(p)v_k^\mu(t), \\ v_i^\mu(t) = v_k^\mu(t), \end{cases}$$

where $t = \varpi(p)$, $p \in \mathscr{U} \cap \mathscr{U}_i \cap \mathscr{U}_k$. π is a section of the subsheaf $\Theta \subset \Pi$ if and only if the "horizontal components" $v_i^\mu(t)$ of $\pi(p)$ vanish identically, and the homomorphism j is given by

$$j: \pi(p) \longrightarrow v(t) = (v_i^1(t), \cdots, v_i^\mu(t), \cdots, v_i^m(t)).$$

Now let $v: t \to v(t) = (v^1(t), \cdots, v^\mu(t), \cdots)$ be a vector field $\in T_M(U) \cong H^0(\mathscr{V}|U, T)$. Then, letting

$$\pi_i: p \longrightarrow \pi_i(p) = (0, \cdots, 0, v^1(t), \cdots, v^\mu(t), \cdots), \ t = \varpi(p), \ p \in \mathscr{U}_i,$$

in terms of the system of local coordinates $(\cdots, z_i^\alpha, \cdots, t^\mu, \cdots)$ on $\mathscr{U}_i$, we obtain a section π_i of Π over $\mathscr{U}_i$ for each i. Clearly we have $j\,\pi_i = v$. Therefore the differences $\theta_{ik} = \pi_k - \pi_i$ are sections of Θ over $\mathscr{U}_i \cap \mathscr{U}_k$, respectively, and $\{\theta_{ik}\}$ forms a 1-cocycle on the nerve of the covering $\mathfrak{U} = \{\mathscr{U}_i\}$ which represents an element $\theta_\mathfrak{U}$ of $H^1(\mathfrak{U}, \Theta)$. Letting $P_\mathfrak{U}: H^1(\mathfrak{U}, \Theta) \to H^1(\mathscr{V}|U, \Theta)$ be the canonical homomorphism, we have, by the definition of δ^*,

$$\delta^*(v) = P_\mathfrak{U}(\theta_\mathfrak{U}).$$

By (5.3), π_k may be written in terms of the system of local coordinates $(\cdots, z_i^\alpha, \cdots, t^\mu, \cdots)$ on $\mathscr{U}_i$ in the form

$$\pi_k(p) = (\cdots, \textstyle\sum_\mu \theta_{ik\mu}^{\ \alpha}(p)v^\mu(t), \cdots, v^1(t), \cdots, v^m(t)).$$

Hence the components $\theta_{ik}^\alpha(p)$ of $\theta_{ik}(p) = \pi_k(p) - \pi_i(p)$ with reference to the system of local coordinates $(z_i^1, \cdots, z_i^\alpha, \cdots, z_i^n)$ on $\mathscr{U}_i$ along the fibres of $\mathscr{V}$ are given by

$$\theta_{ik}^{\alpha}(p) = \sum_{\mu} \theta_{ik\mu}^{\alpha}(p)v^{\mu}(t) = \sum_{\mu} v^{\mu}(t)\frac{\partial h_{ik}^{\alpha}(z_k(p), t)}{\partial t^{\mu}}$$

or

$$\theta_{ik}^{\alpha}(p) = v_t \cdot h_{ik}^{\alpha}(z_k(p), t) , \qquad t = \varpi(p) ,$$

where $v_t\cdot$ denotes the derivation

$$\sum_{\mu} v^{\mu}(t)\frac{\partial}{\partial t^{\mu}} .$$

Thus, *for any vector field* $v : t \to v(t)$ *of* $T_M(U)$, $\rho_U(v) \in H^1(\mathscr{V}|U, \Theta)$ *is the cohomology class of the* 1*-cocycle* $\{\theta_{ik}\}$ *composed of the sections* $\theta_{ik} : p \to \theta_{ik}(p)$ *of* Θ *over* $\mathscr{U}_i \cap \mathscr{U}_k$ *defined by*

(5.4) $\quad \theta_{ik}(p) = (\theta_{ik}^{1}(p), \cdots , \theta_{ik}^{\alpha}(p), \cdots), \quad \theta_{ik}^{\alpha}(p) = v_t \cdot h_{ik}^{\alpha}(z_k(p), t)$.

By passage to the limit we obtain from (5.1) the exact sequence

(5.5) $\quad 0 \longrightarrow \mathscr{H}^0(\Theta) \longrightarrow \mathscr{H}^0(\Pi) \longrightarrow \mathscr{H}^0(T) \xrightarrow{\delta^*} \mathscr{H}^1(\Theta) \longrightarrow \mathscr{H}^1(\Pi) \longrightarrow \cdots$

Since $\mathscr{H}^0(T) \cong T_M$, where T_M is the sheaf of germs of differentiable vector fields on M, we replace $\mathscr{H}^0(T)$ by T_M and denote δ^* in (5.5) by ρ :

(5.6) $\quad \rho : T_M \longrightarrow \mathscr{H}^1(\Theta)$.

Clearly ρ is obtained from the collections of the homomorphisms ρ_U assigned for all small neighborhoods U on M by passage to the limit. More precisely, given a germ $\tilde{v}$ of a differentiable vector field at a point $o \in M$ represented by a differentiable vector field v on a neighborhood U_i of o, $\rho(\tilde{v}) \in \mathscr{H}^1(\Theta)_o$ is defined as the limit: $\rho(v) = \lim_U \rho_U(v)$ for all U, $o \in U \subset U_i$. Clearly ρ_U is linear over the ring $\mathfrak{r}(U)$ of differentiable functions on U and therefore ρ is linear over the sheaf of differentiable functions on M. Since $T_M(U)$ is a finite $\mathfrak{r}(U)$-module, we infer that, *if* ρ *vanishes*, ρ_U *vanishes for a sufficiently small neighborhood* U of each point on M.

Suppose that QT_M is a subsheaf of T_M composed of germs of sections of a sub-bundle of the tangent bundle of M. Then we denote by ρ_Q the restriction of ρ to QT_M :

$(5.6)_Q \quad \rho_Q : QT_M \longrightarrow \mathscr{H}^1(\Theta)$.

In order to simplify our later work, we describe here the exponential map of vector fields.

Given a differentiable vector field π ; $p \to \pi(p)$ on a differentiable manifold $\mathscr{U}$ and a subdomain $\mathscr{U}_1$ of $\mathscr{U}$ whose closure $\overline{\mathscr{U}_1}$ is compact and contained in $\mathscr{U}$, then π determines a unique one-parameter family $\{g(s)| -\varepsilon < s < +\varepsilon\}$ of differentiable homeomorphisms $g(s)$ of $\mathscr{U}_1$ into $\mathscr{U}$ such that, for each $p \in \mathscr{U}_1$, the tangent of the arc $s \to g(p, s)$

$= g(s)(p)$ at $g(p, \mathrm{s})$ coincides with $\pi(g(p, s))$. In fact, in terms of the local coordinates $(x^1, \cdots, x^\nu, \cdots)$ on $\mathscr{U}$, $g(s) : x \to g(x, \mathrm{s})$ is determined by the simultaneous differential equations

$$\frac{d}{ds} g^\nu(x, s) = \pi^\nu(g(x, s)) \ , \quad g^\nu(x, 0) = x^\nu \ .$$

For simplicity we write $g(s) = \exp(s\pi)$.

Now we apply these considerations to $\pi \in H^0(\mathscr{U}, \Pi)$ where $\mathscr{U}$ is a subdomain of $\mathscr{V}$ such that $\mathscr{U} \cap V_t$ is connected for each t. As was mentioned above, in terms of the local coordinates (z^α, t), π is written in the form

$$\pi(z, t) = (\theta^1(z, t), \cdots, \theta^\alpha(z, t), \cdots, v^1(t), \cdots, v^\mu(t), \cdots) \ ,$$

and

$$g(z, t, s) = (g^1(z, t, s), \cdots, g^\alpha(z, t, s), \cdots, f^1(z, t, s), \cdots, f^\mu(z, t, s), \cdots)$$

is determined by

$$\begin{cases} \dfrac{d}{ds} g^\alpha(z, t, s) = \theta^\alpha(g^1(z, t, s), \cdots, g^\beta(z, t, s), \cdots, f^\mu(z, t, s), \cdots) \\ \dfrac{d}{ds} f^\mu(z, t, s) = v^\mu(f^1(z, t, s), \cdots, f^\nu(z, t, s), \cdots) \end{cases}$$

with the initial conditions

$$g^\alpha(z, \mathrm{t}, 0) = z^\alpha, \quad f^\mu(z, t, 0) = t^\mu$$

where we write f^μ for the "horizontal" components of g. It follows that the $f^\mu(z, t, s)$ are independent of $z : f^\mu(z, t, \mathrm{s}) = f^\mu(t, s)$. Thus

$$f(s) : t \longrightarrow f(t, s)$$

defines a differentiable homeomorphism of $\varpi(\mathscr{U}_1)$ into $\varpi(\mathscr{U})$. Obviously we have

$$f(s) = \exp(sv)$$

where $v \in T_M(U)$, $U = \varpi(\mathscr{U})$, is the canonical image of π under the homomorphism $j^* : H^0(\mathscr{U}, \Pi) \to H^0(\mathscr{U}, T) \cong T_M(U)$. We have :

$$g(z, t, s) = (g^1(z, t, s), \cdots, g^\alpha(z, t, s), \cdots, f(t, s)) \ ,$$

$$\frac{d}{ds} g^\alpha(z, t, s) = \theta^\alpha(\cdots, g^\beta(z, t, s), \cdots, f(t, s)) \ .$$

This shows that the $g^\alpha(z, t, s)$ are holomorphic in z. Thus we obtain the following result :

PROPOSITION 5.1. *If $\pi \in H^0(\mathscr{U}, \Pi)$, then $g(s) = \exp(s\pi)$ is a differentiable homeomorphism of $\mathscr{U}_1$ into $\mathscr{U}$ which maps $V_t \cap \mathscr{U}_1$ biregularly into $V_{f(t,s)} \cap \mathscr{U}$, in the complex analytic sense, where*

$$f(t, s) = \exp(sv)(t)\,, \qquad\qquad v = j^*(\pi).$$

Moreover $(p, s) \to g(s)(p)$ *is a differentiable map of* $\mathscr{U}_1 \times (-\varepsilon, \varepsilon)$ *into* $\mathscr{U}$.

Now we consider a neighborhood U of a point on M which is of the form $U = U' \times U''$ where U' is covered by coordinates $t^1, \cdots, t^p$, U'' by coordinates $u^1, \cdots, u^q$, whose center is $0 \in U''$, $p + q = \dim M$, and corresponding to this we denote by QT_U the subsheaf of T_U of germs of vector fields tangent to the sheets defined by $t^1 = \text{constant}, \cdots, t^p = \text{constant}$. Thus the map $\rho_Q : QT_U \to \mathscr{H}^1(\Theta)|U$ is defined.

DEFINITION 5.1. We say that the family $\mathscr{V} \to M$ is locally q-trivial if and only if each point of M has a neighborhood $U = U' \times U''$ for which there is a differentiable map $h : \varpi^{-1}(U) \to \varpi^{-1}(U' \times 0)$ which maps each fibre $V(t, u) = \varpi^{-1}(t, u)$, $(t, u) \in U' \times U''$, biregularly onto $V(t, 0) = \varpi^{-1}(t, 0)$ in the complex analytic sense.

A differentiable family is thus locally q-trivial if and only if each point of M has a neighborhood $U = U' \times U''$ such that $\varpi^{-1}(U)$ is equivalent, in the sense of Definition 1.2, to $\varpi^{-1}(U' \times 0) \times U''$. In particular, if $p = 0$, $q = \dim M$, then $h : \varpi^{-1}(U) \to \varpi^{-1}(0) = V_0$ and $\varpi^{-1}(U)$ is trivial in the sense of Definition 1.3. If $q = \dim M$, we say simply that *the family is locally trivial.*

THEOREM 5.1. *A differentiable family* $\mathscr{V} \to M$ *of compact complex manifolds is locally trivial if and only if the homomorphism*

$$\rho : T_M \to \mathscr{H}^1(\Theta)$$

vanishes.

THEOREM 5.1_Q. *A differentiable family* $\mathscr{V} \xrightarrow{\varpi} M$ *of compact complex manifolds is locally q-trivial if and only if each point of M has a neighborhood of the form* $U = U' \times U''$, $\dim U'' = q$, *such that the corresponding map* $\rho_Q : QT_U \to \mathscr{H}^1(\Theta)|U$ *is the* 0*-homomorphism.*

PROOF. Theorem 5.1 is reduced to a special case of Theorem 5.1_Q. We prove Theorem 5.1_Q by induction on q. If $q = 0$ there is nothing to prove, so assume that the theorem is true in case $\dim U'' = q - 1$. If, for $U = U' \times U''$, $\dim U'' = q$, there is a differentiable map $h : \varpi^{-1}(U) \to \varpi^{-1}(U' \times 0)$ which maps $V(t, u) = \varpi^{-1}(t, u)$ biregularly onto $V(t, 0)$, then ρ_Q is the 0-homomorphism. In fact $\varpi^{-1}(U)$ is equivalent to $\varpi^{-1}(U' \times 0) \times U''$, and therefore every germ $\tilde{v} \in QT_U$ can be lifted up to a germ $\tilde{\pi} \in \Pi$, and, if we denote the element of $\mathscr{H}^0(T)|U \cong T_U$ corresponding to $\tilde{v} \in QT_U \subset T_U$ by the same symbol $\tilde{v}$, we have $j^*(\tilde{\pi}) = \tilde{v}$. It follows that $\rho_Q(\tilde{v}) = \delta^* j^*(\tilde{\pi}) = 0$. Assume conversely that ρ_Q is the 0-homomorphism for $U = U' \times U''$, dim $U'' = q$, and let a be a positive number such that

$$U'_a = \{t | -a < t^\mu < a\} \subset U', \quad U''_a = \{u | -a < u^\nu < a\} \subset U'' .$$

From the exact sequene (5.5) we conclude that the vector field over $U = U' \times U''$ which is tangent to the u^q-lines, that is the field $\partial/\partial u^q$, is the image $j(\pi)$ of an element $\pi \in H^0(\varpi^{-1}(U), \Pi)$, provided that U is chosen sufficiently small. Let $h(p, s) = \exp(-s\pi)$ and write $\varpi^q(p) = u^q$, $p \in U$. Moreover we write

$$U^0 = U'_a \times \{(u^1, \cdots, u^{q-1}, 0) | -a < u^\nu < a\} .$$

We infer from Proposition 5.1 that $(p, s) \to h(p, s)$ is a differentiable map of $\varpi^{-1}(U_a) \times (-2a, 2a)$ into $\varpi^{-1}(U)$ which maps $V(t, u^1, \cdots, u^{q-1}, u^q)$ biregularly onto $V(t, u^1, \cdots, u^{q-1}, u^q - s)$ in the complex analytic sense, provided that a is sufficiently small. Hence, letting $f(p) = h(p, \varpi^q(p))$, we obtain a differentiable map $f\colon \varpi^{-1}(U_a) \to \varpi^{-1}(U^0)$ which induces a biregular map of $V(t, u^1, \cdots, u^{q-1}, u^q)$ onto $V(t, u^1, \cdots, u^{q-1}, 0)$ in the complex analytic sense. By the inductive hypothesis there exists a differentiable map $f_{q-1}\colon \varpi^{-1}(U^0) \to \varpi^{-1}(U'_a \times 0)$ which maps each fibre $V(t, u^1, \cdots, u^{q-1}, 0)$ biregularly onto $V(t, 0, \cdots, 0) = V(t, 0)$. The composite map $f_{q-1} \circ f$ is a differentiable map of $\varpi^{-1}(U_a)$ onto $\varpi^{-1}(U'_a \times 0)$ which induces a biregular map of $V(t, u^1, \cdots, u^q) = V(t, u)$ onto $V(t, 0)$ for each $(t, u) \in U_a$, q. e. d.

If $\mathscr{V} \to M$ is a complex analytic family of complex structures over M (which is then necessarily complex analytic), the bundles $\mathfrak{F}$ and $\mathfrak{G}$ have natural complex-analytic structures in the full sense and, in this case, we may suppose that the sheaves are sheaves of germs of holomorphic sections of the various bundles and therefore sheaves of modules over the local rings of holomorphic functions. The proof of Theorem 5.1 then shows that the theorem remains valid if local q-triviality is interpreted in the complex analytic sense, that is to say with the word "differentiable" in Definition 5.1 replaced by "holomorphic".

6. Deformation spaces; regular families

Let $\mathscr{V} \to M$ be a differentiable family of complex structures with compact fibres, and consider the fibre $V_t = \varpi^{-1}(t)$ over a particular point $t \in M$. The exact cohomology sequence corresponding to the exact sequence $(4.2)_t$ of sheaves is

$$(6.1) \quad 0 \to H^0(V_t, \Theta_t) \to H^0(V_t, \Pi_t) \to H^0(V_t, T_t) \xrightarrow{\delta^*} H^1(V_t, \Theta_t) \to \cdots .$$

Here the homomorphism $\delta^*: H^0(V_t, T_t) \to H^1(V_t, \Theta_t)$ is of particular importance. In view of the canonical isomorphism $H^0(V_t, T_t) \cong (T_M)_t$, we replace $H^0(V_t, T_t)$ by $(T_M)_t$ and consider δ^* as a homomorphism

$$(T_M)_t \longrightarrow H^1(V_t, \Theta_t)$$

which we denote by ρ_t:

(6.2) $$\rho_t : (T_M)_t \longrightarrow H^1(V_t, \Theta_t) .$$

This homomorphism was first introduced, for the case of one-parameter differentiable families, by Frölicher and Nijenhuis [16] using a differential-geometric method. We have the commutative diagram:

$$\begin{tikzcd} T_M \arrow[r, "\rho"] \arrow[d] & \mathscr{H}^1(\Theta) \arrow[d] \\ (T_M)_t \arrow[r, "\rho_t"] & H^1(V_t, \Theta_t) \end{tikzcd}$$

For any tangent vector $v_t \in (T_M)_t$ we call $\rho_t(v_t) \in H^1(V_t, \Theta_t)$ the *infinitesimal deformation of* V_t *along* v_t. We denote by $(\rho_Q)_t$ the restriction of ρ_t to $(QT_M)_t$.

