Lecture 5

De Rham spaces

1. Backgroup: Riemann-Hilbert problem (Hilbert's 21^{st} problem. 1900)

original problem: show the existence of linear systems of differential equations with prescribed monodromy group.

$\Updownarrow$

the map $\underset{\sim\sim}{\text{Fuchsian}}$ equations $\longrightarrow$ equiv. classes of representations is surjective. $\uparrow$

regular singularities i.e. 1^{st} order singularity.

1908. J. Plemelj gave a proof

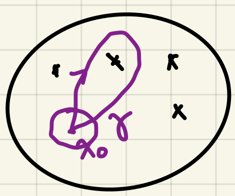
1990. A.A. Bolibrukh found a counterexample to Plemelj's proof

Consider the following linear system of ODEs.

$$\frac{d\Psi(z)}{dz} = A(z)\,\Psi(z) \qquad (*)$$

defined on $\Sigma := \mathbb{CP}^1 - \{z_1, \dots, z_s\}$

$A(z) \in \mathfrak{gl}_n(\mathbb{C}(z))$ is analytic on Σ.



$\mathbb{CP}^1$

Fix $x_0 \in \Sigma$

a fundamental solution to $(*)$ is $Y(z) \in \mathfrak{gl}_n(\mathbb{C}(z))$ with $\det Y(z)\big|_{x_0} \neq 0$.

satisfies $(*)$. always exists on some disc $D_r(x_0)$.

Taking γ a loop based at x_0.

analytic continuity of $Y(z)$ along γ $\rightsquigarrow$ $Y'(z)$

$Y(z) = Y'(z) \cdot g_\gamma \qquad g_\gamma \in GL_n(\mathbb{C})$

Moreover. g_γ only depends on the homotopy class of γ

$\rightsquigarrow$ $(*) \Rightarrow$ $\rho: \pi_1(\Sigma, x_0) \longrightarrow GL_n(\mathbb{C})$
$[\gamma] \longmapsto g_\gamma$

called the monodromy of the ODEs $(*)$.

When $g \geq 2$ $\rightsquigarrow$ it's more "natural" to formulate the problem to the relationship between parallel sections of connections and the monodromies.

$\rightsquigarrow$
" Riemann-Hilbert correspondence ".

Today: an analytic version of <u>Deligne's R-H correspondence</u>.

X alg. variety $/\mathbb{C}$ $\qquad X^{an} := (X(\mathbb{C}), \text{analytic top})$ the associated analytic space
X^{top} underlying topological space

<u>Thm</u> (Deligne) Fix $n \in \mathbb{Z}_{>0}$. equivalence of categories

$$\mathcal{C}^{reg}_{Flat}(X, n) \simeq \mathcal{C}_{Flat}(X^{an}, n) \simeq \mathcal{C}_{Loc}(X^{an}, n)$$
$$\simeq \mathcal{C}_{Rep}(X^{top}, n)$$

algebraic flat connections
with regular singularity

In particular. if X smooth projective variety $/\mathbb{C}$

<u>Cor</u>: $\mathcal{C}_{Flat}(X, n) \simeq \mathcal{C}_{Flat}(X^{an}, n) \simeq \cdots$

§2. Holomorphic structures and Dolbeault operators

$\mathbb{k}=\mathbb{C}$. X complex mfld $\dim_{\mathbb{C}} X = m$

$\pi : E \to X$ smooth complex vector bundle. $\mathrm{rk} = n$

$0 \le k \le 2m$. space of C^∞ k-forms on E:

$$A^k(X, E) := \Gamma(X, \Lambda^k_{\mathbb{C}} X \otimes E)$$
$$= \Gamma(X, \Lambda^k(T^*_{\mathbb{C}} X) \otimes E)$$

$0 \le p, q \le m$. space of C^∞ (p,q)-forms on E

$$A^{p,q}(X, E) := \Gamma(X, \Lambda^{p,q} X \otimes E)$$
$$= \Gamma(X, \Lambda^p(T^*_{1,0} X) \otimes_{\mathbb{C}} \Lambda^q(T^*_{0,1} X) \otimes E)$$

$\Lambda^k_{\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} X$ $\rightsquigarrow$

$$A^k(X, E) = \bigoplus_{p+q=k} A^{p,q}(X, E)$$

<u>Recall</u>:

holo. bundle $\Longleftrightarrow$ transition functions $\{ g_{ij} : U_i \cap U_j \to GL_n \mathbb{C} \}$ are holomorphic

Denote $\mathcal{E}$ as a holomorphic bundle.

Given $\mathcal{E}$ holo. let E the underlying $\mathbb{C}$-bundle.

$\Omega^0(X, E)$: space of holomorphic sections on $\mathcal{E}$.

$s \in A^0(X, E)$ is holomorphic if $\forall x \in X$. $\exists$ $x \in U \subseteq X$ open s.t. under the local trivialization

$$\varphi_U : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^n$$

the map

$$\xi : U \longrightarrow \mathbb{C}^n$$
$$z \longmapsto \xi(z)$$

determined by

$$\begin{aligned} s|_U : U &\to E|_U \xrightarrow{\simeq} U \times \mathbb{C}^n \\ z &\longmapsto (z, \xi(z)) \end{aligned}$$

is holomorphic.

$\Omega^p(X, E)$: space of holomorphic p-forms on E.

$\alpha \otimes s$, $\alpha \in A^{p,0}(X)$. holo.

$s \in A^0(X, E)$ holo.

Prop 2.2 $\pi : E \to X$ cplex bundle. then E is holomorphic iff it admits an operator $\bar\partial_E : A^0(X, E) \to A^{0,1}(X, E)$ that satisfies the following Leibniz rule:

$$\bar\partial_E(fs) = \bar\partial(f) \otimes s + f \bar\partial_E(s)$$

$\forall f \in C^\infty(X, \mathbb{C})$. $s \in A^0(X, E)$

and

$$\bar\partial_E^2 = 0$$

under the natural extension

$$\begin{aligned} \bar\partial_E : A^{p,q}(X, E) &\to A^{p,q+1}(X, E) \\ \underset{A^{p,q}(X)}{\alpha} \otimes \underset{A^0(X,E)}{s} &\longmapsto \bar\partial\alpha \otimes s + (-1)^{p+q} \alpha \wedge \bar\partial_E s \end{aligned}$$

Rmk So defined $\bar\partial_E$ is called a Dolbeault operator.

Pf: Suppose E is holomorphic. of rk n. under a trivialization $(U, \varphi_U : E|_U \simeq U \times \mathbb{C}^n)$.

choose local holomorphic frame $\{s_1, \dots s_n\}$ on this chart.

locally define

$$\begin{aligned} \bar\partial_E : A^0(X, E|_U) &\to A^{0,1}(X, E|_U) \\ s = \textstyle\sum_i f_i s_i &\longmapsto \textstyle\sum_i \bar\partial(f_i)\, s_i \end{aligned}$$

This is well-defined to give a global defined operator.

Indeed. Take $(V, \varphi_V : E|_V \simeq V \times \mathbb{C}^n)$ s.t. $U \cap V \neq \phi$

$\{s_1', \dots s_n'\}$, then $\exists$ holo. functions g_{ij} s.t. on $U \cap V$.

$$s_i' = \sum_j g_{ij} s_j$$

Suppose $s = \sum_i f_i' s_i' \Rightarrow f_i = \sum_j f_j' g_{ji}$

$$\Rightarrow \bar{\partial}_E(s) = \bar{\partial}_E \left(\sum_{i,j} f_i' g_{ij} s_j \right)$$

$$= \sum_{i,j} \bar{\partial}(f_i') g_{ij} s_j$$

$$= \sum_{i,j} \bar{\partial}(f_i') s_i'$$

Conversely. if we have $\bar{\partial}_E$, then $\forall\, x \in X$. $\exists\, x \in U \subseteq X$ open. s.t.

$$E|_U \simeq U \times \mathbb{C}^n$$

we may choose a local frame $\{s_1, \dots s_n\}$ s.t.

$$\bar{\partial}_E(s_i) = 0 \qquad 1 \leq i \leq n$$

need PDE technique! As long as this holds, we can find that the transition function are holo. $\rightsquigarrow$ E is holomorphic bundle.

$\boxtimes$

From now on. write $\mathcal{E} = (E, \bar{\partial}_E)$ for $\bar{\partial}_E$ the Dolbeault operator.

Rmk: $\Omega^p(X, E) = \operatorname{Ker}\left(\bar{\partial}_E : A^{p,0}(X, E) \to A^{p,1}(X, E) \right)$

§3. Smooth and holomorphic connections

$\pi : E \to X$ complex v.b. of rk n

Def 3.1 A C^∞ connection on E is an operator

$$\nabla : A^0(X, E) \longrightarrow A^1(X, E)$$

that satisfies the following Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

$\forall f \in C^\infty(X, \mathbb{C})$, $s \in A^0(X, E)$.

It is clearly that ∇ extends inductively to an operator

$$\nabla : A^k(X, E) \longrightarrow A^{k+1}(X, E)$$
$$\underset{A^k(X,\mathbb{C})}{\alpha} \otimes \underset{A^0(X,E)}{s} \longmapsto d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s)$$

If moreover, $\nabla^2 = 0$, then ∇ is called a flat C^∞ connection, and the pair (E, ∇) is called a C^∞ flat bundle.

Rmk If ∇ is a flat C^∞ connection on E.

let $\nabla^{1,0} := \pi_{1,0} \circ \nabla$

$\nabla^{0,1} := \pi_{0,1} \circ \nabla$

for $\pi_{1,0} : A^1(X, E) \to A^{1,0}(X, E)$, $\pi_{0,1} : A^1(X,E) \to A^{0,1}(X, E)$ proj

$\Rightarrow$

$$\nabla^2 = 0 \iff \begin{cases} (\nabla^{1,0})^2 = 0 \\ (\nabla^{0,1})^2 = 0 \\ [\nabla^{1,0}, \nabla^{0,1}] = 0 \end{cases}$$

$$[\nabla^{1,0}, \nabla^{0,1}] = \nabla^{1,0} \circ \nabla^{0,1} + \nabla^{0,1} \circ \nabla^{1,0}$$

$\rightsquigarrow$ induces a holomorphic bundle $(E, \bar\partial_E := \nabla^{0,1})$.

Conversely, suppose $\mathcal{E} := (E, \bar\partial_E)$ is holo. bundle.

Def 3.3 A holomorphic connection on $\mathcal{E}$ is an operator

$$D : \Omega^0(X, E) \longrightarrow \Omega^1(X, E)$$

that satisfies the Leibniz rule

$$D(fs) = df \otimes s + f D(s)$$

$\forall f \in \mathcal{O}_X(X)$ holomorphic function. $s \in \Omega^0(X, E)$

Similarly, D extends

$$D: \Omega^p(X, E) \to \Omega^{p+1}(X, E) \qquad \forall p$$

if $D^2 = 0$ under extension, then D is called a flat holomorphic connection.
$(\mathcal{E}, D)$ is called a holomorphic flat bundle.

Prop 3.4 Given a flat holomorphic connection on a holo. bundle is equivalent to given a flat C^∞ connection on its underlying complex bundle.

pf.
If we have a holo. flat bundle $(\mathcal{E}, D) = (E, \bar\partial_E, D)$

$\rightsquigarrow$ $\nabla := D + \bar\partial_E$ is flat C^∞ connection.

Conversely, if we have a C^∞ flat bundle (E, ∇), then it follows from the above Remark that

$$\bar\partial_E := \nabla^{0,1} \quad \text{holo. str.}$$

$$D := \nabla^{1,0} \quad \text{is a holo. connection, which is flat.}$$

$\rightsquigarrow$ we get a holomorphic flat bundle $(\mathcal{E} := (E, \bar\partial_E), D)$.

$\blacksquare$

From now on, we write (E, ∇)
$(\mathcal{E}, D)$

Q: When does a holo. bundle admit a flat holo. connection, or equivalently, when does a complex bundle admits a flat C^∞ connection?

When $\dim_{\mathbb{C}} X = 1$ ($n = 1$), this is well-known :

Prop 3.6 (Weil-Atiyah criterion)

A holo. bundle on a cpt connected R.S. admits a holo. connection

$\Updownarrow$ $n=1$

$\{\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$ $\mathcal{E}_i$: indecomposable$\}$ flat holo. conn.

iff each direct summand of it is of degree 0.

Prop 3.7 (Conjecture?)

A holo. bundle $\mathcal{E}$ over a cpt. connected cplex mfld admits a flat holo connection iff each direct summand of it has all chern classes vanishing.

Conj 3.8 $n > 1$.

holomorphic connection induces a flat holo. connection.

§4. Riemann-Hilbert correspondence

$\mathcal{E}$ holo. bundle

$$D : \Omega^0(X, E) \to \Omega^1(X, E) \quad \text{holo. conn.}$$

Recall:

$$\{\text{holo. bundles of rk } n\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{locally free sheaves of } \mathcal{O}_X\text{-modules} \\ \text{of rk } n\end{array}\right\}$$

Viewing $\mathcal{E}$ as a locally free sheaf of $\mathcal{O}_X$-modules, then D can be viewed as a sheaf morphism

$$D : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1$$

$\uparrow$ sheaf of holo. 1-forms on X

that satisfies the Leibniz rule

$$D(fs) = df \otimes s + f\,Ds$$

$f \in \mathcal{O}_X$

$s \in \mathcal{E}$

Thm 4.1 (Riemann-Hilbert correspondence. I)

$\forall n \in \mathbb{Z}_{>0}$. then a $\mathbb{C}$-local system of rk n over X is equivalent to a holomorphic flat bundle of rk n over X.

