QUANTUM SCHUBERT CALCULUS FOR SMOOTH SCHUBERT DIVISORS OF $F\ell_n$

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Abstract. We propose to study the quantum Schubert calculus for Schubert varieties, and investigate the smooth Schubert divisors X of the complete flag variety $F\ell_n$. We provide a Borel-type ring presentation of the quantum cohomology of X. We derive the quantum Monk-Chevalley formula for X by geometric arguments. We also show that the quantum Schubert polynomials for X are the same as that for $F\ell_n$ introduced by Fomin, Gelfand and Postnikov.

1. Introduction

Schubert problems, which count the number of geometric objects with given geometric constraints, are fundamental to enumerative geometry. Here the central objects are flag varieties G/P together with their Schubert subvarieties. The *classical Schubert calculus*, in modern language, is about the study of the integral cohomology ring $H^*(G/P,\mathbb{Z})$. The Schubert classes σ^u of the Schubert varieties form an additive basis of $H^*(G/P,\mathbb{Z})$. A profound understanding of the cohomology ring mainly consists of the following three parts:

(1) A ring representation of the form $H^*(G/P,\mathbb{Z})=\mathbb{Z}[\mathbf{x}]/I$.
(2) A (manifestly positive) formula of the Schubert constants $c_{u,v}^w$ in the cup product $\sigma^u\cup\sigma^v=\sum_w c_{u,v}^w\sigma^w$.
(3) A Schubert polynomial $\mathfrak{S}_u(\mathbf{x})\in\mathbb{Z}[\mathbf{x}]$ that represents the Schubert class σ^u in the aforementioned ring presentation $\mathbb{Z}[\mathbf{x}]/I$.

We refer to the very nice article [BGP25] and the references therein for the progress of classical Schubert calculus with an emphasis on the case $G=SL(n,\mathbb{C})$. With the Gromov-Witten theory introduced in 1990s, the classical cohomology $H^*(G/P,\mathbb{Z})$ can be deformed to the integral (small) quantum cohomology ring $QH^*(G/P,\mathbb{Z})=(H^*(G/P,\mathbb{Z})\otimes\mathbb{Z}[\mathbf{q}],\star)$, by incorporating 3-pointed, gezus-0 Gromov-Witten invariants. In particular, we can write $\sigma^u\star\sigma^v=\sum_{w,\mathbf{d}}N_{u,v}^{w,\mathbf{d}}\sigma^w\mathbf{q}^{\mathbf{d}}$, where $N_{u,v}^{w,\mathbf{0}}=c_{u,v}^w$. There have been extensive studies of the *quantum Schubert calculus*, namely of the quantum versions of the above (1)-(3) for $QH^*(G/P,\mathbb{Z})$ (see e.g. the survey [LL17] and the references therein).

All the Schubert varieties, including flag varieties as special cases, have CW complex structures by Schubert cells. In return, the integral cohomology of a Schubert variety X_w inside G/P is torsion free, and has an additive basis of Schubert classes indexed by the Weyl group elements u satisfying $u\le w$ with respect to the Bruhat order.

Question 1.1. *What is the (extended) Schubert calculus for Schubert varieties?*

The natural inclusion $\iota: X_w\hookrightarrow G/P$ induces a surjective ring homomorphism $\iota^*: H^*(G/P,\mathbb{Z})\to H^*(X_w,\mathbb{Z})$ with kernel $I_w=\sum_{u\not\le w}\sigma^u$. Hence, the (extended) classical

Schubert calculus for Schubert varieties is trivial in the sense that all points (1)-(3) can be reduced to that for the flag varieties, with the price that the ring presentation $H^*(X_w,\mathbb{Z}) = H^*(G/P,\mathbb{Z})/I_w$ being not good enough. We refer to [ALP92, GR02, DMR07, RWY11, DY24] for the study of the ring presentation of the cohomology of Schubert varieties.

The (extended) quantum Schubert calculus for Schubert varieties is highly nontrivial. First of all, we have to restrict to the smooth ones, since there was no Gromov-Witten theory for singular (Schubert) varieties yet. To our knowledge, there have been very few pioneer studies [Pec13, MS19, HKLS25] in different context. The odd symplectic Grassmannian $IG(k,2n+1)$ is a smooth Schubert variety of the symplectic Grassmannian $IG(k,2n+2)$, a flag variety G/P with $G=Sp(2n,\mathbb{C})$. In [Pec13], Pech studied the case $k=2$, which happens to be a general hyperplane section of the complex Grassmannian $Gr(2,2n+1)$. She did a relatively complete quantum Schubert calculus for a non-homogeneous Schubert variety for the first time, by providing a ring presentation, the quantum Pieri formula (a partial formula for the quantum version of point (2), see also [GLLX25]), and the quantum Giambelli formula (i.e. the quantum version of (3)). In [MS19], Mihalcea and Shifler provided the (equivariant) quantum Chevally formula for $IG(k,2n+1)$ by using the curve neighborhood technique [BM15]. In [HKLS25], Hu, Ke, Li and Song provided a ring presentation for the quantum cohomology of the blowup of $Gr(2,n)$ along $Gr(2,n-1)$ for the purpose of studying mirror symmetry, which happens to be a Schubert divisor in a two-step flag variety. The special case when $n=3$ is the blowup of $\mathbb{P}^2$ at point, which has been well studied much earlier. Despite being a very natural extension from the viewpoint of Schubert calculus, the quantum Schubert calculus for smooth non-homogeneous varieties is still largely uncharted territory, with many aspects awaiting exploration.

The complete flag variety $F\ell_n := \{V_1 \le \cdots \le V_{n-1} \le \mathbb{C}^n \mid \dim V_i = i, \forall 1 \le i < n\}$ is the quotient of $G=SL(n,\mathbb{C})$ by the Borel subgroup of upper tiangular matrices in G. Denote by $F_\bullet$ the standard complete flag. Each permutation $w \in S_n$ labels a Schubert variety X_w of dimension $\ell(w)$ defined by ranking condition of the form $X_w = \{V_\bullet \mid \dim(V_i \cap F_j) \ge m(i,j,w), \forall i,j\}$. Note that the permutation $w_0 = n\cdots 21$ in one-line notation is the longest element in S_n, and $s_i := (i,i+1)$, $i<n$, denote the simple transpositions. In this paper, we focus on the Schubert divisor

$$X := X_{w_0 s_{n-1}} = \{V_\bullet \mid F_1 \le V_{n-1}\}.$$

Note $X \cong X_{w_0 s_1}$, while all the other Schubert divisors $X_{w_0 s_i}$, $2 \le i \le n-2$, are singular.

