Intersection Theory

Notation:

- Scheme : algebraic scheme / K (i.e. $X \to \operatorname{Spec} K$ finite type)
- variety : integral scheme
- subvariety : closed subscheme which is variety
- point : closed point.

§ 1.2

- X : var
  $\cup\!|$
  V subvar of codim 1
- $r \in R(X)^*$ ← field of rational function
- $A = \mathcal{O}_{V,X}$ is 1-dim local domain.

localization of $\mathcal{O}_X$ at generic point of V.

Def

$$\boxed{\begin{aligned} \operatorname{ord}_V : R(X)^* &\to \mathbb{Z} \\ r = \frac{a}{b} &\mapsto \operatorname{ord}_V(r) := l_A(A/(a)) - l_A(A/(b)) \end{aligned}}$$

Rmk: if X nonsingular along V, then $\operatorname{ord}_V(\cdot)$ coincide with discrete valuation on $\mathcal{O}_{V,X}$.

This time $(A, m) = (\mathcal{O}_{V,X}, m_V)$
ideal in A is of the form
$\{ m^k = (t^k) \mid k \geq 0 \}$

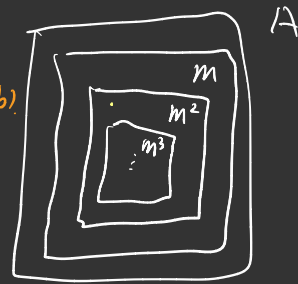
$\mathcal{O}_{V,X}$ is 1-dim regular local ring
$\Updownarrow$
DVR $\Leftrightarrow$ PID

$$v: A \longrightarrow \mathbb{Z} \xrightarrow{\text{extend}} v: R(A) \longrightarrow \mathbb{Z}$$

$$a \longmapsto \max_k \{ k \geq 0 \mid a \in m^k \} \qquad \frac{a}{b} \longmapsto v(a)-v(b)$$

$$\underbrace{a}_{} = ut^k \longmapsto k$$

unit



Def of length:

- M: A-mod.

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_r = 0.$$

If M_i / M_{i+1} are simple modules, then $l_A(M) := r$.

$l_A(M)$ is independent with choice of chain.

By Jordan Hölder Theorem: appearance of M_i/M_{i+1} independent with the choice of chain.

Stack project:

Lemma 10.52.10. *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) M *is simple,*

(2) $\text{length}_R(M) = 1$, *and*

(3) $M \cong R/\mathfrak{m}$ *for some maximal ideal* $\mathfrak{m} \subset R$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 10.52.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. $\square$

M

Lemma 10.52.5. *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. □

Lemma 10.52.6. *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m}M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak{m}$ vector space. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 10.52.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module.

Lem 10.52.6

- $R \supseteq m$ maximal ideal.
- $mM = 0$
- $R \longrightarrow R/m$, M can be consider as R/m-mod.

By Lem 10.52.5, $l_R(M) = l_{R/m}(M) = \dim_{R/m}(M)$.

Lem A.1.1

$0 \to M' \to M \to M'' \to 0$ exact sequence of finite length A-mod.

Then $l_A(M) = l_A(M') + l_A(M'')$.

Lem A.1.2

If M has finite length, then $l_A(M) = \sum_{\substack{m \subseteq A \\ \text{maximal}}} l_{A_m}(M_m)$.

Sketch: Assume $\boxed{M_i / M_{i+1} \simeq A/m'}$ for some i. when localize at m'', then $(A/m')_{m''} = 0$. ($\because$ m' contain a unit element in local ring $A_{m''}$)

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \cdots \supsetneq M_r = 0.$$

$$0 \to M_1 \to M_0 \to M_0/M_1 \to 0$$

$$0 \to M_2 \to M_1 \to M_1/M_2 \to 0.$$

Then

add
$$\begin{cases} \boxed{l(M_0)} = \cancel{l(M_1)} + l(M_0/M_1) \\ \cancel{l(M_1)} = \cancel{l(M_2)} + l(M_1/M_2) \\ \quad\vdots \end{cases}$$

Lem A.1.3

- $\varphi : A \to B$ local homomorphism of local ring.
- $[B/m_B : A/m_A] = d$
- M : B-mod

Then M is finite length of B-mod iff

M is finite length of A-mod & $d < \infty$.