It will sometimes be convenient to bear in mind the following remark, namely: let $\mathscr{V}' \to M'$ be the family of complex structures over M' induced from $\mathscr{V} \to M$ by a differentiable map $f: M' \to M$ and let $t = f(t')$; then the triangle

$$\begin{tikzcd} (T_{M'})_{t'} \arrow[rr, "f_*"] \arrow[dr, "\rho'_{t'}"'] & & (T_M)_t \arrow[dl, "\rho_t"] \\ & H^1(V_t, \Theta_t) & \end{tikzcd}$$

is commutative.

DEFINITION 6.1. The differentiable family $\mathscr{V} \to M$ will be called regular if and only if $\dim H^1(V_t, \Theta_t)$ is the same for all points $t \in M$.

Assume that $\mathscr{V} \xrightarrow{\varpi} M$ is a regular family. By Propositions 2.5, 2.6, 2.7, $\mathfrak{H} = \cup_{t \in M} H^1(V_t, \Theta_t)$ forms a differentiable complex vector bundle over M and $H^1(\varpi^{-1}(U), \Theta)$ is isomorphic to the space of differentiable sections of $\mathfrak{H}|U = \cup_{t \in U} H^1(V_t, \Theta_t)$. Moreover we have the commutative diagram:

$$\begin{tikzcd}[column sep=small] H^1(\varpi^{-1}(U), \Theta) \arrow[dr, "r_t"'] & \cong & \mathfrak{H}|U \arrow[dl, "r_t"] \\ & H^1(V_t, \Theta_t) & \end{tikzcd}$$

Hence we obtain the following

THEOREM 6.1. *In case the family* $\mathscr{V} \to M$ *is regular,*

$$\mathfrak{H} = \cup_{t \in M} H^1(V_t, \Theta_t)$$

forms a differentiable complex vector bundle over M, *and* $\mathscr{H}^1(\Theta)$ *is isomorphic to the sheaf* $D(\mathfrak{H})$ *over* M *of germs of differentiable sections of* $\mathfrak{H}$; *moreover the diagram*

$$\begin{tikzcd} T_M \arrow[r, "\rho"] \arrow[d, "r_t"] & \mathscr{H}^1(\Theta) \cong D(\mathfrak{H}) \arrow[d, "r_t"] \\ (T_M)_t \arrow[r, "\rho_t"] & H^1(V_t, \Theta_t) \end{tikzcd}$$

is commutative.

Let $U = U' \times U''$, QT_U and ρ_Q have the same meanings as in Section 5. We infer from Theorem 6.1 that $\rho_Q = 0$ if and only if $(\rho_Q)_t = 0$ for all $t \in U$. Hence, by Theorems 5.1, 5.1_Q, we obtain

THEOREM 6.2. *A regular differentiable family* $\mathscr{V} \to M$ *is locally trivial if and only if* $\rho_t = 0$ *for all points* $t \in M$.

THEOREM 6.2_Q. *A regular differentiable family* $\mathscr{V} \to M$ *is locally q-trivial if and only if each point of* M *has a neighborhood of the form* $U = U' \times U''$, dim $U'' = q$, *such that* $(\rho_Q)_t = 0$ *for all* $t \in U$.

We remark that the hypothesis of regularity in Theorems 6.2, 6.2_Q is essential. In fact there is an example of a differentiable family $\mathscr{V} \to M$ of complex structures for which $\rho_t = 0$ for all $t \in M$ but which is not locally trivial (see Section 15).

THEOREM 6.3 (Frölicher and Nijenhuis). *Let* $\mathscr{V} \xrightarrow{\varpi} M$ *be a differentiable family of complex structures with compact fibres. If* $H^1(V_o, \Theta_o) = 0$ *for* $V_o = \varpi^{-1}(o)$, $o \in M$, *there exists a neighborhood* U *of* o *on* M *such that* $\varpi^{-1}(U)$ *is trivial.*

PROOF. By Theorem 2.1 (upper semi-continuity), $H^1(V_o, \Theta_o) = 0$ implies that $H^1(V_t, \Theta_t) = 0$ for $t \in U$, provided that U is sufficiently small. Hence, by the above result, $\varpi^{-1}(U)$ is trivial, q. e. d.

Theorem 6.3 shows that, if $H^1(V_o, \Theta_o) = 0$, the complex structure of V_o is *rigid* in the sense that it cannot be changed by any small deformation. For the case of one-parameter differentiable families, Frölicher and Nijenhuis [16] proved this theorem by using a differential-geometric method.

DEFINITION 6.2. The image $\rho_t((T_M)_t) \subset H^1(V_t, \Theta_t)$ will be called the space of infinitesimal deformations determined by the imbedding of V_t as a fibre in the differentiable family of complex structures.

Given a complex analytic manifold V_o, denote by Θ_o the sheaf of germs of holomorphic sections of its tangent bundle. Each imbedding of V_o as fibre in a differentiable family of complex structures defines a subspace of $H^1(V_o, \Theta_o)$, namely the space of infinitesimal deformations for that imbedding. A space of infinitesimal deformations in $H^1(V_o, \Theta_o)$ determined in this way will be called *maximal* if it is not a proper subspace of a space of infinitesimal deformations determined by another imbedding of V_o as fibre in a differentiable family.

DEFINITION 6.3. A subspace of $H^1(V_o, \Theta_o)$ will be called a deformation space if it is a maximal space of infinitesimal deformations.

DEFINITION 6.4. Given a differentiable family $\mathscr{V}$ of complex structures over M, we say that the family $\mathscr{V}$ is effectively parametrized at $t \in M$ if and only if the map $\rho_t : (T_M)_t \to H^1(V_t, \Theta_t)$ is injective. If $\mathscr{V}$ is effectively parametrized at every point $t \in M$, we say that the family $\mathscr{V}$ is effectively parametrized.

The following proposition follows almost immediately from the definitions :

PROPOSITION 6.1. *Any deformation space in $H^1(V_o, \Theta_o)$ is the space of infinitesimal deformations determined by the imbedding of V_o as fibre over the point $o \in M$ in a differentiable family which is effectively parametrized at o.*

At this point it is perhaps useful to compare some of the concepts introduced in the above definitions with somewhat similar notions in classical algebraic geometry. We remark, first of all, that the word "complete" in algebraic geometry usually signifies "maximal" (in some sense). Analogues in classical geometry of (1) differentiable family of complex structures, (2) space of infinitesimal deformations, (3) complete family of complex structures, and (4) deformation space, are respectively (1′) continuous system, (2′) characteristic linear system, (3′) complete continuous system (= maximal continuous system), and (4′) characteristic linear system of a complete continuous system. Finally the analogue of the statement $\rho_t((T_M)_t) = H^1(V_t, \Theta_t)$ is "the characteristic linear system of a complete continuous system is complete'' (compare Zariski [41]). Later we shall see that there are more substantial relations between these two sets of concepts (see Sections 12, 13, 14).

As was pointed out in Section 4, the multiplication in a sheaf arising from the Poisson bracket of vector fields induces in the cohomology a structure of graded Lie algebra. In particular,

$$H^*(V_o, \Theta_o) = \sum_{q=0}^{n} H^q(V_o, \Theta_o)$$

is a graded Lie algebra. We say that a linear subspace S of $H^*(V_o, \Theta_o)$ is an abelian Lie algebra if $[S, S] = 0$, that is if the bracket of any two elements of S vanishes. We have :

PROPOSITION 6.2. *Any deformation space in $H^1(V_o, \Theta_o)$ is an abelian Lie algebra.*

PROOF. Let $\mathfrak{U}_o$ be a sufficiently fine finite open covering of V_o, and let $\{\theta_{ik}\}$, $\{\varphi_{ik}\}$ be two 1-cocycles on the nerve of $\mathfrak{U}_o$ with coefficients in Θ_o. Then

$$[\theta, \varphi]_{ijk} = \frac{1}{2} \{[\theta_{ij}, \varphi_{jk}] + [\varphi_{ij}, \theta_{jk}]\} \tag{6.3}$$

is a 2-cocycle on the nerve of $\mathfrak{U}_o$ with coefficients in Θ_o, and, if $\{\theta_{ik}\}$, $\{\varphi_{ik}\}$ represent classes $\theta, \varphi \in H^1(V_o, \Theta_o)$, then $[\theta, \varphi]_{ijk}$ represents a class $[\theta, \varphi] \in H^2(V_o, \Theta_o)$ which, by definition, is the bracket of θ, φ in the graded Lie algebra.

Now suppose that $D \subset H^1(V_o, \Theta_o)$ is a deformation space. Then $D = \rho_o((T_M)_o)$ for some imbedding of V_o as fibre over the point $o \in M$ in a differentiable family $\mathscr{V} \to M$. Let $v_o \in (T_M)_o \cong H^0(V_o, T_o)$. Then $v_o = r_o(\tilde{v})$ where $\tilde{v}$ is a germ of T_M at the point $o \in M$. Let $\tilde{v}$ be represented by the vector field v over a neighborhood U of o, and let $\mathfrak{U} = \{\mathscr{U}_i\}$ be a locally finite open covering of $\varpi^{-1}(U)$. If the covering is chosen sufficientiy fine, there exists a section π_i of Π over $\mathscr{U}_i$ such that $j^*(\pi_i) = v \mid \varpi(\mathscr{U}_i)$ where j^* is the map defined by the exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(\varpi^{-1}(U), \Theta) \longrightarrow H^0(\varpi^{-1}(U), \Pi) \xrightarrow{j^*} H^0(U, T_M) \\ \xrightarrow{\delta^*} H^1(\varpi^{-1}(U), \Theta) \longrightarrow \cdots . \end{aligned} \tag{6.4}$$

Let $\theta_{ij} = \pi_j - \pi_i$ in $\mathscr{U}_i \cap \mathscr{U}_j$; by definition of the coboundary map δ^* in (6.4), $\{\theta_{ij}\}$ is a 1-cocycle on the nerve of the covering $\mathfrak{U}$ which represents $\theta_U = \delta^*(v) \in H^1(\varpi^{-1}(U), \Theta)$. Now define $\mu_{ij} = [\pi_i, \pi_j]$. Since $j^*(\mu_{ij}) = j^*([\pi_i, \pi_j]) = [j^*\pi_i, j^*\pi_j] = [v, v] = 0$, we see that $\{\mu_{ij}\}$ is a 1-cochain on the nerve of $\mathfrak{U}$ with coefficients in Θ. We have

$$\begin{aligned}(\delta\mu)_{ijk} &= [\pi_j, \pi_k] - [\pi_i, \pi_k] + [\pi_i, \pi_j] = [\theta_{ij}, \pi_k] + [\pi_i, \pi_j] \\ &= [\theta_{ij}, \pi_k] + [\pi_j, \pi_j] - [\theta_{ij}, \pi_j] = [\theta_{ij}, \theta_{jk}] ,\end{aligned}$$

that is, by (6.3),

$$[\theta_U, \theta_U]_{ijk} = (\delta\mu)_{ijk} .$$

Therefore $[\theta_U, \theta_U]$ is the 0-class of $H^2(\varpi^{-1}(U), \Theta)$ (provided that U is sufficiently small) and it follows that $[\delta^*(\tilde{v}), \delta^*(\tilde{v})]$ is zero in $\mathscr{H}^2(\Theta)$. Finally the multiplication induced by the Poisson bracket in the cohomology with coefficients in Θ obviously commutes with the restriction map r_o, that is the diagram

$$\begin{array}{ccc} \mathscr{H}^*(\Theta) \otimes \mathscr{H}^*(\Theta) & \longrightarrow & \mathscr{H}^*(\Theta) \\ \downarrow{\scriptstyle r_o} & & \downarrow{\scriptstyle r_o} \\ H^*(V_o, \Theta_o) \otimes H^*(V_o, \Theta_o) & \longrightarrow & H^*(V_o, \Theta_o) \end{array}$$

is commutative. Hence, if $\theta_o = \rho_o(v_o)$, then $[\theta_o, \theta_o]$ vanishes in $H^2(V_o, \Theta_o)$.

Now let $\theta, \varphi \in D$. Since

$$[\theta + \varphi, \theta + \varphi] = [\theta, \theta] + 2[\theta, \varphi] + [\varphi, \varphi] ,$$

we infer that $[\theta, \varphi] = 0$ for $\theta, \varphi \in D$, q. e. d.

By Proposition 6.2 a class $\theta_o \in H^1(V_o, \Theta_o)$ such that $[\theta_o, \theta_o] \neq 0$ in $H^2(V_o, \Theta_o)$ cannot belong to any deformation space. (This is proved by Frölicher and Nijenhuis using a differential-geometric method.)

DEFINITION 6.5. A class $\theta_o \in H^1(V_o, \Theta_o)$ for which $[\theta_o, \theta_o] \neq 0$ in $H^2(V_o, \Theta_o)$ will be called *obstructed*.

An example of a V_o for which $H^1(V_o, \Theta_o)$ contains obstructed elements will be given in Section 16.

CHAPTER III. GENERAL THEORY OF DEFORMATIONS OF COMPLEX FIBRE BUNDLES

7. Deformations of complex fibre bundles

Let $\mathscr{V} \xrightarrow{\varpi} M$ be a differentiable family of compact complex structures, and let $\mathscr{B} \to \mathscr{V}$ be a differentiable family of complex fibre bundles over $\mathscr{V}$, $\mathscr{P} \to \mathscr{V}$ the associated family of complex principal bundles. We then have the fundamental sheaf diagram $(4.2)_P$ for the family $\mathscr{P} \to \mathscr{V} \to M$ which gives rise to the following exact commutative diagram of cohomology sheaves over M (compare the beginning of Section 5):

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathscr{H}^1(\Sigma) \arrow[r] \arrow[u] & \mathscr{H}^1(\Gamma) \arrow[r] \arrow[u] & \mathscr{H}^1(T) \arrow[r] \arrow[u] & \mathscr{H}^2(\Sigma) \arrow[r] \arrow[u] & \mathscr{H}^2(\Gamma) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathscr{H}^1(\Xi) \arrow[r] \arrow[u] & \mathscr{H}^1(\Xi) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \mathscr{H}^2(\Xi) \arrow[r] \arrow[u] & \mathscr{H}^2(\Xi) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathscr{H}^0(\Theta) \arrow[r] \arrow[u] & \mathscr{H}^0(\Pi) \arrow[r] \arrow[u, "\sigma"'] & T_M \arrow[r, "\rho"] \arrow[u] & \mathscr{H}^1(\Theta) \arrow[r] \arrow[u] & \mathscr{H}^1(\Pi) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathscr{H}^0(\Sigma) \arrow[r] \arrow[u] & \mathscr{H}^0(\Gamma) \arrow[r] \arrow[u] & T_M \arrow[r, "\eta"] \arrow[u] & \mathscr{H}^1(\Sigma) \arrow[r] \arrow[u] & \mathscr{H}^1(\Gamma) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathscr{H}^0(\Xi) \arrow[r] \arrow[u] & \mathscr{H}^0(\Xi) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \mathscr{H}^1(\Xi) \arrow[r] \arrow[u] & \mathscr{H}^1(\Xi) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Here we have made use of the isomorphism $\mathscr{H}^0(T) \cong T_M$ where T_M is the sheaf of germs of differentiable sections of the tangent bundle of M. We call this diagram the "deformation diagram" for the family of complex bundles. In addition to the map ρ (discussed in Section 5) we have two coboundary maps

$$\eta : T_M \longrightarrow \mathscr{H}^1(\Sigma), \quad \sigma : \mathscr{H}^0(\Pi) \longrightarrow \mathscr{H}^1(\Xi) .$$

The map η is the analogue for bundles of the map ρ for structures. We show that the various results obtained for ρ in Section 5 are also valid for η; this is to be expected since η corresponds to the map ρ for the family $\mathscr{P} \to M$ of complex structures with the additional requirement that the structures of complex fibre bundles be preserved in the deformation.

As in Section 5, we consider a neighborhood on M of the form $U = U' \times U''$, where U' is covered by coordinates $t^1, \cdots, t^p$, U'' by cooordinates $u^1, \cdots, u^q$, and denote by QT_U the subsheaf of T_U of germs of vector fields tangent to the sheets $t \times U''$, $t \in U'$. Corresponding to this we denote by η_Q the coboundary map : $QT_U \to \mathscr{H}^1(\Sigma)|U$.