Namely. we have the following one-to-one correspondence of categories:

$$\mathcal{C}_{Loc}(X.n) \simeq \mathcal{C}_{Flat}(X.n)$$

for

$\mathcal{C}_{Flat}(X,n)$: the category of holo. flat bundles of rk n over X

morphism: $(\mathcal{E}.D_\mathcal{E})$ & $(\mathcal{F}.D_\mathcal{F})$ flat bundles

a morphism is a morphism $f: \mathcal{E} \to \mathcal{F}$ s.t.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[d, "D_\mathcal{E}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F} \arrow[d, "D_\mathcal{F}"] \\ \mathcal{E}\otimes_{\mathcal{O}_X}\Omega_X^1 \arrow[r, "f\otimes \mathrm{id}_{\Omega_X^1}"'] & \mathcal{F}\otimes_{\mathcal{O}_X}\Omega_X^1 \end{tikzcd}$$

Pf: "$\Rightarrow$"

L local system of rk n.

then $\mathcal{E} := L\otimes_\mathbb{C}\mathcal{O}_X$ locally free sheaf of $\mathcal{O}_X$-modules of rk n

$D := 1\otimes d$ for $d: \mathcal{O}_X \to \Omega_X^1$ the usual exterior diff. operator

$\Rightarrow D: \mathcal{E} \to \mathcal{E}\otimes\Omega_X^1$ flat.

And moreover. $L = \mathrm{Ker}(D) := \ker(L\otimes_\mathbb{C}\mathcal{O}_X \xrightarrow{1\otimes d} L\otimes\Omega_X^1) = L\otimes\mathbb{C} \simeq L$

"sheaf of parallel sections".

$\Leftarrow$ Given a holo. flat bundle $(\mathcal{E}.D)$. define $L^D := \mathrm{Ker}(D) \subset \mathcal{E}$

which is a locally free subsheaf. Claim L^D is a local system.

$\mathcal{E} \simeq L^D\otimes\mathcal{O}_X$. $(\mathcal{E}.D) \simeq (L^D\otimes\mathcal{O}_X.\ 1\otimes d)$

To show L^D is a locally constant sheaf. it suffices to show the transition function are locally constant.

by definition., sections of L are localy of D-parallel. chose local frame $\{s_1, \dots s_n\}$. $\{s_1', \dots, s_n'\}$ $\rightsquigarrow$

$$L|_{U_i} \cong U_i \times \mathbb{C}^n \qquad U_i \cap U_j \neq \phi$$

$$L|_{U_j} \simeq U_j \times \mathbb{C}^n$$

$\Rightarrow$ on $U_i \cap U_j$.

$$s_i = \sum_j g_{ij} s_j' \qquad 1 \leq i \leq n$$

for $g_{ij}: U_i \cap U_j \to Gl_n\mathbb{C}$ transition functions

$\Rightarrow$

$$0 = Ds_i = \sum_j dg_{ij} \otimes s_j' + \sum_j g_{ij} Ds_j'$$

$$= \sum_j dg_{ij} \otimes s_j'$$

$\rightsquigarrow$ $dg_{ij} = 0 \qquad 1 \leq i, j \leq n$

$\rightsquigarrow$ L^D is a localy constant sheaf.

∎

Already shown:

$$\mathcal{C}_{Loc}(X, n) \simeq \mathcal{C}_{Rep}(X, n)$$

Thm 4.2 (Riemann-Hilbert correspondence. II).

For $n \in \mathbb{Z}_{>0}$. then a holo. flat bundle of rk n over X is equivalent. to a fundamental group representation into $Gl_n(\mathbb{C})$.

Namely, we have the following one-to-one correspondence between categories

$$\mathcal{C}_{Flat}(X, n) \simeq \mathcal{C}_{Rep}(X, n)$$

Pf.... $\Leftarrow$ Given $\rho: \pi_1(X, x) \to Gl_n\mathbb{C}$,

$$E_\rho := \tilde{X} \times_\rho \mathbb{C}^n = \tilde{X} \times \mathbb{C}^n /_\rho$$

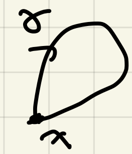
$\gamma \cdot (\tilde{x}, v) := (\gamma \tilde{x}, \rho(\gamma) v)$

$\tilde{X}/\pi_1(X,x) \simeq X \rightsquigarrow E_\rho$ admits a trivial flat connection.

"$\Leftarrow$" $(\mathcal{E}, D)$ holo. flat bundle. (E, ∇) associated C^∞ flat bundle.

Consider the parallel transport w.r.t. ∇:

$\forall \gamma : [0,1] \to X$ based at x, consider



$$\nabla(s)(\gamma'(t)) = 0 \qquad (**)$$

Taking local frame $\{s_1, \dots s_n\}$. under the local frame. then we have

$$\nabla s_i = A_{ij} s_j \quad \text{for } A_{ij} \text{ 1-form.}$$

$s = \sum_i f_i s_i$, then $(**)$

$$\Leftrightarrow \quad \frac{d f_i(\gamma(t))}{dt} + \sum_{j=1}^{n} A_{ij}(\gamma'(t)) f_i(\gamma(t)) = 0 \qquad 1 \leq i \leq n \qquad (***)$$

By standard theory on the existence and uniqueness of solutions to linear systems of ODEs. $\forall$ initial values $f_i \in C^\infty(X, \mathbb{C})$ with

$$f_i(\gamma(0)) = f_i \qquad 1 \leq i \leq n$$

then $\exists$ a unique smooth solution to $(***)$.

$\Updownarrow$

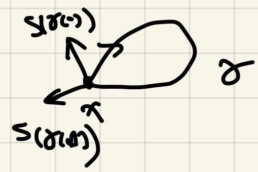
$\forall$ initial $v \in E_{\gamma(0)} = E_x$ with $s(\gamma(0)) = v$

then $\exists$ a unique section s along γ to $(**)$..

and the solution is independent on the homotopy class of γ

then

$$v = s(\gamma(0)) \longrightarrow s(\gamma(1))$$



$\Rightarrow$ uniquely determines an $\underbrace{\text{element}}_{g_\gamma}$ in $GL_n(E_x) \simeq GL_n \mathbb{C}$.

$$\rightsquigarrow \quad \rho_\nabla : \pi_1(X,x) \to GL_n\mathbb{C}$$
$$[\gamma] \mapsto g_\gamma .$$

called the monodromy representation of (E,∇) ☒

§5. Some preliminaries for later use

$E \xrightarrow{\pi} X$ C^∞ cplex bundle. a hermitian metric on E is
$$h : A^0(X,E) \times A^0(X,E) \to A^0(X,\mathbb{C})$$
which is an inner product on each fiber.
$$h_x : E_x \times E_x \to \mathbb{C}$$

(1) $h(\lambda v, w) = \lambda h(v,w)$

(2) $h(v,w) = \overline{h(w,v)}$

(3) $h(v,v) \geq 0$ & $h(v,v) = 0 \Leftrightarrow v = 0$

extends to
$$h : A^p(X,E) \times A^q(X,E) \to A^{p+q}(X,\mathbb{C})$$
$$(\alpha\otimes u, \quad \beta\otimes v) \mapsto \alpha\wedge\overline{\beta}\, h(u,v)$$

(E,∇) C^∞ flat bundle. $\overline{\partial}_E := \nabla^{0,1}$. with a hermitian metric h

Prop 5.1 There is a unique $(1,0)$-type connection ∂_h s.t. $D_h := \partial_h + \overline{\partial}_E$ is a metric connection. i.e. D_h is C^∞ connection that satisfies
$$d(h(u,v)) = h(D_h(u), v) + h(u, D_h(v)) \qquad (5.1)$$
$$\forall u,v \in A^0(X,E)$$

∂_h is $(1,0)$-type :
$$\partial_h : A^0(X,E) \to A^{1,0}(X,E)$$
that satisfies the ∂-twisted Leibniz rule.

Rmk: Such a metric connection is called a Chern connection.

Pf.

(1) Uniqueness:

$E|_U \simeq U \times \mathbb{C}^n$.

$s_U := (s_1, \dots, s_n)$ local holo. frame.

$\omega_U = (\omega_\alpha^\beta)_{1 \le \alpha, \beta \le n}$ the local conn. form of D_h

$$\left(\text{i.e. } \underset{\partial_h + \bar\partial_E}{\overset{\|}{D_h}} s_\alpha = \sum_\beta \omega_\alpha^\beta s_\beta \right) \qquad (5.2)$$
$$= \partial_h s_\alpha$$

$\Rightarrow \omega_\alpha^\beta$ is type $(1,0)$.

(5.2) has a matrix expression

$$D_h s_U = s_U \omega_U$$

$$\Rightarrow \quad \partial h(s_\alpha, s_\beta) = h(\partial_h s_\alpha, s_\beta) + \cancel{h(s_\alpha, \bar\partial_E s_\beta)}$$
$$= h(D_h s_\alpha, s_\beta)$$

$$\Rightarrow \quad \partial H_U = (\omega_U)^T H_U$$

$$\Rightarrow \quad \omega_U = (\partial H_U \cdot H_U^{-1})^T$$

i.e. the connection form is uniquely determined by H_U.

(2) existence: locally defined is global defined.

$U \cap V \neq \phi$

$s_U = (s_1, \dots s_n)$

$s_V = (s_1', \dots, s_n')$ local holo. frame.

$$s_V = s_U g_{UV} \qquad g_{UV} : U \cap V \to GL_n\mathbb{C} \text{ holo.}$$

$$\Rightarrow \quad s_U g_{UV} \omega_V = s_V \omega_V = D_h s_V$$

$$= D_h(s_U g_{UV})$$
$$= D_h(s_U) g_{UV} + s_U \, d g_{UV}$$
$$= s_U \omega_U g_{UV} + s_U \, d g_{UV}$$

$$\Rightarrow \quad \omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} \, d g_{UV}$$

on the other hand, H_U, H_V are related by

$$H_V = g_{UV}^{T} H_U \overline{g_{UV}}$$

$$\Rightarrow \quad \omega_V = (\partial H_V \cdot H_V^{-1})^T = \cdots$$

$$= g_{UV}^{-1} d g_{UV} + g_{UV}^{-1} \omega_U g_{UV}$$

⇝ globaly defined

⇝ well-defined.

$\boxtimes$

Lecture 6

Dolbeault spaces

Recall:

the categorical correspondence given by the Riemann-Hilbert correspondence:

$$\mathcal{C}_{Loc}(X,n) \simeq \mathcal{C}_{Flat}(X,n)$$

Today: define a new category so that it is equivalent to a full subcategory of $\mathcal{C}_{Flat}(X,n)$

$$\begin{array}{ccc} & & \subsetneq \mathcal{C}_{Flat}(X,n) \\ \mathcal{C}_{dR}(X,n) & \simeq & \mathcal{C}_{Dol}(X,n) \end{array}$$

Setting: (X,ω) cpt Kähler mfd $\dim_{\mathbb{C}} X = m$

In particular, smooth proj. mfd.

§1. Higgs bundles and harmonic metric

Def 1.1 $\mathcal{E} := (E, \bar{\partial}_E)$ be a holo. vector bundle. a Higgs field is a morphism of holo. bundles (sheaf morphism)

$$\varphi: \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$$

s.t.

$$\varphi \wedge \varphi = 0$$

in $\mathrm{End}(\mathcal{E}) \otimes \Omega_X^2$.

$$\mathcal{E} \xrightarrow{\varphi} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\varphi \otimes \mathrm{id}_{\Omega_X^1}} \mathcal{E} \otimes \Omega_X^1 \otimes \Omega_X^1 \xrightarrow{\wedge} \mathcal{E} \otimes \Omega_X^2$$

$(\mathcal{E}, \varphi)$ or $(E, \bar{\partial}_E, \varphi)$ is called a Higgs bundle.

Rmk. A Higgs field φ is a holomorphic section $\varphi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes \Omega_X^1)$ s.t. $\varphi \wedge \varphi = 0$.

The above definition has a C^∞ interpretation:

E smooth complex vector bundle. define

$$D'': A^0(X, E) \to A^1(X, E)$$

satisfies

(1) $\bar{\partial}$-Leibniz rule:

$$D''(fs) = \bar{\partial} f \otimes s + f D''(s) \qquad \forall f \in C^\infty(X, \mathbb{C}),\ s \in A^0(X, E)$$

(2) integrability condition:

$$(D'')^2 = 0$$

under natural extension

$$D'' : A^k(X, E) \to A^{k+1}(X, E)$$

Indeed, decompose D'' into different types:

$$D'' := \varphi + \bar{\partial}_E$$

then

$$(D'')^2 = 0 \iff \begin{cases} (\bar{\partial}_E)^2 = 0 \\ \bar{\partial}_E \varphi = 0 \leftarrow \bar{\partial}_E \circ \varphi + \varphi \circ \bar{\partial}_E \\ \varphi \wedge \varphi = 0 \end{cases}$$

(1) & (2) $\iff$ $(E, \bar{\partial}_E, \varphi)$ is Higgs bundle.

Ex.: $n = 1$ i.e. X cpt. R.S. $g \geq 2$

(1) $K_X := T^*_{1,0} X$ holo. cotangent bundle $\quad K_X = \Omega_X^1 = \Lambda^1(T^*_{1,0}X) = T^*_{1,0}X$

line bundle.

choose a square root $K_X^{\frac{1}{2}}$ of K_X $\quad (2^{2g}$ choices)

then the following is a Higgs bundle.

$$\left(\mathcal{E} = K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix} \right)$$

$$1 : K_X^{\frac{1}{2}} \to K_X^{-\frac{1}{2}} \otimes K_X$$

$$q_2 : K_X^{-\frac{1}{2}} \to K_X^{\frac{1}{2}} \otimes K_X$$

i.e. $q_2 \in H^0(X, K_X^2)$ holo. quadratic differential

Hitchin: space of q_2, i.e. $H^0(X, K_X^2)$ parametrizes the Teichmüller space of the underlying oriented smooth surface S of X

(2) $\left(\mathcal{E} = K_X \oplus \mathcal{O}_X \oplus K_X^{-1},\ \varphi = \begin{pmatrix} 0 & 0 & q_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right)$ q_2 q_2

$$q_3 : K_X^{-1} \to K_X \otimes K_X \quad \text{i.e. } q_3 \in H^0(X, K_X^3)$$

holo. cubic differential

(3) In general. $\mathcal{E} = \mathrm{Sym}^{n-1}(K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}}) = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \cdots \oplus K_X^{-\frac{n-1}{2}}$. rk n

$$\varphi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ 1 & 0 & q_2 & \cdots & q_{n-1} \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

$$q_i : K_X^{\overrightarrow{\frac{n-2i+1}{2}}} \to K_X^{\frac{n-1}{2}} \otimes K_X \quad \text{i.e. } q_i \in H^0(X, K_X^i)$$

holo. i-th differential

<u>Hitchin</u>: if we regard $q := (q_2, \ldots q_n) \in \bigoplus_{i=2}^{n} H^0(X, K_X^i)$, then the Higgs bundles $(\mathcal{E}, \varphi)$ considered above are located in the image of a section of Hitchin morphism, called a Hitchin section.