Moreover, $l_A(M) = d \cdot l_B(M)$

Pf: Reduced to case $M \simeq B/m_B$.

Example A.1.1

- $k \hookrightarrow A$ local ring
- residue field $A/\mathfrak{m} \simeq k$

$\Big\}$ i.e. $\mathfrak{m}$ is k-rational point in $X = \operatorname{Spec} A$.

Then $l_A(M) = l_k(M) = \dim_k M$.

Example: $A = \mathbb{C}[t]_{(t)}$, $a = t^3$

$$l_A(A/(a)) \underset{\text{Ex A.1.1}}{=} l_k(A/(a)) = \dim_k (A/(a)) = \dim \langle 1, t, t^2 \rangle_k$$

$= 3$.

Example 2: $A = \mathbb{C}[t]_{(t)}$, $a = t^3 - t^2 = t^2(t-1)$

$V_0 := \{t=0\} \subseteq \mathbb{A}^1_{\mathbb{C}}$

$V_1 := \{t=1\} \subseteq \mathbb{A}^1_{\mathbb{C}}$

$$\begin{aligned}
\operatorname{ord}_{V_0}(a) = l_{V_0}\left(A/(a)\right) &= l_{V_0}\left(A/(t^2(t-1))\right) \\
&= l_{V_0}\left(A/(t^2)\right) \\
&= \dim_k\left(A/(t^2)\right) \\
&= 2.
\end{aligned}$$

$$\begin{aligned}
\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[t]}{t^2(t-1)}\right) = 3 = 2+1 &= \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[t]}{t^2(t-1)}\right)_{(t)} + \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[t]}{t^2(t-1)}\right)_{(t-1)} \\
&= \operatorname{ord}_{V_0}(a) + \operatorname{ord}_{V_1}(a)
\end{aligned}$$

$\|?$

$$\boxed{l_{\mathbb{C}[t]}\left(\frac{\mathbb{C}[t]}{t^2 - t^3}\right)}$$

localization prop.

✓

Def A.2

- $\varphi: M \to M$ A-mod homomorphism of f.g. A-mod.

$$
\begin{array}{ccccccccccc}
0 \to & \operatorname{Ker}\varphi & \to & M & \xrightarrow{\varphi} & M & \to & \operatorname{coker}\varphi & \to 0 \\
& \| & & & & & & \| & \\
& {}_{\varphi}M & & & & & & M_{\varphi} &
\end{array}
$$

define $e_A(\varphi, M) := l_A(M_\varphi) - l_A({}_\varphi M)$

If both finite.

Rmk: In application, φ usually is $\cdot a: M \to M$, $x \mapsto ax$ for some $a \in A$ non zero-divisor, then $e_A(\cdot a, M) = l_A(M/aM)$

lemma.5

- $M \xrightarrow{\psi} M \xrightarrow{\varphi} M$ A-mod hom.

If two of $e_A(\varphi, M), e_A(\psi, M), e_A(\varphi\psi, M)$ is defined, so is the third, moreover,

$$e_A(\varphi\psi, M) = e_A(\varphi, M) + e_A(\psi, M)$$

Rmk: $\begin{cases} \varphi = \cdot a \\ \psi = \cdot b \end{cases}$

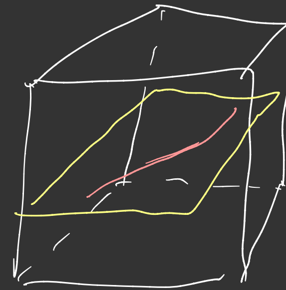
$M = A$: 1-dim domain

$$\begin{array}{ccccc} e_A(\cdot a \cdot b, A) & = & e_A(\cdot a, A) & + & e_A(\cdot b, A) \\ \| & & \| & & \| \\ l_A(A/(ab)) & & l_A(A/(a)) & & l_A(A/(b)) \\ \| & & \| & & \| \\ \mathrm{ord}_A(ab) & & \mathrm{ord}_A(a) & & \mathrm{ord}_A(b) \end{array}$$

$\therefore$ $\mathrm{ord}_A : R(A)^* \longrightarrow \mathbb{Z}$ group homomorphism.

$$\frac{a}{b} \longmapsto \mathrm{ord}_A(a) - \mathrm{ord}_A(b)$$

§ 1.3:



Def:

- X : Sch
- $Z_k X :=$ group of k-cycle : finite formal sum $\sum n_i [V_i]$

  $n_i \in \mathbb{Z}$, $V_i \subset X$ k-dim subvar.

