A question on base locus of real quadratic forms for n=2,4,8,16

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0 The statement and remarks

A conjecture that is formulated by us is introduced in this notes. We first state it and give some explanations on its background. For the sake of simplicity, but with somewhat abuse of notation, we say a vector space of $n \times n$ matrices W_n (resp. $n \times n$ symmetric matrices $W_{n,sym}$) is **intvertible** if each nonzero element in W_n (resp. $W_{n,sym}$) is invertible. We shall denote $\mathbb{R}^{n\times n}$ the space of $n \times n$ real matrices, and $\mathbb{R}^{n\times n}_{sym}$ the space of $n \times n$ symmetric real matrices.

Conjecture 0.1 (Quadrics base locus conjecture)**.** *For $n = 2, 4, 8, 16$, let $W_{n,sym}$ be an invertible linear subspace of $\mathbb{R}^{n\times n}_{sym}$ with dimension $\dim(W_{n,sym}) = \frac{1}{2}n + 1$. Take any basis $\{A_1, \cdots, A_{\frac{1}{2}n+1}\}$ of $W_{n,sym}$. Then,*

$$\begin{cases} \mathbf{x}^T A_1 \mathbf{x} & = 0 \\ \cdots \\ \mathbf{x}^T A_{\frac{1}{2}n+1} \mathbf{x} & = 0 \end{cases}$$

has no solution for $\mathbf{x} \in \mathbb{R}^n - \{0\}$.

Remark 0.2.

1. There are other ways to phrase the conjecture. First, instead of considering a system of quadratic equations, one can state that the quatic (degree 4) equation

$$\sum_{1\le i\le \frac{1}{2}n+1} (\mathbf{x}^T A_i \mathbf{x})^2 = 0 \quad \text{has no nonzero real solution.}$$

   We can also consider the *linear system of subvarieties*. The conjecture is equivalent to the following.

   For $n = 2, 4, 8, 16$, if the $\dfrac{1}{2}n$ dimensional linear system $\mathcal{L}$ of quadratic hypersurfaces (quadrics) in $\mathbb{R}\mathbf{P}^{n-1}$ has no singular member, then $\mathcal{L}$ has no real base locus.

   From this point of view, we see that even a basis is chosen when phrasing the conjecture, the validity of it **does not depend on the basis**, only possibly depends on the linear space $W_{n,sym}$.

   Notice also for complex projective variety, any nonzero dimensional linear system $\mathcal{L}$ of quadrics will have singular member, since the space of all quadrics in $\mathbf{P}^{n-1}$ is $\mathbf{P}^{\frac{n(n+1)}{2}-1}$. Then $\mathcal{L} \subset \mathbb{C}\mathbf{P}^{\frac{n(n+1)}{2}-1}$ has to intersect the hypersurface $\det(A) = 0$, with A being symmetric matrices. The real case will be very different, as we shall explain as follows.

2. Such a $\frac{1}{2}n + 1$ dimensional W (whose each nonzero element is invertible) exists. It can be constructed as the space of matrices of the form $\begin{bmatrix} rI & A \\ A^T & rI \end{bmatrix}$ for $r \in \mathbb{R}$, $A \in \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (as matrix representation). The fact that $\frac{1}{2}n + 2$ dimensional such W **does not exist** involves homotopy theory and Bott periodicity, as proved in [1] [2] [3].

Proposition 1. *Conjecture 0.1 holds for $n = 2, 4$.*

Proof. The $n = 2$ case can be argued by simple linear algebra. Suppose that $\mathbf{x}_0 \in \mathbb{R}^2$ satisfies $\mathbf{x}_0^T A_1 \mathbf{x}_0 = 0$, $\mathbf{x}_0^T A_2 \mathbf{x}_0 = 0$ with $\{A_1, A_2\}$ being a basis of $W_{2,sym}$. Then for any $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathbf{x}_0^T(\lambda_1 A_1 + \lambda_2 A_2)\mathbf{x}_0 = 0$, which means the vector $(\lambda_1 A_1 + \lambda_2 A_2)\mathbf{x}_0$ is perpendicular to $\mathbf{x}_0$ in $\mathbb{R}^2$. Thus $\lambda_1 A_1 \mathbf{x}_0$ and $\lambda_2 A_2 \mathbf{x}_0$ are collinear, which implies the existence of λ_1, λ_2 not all zero but $(\lambda_1 A_1 + \lambda_2 A_2)\mathbf{x}_0 = 0$. This contradicts the assumption that $(\lambda_1 A_1 + \lambda_2 A_2) \in W_{2,sym}$ is invertible.