DEFINITION 7.1. We say that a differentiable family of complex bundles is locally q-trivial if and only if each point of M has a neighborhood $U = U' \times U''$, $\dim U'' = q$, for which there exists a differentiable bundle map h: $\mathscr{B} \,|\, \varpi^{-1}(U) \to \mathscr{B} \,|\, \varpi^{-1}(U' \times 0)$ which is a biregular holomorphic bundle map of each restricted bundle $B(t, u) \to V(t, u)$ onto $B(t, 0) \to V(t, 0)$ where

$$V(t, u) = \varpi^{-1}(t, u),\ B(t, u) = \mathscr{B} \mid V(t, u), \qquad (t, u) \in U' \times U'' .$$

If $q = \dim M$, we say simply that the family of bundles is locally trivial.

THEOREM 7.1. *Let $\mathscr{V} \to M$ be a family of complex structures with compact fibres and let $\mathscr{B} \to \mathscr{V} \to M$ be a differentiable family of complex bundles. The family $\mathscr{B} \to M$ is locally q-trivial if and only if there exists a decomposition $U = U' \times U''$, $\dim U'' = q$, of each sufficiently small neighborhood U on M such that the corresponding η_Q is the* 0*-homomorphism.*

The proof is essentially the same as that of the analogous Theorem 5.1_Q; it will therefore be omitted.

We note that $\mathscr{H}^0(\Gamma)$ corresponds to $\mathscr{H}^0(\Pi)$ in the proof of Theorem 5.1_Q, and, given γ representing a germ in $\mathscr{H}^0(\Gamma)$, the map $\exp(s\gamma)$ corresponding to $\exp(s\pi)$ (in the proof of Theorem 5.1_Q) is defined as follows. We recall that Γ is the sheaf of differentiable sections of the bundle $\mathfrak{R}$ which are holomorphic along the fibres of $\mathscr{V}$ and that $\mathfrak{R} = \mathfrak{E}_P/G$. Thus there is a canonical map $\mathfrak{E}_P \to \mathfrak{R}$. We pick up the G-invariant inverse image β of the section γ and form the map $\exp(s\beta)$ which, however, we denote simply by $\exp(s\gamma)$.

REMARK. If $\mathscr{B} \to \mathscr{V}$ is a complex analytic family of bundles, we may redefine the various bundles and sheaves to have natural complex-analytic structures. In this case Theorem 7.1 remains valid with local q-triviality interpreted in the complex analytic sense, that is with the word "differentiable" which occurs twice in Definition 7.1 replaced by "complex analytic". Compare Section 18.

Assume that $\mathscr{V} = V_o \times M$, where V_o is a compact complex-analytic manifold, in which case $\mathscr{B} \to \mathscr{V}$ is a family of fibre bundles over a fixed complex manifold V_o. The map η (see the deformation diagram) can be lifted to a *unique* map

$$\tau : T_M \longrightarrow \mathscr{H}^1(\Xi) . \tag{7.1}$$

In fact, since $\mathscr{V} = V_o \times M$, the sheaf Π splits : $\Pi \cong \Theta \oplus T$. Let

$$\Gamma' = \alpha^{-1}(T),\ T \subset \Theta \oplus T ,$$

where $\alpha : \Gamma \to \Pi$ is the map in the fundamental sheaf diagram $(4.2)_P$. Using the commutativity of the upper left square of the fundamental sheaf diagram, we extract the following exact commutative diagram :

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & T \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Xi \arrow[r] \arrow[u] & \Gamma' \arrow[r] \arrow[u, "\alpha"'] & T \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Xi \arrow[r] \arrow[u] & \Xi \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Thus η can be lifted back to the map $\tau = \delta^* : T_M \to \mathscr{H}^1(\Xi)$ which arises from the exact sequence

$$0 \longrightarrow \Xi \longrightarrow \Gamma' \longrightarrow T \longrightarrow 0\ ,$$

q. e. d.

We now consider the "infinitesimal-deformation diagram" determined by the imbedding of the fibre in the family, namely the exact commuta-

$$\begin{tikzcd}[column sep=small]
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & H^1(V_o, \Sigma_o) \arrow[r] \arrow[u] & H^1(V_o, \Gamma_o) \arrow[r] \arrow[u] & H^1(V_o, T_o) \arrow[r] \arrow[u] & H^2(V_o, \Sigma_o) \arrow[r] \arrow[u] & H^2(V_o, \Gamma_o) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H^1(V_o, \Xi_o) \arrow[r] \arrow[u] & H^1(V_o, \Xi_o) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & H^2(V_o, \Xi_o) \arrow[r] \arrow[u] & H^2(V_o, \Xi_o) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H^0(V_o, \Theta_o) \arrow[r] \arrow[u] & H^0(V_o, \Pi_o) \arrow[r] \arrow[u, "\sigma_o"'] & (T_M)_o \arrow[r, "\rho_o"] \arrow[u] & H^1(V_o, \Theta_o) \arrow[r] \arrow[u] & H^1(V_o, \Pi_o) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H^0(V_o, \Sigma_o) \arrow[r] \arrow[u] & H^0(V_o, \Gamma_o) \arrow[r] \arrow[u] & (T_M)_o \arrow[r, "\eta_o"] \arrow[u] & H^1(V_o, \Sigma_o) \arrow[r] \arrow[u, "\kappa_o"'] & H^1(V_o, \Gamma_o) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & H^0(V_o, \Xi_o) \arrow[r] \arrow[u] & H^0(V_o, \Xi_o) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & H^1(V_o, \Xi_o) \arrow[r] \arrow[u] & H^1(V_o, \Xi_o) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & {} \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

tive cohomology diagram corresponding to $(4.2)_{P,t}$. For simplicity we denote the point t of M by o: then the infinitesimal-deformation diagram is obtained from the deformation diagram by the restriction r_o^* to the fibre $V_o = \varpi^{-1}(o)$.

DEFINITION 7.2. A differentiable family of complex bundles will be called regular if and only if dim $H^1(V_t, \Sigma_t)$ is the same for all points $t \in M$.

We have:

THEOREM 7.2. *A regular differentiable family of complex bundles is locally q-trivial if and only if each point of M has a neighborhood*

$$U = U' \times U'', \ \dim\ U'' = q\ ,$$

such that $(\eta_Q)_t = 0$ for all $t \in U$ where $(\eta_Q)_t : (QT_M)_t \to H^1(V_t, \Sigma_t)$.

Let U be a neighborhood of the point $o \in M$. We say that $\mathscr{B}\,|\varpi^{-1}(U)$ is trivial (compare Definition 7.1) if and only if there is a differentiable map $h : \mathscr{B} \mid \varpi^{-1}(U) \to B_o = \mathscr{B} \mid \varpi^{-1}(o)$ which, for each $t \in U$, is an isomorphism of $B_t \to V_t$ onto $B_o \to V_o$ in the category of complex-analytic fibre bundles.

THEOREM 7.3. *If $H^1(V_o, \Sigma_o) = 0$, there exists a neighborhood U of the point o in M such that $\mathscr{B}\,|\varpi^{-1}(U)$ is trivial.*

The proofs of Theorems 7.2 and 7.3 are entirely similar to the proofs of the analogous Theorems 6.2_Q and 6.3.

THEOREM 7.4. *Consider a family of complex bundles $\mathscr{B}$ over a fixed complex manifold V_o, i. e. $\mathscr{V} = V_o \times M$, and suppose that $H^1(V_o, \Xi_o) = 0$. Then there is a neighborhood U of o such that $\mathscr{B}\,|\varpi^{-1}(U)$ is trivial.*

PROOF. Since $H^1(V_o, \Xi_o) = 0$, the principle of upper semi-continuity (Theorem 2.1) implies that there is a neighborhood U of o in M such that $H^1(V_t, \Xi_t) = 0$ for all $t \in U$. We then infer from Theorem 2.2, part (ii), that $H^1(\varpi^{-1}(U), \Xi) = 0$. Since $\mathscr{V} = V_o \times M$, the theorem follows from (7.1) and Theorem 7.2.

The following definitions are completely analogous to those of Section 6; the only difference is that the fibre here possesses the additional structure of fibre bundle.

DEFINITION 7.3. The image $\eta_o((T_M)_o) \subset H^1(V_o, \Sigma_o)$ will be called the space of infinitesimal bundle deformations determined by the imbedding of the bundle $B_o \to V_o$ as fibre in the family $\mathscr{B} \to M$ where

$$V_o = \varpi^{-1}(o), \; B_o = \mathscr{B} \,|\, V_o .$$

We call a space of infinitesimal bundle deformations *maximal* if it is not a proper subspace of a space of infinitesimal bundle deformations determined by some other imbedding of $B_o \to V_o$ as fibre in a family.

DEFINITION 7.4 A subspace of $H^1(V_o, \Sigma_o)$ will be called a bundle deformation space if it is a maximal space of infinitesimal bundle deformations.

DEFINITION 7.5. We say that a family of bundles is effectively parametrized at $o \in M$ if and only if the map $\eta_o : (T_M)_o \to H^1(V_o, \Sigma_o)$ is injective. If the family is effectively parametrized at every point of M, we say that the family is effectively parametrized.

As in Section 6 we have :

PROPOSITION 7.1. *Any bundle deformation space in $H^1(V_o, \Sigma_o)$ is the space of infinitesimal bundle deformations determined by the imbedding of $B_o \to V_o$ as fibre over the point $o \in M$ in a differentiable family of bundles which is effectively parametrized at o.*

We recall that the multiplication in a sheaf arising from the Poisson bracket induces in the cohomology a structure of graded Lie algebra. In particular, $H^*(V_o, \Sigma_o)$ and $H^*(V_o, \Xi_o)$ are graded Lie algebras.

PROPOSITION 7.2. *Any bundle deformation space in $H^1(V_o, \Sigma_o)$ is an abelian Lie algebra.*

The proof is the same as that of the analogous Proposition 6.2.

In a similar manner we may define the spaces of infinitesimal bundle deformations in $H^1(V_o, \Xi_o)$ which correspond to deformations leaving the base space fixed. Namely, in this case there is a map $\tau : T_M \to \mathscr{H}^1(\Xi)$ (see (7.1)) and the image $\tau_o((T_M)_o) \subset H^1(V_o, \Xi_o)$ will be called the space of infinitesimal bundle deformations leaving the base space V_o fixed which is determined by the imbedding of the bundle $B_o \to V_o$ as fibre in the family $\mathscr{B} \to \mathscr{V} \to M$ where $\mathscr{V} = V_o \times M$. A subspace of $H^1(V_o, \Xi_o)$ will be called a bundle deformation space corresponding to a fixed base space V_o if it is a maximal space of infinitesimal bundle deformations determined in the above manner. Then the analogues of Propositions 7.1 and 7.2 are true for $H^1(V_o, \Xi_o)$. An element $\xi \in H^1(V_o, \Xi_o)$ which satisfies $[\xi, \xi] \neq 0$ will be called *obstructed*. The existence of obstructed elements will be established by an example in Section 17.

Finally we discuss the map σ. Suppose that there is a non-trivial element π of $H^0(\varpi^{-1}(U), \Pi)$, $U \subset M$, and assume further that $\sigma(\pi) = 0$ in $H^1(\varpi^{-1}(U), \Xi)$. Then π is the image of some element $\gamma \in H^0(\varpi^{-1}(U), \Gamma)$. There are two cases :

CASE 1. The image of π in $H^0(U, T_M)$ is the identically vanishing vector field over U. Then π is the image of an element $\theta \in H^0(\varpi^{-1}(U), \Theta)$ and $g = \exp(s\theta)$ defines an automorphism of $\varpi^{-1}(U)$ which is a biregular complex-analytic map of each fibre onto itself and, for sufficiently small U, there is at least one bundle automorphism $\mathscr{P} \mid \varpi^{-1}(U) \to \mathscr{P} \mid \varpi^{-1}(U)$ which induces $g : \varpi^{-1}(U) \to \varpi^{-1}(U)$.

CASE 2. The image v of π in $H^0(U, T_M)$ does not vanish identically. Let U_o be a neighborhood of o, $\bar{U}_o \subset U$, and let $h = \exp(s\pi)$, $h = h(p, s)$, $p \in \varpi^{-1}(U_o)$, $h(p, o) = p$. For each sufficiently small s, $h : \varpi^{-1}(U_o) \to \varpi^{-1}(U)$ is a homeomorphism and, since $\sigma(\pi) = 0$ in $H^1(\varpi^{-1}(U), \Xi)$, there is at least one bundle homeomorphism h_P: $\mathscr{P} |\varpi^{-1}(U_o) \to \mathscr{P} |\varpi^{-1}(U)$ which induces h.

Thus, in any case, the vanishing of the coboundary map $\sigma : \mathscr{H}^0(\Pi) \to \mathscr{H}^1(\Xi)$ means that germs of homeomorphisms of neighborhoods of fibres of $\mathscr{V}$ are induced by germs of bundle homeomorphisms of the restrictions of $\mathscr{P}$ to these fibre neighborhoods in $\mathscr{V}$ if there exist non-trivial homeomorphisms of neighborhoods of $\mathscr{V}$. In Section 9 we shall see that, in the case where $\mathscr{P}$ is the associated principal bundle of the bundle $\mathfrak{F}$ along the fibres of $\mathscr{V}$, the coboundary map σ vanishes identically.

CHAPTER IV. DIFFERENTIAL GEOMETRY ON A FAMILY OF COMPLEX STRUCTURES

8. Connections of differentiable families

In this section we discuss the notion of "connection" for differentiable families of complex structures. The results of this section will not be used later.

We apply to the sequence (4.2) the functor Hom $(T, \cdots)$, where Hom $= \mathrm{Hom}_O$ is understood in the sense of O-modules. Since this functor is covariant in its second argument, and since the sheaves are locally free, we obtain the exact sequence

$$(8.1) \quad 0 \longrightarrow \mathrm{Hom}\ (T, \Theta) \longrightarrow \mathrm{Hom}\ (T, \Pi) \longrightarrow \mathrm{Hom}\ (T, T) \longrightarrow 0\ .$$

The corresponding exact cohomology sequence is :

$$(8.2) \quad \begin{aligned} 0 \longrightarrow H^0(\mathscr{V}, \mathrm{Hom}\ (T, \Theta)) \longrightarrow H^0(\mathscr{V}, \mathrm{Hom}\ (T, \Pi)) \longrightarrow \\ H^0(\mathscr{V}, \mathrm{Hom}\ (T, T)) \longrightarrow H^1(\mathscr{V}, \mathrm{Hom}\ (T, \Theta)) \longrightarrow \cdots\ . \end{aligned}$$

The sequence (4.2) defines an extension of T by Θ ; an extension is an O-module Π and two O-homomorphisms $i : \Theta \to \Pi$, $j : \Pi \to T$ such that (4.2) is exact. Two extensions Π and Π_1 are said to be equivalent if the diagram

$$\begin{tikzcd}
& \Pi \arrow[dr, "j"] \arrow[dd, "k"'] & \\
\Theta \arrow[ur, "i"] \arrow[dr, "i_1"'] & & T \\
& \Pi_1 \arrow[ur, "j_1"'] &
\end{tikzcd}$$

is commutative and k is an isomorphism. In $H^0(\mathscr{V}, \operatorname{Hom}(T, T))$ there is a canonical element which is the identity map $\iota : T \to T$, and $\delta^*(\iota) \in H^1(\mathscr{V}, \operatorname{Hom}(T, \Theta))$ is called the characteristic class, or obstruction, of the extension (4.2). If this class vanishes, ι is in the kernel of δ^*, therefore in the image of $j^* : H^0(\mathscr{V}, \operatorname{Hom}(T, \Pi)) \to H^0(\mathscr{V}, \operatorname{Hom}(T, T))$ and there exists an O-homomorphism $\tau : T \to \Pi$ such that $j \circ \tau$ is the identity map of T. This is equivalent to the statement that the sequence (4.2) splits ; that is

$$\Pi = \Theta \bigoplus \tau(T) . \tag{8.3}$$

As is immediately verified, the obstruction is the same for two equivalent extensions. Moreover, since the sheaves are locally free, it is well known that the classes of equivalent extensions of T by Θ are in one-one correspondence with the elements of $H^1(\mathscr{V}, \operatorname{Hom}(T, \Theta))$.

Suppose that the characteristic class $\delta^*(\iota)$ vanishes ; then (8.3) defines a "connection" for the differentiable family $\mathscr{V} \to M$ of complex structures. Namely, an element of Π is horizontal if its projection on Θ vanishes, vertical if its projection on $\tau(T)$ vanishes. A connection is equivalent to the existence of a Θ-valued "connection form" $\omega : \Pi \to \Theta$ whose restriction to Θ is the "Maurer-Cartan form" (identity map of Θ). A connection for a differentiable family is therefore analogous to a connection in the sense of classical differential geometry. However, in contrast to the classical one, whose existence follows from the fact that the sheaves are fine, a connection for a differentiable family exists only if the family is locally trivial, since the existence of a connection implies that the map ρ is zero.

9. Associated sequences

In this section we examine the structure of the associated sheaf diagram of a differentiable family $\mathscr{V} \to M$ of complex structures. We recall that the associated sheaf diagram is the fundamental sheaf diagram $(4.2)_P$ for the special case where $\mathscr{P} \to \mathscr{V}$ is the principal bundle of the bundle $\mathfrak{F}$ along the fibres of $\mathscr{V} \to M$. The results of this section will not be used later.