- k-cycle rational equivalent to 0 : $\alpha \sim 0$.

  if $\exists$ finite many $W_i \subset X$ subvar of dim $k+1$, $r_i \in R(W_i)^*$,

  such that $\alpha = \sum_i [\operatorname{div}(r_i)] = \sum_i \sum_V \operatorname{ord}_V(r_i)[V]$.

- $\operatorname{Rat}_k(X) := \{\alpha \in Z_k X \mid \alpha \sim 0\} \subseteq Z_k X$ subgroup.
- $A_k X := Z_k X / \operatorname{Rat}_k X$
- $A_* X = \bigoplus_{k=0}^{\dim X} A_k X = Z_* X / \operatorname{Rat}_* X$

Assume

E : genus > 0 proj curve.

If $p \sim q$, then

$\exists f \in R(E)^*$, s.t $\operatorname{div}(f) = p - q$.

$$E \xrightarrow{f} \mathbb{P}^1 \quad \deg 1.$$
$$p \longmapsto 0$$
$$q \longmapsto \infty$$

$\Rightarrow E \simeq \mathbb{P}^1$ 矛盾.

§1.4 pushforward of cycles

Def: $f: X \longrightarrow Y$ proper

closed integral subvar $V \subseteq X$, $V \longrightarrow f(V) =: W \subseteq Y$

define $\deg(V/W) := \begin{cases} [R(V):R(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V \end{cases}$

define $f_*: Z_k X \longrightarrow Z_k Y$

$[V] \longmapsto f_*[V] := \deg(V/W)[W]$

Prop 1.4

$f: X \underset{\text{proper}}{\overset{\text{surj}}{\longrightarrow}} Y$, $r \in R(X)^*$

Then

(a) $f_*[\operatorname{div}(r)] = 0$ $\dim Y < \dim X$

(b) $f_*[\operatorname{div}(r)] = [\operatorname{div} N(r)]$ $\dim Y = \dim X$

projection formula:

$f^* D \cdot C = D \cdot f_* C$

where $N(r)$ is determinant of $\cdot r: R(X) \longrightarrow R(X)$ as $R(Y)$-linear hom

Therefore $f_*: A_k X \longrightarrow A_k Y$ is well-defined.

pf of Therefore: If $\alpha \sim 0 \in Z_k X$, $\exists\, W_i \subseteq X$ $\dim k+1$, $r_i \in R(W_i)^*$

$\alpha = \sum [\operatorname{div} r_i]$, $f|_{W_i}: W_i \longrightarrow f(W_i) \subseteq Y$.

Then by prop 1.4, $f_* \alpha = \sum f|_{W_i *} [\operatorname{div} \gamma_i]$
$$= \sum \{ [\operatorname{div}[N(\gamma_i)]] \text{ 或 } 0 \}.$$

pf of prop 1.4

(Case 1) $f: \mathbb{P}^1_K \to Y = \operatorname{Spec} K$.

$$R(X) = K(t), \quad \gamma \in R(X)^*$$

can Assume γ is irr polynomial in $K[t]$ of deg d.

then $[\operatorname{div}(\gamma)] = [P] - d[P_\infty]$

prime ideal (γ)

Now $[R(p) = K[t]/p : K] = d$

$$\therefore f_*[\operatorname{div}(\gamma)] = f_*[P] - d f_*[P_\infty]$$
$$= d[Y] - d[Y]$$
$$= 0$$