The $n = 4$ case is known in [8, last line of Table 1], which resembles the historical research in the nineteenth century on the 28 bitangents of the quartic curves in $\mathbf{P}^2$ (both complex and real). There is also another proof using **euler characteristic** due to Sergey Galkin. □

1 Background

The equation we are considering is **von Karman system** over a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} \text{Find} \quad & v : \Omega \longrightarrow \mathbb{R}, \quad w : \Omega \longrightarrow \mathbb{R}^n \\ \text{satisfying} \quad & \frac{1}{2}\nabla v \otimes \nabla v + \frac{1}{2}(\nabla W + (\nabla w)^T) = A \end{aligned} \tag{1.1}$$

for given $A \in C^\infty(\Omega, \mathbb{R}^{n\times n}_{sym})$.

It is not hard to see its relation to **Nash (local) embedding of codimension** 1. Borrowing convex iteration scheme for Nash $C^{1,\alpha}$ embedding (Nash-Kuiper) [5], Conti-De Lellis-Székelyhidi. [7] and Cao-Székelyhidi [4] showed that the existence of the higher regularity solution $v \in C^{1,\alpha}$, where the highest achievable α is obtained as

$$\alpha = \begin{cases} \frac{1}{1+n(n+1)} & \text{in [7]} \\ \frac{1}{5} & \text{in [4]} \end{cases}$$

and it was known that the nonexistence of $\alpha > \frac{1}{2}$.

Our project is curious about: how come Cao-Székelyhidi's method not applicable for higher dimensions? In their paper, it is claimed that they directly apply *fluid equation methods* (which usually provide many tools in dim 2 or 3). Following their methods in [4], the **decomposition equation** we are facing is:

$$\text{For any } D \in C^{j,\alpha}(\Omega, \mathbb{R}^{n\times n}_{sym}), \text{ find the } \Phi \in C^{j+1,\alpha}(\Omega, \mathbb{R}^n) \text{ satisfying}$$
$$D + \frac{1}{2}(\nabla\Phi + (\nabla\Phi)^T) = \sum_{i}^{n(n+1)/2-\Xi} a_i^2 \xi_i \otimes \xi_i$$

for some $a_i \in C^{j,\alpha}(\Omega, \mathbb{R})$, $\xi_i \in \mathbb{R}^n$ and $\Xi \geq 1$. Namely, we are using $\frac{1}{2}(\nabla\Phi + (\nabla\Phi)^T)$ to decompose symmetric functional D into rank 1 symmetric constant matrices $\xi_i \otimes \xi_i$ with functional coefficients a_i, and the $\xi_i \otimes \xi_i$ in need is fewer than $\frac{n(n+1)}{2}$. In fact, **the fewer the rank 1 matrices needed in decomposition, the higher regularity of solution in** (1.1) **we can construct**. Our goal is to find the largest possible Ξ for each dimension n

2 Intersections in real projective spaces

Using the theory of elliptic system of PDEs and with some careful treatment, we reduce the problem of finding optimal Ξ to the following problem.

Problem 2.1. *Consider an n_L dimensional subspace L of the space of real symmetric matrices $\mathbb{R}^{n\times n}_{sym}$. Let L satisfy that*

1. *every nonzero element of L is invertible after scaling the diagonal by 2, and*

2. *for the dual space $L^\vee$, there exist primitive matrices $\xi_1 \otimes \xi_1, \cdots, \xi_{\frac{n(n+1)}{2}-n_L} \otimes \xi_{\frac{n(n+1)}{2}-n_L} \in \mathbb{R}^{n\times n}_{sym}$ that span $L^\vee$.*

For each n, find an L that maximizes n_L.