Denote the following $n \times n$ matrices

$$\begin{pmatrix} x_1 & q_1 & & & & \\ -1 & x_2 & q_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & x_{n-2} & q_{n-2} & \\ & & & -1 & x_{n-1} & q_{n-1} \\ & & & & -1 & x_n \end{pmatrix} \qquad \begin{pmatrix} x_1 & q_1 & & & & \\ -1 & x_2 & q_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & x_{n-2} & q_{n-2} & -q_{n-1}q_{n-2} \\ & & & -1 & x_{n-1} & -q_{n-1}x_{n-1} \\ & & & & -1 & x_n \end{pmatrix}$$

as $M_{F\ell_n}$ and $M_{X_{w_0 s_{n-1}}}$ respectively. Write

$$\det(I_n + \lambda M_{F\ell_n}) = \sum_{i=0}^{n} E_i^n \lambda^i, \qquad \det(I_n + \lambda M_{X_{w_0 s_{n-1}}}) = \sum_{i=0}^{n} \hat{E}_i^n \lambda^i. \tag{1.1}$$

The coefficients E_i^n, $\hat{E}_i^n$ may be viewed as quantizations of the i-th elementary symmetric polynomial $e_i^n(x_1, \cdots, x_n)$. Moreover, we notice $\hat{E}_1^n = E_1^n$ and $\hat{E}_n^n = (x_n - q_{n-1})E_{n-1}^{n-1}$, while the difference between $\hat{E}_i^n$ and E_i^n is a bit involved for $1 < i < n$. As shown by Givental and Kim [GK95], there is a canonical ring isomorphism

$$\Phi_q : QH^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \cdots, x_n, q_1, \cdots, q_{n-1}]/(E_1^n, \cdots, E_n^n).$$

Its classical limit at $\mathbf{q} = \mathbf{0}$ gives Borel's ring isomorphism $\Phi : H^*(F\ell_n, \mathbb{Z}) \xrightarrow{\sim} \frac{\mathbb{Z}[x_1, \cdots, x_n]}{(e_1^n, \cdots, e_n^n)}$ [Bo53], where $e_i^n = e_i^n(x)$ denotes the i-th elementary symmetric polynomial in variables $x_1, \cdots, x_n$. As the first main result of this paper, we obtain a similar quantum ring presentation for X.

Theorem 1.2 (Borel-type ring presentation)**.** *There is a canonical ring isomorphism*

$$\Psi_q : QH^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \cdots, x_n, q_1, \cdots, q_{n-1}] \Big/ \big(\hat{E}_1^n, \cdots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}\big).$$

In particular, we obtain the canonical ring isomorphism $\Psi : H^*(X, \mathbb{Z}) \xrightarrow{\sim} \frac{\mathbb{Z}[x_1, \cdots, x_n]}{(e_1^n, \cdots, e_{n-1}^n, e_{n-1}^{n-1})}$. in [GR02, RWY11] by taking the classical limit at $\mathbf{q} = \mathbf{0}$. By "canonical" above, we mean that x_i represents the first Chern class of a specific tautological line bundle (see Equation (2.8) for $F\ell_n$ and Equation (2.10) for X).

For any $u \in S_n$, the Schubert class σ^u in $H^{2\ell(u)}(F\ell_n, \mathbb{Z})$ is given by the Poincaré dual of the homology class $[X_{w_0 u}]$ in $H_{2\ell(w_0) - 2\ell(u)}(F\ell_n, \mathbb{Z})$. The pullback Schubert classes $\{\xi^u := \iota^*(\sigma^u)\}_{u \le w_0 s_{n-1}}$ form an additive basis of $H^*(X, \mathbb{Z})$, and the divisor classes ξ^{s_i}'s generate $QH^*(X, \mathbb{Z})$ as a $\mathbb{Z}[\mathbf{q}]$-algebra. Whenever referring to a transposition $t_{ij} = (i, j)$, we always assume $i < j$. We say $u \lessdot_k ut_{ij}$ (resp. $u \lessdot_k^q ut_{ij}$) in the (quantum) k-Bruhat order, if both $i \le k < j$ and $\ell(ut_{ij}) = \ell(u) + 1$ (resp. $\ell(ut_{ij}) = \ell(u) - \ell(t_{ij})$) hold. As the second main result of this paper, we obtain the following quantum Monk-Chevalley formula, in analogy with the quantum Monk's formula for $QH^*(F\ell_n, \mathbb{Z})$ [FGP97] or more generally the quantum Chevalley formula for $QH^*(G/P, \mathbb{Z})$ [FW04]. It is a special case of the quantum version of point (2) for X but sufficient to determine the all the structure coefficients in principle.

Theorem 1.3 (Quantum Monk-Chevalley formula)**.** *Let $1 \le k \le n-1$ and $u \in S_n$ with $u \le w_0 s_{n-1}$. In $QH^*(X, \mathbb{Z})$, we have*

$$\xi^{s_k} \star \xi^u = \sum \xi^{ut_{ab}} + \sum \xi^{ut_{ab}} q_a \cdots q_{b-1} + \sum \xi^w q_a \cdots q_{n-1} - \delta_{k,n-1} \xi^u q_{n-1},$$

where the first sum is over $u \lessdot_k ut_{ab} \le w_0 s_{n-1}$, the second sum is over $u \lessdot_k^q ut_{ab}$ with $b < n$, and the third sum is over (w, a) that satisfies $wt_{an} \lessdot_k^q w$ and $u \lessdot_{n-1} wt_{an} \in S_n$.

By Remark 4.12, the permutation w in the above theorem does satisfy $w \le w_0 s_{n-1}$, so does for ut_{ab} in the second sum.

Equivalently, the quantum Monk-Chevalley formula for X can be read off directly from that for $F\ell_n$: the first two sums are the cut-off of the quantum product $\sigma^{s_k} \star \sigma^u$, and the third sum is about the quantum terms involving q_{n-1} and appearing in $\sigma^{s_k} \star (\sigma^u \cup \sigma^{s_{n-1}})$. We emphasize that the minus sign in the fourth part is necessary and has a geometric explanation, which gives rise to a key ingredient in our geometric arguments. Intuitively, a smooth curve with two marked points of degree d with $d_{n-1} = 1$ lies on X if and only if the two marked points are both on X. For stable maps, an additional correction term must be taken into consideration, due to the presence of nodal curves. The fourth part may or may not be canceled by the third part.

Example 1.4. *For the Schubert divisor $X_{w_0 s_3}$ of $F\ell_4$, we have*

$$\xi^{s_3 s_2} \star \xi^{s_3} = \xi^{s_3 s_2 s_3} + 0 + (q_3 \xi^{s_3 s_2} + q_2 q_3) - q_3 \xi^{s_3 s_2} = \xi^{s_3 s_2 s_3} + q_2 q_3;$$
$$\xi^{s_1 s_3} \star \xi^{s_3} = \xi^{s_2 s_1 s_3} + 0 + (q_3 \xi^{s_1 s_2} + q_3 \xi^{s_2 s_1}) - q_3 \xi^{s_1 s_3}.$$

To show the vanishing of Gromov-Witten invariants of higher degrees, we mainly use the curve neighborhood technique developed by Buch and Mihalcea [BM15], with a special treatment by more involved geometric arguments for degrees of the form $(0, \cdots, 0, 1, \cdots, 1, 2)$.