(Case 2) f is finite morphism

Lemma A.3. *Let A be a one-dimensional domain with quotient field K. Let $\varphi: M \to M$ be an endomorphism of a finitely generated A-module, and let φ_K be the induced endomorphism of $M_K = M \otimes_A K$. If* $\det(\varphi_K) \neq 0$, *then*

$$e_A(\varphi, M) = \operatorname{ord}_A(\det(\varphi_K)).$$

Proof. By Lemmas A.2.1 and 2.4, we may replace M by M/M', where M' is the torsion submodule of M; i.e., we may assume M imbeds in $M \otimes_A K$. Choosing a basis for $M \otimes_A K$ from elements in M, one constructs a free submodule F of M with $F \otimes_A K = M \otimes_A K$. Choose a common denominator a for a matrix for φ, so that $a\varphi(F) \subset F$. Since $(M/F) \otimes_A K = 0$, M/F has finite length, so

$$e_A(a\varphi, M) = e_A(a\varphi, F)$$

by Lemmas A.2.1 and 2.4 again. Similarly $e_A(a, M) = e_A(a, F)$. Then, using Lemma A.2.5,

$$e_A(a\varphi, M) = e_A(a, M) + e_A(\varphi, M) = e_A(a, F) + e_A(\varphi, M) = \operatorname{ord}_A(a^n) + e_A(\varphi, M),$$

where n is the rank of F. By Lemma A.2.6,

$$e_A(a\varphi, F) = \operatorname{ord}_A(\det(a\varphi)) = \operatorname{ord}_A(a^n) + \operatorname{ord}_A(\det(\varphi_K)).$$

Comparing these equations gives the lemma. □

general case (a)

can assume $\dim(Y) = \dim X - 1$, $K := R(Y)$, $g \in R(\tilde{X}_K)$

$$f_{K*}[\operatorname{div}(r)]$$
$$= f_{K*} h_*[\operatorname{div} \tilde{r}]$$

I guess $\tilde{r} = \sqrt[n]{r}$.

$$= p_* g_* [\operatorname{div} \tilde{r}]$$
$$= p_* \begin{cases} 0 & \text{if } g \text{ constant} \\ [\operatorname{div} N(\tilde{r})] & \text{if } g \text{ is finite} \end{cases}$$

for curve

$$\begin{tikzcd}
& & \tilde{X}_K \ \tilde{r} \arrow[dll, "g"'] \arrow[d, "h", "\text{normalization}"'] & \\
\mathbb{P}^1_K \ N(\tilde{r}) \arrow[drr, "p"'] & & X_K = X \times_Y K \arrow[r, "i_X"] \arrow[d, "f_K"] & X \arrow[d, "f"] \\
& & \operatorname{Spec} R(Y) \arrow[r] & Y
\end{tikzcd}$$

$R(X_K)^* \ni r$, $r \in R(X)^*$

case 1

$= 0$

$$i_{X*}[\operatorname{div}(r)] = [\operatorname{div}(r)]$$

general case of (b)

$\dim X = \dim Y$
$\dim \tilde{X} = \dim \tilde{Y}$
$\dim W$

$R(W) = R(\tilde{X})$

general fiber is a point $\Rightarrow g$ birational

fiber connected

stein decomposition

$$\begin{tikzcd}
& W \ \tilde{r} \arrow[dr, "h"] & & \\
\tilde{X} \arrow[ur, "g"] \arrow[rr, "\tilde{f}"] & & \tilde{Y} \arrow[d] & N(\tilde{r}) \\
X \arrow[rr, "f"] & & Y & N(N\tilde{r})
\end{tikzcd}$$

finite

$\sqrt[n]{r} = \tilde{r}$

normalization

r

$$f_*[\operatorname{div}(r)] = [\operatorname{div}(N(N\tilde{r}))]$$

§2.2

$$\mathrm{Div}(X) \longrightarrow Z_{n-1}(X)$$

group of Cartier divisor

$$D = (U_\alpha, f_\alpha) \longmapsto [D] := \sum_{V:\text{ codim } 1} \mathrm{ord}_V D \cdot [V]$$

$f_\alpha \in R(U_\alpha)$

finite sum.

$$:= \sum_{V:\text{ codim } 1} \mathrm{ord}_V (f_\alpha) \cdot [V]$$

Induce $\rightsquigarrow$ $\mathrm{Pic}(X) = \mathrm{Div}(X) / \sim$ linear equ $\longrightarrow A_{n-1}(X) = Z_{n-1}(X) / \sim$ rational equ.