Let P be a principal bundle over X with structure group $G = GL(n, \boldsymbol{C})$, and let P be defined by $\{g_{ik}\} \in H^1(\mathfrak{U}, G)$ where $\mathfrak{U} = \{U_i\}$ is an open covering of X. Then $P|U_i = \{(g_i, x_i)\}$, $g_i = g_{ik}(x)g_k$, where $g_i, g_k \in G$ and where $x_i = (x_i^1, \cdots, x_i^\nu, \cdots)$ are local coordinates on X covering U_i. Denote by $\mathfrak{F}_P$ the bundle along the fibres of P (bundle over P of tangent vectors along the fibres of P), and denote by $\mathfrak{E}_P$ the tangent bundle of P. Then $\mathfrak{F}_P|U_i = \{(g_i, x_i, \lambda_i)\}$ where $\lambda_i \in \mathfrak{g}$, the Lie algebra of G. Since λ_i is left invariant, $(g_i, x_i, \lambda_i) = (g_k, x_k, \lambda_k)$ if and only if $g_i = g_{ik}(x)g_k$, $\lambda_i = \lambda_k$, where x_i, x_k are different coordinates for the same point of X in $U_i \cap U_k$. Furthermore, $\mathfrak{E}_P|U_i = \{(g_i, x_i, \lambda_i, u_i)\}$ where

$$\lambda_i \in \mathfrak{g},\ u_i = \sum u_i^\mu \frac{\partial}{\partial x^\mu}\ ;$$

the rule of transformation in $U_i \cap U_k$ is

$$\begin{cases} g_i = g_{ik}(x)\cdot g_k \\ \lambda_i = \lambda_k + g_k^{-1}\cdot g_{ik}(x)^{-1}\cdot \dfrac{\partial g_{ik}(x)}{\partial x_k^\nu}\cdot g_k \cdot u_k^\nu \\ u_i^\mu = \dfrac{\partial x_i^\mu}{\partial x_k^\nu}\, u_k^\nu\ . \end{cases}$$

For simplicity we use here the classical summation convention. Finally we have the exact sequence of bundles over P

$$(9.1)\qquad 0 \longrightarrow \mathfrak{F}_P \longrightarrow \mathfrak{E}_P \longrightarrow \mathfrak{E}_P/\mathfrak{F}_P \longrightarrow 0$$

where $\mathfrak{E}_P/\mathfrak{F}_P|U_i = \{g_i, x_i, u_i)\}$.

Now G operates by right translation on P and therefore operates by right translation on the sequence (9.1). The right action of $g \in G$ is defined by $(g_i, x_i, \lambda_i)\cdot g = (g_i\cdot g, x_i, \mathrm{Ad}\,(g^{-1})\lambda_i)$, where $\mathrm{Ad}\,(g^{-1})\lambda_i = g^{-1}\cdot\lambda_i\cdot g$. Let $\mathfrak{L} = \mathfrak{F}_P/G$, $\mathfrak{R} = \mathfrak{E}_P/G$. Since $(\mathfrak{E}_P/\mathfrak{F}_P)/G$ is isomorphic to the tangent bundle $\mathfrak{E}$ of X, we obtain the exact sequence of bundles over X (compare [4]):

$$(9.2)\qquad 0 \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{E} \longrightarrow 0\ .$$

As representative modulo G of $(g_i, x_i, \lambda_i, u_i) \in \mathfrak{E}_P|U_i$, take $(1, x_i, \lambda_i, u_i)$ and denote it by (x_i, λ_i, u_i). Then $(x_i, \lambda_i, u_i) = (x_k, \lambda_k, u_k)$ if and only if

$$(9.3)\qquad \begin{cases} \lambda_i = \mathrm{Ad}\,(g_{ik}(x))\cdot\lambda_k + \dfrac{\partial g_{ik}(x)}{\partial x_k^\nu}\cdot g_{ik}(x)^{-1}\cdot u_k^\nu \\ u_i^\mu = \dfrac{\partial x_i^\mu}{\partial x_k^\nu}\cdot u_k^\nu\,. \end{cases}$$

Thus: $\mathfrak{L}|U_i = \{(x_i, \lambda_i)\}$ where $\lambda_i = \mathrm{Ad}\ (g_{ik}(x))\lambda_k$ in $U_i \cap U_k$; $\mathfrak{R}|U_i = \{(x_i, \lambda_i, u_i)\}$ where the law of transformation is (9.3); $\mathfrak{E}|U_i = \{(x_i, u_i)\}$.

There are two cases in which we are interested: (1) P is the principal bundle associated with the (holomorphic) tangent bundle of a complex analytic manifold V; (2) the principal bundle is $\mathscr{P}$, the associated principal bundle of the bundle $\mathfrak{F}$ along the fibres of a differentiable family $\mathscr{V} \to M$ of complex structures. In case (1) we denote respectively by Ξ, Σ, Θ the sheaves of germs of holomorphic sections of the bundles $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{E}$ of the sequence (9.2); in this case the sheaves are sheaves of Ω-modules where Ω is the sheaf of local rings of holomorphic functions on V. In case (2) we denote by Ξ, Γ, Π respectively the sheaves of the sequence (4.4); these are sheaves of O-modules. The fact that we have used the same symbol Ξ in both cases will not lead to confusion. Thus we have the following two exact sequences:

$$(9.4)\qquad 0 \longrightarrow \Xi \longrightarrow \Sigma \xrightarrow{j} \Theta \longrightarrow 0 \qquad \text{Case 1}$$

$$(9.5)\qquad 0 \longrightarrow \Xi \longrightarrow \Gamma \xrightarrow{k} \Pi \longrightarrow 0 \qquad \text{Case 2}$$

Proposition 9.1. *The sequence* (9.4) *splits over* $\boldsymbol{C}$; (9.5) *splits over* O.

The statement that (9.4) splits over $\boldsymbol{C}$ means that it splits as a sequence of sheaves of $\boldsymbol{C}$-modules, but it does not in general split over Ω.

Proof. Case 1 (sequence 9.4)). In this case $g_{ik}(x) = (g_{ik,}{}^{\alpha}_{\beta}(x))$ where $g_{ik,}{}^{\alpha}_{\beta}(x) = \partial x_i^\alpha / \partial x_k^\beta$. We assert that there exists a $\boldsymbol{C}$-homomorphism $\tau : \Theta \to \Sigma$ such that the triangle

$$\begin{tikzcd} \Sigma \arrow[rr, "j"] & & \Theta \\ & \Theta \arrow[ul, "\tau"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

is commutative where $\iota : \Theta \to \Theta$ is the identity map and where the map $\tau : \Theta \to \Sigma$ is defined by

$$\theta(x_i) = (x_i,\, u_i(x)) \longrightarrow \sigma(x_i) = (x_i,\, \lambda_i(x),\, u_i(x))\ , \quad \lambda_i(x) = \frac{\partial u_i}{\partial x_i} = \left(\frac{\partial u_i^\alpha}{\partial x_i^\beta}\right).$$

In fact, it follows from $u_i^\alpha(x) = g_{ik,\beta}^{\ \alpha}(x)\cdot u_k^\beta(x)$ that

$$\begin{aligned}\frac{\partial u_i^\alpha}{\partial x_i^\beta} &= g_{ik,\nu}^{\ \alpha}\frac{\partial x_k^\gamma}{\partial x_i^\beta}\cdot\frac{\partial u_k^\nu}{\partial x_k^\gamma} + \frac{\partial g_{ik,\nu}^{\ \alpha}}{\partial x_k^\gamma}\cdot\frac{\partial x_k^\gamma}{\partial x_i^\beta}\cdot u_k^\nu \\ &= g_{ik,\nu}^{\ \alpha}\frac{\partial u_k^\nu}{\partial x_k^\gamma}\, g_{ki,\beta}^{\ \gamma} + \frac{\partial g_{ik,\gamma}^{\ \alpha}}{\partial x_k^\nu}\, g_{ki,\beta}^{\ \gamma}\cdot u_k^\nu\end{aligned}$$

or

$$\frac{\partial u_i}{\partial x_i} = \mathrm{Ad}\,(g_{ik})\frac{\partial u_k}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_k^\nu}\cdot g_{ik}^{-1}u_k^\nu\,,$$

q. e. d.

It follows in particular that, in the exact cohomology sequence corresponding to (9.4), all coboundary maps $\delta^*: H^q(V,\Theta)\to H^{q+1}(V,\Xi)$ vanish for $q=0,1,2,\cdots$. The first assertion of Proposition (9.1) is equivalent to the statement that, in the map $\delta^*:\ H^0(V,\mathrm{Hom}_C(\Theta,\Theta))\to H^1(V,\mathrm{Hom}_C(\Theta,\Xi))$, the canonical element $\iota\in H^0(V,\mathrm{Hom}_C(\Theta,\Theta))$ maps into $\delta^*(\iota)=0$, but this does not imply that the map

$$H^0(V,\mathrm{Hom}_\Omega(\Theta,\Theta))\longrightarrow H^1(V,\mathrm{Hom}_\Omega(\Theta,\Xi))$$

is trivial.

REMARK. The map $\tau:\Theta\to\Sigma$ may also be constructed very simply as follows. Namely, let $u\in H^0(U,\Theta)$, and let $U'\subset U$, $\overline{U'}\subset U$. For $x_0\in U'$, let $x=x(x_0,s)=\exp(su)(x_0)$; for each sufficiently small value of s we thus obtain a homeomorphism $U'\longrightarrow V'\subset U$ and

$$\partial^2x_i^\mu(x_0,s)/\partial s\partial x_0^\nu|_{s=0} = \partial u^\mu/\partial x^\nu|_{x=x_0}\,.$$

This homeomorphism induces a map h of the (holomorphic) tangent bundle of V restricted to U' and it induces a bundle homeomorphism $h^P: P|U'\to P|V'$ of the associated principal bundle P which is defined by $(x_0,g_0)\to(x(x_0,s),\,g_0\cdot(\partial x/\partial x_0))$ where $\partial x/\partial x_0$ is the matrix $(\partial x^\mu/\partial x_0^\nu)$. Finally $h^P=\exp(s\sigma)$ where $\sigma(x_0)=(x_0,\lambda(x_0),u(x_0))$, $\lambda(x_0)=\partial u/\partial x_0$, $x_0\in U'$. Since U' is an arbitrary subneighborhood of U, $\overline{U'}\subset U$, we obtain again the map τ described above.

CASE 2 (sequence (9.5)). In this case $\mathscr{P}\to\mathscr{V}$ is the principal bundle and $x_i=(z_i,t_i)$ where $z_i=(z_i^\alpha)$ are fibre coordinates of $\mathscr{V}$ and t_i are coordinates on M, $u_i=(\theta_i,v_i)$ where $\theta_i=(\theta_i^\alpha)$ are tangential to the fibres of $\mathscr{V}$ and v_i are tangential to M. In this case $g_{ik}(z,t)=g_{ik,\beta}^{\ \alpha}(z,t)$ where $g_{ik,\beta}^{\ \alpha}(z,t)=\partial z_i^\alpha/\partial z_k^\alpha$. The bundles of the sequence (9.2) have coordinates as follows: $\mathfrak{L}|\mathscr{U}_i=\{(z_i,t_i,\lambda_i)\}$; $\mathfrak{S}|\mathscr{U}_i=\{(z_i,t_i,\theta_i,v_i)\}$; $\mathfrak{R}|\mathscr{U}_i=\{(z_i,t_i,\lambda_i,\theta_i,v_i)\}$ where

$$\begin{cases} \lambda_i = \mathrm{Ad}\,(g_{ik}) \cdot \lambda_k + \theta_k^\alpha \dfrac{\partial g_{ik}}{\partial z_k^\alpha} g_{ik}^{-1} + v_k^\nu \dfrac{\partial g_{ik}}{\partial t_k^\nu} g_{ik}^{-1} \\ \theta_i^\alpha = g_{ik,\beta}^{\ \alpha} \cdot \theta_k^\beta + \dfrac{\partial z_i^\alpha}{\partial t_k^\nu} v_k^\nu \\ v_i^\mu = \dfrac{\partial t_i^\mu}{\partial t_k^\nu} \cdot v_k^\nu . \end{cases}$$

Define $\tau : \Pi \to \Gamma$ as follows : $\pi(z, t) = (z_i, t_i, \theta_i(z, t), v_i(t)) \longrightarrow \gamma(z, t) = (z_i, t_i, \partial\theta_i(z, t)/\partial z_i, \theta_i(z, t), v_i(t))$. Then we have the commutative triangle

$$\begin{tikzcd} \Gamma \arrow[rr, "k"] & & \Pi \\ & \Pi \arrow[ul, "\tau"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

where $\iota : \Pi \to \Pi$ is the identity. Since, in this case, the map $\tau : \Pi \to \Gamma$ involves differentiation only with respect to the coordinates z_i along the fibres of $\mathscr{V}$, τ is linear over O and hence the split is over O ; namely $\Gamma = \Xi \bigoplus \tau(\Pi)$ in the sense of O-modules, q. e. d.

Proposition 9.1 together with the associated sheaf diagram $(4.2)_P$ implies that the sequence

$$(9.6) \qquad 0 \longrightarrow \Xi \longrightarrow \Sigma \longrightarrow \Theta \longrightarrow 0$$

also splits over O. The associated sheaf diagram therefore contributes little additional information concerning the structure of a family $\mathscr{V} \to M$, as might be expected.

We remark that the above splits do not define connections in the classical sense for these bundles; we shall discuss a special type of classical connection in the next section.

10. Differentiable families of connections

In Section 8 we defined connections for differentiable families of complex structures and showed that a connection for a differentiable family exists only if the family is locally trivial. In this section we consider differentiable families of connections defined on the fibres of a family $\mathscr{P} \to M$ of complex principal bundles. The results of this section will be used later only for the case where $\mathscr{P}$ is a $\boldsymbol{C}^*$-bundle (principal bundle of a complex line bundle).

Let $\mathscr{V} \xrightarrow{\varpi} M$ be a differentiable family of complex structures and let $\mathscr{P} \to \mathscr{V}$ be a differentiable family of complex principal bundles over $\mathscr{V}$ with the structure group G. Then (compare Section 1) $\mathscr{P} \to \mathscr{V} \to M$ is a differentiable family $\mathscr{P} \to M$ of complex structures whose fibre over a point $t \in M$ is a complex-analytic principal bundle $P_t \to V_t$ where $V_t = \varpi^{-1}(t)$, $P_t = \mathscr{P} | V_t$. We denote by J the complex structure tensor (see Section 1) of the family $\mathscr{P} \to M$.

Now let $\mathfrak{F}_+$ be the bundle over $\mathscr{P}$ of tangent vectors along the fibres of $\mathscr{P} \to M$ (compare Section 1). We say that a vector of $\mathfrak{F}_+$ over the point $p \in \mathscr{P}$ is vertical if and only if it goes into zero under the map induced from the projection $\mathscr{P} \to \mathscr{V}$.

DEFINITION 10.1. A differentiable family of connections is a direct-sum decomposition $\mathfrak{F}_+ = H(\mathfrak{F}_+) \oplus V(\mathfrak{F}_+)$ where $V(\mathfrak{F}_+)_p$ (restriction of the bundle $V(\mathfrak{F}_+)$ to the point $p \in \mathscr{P}$) is the linear space of vertical vectors at the point $p \in \mathscr{P}$ and where the decomposition is invariant under the right action of G.

A differentiable family of connections plainly defines, by restriction, a connection in the usual sense on each fibre $P_t \to V_t$, $t \in M$.

We call $H(\mathfrak{F}_+)_p$ the space of horizontal vectors at the point $p \in \mathscr{P}$, and we observe that the projection $\varpi_P : \mathscr{P} \to \mathscr{V}$ induces an isomorphism of $H(\mathfrak{F}_+)_p$ onto the space of vectors tangent to the fibre of $\mathscr{V}$ passing through the point $\varpi_P(p) \in \mathscr{V}$. We denote by H_p the space $H(\mathfrak{F}_+)_p$ of horizontal vectors at $p \in \mathscr{P}$.

For any point $p \in \mathscr{P}$, let $J_p = J|p$ denote the linear transformation on the tangent space (along the fibre of $\mathscr{P} \to M$) at p which is defined by the restriction of J to the point p. Then : (i) the complex structure is invariant under right translation by G, that is $R_g \circ J_p = J_{p \cdot g}$ where R_g is the map induced by right translation by $g \in G$ and where $p \cdot g$ denotes the right translation of p by g ; (ii) J_p leaves invariant the vertical subspace of the tangent space (along the fibre of $\mathscr{P} \to M$) at p, and J_p restricted to the vertical subspace induces an almost-complex structure on the fibre of $\mathscr{P} \to \mathscr{V}$ passing through p which agrees with the complex structure of G.

Assume that G is semi-simple. Given a reduction of the structure group G of $\mathscr{P} \to \mathscr{V}$ to a maximal compact subgroup K, let $\mathscr{P}_K$ be the subbundle of $\mathscr{P}$ with group K.

THEOREM 10.1. *For every reduction of $\mathscr{P}$ to a sub-bundle $\mathscr{P}_K$ there exists a unique differentiable family of connections satisfying the following two conditions* :

(a) *for* $p \in \mathscr{P}$, $J_p(H_p) = H_p$;

(b) *for* $p \in \mathscr{P}$, H_p *is contained in the subspace of the tangent space* (*along the fibre of* $\mathscr{P} \to M$) *at* p *which is tangent to* $\mathscr{P}_K$.

In the case of a single complex-analytic fibre bundle (case where M is reduced to one point), this theorem, due to I. M. Singer (unpublished), is a generalization of the well known special case where $G = GL(q, \boldsymbol{C})$, $K = U(q)$ (see e. g., Nakano [31]). The proof of Theorem 10.1 is essentially the same as that for a single complex-analytic bundle and will therefore be omitted.