Remark 1.5. *We anticipate that our approach to proving Theorem 1.3 could be applied to more general cases and in the equivariant quantum cohomology setting. For instance, we can already obtain the equivariant quantum Monk-Chevalley formula for X immediately. Indeed, our arguments are purely geometrical and all the involved morphisms are torus-equivariant. As a direct consequence of our proof, Theorem 4.2 holds equivariantly by simply treating $N_{v,u}^{w,d}$ as equivariant quantum Schubert structure constants for $F\ell_n$. Therefore, the equivariant quantum extension of Theorem 1.3 is simply obtained by adding into it the single term $(\omega_k - u(\omega_k))\xi^u$, where ω_i's denote the fundamental weights.*

The Schubert polynimals $\mathfrak{S}_w \in \mathbb{Z}[x_1, \cdots, x_n]$, $w \in S_n$, were introduced by Lascoux and Schützenberger [LS82], which satisfy $\Phi(\sigma^w) = [\mathfrak{S}_w]$ in Borel's presentation of $H^*(F\ell_n, \mathbb{Z})$. Moreover, every $\mathfrak{S}_w$ admits a unique linear expansion $\mathfrak{S}_w = \sum \alpha_{i_1 \ldots i_{n-1}} e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1}$. In [FGP97], Fomin, Gelfand and Postnikov introduced the quantum Schubert polynomial

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \ldots i_{n-1}} E_{i_1}^1 E_{i_2}^2 \cdots E_{i_{n-1}}^{n-1}.$$

They showed that $\Phi_q(\sigma^w) = [\mathfrak{S}_w^q]$ under the aforementioned canonical ring isomorphism Φ_q for $QH^*(F\ell_n, \mathbb{Z})$. As the third main result of this paper, we obtain the following. Recall the ring isomorphism Ψ_q in Theorem 1.2.

Theorem 1.6 (Quantum Schubert polynomials)**.** *For any $w \le w_0 s_{n-1}$, we have*

$$\Psi_q(\xi^w) = [\mathfrak{S}_w^q].$$

It is the key ingredient in our proof that we use a transition equation (in Proposition 5.3) for quantum Schubert polynomials. This says that quantum Schubert polynomials can be completely determined by a very small part of the quantum Monk-Chevalley formula. This powerful idea was noticed and applied early in [LOTRZ25, Theorem 4 and Remark 3.15] in the study of quantum double Schubert polynomials for $F\ell_n$.

On the one hand, even though the quantum (resp. classical) Schubert polynomial of the class ξ^w for X is the same as that for the class σ^w for $F\ell_n$, the situation for the quantum cohomology is much more non-trivial. As we can already see from Theorem 1.3, the expansion of $\mathfrak{S}_i^q \mathfrak{S}_u^q$ for X is not a cut-off from that for $F\ell_n$. Consequently, the naive linear map $\sigma^u \mapsto \iota^*(\sigma^u) = \xi^u$ cannot give rise to a ring homomorphism $QH^*(F\ell_n, \mathbb{Z}) \to QH^*(X, \mathbb{Z})$, in contrast to the classical situation.

On the other hand, we do be able to obtain a ring homomorphism as follows.

Theorem 1.7 (Quantum Lefschetz hyperplane principle). *There is a ring homomorphism $\iota_q^*: QH^*(F\ell_n,\mathbb{Z})\to QH^*(X,\mathbb{Z})$, defined by $\sigma^{s_i}\mapsto \xi^{s_i}$ and $q_i\mapsto q_i$ for $1\le i\le n-2$, and*

$$\sigma^{s_{n-1}}\mapsto \xi^{s_{n-1}}+q_{n-1},\qquad q_{n-1}\mapsto q_{n-1}\xi^{s_{n-1}}+q_{n-1}^2.$$

The above theorem can be viewed as a precise example of the new formulation of quantum Lefschetz hyperplane principle by Galkin and Iritani [GI], see also [GLLX25, Proposition 1.10]. It exhibits a surprising functoriality of quantum cohomology with respect to the inclusion $\iota: X\hookrightarrow F\ell_n$ of the hypersurface X, which is lacking in general.

The quantum Schubert calculus for Schubert varieties is also of great importance beyond the scope of enumerative geometry. A profound insight, due to Dale Peterson [Pet97], states that the spectrum of the quantum cohomology ring $QH^*(G/P)=QH^*(G/P,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$ for all P can be assembled into the Peterson variety in the Langlands dual complete flag variety (see [Rie03, Chow22]). Moreover, each Peterson strata corresponds to $QH^*(G/P)$ for some P, and arises from the critical points of a holomorphic function defined on an open Richardson variety by Rietsch [Rie08]. In the forthcoming work [LRY], Li, Rietsch and Yang define a holomorphic function for any (possibly singular) Schubert variety, and propose an extended Peterson program, generalizing the framework for flag varieties as well as the recent study [RW24] of singular Schubert varieties in complex Grassmannians. Our current paper was strongly motivated by one conjecture in the extended Peterson program that the quantum cohomology ring of a smooth Fano Schubert variety should be isomorphic to the Jacobi ring of the corresponding holomorphic function defined therein. In fact, the ring presentation in Theorem 1.2 was first predicted in [LRY] in the wider Peterson program initiated there, making our proof a bit easier for having known the expectation in advance. Our Theorem 1.2 will further be applied there, serving as the first important supporting evidence to Peterson program. In conclusion, we would raise our arms and shout:

It is the right time to develop quantum Schubert calculus for smooth Schubert varieties from various perspectives!

This paper is organized as follows. In Section 2, we introduce necessary background. In Section 3, we derive the ring presentation of $QH^*(X,\mathbb{Z})$ by investigating the quantum differential equations. In Section 4, we provide the quantum Monk-Chevalley formula, by investigating the moduli space of stable maps of certain degrees and using curve neighborhood technique for higher degrees. Finally in Section 5, we show that the quantum Schubert polynomials for X are the same as that for $F\ell_n$, by using induction based on a transition equation for quantum Schubert polynomials.

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2. Notations

We review some background, and refer to [BB05, GR02, CK99] for more details.

2.1. **Combinatorics of S_n.** The symmetric group S_n is generated by simple reflections $s_i := (i, i+1)$, $1 \leq i < n$. Every element $w \in S_n$ can be denoted as $w(1) \cdots w(n)$ in one-line notation, and is of length $\ell(w)$ that counts the inversion set $\{(i,j) \mid i < j,\ w(i) > w(j)\}$. The (unique) longest element in S_n is given by $w_0 := n \cdots 21$. For any permutation w, we have $\ell(w_0 w) = \ell(w_0) - \ell(w)$.