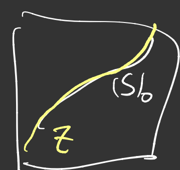
may not inj or surj.

example in [Har II §6]

<u>Def</u>:

- pseudo-divisor (L, Z, s)

  L: line bundle; Z: closed subset in X; s: section of L nowhere vanishing on $X - Z$.



- $(L, Z, s) = (L', Z', s')$ iff

① $L \simeq L'$

② $Z = Z'$

③ $L \xrightarrow[\simeq]{\exists\, \sigma} L'$, such that $L|_{X-Z} \xrightarrow[\simeq]{\sigma|_{X-Z}} L|_{X'-Z'}$

$$s \longmapsto s'$$

<u>Rmk</u>: if s nowhere vanishing on X, then

$$\varphi : \mathcal{O}_X|_U \xrightarrow{\simeq} L|_U$$
$$1|_U \longmapsto s|_U$$

Rmk:

① If $Z = X$, then pseudo-divisor is isomorphism classes of line bundle.

② Given Cartier div D, it determines pseudo-div

$$\left(\mathcal{O}_X(D),\ \underset{\text{supp } D^+ \cup \text{supp } D^-}{\overset{\shortparallel}{\text{supp } D}} = |D|,\ s_D \right)$$

locall is f_α if $D = (U_\alpha, f_\alpha)$

Def Cartier divisor D <u>represends</u> a pseudo divisor (L, Z, s) iff

① $|D| \subset Z$

② $\mathcal{O}_X(D) \xrightarrow[\simeq]{\exists \sigma} L$, such that $\mathcal{O}_X(D)|_{X-Z} \xrightarrow{\simeq} L|_{X-Z}$, $s_D \longmapsto s$

Lem 2.2

$\forall$ pseudo-div (L, Z, s) is represented by Cartier divisor.

(a) If $Z \neq X$, D is uniquely determined.

(b) If $Z = X$, D is determined up to linear equivalence.

Pf: (existence)

- $g_{\alpha\beta}$: transition function of L
- $f_\alpha := g_{\alpha\alpha_0}$ for some fixed α_0.

then $\frac{f_\alpha}{f_\beta} = \frac{g_{\alpha\alpha_0}}{g_{\beta\alpha_0}} = g_{\alpha\beta} \in R^*(U_\alpha \cap U_\beta)$

$\therefore (U_\alpha, f_\alpha)$ is Cartier divisor D with $O_X(D) \simeq L$.

If $Z \neq X$, $U := X - Z$, section s locally given by regular function s_α on $U \cap U_\alpha$, s.t. $s_\alpha = g_{\alpha\beta} s_\beta$.

let s_D determined by f_α, then $\frac{s_\alpha}{f_\alpha} = \frac{g_{\alpha\beta} s_\beta}{g_{\alpha\beta} f_\beta} = \frac{s_\beta}{f_\beta}$

Then $\exists\, r \in R(X)^*$, s.t. $r = \frac{s_\alpha}{f_\alpha}$ on $U \cap U_\alpha$ for all α.

let $D' := D + \operatorname{div}(r)$, then

$(O_X(D'), |D'|, s_{D'})$ represends (L, Z, s)

locally $\| \quad f_\alpha \cdot r = f_\alpha \cdot \frac{s_\alpha}{f_\alpha} = s_\alpha$.

Uniqueness when $U \neq \phi$, if $s_{D'} = s_D$, then $f'_\alpha = f_\alpha$ on U, hence on X, thus $D' = D$

$\square$

Def 2.2.2

- $D=(L,Z,s)$ pseudo-div rep by Cartier div $\tilde{D}$

define $[D]:=[\tilde{D}]\in A_{n-1}(|D|)$

Weil divisor class

$=$

- $D+D':=(L\otimes L', Z\cup Z', s\otimes s')$

$-D:=(L^{-1}, Z, \frac{1}{s})$

- $f: X'\longrightarrow X$

$D=(L,Z,s)$

$f^*D:=(f^*L, f^{-1}(Z), f^*s)$

Rmk: Assume $Z\neq X$, D is rep by Cartier div $\tilde{D}$

① $f^{-1}Z\neq X'$, f^*D is uniquely rep by $f^*\tilde{D}$

② $f^{-1}Z=X'$, then $f^*D=(f^*L, X', f^*s=0)$ is only determined up to linear equivalence.