DEFINITION 10.2. Given a family of connections for $\mathscr{P} \to \mathscr{V} \to M$, the covariant derivative D on $\mathscr{P}$ is defined to be $H^* \circ d$ where d is the exterior differential along the fibres of $\mathscr{P} \to M$ and where H^*, the transpose of H, is the projection of the differential forms along the fibres of $\mathscr{P} \to M$ onto their horizontal components.

A family of connections for $\mathscr{P} \to \mathscr{V} \to M$ gives rise to a connection form ω which is a differential form of degree 1 along the fibres of $\mathscr{P} \to M$ with values in $\mathfrak{g}$, the Lie algebra of G. The form ω is characterized by the following two properties:

(1) If i_q^* is the transposed homeomorphism of the injection i_q of the fibre $G_q = \varpi_P^{-1}(q)$, $q \in \mathscr{V}$, into $\mathscr{P}$, then $i_q^*\omega$ is the Maurer-Cartan form of G_q (identity map of the Lie algebra).

(2) $\omega \circ R_g = \mathrm{Ad}\,(g^{-1})\omega$.

Let Q be the natural isomorphism of the Lie algebra of G onto the Lie algebra of vertical vector fields of $\mathscr{P}$. Then ω is defined as follows: for $u \in (\mathfrak{F}_+)_p$, $p \in \mathscr{P}$, $\omega(u) = \lambda$ where λ is the element of the Lie algebra of G such that $Q\lambda$ coincides with the vertical component of u. Conversely, given a form ω along the fibres of $\mathscr{P} \to M$ which satisfies (1) and (2), a connection is obtained by defining $H_p = \{u \in (\mathfrak{F}_+)_p | \omega(u) = 0\}$.

DEFINITION 10.3. The curvature Ω of a differentiable family of connections for $\mathscr{P} \to \mathscr{V} \to M$ is the covariant derivative of the connection form: $\Omega = D\omega = H^* \circ d\omega$.

THEOREM 10.2. *The curvature* Ω *of the connection of Theorem* 10.1 *corresponding to a reduction of* $\mathscr{P}$ *to a sub-bundle* $\mathscr{P}_K$ *is a* $\mathfrak{g}$-*form along the fibres of* $\mathscr{P} \to M$ *of type* (1, 1). *That is, for any two vectors* $u, v \in (\mathfrak{F}_+)_p$, $\Omega(u, v) = \Omega(J_p(u), J_p(v))$.

In the case of a single complex-analytic principal bundle (M a point), this theorem, also due to Singer (unpublished), generalizes the well known special case where $G = GL(q, \boldsymbol{C})$, $K = U(q)$ (Nakano [31]). The proof of Theorem 10.2 is a straightforward calculation, which is essentially the same as the calculation for a single complex-analytic bundle ; it will therefore be omitted.

These theorems are of particular interest in the case where

$$G = GL(q, \boldsymbol{C}), K = U(q) .$$

In fact, let $\mathscr{B} \to \mathscr{V}$ be a differentiable family of complex vector bundles over $\mathscr{V}$ with fibre $\boldsymbol{C}^q$ and structure group $G = GL(q, \boldsymbol{C})$. Then any hermitian metric along the fibres of $\mathscr{B}$ (Definition 2.2) defines. by Theorem 10.1, a unique connection whose curvature form, by Theorem 10.2, is a $\mathfrak{g}$-form ($\mathfrak{g}$ the Lie algebra of $G = GL(q, \boldsymbol{C})$) of type (1, 1) along the fibres of $\mathscr{P} \to M$ where $\mathscr{P}$ is the family of principal bundles associated to $\mathscr{B}$. In the case $q = 1$, $GL(1, \boldsymbol{C}) = \boldsymbol{C}^*$ (multiplicative group of complex numbers) and the Lie algebra $\mathfrak{g}$ of $\boldsymbol{C}^*$ is $\boldsymbol{C}$. In this case $\mathscr{B}$ is a family of complex line bundles over $\mathscr{V}$ with fibre $\boldsymbol{C}$ and the curvature of the connection induces a form γ of type (1, 1) along the fibres of $\mathscr{V}$. In terms of local coordinates (z, t) on $\mathscr{V}$, $\gamma = i \sum \gamma_{\alpha\bar{\beta}} (z, z, t) dz^{\alpha} \wedge dz^{\beta}$. We say that γ is positive and we write $\gamma > 0$ if the hermitian form $\sum \gamma_{\alpha\bar{\beta}} (z, \bar{z}, t) u^{\alpha} \bar{u}^{\beta}$ in n variables $u^1, \cdots, u^n$ is positive definite at each point of $\mathscr{V}$.

DEFINITION 10.4. We say that a family is a differentiable family over $\mathscr{V}$ of positive complex line bundles if and only if there is a connection with positive γ.

It is clear that a family is a family of positive complex line bundles only if its restriction to each fibre of $\mathscr{V}$ is positive in the sense of Kodaira [23] (see also [21]).

CHAPTER V. DEFORMATIONS OF COMPLEX STRUCTURES UNDER ADDITIONAL RESTRICTIONS

11. Number of moduli

In this section we introduce the concept of number of moduli of a compact complex manifold. For this purpose we first consider an effectively parametrized complete complex analytic family, i. e. a complex analytic family $\mathscr{V} = \{V_t | t \in M\}$ which is complete in the sense of Definition 1.7 and is such that $\rho_t : (T_M)_t \to H^1(V_t \Theta_t)$ is injective for each point $t \in M$.

PROPOSITION 11.1. *Let $\mathscr{V} = \{V_t | t \in M\}$ and $\mathscr{V}' = \{V'_{t'} | t' \in M'\}$ be effectively parametrized complete complex analytic families of deformations of one and the same compact complex manifold $V_o = V'_{o'}$. Then there exist neighborhoods U of o on M and U' of o' on M' such that the restrictions $\mathscr{V} | U$ and $\mathscr{V}' | U'$ are equivalent. Moreover the equivalence is induced by a biregular holomorphic map f of U onto U'.*

PROOF. By hypothesis there exists a neighborhood U of o and a differentiable map f of U into M' with $f(o) = o'$ such that $\mathscr{V}|U$ coincides with the family $\{V'_{f(t)}|t \in U\}$ induced from $\mathscr{V}'$. For a moment we denote by R_t [or $R'_{t'}$] the *real* tangent space of M [or M'] at t [or t']. Let $t' = f(t)$ and denote by f_{t*} the homomorphism $R_t \to R'_{t'}$ induced by f. Then we have the commutative diagram

$$\begin{array}{ccc} R_t & \xrightarrow{\quad f_{t*} \quad} & R'_{t'} \\ {\scriptstyle \rho_t}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho'_{t'}} \\ H^1(V_t, \Theta_t) & = & H^1(V'_{t'}, \Theta'_{t'}) \end{array} \tag{11.1}$$

(see Section 6). Moreover ρ_t and $\rho'_{t'}$ are injective, since $\mathscr{V}$ and $\mathscr{V}'$ are effectively parametrized. It follows that f_{t*} is injective. Hence f is a differentiable homeomorphism of U into M'. Since our hypothesis is symmetric in M and M', we infer that M and M' have the same dimension and consequently $U' = f(U)$ is a neighborhood of o' on M'. Clearly

$$\mathscr{V}|U = \{V'_{f(t)}|t \in U\}$$

is equivalent to $\mathscr{V}'|U'$.

To prove that f is holomorphic, we observe the canonical decomposition

$$R_t \otimes_{\boldsymbol{R}} \boldsymbol{C} = T_t \oplus \overline{T}_t\ ,$$

where T_t is the holomorphic tangent space and $\overline{T}_t$ is its conjugate. Since $\mathscr{V}$ is a complex analytic family, $\rho_t(\overline{T}_t) = 0$ and $\rho_t : T_t \longrightarrow H^1(V_t, \Theta_t)$ is injective. Similarly we have $R'_{t'} \otimes_{\boldsymbol{R}} \boldsymbol{C} = T'_{t'} \oplus \overline{T}'_{t'}$, $\rho'_{t'}(\overline{T}'_{t'}) = 0$, and $\rho'_{t'} : T'_{t'} \longrightarrow H^1(V'_{t'}, \Theta'_{t'})$ is injective. Now it follows from (11.1) that

$$\rho'_{t'} f_t^*(\overline{T}_t) = \rho_t(\overline{T}_t) = 0\ .$$

Hence we get $f_t^*(\overline{T}_t) \subseteqq \overline{T}'_{t'}$. This proves that f is holomorhic, q. e. d.

By making certain assumptions on the postulation formulae, Noether [33] has shown already in 1888 that the *number of moduli* m of the regular algebraic surfaces with the arithmetic genus $p_a > 3$ and the linear genus $p^{(1)} > 5$ is given by

$$m = 10(p_a + 1) - 2(p^{(1)} - 1)\ . \tag{11.2}$$

The number of moduli of an algebraic curve of a given genus p is defined to be the dimension of the irreducible algebraic system consisting of all *birationally distinct* algebraic curves of genus p and, for $p \geqq 2$, this number of moduli is equal to $3p - 3$. The number of moduli of an algebraic surface is supposed to be defined in a similar manner (see

Zariski [41], p. 97). However it would seem to be impossible in general to apply a similar definition of number of moduli to compact complex manifolds (see Section 14, (γ); cf. also Zariski, loc. cit., where the ambiguity involved in the classical definition has been pointed out). Here we propose tentatively the following definition :

DEFINITION 11.1. Let V_o be a compact complex manifold. If there exists an effectively parametrized complete complex analytic family $\mathscr{V} = \{V_t \mid t \in M\}$ of deformations V_t of V_o, then we call $m = \dim_C M$ the number of moduli of V_o and denote it by $m(V_o)$:

$$m(V_o) = \dim_C M \ . \tag{11.3}$$

It is obvious by the above Proposition 11.1 that $m(V_o)$ is uniquely determined by V_o. Moreover, by hypothesis, since the homomorphism $\rho_o : (T_M)_o \to H^1(V_o, \Theta_o)$ is injective, we have the *inequality*

$$m(V_o) \leqq \dim H^1(V_o, \Theta_o) \ . \tag{11.4}$$

In several respects our concept of number of moduli is different from the classical one mentioned above. First, we do not define the number of moduli $m(V)$ for manifolds V which cannot be imbedded as fibres in effectively parametrized complete complex analytic families. In fact, for some examples of such manifolds, the number of moduli does not make sense (see Section 15). Second, $m(V)$ is concerned with the complex structure of V and therefore $m(V)$ is not birationally invariant. Third, $m(V)$ is the number of moduli of V considered as a *complex manifold.* Therefore, even in case V is algebraic, $m(V)$ may be greater than the number of moduli of V known in classical algebraic geometry.

It will be shown in Section 14 that, for some simple types of manifolds, e.g. complex tori, hypersurfaces in projective space and in abelian varieties, *the equality*

$$m(V_o) = \dim H^1(V_o, \Theta_o) \tag{11.5}$$

holds.

Now we consider the case in which V_o is an *algebraic* surface and discuss the relation between $\dim H^1(V_o, \Theta_o)$ and Noether's formula. Replacing the linear genus $p^{(1)}$ by $c_1^2 + 1$, c_1 being the first Chern class of V_o, we put Noether's formula in the form :

$$m(V_o) = 10(p_a + 1) - 2c_1^2 \ . \tag{11.6}$$

c_1^2 is not a birational invariant, but $c_1^2 = p^{(1)} - 1$ if V is a minimal model. Computing

$$\chi(V_o, \Theta_o) = \sum_{q=0}^{2} (-1)^q \dim H^q(V_o, \Theta_o)$$

with the help of Hirzebruch's formula (see Hirzebruch [17], p. 148), we find that

$$\begin{aligned}(11.7)\qquad \dim H^1(V_o, \Theta_o) = 10(p_a + 1) - 2c_1^2 + \dim H^0(V_o, \Theta_o) \\ + \dim H^2(V_o, \Theta_o)\,.\end{aligned}$$

Thus *Noether's formula gives the "principal part" of* $\dim H^1(V_o, \Theta_o)$. The term $\dim H^0(V_o, \Theta_o)$ gives the dimension of the Lie group of global analytic automorphisms of V_o. Noether assumed that $\dim H^0(V_o, \Theta_o) = 0$. *The term* $\dim H^2(V_o, \Theta_o)$ *is a birational invariant*, as will be shown below.

Returning to the general case, we consider a complete complex analytic family $\mathscr{V} = \{V_t \mid t \in M\}$ of deformations of a compact complex manifold V_o which is q-trivial in a neighborhood of o in the following sense: There is a system of local complex coordinates $(t^1, \cdots, t^q, \cdots, t^m)$ with the center o, a coordinate neighborhood $U = \{t \mid |t^k| < \varepsilon\,(1 \leqq k \leqq m)\}$, a coordinate plane $U' = \{t \mid t \in U, t^1 = \cdots = t^q = 0\}$, and a differentiable map $h: \mathscr{V} | U \to \mathscr{V} | U'$ which maps each fibre V_t of $\mathscr{V} | U$ biregularly onto $V_{\pi(t)}$, where π denotes the canonical projection $\pi: U \to U'$. (See Section 5. Here q denotes the complex dimension.)

PROPOSITION 11.2. *If the restriction* $\mathscr{V} | U'$ *is effectively parametrized, then the number of moduli* $m(V_o)$ *is given by*

$$(11.8)\qquad m(V_o) = \dim_C M - q\ .$$

PROOF. It is clear that the completeness of $\mathscr{V} | U$ implies the completeness of $\mathscr{V} | U'$. Hence we obtain

$$m(V_o) = \dim_C U' = \dim_C M - q\ ,$$

q.e.d.

In order to determine the number of moduli of a given compact complex manifold V_0, it is necessary to find a complete complex analytic family of deformations of V_o; however this is quite difficult in some cases, in particular in case V_o is non-Kähler. In this connection we state the following proposition which follows immediately from (11.1).

PROPOSITION 11.3. *Assume that the number of moduli* $m(V_o)$ *of* V_o *can be defined. If there exists a differentiable family* $\mathscr{V} = \{V_t \mid t \in M\}$ *of deformations* V_t *of* V_o *such that the map* $\rho_o: (T_M)_o \to H^1(V_o, \Theta_o)$ *is surjective, then we have*

$$m(V_o) = \dim H^1(V_o, \Theta_o)\ .$$

REMARKS. For a moment we drop the subscript o and write V, Θ for V_o, Θ_o. Let K be the canonical bundle of V. The characteristic class $c(K)$ of K is equal to $-c_1$, where c_1 is the first Chern class of V. By the duality theorem, we have

(11.9) $$\dim H^q(V, \Theta) = \dim H^{n-q}(V, \Omega^1(K)) ,$$

where $n = \dim V$ and $\Omega^1(K)$ denotes the sheaf of germs of holomorphic 1-forms with coefficients in K. Letting F be an arbitrary complex line bundle over V, we consider $H^q(V, \Omega^p(F))$. We have

(11.10) $$H^q(V, \Omega^p(F^{-1})) \cong H^{n-q}(V, \Omega^{n-q}(F))$$

(see Kodaira [21], formula (3)). We know that, if the characteristic class $c(F)$ of F is sufficiently positive, the cohomology $H^q(V, \Omega^p(F^{-1}))$ vanishes for $q \leqq n-1$ (Kodaira [21], Theorem 1). The following useful criteria for the vanishing of the cohomology are due to Akizuki and Nakano [2], Theorems 1 and 1′:

(11.11) $$H^q(V, \Omega^p(F)) = 0 \qquad \text{for } p + q \leqq n - 1, \text{ if } c(F) < 0,$$

(11.12) $$H^q(V, \Omega^p(F)) = 0 \qquad \text{for } p + q \geqq n + 1, \text{ if } c(F) > 0.$$

It has been pointed out by Nakano that, combined with (11.9), these criteria give the following

THEOREM (Nakano).

(i) *If* $c_1 < 0$, *then* $H^0(V, \Theta) = 0$;

(ii) *If* $c_1 > 0$, *then* $H^q(V, \Theta) = 0$ *for* $q \geqq 2$.

The first part of this theorem asserts that V *admits no continuous group of analytic automorphisms if the first Chern class* c_1 *of* V *is negative.* This is a generalization to higher dimensional manifolds of a well known theorem to the effect that any compact Riemann surface of genus $p \geqq 2$ admits no continuous group of analytic automorphisms.

12. Differentiable families of submanifolds of a fixed manifold

Let W be a compact complex manifold.

DEFINITION 12.1. By a differentiable (or complex analytic) family of (complex) submanifolds of W we mean a differentiable (or complex analytic) family $\mathscr{V} \xrightarrow{\varpi} M$ of complex structures such that each fibre $V_t = \varpi^{-1}(t)$ is a complex submanifold of W together with a differentiable (or holomorphic) map Φ of $\mathscr{V}$ into W whose restriction to each fibre V_t is the inclusion map $V_t \to W$. With reference to a fibre $V_o = \varpi^{-1}(o)$, $o \in M$, we call $\mathscr{V}$ a differentiable (or complex analytic) family of deformations V_t of V_o in W.

We note that the map Φ may not be a homeomorphism. In case it is necessary to indicate the map Φ explicitly we write $(\mathscr{V}, \Phi)$ or $(\mathscr{V} \to M, \Phi)$ for $\mathscr{V} \to M$.