$\Updownarrow$

$$\rho : \pi_1(X, x) \to SL(n, \mathbb{R})$$

<u>Now</u>. $(E, \bar{\partial}_E, \varphi)$ a Higgs bundle. h a given hermitian metric. recall

$$g_{\mathbb{C}} : A^k(X, \mathbb{C}) \times A^k(X, \mathbb{C}) \to \mathbb{C}$$

$$(\alpha, \beta) \mapsto g_{\mathbb{C}}(\alpha, \beta)\, \mathrm{vol}_g := \alpha \wedge * \bar{\beta}$$

$\rightsquigarrow$ induces an inner product on bundle-valued k-forms:

$$\langle\ \rangle_{g,h} : A^k(X, E) \times A^k(X, E) \to A^0(X, \mathbb{C})$$

$$(\alpha\otimes u,\ \beta\otimes v) \longmapsto g_{\mathbb{C}}(\alpha,\beta)\cdot h(u,v)$$

$$\underset{A^k(X,\mathbb{C})}{\alpha} \quad \underset{A^0(X,E)}{u}$$

Chern connection:

$$\bar\partial_E \ \& \ h \ \rightsquigarrow \ \partial_h \qquad \partial\text{-type operator}$$

∂-Leibniz rule.

$$\varphi \ \& \ h \ \rightsquigarrow \ \varphi^{*h} \quad \text{via}$$

$$h(\varphi(u), v) = h(u, \varphi^{*h}(v))$$

adjoint operator of φ

locally. Choose $\{z_1, \dots z_m\}$. $\varphi = \sum_{\alpha=1}^{m} \varphi_\alpha dz_\alpha$

$$\Rightarrow \varphi^{*h} = \sum_{\alpha=1}^{m} \varphi_\alpha^{*h} d\bar z_\alpha$$

where $\varphi_\alpha^{*h} = \bar H^{-1} \cdot \bar\varphi_\alpha^T \cdot \bar H$ for H the Hermitian matrix corresponds to h

Introduce

$$\nabla_h := \underset{(\partial_h + \bar\partial_E)}{\underset{\parallel}{D_h}} + \varphi + \varphi^{*h}$$

It's easy to check ∇_h is C^∞ connection on E.

$$\nabla_h : A^0(X,E) \to A^1(X,E)$$

$$\text{s.t.} \quad \nabla_h(fs) = df \otimes s + f \nabla_h(s)$$

Def 2.2 h on $(E, \bar\partial_E, \varphi)$ is a pluri-harmonic metric if the induced C^∞ connection ∇_h is flat, i.e. $\nabla_h^2 = 0$

$\rightsquigarrow$ $(E, \bar\partial_E, \varphi, h)$ harmonic Higgs bundle.

Rmk: By definition

$$\nabla_h^2 = 0 \iff (D_h + \varphi + \varphi^{*h})^2 = 0$$

$$\iff \begin{cases} D_h^2 + [\varphi, \varphi^{*h}] = 0 \\ \partial_h \varphi = 0 \left(= \bar{\partial}_E (\varphi^{*h})\right) \end{cases} \qquad (*)$$

since $(\partial_h \varphi)^{*h} = \bar{\partial}_E(\varphi^{*h})$

$(*)$ is called the Hitchin's self-duality equation.

$$(D_h + \varphi + \varphi^{*h})^2 = D_h^2 + \varphi \wedge \varphi + \varphi^{*h} \wedge \varphi^{*h} + [\varphi, \varphi^{*h}]$$

$$\overset{\shortparallel}{\partial_h + \bar{\partial}_E}$$

$$+ \partial_h(\varphi) + \partial_h(\varphi^{*h}) + \bar{\partial}_E(\varphi) + \bar{\partial}_E(\varphi^{*h})$$

$$\# \quad D_h^2 = (\partial_h + \bar{\partial}_E)^2 = \partial_h^2 + \bar{\partial}_E^2 + [\partial_h, \bar{\partial}_E]$$

In particular, if $m=1$, then

$$(*) \iff D_h^2 + [\varphi, \varphi^{*h}] = 0$$

Def 2.3 h is harmonic if $\Lambda_\omega F_h = 0$ for $F_h := \nabla_h^2$

h is Hermite-Einstein if $\Lambda_\omega F_h = \lambda \cdot \mathrm{Id}_E$

Rmk: Easy to check

$$\lambda = -2\pi\sqrt{-1}\,\frac{\deg E}{\mathrm{rk}(E)\cdot \mathrm{vol}_g(X)}$$

Rmk: By definition:

pluri-harmonic $\Rightarrow$ harmonic $\Rightarrow$ Hermite-Einstein

Hermite-Einstein & $c_1(E) = 0 \Rightarrow$ harmonic.

Q: When is a harmonic metric pluri-harmonic?

Recall total chern class of E can be defined via any connection ∇

$$\det\left(\mathrm{Id} + \frac{\sqrt{-1}}{2\pi} F_\nabla\right) = 1 + c_1(E) + c_2(E) + \cdots$$

$c_i(E) \in H^{2i}(X, \mathbb{Z})$

Prop 2.4 A harmonic metric is pluri-harmonic if and only if $ch_2(E) = 0$.

pf. "$\Leftarrow$" ✓

"$\Rightarrow$" express $ch_2(E)$ in terms F_h as

$$-\int_X ch_2(E) \wedge \frac{\omega^{m-2}}{(m-2)!} = \int_X \left(c_2(E) - \frac{1}{2} c_1^2(E) \right) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= -\int_X \mathrm{Tr}\left(\left(\frac{\sqrt{-1}}{2\pi} F_h \right)^2 \right) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= \frac{1}{8\pi^2} \int_X \mathrm{Tr}(F_h \wedge F_h) \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$F_h = \mathcal{D}_h^2 = (D_h + \varphi + \varphi^{*h})^2$$

$$= (D_h^2 + [\varphi, \varphi^{*h}]) + (\partial_h \varphi + \bar{\partial}_E(\varphi^{*h}))$$

$$=: F_1 + F_2$$

$\Rightarrow$ $(F_1)^{*h} = -F_1$ since $(D_h^2)^{*h} = -D_h^2$

$(F_2)^{*h} = F_2$ $\quad ([\varphi, \varphi^{*h}])^{*h} = -[\varphi, \varphi^{*h}]$

$(\partial_h(\varphi))^{*h} = \bar{\partial}_E(\varphi^{*h})$

$F_1 =: F_{01} + F_{11}$

for F_{01} primitive part of F_1, i.e. $F_{01} = F_1 - \frac{\omega}{m} \Lambda_\omega F_1$

$(\Lambda_\omega F_{01} = 0)$

$$= \left(F_1 - \frac{\omega}{m} \Lambda_\omega F_1\right) + \frac{\omega}{m} \Lambda_\omega F_1$$

orthogonal decomposition

Apply $*$:

$$* F_{01} = -F_{01} \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$* F_{11} = F_{11} \wedge \frac{\omega^{m-2}}{(m-1)!}$$

$$* F_2 = F_2 \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$\Rightarrow$

$$* F_h^{*h} = *(-F_{01} - F_{11} + F_2)$$

$$= F_h \wedge \frac{\omega^{m-2}}{(m-2)!} - \frac{m}{m-1} F_{11} \wedge \frac{\omega^{m-2}}{(m-2)!}$$

$$= F_h \wedge \frac{\omega^{m-2}}{(m-2)!} - m * F_{11}^{*h}$$

$$\Rightarrow \int_X \mathrm{Tr}(F_h \wedge F_h) \wedge \frac{\omega^{m-2}}{(m-2)!} = \| F_h \|_{L^2}^2 - m \| F_{11} \|_{L^2}^2$$
$$= \| F_h \|_{L^2}^2 - \| \Lambda_\omega F_h \|_{L^2}^2$$

$\Rightarrow$

$$\Lambda_\omega F_h = 0 \quad \& \quad ch_2(E) = 0 \iff F_h = 0$$

i.e.

$$h \text{ harmonic} \quad \& \quad ch_2(E) = 0 \iff h \text{ pluri-harmonic.}$$

$\boxed{//}$

Cor 2.5 $m = 1$

pluri-harmonic = harmonic

Lem 2.6 (Kähler identities for Higgs bundles)

$(E, \bar\partial_E, \varphi)$ Higgs bundle, h hermitian metric

let $D_h' := \partial_h + \varphi^{*h}$

$D_h'' := \bar\partial_E + \varphi$

$$\Rightarrow \quad (D_h')^* = \sqrt{-1}\,[\Lambda_\omega, D_h'']$$

$$(D_h'')^* = -\sqrt{-1}\,[\Lambda_\omega, D_h']$$

§3. Flat bundles and pluri-harmonic metrics

(E, ∇) C^∞ flat bundle, h hermitian metric.

$\Rightarrow$ ∇ decomposes uniquely as

$$\nabla = D_h + \Phi_h$$

for
- D_h unitary connection w.r.t. h
- $\Phi_h \in A^1(X, \mathrm{End}(E))$ is self-adjoint, i.e. $\Phi_h^{*h} = \Phi_h$

Indeed. Ψ_h is defined as

$$h(\Psi_h(u),v) = \frac{1}{2}\Big(h(\nabla(u),v) + h(u,\nabla(v)) - dh(u,v)\Big)$$

$$\forall\ u,v \in A^0(X,E)$$

$\Rightarrow$

$$h(\Psi_h(v),u) = \frac{1}{2}\Big(h(\nabla(v),u) + h(v,\nabla(u)) - dh(v,u)\Big)$$

$\Rightarrow$ $$h(u,\Psi_h(v)) = \frac{1}{2}\Big(h(u,\nabla(v)) + h(\nabla(u),v) - dh(u,v)\Big)$$

$\Rightarrow$ $\Psi_h^{*h} = \Psi_h$

Similary, one checks

$$h(D_h(u),v) + h(u,D_h(v)) = dh(u,v)$$

Decomposes D_h & Ψ_h furthermore into different types

$$D_h := \partial_h + \bar\partial_h \qquad \Psi_h := \varphi_h + \varphi_h^{*h}$$

introduce

$$D_h' = \partial_h + \varphi_*^{*h} \qquad D_h'' = \bar\partial_h + \varphi_h$$

$\Rightarrow$ one checks

$$D_h'': A^0(X,E) \to A^1(X,E) \quad \text{is a } \bar\partial\text{-operator.}$$

$$D_h''(fs) = \bar\partial f \otimes s + f\, D_h''(s).$$

define $G_h := (D_h'')^2$. then G_h is $C^\infty(X,\mathbb{C})$-linear operator

i.e. $\mathrm{End}(E)$-valued 2-form

called the pseudo-curvature.

Def 2.1 A hermitian metric h on a flat bundle (E,∇) is called a

harmonic metric if

$$\Lambda_\omega G_h = 0$$

pluri-harmonic metric if

$$G_h = 0$$

i.e. $(D_h'')^2 = 0 \iff (\bar\partial_h + \varphi_h)^2 = 0$

$\iff (E, \bar\partial_h, \varphi_h)$ defines a Higgs bundle.

when h pluri-harmonic, we call (E, ∇, h) a harmonic flat bundle.

Lem 2.2 (Kähler identities for flat bundles)

(E, ∇) flat bundle. h hermitian metric

define $D_h^c := D_h'' - D_h'$

$$= (\bar\partial_h + \varphi_h) - (\partial_h + \varphi_*^{*h})$$

$\Rightarrow$

$$(D_h^c)^* = -\sqrt{-1}\,[\Lambda_\omega, \nabla]$$

$$(\nabla)^* = \sqrt{-1}\,[\Lambda_\omega, D_h^c].$$

Q: When is a harmonic metric on (E, ∇) pluri-harmonic ?

Prop 2.3 When the base (X, ω) cpt Kähler mfld.

then h is pluri-harmonic $\iff$ h is harmonic

i.e. $G_h = 0 \iff \Lambda_\omega G_h = 0$

pf: Claim: $(\overbrace{D_h'}^{=(\partial_h + \varphi_*^{*h})})^2 = -(D_h'')^2$

Indeed, decompose ∇ into different types

$$\nabla = d' + d''$$

$h \rightsquigarrow$ d' determines a δ_h'' s.t. $d' + \delta_h''$ unitary

d'' determines a δ_h' s.t. $d'' + \delta_h'$ unitary

$\Rightarrow$ $\partial_h = \frac{1}{2}(d' + \delta_h')$ $\qquad \bar\partial_h = \frac{1}{2}(d'' + \delta_h'')$

$\varphi_h = \frac{1}{2}(d' - \delta_h')$ $\qquad \varphi_*^{*h} = \frac{1}{2}(d'' - \delta_h'')$

then $(D_h')^2 = -(D_h'')^2$ follows.

$$\Rightarrow \quad 0 = \nabla^2 = (D_h' + D_h'')^2 = (D_h')^2 + (D_h'')^2 + D_h' \circ D_h'' + D_h'' \circ D_h'$$
$$= D_h' D_h'' + D_h'' D_h'$$

$$\Rightarrow \quad (D_h^c)^2 = (D_h'' - D_h')^2$$
$$= -(D_h'' D_h' + D_h' D_h'')$$
$$= 0$$

$$\left\{ \begin{array}{l} \nabla = D_h' + D_h'' \\ D_h^c = D_h'' - D_h' \end{array} \right.$$

$$\Rightarrow \quad G_h = (D_h'')^2 = \frac{1}{4}(\nabla + D_h^c)^2$$
$$= \frac{1}{4}(D_h^c \nabla + \nabla D_h^c)$$

$\Rightarrow$ Bianchi identities for G_h:

$$D_h^c G_h = [D_h^c, G_h] = 0$$
$$\nabla G_h = [\nabla, G_h] = 0$$

Apply Kähler identities for flat bundles (Lem 2.2)

$$\Rightarrow \quad (\nabla)^* G_h = \sqrt{-1}\,[\Lambda_\omega, D_h^c]\, G_h$$
$$= 0 \qquad (\because \Lambda_\omega G_h = 0)$$

On the other hand, one checks

$$D_h^c = d'' - d' + 2(\varphi_h - \varphi_h^{*h})$$

$$\nabla = d' + d''$$
$$\nabla^2 = 0 \Leftrightarrow \left\{ \begin{array}{l} (d')^2 = 0 = (d'')^2 \\ d'd'' + d''d' = 0 \end{array} \right.$$

$$\Rightarrow \quad 4G_h = D_h^c \nabla + \nabla D_h^c$$
$$= 2\nabla(\varphi_h - \varphi_h^{*h})$$

i.e.

$$G_h = \frac{1}{2}\nabla(\varphi_h - \varphi_h^{*h})$$

Consequently,

$$\Rightarrow \quad \|G_h\|_{L^2}^2 = \int_X |G_h|_{g,h}^2 \frac{\omega^m}{m!}$$

$$= \int_X \langle G_h, G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \frac{1}{2} \int_X \langle \nabla(\varphi_h - \varphi_h^{*h}), G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \frac{1}{2} \int_X \langle \varphi_h - \varphi_h^{*h}, (\nabla)^* G_h \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

$$\Rightarrow G_h = 0$$

$\boxtimes$

Rmk: The Kählerian condition can be relaxed.