It is natural to consider Problem 2.1 under projectification and in the language of intersection. Denote

$$Y := \{[y_{ij}] \in \mathbf{P}(\mathbb{R}^{n\times n}_{sym}) \,\big|\det\big((y_{ij}) + \mathrm{diag}(y_{ii})\big) = 0\} \subset \mathbf{P}(\mathbb{R}^{n\times n}_{sym}),$$

thus Y is a degree n hypersurface in $\mathbf{P}(\mathbb{R}^{n\times n}_{sym}) \cong \mathbb{R}\mathbf{P}^{\frac{n(n+1)}{2}}$. Additionally, denote

$$Z := \{[w_1^2, \cdots, w_n^2, w_1w_2, \cdots, w_{n-1}w_n] \,\big|[w_1, \cdots, w_n] \in \mathbb{R}\mathbf{P}^{n-1}\} \subset \mathbf{P}(\mathbb{R}^{n\times n}_{sym})^\vee,$$

namely Z is the image of $\mathbb{R}\mathbf{P}^{n-1}$ in $\mathbf{P}(\mathbb{R}^{n\times n}_{sym})^\vee$ via **Veronese embedding** $\iota : \mathbb{R}\mathbf{P}^{n-1} \longrightarrow \mathbf{P}(\mathbb{R}^{n\times n}_{sym})^\vee$. We denote $\mathbf{L} = \mathbf{P}(L)$ and recall that $\mathbf{P}(L^\vee) = \mathbf{L}^\vee \subset \mathbf{P}(\mathbb{R}^{n\times n}_{sym})^\vee$. Recall that a variety $X \subset \mathbb{R}\mathbf{P}^n$ is said to be **nondegenerate** if it is not contained in any hyperplane of $\mathbb{R}\mathbf{P}^n$, and it is equivalent to the existence of $n+1$ points in X that span $\mathbb{R}\mathbf{P}^n$. Thus the following is an equivalent description of Problem 2.1.

Problem 2.2. *Consider an $n_\mathbf{L}$ dimensional linear subspace $\mathbf{L}$ of $\mathbf{P}(\mathbb{R}^{n\times n}_{sym})$, and its dual space $\mathbf{L}^\vee$. Let $\mathbf{L}$ satisfy the following conditions on $Y \subset \mathbf{P}(\mathbb{R}^{n\times n}_{sym})$ and $Z \subset \mathbf{P}(\mathbb{R}^{n\times n}_{sym})^\vee$:*

1. *$\mathbf{L} \cap Y = \varnothing$, and*
2. *$\mathbf{L}^\vee \cap Z$ is nondegenerate in $\mathbf{L}^\vee$.*

For each n, find an $\mathbf{L}$ that maximizes $n_\mathbf{L}$.

Remark 2.3. *Recall that for a variety X, its **dual variety** $X^\vee$ is defined as the closure of the locus of tangent hyperplanes to X at smooth points. It is a classical result that $Y^\vee = Z$, since Y is actually the **discriminant** of quadratic forms.*

Definition 2.4 (Radon-Hurwitz number)**.** *For $n \geq 1$, write $n = 2^{4a+b}(2c+1)$ with integers $0 \leq a$, $0 \leq b \leq 3$, $0 \leq c$. Then **Radon-Hurwitz number** $\rho(n)$ is defined as*

$$\rho(n) = 8a + 2^b. \tag{2.1}$$

It is known to Randon and Hurwitz in early 20th century on how to construct the $\rho(n)$ many linearly independent invertible $n \times n$ matrices with any linear combination of them are still invertible. Particularly, it can be taken as real matrix representation of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ for $n = 1, 2, 4, 8$.

n	1	2	3	4	5	6	7	8	$\cdots$	16	$\cdots$
$\rho(n)$	1	2	1	4	1	2	1	8	$\cdots$	9	$\cdots$

Table 1: first few values of $\rho(n)$.

Proposition 2.5. *Consider the following existence statements on ℓ.*

i. *The space of $n \times n$ real matrices has an ℓ-dimensional invertible subspace W_n.*
ii. *$\mathbb{R}^n$ admits a $\mathrm{Cl}_{\ell-1}$ representation.*
iii. *There are $\ell - 1$ independent vector fields on sphere S^{n-1}.*
iv. *The space of $2n \times 2n$ symmetric real matrices has an $\ell + 1$-dimensional invertible subspace $W_{2n,sym}$.*
v. *The space of $16n \times 16n$ matrices has an $\ell + 8$-dimensional invertible subspace W'_{16n}.*

Then (i) $\Longrightarrow$ (iii), (ii) $\Longrightarrow$ (iii), (i) $\Longrightarrow$ (iv) *and* (i)$\Longrightarrow$ (v).

Proof. (ii) implies (iii) as proved in [6, Theprem 7.1, Chapter 1].