§2.3 Intersection with divisors.

Def 2.3

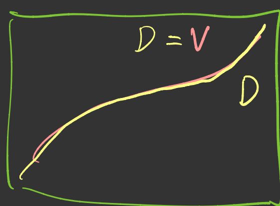
- $V \overset{j}{\hookrightarrow} X$ k-dim closed subvar
- D : pseudo-div

Define $D\cdot[V] \overset{\text{denote}}{=\!=} D\cdot V := [j^*D] \in A_{k-1}(|D|\cap V)$

Intersection.

$j^*D = j^*\mathcal{O}_X(D) \in \operatorname{Pic} V$

$[j^*\mathcal{O}_X(D)] \quad A_{k-1}(V)$



Prop 2.3 α, α' k-cycles on X.

(a) $D\cdot(\alpha+\alpha') = D\cdot\alpha + D\cdot\alpha' \in A_{k-1}(|D|\cap(|\alpha|\cup|\alpha'|))$

(b) $(D+D')\alpha = D\cdot\alpha + D'\cdot\alpha \in A_{k-1}((|D|\cup|D'|)\cap|\alpha|)$

(c) projection formula $f: \overset{\alpha}{X'} \to X^{D}$ proper

$g = f|_{f^{-1}(|D|\cap f|\alpha|)} : f^{-1}(|D|\cap f|\alpha|) \to |D|\cap f|\alpha|$

Then $g_*(f^*D\cdot\alpha) = D\cdot f_*(\alpha) \in A_{k-1}(|D|\cap f|\alpha|)$

$A_{k-1}(X)$

(d) compatible with flat pull-back. $f^*: A_k(X) \to A_{k+n}(X')$

$\alpha \mapsto f^*\alpha$

$f: X' \to X$ (α, D on X) flat rel dim n.

$g := f|_{f^{-1}(|D| \cap |\alpha|)} \quad f^{-1}(|D| \cap |\alpha|) \to |D| \cap |\alpha|$

Then $\quad f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \in A_{k+n-1}(f^{-1}(|D| \cap |\alpha|))$

(e) If $\mathcal{O}_X(D) \simeq \mathcal{O}_X$, then $D \cdot \alpha = 0 \in A_{k-1}(|\alpha|)$

(e) $\left(\because [j^*D] = [\mathcal{O}_\alpha] = 0 \in A_{k-1}(|\alpha|) \right.$

$\left. \mathcal{O}_\alpha = 0 \in \operatorname{Pic} X \right)$

Sketch of (c): $\quad f: X' \to X$ ($\alpha = V$ on X', D on X) proper $d = [R(V) : R(f(V))]$

Can assume $\alpha = [V]$ with $V = X'$, to show.

$f_*(f^*D \cdot \alpha) := f_*[f^*D] \overset{\text{want}}{=\!=} \deg(X'/X)[D] = [\deg(X'/X) \cdot D]$

$=: f_*\alpha \cdot D$

Locally: $D = \operatorname{div} r$, then

$\text{LHS} = f_*[\operatorname{div}(f^*r)] = [\operatorname{div} N(f^*r)] = [\operatorname{div}(r^d)]$

$= d[\operatorname{div}(r)] = \text{RHS}$

$\square$

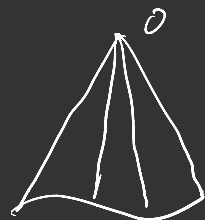
§6 Intersection products

§4.