Whenever referring to a transposition $t_{ab} := (a, b)$, we always assume $a < b$ in this paper. Let $u, w \in S_n$. We say $u \lessdot w$ (i.e. u is covered by w), if there exists t_{ab}, such that $w = ut_{ab}$ and $\ell(w) = \ell(u) + 1$. Then $u \leq w$ in the Bruhat order, if u can be transformed into w by a series of transpositions t_{ij} that each increase the inversion number by 1. Note that $u \leq w$ holds if and only if for any $1 \leq j \leq n$, the increasingly sorted list of $u(1), \cdots, u(j)$ is less than or equal to that of $w(1), \cdots, w(j)$ in the usual partial order. In particular, we have

$$u \leq w_0 s_{n-1} = n \cdots 4312 \iff u(n) \neq 1. \tag{2.1}$$

Note $\ell(t_{ab}) = 2b - 2a - 1$. The quantum k-Bruhat cover $\lessdot^q$ is defined by

$$u \lessdot^q ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) - \ell(t_{ab}).$$

For $1 \leq k < n$, we can further define the (quantum) k-Bruhat cover by

$$u \lessdot_k ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) + 1 \text{ and } a \leq k < b; \tag{2.2}$$

$$u \lessdot_k^q ut_{ab} \quad \text{if and only if} \quad \ell(ut_{ab}) = \ell(u) - \ell(t_{ab}) \text{ and } a \leq k < b. \tag{2.3}$$

2.2. **Schubert varieties of $F\ell_n$.** Consider the complete flag variety

$$F\ell_n := \{V_1 \leq \cdots \leq V_{n-1} \leq \mathbb{C}^n \mid \dim V_i = i, \forall\ 1 \leq i < n\}.$$

Let $F_\bullet \in F\ell_n$ be the standard complete flag in $\mathbb{C}^n$. For any permutation $w \in S_n$, we define the Schubert cell $X_w^\circ \subseteq F\ell_n$ (associated to $F_\bullet$) by the following rank conditions.

$$X_w^\circ := \{V_\bullet \in F\ell_n | \dim(V_p \cap F_q) = \sharp\{k \in \mathbb{Z}_{>0} | k \leq p, w(k) \leq q\}, \forall\ 1 \leq p, q \leq n\}.$$

Then $X_w^\circ \cong \mathbb{C}^{\ell(w)}$, and $X_u^\circ \cap X_w^\circ = \emptyset$ for any $u \neq w$. The Schubert variety X_w is given by

$$X_w := \overline{X_w^\circ} = \{V_\bullet \in F\ell_n | \dim(V_p \cap F_q) \geq \sharp\{k \in \mathbb{Z}_{>0} | k \leq p, w(k) \leq q\}, \forall\ 1 \leq p, q \leq n\},$$

and admits a cell decomposition by Schubert cells (with respect to the Bruhat orders):

$$X_w = \bigsqcup_{u \leq w} X_u^\circ. \tag{2.4}$$

In particular, $F\ell_n$ is the (biggest) Schubert variety X_{w_0}. The Schubert divisors are given by X_{ws_i}, $1 \leq i \leq n-1$. Throughout this paper, we will focus on the Schubert divisor $X_{w_0 s_{n-1}}$, whose defining rank conditions can be reduced to the single one:

$$X := X_{w_0 s_{n-1}} = \{V_\bullet \in F\ell_n | F_1 \subset V_{n-1}\}. \tag{2.5}$$

Let $pr : F\ell_n \to Gr(n-1, n)$ be the natural projection that sends $V_\bullet$ to V_{n-1}. Then the divisor X can be viewed as the zero locus of a section of the line bundle over $F\ell_n$,

$$\mathcal{L}_{\varpi_{n-1}} := pr^* \mathcal{O}_{Gr(n-1,n)}(1). \tag{2.6}$$

We notice that $X_{w_0 s_1} \cong X_{w_0 s_{n-1}}$ is smooth, isomorphic to an $F\ell_{n-1}$-bundle over $\mathbb{P}^{n-2}$. while $X_{w_0 s_i}$'s are all singular for $1 < i < n-1$.

2.2.1. *Topology of $F\ell_n$.* For $w \in S_n$, denote by $\sigma^w \in H^{2\ell(w)}(F\ell_n, \mathbb{Z})$ the Schubert class defined by the Poincaré dual of the homology class $[X_{w_0w}] \in H_{2\ell(w_0w)}(F\ell_n, \mathbb{Z})$. We have

$$(\sigma^{w_0u}, \sigma^w) := \int_{F\ell_n} \sigma^{w_0u} \cup \sigma^w = \langle [X_u], \sigma^w \rangle = \delta_{u,w}.$$

Here $(\cdot,\cdot)$ denotes the Poincaré pairing for $F\ell_n$, and $\langle \cdot, \cdot \rangle$ (resp. $\delta_{u,w}$) denote the Kronecker pairing (resp. symbol). It follows from Equation (2.4) that $H^*(F\ell_n, \mathbb{Z}) = \bigoplus_{w\in S_n} \mathbb{Z}\sigma^w$.

Denote the i-th elementary symmetric polynomial in variables $x_1, \cdots, x_n$ as

$$e_i^n = e_i^n(x_1, \cdots, x_n). \tag{2.7}$$

The following result was due to Borel [Bo53].

Proposition 2.1. *There is a canonical ring isomorphism*

$$\Phi : H^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, x_2. \cdots, x_n] \big/ \big(e_1^n, e_2^n, \cdots, e_n^n\big) .$$

Denote $V_0 = \{0\}$ and $V_n = \mathbb{C}^n$. Let $\mathcal{S}_i$ be the i-th tautological subbundle of $F\ell_n$, $0 \le i \le n$, namely the fiber of $\mathcal{S}_i$ at a point $V_\bullet \in F\ell_n$ is given by the vector space V_i. By "canonical" in the above proposition, we mean

$$\Phi^{-1}([x_i]) = c_1((\mathcal{S}_i/\mathcal{S}_{i-1})^\vee) \in H^2(F\ell_n, \mathbb{Z}). \tag{2.8}$$

2.2.2. *Topology of X.* It also follows from the cell decomposition *Equation* (2.4) that $H^*(X, \mathbb{Z})$ is torsion free and has an additive basis $\{PD([X_w])\}_{w\le w_0s_{n-1}}$. Let $\{\xi^w\}_{w\le w_0s_{n-1}}$ denote the dual basis with respect to Poincaré pairing, namely $(PD([X_u]), \xi^w) = \langle [X_u], \xi^w\rangle = \delta_{u,w}$ for any $u, w \le w_0s_{n-1}$. Induced from the natural inclusion map $\iota : X \hookrightarrow F\ell_n$, we have a surjective ring homomorphism

$$\iota^* : H^*(F\ell_n, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z}); \quad \iota^*(\sigma^w) = \begin{cases} \xi^w, & \text{if } w \le w_0s_{n-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

We therefore call ξ^w's the pullback Schubert classes. Moreover, the following ring presentation was obtained in [GR02, RWY11].