Given a differentiable family $\mathscr{V}$ of submanifolds of W, let $\mathfrak{L}$ be the vector bundle over $\mathscr{V}$ induced from the holomorphic tangent bundle of

W by the map $\Phi: \mathscr{V} \to W$ and let Ξ be the sheaf of germs of differentiable sections of $\mathfrak{L}$ whose restrictions to the fibres of $\mathscr{V}$ are holomorphic. Moreover let

$$0 \longrightarrow \Theta \longrightarrow \Pi \longrightarrow T \longrightarrow 0$$

be the fundamental sequence of sheaves for the family $\mathscr{V} \to M$. For a moment we denote by $\mathscr{D}$ the sheaf of germs of differentiable vector fields on $\mathscr{V}$. Clearly Φ induces a homomorphism of $\mathscr{D}$ into the sheaf of germs of differentiable sections of $\mathfrak{L} \oplus \bar{\mathfrak{L}}$. Hence, by projection, we obtain a homomorphism Φ_* of $\mathscr{D}$ into the sheaf of germs of differentiable sections of $\mathfrak{L}$ which maps the subsheaf Π of $\mathscr{D}$ into Ξ. Moreover Φ_* maps the subsheaf Θ of Π *isomorphically* into Ξ. Identifying Θ with $\Phi_*(\Theta)$ we consider Θ as a subsheaf of Ξ and form the factor sheaf $\Psi = \Xi/\Theta$. Then we obtain the exact commutative diagram

(12.1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta & \longrightarrow & \Pi & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \Phi_* \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta & \xrightarrow{\iota} & \Xi & \xrightarrow{\mu} & \Psi & \longrightarrow & 0
\end{array}$$

where $\|$ denotes the identity map. Letting $\mathscr{H}^q(\Theta)$, $\mathscr{H}^q(\Pi)$, $\cdots$ have the same meaning as in Section 5, we get from (12.1) the corresponding exact commutative diagram

(12.2)
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T_M & \xrightarrow{\rho} & \mathscr{H}^1(\Theta) & \longrightarrow & \mathscr{H}^1(\Pi) & \longrightarrow & \cdots \\
 & & \rho_d \downarrow & & \| & & \downarrow & & \\
\cdots & \longrightarrow & \mathscr{H}^0(\Psi) & \xrightarrow{\delta^*} & \mathscr{H}^1(\Theta) & \xrightarrow{\iota^*} & \mathscr{H}^1(\Xi) & \longrightarrow & \cdots
\end{array}$$

where $\mathscr{H}^0(T)$ is replaced by $T_M \cong \mathscr{H}^0(T)$. We call the image $\rho_d(T_M)$ in $\mathscr{H}^0(\Psi)$ the sheaf of germs of displacements.

For each point $t \in M$ we denote by Ξ_t, Ψ_t the restrictions (in the sense defined in Section 4) of Ξ, Ψ to V_t. Ξ_t coincides with the restriction of the sheaf of germs of holomorphic vector fields over W to the submanifold V_t. Hence Ξ_t and $\Psi_t = \Xi_t/\Theta_t$ are uniquely determined by the imbedding of V_t in W and are independent of the family $\mathscr{V}$ containing V_t as a fibre. Obviously we have the exact commutative diagram

(12.3)
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathscr{H}^0(\Psi) & \xrightarrow{\delta^*} & \mathscr{H}^1(\Theta) & \xrightarrow{\iota^*} & \mathscr{H}^1(\Xi) & \longrightarrow & \cdots \\
 & & r_t^* \downarrow & & r_t^* \downarrow & & r_t^* \downarrow & & \\
\cdots & \longrightarrow & H^0(V_t, \Psi_t) & \xrightarrow{\delta^*} & H^1(V_t, \Theta_t) & \xrightarrow{\iota_t^*} & H^1(V_t, \Xi_t) & \longrightarrow & \cdots
\end{array}$$

where r_t denotes the restriction map to V_t. Letting

$$\rho_t = r_t^*\rho\,, \qquad \rho_{d,t} = r_t^*\rho_d\,,$$

we obtain therefore the commutative diagram

(12.4)
$$\begin{tikzcd} & (T_M)_t \arrow[dl, "\rho_{d,t}"'] \arrow[dr, "\rho_t"] & \\ H^0(V_t, \Psi_t) \arrow[rr, "\delta^*"] & & H^1(V_t, \Theta_t) \end{tikzcd}$$

It follows that

(12.5)
$$\iota_t^*\rho_t = 0\,.$$

The image $\rho_{d,t}(v)$ of a vector $v \in (T_M)_t$ will be called the *infinitesimal displacement* (of V_t) along v. Moreover we call $\rho_t(v)$, $v \in (T_M)_t$, an infinitesimal deformation of V_t *in* W.

DEFINITION 12.2. We denote by $H^1_W(V_t, \Theta_t)$ the kernel of the homomorphism $\iota_t^* : H^1(V_t, \Theta_t) \to H^1(V_t, \Xi_t)$.

The above formula (12.5) shows that every infinitesimal deformation of V_t *in* W belongs to $H^1_W(V_t, \Theta_t)$. Obviously we have

$$H^1_W(V_t, \Theta_t) = \delta^* H^0(V_t, \Psi_t)\,.$$

Let $(\mathscr{V} \to M, \Phi)$ be a differentiable family of submanifolds of W and let $\mathscr{V}' \to M'$ be the differentiable family induced from $\mathscr{V} \to M$ by a map $f: M' \to M$. By definition there is a canonical differentiable map $h: \mathscr{V}' \to \mathscr{V}$ which maps each fibre $V'_{t'}$ of $\mathscr{V}'$ biregularly onto the fibre $V_{f(t')}$ of $\mathscr{V}$. Letting $\Phi' = \Phi \circ h$, we infer therefore that $(\mathscr{V}' \to M', \Phi')$ forms a differentiable family of submanifolds of W. Thus any family induced from a family of submanifolds of W is again a family of submanifolds of W.

DEFINITION 12.3. A complex analytic family $\mathscr{V} = \{V_t \mid t \in M\}$ of submanifolds of W is called complete relative to W at $t \in M$ if, for any differentiable family $\mathscr{V}' = \{V'_{t,s} \mid s \in N\}$ of deformations $V'_{t,s}$ of $V_t = V'_{t,o'}$ in W, there exists a neighborhood U' of o' on N such that the restriction $\mathscr{V}' \mid U'$ is a family induced from $\mathscr{V}$. Moreover we say that $\mathscr{V}$ is complete relative to W if $\mathscr{V}$ is complete relative to W at each point t of M.

DEFINITION 12.4. Let V_o be a complex submanifold of W. If there exists an effectively parametrized complex-analytic family $\mathscr{V} \to M$ of deformations of V_o in W which is complete relative to W, then we call $\dim_C M$ the number of moduli of V_o relative to W and denote it by $m_W(V_o)$:

(12.6) $$m_W(V_o) = \dim_C M .$$

Since Proposition 11.1 is valid also for effectively parametrized relatively complete complex analytic families, $m_W(V_o)$ is uniquely determined by V_o and the inclusion map $V_o \to W$. Moreover we infer from (12.4) that *the inequality*

(12.7) $$m_W(V_o) \leqq \dim H^1_W(V_o, \Theta_o)$$

holds.

In this section we ask whether the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ holds in the special case in which W is an algebraic manifold of dimension $n+1 \geqq 2$ imbedded in a projective space and V_o is a submanifold of dimension n, and we derive the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ from the theorem of completeness of characteristic linear systems of complete continuous systems under an additional restriction on V_o.

Let $\mathscr{F}_o = [V_o]$ be the complex line bundle over W determined by the *divisor* V_o and let F_o be the restriction of $\mathscr{F}_o$ to V_o. Clearly the factor sheaf $\Psi_o = \Xi_o/\Theta_o$ is isomorphic to the sheaf $\Omega(F_o)$ of germs of holomorphic sections of F_o and therefore *we may identify* Ψ_o *with* $\Omega(F_o)$. Thus we obtain the exact sequence

(12.8) $$0 \longrightarrow \Theta_o \xrightarrow{\iota_o} \Xi_o \xrightarrow{\mu_o} \Omega(F_o) \longrightarrow 0 .$$

The homomorphism μ_o is described explicitly as follows: Let $\{W_i\}$ be a finite covering of W by small open sets and let $S_i(w) = 0$ be the minimal local equation of V_o in W_i, where w denotes a "variable" point on W. Then the line bundle $\mathscr{F}_o = [V_o]$ is defined by the system $\{f_{ik}(w)\}$ of transition functions $f_{ik}(w) = S_i(w)/S_k(w)$ and therefore F_o is defined by $\{f_{ik|o}(z)\}, f_{ik|o}(z) = f_{ik}(\Phi(z, o))$, where $z \to \Phi(z, o)$ is the inclusion map $V_o \to W$. For an arbitrary germ ξ of the sheaf Ξ_o over a point $z \in V_o$, we set

$$(\xi \cdot S_i)(z) = \sum_{\nu=1}^{n+1} \xi^\nu (\partial S_i(w)/\partial w^\nu)_{w=\Phi(z,o)}$$

where $(\xi^1, \cdots, \xi^\nu, \cdots, \xi^{n+1})$ denote the components of ξ with respect to the system of local coordinates $(w^1, \cdots, w^\nu, \cdots, w^{n+1})$ on W. We have

$$(\xi \cdot S_i)(z) = f_{ik|o}(z)(\xi \cdot S_k)(z) ;$$

thus $(\xi \cdot S_i)(z)$ represents a germ of $\Omega(F_o)$ over z which we denote by $\xi \cdot S$. Then *the homomorphism* μ_o *is given by*

(12.9) $$\mu_o : \xi \longrightarrow \mu_o(\xi) = \xi \cdot S .$$

It is well known that the set $\mathscr{C}$ of all effective divisors $V \sim V_o$ on W, where $\sim$ denotes homology with coefficients in $\boldsymbol{Z}$, forms an *algebraic system*, i.e. there exist a (possibly reducible and singular) projective variety N and a *one-to-one algebraic correspondence* $t \to V_t$ between N and $\mathscr{C}$ (see Weil [39], p. 887). If the point o corresponding to V_o is a simple point of N, then, for a sufficiently small neighborhood U of o on N, $\mathscr{C} \mid U = \{V_t \mid t \in U\}$ is a complex analytic family of deformations of V_o in W, and therefore the homomorphism

$$\rho_{d,o} : (T_U)_o \longrightarrow H^0(V_o, \Psi_o) = H^0(V_o, \Omega(F_o))$$

is defined. *The linear subsystem of the complete linear system* $|F_o|$ *corresponding to the linear subspace* $\rho_{d,o}((T_U)_o)$ *of* $H^0(V_o, \Omega(F_0))$ *coincides with the characteristic linear system of* $\mathscr{C}$ *on* V_o. This can be verified as follows: Let $S_i(w, t) = 0$ be the minimal local equation of V_t, $t \in U$, in W_i, where $S_i(w, t)$ is holomorphic in both variables w, t and $S_i(w, o) = S_i(w)$. For each vector $v \in (T_U)_o$, we set

$$\psi_i(z, v) = (v \cdot S_i)(\Phi(z, o), o) = \sum_h v^h \left(\frac{\partial S_i(w, t)}{\partial t^h}\right)_{t=o, w=\Phi(z,o)}$$

where $(t^1, \cdots, t^h, \cdots)$ is a system of local coordinates with the center o which covers U and $(v^1, \cdots, v^h, \cdots)$ are the corresponding components of v. Then we have

$$\psi_i(z, v) = f_{ik|o}(z) \cdot \psi_k(z, v) .$$

Thus $\psi_i(z, v)$ represents a holomorphic section of $\Omega(F_o)$ which we denote by $\psi(v)$. The characteristic linear system of $\mathscr{C}$ on V_o is, by definition, the linear subsystem of $|V_o|$ corresponding to the linear subspace of $H^0(V_o, \Omega(F_o))$ consisting of all $\psi(v)$, $v \in (T_U)_o$ (see Kodaira [24], p. 738). In order to show that

$$\rho_{d,o}(v) = -\psi(v) ,$$

we consider the commutative diagram

$$\begin{array}{ccc} \Pi_o & \longrightarrow & T_o \\ \Phi_{*o} \downarrow & & \downarrow \\ \Xi_o & \xrightarrow{\mu_o} & \Omega(F_o) \end{array} \tag{12.10}$$

which is obtained from (12.1) by restriction to V_o. Let $\{\mathscr{U}_i\}$ be a covering of $\mathscr{C} \mid U$ by coordinate neighborhoods $\mathscr{U}_i$ with the systems of coordinates of the form $(z_i^1, \cdots, z_i^\alpha, \cdots, z_i^n, t^1, \cdots, t^h, \cdots)$ (see Section 1). In terms of the system of coordinates on $\mathscr{U}_i$, any germ π of the sheaf Π_o over a point on $V_o \cap \mathscr{U}_i$ is written in the form

$$\pi = (\theta_i^1(z), \cdots, \theta_i^\alpha(z), \cdots, v^1, \cdots, v^h, \cdots) .$$

We write the restriction Φ_i of the map $\Phi : \mathscr{C} \mid U \to W$ to $\mathscr{U}_i$ in the form

$$w^\nu = \Phi_i^\nu(z_i, t) .$$

Then we have

$$\Phi_{*o}(\pi) = \xi = (\xi^1, \cdots, \xi^\nu, \cdots, \xi^{n+1}) ,$$

where

$$\xi^\nu = \sum_{\alpha=1}^n \theta_i^\alpha(z) \frac{\partial \Phi_i^\nu(z_i, o)}{\partial z_i^\alpha} + \sum_h v^h \left(\frac{\partial \Phi_i^\nu(z_i, t)}{\partial t^h} \right)_{t=o} .$$

Suppose for simplicity that $\Phi_i(\mathscr{U}_i) \subset W_i$. Since $S_i(\Phi_i(z, o)) = 0$, we have

$$\sum_\nu \frac{\partial \Phi_i^\nu(z_i, o)}{\partial z_i^\alpha} \left(\frac{\partial S_i(w)}{\partial w^\nu} \right)_{w=\Phi(z,o)} = 0 .$$

Hence we get

$$(\xi \cdot S_i)(z) = \sum_h v^h \left(\frac{\partial S_i(\Phi_i(z_i, t))}{\partial t^h} \right)_{t=o} , \qquad \text{for } \xi = \Phi_{*o}(\pi),$$

while $S_i(w) = S_i(w, o)$ and $S_i(\Phi_i(z_i, t), t) = 0$. Consequently we obtain

$$(\xi \cdot S_i)(z) = -\psi_i(z, v) , \qquad \text{for } \xi = \Phi_{*o}(\pi).$$

Comparing this with (12.9) and (12.10), we infer that the homomorphism $\rho_{d,o} : (T_U)_o = H^0(V_o, T_o) \to H^0(V_o, \Omega(F_o))$ is given by $\rho_{d,o}(v) = -\psi(v)$.

It follows from the above result that the characteristic linear system of $\mathscr{C}$ on V_o is complete if and only if $\rho_{d,o} : (T_U)_o \to H^0(V_o, \Omega(F_o))$ is surjective. On the other hand, if $\rho_{d,o}$ is surjective, then, by (12.4), $\rho_o : (T_U)_o \to H_W^1(V_o, \Theta_o)$ is also surjective, and therefore $m_W(V_o) = \dim H_W^1(V_o, \Theta_o)$ if $m_W(V_o)$ can be defined. Thus we obtain the following

PROPOSITION 12.1. *If the characteristic linear system of $\mathscr{C}$ on V_o is complete, then the homomorphism $\rho_o : (T_U)_o \to H_W^1(V_o, \Theta_o)$ is surjective ; if, moreover, $m_W(V_o)$ is defined, the equality $m_W(V_o) = \dim H_W^1(V_o, \Theta_o)$ holds.*

The theorem of completeness of the characteristic linear systems of complete continuous systems (Kodaira [24]) asserts that, *if $H^1(W, \Omega(V_o)) = 0$, where $\Omega(V_o)$ denotes the sheaf over W of germs of meromorphic functions which are multiples of $-V_o$, the point o corresponding to V_o is a simple point of N and $\rho_{d,o}$ gives an isomorphism*

(12.11) $$(T_N)_o \cong H^0(V_o, \Omega(F_o)) .$$

We note that $\Omega(V_o) = \Omega(\mathscr{F}_o)$. In the special case where $H(W, \Omega) = 0$, the isomorphism (12.11) holds for *every* non-singular n-dimensional submanifold V_o of W. In fact, in this case, $\mathscr{C}$ coincides with the complete linear system $|V_o|$ on W (therefore N is a projective space), and the isomorphism (12.11) follows immediately from the exact sequence

$$0 \longrightarrow H^0(W, \Omega) \longrightarrow H^0(W, \Omega(\mathscr{F}_o)) \longrightarrow H^0(V_o, \Omega(F_o)) \longrightarrow 0 .$$

Consequently we obtain the following

THEOREM 12.1. *If either* $H^1(W, \Omega(V_o)) = 0$ *or* $H^1(W, \Omega) = 0$, *then the equality*

$$m_W(V_o) = \dim H^1_W(V_o, \Theta_o) \tag{12.12}$$

holds, provided that $m_W(V_o)$ *is defined.*

THEOREM 12.2. *If* $H^1(W, \Omega(V_o)) = 0$, *then the set of all non-singular submanifolds* $V \sim V_o$ *of* W *satisfying* $H^1(W, \Omega(V)) = 0$ *forms a complex analytic family of submanifolds of* W *which is complete relative to* W.