To see (i) implies (iii), we define the $\ell - 1$ vector fields at each unit vector $\varepsilon \in S^{n-1} \subset \mathbb{R}^n$ to be $\Pi_{\varepsilon^\perp}(A_1^{-1}A_i\varepsilon) := A_1^{-1}A_i\varepsilon - \langle A_1^{-1}A_i\varepsilon, \varepsilon\rangle\varepsilon$ for $2 \le i \le \ell$, where $\{A_1, \cdots, A_\ell\}$ is a basis of invertible W_n, and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$. Assume that these vector fields are not independent, namely for some not all zero c_i and some $\varepsilon_0 \in S^{n-1}$, one has $\sum_{i\ge 2} c_i \Pi_{\varepsilon_0{}^\perp}(A_1^{-1}A_i\varepsilon_0) = 0$, which is equivalent to

$$\sum_{i\ge 2} c_i(A_i\varepsilon_0) - (\sum_{i\ge 2} c_i\langle A_1^{-1}A_i\varepsilon, \varepsilon\rangle)A_1\varepsilon_0 = (\sum_{i\ge 2} c_iA_i - \Big(\sum_{i\ge 2} c_i\langle A_1^{-1}A_i\varepsilon, \varepsilon\rangle\Big)A_1)\varepsilon_0 = 0. \qquad (2.2)$$

Thus it contradicts to that any nonzero linear combination of A_i are nonsingular.

That (i) implies (iv) is due to [2], one only needs to construct the $2n \times 2n$ matrices as $\begin{bmatrix} rI & A \\ A^T & -rI \end{bmatrix}$ for any $r \in \mathbb{R}$ and $A \in W_n$ from (i). Compute the determinant of block matrices(we shall show in the next proposition), we see constructed matrices form invertible $W_{2n,sym}$.

[2] also used a construction to show (i) implies (v), which is formed by

$$A \oplus \iota I_8 := A \otimes I_8 + I_{2n} \otimes \iota I_8,$$

where ι is the matrix representation of the 7 imaginary units of $\mathbb{O}$, hence $\dim(W'_{16n}) = \dim(W_{2n,sym}) + 7$. The $\oplus$ of matrices is known as Kronecker sum, and as one basic property of $\oplus$, each eigenvalue of $A \oplus \iota I_8$ is a summand of one eigenvalue of A and one eigenvalue of ιI_8. Since eigenvalues of ιI_8 are all purely imaginary, and eigenvalues of symmetric A are all reals, we see $A \oplus \iota I_8$ only has nonzero eigenvalues, hence is invertible. □

Proposition 2.6 (Block matrix determinant lemma)**.** *Consider* $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *with* A, B, C, D *being* $n \times n$ *matrices,* A *being invertible. Then*

$$\det(M) = \det(A)\det(D - CA^{-1}B)$$

Proof. The spirit is similar with the Gauss elimination. In fact, via simple matrix multiplication, one notices

$$\det(\begin{bmatrix} A & 0 \\ C & D - CA^{-1}B \end{bmatrix}) = \det(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) \cdot \det(\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}),$$

the proposition gets proved. □

The above simple proposition ensures the existence of $\rho(n) - 1$ many vector fields on S^{n-1}. The upper bound is proved by Adams in his remarkable work [1] via homotopy theory and Bott periodicity.

Proposition 2.7 (Adams [1])**.** *There exist no* ℓ *independent vector fields on sphere* S^{n-1}.

Collecting above results, we can have the complete answer for our Problem 2.2, namely

Corollary 2.8. *Let* $\mathbf{L}$ *only satisfy the first condition*

1. $\mathbf{L} \cap Y = \varnothing$,

then the maximal possible dimension of $\mathbf{L}$ *is* $\rho(\frac{1}{2}n) + 1$, *where* $\rho(\frac{1}{2}n)$ *is taken to be* 0 *if* n *is odd.*

The only part left is to see whether $\mathbf{L}^\vee \cap Z$ is nondegenerate in $\mathbf{L}^\vee$. In fact, this is the case as long as $\mathbf{L}^\vee \cap Z \neq \varnothing$. Via our construction, it is easy to see the intersection is nonempty for any $n \neq 2, 4, 8, 16$. The final last piece of the puzzle is to consider $n = 2, 4, 8, 16$ as special cases. Not only the aforementioned construction of $\rho(\frac{1}{2}n) + 1 = \frac{1}{2}n + 1$ dimensional invertible vector space gives empty intersection, but we tried multiple numberical experiments, they all having no real solution.

After having surveyed some literature in quadratic forms and real algebraic geometry, and having consulted some experts in real algebraic geometry, we present the Conjecture 0.1.

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