Proposition 2.2. *There is a canonical ring isomorphism*

$$\Psi : H^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, x_2. \cdots, x_n] \big/ \big(e_1^n, e_2^n, \cdots, e_{n-1}^n, e_{n-1}^{n-1}\big) .$$

Here by "canonical", we mean

$$\Psi^{-1}([x_i]) = c_1((\iota^*\mathcal{S}_i/\iota^*\mathcal{S}_{i-1})^\vee) \in H^2(X, \mathbb{Z}). \tag{2.10}$$

2.3. **Quantum cohomology.** To simplify the descriptions, we only review the necessary notions for $Y \in \{F\ell_n, X\}$, which is Fano with the first Chern class

$$c_1(Y) = \begin{cases} 2\sigma^{s_1} + 2\sigma^{s_2} + \cdots + 2\sigma^{s_{n-2}} + 2\sigma^{s_{n-1}}, & \text{if } Y = F\ell_n, \\ 2\xi^{s_1} + 2\xi^{s_2} + \cdots + 2\xi^{s_{n-2}} + \xi^{s_{n-1}}, & \text{if } Y = X. \end{cases} \tag{2.11}$$

(See [LRY25] for a criterion for a factorial Schubert variety of general Lie type being Fano, as well as a formula of the first Chern class.) Uniformly denote by γ^w the cohomology class σ^w (resp. ξ^w) labeled by an appropriate permutation w for $Y = F\ell_n$ (resp. X).

Let $\overline{\mathcal{M}}_{0,m}(Y, d)$ denote the moduli space of m-pointed, genus-0 stable maps $(f : C \to Y; \mathrm{pt}_1, \cdots, \mathrm{pt}_m)$ of degree $d \in H_2(Y; \mathbb{Z})$. For each i, let $ev_i : \overline{\mathcal{M}}_{0,m}(Y, d) \longrightarrow Y$ denote the

i-th evaluation map, which sends $(f : C \to Y; \mathrm{pt}_1, \cdots, \mathrm{pt}_m)$ to $f(\mathrm{pt}_i)$; let $\mathcal{L}_i$ denote the i-th universal cotangent line bundle on $\overline{\mathcal{M}}_{0,m}(Y, d)$. The gravitational Gromov-Witten invariant, associated to nonnegative integers a_i and cohomology classes $\gamma^{u_i} \in H^*(Y) = H^*(Y, \mathbb{C})$, is defined by

$$\langle \tau_{a_1}\gamma^{u_1}, \tau_{a_2}\gamma^{u_2}, \cdots, \tau_{a_m}\gamma^{u_m}\rangle^Y_{0,m,d} := \int_{[\overline{\mathcal{M}}_{0,m}(Y,d)]^{\mathrm{vir}}} \prod_{i=1}^{m} (c_1(\mathcal{L}_i)^{a_i} \cup ev_i^*(\gamma^{u_i})).$$

Here $[\overline{\mathcal{M}}_{0,m}(Y, d)]^{\mathrm{vir}}$ denotes the virtual fundamental class, which is of expected dimension $\dim Y + m - 3 + \int_d c_1(Y)$. In particular, Gromov-Witten invariants of degree d are vanishing unless d belongs to the Mori cone $\overline{\mathrm{NE}}(Y) = \bigoplus_{j=1}^{n-1} \mathbb{Z}_{\geq 0}[X_{s_j}] \subset H_2(Y, \mathbb{Z})$ of effective curve classes of Y, which we simply denote as $d \geq 0$.

Let $\{\gamma_u\}_u \subset H^*(Y)$ be the dual basis of $\{\gamma^u\}_u$ with respect to the Poincaré pairing. The (small) quantum cohomology $QH^*(Y) = (H^*(Y) \otimes \mathbb{C}[\mathbf{q}], \star)$ is a $\mathbb{C}[\mathbf{q}]$-algebra with the (small) quantum product defined by

$$\gamma^u \star \gamma^v = \sum_{d \in \overline{\mathrm{NE}}(Y)} \sum_w \langle \gamma^u, \gamma^v, \gamma_w\rangle^Y_{0,3,d} \gamma^w q^d, \tag{2.12}$$

where $q^d := q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ for $d = \sum_{j=1}^{n-1} d_j[X_{s_j}] \in \overline{\mathrm{NE}}(Y)$. This does happen for $Y = F\ell_n$, so does for $Y = X$ as we will see. The quantum variable q_j is of degree

$$\deg q_j = \int_{[X_{s_j}]} c_1(Y) = \begin{cases} 1, & \text{if } Y = X \text{ and } j = n-1 \text{ both hold,} \\ 2, & \text{otherwise.} \end{cases} \tag{2.13}$$

Whenever $H^*(Y, \mathbb{Z}) \otimes \mathbb{Z}[\mathbf{q}]$ is closed under the quantum multiplication $\star$, we call it the integral quantum cohomology, simply denoted as $QH^*(Y, \mathbb{Z})$.

3. A Borel-type ring presentation of $QH^*(X)$

For $Y \in \{F\ell_n, X\}$, we denote

$$\mathcal{H}(Y) := H^*(Y) \otimes_{\mathbb{C}} \mathbb{C}[\hbar][\![\hbar^{-1}]\!][\![q_1, \cdots, q_n]\!]. \tag{3.1}$$

In this section, we will introduce Givental's J-function $J^Y(t, \hbar)$ of Y, viewed as an element in $\mathcal{H}(Y)$. We then prove the Borel-type ring presentation of $QH^*(X)$ in Theorem 1.2, by finding quantum differential equations that annihilate $J^X(t, \hbar)$.

At the beginning and the end of this section, we will identify both first Chern classes in Equation (2.8) and Equation (2.10) as x_i by abuse of notation. Then $\gamma^{s_j} = x_1 + x_2 + \cdots + x_j$ for $1 \leq j \leq n-1$. Denoting $t = \sum_{i=1}^n t_i x_i \in H^2(Y)$, we view the quantum variables q_j as functions on $H^2(Y)$, by letting $q_j = e^{\int_{[X_{s_j}]} t} = e^{t_j - t_{j+1}}$. Then it follows from $\delta_{i,j} = \int_{[X_{s_i}]} \gamma^{s_j}$ that $q^d = e^{\int_d t} = e^{\int_d \sum_i t_i(\gamma_i - \gamma_{i-1})} = e^{\int_d \sum_i (t_i - t_{i+1})\gamma_i} = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

From Proposition 3.3 until Theorem 3.10, we will use $\iota^* x_i$ for X to distinguish the use of x_i for $F\ell_n$, in order to avoid confusions. For any $f(x, \hbar) \in \mathcal{H}(F\ell_n)$, by $f(\iota^* x, \hbar)$ we mean the element in $\mathcal{H}(X)$ simply obtained from $f(x, \hbar)$ by replacing all x_i with $\iota^* x_i$. In particular, by $\iota^* t$ we mean $\sum_i t_i \iota^* x_i$.