PROOF. Let $\mathscr{C} = \{V_t \mid t \in N\}$ be the algebraic system of all effective divisors $V_t \sim V_o$ on W. It can be shown that the subset M of all points t on N for which V_t is non-singular and satisfies $H^1(W, \Omega(V_t)) = 0$ is a connected non-singular open manifold (Kodaira [24], Theorem 4.1). Thus the set $\mathscr{C} \mid M = \{V_t \mid t \in M\}$ forms a complex analytic family of submanifolds of W. To prove that $\mathscr{C} \mid M$ is complete relative to W at an arbitrary point, say o on M, let $\mathscr{V}' = \{V'_s \mid s \in L\}$ be a differentiable family of deformations V'_s of $V_o = V'_{o'}$ in W. Since $V'_s \sim V_o$, V'_s can be written in the form $V'_s = V_{t(s)}$, where $s \to t(s)$ is a *continuous* map of L into N. It suffices therefore to show that the map $s \to t(s)$ is differentiable in a neighborhood of $s = o'$. Let F_t be the restriction to V_t of the complex line bundle $\mathscr{F}_t = [V_t]$ on W. Moreover let $S_i(w, t)$ have the same meaning as above and let

$$\psi_{ti}(w, v) = (v \cdot S_i)(w, t)\ , \quad \text{for } v \in (T_M)_t,\ w \in V_t,\ t \in M.$$

Then $\psi_{ti}(w, v)$ represents an element $\psi_t(v)$ of $H^0(V_t, \Omega(F_t))$ and $v \to \psi_t(v)$ gives an isomorphism: $(T_M)_t \cong H^0(V_t, \Omega(F_t))$, as (12.11) shows. Hence we have

$$(v \cdot S_i)(w, t) \neq 0, \qquad \text{for general points } w \in V_t, \tag{12.13}$$

provided that $v \neq 0$. Now $t(s)$ is determined by the simultaneous equations

$$S_i(w, t(s)) = 0\ , \qquad \text{for } w \in V'_s = V_{t(s)}.$$

Hence, by (12.13), $t(s)$ is differentiable in s, q.e.d.

Now we derive an exact diagram. Changing the meaning of the symbol Ξ, we denote by Ξ the sheaf of germs of holomorphic sections of the tangent bundle *of* W instead of the bundle *induced on* $\mathscr{V}$. Clearly the restriction Ξ_t of Ξ to a submanifold V_t holds the same meaning as before. Let $\mathscr{F}_t = [V_t]$ and let F_t be the restriction of $\mathscr{F}_t$ to V_t. Then we have the exact sequences

$$0 \longrightarrow \Theta_t \xrightarrow{\iota_t} \Xi_t \xrightarrow{\mu_t} \Omega(F_t) \longrightarrow 0\ , \tag{12.14}$$

$$0 \longrightarrow \Xi \otimes \mathscr{F}_t^{-1} \longrightarrow \Xi \xrightarrow{r_t} \Xi_t \longrightarrow 0\ , \tag{12.15}$$

(12.16) $$0 \longrightarrow \Omega \longrightarrow \Omega(\mathscr{F}_t) \xrightarrow{r_t} \Omega(F_t) \longrightarrow 0$$

where $\Xi \otimes \mathscr{F}_t^{-1}$ denotes the sheaf of germs of holomorphic sections of the tensor product of the tangent bundle of W and the line bundle $\mathscr{F}_t^{-1} = [-V_t]$. The corresponding exact cohomology sequences give the exact diagram (12.17) shown on the following page.

In what follows we consider the case in which $|V_o|$ is sufficiently ample (i.e. the characteristic class $c(\mathscr{F}_o)$ of $\mathscr{F}_o = [V_o]$ is sufficiently positive). In this case the parameter variety N of the algebraic system $\mathscr{C} = \{V_t \,|\, t \in N\}$ of all effective divisors $V_t \sim V_o$ is a non-singular algebraic manifold (Kodaira [24], Theorem 3.1) and $H^1(W, \Omega(V_t)) = 0$ for all $V_t \in \mathscr{C}$. Letting M be the open subset of N of all t for which the V_t are non-singular submanifolds, we infer from Theorem 12.2 that $\mathscr{C}\,|M$ is a complex analytic family which is complete relative to W. We denote $\mathscr{C}\,|M$ by $\mathscr{V}$. We have

$$H^q(W, \Xi \otimes \mathscr{F}_t^{-1}) = 0\,, \qquad \text{for } 0 \leqq q \leqq n\,,$$
$$H^q(W, \Omega(\mathscr{F}_t)) = 0, \qquad \text{for } q \geqq 1\,,$$

where $\mathscr{F}_t = [V_t]$, $V_t \in \mathscr{V} = \mathscr{C}\,|M$ (see Kodaira [21]). Moreover the first Chern class of $V_t \in \mathscr{V}$ is negative and therefore, by a theorem of Nakano (see Section 11), $H^0(V_t, \Theta_t) = 0$. Hence we obtain from (12.17) and (12.4) the exact commutative diagram

(12.18)
$$\begin{tikzcd}
 & 0 & & & \\
0 \arrow[r] & H^0(V_t, \Xi_t) \arrow[u] \arrow[r, "\mu_t^*"] & H^0(V_t, \Omega(F_t)) \arrow[u] \arrow[r, "\delta^*"] & H^1_W(V_t, \Theta_t) \arrow[r] & 0 \\
 & H^0(W, \Xi) \arrow[u, "r_t^*"] & (T_M)_t \arrow[u, "\rho_{a,t}"] \arrow[ur, "\rho_t"'] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

In case $H^0(W, \Xi) = 0$, we infer from this diagram that ρ_t gives an isomorphism : $(T_M)_t \cong H^1_W(V_t, \Theta_t)$. Thus we obtain the following

THEOREM 12.3. *Let W be an algebraic manifold of dimension $n+1 \geqq 2$ which admits no continuous group of analytic automorphisms and let V_o be a non-singular submanifold of W of dimension n such that $|V_o|$ is sufficiently ample. Then the set $\mathscr{V}$ of all non-singular submanifolds $V_t \sim V_o$ on W forms an effectively parametrized complex-analytic family which is complete relative to W and the number of relative moduli $m_W(V_t)$ of each member $V_t \in \mathscr{V}$ is equal to* $\dim H^1_W(V_t, \Theta_t)$.

Now we consider the case in which $H^0(W, \Xi) \neq 0$. Let $\mathfrak{G}$ be the con-

(12.17)

$$\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & \\
 & & H^1(W, \Xi \otimes \mathscr{F}_t^{-1}) \arrow[u] & H^1(W, \Omega) \arrow[u] & & H^2(W, \Xi \otimes \mathscr{F}_t^{-1}) \arrow[u] & \\
0 \arrow[r] & H^0(V_t, \Theta_t) \arrow[r] & H^0(V_t, \Xi_t) \arrow[r, "\mu_t^*"] \arrow[u] & H^0(V_t, \Omega(F_t)) \arrow[r] \arrow[u] & H^1(V_t, \Theta_t) \arrow[r] & H^1(V_t, \Xi_t) \arrow[r] \arrow[u] & \cdots \\
 & & H^0(W, \Xi) \arrow[u] & H^0(W, \Omega(\mathscr{F}_t)) \arrow[u] & & H^1(W, \Omega) \arrow[u] & \\
 & & H^0(W, \Xi \otimes \mathscr{F}_t^{-1}) \arrow[u] & H^0(W, \Omega) \arrow[u] & & H^1(W, \Xi \otimes \mathscr{F}_t^{-1}) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

nected component of the identity of the complex Lie group of all analytic automorphisms of W and let γ be an element of $\mathfrak{G}$. Clearly $V_t \in \mathscr{V}$ implies $\gamma V_t \in \mathscr{V}$ and therefore γ induces an automorphism $t \to \gamma t$ of M such that $V_{\gamma t} = \gamma V_t$. Thus $\mathfrak{G}$ acts on M. Moreover we infer in the same manner as in the proof of Theorem 12.2 that $(t, \gamma) \to \gamma t$ is a holomorphic map of $M \times \mathfrak{G}$ onto M. Letting $\mathfrak{g}$ be the infinitesimal group of $\mathfrak{G}$, we have therefore the homomorphism

$$\lambda_t : \mathfrak{g} \longrightarrow (T_M)_t$$

induced by the action of $\mathfrak{G}$ on M. Moreover $\mathfrak{g}$ can be canonically identified with $H^0(W, \Xi)$ and the diagram

$$\begin{array}{ccc} H^0(V_t, \Xi_t) & \xrightarrow{\mu_t^*} & H^0(V_t, \Omega(F_t)) \\ r_t^* \uparrow & & \uparrow \rho_{d,t} \\ H^0(W, \Xi) = \mathfrak{g} & \xrightarrow{\lambda_t} & (T_M)_t \end{array}$$

is commutative. Hence we infer from (12.18) that the sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\lambda_t} (T_M)_t \xrightarrow{\rho_t} H^1_W(V_t, \Theta_t) \longrightarrow 0 \tag{12.19}$$

is exact. It follows that, for each point $t \in M$, the map $\gamma \to \gamma t$ of $\mathfrak{G}$ into M is locally biregular. Let $\mathfrak{U}$ be a small "spherical" neighborhood of 1 on $\mathfrak{G}$ and let U_t' be a complex coordinate plane on M containing t with $\dim U_t' = \dim M - \dim \mathfrak{G}$ which is transverse to the "orbit" $\mathfrak{U}t$. Then

$$U = \mathfrak{U} \cdot U' = \{\gamma t' \mid \gamma \in \mathfrak{U}, t' \in U'\}$$

is a neighborhood of t on M. We define a map $h : \mathscr{V} | U \to \mathscr{V} | U'$ by

$$h(w) = \gamma^{-1} w , \qquad \text{for } w \in V_{\gamma t'},$$

where $\gamma^{-1}w$ is the image of the point $w \in V_{\gamma t'} \subset W$ under the automorphism γ^{-1} of W. Clearly h is holomorphic and h maps $V_{\gamma t'}$ biregularly onto $V_{t'}$. Thus $\mathscr{V} | U$ is induced from $\mathscr{V} | U'$, while $\mathscr{V} | U$ is complete relative to W. It follows that $\mathscr{V} | U'$ is also complete relative to W. On the other hand we infer from (12.19) that $\mathscr{V} | U'$ is effectively parametrized. Consequently $m_W(V_t) = \dim U'$ is defined and is equal to $\dim H^1_W(V_t, \Theta_t)$. Thus we obtain the following.

THEOREM 12.4. *Let W be an algebraic manifold of dimension $n + 1 \geqq 2$ and let V_o be a non-singular submanifold of W of dimension n. If $|V_o|$ is sufficiently ample, then the number of moduli $m_W(V_o)$ of V_o relative to W exists and the equality $m_W(V_o) = \dim H^1_W(V_o, \Theta_o)$ holds.*

Finally we prove the following.

THEOREM 12.5. *Let W be an algebraic manifold of dimension $n + 1 \geqq 2$ and let $\{V's - 1 < s < 1\}$ be a one-parameter differentiable family of* *submanifolds $V's$ of W of dimension n such that $|V's|$ is sufficiently ample. If the infinitesimal deformation $\rho s\,(d/ds)$ of the family vanishes indentically, then there exists a one-parameter differentiable family $\{\gamma_s \mid -1 < s < 1\}$ of analytic automorphisms γ_s of W such that $V'_s = \gamma_s V'_0$.*

PROOF. Let $\mathscr{V} = \mathscr{C}\,|M = \{V_t \mid t \in M\}$ be the family of all non-singular submanifolds $V_t \sim V'_o$ on W. Clearly we can write $Vs = V_{t(s)}$, where $t(s)$ depends differentiably on s. Let $v_s = dt(s)/ds$ be the tangent vector of the differentiable arc $s \to t(s)$ on M. We have

$$\rho_{t(s)}(V_s) = \rho_s\left(\frac{d}{ds}\right) = 0\,.$$

In view of (12.19), there exists therefore $g_s \in \mathfrak{g}$ such that $v_s = \lambda_{t(s)}(g_s)$. Letting $s \to \gamma_s \in \mathfrak{G}$ be the solution of the differential equation

$$\frac{d}{ds}\gamma_s = g_s \cdot \gamma_s\,, \qquad \gamma_0 = 1\,,$$

we have

$$\frac{d}{ds}(\gamma_s t) = \lambda_{\gamma_s t}(g_s)\,.$$

Comparing this with

$$\frac{d}{ds}t(s) = \lambda_{t(s)}(g_s)\,,$$

we infer that $t(s) = \gamma_s t(0)$. Hence we obtain $V'_s = V_{t(s)} = \gamma_s V_{t(0)} = \gamma_s V'_0$, q.e.d.

13. Deformations of holomorphic maps into projective spaces

Let $\mathscr{V} = \{V_t \mid t \in M\}$ be a differentiable family of deformations V_t of a compact complex manifold V_o and let Φ_o be a holomorphic map of V_o into a projective space $P = P_l(C)$ of complex dimension l such that the image $\Phi_o(V_o)$ is not contained in any hyperplane on P. In what follows U will denote a sufficiently small "spherical" neighborhood of o on M. Our purpose in this sections is to show that, under certain conditions, Φ_o can be extended to a differentiable map Φ of $\mathscr{V}|U$ into P which is *holomorphic on each fibre V_t, $t \in U$.*

First we fix our notations. Denote by $\zeta = (\zeta_o, \zeta_1, \cdots, \zeta_l)$ a point on P, where $(\zeta_o, \zeta_1, \cdots, \zeta_l)$ are the homogeneous coordinates of ζ, and let $\mathfrak{U}_\nu = \{\zeta \mid \zeta_\nu \neq 0\}$. Let $e_{\lambda\nu}(\zeta) = \zeta_\nu/\zeta_\lambda$ for $\zeta \in \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ and denote by E the complex line bundle over P defined by $\{e_{\lambda\nu}(\zeta)\}$ with respect to the covering $\{\mathfrak{U}_\nu\}$. Since U is "sufficiently small", we may assume that $\mathscr{V}|U$ is covered by a finite number of "coordinate neighborhoods"

$$\mathscr{U}_j = \{(z_j^1, \cdots, z_j^n, t) \mid \Sigma_\alpha |z_j^\alpha|^2 < 1, t \in U\}$$

(see Section 1). Take a covering $\{\mathfrak{U}'_\nu \mid \nu = 0, 1, \cdots, l\}$ of P such that the closures of $\mathfrak{U}'_\nu$ are contained in $\mathfrak{U}_\nu$, resp., and, for each j, we choose $\nu(j)$ such that $\Phi_o(\mathscr{U}_j \mid V_o) \subset \mathfrak{U}'_{\nu(j)}$. Then, letting

$$e_{ojk}(z) = e_{\lambda(j)\lambda(k)}(\Phi_o(z)), \quad \text{for } z \in \mathscr{U}_j \cap \mathscr{U}_k \cap V_o,$$

we obtain a system $\{e_{ojk}(z)\}$ defining the induced bundle $E_o = \Phi_o^*(E)$ over V_o with respect to the covering $\{\mathscr{U}_j \mid V_o\}$. Let

$$\Phi_o(z) = (\varphi_{o0j}(z), \cdots, \varphi_{o\lambda j}(z), \cdots, 1_{o\nu(j)j}, \cdots, \varphi_{olj}(z)), \quad \text{for } z \in \mathscr{U}_j \mid V_o, \tag{13.1}$$

where $1_{o\nu(j)j}$ indicates that the $\nu(j)^{\text{th}}$ coordinate $\varphi_{o\nu(j)j}(z)$ equals 1. Clearly $\varphi_{o\lambda j}(z)$ are holomorphic functions of $z \in \mathscr{U}_j \mid V_o$ and

$$\varphi_{o\lambda j}(z) = e_{ojk}(z) \cdot \varphi_{o\lambda k}(z), \quad \text{for } z \in \mathscr{U}_j \cap \mathscr{U}_k \cap V_o;$$

thus $\varphi_{o\lambda} : z \to \varphi_{o\lambda}(z) = (z, \varphi_{o\lambda j}(z))$ is a holomorphic section of E_o over V_o. Under these circumstances we write the above formula (13.1) in the form

$$\Phi_o(z) = (\varphi_{o0}(z), \cdots, \varphi_{o\lambda}(z), \cdots, \varphi_{ol}(z)). \tag{13.2}$$

Now suppose that there is an extension Φ of Φ_o to $\mathscr{V} \mid U$ and let $\mathscr{E} = \Phi^*(E)$ be the bundle over $\mathscr{V} \mid U$ induced from E by the map $\Phi : \mathscr{V} \mid U \to P$. Clearly $\mathscr{E}$ is complex analytic on each fibre V_t and Φ is written in the form

$$\Phi(p) = (\varphi_0(p), \cdots, \varphi_\lambda(p), \cdots, \varphi_l(p)), \tag{13.3}$$

where $\varphi_\lambda : p \to \varphi_\lambda(p)$ are differentiable sections of $\mathscr{E}$ which are holomorphic on each fibre V_t. Moreover $\mathscr{E}$, φ_λ are extensions of E_o, $\varphi_{o\lambda}$, respectively, i.e. $\mathscr{E} \mid V_o = E_o$, $\varphi_\lambda \mid V_o = \varphi_{o\lambda}$. Conversely if we have an extension $\mathscr{E}$ of E_o to $\mathscr{V} \mid U$ which is complex analytic on each fibre V_t and extensions of $\varphi_{o\lambda}$ to differentiable sections φ_λ of $\mathscr{E}$ which are holomorphic on each fibre V_t, then

$$\Phi : p \longrightarrow \Phi(p) = (\varphi_0(p), \cdots, \varphi_\lambda(p), \cdots, \varphi_l(p))$$

is an extension of Φ_o to $\mathscr{V} \mid U$ which is holomorphic on each fibre V_t.