In conclusion, the flatness condition $\nabla^2 = 0$ is more rigid constraint than the Higgs condition $(\bar\partial_E + \varphi)^2 = 0$.

§3. Equivalence of categories

Fix $n \in \mathbb{Z}_{>0}$.

$\mathcal{C}_{Dol}(X, n)$: category of harmonic Higgs bundles of rk n

$\mathcal{C}_{dR}(X, n)$: category of harmonic flat bundles of rk n

Thm 3.1. There is one-to-one correspondence

$$\Xi_X : \mathcal{C}_{Dol}(X, n) \xrightarrow{\sim} \mathcal{C}_{dR}(X, n)$$

$$(E, \bar\partial_E, \varphi, h) \mapsto (E, \nabla_h := D_h + \varphi + \varphi^{*h}, h)$$

Moreover, such equivalence of categories preserves direct sums, duals, and tensor products, and it is functorial, i.e. $f : Y \to X$ morphism of proj. mflds (or Kähler mflds), then f^* induces

$$\Xi_Y := f^*\Xi_X : \mathcal{C}_{Dol}(Y,n) \to \mathcal{C}_{dR}(Y,n)$$

that is.

$$\Xi_Y \circ f^*(E, \bar\partial_E, \varphi, h) = f^* \circ \Xi_X(E, \bar\partial_E, \varphi, h).$$

pf. Ξ_X is well-defined: $(E, \bar\partial_E, \varphi, h)$ harmonic Higgs

$\Rightarrow \nabla_h$ flat & h is a harmonic metric for (E, ∇_h)

To show the equivalence of cats. need to show Ξ_X maps morphisms in $\mathcal{C}_{Dol}(X,n)$ to morphisms in $\mathcal{C}_{dR}(X,n)$

$$(E, \bar\partial_E, \varphi, h) \xmapsto{\Xi_X} (E, \nabla_h, h)$$

$H^0_{Dol}(E, D'')$ space of D''-flat sections of E i.e. $s \in A^0(X,E)$ with $D''(s) = 0$

$H^0_{dR}(E, \nabla_h)$: space of ∇_h flat sections of E.

lem 2.2 $D''(s) = 0 \iff \nabla_h(s) = 0$

i.e.

$$H^0_{Dol}(E, D'') \simeq H^0_{dR}(E, \nabla_h)$$

pf.

"$\Rightarrow$" $D''s = 0$. to show $\nabla_h(s) = 0$. we need to show $D'_h(s) = 0$

Indeed. use Kähler identities for Higgs bundles.

$$\Rightarrow (D'_h)^* D'_h(s) = \sqrt{-1}\Lambda_\omega D'' D'_h(s) = -\sqrt{-1}\Lambda_\omega D'_h D''_h(s) = 0$$

$$\Rightarrow \|D'_h(s)\|^2_{L^2} = \int_X \langle (D'_h)^*(D'_h(s)), s \rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

"$\Leftarrow$" $\nabla_h(s) = 0$. $(\nabla_h = D'_h + D'')$

use Kähler identities for flat bundles.

$$\Rightarrow \|D_h^c(s)\|_{L^2}^2 = \int_X \langle (D_h^c)^* D_h^c(s), s\rangle_{g,h} \frac{\omega^m}{m!}$$

$$= \int_X \langle -\sqrt{-1}\,\Lambda_\omega \nabla_h D_h^c(s), s\rangle_{g,h} \frac{\omega^m}{m!}$$

$$= 0$$

the last equality is due to.

$$\begin{aligned}
\nabla_h \circ D_h^c &= (D'' + D_h') \circ (D'' - D_h') \\
&= (D'')^2 - (D_h')^2 - D'' \circ D_h' + D_h' \circ D'' \\
&= -D'' \circ D_h' + D_h' \circ D'' \\
&= -(D'' - D_h') \circ (D'' + D_h') \\
&= -D_h^c \circ \nabla_h
\end{aligned}$$

$$\Rightarrow D''(s) = \frac{1}{2}(\nabla_h + D_h^c)(s) = 0$$

$\boxtimes$

So Ξ_X maps morphisms in $\mathcal{C}_{\mathrm{Dol}}(X,h)$ to morphisms in $\mathcal{C}_{\mathrm{DR}}(X,h)$

as long as we show.

dual of harmonic is still harmonic

tensor product of harmonic is still harmonic.

<u>dual</u>:

$(E, \bar\partial_E, \varphi)$ Higgs. $D_E'' = \bar\partial_E + \varphi$

dual Higgs bundle. E^*.

- $\bar\partial$-type operator $D_{E^*}'' : A^0(X, E^*) \to A^1(X, E^*)$ define as

$$D_{E^*}^\vee(u)(v) := -u(D_E''(v)) + \bar\partial(u(v))$$

$u \in A^0(X, E^*)$

$v \in A^0(X, E)$

extends $\quad D^{h}_{E^*} : A^k(X, E^*) \to A^{k+1}(X, E^*)$.

satisfies $\quad (D^h_{E^*})^2 = 0$.

$\rightsquigarrow$ dual Higgs bundle.

holo. str. $\quad \bar\partial_{E^*}(u)(v) = -u(\bar\partial_E(v)) + \bar\partial(u(v))$

$$\varphi_{E^*}(u)(v) = -u(\varphi_E(v))$$

given h on $(E, \bar\partial_E, \varphi)$. $\rightsquigarrow$ h^* on $(E^*, \bar\partial_{E^*}, \varphi_{E^*})$ as

$$h^*(u_1, u_2) := u_1((u_2)^+_h)$$

for $(u_2)^+_h \in A^0(X, E)$ determined by u_2 & h via

$$u_2(v) = h(v, (u_2)^+_h)$$

Lem 3.3. $(1,0)$-part of the Chern connection of $\bar\partial_{E^*}$ is related to ∂_h as

$$\partial_{h^*}(u)(v) = -u(\partial_h(v)) + \partial(u(v))$$

Lem 3.4 $(\varphi_{E^*})^{*h^*}$ satisfies

$$(\varphi_{E^*})^{*h^*}(u)(v) = -u((\varphi_E)^{*h}(v))$$

Prop 3.5 Dual of harmonic Higgs bundle is still a harmonic Higgs bundle.

pf.

$$F_{h^*}(u)(v) = \nabla_{h^*} \circ \nabla_{h^*}(u)(v)$$

$$= \nabla_{h^*}(u)(\nabla_h(v)) + d(\nabla_{h^*}(u)(v))$$

$$= -u(\underbrace{\nabla_h \circ \nabla_h}(v)) + d(u(\nabla_h(v))) + d(-u(\nabla_h(v)) + d(u(v)))$$

$$= -u F_h(v)$$

$$= 0$$

$\boxtimes$

define inverse functor

$$\Xi_X' : \mathcal{C}_{\mathrm{dR}}(X, n) \to \mathcal{C}_{\mathrm{Dol}}(X, n)$$

$$(E, \nabla, h) \mapsto (E, \bar{\partial}_h, \varphi_h, h)$$

satisfies

$$\Xi_X \circ \Xi_X' = \mathrm{id}_{\mathcal{C}_{\mathrm{dR}}}. \qquad \Xi_X' \circ \Xi_X = \mathrm{id}_{\mathcal{C}_{\mathrm{Dol}}}.$$

$\boxed{\text{VK}}$

Rmk. From now on, we will directly call harmonic bundles.

$$(E, \bar{\partial}_E, \varphi, h). \quad \text{or} \quad (E, \nabla, D'', h)$$

Q: Under which conditions that a Higgs bundle (resp. flat bundle) admit a pluri-harmonic metric ?

A: Need stability!

Lecture 7

Kobayashi-Hitchin correspondence

(existence of pluri-harmonic metrics = stability + top. cond.)

Recall: categorical correspondence:

$$\mathcal{C}_{Dol}(X,n) \simeq \mathcal{C}_{dR}(X,n)$$

harmonic Higgs $\qquad\qquad$ harmonic flat

Q: When does a Higgs bundle (resp. flat bundle) admit a pluri-harmonic metric?

Today: solve **Q**!

Setting: (X,ω) cpt Kähler mfld. $\dim_{\mathbb{C}} X = m$

Main thm.
(1) A Higgs bundle over X admits a pluri-harmonic metric iff it is polystable with $c_1 = 0 = ch_2$.

Moreover, such metric, if exists, is unique up to scalar multiplication. (on each stable component)

(2) A flat bundle over X admits a pluri-harmonic metric iff it is semisimple. Moreover, ...

Part I. Kobayashi-Hitchin correspondence for Higgs bundles

§1. Stability for Higgs bundles

Def 1.1 A Higgs bundle $(\underbrace{E,\bar\partial_E}_{\mathcal{E}},\varphi)$ is called slope-stable (resp. slope-semistable) if for $\forall$ proper torsion-free coherent subsheaf $\mathcal{F}\subset\mathcal{E}$ s.t.

(1) $0<\mathrm{rk}\,\mathcal{F}<\mathrm{rk}\,\mathcal{E}$

(2) φ-invariant: $\varphi(\mathcal{F})\subset\mathcal{F}\otimes\Omega^1_X$ $\qquad \leftarrow \quad \varphi:\mathcal{E}\to\mathcal{E}\otimes\Omega^1_X$

the following inequality holds

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} \underset{(\le)}{<} \frac{\deg(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} =: \mu(\mathcal{E})$$

It is slope-polystable if it decomposes as the direct sum of slope-stable Higgs bundle of the same slope.

Rmk (1) $m > 1$. torsion-free sheaf $\mathcal{F}$ is locally free on $X \setminus \mathrm{Sing}(\mathcal{F})$

singular set. codim ≥ 2

(2) $m > 1$. there are many stability conditions.

e.g. Gieseker stability defined by reduced Hilbert polynomials

$$p(\mathcal{F}) := \frac{\chi(\mathcal{F}(m))}{a_d} \qquad m \geq 0 \qquad \dim \mathcal{F} = d, \quad \chi(\mathcal{F}(m)) = \sum_{i=0}^{d} a_i \cdot m^i$$

Bridgeland stability on $D^b(X)$ ...

From now on. we only use stable / semistable / polystable to denote slope stability.

(3) To check stability it's enough to check the inequality for saturated subsheaves.

i.e. $\mathcal{E}/\mathcal{F}$ torsion-free.

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}/\mathcal{F} \to 0$$

$$\mu(\mathcal{F}) < (\leq)\ \mu(\mathcal{E}) \iff \mu(\mathcal{E}) \underset{(\leq)}{<} \mu(\mathcal{E}/\mathcal{F})$$

$$\deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{E}/\mathcal{F})$$

$$\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F}) + \mathrm{rk}(\mathcal{E}/\mathcal{F})$$

$T \subset \mathcal{E}/\mathcal{F}$ torsion part. i.e. $\left(\mathcal{E}/\mathcal{F}\right)/T$ torsion-free. $\Rightarrow \deg(T) \geq 0$

$$\Rightarrow (\mathcal{E}/\mathcal{F})_{tf} := \left(\mathcal{E}/\mathcal{F}\big/ T\right) \text{ has deg } \leq \deg(\mathcal{E}/\mathcal{F})$$

Ex $m = 1$ $g \geq 2$

$$\left(\mathcal{E} = K_X^{\frac{1}{2}} \oplus K_X^{-\frac{1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}\right) \qquad q_2 \in H^0(X, K_X^2)$$

$\otimes K_X$

Claim: $(\mathcal{E}, \varphi)$ is stable.

Indeed. if $q_2 = 0$. the only φ-inv. proper subbundle is $K_X^{-\frac{1}{2}}$. $\deg = 1-g < 0 = \deg \mathcal{E}$

if $q_2 \neq 0$. there is no φ-inv. proper subbundle.

In general. $\left(\mathcal{E} = K_X^{\frac{n-1}{2}} \oplus K_X^{\frac{n-3}{2}} \oplus \cdots \oplus K_X^{-\frac{n-1}{2}},\ \varphi = \begin{pmatrix} 0 & q_2 & \cdots & q_n \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & q_2 \\ & & 1 & 0 \end{pmatrix}\right)$ is stable.

As harmonic + $ch_2 = 0 \iff$ pluri-harmonic

To show Thm (1), it suffices to show harmonic $\iff$ polystable + $c_1 = 0$

Thm 1.2 $(E, \bar\partial_E, \varphi)$ admits a pluri-harmonic metric

$$\Rightarrow (E, \bar\partial_E, \varphi) \text{ polystable } + c_1 = 0 = ch_2$$

Pf. By definition, if h is a pluri-harmonic metric on $(E, \bar\partial_E, \varphi)$.