3.1. **Givental's J-function of X.** The (small) quantum connection acts on the trivial $H^*(Y)$-bundle over $H^2(Y)\times\mathbb{C}^*$. Its derivations along the $H^2(Y)$-direction are given by

$$\nabla_{\frac{\partial}{\partial t_i}} := \frac{\partial}{\partial t_i} + \frac{1}{\hbar}x_i\star,\ 1\le i\le n,$$

where $\hbar$ is the coordinate of $\mathbb{C}^*$. This quantum connection is flat, with the fundamental solution $L(t,\hbar)$ to the quantum differential equation $\nabla_{\frac{\partial}{\partial t_i}}(L(t,\hbar)\alpha)=0$ given by

$$L(t,\hbar)\alpha := e^{-\frac{t}{\hbar}}\alpha + \sum_d\sum_w \langle \frac{e^{-\frac{t}{\hbar}}\alpha}{-\hbar - c_1(\mathcal{L}_1)}, \gamma_w\rangle^Y_{0,2,d}\gamma^w q^d.$$

Here $\alpha\in H^*(Y)$, and we take the expansion $\frac{1}{-\hbar-c_1(\mathcal{L}_1)} = \sum_{i=0}(-1)^{i+1}\hbar^{-i-1}(c_1(\mathcal{L}_1))^i$.

Definition 3.1. *The Givental's (small) J-function of Y is defined by*

$$J^Y(t,\hbar) := L(t,\hbar)^{-1}(\mathbf{1}) = e^{\frac{t}{\hbar}}\sum_d\sum_w\langle \mathbf{1}, \frac{\gamma_w}{\hbar - c_1(\mathcal{L}_1)}\rangle^Y_{0,2,d}\ \gamma^w q^d;$$

$$J^Y(t_0,t,\hbar) := e^{\frac{t_0\mathbf{1}}{\hbar}}J^Y(t,\hbar).$$

Here $\mathbf{1}=\gamma^{\mathrm{id}}$ is the identity element in $H^*(Y)$.

Proposition 3.2 ([BCK08, Theorem 1])**.** *The Givental's J-function of $F\ell_n$ is given by*

$$J^{F\ell_n}(t,\hbar) = e^{\frac{t}{\hbar}}(\sum_{d\ge 0}q^d J_d^{F\ell_n}(x,\hbar)), \qquad \textit{where}$$

$$J_d^{F\ell_n}(x,\hbar) = \sum_{\sum_j d_{i,j}=d_i}\Big(\prod_{i=1}^{n-1}\prod_{1\le j<j'\le i}(-1)^{d_{i,j}-d_{i,j'}}\frac{x_j - x_{j'} + (d_{i,j}-d_{i,j'})\hbar}{x_j - x_{j'}}\Big)$$
$$\Big(\prod_{i=1}^{n-2}\prod_{\substack{1\le j\le i\\ 1\le j'\le i+1}}\frac{\prod_{k=-\infty}^0(x_j - x_{j'}+k\hbar)}{\prod_{k=-\infty}^{d_{i,j}-d_{i+1,j'}}(x_j-x_{j'}+k\hbar)}\Big)\prod_{1\le j\le n-1}\frac{1}{\prod_{k=1}^{d_{n-1,j}}(x_j+k\hbar)^n}.$$

Here the sum is over nonnegative integers $d_{i,j}$, with $1\le j\le i\le n-1$.

Proposition 3.3. *The Givental's J-function of X is given by*

$$J^X(\iota^*t,\hbar) = e^{-\frac{q_{n-1}}{\hbar}}I_{F\ell_n,X}(\iota^*t,\hbar),$$

where

$$I_{F\ell_n,X}(t,\hbar) := e^{\frac{t}{\hbar}}\Big(\sum_{d\ge 0}q^d J_d^{F\ell_n}(x,\hbar)\prod_{k=1}^{\int_d\sigma^{s_{n-1}}}(-x_n+k\hbar)\Big)$$

with the contribution from the product $\prod_{k=1}^{0}$ read off as multiplication by $-x_n$ by convention.

Proof. Expanding $I_{F\ell_n,X}(t,\hbar)$ with respect to the variable $\hbar$, we obtain

$$I_{F\ell_n,X}(t,\hbar) = 1 + \frac{1}{\hbar}(q_{n-1}\mathbf{1} + t) + O(\hbar^{-2}).$$

Recall from Equation (2.6) that the smooth Schubert divisor X is realized as the zero locus of a section of the line bundle $\mathcal{L}_{\varpi_{n-1}}$ over $F\ell_n$ with $c_1(\mathcal{L}_{\varpi_{n-1}}) = \sigma^{s_{n-1}} = -x_n$. By using the quantum Lefschetz theorem [CG07, Corollary 7] with respect to $F\ell_n$ and X[1], we obtain

$$J^X(q_{n-1}\mathbf{1} + \iota^*t, \hbar) = I_{F\ell_n,X}(\iota^*t, \hbar) = e^{\frac{\iota^*t}{\hbar}}(\sum_{d\geq 0} q^d J_d^{F\ell_n}(\iota^*x, \hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}} (-\iota^*x_n + k\hbar)).$$

This implies $J^X(\iota^*t, \hbar) = e^{-\frac{q_{n-1}}{\hbar}} I_{F\ell_n,X}(\iota^*t, \hbar)$. □

3.2. **Quantum differential equations of X.** In this subsection, we will investigate quantum differential equations of X, namely differential operators that annihilate $J^X(t, \hbar)$.

For any polynomial $P(x, q) \in \mathbb{C}[x, q]$, we write it in the fixed order with respect to $x_1, \cdots, x_n, q_1, \cdots, q_{n-1}$, namely

$$P(x, q) = \sum_{i_I, d} A_{i_I}^d x^{i_I} q^d$$

with $x^{i_I} := x_1^{i_1} \cdots x_n^{i_n}$, $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ and the coefficients $A_{i_I}^d \in \mathbb{C}$. Take the conventions:

$$\hbar\frac{\partial}{\partial t_{[1,m]}} := (\hbar\frac{\partial}{\partial t_1}, \hbar\frac{\partial}{\partial t_2} \cdots, \hbar\frac{\partial}{\partial t_m}), \qquad q_{[1,m]} := (q_1, q_2, \cdots, q_m), \tag{3.2}$$

where $q_j = e^{t_j - t_{j+1}}$ whenever it is treated as an operator. We then denote

$$P(\hbar\frac{\partial}{\partial t}, q) = P(\hbar\frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-1]}) := \sum_{i_I, d} A_{i_I}^d (\hbar\frac{\partial}{\partial t_1})^{i_1} \cdots (\hbar\frac{\partial}{\partial t_n})^{i_n} q_1^{d_1} \cdots q_{n-1}^{d_{n-1}},$$

and call the differential operator $P(\hbar\frac{\partial}{\partial t}, q)$ *the quantization of* $P(x, q)$.