<u>Cone</u>: X: sch, $S^{\cdot}$: sheaf of graded $\mathcal{O}_X$-algebra

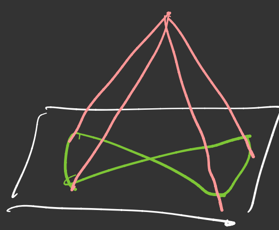
$$C := \operatorname{Spec}(S^{\cdot})$$

Ex: $\mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$



projective compeletion.

$$\overline{C}_{\mathbb{P}^n} = \mathbb{P}\mathrm{roj}(S^{\cdot}[z])$$



Ex: $C = (x^2 - yt = 0) \subseteq \mathbb{A}^3$.

$$S^{\cdot} = \frac{k[x,y,t]}{(x^2-yt)}$$

$$S^{\cdot}[z] = \frac{k[x,y,t,z]}{(x^2-yt)}$$

$$\overline{C}_{\mathbb{P}^3} = \mathbb{P}\mathrm{roj}(S^{\cdot}[z]) = \mathrm{proj}\ k[x,y,t,z]/(x^2-yt)$$

Normal cone $X \underset{\text{closed}}{\hookrightarrow} Y$ proj sch defined by ideal sheaf J, $C_X Y := \operatorname{Spec} \bigoplus_{n=0}^{\infty} \frac{J^n}{J^{n+1}}$.

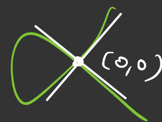
$\boxed{\text{Statement}}$ when X is point, $C_X Y$ is tangent cone.

Def: $X \hookrightarrow Y$ codimension d, then $\forall p \in X$, $\exists U \subset Y$ open ($p \in U$), s.t. X defined by I: gen by length d regular seq.

$$\boxed{\text{regular imbedding} \Longleftrightarrow \text{local complete intersection ?}}$$

Ex $X = (0,0) \subset Y = (x^3 + x^2 - y^2 = 0)$

$$(\mathcal{O}_{Y,x}, m) = \left(\left(\frac{k[x,y]}{(x^3+x^2-y^2)} \right)_{(x,y)}, \ (x,y)_{(x,y)} \right)$$



$$C_X Y = \operatorname{Spec} \bigoplus_{n=0}^{\infty} \frac{m^n}{m^{n+1}}$$

$$= \operatorname{Spec} \underset{\substack{\cong \\ k}}{\frac{\mathcal{O}_{Y,x}}{m}} \oplus \underset{\bar{x},\, \bar{y}}{\boxed{\frac{m}{m^2}}} \oplus \underset{\bar{x}^2,\, \bar{y}^2,\, \bar{x}\bar{y}}{\boxed{\frac{m^2}{m^3}}} \oplus \cdots$$

$$\bar{x}^2 - \bar{y}^2 \equiv \bar{x}^2 - \bar{y}^2 - \bar{x}^3 \mod m^3$$

$$= 0.$$

$$= \operatorname{Spec} \frac{k[\bar{x}, \bar{y}]}{(\bar{x}^2 - \bar{y}^2)} = \operatorname{Spec} \frac{k[\bar{x}, \bar{y}]}{(\bar{x}+\bar{y})(\bar{x}-\bar{y})}$$

Tangent cone:

$X = \operatorname{Spec} A/I \subset Y$

Ψ

0

$y^2 - x^3 = 0$

tangent cone $(y^2 = 0)$

$I = (f = 0)$ where $f = a_{11}x^2 + a_{12}xy + a_{22}y^2 + (\text{higher term})$

then tangent cone at 0 is $\operatorname{Spec} A / (a_{11}x^2 + a_{12}xy + a_{22}y^2)$

if $0 \in X$ smooth,

$A = k[x,y]$ then $f = a_1 x + b_1 y + (\text{higher terms})$

$$k[x,y]/(a_1x + b_1y) \simeq \mathbb{A}^1$$

$$\dim_{A_p/m_p} m_p/m_p^2 = 1$$

Construction of Intersection product §6.1

$N := g^* C_X Y$

$$\begin{tikzcd} W = X \times_Y V \arrow[r, "i"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

f morphism

i: closed regular imbedding defined by J

W defined by $\mathcal{J}$ in V.

if f closed imbedding then W may be $X \cap V$.

$$\begin{tikzcd} \dfrac{A/I}{J \otimes A/I} = A/J \otimes A/I & A/I \arrow[l] \\ A/J \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$J \otimes A/I \to \mathcal{J} = J \cdot A/I$