Then $\nabla_h := D_h + \varphi + \varphi^{*h}$ is flat $\Rightarrow c_i = 0$

$\mathcal{F} \subset \mathcal{E}$ saturated subsheaf that is φ-inv. $(0 < \mathrm{rk}\,\mathcal{F} < \mathrm{rk}\,\mathcal{E})$

$Q := \mathcal{E}/\mathcal{F}$.

$$0 \to \mathcal{F} \underset{\pi}{\leftrightarrows} \mathcal{E} \to Q \to 0$$

given h

$\Rightarrow$ on $\overset{X'}{\overset{\shortparallel}{X \setminus S'}}$, $\mathcal{E}$ splits

$$\mathcal{E} \overset{C^\infty}{\simeq} \mathcal{F} \oplus Q \quad \text{(not necessarily holo.)}$$

denote $\pi: \mathcal{E} \to \mathcal{F}$ projection.

$\mathrm{Id} - \pi: \mathcal{E} \to Q$

Aim write ∇_h in terms of π, D'', D'_h . . .

F_{∇_h} . . .

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in A^0(X', \mathcal{E})$. $u_1 \in A^0(X', \mathcal{F})$, $u_2 \in A^0(X', Q)$

write

$$\nabla_h \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \pi \circ \nabla_h & \beta \\ \alpha & (\mathrm{Id}-\pi) \circ \nabla_h \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\alpha \in A^1(X', \mathrm{Hom}(\mathcal{F}, Q))$

$\beta \in A^1(X', \mathrm{Hom}(Q, \mathcal{F}))$ $\quad$ 2nd fundamental forms

• $\alpha(u_1) = (\mathrm{Id} - \pi) \circ \nabla_h(u_1)$ $\quad \partial_h + \varphi^{*h}$ $\quad = \bar\partial_E + \varphi$

$= (\mathrm{Id} - \pi) \circ (D'_h + D'')(u_1)$

$$= (Id-\pi)\circ D_h'(u_1)$$

$$= D_h'\circ\pi(u_1) - \pi\circ D_h'(u_1) \qquad D_h'(\pi) = D_h'\circ\pi - \pi\circ D_h'$$

$$= D_h'(\pi)(u_1) \qquad \text{i.e. } \alpha = D_h'(\pi)$$

- $\beta(u_2) = \nabla_h(u_2) - (Id-\pi)\circ\nabla_h(u_2)$

$$= \pi\circ\nabla_h(u_2)$$

$$= \pi\circ(D_h'+D'')(u_2)$$

claim $D_h'(u_2) = 0$. indeed.

$$0 = h(\bar\partial_E(u_1), u_2) = -h(u_1, \partial_h(u_2))$$

$$0 = h(\varphi(u_1), u_2) = h(u_1, \varphi^{*h}(u_2))$$

$$\Rightarrow \quad h(u_1, D_h'(u_2)) = 0$$

$$\Rightarrow \beta(u_2) = \pi\circ D''(u_2)$$

$$= (\pi\circ D'' - D''\circ\pi)(u_2)$$

$$= -D''(\pi)(u_2) \qquad \text{i.e. } \beta = -D''(\pi)$$

$$\Rightarrow \quad \nabla_h\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\pi\circ\nabla_h & -D''(\pi)\\ D_h'(\pi) & (Id-\pi)\circ\nabla_h\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$\Rightarrow \quad F_h\begin{pmatrix}u_1\\u_2\end{pmatrix} = \nabla_h^2\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}F_{\pi\circ\nabla_h} - D''(\pi)\circ D_h'(\pi) & *_1\\ *_2 & F_{(Id-\pi)\circ\nabla_h} - D_h'(\pi)\circ D''(\pi)\end{pmatrix}$$

In other words. let $\nabla_{\mathcal{F}} := \pi\circ\nabla_h$

$$\Rightarrow \pi\circ F_h\circ\pi = F_{\nabla_{\mathcal{F}}} - D''(\pi)\circ D'(\pi)$$

$$\Rightarrow \pi\circ\Lambda_\omega F_h\circ\pi = \Lambda_\omega F_{\nabla_{\mathcal{F}}} - \Lambda_\omega D''(\pi)D'(\pi)$$

$$\Rightarrow \deg(\mathcal{F}) = \frac{\sqrt{-1}}{2\pi}\int_X \operatorname{Tr}(\Lambda_\omega F_{\nabla_{\mathcal{F}}})\frac{\omega^m}{m!}$$

$$= \frac{\sqrt{-1}}{2\pi}\int_X \mathrm{Tr}\,(\pi\,\Lambda\omega F_h)\,\frac{\omega^m}{m!} + \frac{\sqrt{-1}}{2\pi}\int_X \mathrm{Tr}\left(\Lambda\omega D''(\pi)\, D'(\pi)\right)\frac{\omega^m}{m!}$$

$$= -\|D''(\pi)\|_{L^2}^2 \leq 0$$

i.e. $\deg(\mathcal{F}) \leq \deg(\mathcal{E})$ "=" $\Leftrightarrow D''(\pi) = 0$

$$\Leftrightarrow (\bar\partial_E + \varphi)(\pi) = 0$$

$$\pi : \mathcal{E} \to \mathcal{F}$$

i.e. $\mathcal{F} \subset \mathcal{E}$ φ-inv. holo. subbundle.

i.e. $\mathcal{E} \simeq \mathcal{F} \oplus Q$ holo. splitting & preserves Higgs fields

Continue. after finite steps. will stop.

$$\rightsquigarrow (\mathcal{E}, \varphi) = \bigoplus_{i=1}^{\ell} (\mathcal{E}_i, \varphi_i) \qquad \text{polystable.}$$

$\boxtimes$

$$\bar\partial_E = \begin{pmatrix} \bar\partial_F & \alpha \\ 0 & \bar\partial_Q \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \varphi_F & \beta \\ 0 & \varphi_Q \end{pmatrix}$$

$\nabla_h \quad F_h \quad \cdots$

$$\deg(\mathcal{E}) = \deg(\mathcal{F}) + \|\alpha\|_{L^2}^2 + \|\beta\|_{L^2}^2$$

§2. Existence of pluri-harmonic metrics.

Thm 2.1 (Uhlenbeck–Yau, Popovici (simplified proofs))

$(E, \bar\partial_E)$ holo. v.b. over X $\quad h$ hermitian metric.

$\pi \in W^{1,2}(X, \mathrm{End}(E))$ be a section s.t.

(1) $\pi^2 = \pi^{*h} = \pi$

(2) $(\mathrm{Id} - \pi)\circ \bar\partial_E(\pi) = 0$

holds almost everywhere.

$\Rightarrow \exists\ \mathcal{F} \subset \mathcal{E}$ coherent subsheaf & S codim ≥ 2 s.t.

(1) $\pi \in C^\infty(X\setminus S, \mathrm{End}(E))$

(2) $\pi^2 = \pi^{*h} = \pi$ & $(\mathrm{Id}-\pi)\circ \bar\partial_E(\pi) = 0$ holds on $X\setminus S$

(3) $\mathcal{F}|_{X\setminus S} = \pi_{X\setminus S}(\mathcal{E}|_{X\setminus S})$ holo. subbundle of $\mathcal{E}|_{X\setminus S}$

Rmk holds for $(E, \bar\partial_E, \varphi)$ & h

$$\bar\partial_E \rightsquigarrow D'' = \bar\partial_E + \varphi.$$

Thm 2.2 (Hitchin ($n=1$, $r=2$), Simpson (general))

Polystable + $c_1 = 0 \Rightarrow$ there exists a harmonic metric.

Fix K. $H(t) = K\cdot h(t)$ $h(t) \in \mathrm{Aut}(E)$ s.t. $h(t) = h(t)^{*K}$

$\Rightarrow h = K^{-1}\cdot H$

Lem 2.3 (1) $D'_H = D'_K + h^{-1}D'_K(h)$

(2) $\sqrt{-1}\Lambda_\omega(F_H) = \sqrt{-1}\Lambda_\omega(F_K) + \sqrt{-1}\Lambda_\omega D''(h^{-1}D'_K(h))$

Introduce for $\forall$ h_0 hermitian metric

$$\Delta'_{h_0} := (D'_{h_0})^* D'_{h_0} + D'_{h_0}(D'_{h_0})^*$$

Cor 2.4

$$\Delta'_K(h) = h\sqrt{-1}\Lambda_\omega(F_H - F_K) + \sqrt{-1}\Lambda_\omega\left(D''(h)h^{-1}D'_K(h)\right)$$

pf. KI for Higgs.

$$\Delta'_K(h) = (D'_K)^* D'_K(h) + D'_K \underbrace{(D'_K)^*(h)}_{=0}$$

$$= \sqrt{-1}\Lambda_\omega D'' D'_K(h)$$

$$\overset{\text{Lem 2.3 (2)}}{=} \cdots$$

$\square$

Donaldson heat flow:

$$H(0) = K$$

$$H(t) = K \cdot h(t) \qquad \det(h) = 1$$

$$H^{-1} \frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega F_H \qquad (*)$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} = K^{-1}\frac{\partial H}{\partial t} = K^{-1} H \cdot H^{-1} \frac{\partial H}{\partial t}$$

$$= h \cdot H^{-1} \frac{\partial H}{\partial t}$$

$$\overset{(*)}{=} -h\sqrt{-1}\Lambda_\omega F_H.$$

$\Rightarrow$ Cor +

$$\left(\frac{\partial}{\partial t} + \Delta'_K\right)(h) = -h\sqrt{-1}\Lambda_\omega F_K + \sqrt{-1}\Lambda_\omega \left(D''(h)h^{-1}D'_K(h)\right) \qquad (**)$$

Donaldson functional:

$$M : \mathrm{Herm}(E) \times \mathrm{Herm}(E) \to \mathbb{R}$$

$$\frac{d}{dt} M(H,K) = \int_X \mathrm{Tr}\left(\sqrt{-1}\Lambda_\omega F_H \cdot H^{-1}\frac{\partial H}{\partial t}\right)\frac{\omega^m}{m!}$$

Lem 2.5 $\quad M(H,K) = \int_0^1 \int_X \mathrm{Tr}(\cdots)\frac{\omega^m}{m!}\,dt$ is path-independent.

$\forall$ $H(t)$, $H'(t)$. with

$$\begin{cases} H(0) = H'(0) = K \\ H(1) = H(1) \\ \det(H) = \det(K) = \det(H') \end{cases}$$

$$\Rightarrow M(H,K) = M(H',K)$$

Hence choose a special path $H(t) = K \cdot e^{ts}$ $\quad \mathrm{Tr}(s) = 0$

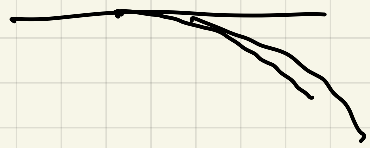
$$H^{-1}\frac{\partial H}{\partial t} = s$$

$$\Rightarrow M(H,K) = \int_X \mathrm{Tr}(\sqrt{-1}\Lambda_\omega F_K \cdot s)\frac{\omega^m}{m!} + \int_X (\Psi(s)D''s, D's)\frac{\omega^m}{m!}$$

idea: along heat flow.

$$\frac{d}{dt} M(H(t), H(0)) = -\|\Lambda_\omega F_H\|^2_{L^2}$$

$$\leq 0$$



Initial value

$$M(H(0), H(0)) = 0$$

Thm 2.6 (Simpson's main estimate)

$$H = Ke^s \qquad \mathrm{Tr}(s). \qquad \sup_X |s| < \infty$$

if $(E, \bar\partial_E, \varphi)$ stable with $\sup_X |\Lambda_\omega F_H| \leq C$

$\Rightarrow$ $M(H, K)$ is bdd from below.

$$\sup_X |s| \leq C_1 + C_2 M(H, K)$$

Pf need Uhlenbeck-Yau's construction to show. !

Next. $\sup_X |\Lambda_\omega (F_H)| \leq C$

$$\frac{d}{dt} \underbrace{\Lambda_\omega F_H} = -\sqrt{-1} \Lambda_\omega D'' D'_H (\Lambda_\omega F_H)$$

$$\Rightarrow \frac{d}{dt} |\Lambda_\omega F_H|^2 = 2\sqrt{-1}\, \mathrm{Tr}(\Lambda_\omega F_H \cdot \Lambda_\omega D'' D'_H (\Lambda_\omega F_H))$$

$$\Delta' |\Lambda_\omega F_H|^2 = \cdots$$

$$= -2\sqrt{-1}\, \mathrm{Tr}\left(\Lambda_\omega F_H \cdot \Lambda_\omega D'' D'_H (\Lambda_\omega F_H)\right) - 2|D''(\Lambda_\omega F_H)|^2$$

$\Rightarrow$

$$\left(\frac{d}{dt} + \Delta'\right) |\Lambda_\omega F_H|^2 = -2|D'' \Lambda_\omega F_H|^2$$

$$\leq 0$$

$\to$ parabolic PDE $\quad \sup_X |\Lambda_\omega F_H| \leq C.$

Left: long existence of solutions to heat flow.

$$\frac{d^2}{dt^2} M(H(t), K) \geq 0$$

$\Rightarrow$ $\frac{d}{dt} M(H, K) \to 0$ $\quad t \to \infty$

i.e. $\Lambda_\omega F_{H_t} \to 0$ in L^2

taking $S_t \to S_\infty$ in $W^{2,p}$ $\quad$ ($H_t = K \cdot e^{S(t)}$)

$\underline{\underline{H_\infty}} := K \cdot h_\infty$
$= K \cdot e^{S_\infty}$

$\Rightarrow$

$$\hat{\Delta}_K (h_\infty) := -h_\infty \sqrt{-1} \Lambda_\omega F_K + \sqrt{-1} \Lambda_\omega D''(h_\infty) h_\infty^{-1} D'_K(h_\infty).$$

elliptic regularity to show h_∞ is $\underline{C^\infty}$

$\rightsquigarrow$ $H_\infty = K \cdot h_\infty$ satisfies

$\underline{\Lambda_\omega F_{H_\infty} = 0}$ $\quad$ i.e. H_∞ harmonic.

uniqueness:
suppose stable.

$$\Lambda_\omega F_{H_1} = \Lambda_\omega F_{H_2} = 0 \qquad h = H_1^{-1} \cdot H_2$$

$$\Rightarrow \frac{1}{2} \Delta(\mathrm{Tr}\, h) = \Delta'(\mathrm{Tr}\, h)$$
$$= -|D''(h) h^{-\frac{1}{2}}|^2$$
$$\leq 0$$

$\Rightarrow$ $\mathrm{Tr}\, h$ subharmonic $\overset{MP}{\Rightarrow}$ $\mathrm{Tr}\, h = C$

$D''(h) = 0 = D'(h)$

i.e. h is a holo. morphism of stable ~~Higgs~~ bundle.

stable $\Rightarrow$ simple i.e. $\mathrm{Aut} \simeq \mathbb{C} \cdot \mathrm{Id}$.