Recall in Equation (1.1) that $E_i^n = E_i^n(x, q)$ are the coefficients of the expansion of the polynomial $\det(I_n + \lambda M_{F\ell_n}) = \sum_i E_i^n \lambda^i$ in variable λ, with respect to the matrix

$$M_{F\ell_n} = \begin{pmatrix} x_1 & q_1 & & & & \\ -1 & x_2 & q_2 & & & \\ & -1 & x_3 & q_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & x_{n-1} & q_{n-1} \\ & & & & -1 & x_n \end{pmatrix}.$$

Proposition 3.4 ([GK95, Theorem 4])**.** *Let $D_i^n(\hbar\frac{\partial}{\partial t}, q)$ be the quantization of $E_i^n(x, q)$. Then $D_i^n(\hbar\frac{\partial}{\partial t}, q)(J^{F\ell_n}(t, \hbar)) = 0$ for any $1 \leq i \leq n$.*

Definition 3.5. *Viewing $q_j = e^{t_j - t_{j+1}}$, we define the operators*

$$\mathbb{T}, \mathbb{S} : \mathcal{H}(F\ell_n) \longrightarrow \mathcal{H}(X),$$

$$\mathbb{T}(\sum_{d\geq 0} q^d f_d(x, \hbar)) := \sum_{d\geq 0} q^d f_d(\iota^*x, \hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}} (-\iota^*x_n + k\hbar),$$

[1]The function $I_{F\ell_n,X}$ in [CG07, Section 9] is a multiple of ours by $\hbar$ with the identification $z = \hbar$.

$$\mathbb{S}(\sum_{d\geq 0} q^d f_d(x,\hbar)) := \sum_{d\geq 0} q^d f_d(\iota^* x,\hbar) \prod_{k=1}^{\int_d \sigma^{s_{n-1}}-1} (-\iota^* x_n + k\hbar),$$

where the product $\prod_{k=1}^{0}$ *(resp.* $\prod_{k=1}^{-1}$*) is read off as multiplication by* $-\iota^* x_n$ *(resp. by* 1*).*

Lemma 3.6. *As operators that send* $\mathcal{H}(F\ell_n)$ *to* $\mathcal{H}(X)$*, we have*

$$[\hbar\frac{\partial}{\partial t_k}, \mathbb{T}] = [\hbar\frac{\partial}{\partial t_k}, \mathbb{S}] = [q_l, \mathbb{T}] = [q_l, \mathbb{S}] = 0$$

for any $1 \leq k \leq n$ *and* $1 \leq l \leq n-2$*. Moreover, we have*

$$-\hbar\frac{\partial}{\partial t_n} \circ q_{n-1} \circ \mathbb{T} = \mathbb{T} \circ q_{n-1}, \quad q_{n-1} \circ \mathbb{T} = \mathbb{S} \circ q_{n-1}, \quad \mathbb{T} = -\mathbb{S} \circ \hbar\frac{\partial}{\partial t_n}, \tag{3.3}$$

as operators acting on $e^{t/\hbar}\sum_{d\geq 0} q^d f_d(x,\hbar)$ *with* $\frac{\partial f_d}{\partial t_i} = 0$ *for* $1 \leq i \leq n$ *held for any* d*.*

Proof. Note that $\int_d \sigma^{s_{n-1}} = d_{n-1}$ and that the operators $\mathbb{T}$ and $\mathbb{S}$ are effected only by the power d_{n-1} of $q_{n-1} = e^{t_{n-1}-t_n}$. Thus the identities in the first half of the statement hold.

The last identity in Equation (3.3) follows directly from the definition of the operators $\mathbb{S}, \mathbb{T}$.

$$\begin{aligned} q_{n-1} \circ \mathbb{T}(e^{t/\hbar}\sum_{d\geq 0} q^d f_d(x,\hbar)) &= e^{\iota^* t/\hbar} \sum_{d=(d_1,\cdots,d_{n-1}), d_{n-1}\geq 1} q^d g_d(\iota^* x,\hbar) \prod_{k=1}^{d_{n-1}-1} (-\iota^* x_n + k\hbar) \\ &= \mathbb{S} \circ q_{n-1}(e^{t/\hbar}\sum_{d\geq 0} q^d f_d(x,\hbar)), \end{aligned}$$

where $g_d(\iota^* x,\hbar) = f_{(d_1,d_2,\cdots,d_{n-1}-1)}(\iota^* x,\hbar)$. Thus the second identity in Equation (3.3) holds. Denoting $g_d(x,\hbar) = f_{(d_1,d_2,\cdots,d_{n-1}-1)}(x,\hbar)$, we have

$$\begin{aligned} \mathbb{T} \circ q_{n-1}(e^{t/\hbar}\sum_{d\geq 0} q^d f_d(x,\hbar)) &= \mathbb{T}(e^{t/\hbar} \sum_{d=(d_1,\cdots,d_{n-1}), d_{n-1}\geq 1} q^d g_d(x,\hbar)) \\ &= e^{\iota^* t/\hbar} \sum_{d=(d_1,\cdots,d_{n-1}), d_{n-1}\geq 1} q^d g_d(\iota^* x,\hbar) \prod_{k=1}^{d_{n-1}} (-\iota^* x_n + k\hbar) \\ &= -\hbar\frac{\partial}{\partial t_n} \circ q_{n-1} \circ \mathbb{T}(e^{t/\hbar}\sum_{d\geq 0} q^d f_d(x,\hbar)) \end{aligned}$$

Therefore the first identity in Equation (3.3) holds as well. $\square$

Proposition 3.7. *For the quantizations* $D_i^n(\hbar\frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-1]})$ *of* $E_i^n(x,q)$*,*

$$D_i^n(\hbar\frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar\frac{\partial}{\partial t_n}) \circ q_{n-1})(I_{F\ell_n,X}(\iota^* t,\hbar)) = 0.$$

holds for $1 \leq i \leq n-1$*. Moreover, we have*

$$(-D_{n-1}^{n-1} + D_{n-2}^{n-2} q_{n-1})(I_{F\ell_n,X}(\iota^* t,\hbar)) = 0. \tag{3.4}$$

Proof. It follows directly from the definition that $\mathbb{T}(J^{F\ell_n}(t,\hbar)) = I_{F\ell_n,X}(\iota^*t,\hbar)$. By Proposition 3.4 and Lemma 3.6, for $1 \le i \le n-1$, we obtain

$$D_i^n(\hbar\frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar\frac{\partial}{\partial t_n})\circ q_{n-1})(\mathbb{T}(J^{F\ell_n}(t,\hbar))) = \mathbb{T}\circ D_i^n(\hbar\frac{\partial}{\partial t}, q)(J^{F\ell_n}(t,\hbar)) = 0.$$

By Lemma 3.6, all $q_l, \frac{\partial}{\partial t_k}$ commute with $\mathbb{T}, \mathbb{S}$ for $1 \le l \le n-2, 1 \le k \le n-1$. It follows that $[D_a^a, \mathbb{S}] = 0$ and $[D_a^a, \mathbb{T}] = 0$ for $a \in \{n-1, n-2\}$. Using the identities in Equation (3.3), we have

$$\begin{aligned}(-D_{n-1}^{n-1} + D_{n-2}^{n-2}q_{n-1})(I_{F\ell_n,X}(\iota^*t,\hbar)) &= (-D_{n-1}^{n-1} + D_{n-2}^{n-2}q_{n-1})(\mathbb{T}(J^{F\ell_n}(t,\hbar)))\\ &= (\mathbb{S}\circ(\hbar\frac{\partial}{\partial t_n})\circ D_{n-1}^{n-1} + \mathbb{S}\circ D_{n-2}^{n-2}\circ q_{n-1})(J^{F\ell_n}(t,\hbar))\\ &= \mathbb{S}(D_n^n(J^{F\ell_n}(t,\hbar))) = 0.\end{aligned}$$