$0 \to J \to A \to A/J \to 0$

$\otimes A/I$

$J \otimes A/I \xrightarrow{f} A/I \xrightarrow{g} A/J \otimes A/I \to 0$

$\operatorname{Im} f$: $J \cdot A/I = \ker g$

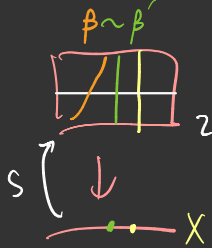
$$\bigoplus_n f^*\left(J^n / J^{n+1}\right) \twoheadrightarrow \bigoplus_n \left(\mathcal{J}^n / \mathcal{J}^{n+1}\right)$$

$$\begin{tikzcd} C_W V \arrow[rr, hook, "\text{closed imbedding}"] \arrow[dr] & & N \arrow[dl, "\text{flat}"] \\ & W & \end{tikzcd}$$

cone: $C_W V$

N: vector bundle

Thm 3.3 (a)



$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array} \quad E \text{ rk } r \text{ vect bundle}$$

Then flat pull back $\pi^*: A_{k-r}X \xrightarrow[\text{iso}]{\sim} A_k E$

Def 3.3:

$$s^*: A_k E \longrightarrow A_{k-r} X$$
$$\beta \longmapsto (\pi^*)^{-1}(\beta)$$

Def:

$$\underbrace{X \cdot V = X \cdot_Y V = i^![V]}_{\text{notation}} := s^*[C] \in A_* W$$

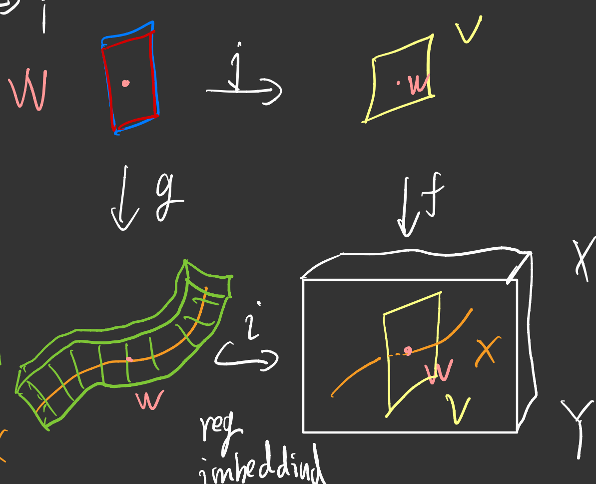
$$\begin{array}{ccc} W = X \times_Y V & \longrightarrow & V \\ \downarrow & & \downarrow f \\ X & \xhookrightarrow[i]{} & Y \end{array}$$

cone $\quad$ bundle

$$C = C_W V \hookrightarrow N \supset [C]$$
$$\searrow \quad \swarrow \curvearrowright s: \text{zero section}$$
$$W$$

$N = g^* C_X Y$

$C_W V$



$C_X Y$ vect bun

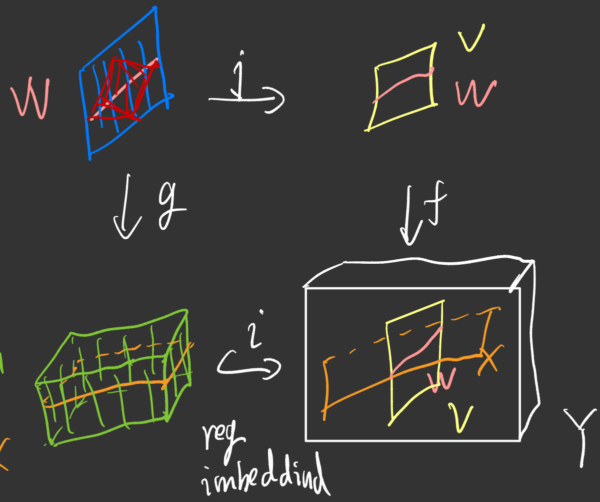
X

$$X \cdot V = s^*[C] = W$$
$$\in A_0 W$$

$N = g^* C_X Y$

$C_W V$

$C_X Y$ vect bun



$$X \cdot V := s^*[C]$$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr] & N \arrow[d, "\pi"'] \\ & W \arrow[u, bend right, "s"'] \end{tikzcd}$$