$\Rightarrow h = G'$.

$\Rightarrow$ uniqueness.

$\boxtimes$

Part II: Kobayashi-Hitchin correspondence for flat bundles

§3. Stability of flat bundles.

Def 3.1 A holo. flat bundle $(\mathcal{E}, D)$ over X is

- irreducible/simple if it has no non-trivial proper holo. flat subbundle.

$$(\mathcal{F}, D_{\mathcal{F}}) \text{ s.t. } \mathcal{F} \subset \mathcal{E} \text{ holo subbundle}$$
$$D(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$$
$$D_{\mathcal{F}} := D|_{\mathcal{F}}$$

- semisimple / completely reducible if it decomposes as the direct sum of irreducible ones

$$(\mathcal{E}, D) = \bigoplus_{i=1}^{\ell} (\mathcal{E}_i, D_i) \quad \text{for each } (\mathcal{E}_i, D_i) \text{ is irred.}$$

Rmk (1) similarly for (E, ∇)

(2) Not like Higgs bundles, we only consider subbundles for stability.

Lem 3.2 (Katz) Any coherent sheaf of $\mathcal{O}_X$-modules with integrable connection is locally free.

Lem 3.3 (André) Any coherent sheaf of $\mathcal{O}_X$-modules with a connection is locally free

Thm 3.4 (E, ∇) flat, harmonic metric $\Rightarrow$ semisimple.

Pf. h (E, ∇)

$\rightsquigarrow$ $(E, \bar\partial_h, \varphi_h)$ also admits h as a pluri-harmonic metric.

$\Rightarrow$ $(E, \bar\partial_h, \varphi_h)$ polystable $c_i = 0$

i.e. $(E, \bar\partial_h, \varphi_h) = \bigoplus_{i=1}^{\ell} (E_i, \bar\partial_{E_i}, \varphi_i)$

stable. $h_i = h|_{E_i}$ $c_i = 0$

$\rightsquigarrow$ $(E, \nabla) = \bigoplus_{i=1}^{\ell} (E_i, \nabla_i)$

$\nabla_i = \bar\partial_{E_i} + \partial_{h_i} + \varphi_i + \varphi_i^{*h_i}$

direct sum of irreducible flat bundles. $\boxtimes$

§4. Existence of pluri-harmonic metrics

Thm 4.1 (Donaldson ($m=1$, $n=2$). Corlette (general))

semisimple for (E, ∇) $\Rightarrow$ $\exists$ harmonic metric.

! up to scalar multiplication..

Pf ① Modify $H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega F_H$

to $H^{-1}\frac{\partial H}{\partial t} = -\sqrt{-1}\Lambda_\omega G_H$

$\rightsquigarrow$ ...

Pf ② R-H.

$\rho : \pi_1(X, x) \to GL(n, \mathbb{C}) \longleftrightarrow (E, \nabla)$

irreducible / simple $\longleftrightarrow$ irreducible / simple

semisimple $\longleftrightarrow$ semisimple

$U(n) \subset GL(n, \mathbb{C})$ maximal cpt.

$\mathfrak{u}(n)$ $\mathfrak{gl}(n, \mathbb{C})$

$\rightsquigarrow \quad \mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}$

$\rightarrow$ Killing form B on $\mathfrak{gl}(n,\mathbb{C})$ is positive-define on $\mathfrak{p}$

$\rightsquigarrow$ $GL(n,\mathbb{C})/U(n)$ is a Riemann mfd.

$$\left(T_{eU(n)} GL(n,\mathbb{C})/U(n) \simeq \mathfrak{p} \right)$$

$\mathcal{M} := \{ M \in GL(n,\mathbb{C}) : M \text{ positive-define}, \ M^* = M \}$

space of hermitian matrices

$GL(n,\mathbb{C}) \curvearrowright \mathcal{M}$

$$(g \cdot M) := (g^{-1})^* M g^{-1}$$

$\mathcal{M} \simeq GL(n,\mathbb{C})/U(n)$ via

$$g \cdot U(n) \longmapsto (g^{-1})^* g^{-1}$$

regarding $GL(n,\mathbb{C}) \curvearrowright GL(n,\mathbb{C})/U(n)$

$$g \cdot M := (g^{-1})^* M g^{-1}$$

$\rho : \pi_1(X, x) \to GL(n,\mathbb{C}) \quad \longleftrightarrow \quad (E := \tilde{X} \times_\rho \mathbb{C}^n, \ \nabla = d)$

Prop4.1 h on $(E, \nabla) \iff \rho$-equivariant map $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

i.e.
$$h_\rho(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot h_\rho(\tilde{x})$$
$$= (\rho(\gamma)^{-1})^* \cdot h_\rho(\tilde{x}) \cdot (\rho(\gamma))^{-1}$$
$$\forall \gamma \in \pi_1(X, x), \ \tilde{x} \in \tilde{X}$$

Pf:

"$\Leftarrow$" $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

$$I(X.E) = I(X.\tilde{X}\times_\rho \mathbb{C}^n) \Leftrightarrow C^\infty(u: \tilde{X} \to \mathbb{C}^n,\ \rho\text{-equiv. map})$$

i.e. $u(\gamma.\tilde{x}) = \rho(\gamma)u(\tilde{x})$

define h via

$$h(u.u')(\tilde{x}) := \langle u(\tilde{x}), h_\rho(\tilde{x})u'(\tilde{x})\rangle_{\mathbb{C}^n}$$

$$= u(\tilde{x})^T \overline{h_\rho(\tilde{x})u'(\tilde{x})}$$

"$\Rightarrow$" h on $(E.\nabla)$. define $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$

via

$$h(u.u')(\tilde{x}) = \langle u(\tilde{x}), h_\rho(\tilde{x})u'(\tilde{x})\rangle_{\mathbb{C}^n}$$

check h_ρ is ρ-equiv. $\boxtimes$

Recall.

self-adjoint w.r.t. h

$$\nabla = D_h + \Psi_h$$

unitary conn. w.r.t. h

Prop 4.3 $\Psi_h = -\frac{1}{2}h_\rho^{-1}\cdot dh_\rho$

pf.

$$dh(u,v) = h(D_h(u),v) + h(u, D_h(v))$$

$$\parallel$$

$$d(\langle u.h_\rho v\rangle_{\mathbb{C}^n}) = d(u^T\overline{h_\rho v})$$

$$= \cdots$$

$$= \underset{\substack{\parallel \\ (D_h+\Psi_h)}}{h(du.v)} + h(u.h_\rho^{-1}(dh_\rho v)) + \underset{\substack{\parallel \\ D_h+\Psi_h.}}{h(u,dv)}$$

Prop 4.4. $h_\rho : \tilde{X} \to GL(n,\mathbb{C})/U(n)$ is harmonic $\Leftrightarrow$ h is harmonic metric. $\boxtimes$

$\Leftrightarrow \Lambda_\omega G_h = 0$

h_ρ is critical pt of

$$E(h_\rho) := \frac{1}{2}\int_X |dh_\rho|^2 \frac{\omega^m}{m!}$$

i.e. h_ρ satisfies EL eqn.

$$d_\nabla^* d h_\rho = 0$$

Pf. Prop 4.3 $\Phi_h = -\frac{1}{2} h_\rho^{-1} d h_\rho$

$$\Rightarrow \quad E(h_\rho) = 4\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$$

find EL equation for $\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$

$$\tilde{h} = h \cdot e^{s(t)} \qquad e^{s(t)} \in \mathrm{Aut}(E). \qquad s(0) = 0$$

$$\Rightarrow \quad \delta_{\tilde{h}}' = \delta_h' + \delta_h'(s)$$

$$\delta_{\tilde{h}}'' = \delta_h'' + \delta_h''(s)$$

$$\Rightarrow \quad \varphi_{\tilde{h}} = \frac{1}{2}(d' - \delta_{\tilde{h}}') = \varphi_h - \frac{1}{2}\delta_h'(s)$$

$$\varphi_{\tilde{h}}^{*\tilde{h}} = \frac{1}{2}(d'' - \delta_{\tilde{h}}'') = \varphi_h^{*h} - \frac{1}{2}\delta_h''(s)$$

$$\Rightarrow \quad \int_X |\Phi_{\tilde{h}}|^2 \frac{\omega^m}{m!} = \int_X \langle \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^{*\tilde{h}} . \ \varphi_{\tilde{h}} + \varphi_{\tilde{h}}^{*\tilde{h}} \rangle_{g.\tilde{h}} \frac{\omega^m}{m!}$$

$$= \int_X |\Phi_h|^2 \frac{\omega^m}{m!} - \int_X \langle (\delta_h')^*(\varphi_h) + (\delta_h'')^*(\varphi_h^{*h}) . s \rangle$$

$$+ \frac{1}{4}\int_X |\delta_h'(s) + \delta_h''(s)|^2 \frac{\omega^m}{m!}$$

$\Rightarrow$ EL for $\int_X |\Phi_h|^2 \frac{\omega^m}{m!}$ is

$$(\delta_h')^*(\varphi_h) + (\delta_h'')^*(\varphi_h^{*h}) = 0 \qquad (***)$$

To show $(***) \Leftrightarrow \Lambda_\omega G_h = 0$

KI :

$$(d')^* = \sqrt{-1}[\Lambda_\omega, \delta_h'']$$

$$(d'')^* = -\sqrt{-1}[\Lambda_\omega, \delta_h']$$

$$(\delta_h')^* = \sqrt{-1}[\Lambda_\omega, d'']$$

$$(\delta_h'')^* = -\sqrt{-1}\,[\Lambda_\omega, d']$$

$$\overset{(***)}{\Rightarrow}\quad 0 = \Lambda_\omega\left(d''(\varphi_h) - d'(\varphi_h^{*_h})\right)$$

$$= \Lambda_\omega\left(\bar\partial_h(\varphi_h) + \cancel{[\varphi_h^{*_h}, \varphi_h]} - \partial_h(\varphi_h^{*_h}) - \cancel{[\varphi_h, \varphi_h^{*_h}]}\right)$$

$$= \Lambda_\omega\left(\bar\partial_h(\varphi_h) - \underbrace{\partial_h(\varphi_h^{*_h})}\right)$$

$$= 2\Lambda_\omega\left(\bar\partial_h(\varphi_h)\right)$$

$$= 2\Lambda_\omega G_h$$

$$\bar\partial_h(\varphi_h) = -\partial_h(\varphi_h^{*_h})$$

$$G_h = (D_h'')^2$$

$$= (\bar\partial_h + \varphi_h)^2$$

$$= \bar\partial_h(\varphi_h) + (2,0) + (0,2)$$

$\blacksquare$

Left: when is h_ρ harmonic map.

Thm 4.5 h_ρ harmonic map $\Leftrightarrow$ $\rho : \pi_1(X, x) \to GL(n, \mathbb{C})$ semisimple

$$\begin{cases} \dfrac{\partial h_{\rho,t}}{\partial t} = -d_\nabla^* d h_{\rho,t} \\ h_{\rho,0} = h_0 \end{cases}$$

$h_{\rho,\infty}$ is the solution. C^∞

§5. Conclusion

Algebraic description of $\mathcal{C}_{Dol}(X, n)$ & $\mathcal{C}_{dR}(X, n)$

By abuse of notations. use $\mathcal{C}_{Dol}(X, n)$: category of polystable Higgs bundle + $c_i = 0$ $rk = n$

$\mathcal{C}_{dR}(X, n)$: category of semisimple flat bundles $rk = n$.

Thm 5.1 (Nonabelian Hodge correspondence. categorical version)

$$\mathcal{C}_{Dol}(X,n) \simeq \mathcal{C}_{dR}(X,n) \quad \left(\simeq \mathcal{C}^{s.s}_{Loc}(X,n) \right)$$

Rmk. In fact, this can be generalized to semistable Higgs bundles.

Thm 5.2 (Simpson (projective). Nie-Zhang (Kähler mflds))

$$\mathcal{C}_{Higgs}(X,n) \simeq \mathcal{C}_{Flat}(X,n)$$

category of semistable Higgs bundles of rk n

category of flat bundles of rk n

Simpson idea: semistable + $c_i = 0$ are extension of stable + $c_i = 0$.

depends on Metha-Ramanathan hyperplane restriction thm.