Let $\mathfrak{D}$ be the sheaf over $\mathscr{V} \mid U$ of germs of differentiable functions which are holomorphic on each fibre V_t and let $\mathfrak{D}^*$ be the multiplicative sheaf over $\mathscr{V} \mid U$ of germs of non-vanishing differentiable functions which are holomorphic on each fibre V_t. Then (see Section 1) we have the exact sequence

$$0 \longrightarrow \boldsymbol{Z} \xrightarrow{\iota} \mathfrak{D} \xrightarrow{\varepsilon} \mathfrak{D}^* \longrightarrow 0$$

where ι is the canonical inclusion map and ε is the exponential map :

$f \to \exp(2\pi i f)$, $f \in \mathfrak{D}$. This leads to the corresponding exact cohomology sequence

$$\cdots \longrightarrow H^1(\mathscr{V}|U, \mathfrak{D}) \xrightarrow{\varepsilon^*} H^1(\mathscr{V}|U, \mathfrak{D}^*) \xrightarrow{\delta^*} H^2(\mathscr{V}|U, \boldsymbol{Z}) \tag{13.4}$$
$$\xrightarrow{\iota^*} H^2(\mathscr{V}|U, \mathfrak{D}) \longrightarrow \cdots .$$

By Proposition 1.1, $H^1(\mathscr{V}|U, \mathfrak{D}^*)$ can be identified canonically with the group of equivalence classes of differentiable complex line bundles $\mathscr{F}$ over $\mathscr{V}|U$ which are complex analytic on each fibre V_t. In what follows we call $\mathscr{F} \in H^1(\mathscr{V}|U, \mathfrak{D}^*)$ simply a complex line bundle over $\mathscr{V}|U$. Since we are concerned with a small "spherical" neighborhood U of o, $\mathscr{V}|U$ is *differentiably* a product space: $\mathscr{V}|U = X \times U$ and thus $H^q(\mathscr{V}|U, \boldsymbol{Z})$ is canonically isomorphic to $H^q(X, \boldsymbol{Z})$; thus we may replace $H^q(\mathscr{V}|U, \boldsymbol{Z})$ in (13.4) by $H^q(X, \boldsymbol{Z})$. Now let Ω_t be the sheaf over V_t of germs of holomorphic functions and let Ω_t^* be the multiplicative sheaf over V_t of non-vanishing holomorphic functions. Obviously we have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \mathfrak{D} & \longrightarrow & \mathfrak{D}^* & \longrightarrow & 0 \\
 & & {\scriptstyle r_t}\downarrow & & {\scriptstyle r_t}\downarrow & & {\scriptstyle r_t}\downarrow & & \\
0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \Omega_t & \longrightarrow & \Omega_t^* & \longrightarrow & 0
\end{array}$$

where r_t denotes the restriction map. Since $H^q(V_t, \boldsymbol{Z}) = H^q(X, \boldsymbol{Z})$, we obtain from this the exact commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \to & H^1(\mathscr{V}|U, \mathfrak{D}) & \xrightarrow{\varepsilon^*} & H^1(\mathscr{V}|U, \mathfrak{D}^*) & \xrightarrow{\delta^*} & H^2(X, \boldsymbol{Z}) & \xrightarrow{\iota^*} & H^2(\mathscr{V}|U, \mathfrak{D}) & \to \cdots \\
 & \downarrow{\scriptstyle r_t^*} & & \downarrow{\scriptstyle r_t^*} & & \| & & \downarrow{\scriptstyle r_t^*} & \\
\cdots \to & H^1(V_t, \Omega_t) & \xrightarrow{\varepsilon_t^*} & H^1(V_t, \Omega_t^*) & \xrightarrow{\delta^*} & H^2(X, \boldsymbol{Z}) & \xrightarrow{\iota_t^*} & H^2(V_t, \Omega_t) & \to \cdots
\end{array} \tag{13.5}$$

where $\|$ denotes the identity map.

PROPOSITION 13.1. *Assume that* dim $H^2(V_t, \Omega_t)$ *is independent of* t. *Given an element* $c \in H^2(X, \boldsymbol{Z})$, *there exists a complex line bundle* $\mathscr{F} \in H^1(\mathscr{V}|U, \mathfrak{D}^*)$ *with* $\delta^*(\mathscr{F}) = c$ *if and only if* $\iota_t^*(c) = 0$ *for all* $t \in U$.

PROOF. It is clear by (13.5) that $\mathscr{F}$ with $\delta^*(\mathscr{F}) = c$ exists if and only if $\iota^*(c) = 0$, while by Theorem 2.2, (ii), $\iota^*(c) = 0$ if and only if $r_t^*\iota^*(c) = 0$ for all $t \in U$, and moreover $\iota_t^*(c) = r_t^*\iota^*(c)$ by (13.5). Hence we obtain our proposition.

PROPOSITION 13.2. *Assume that* dim $H^2(V_t, \Omega_t)$ *is independent of* t. *Let* E_o *be a complex line bundle over* V_o *and let* $c = \delta^*(E_o)$ *be the characteristic class of* E_o. *There exists an extension* $\mathscr{E} \in H^1(\mathscr{V}|U, \mathfrak{D}^*)$ *of* E_o *to* $\mathscr{V}|U$ *if and only if* $\iota_t^*(c) = 0$ *for all* $t \in U$.

PROOF. Suppose such an extension $\mathscr{E}$ exists. Then we have $c = \delta^*(E_0) = \delta^*(\mathscr{E})$ by (13.5), and hence $\iota_t^*(c) = r_t^*\iota^*(c) = r_t^*\iota^*\delta^*(\mathscr{E}) = 0$.

Suppose that $\iota_t^*(c) = 0$. By the above Proposition 13.1 there exists $\mathscr{F} \in H^1(\mathscr{V}|U, \mathfrak{D}^*)$ with $\delta^*(\mathscr{F}) = c$. Let $F_o = r_o^*(\mathscr{F})$. Then we have $\delta^*(E_o - F_o) = 0$ and therefore there exists an element $\eta_o \in H^1(V_o, \Omega_0)$ such that $\varepsilon_o^*(\eta_o) = E_o - F_o$. By Theorem 2.2, (i), the homormorphism $r_o^* : H^1(\mathscr{V}|U, \mathfrak{D}) \to H^1(V_o, \Omega_o)$ is surjective. Hence there exists an element $\eta \in H^1(\mathscr{V}|U, \mathfrak{D})$ such that $r_o^*(\eta) = \eta_o$. Now let

$$\mathscr{E} = \mathscr{F} + \varepsilon^*(\eta)\,.$$

Then we have, by using (13.5),

$$\mathscr{E} \mid V_o = r_o^*(\mathscr{E}) = r_o^*(\mathscr{F}) + r_o^*\varepsilon^*(\eta) = F_o + \varepsilon_o^*(\eta_o) = E_o\,.$$

Thus $\mathscr{E}$ is an extension of E_o, q.e.d.

Now let $E_o = \Phi_o^*(E)$ be the complex line bundle over V_o induced from E by the map $\Phi_o : V_o \to P$ and let $c = \delta^*(E_o)$ be the characteristic class of E_o.

THEOREM 13.1. *Assume that* $\dim H^2(V_t, \Omega_t)$ *is independent of* t. *If* $\iota_t^*(c) = 0$ *for all* $t \in U$ *and if* $H^1(V_o, \Omega(E_o)) = 0$, *then* Φ_o *can be extended to a differentiable map* Φ *of* $\mathscr{V}|U$ *into* P *which is holomorphic on each fibre* V_t, *provided that the neighborhood* U *of* o *is sufficiently small.*

PROOF. By Proposition 13.2, we see that there exists an extension $\mathscr{E} \in H^1(\mathscr{V}|U, \mathfrak{D}^*)$ of E_o. Let $E_t = \mathscr{E} \mid V_t$. By the principle of upper semi-continuity, we have

$$\dim H^1(V_t, \Omega(E_t)) \leqq \dim H^1(V_o, \Omega(E_o)) = 0\,, \qquad \text{for } t \in U,$$

provided that U is sufficiently small. Thus $H^1(V_t, \Omega(E_t)) = 0$ for $t \in U$, and therefore, by Theorem 2.2, (i), $r_o^* : H^0(\mathscr{V}|U, \mathfrak{D}(\mathscr{E})) \to H^0(V_o, \Omega(E_0))$ is surjective. Let $\Phi_o(z) = (\varphi_{o0}(z), \cdots, \varphi_{o\lambda}(z), \cdots, \varphi_{ol}(z))$, where $\varphi_{o\lambda} : z \to \varphi_{o\lambda}(z)$ are holomorphic sections of F_o (see (13.2)). Then there exist elements $\varphi_\lambda \in H^0(\mathscr{V}|U, \mathfrak{D}(\mathscr{E}))$ such that $\varphi_\lambda | V_o = r_o^*(\varphi_\lambda) = \varphi_{o\lambda}$, and clearly

$$p \longrightarrow \Phi(p) = (\varphi_0(p)\,, \cdots, \varphi_\lambda(p)\,, \cdots, \varphi_l(p))\,, \qquad p \in \mathscr{V}|U,$$

defines an extension Φ of Φ_o to $\mathscr{V}|U$ which is holomorphic on each fibre V_t, q.e.d.

We note that, in case V_o admits a Kähler metric, the assumption "$\dim H^2(V_t, \Omega_t)$ is independent of $t \in U$" is automatically satisfied for small U, since, by Theorem 3.1, V_t is a Kähler manifold for $t \in U$ (cf. Kodaira and Spencer [28]).

For applications it is convenient to have sufficient conditions for the existence of the extension Φ of Φ_o which refer to the fibre V_o only.

THEOREM 13.2. *If $H^2(V_o, \Omega_o) = 0$ and if $H^1(V_o, \Omega(E_o)) = 0$, then $\Phi_o : V^o \to P$ can be extended to a differentiable map $\Phi : \mathscr{V} | U \to P$ which is holomorphic on each fibre V_t, provided that the neighborhood U of o is sufficiently small.*

Proof is clear by the theorem of upper semi-continuity.

THEOREM 13.3. *Assume that V_o admits a Kähler metric. If the characteristic class $c = c(E_o)$ of $E_o = \Phi_o^*(E)$ is a rational multiple of the first Chern class c_1 of V_o and if $H^1(V_o, \Omega(E_o)) = 0$, then $\Phi_o : V_o \to P$ can be extended to a differentiable map $\Phi : \mathscr{V} | U \to P$ which is holomorphic on each fibre V_t, provided that U is sufficiently small.*

PROOF. It suffices to show that $\iota_t^*(c) = 0$ for $t \in U$. Let $\mathscr{K}$ be the complex line bundle over $\mathscr{V} | U$ defined by the system $\{J_{ik}\}$ where $J_{ik} = \det(\partial z_k / \partial z_i)$ with respect to the covering $\{\mathscr{U}_i\}$ of $\mathscr{V} | U$, and $\mathscr{U}_i$ are the coordinate neighborhoods mentioned above, and let

$$c_1 = -\delta^*(\mathscr{K}) \in H^2(X, \boldsymbol{Z}) .$$

We note that c_1 is the first Chern class of V_t for each $t \in U$. In fact, $K_t = r_t^*(\mathscr{K})$ is the canonical bundle on V_t and it follows from (13.5) that $c_1 = -\delta^*(K_t)$. Now, by hypothesis, there exists a pair of integers h, $m \neq 0$ such that $m \cdot c = h \cdot c_1$. Since $\iota^*(c_1) = \iota^*\delta^*(\mathscr{K}) = 0$, we get, by (13.5), $m \cdot \iota_t^*(c) = h\iota_t^*(c_1) = 0$ and hence $\iota_t^*(c) = 0$, q.e.d.

In conclusion we prove the following result, a generalization of Kodaira's theorem [23]:

THEOREM 13.4. *Let $\mathscr{V} \to M$ be a differentiable family of complex structures with compact fibres and suppose that $\mathscr{E} \to \mathscr{V} \to M$ is a differentiable family over $\mathscr{V}$ of positive complex line bundles (Definition 10.4) Then, given any subdomain N of M which is differentiably contractible to a point and has a compact closure in M, there exists an integer l depending on N and a differentiable map $\Phi : \mathscr{V} | N \to P_l(\boldsymbol{C})$ which is biregular on each fibre of $\mathscr{V} | N$ in the complex analytic sense..*

PROOF. Given a point $t \in M$, let $E_t = \mathscr{E} | V_t$ be the restriction of $\mathscr{E}$ to the fibre $V_t = \varpi^{-1}(t)$ of $\mathscr{V}$ and write $\mathscr{E}^h$, E_t^h respectively for the tensor products of $\mathscr{E}$ with itself h times and of E_t with itself h times. Moreover let c_1 be the first Chern class of V_t. Since we know that $H^1(V_t, \Omega(E_t^h))$ vanishes if $c(E_t^h) + c_1$ is positive on V_t (Kodaira [21], Theorem 3), it follows that, for any domain U with compact closure in M, there exists an integer $h(U)$ such that

$$H^1(V_t, \Omega(E_t^h)) = 0 , \qquad \text{for } t \in U ,\ h \geqq h(U) .$$

Then, by Theorem 2.3, $\dim H^0(V_t, \Omega(E_t^h))$ is independent of $t \in U$ and therefore, by Proposition 2.7, $\cup_{t \in U} H^0(V_t, \Omega(E_t^h))$ forms a differentiable complex vector bundle such that $H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{E}^h))$ coincides with the space of differentiable sections of $\cup_{t\in U}H^0(V_t, \Omega(E_t^h))$. If, moreover, U is differentiably contractible to a point, the bundle $\cup_{t\in U}H^0(V_t, \Omega(E_t^h))$ is trivial (by a theorem of Feldbau), and therefore we can choose $\varphi_\lambda^{(h)} \in H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{E}^h))$, $\lambda = 0, 1, \cdots, l$, in such a way that the restrictions $\varphi_{\lambda t}^{(h)} = \varphi_\lambda^{(h)}|V_t$ $(\lambda = 0, 1, \cdots, l)$ form a base of $H^0(V_t, \Omega(E_t^h))$ for each $t \in U$.

For each point $s \in M$, let $\{\psi_{s0}^{(k)}, \cdots, \psi_{s\lambda}^{(k)}, \cdots, \psi_{sl}^{(k)}\}$ be a base of $H^0(V_s, \Omega(E_s^k))$ and let

$$\Psi_s^{(k)} : p \longrightarrow (\psi_{s0}^{(k)}(p), \cdots, \psi_{sl}^{(k)}(p))$$

be the corresponding "meromorphic" map of V_s into $P_l(C)$, where l depends on s and k. Since E_s is a positive complex line bundle, there exists an integer k_s such that, for any integer $k \geqq k_s$, $\Psi_s^{(k)}$ is a biregular holomorphic map (of V_s into $P_l(C)$). Now we show that there exist a neighborhood $U(s)$ of s and an integer j_s such that $\Psi_t^{(k)}$ is biregular holomorphic for $t \in U(s)$, $k \geqq j_s$. Take a "spherical" neighborhood U of s and let

$$m_s = \max\{k_s, h(U)\} .$$

Moreover let h be an integer $\geqq m_s$. Then, by the above result, we may suppose that, for $t \in U$,

$$\psi_{t\lambda}^{(h)} = \varphi_\lambda^{(h)}|V_t , \qquad \varphi_\lambda^{(h)} \in H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{E}^h)) ,$$

while $\Psi_s^{(h)}$ is biregular holomorphic. Hence there exists a neighborhood $U^{(h)} \subset U$ of s such that $\Psi_t^{(h)}$ is biregular holomorphic for $t \in U^{(h)}$. It is clear that, if $\Psi_t^{(h)}$ and $\Psi_t^{(k)}$ are biregular holomorphic, $\Psi_t^{(h+k)}$ is also biregular holomorphic. Consequently, letting $j_s = 2m_s$ and

$$U(s) = \cap_{m_s \leqq h \leqq j_s} U^{(h)} ,$$

we infer that $\Psi_t^{(h)}$ is biregular holomorphic for $t \in U(s)$, $h \geqq j_s$.

Let N be the subdomain of M satisfying the conditions stated in the theorem and let h be a sufficiently large integer. We choose $\varphi_\lambda^{(h)} \in H^0(\mathscr{V}|U, \mathfrak{O}(\mathscr{E}^h))$, $\lambda = 0, 1, \cdots, l$, such that $\varphi_{\lambda t}^{(h)} = \varphi_\lambda^{(h)}|V_t$ form a base of $H^0(V_t, \Omega(E_t^h))$ for each $t \in N$. Then, by the above result,

$$\Phi : p \longrightarrow (\varphi_0^{(h)}(p), \cdots, \varphi_\lambda^{(h)}(p), \cdots, \varphi_l^{(h)}(p))$$

is a differentiable map of $\mathscr{V}|N$ into $P_l(C)$ which is biregular holomorphic on each fibre V_t, $t \in N$, q.e.d.