Lecture 8

Moduli spaces and Hitchin morphisms

§1. Moduli spaces

$k = \mathbb{C}$

X smooth irreducible projective variety $/\mathbb{C}$

$(\mathrm{sch}/\mathbb{C})$: category of $\mathbb{C}$-schemes

(set) : category of sets

Def 1.1 We introduce the following 3 moduli functors: $(n \in \mathbb{Z}_{>0})$

(1) Betti moduli functor:

$$\tilde{M}_B(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{set})$$

via

- $\tilde{M}_B(X,n)(S) := \left\{ E \to X \times S \;\middle|\; \begin{array}{l} \cdot E \text{ locally constant sheaf of } \mathbb{C}\text{-v.s. } \mathrm{rk} = n \\ \cdot \text{ flat over } S \end{array} \right\} \Big/ \sim_S$

where $\sim_S$ means isomorphism:

$$E \sim_S E' \iff E \simeq E' \otimes \pi_S^* L \quad \text{for } L \to S \text{ line bundle}$$

$$\pi_S : X \times S \to S$$

- $\tilde{M}_B(X,n)(f) := f^* : \tilde{M}_B(X,n)(T) \to \tilde{M}_B(X,n)(S)$, where $f : S \to T$ and $f^* = (\mathrm{id}_X \times f)^*$

(2) De Rham moduli functor:

$$\tilde{M}_{dR}(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{set})$$

via

- $\tilde{M}_{dR}(X,n)(S) := \left\{ (\mathcal{E}, \nabla) \to X \times S : \begin{array}{l} \cdot\ \mathcal{E} \text{ vector bundle of } \mathrm{rk}\, n \\ \cdot \text{ flat over } S \\ \cdot\ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X \times S} \text{ integrable conn.} \end{array} \right\} \Big/ \sim_S$

(3) Dolbeault moduli functor:

$$\tilde{M}_{Dol}(X,n) : (\mathrm{Sch}/\mathbb{C})^{op} \to (\mathrm{Set})$$

via

$$\cdot\ \widetilde{M}_{Dol}(X,x,n)(S) := \left\{ (\mathcal{E},\varphi) \to X\times S \ \middle|\ \begin{array}{l} \cdot\ (\mathcal{E}_s,\varphi_s)\ p\text{-semistable Higgs bundles of rk } n \\ \cdot\ \text{flat over } S \\ \cdot\ \forall s\in S(\mathbb{C}).\ c_i(\mathcal{E}_s)=0 \text{ in } H^2,\quad \mathcal{E}_s = \mathcal{E}|_{X\times\{s\}} \\ \cdot\ P(\mathcal{E}) := P(\mathcal{E}(m)) = nP(\mathcal{O}_X) \end{array} \right\} \Big/ \sim_S$$

- p-stability: $\dfrac{P(\mathcal{E}(m))}{a_d}$
- $c_i=0$. p-semi-stability $=$ μ-semistability

  torsion free $\Rightarrow$ locally free.

Def 1.2 We also introduce the following 3 framed moduli functors:

(1) framed Betti ... :

$$\widetilde{R}_B(X,x,n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

$$\cdot\ \widetilde{R}_B(X,x,n)(S) := \left\{ (E,\beta) : \begin{array}{l} \cdot\ x\in X \\ \cdot\ E\ \dots \\ \cdot\ \beta : E|_{\{x\}\times S} \simeq \mathcal{O}_S^n \end{array} \right\} \Big/ \sim_S$$

(2) framed deRham ... :

$$\widetilde{R}_{dR}(X,x,n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

$$\cdot\ \widetilde{R}_{dR}(X,x,n)(S) := \left\{ (\mathcal{E},\nabla,\beta) : \begin{array}{l} \cdot\ x\in X \\ \cdot\ (\mathcal{E},\nabla)\ \dots \\ \cdot\ \beta : \mathcal{E}|_{\{x\}\times S} \simeq \mathcal{O}_S^n \end{array} \right\} \Big/ \sim_S$$

(3) framed Dolbeault ... :

$$\widetilde{R}_{Dol}(X,x,n) : (Sch/\mathbb{C})^{op} \to (Set)$$

via

$$\cdot\ \widetilde{R}_{Dol}(X,x,n)(S) := \left\{ (\mathcal{E},\varphi,\beta) : \begin{array}{l} \dots \\ \dots \\ \dots \end{array} \right\} \Big/ \sim_S$$

Def 1.3: Let $\mathcal{M} : (Sch/\mathbb{C})^{op} \to (Set)$ moduli functor i.e.

$S \mapsto$ iso. classes of S-flat family of ...

$f \mapsto f^* = (id\times f)^*$

A <u>fine moduli space</u> of $\mathcal{M}$ is a scheme $M \in Sch/\mathbb{C}$ s.t. M <u>represents</u> $\mathcal{M}$

A <u>coarse moduli space</u> of $\mathcal{M}$ is $M \in Sch/\mathbb{C}$ s.t. M <u>corepresents</u> $\mathcal{M}$

- represent: functor iso. $\eta : \mathcal{M} \xrightarrow{\sim} h_M := Hom(-, M)$
  - if exists, is unique up to iso.
  - $\exists$ universal family. element corresponds to $id_M \in h_M(M)$

$$\mathcal{M}(M) \overset{\eta_M}{\underset{\sim}{\Longleftrightarrow}} h_M(M) = Hom(M, M)$$
$$\mathcal{U} = \eta_M^{-1}(id_M) \longleftarrow id_M$$

- corepresent: . bijective: $\mathcal{M}(Spec\,\mathbb{C}) \xrightarrow{\eta_{Spec\,\mathbb{C}}} h_M(Spec\,\mathbb{C})$

  i.e. $\mathbb{C}$-pts in $M \Longleftrightarrow$ iso. classes of ...

  - universal property: $\forall$ natural trans. $\nu : \mathcal{M} \to h_T = Hom(-, T)$

    uniquely factors through $\eta : \mathcal{M} \to h_M$

$$\begin{tikzcd} & h_M \arrow[d, "h_f"] \\ \mathcal{M} \arrow[ur, "\eta"] \arrow[r, "\nu"'] & h_T \end{tikzcd} \quad \exists! f : M \to T$$

if exists, is unique up to iso.

Ex

(1) Quot scheme.

$X \in Sch/\mathbb{C}$, $\mathcal{F} \to X$ coherent sheaf

$\underline{Quot}_X(\mathcal{F}) : (Sch/\mathbb{C}) \to (Set)$

$$S \mapsto \{ q_S : \pi_S^* \mathcal{F} \twoheadrightarrow \mathcal{E} \} / \sim_S$$

$\exists$ fine moduli space, $Quot_X(\mathcal{F})$.

(2) Hilbert scheme: $X \in Sch/\mathbb{C}$

$\underline{Hilb}_X : (Sch/\mathbb{C}) \to (Set)$

$$S \mapsto \{ Y \subset X \times S \text{ closed subscheme flat over } S \} / \sim$$

$\exists$ fine moduli space. Hilb_X

But, the pathological behaviours s.t. no existence of moduli spaces:

(1) unboundedness: $\nexists\ \mathcal{F} \in \mathcal{M}(S)$ s.t. $\mathcal{F}|_{X\times\{s\}} \cong \cdots\cdot$

(2) jump phenomenon: $\exists\ \mathcal{F} \in \mathcal{M}(S)$ s.t. $\cdot\ \mathcal{F}|_{X\times\{s_1\}} \sim \mathcal{F}|_{X\times\{s_2\}}$ $\quad \forall s_1, s_2 \neq s_0$

$\cdot\ \mathcal{F}|_{X\times\{s\}} \nsim \mathcal{F}|_{X\times\{s_0\}}$ $\quad \forall s \neq s_0$

one direction: add stability s.t. family becomes "small".

Thm 1.4 (Simpson)

(1) $\exists$ fine moduli spaces $R_B(X,x,n)$, $R_{dR}(X,x,n)$, $R_{Dol}(X,x,n)$ for the functors $\tilde{R}_B(X,x,n)$, $\tilde{R}_{dR}(X,x,n)$, $\tilde{R}_{Dol}(X,x,n)$, resp.

they are quasi-proj.

(2) $\exists$ coarse moduli spaces $M_B(X,n)$, $M_{dR}(X,n)$, $M_{Dol}(X,n)$ for the functors $\tilde{M}_B(X,n)$, $\tilde{M}_{dR}(X,n)$, $\tilde{M}_{Dol}(X,n)$, resp.

they are quasi-proj.

Moreover, $M_B(X,n)$ is exactly the one we constructed before via affine GIT.

(3) $GL(n,\mathbb{C}) \curvearrowright R_B(X,x,n)$, $R_{dR}(X,x,n)$, $R_{Dol}(X,x,n)$

s.t.

$$R_B(X,x,n) \longrightarrow M_B(X,n)$$
$$R_{dR}(X,x,n) \longrightarrow M_{dR}(X,n)$$
$$R_{Dol}(X,x,n) \longrightarrow M_{Dol}(X,n)$$

are GIT quotients.
independent of $x \in X$

(4) By GIT, closed pts of M_B parametrizes closed orbits in R_B

M_{dR} $\quad$ - - $\quad$ M_{dR}

M_{Dol} $\quad$ - - $\quad$ M_{Dol}

i.e. iso. classes of polystable ...

or S-equivalent classes of semi-stable ...

Thm 1.5 (Nonabelian Hodge correspondence, moduli version. Simpson)

(1) Complex analytic iso (i.e. iso. as $\mathbb{C}$-analytic spaces)

$$M_B^{(an)}(X,n) \simeq M_{dR}^{(an)}(X,n)$$

(2) Real analytic iso. (i.e. homeor as top. spaces)

$$M_{dR}^{(top)}(X,n) \simeq M_{Dol}^{(top)}(X,n)$$

pf (sketch)

(1)
$$\begin{tikzcd} R_B^{(an)}(X,x,n) \arrow[r] \arrow[d, "//GL(n,\mathbb{C})"'] \arrow[dr, phantom, "\circlearrowright"] & R_{dR}^{(an)}(X,x,n) \arrow[d, "//GL(n,\mathbb{C})"] \\ M_B(X,n) \arrow[r] & M_{dR}(X,n) \end{tikzcd}$$

key step: show $R_B^{(an)}(X,x,n)$ and $R_{dR}^{(an)}(X,x,n)$ represent the same analytic moduli functor:

$$R^{(an)}(X,x,n) : (\mathrm{Aan})^{op} \to (\mathrm{Set})$$

$$S^{an} \mapsto \left\{ (\mathcal{F}, \beta) : \begin{array}{l} \cdot\, x \in X \\ \cdot\, \mathcal{F} \to X^{an} \times S^{an} \text{ locally free sheaf of } \mathcal{O}_{X^{an}\times S^{an}}\text{-modules} \\ \cdot\, \beta : \mathcal{F}|_{\{x\}\times S} \simeq \mathcal{O}_{S^{an}}^n \end{array} \right\} \Big/ \sim$$

$$\Rightarrow R_B^{(an)}(X,x,n) \simeq R_{dR}^{(an)}(X,x,n)$$

$$\Rightarrow M_B^{(an)}(X,n) \simeq M_{dR}^{(an)}(X,n)$$

(2) let $R_{Dol}^{J}(X,x,n) \subset R_{DR}(X,x,n)$ consists of:

- (E,φ,β) : $\cdot\,(E,\varphi) \to X$ Higgs bundles with pluri-harmonic metric h
  - $\cdot\,\beta : E_x \simeq \mathbb{C}^n$

- $\beta(h) \simeq J$ ← standard Em. product on $\mathbb{C}^n$.

i.e. consists of Higgs bundles admits pluri-harmonic metric compatible with the frame.

Similarly, define $R^J_{dR}(X, x, n) \subset R_{dR}(X, x, n)$

$$\begin{array}{ccc} R^J_{Dol}(X, x, n) & \longrightarrow & R^J_{dR}(X, x, n) \\ /U(n) \downarrow & & \downarrow /U(n) \\ M^{(top)}_{Dol}(X, n) & \longrightarrow & M^{(top)}_{dR}(X, n) \end{array}$$

- Nonabelian Hodge of categorical version provides homeo.

$$R^J_{Dol}(X, x, n) \simeq R^J_{dR}(X, x, n)$$

- show $R^J_{Dol/dR}(X, x, n) \to M^{(top)}_{Dol/dR}(X, n)$ proper.

Dol:

$(E_i, \bar\partial_{E_i}, \varphi_i, \beta_i)$ sequence of compatible frame Higgs bundles lying over some cpt $K \subset M_{Dol}$.

show. it converges.

- $\Rightarrow M^{(top)}_{Dol}(X, n) \simeq M^{(top)}_{dR}(X, n)$

$\boxtimes$

Ex ($n=1, m=1$) rk$=1$, dim $=1$ X cpt. R.S. $g \geq 2$. $x \in X$

- $M_B(X, 1) = \mathrm{Hom}(\pi_1(X, x), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$ affine
- $M_{Dol}(X, 1) = \{ (L, \varphi) : L \in \mathrm{Jac}(X),\ \varphi \in H^0(X, K_X) \}$

$\simeq \mathrm{Jac}(X) \times H^0(X, K_X)$

$\simeq T^*\mathrm{Jac}(X)$

- $M_{dR}(X, 1) \to \mathrm{Jac}(X)$ affine bundle. i.e. fibers are affine spaces modelled $H^0(X, K_X)$

twisted cotangent bundle

$\mathbb{C}^* \simeq \mathbb{R}^{>0} \times S^1$

§2. $\mathbb{C}^*$-action and Hitchin morphism.

$M_{Dol}(X,n)$ admits an algebraic action of $\mathbb{G}_m = \mathbb{C}^*$:

$$\mathbb{C}^* \times M_{Dol}(X,n) \longrightarrow M_{Dol}(X,n)$$

$$(t, (\mathcal{E},\varphi)) \longmapsto (\mathcal{E}, t\varphi)$$

does not change the stability.

Denote by $M_{Dol}(X,n)^{\mathbb{C}^*}$ the set of $\mathbb{C}^*$-fixed pts.

<u>Lem 2.1</u> $(E,\bar\partial_E,\varphi) \in M_{Dol}(X,n)^{\mathbb{C}^*} \Longleftrightarrow (E,\bar\partial_E,\varphi)$ is a system of Hodge bundles

i.e. $\left((E,\bar\partial_E) = \bigoplus_{i=1}^{\ell} (E_i, \bar\partial_{E_i}),\ \varphi = \begin{pmatrix} 0 & & & \\ \varphi_1 & 0 & & \\ & \ddots & \ddots & \\ & & \varphi_{\ell-1} & 0 \end{pmatrix} \right)$

$$\varphi_i : \mathcal{E}_i \to \mathcal{E}_{i+1} \otimes_{\mathcal{O}_X} \Omega^1_X$$

pf $[(E,\bar\partial_E,\varphi)] = [(E,\bar\partial_E, t\varphi)]$ for some $t \in \mathbb{C}^* \setminus U(1)$
not root of unity

$\Longleftrightarrow (E,\bar\partial_E,\varphi)$ system of Hodge bundles

"$\Leftarrow$" If $(E,\bar\partial_E,\varphi)$ is, then

$$g_t = \begin{pmatrix} t^a Id_{E_1} & & & \\ & t^{a+1} Id_{E_2} & & \\ & & \ddots & \\ & & & t^{a+\ell-1} Id_{E_\ell} \end{pmatrix}, \ \forall a \in \mathbb{Z}$$

$$\Rightarrow \begin{cases} g_t \circ \bar\partial_E \circ g_t^{-1} = \bar\partial_E \\ g_t \circ \varphi \circ g_t^{-1} = t\varphi \end{cases}$$

"$\Rightarrow$" If $[(E,\bar\partial_E,\varphi)] = [(E,\bar\partial_E,t\varphi)]$

$\Rightarrow \exists\, g_t$ s.t. $\begin{cases} g_t \circ \bar\partial_E \circ g_t^{-1} = \bar\partial_E \\ g_t \circ \varphi \circ g_t^{-1} = t\varphi \end{cases}$ $\Rightarrow \bar\partial_E(g_t) = 0$ i.e. g_t holo.

consider the $\det(\lambda \mathrm{Id} - g_t)$. coefficents are holo. functions on X

$\rightsquigarrow$ coefficents are constant.

$\rightsquigarrow$ eigenvalues of g_t are constant.

$$\Rightarrow \mathcal{E} = \bigoplus_\lambda \mathcal{E}_\lambda \quad \text{for } \mathcal{E}_\lambda := \ker\left((g_t - \lambda \mathrm{Id})^n\right)$$

$$g_t \circ \varphi \circ g_t^{-1} = t\varphi \Leftrightarrow g_t \circ \varphi = t\varphi \circ g_t$$

$$\Rightarrow (g_t - t\lambda \mathrm{Id})^n \circ \varphi = t^n \varphi \circ (g_t - \lambda \mathrm{Id})^n$$

$$\Rightarrow \varphi : \mathcal{E}_\lambda \to \mathcal{E}_{t\lambda}$$

i.e. $\lambda, t\lambda, t^2\lambda, \dots$

$t \in \mathbb{C}^* \setminus U(1)$. $\lambda, t\lambda, \dots, t^{\ell-1}\lambda$ are eigenvalues

but $t^{-1}\lambda$, $t^\ell\lambda$ not eigenvalues

$$\rightsquigarrow (\mathcal{E}, \varphi) = \left(\bigoplus_{i=1}^{\ell} \mathcal{E}_i, \ \varphi = \begin{pmatrix} 0 & & & \\ \varphi_1 & \ddots & & \\ & \ddots & \ddots & \\ & & \varphi_{\ell-1} & 0 \end{pmatrix} \right)$$

$\boxtimes$

From now on. work on $m=1$. i.e. X cpt. R.S. $g \geq 2$

Def 2.2 The affine space $A := \bigoplus_{i=1}^n H^0(X, \mathrm{Sym}^i K_X) = \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$ is called the Hitchin base.

Hitchin map $h : M_{Dol}(X, n) \longrightarrow A$

$$(\mathcal{E}, \varphi) \mapsto \text{coefficents of } \det(\lambda \mathrm{Id} - \varphi)$$

namely.

$$(\mathcal{E}, \varphi) \mapsto (a_1, \dots, a_n) \qquad a_i = (-1)^i \operatorname{Tr}(\Lambda^i \varphi)$$

Rmk. (1) $\dim A \overset{\text{Riemann-Roch}}{=} n^2(g-1) + 1 = \frac{1}{2} \dim M_{Dol}(X, n)$

(2) h can be defined for X has $m > 1$

$$h : M_{Dol}(X, n) \to A := \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i \Omega_X^1)$$

(3) Motivation for h in Lie theory (especially for G)

G reductive gp $/\mathbb{C}$ $\qquad \mathfrak{g} = \mathrm{Lie}(G)$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalg.

W Weyl group associated to $\mathfrak{h}$ $\qquad W \curvearrowright \mathfrak{h}$

$\qquad\qquad G \curvearrowright \mathfrak{g}$

Chevalley's restriction thm:

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\simeq} \mathbb{C}[\mathfrak{h}]^W$$

$\rightsquigarrow$

$\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\!\!/G \simeq \mathfrak{h}/W$ (Chevalley morphism)

choose homog. poly. $p_1, \dots p_k$ of $d_1, \dots, d_k$ as deg. for $\mathbb{C}[\mathfrak{h}]^W$

as basis

$\rightsquigarrow$ iso $\mathfrak{h}/W \otimes K_X \simeq \bigoplus_{i=1}^{k} K_X^{\otimes d_i}$

$\rightsquigarrow$ induces a map $\quad M_{Dol}(X, n) \longrightarrow \bigoplus_{i=1}^{k} H^0(X, K_X^{\otimes d_i})$

$\qquad\qquad (\mathcal{E}, \varphi) \longmapsto (p_1(\varphi), \dots p_k(\varphi))$

Thm 2.3 (Hitchin ($n=1$), Simpson ($n>1$), Nitsure ($n=1$, L-twisted))

$\varphi : \mathcal{E} \to \mathcal{E} \otimes L$

h is proper

Pf $\quad (\mathcal{E}, \bar\partial_E, \varphi)$ harmonic bundle. h harmonic metric

Lem 2.4 Given C_1. $\exists C_2$ s.t. all eigenvalue of φ is bounded

$$|\lambda_i|_g \le C_1$$

$$\Rightarrow \quad |\varphi|_{g,h} \le C_2$$

To show h proper, it suffices to show any sequence of harmonic bundles

$(\mathcal{E}_i, \varphi_i, h_i)$ lying in the inverse of some cpt $K \subset A$.

it has a convergent subsequence.

By lemma. $\Rightarrow$ $|\varphi_i| \le C_2$

$$F_{h_i} + [\varphi_i, \varphi_i^{*h_i}] = 0 \Rightarrow F_{h_i} = -[\varphi_i, \varphi_i^{*h_i}]$$

$$\overset{\shortparallel}{F_{D_{h_i}}} \qquad\qquad |F_{h_i}| \le C'$$

By Uhlenbeck's weakly compactness thm.

$\Rightarrow$ $\exists$ unitary connection $\partial_h + \bar\partial$, subsequence $\{i'\}$, C^∞-auto. $g_{i'}$, φ

s.t.

$$g_{i'}^*(h_i) = h$$

$$g_{i'}^*(\bar\partial_{i'}) - \bar\partial, \quad g_{i'}^*(\partial_{h_{i'}}) - \partial_h, \quad g_{i'}^*(\varphi_{i'}) - \varphi.$$

$\longrightarrow 0$ weakly in $W^{1,p}$

$\longrightarrow 0$ strongly in L^p

$\Rightarrow$ $\bar\partial\varphi = 0$

$(\varphi \wedge \varphi = 0)$

$\bar\partial\varphi = 0$

$\Rightarrow$ $(\bar\partial, \varphi)$ Higgs bundle. h is harmonic metric for $(\bar\partial, \varphi)$

$$\nabla_h^2 = (\partial_h + \bar\partial + \varphi + \varphi^{*h})^2 = 0$$

i.e. $(\bar\partial_{i'}, \varphi_{i'}, h_i) \rightarrow (\bar\partial, \varphi, h)$ weakly ~~strongly~~ in $W^{1,p}$

strongly in L^p.

Elliptic regularity $\Rightarrow$ C^∞ convergent.

$\boxtimes$

Cor 2.5. h is surjective.

Rmk. Not true for $n > 1$.

Pf (Beauville - Narasimhan. Ramanan)

$$T^*N^s_{(X,n)} \longrightarrow A \quad \text{dominant.}$$

$\cap$ $\leftarrow$ open dense.

$M_{Dol(X,n)}$ $\qquad N^s(X,n)$ stable vect. b. moduli space

~~f~~ proper $\Rightarrow$ h is surjective. ∎

Rmk: h is $\mathbb{C}^*$-equivariant in the sense that:

$$h(\mathcal{E},\varphi) = (a_1, \dots, a_n)$$

$$h(t.(\mathcal{E},\varphi)) = (ta_1, \dots, t^n a_n)$$

$\mathbb{C}^* \curvearrowright \mathbb{A}$ weighted action

Cor 2.6 $\forall (\mathcal{E},\varphi) \in M_{Dol}(X,n)$, $\lim_{t\to 0} t.(\mathcal{E},\varphi)$ exists in $M_{Dol}(X,n)$

$\forall a \in \mathbb{A}$

$\lim_{t\to 0} t.a = 0$ h is $\mathbb{C}^*$-equiv. & proper $\Rightarrow$ $\lim_{t\to 0} t.(\mathcal{E},\varphi) \in h^{-1}(0)$

As $\lim_{t\to 0} t.(\mathcal{E},\varphi)$ is $\mathbb{C}^*$-fixed pt.

$\overset{\text{lem}}{\Longrightarrow}$ $M_{Dol}(X,n)^{\mathbb{C}^*}$ systems of Hodge bundles.

$$\forall (\mathcal{E},\varphi) \in M_{Dol}^{\mathbb{C}^*} \quad \lim_{t\to 0} t(\mathcal{E},\varphi) = (\mathcal{E},\varphi)$$

$$\Rightarrow \quad M_{Dol}(X,n)^{\mathbb{C}^*} = \left\{ \lim_{t\to 0} t.(\mathcal{E},\varphi) \ : \ (\mathcal{E},\varphi) \in M_{Dol}(X,n) \right\}$$

Bialynicki-Birula:
got stratification of $M_{Dol}(X,n)$ into locally closed subsets

§3. Spectral correspondence (BNR correspondence)

As h is surjective.

Q: What is $h^{-1}(a)$ for general $a \in \mathbb{A}$?

Def 3.1 $\forall a = (a_1, \dots, a_n) \in \mathbb{A}$. the associated spectral curve is

$$\Sigma_a : \lambda^n + \sum_{i=1}^n a_i \lambda^{n-i} = 0 \quad \subset T^*X$$

In particular. $\Sigma_{(\xi,\varphi)} := \Sigma_{h(\xi,\varphi)}$

n-sheeted cover:

$\Sigma_a \subset \mathrm{Tot}(K_X)$

$\downarrow \pi_a$

X

Rmk: Σ_a may be singular, reducible, non-reduced.

BNR: set $U := \{a \in \mathcal{A} : \Sigma_a \text{ smooth}\} \subset \mathcal{A}$ dense open

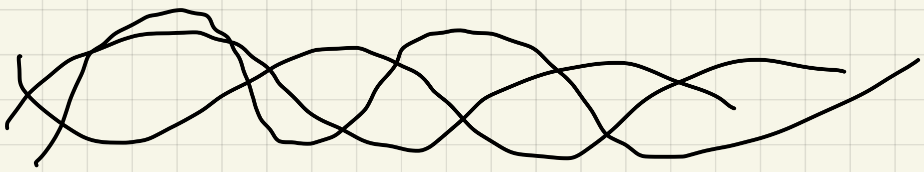
generic spectral curve is smooth.

Smoothness can be characterized via

$$\Delta_a := \prod_{i>j} (\lambda_i - \lambda_j)^2 \in H^0(X, K_X^{n(n-1)})$$

zeros of Δ_a

if Δ_a has only simple zero, then Σ_a is smooth.



Prop 3.2 For a generic Σ_a, it has $2n(n-1)(g-1)$ ramification pts

and genus $g(\Sigma_a) = n^2(g-1)+1 = \frac{1}{2}\dim M_{Dol}(X,n)$

pf.

$\# = \deg(K_X^{n(n-1)}) = 2n(n-1)(g-1)$

Riemann-Hurwitz formula:

$\pi_a : \Sigma_a \to \Sigma$ n-sheeted cover, ramif. index 2

$$\chi(\Sigma_a) = n \cdot \chi(X) - \#$$

$$\overset{\shortparallel}{2-2g(\Sigma_a)} \quad = n\cdot(2-2g) - 2n(n-1)(g-1)$$

$$\Rightarrow g(\Sigma_a) = n^2(g-1)+1.$$

$\boxed{}$

Thm 3.7 (BNR corresp. generic)

For generic Σ_a. we have the following bijective corresp.

$$\mathrm{Pic}^d(\Sigma_a) \longleftrightarrow h^{-1}(a)$$

$\uparrow$ iso. classes of line bundles of deg $d = n(n-1)(g-1)$

Higgs bundles in $h^{-1}(a)$ are stable

pf:

$L \in \mathrm{Pic}^d(\Sigma_a)$

$E := \pi_{a*}L \to X$ rk n bundle.

$\pi_a : \Sigma_a \to X$ ($K_X \downarrow$) $\qquad \pi_a^* K_X \qquad \pi^* K_X$ for $\pi : \mathrm{Tot}(K_X) \to X$

tautological section σ. $\qquad \sigma(y) = y$

$\sigma_a := \sigma|_{\Sigma_a} \in \pi_a^* K_X$

$$L \xrightarrow{\cdot\sigma_a} L \otimes_{\mathcal{O}_{\Sigma_a}} \pi_a^* K_X \quad \rightsquigarrow \quad \pi_{a*}L \xrightarrow{\varphi} \pi_{a*}L \otimes_{\mathcal{O}_X} K_X$$

$(E.\varphi)$

Riemann-Roch.

$$\underset{\substack{\| \\ \deg(E) + n(1-g)}}{\chi(X.\ \overset{\substack{E \\ \|}}{\pi_{a*}L})} = \underset{\substack{\| \\ d + (1-g)}}{\chi(\Sigma_a.\ L)}$$

$$\Rightarrow d = n(n-1)(g-1)$$

if $(E.\varphi)$ not stable. then $\det(\lambda Id - \varphi)$ reducible

$\Rightarrow \Sigma_a$ reducible. $\lightning$

$\boxed{}$

Thm 3.4 (BNR)

Σ_a integral curve (irreducible + reduced) (can singular)

$$\overline{\mathrm{Pic}^d(\Sigma_a)} \longleftrightarrow h^{-1}(a)$$

$\uparrow$

iso. classes of torsion-free rk 1 sheaves on Σ_a

compactification of $\mathrm{Pic}^d(\Sigma_a)$ by adding torsion-free but not invertible sheaves of rk 1.