Here the third equality holds by noting $D_n^n = (\hbar\frac{\partial}{\partial t_n})D_{n-1}^{n-1} + D_{n-2}^{n-2}q_{n-1}$. □

Lemma 3.8. *Both of the following identities hold.*

$$(\hbar\frac{\partial}{\partial t_{n-1}} + q_{n-1})J^X(\iota^*t,\hbar) = e^{-\frac{q_{n-1}}{\hbar}}(\hbar\frac{\partial}{\partial t_{n-1}})I_{F\ell_n,X}(\iota^*t,\hbar),$$
$$(\hbar\frac{\partial}{\partial t_n} - q_{n-1})J^X(\iota^*t,\hbar) = e^{-\frac{q_{n-1}}{\hbar}}(\hbar\frac{\partial}{\partial t_n})I_{F\ell_n,X}(\iota^*t,\hbar).$$

Proof. By Proposition 3.3, $J^X(\iota^*t,\hbar) = e^{-\frac{q_{n-1}}{\hbar}}I_{F\ell_n,X}(\iota^*t,\hbar)$. Then the statement follows from direct calculations by Leibniz rule:

$$(\hbar\frac{\partial}{\partial t_{n-1}})(J^X) = (-q_{n-1})e^{-\frac{q_{n-1}}{\hbar}}I_{F\ell_n,X} + e^{-\frac{q_{n-1}}{\hbar}}(\hbar\frac{\partial}{\partial t_{n-1}})(I_{F\ell_n,X}),$$
$$(\hbar\frac{\partial}{\partial t_n})(J^X) = q_{n-1}e^{-\frac{q_{n-1}}{\hbar}}I_{F\ell_n,X} + e^{-\frac{q_{n-1}}{\hbar}}(\hbar\frac{\partial}{\partial t_n})(I_{F\ell_n,X}).$$

□

3.3. **Proofs of Theorem 1.2 and Theorem 1.7.** As shown in [ST97, Proposition 2.2] by Siebert and Tian, the quantum cohomology $QH^*(X)$ of the Fano manifold X is of the form $\mathbb{C}[x,q]/I_q$, provided that $H^*(X) = \mathbb{C}[x]/I$ and I_q is generated by the corresponding quantized relations in $QH^*(X)$ of the generators of I for $H^*(X)$. Such relations can be found out, by using the following well-known way due to Givental.

Proposition 3.9 ([Gi96, Corollary 6.4])**.** *If a differential operator $P(\hbar\frac{\partial}{\partial t_i}, e^{t_i-t_{i+1}}, \hbar)$ satisfies the equation $P(\hbar\frac{\partial}{\partial t_i}, e^{t_i-t_{i+1}}, \hbar)(J^Y(t,\hbar)) = 0$, then $P(x_i, q_i, 0) = 0$ in $QH^*(Y)$.*

We will first describe $QH^*(X)$, and then prove Theorem 1.2 as a consequence, by using [FP97, Proposition 11], which is a variation of [ST97, Proposition 2.2].

Theorem 3.10. *The quantum cohomology ring of X is canonically given by*

$$QH^*(X) = \mathbb{C}[\iota^*x_1,\cdots,\iota^*x_n,q_1,\cdots,q_{n-1}]\Big/\left(\chi_1,\chi_2,\cdots,\chi_{n-1},\frac{\chi_n}{\iota^*x_n - q_{n-1}}\right).$$

Here $\chi_j := \chi_j(\iota^* x, q)$ *are given by* $\det(I_n + \lambda \tilde{M}_X) = \sum_{i=0}^n \chi_i \lambda^i$ *with respect to the matrix*

$$\tilde{M}_X = \begin{pmatrix} \iota^* x_1 & q_1 & & & & \\ -1 & \iota^* x_2 & q_2 & & & \\ & -1 & \iota^* x_3 & q_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & \iota^* x_{n-1} + q_{n-1} & q_{n-1}(-\iota^* x_n + q_{n-1}) \\ & & & & -1 & \iota^* x_n - q_{n-1} \end{pmatrix}.$$

Proof. By Proposition 3.7 and Lemma 3.8, for any $1 \le i \le n-1$, we have

$$\begin{aligned} 0 &= e^{-\frac{q_{n-1}}{\hbar}} D_i^n(\hbar \frac{\partial}{\partial t_{[1,n]}}, q_{[1,n-2]}, (-\hbar \frac{\partial}{\partial t_n}) \circ q_{n-1})(I_{F\ell_n, X}) \\ &= D_i^n\Big(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, \hbar \frac{\partial}{\partial t_{n-1}} + q_{n-1}, \hbar \frac{\partial}{\partial t_n} - q_{n-1}, q_{[1,n-2]}, (-(\hbar \frac{\partial}{\partial t_n} - q_{n-1}) \circ q_{n-1}) + \hbar G_1 + \hbar^2 G_2\Big)(J^X) \end{aligned}$$

where $G_1 = G_1(\hbar \frac{\partial}{\partial t}, q)$ and $G_2 = G_2(\hbar \frac{\partial}{\partial t}, q)$ are differential operators. Therefore by using Proposition 3.9, we have

$$D_i^n(\iota^* x_1, \cdots, \iota^* x_{n-2}, \iota^* x_{n-1} + q_{n-1}, \iota^* x_n - q_{n-1}, q_1, q_2, \cdots, q_{n-2}, q_{n-1}(-\iota^* x_n + q_{n-1})) = 0$$

in $QH^*(X)$. That is, $\chi_i(\iota^* x, q) = 0$ holds in $QH^*(X)$.

The next two equalities follow directly from Lemma 3.8 and Proposition 3.7 respectively.

$$\begin{aligned} &\big(- D_{n-1}^{n-1}(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, \hbar \frac{\partial}{\partial t_{n-1}} + q_{n-1}, q_{[1,n-2]}) + D_{n-2}^{n-2}(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, q_{[1,n-2]}) q_{n-1} + \hbar H_1 \big)(J^X) \\ &= e^{-\frac{q_{n-1}}{\hbar}} \big(- D_{n-1}^{n-1}(\hbar \frac{\partial}{\partial t_{[1,n-1]}}, q_{[1,n-2]}) + D_{n-2}^{n-2}(\hbar \frac{\partial}{\partial t_{[1,n-2]}}, q_{[1,n-2]}) q_{n-1} \big)(I_{F\ell_n, X}) = 0, \end{aligned}$$

where $H_1 = H_1(\hbar \frac{\partial}{\partial t}, q)$ is a differential operator. Therefore, by Proposition 3.9, in $QH^*(X)$ we have

$$-E_{n-1}^{n-1}(\iota^* x_1, \cdots, \iota^* x_{n-2}, \iota^* x_{n-1} + q_{n-1}, q_1, \cdots, q_{n-2}) + E_{n-2}^{n-2}(\iota^* x_1, \cdots, \iota^* x_{n-2}, q_1, \cdots, q_{n-2}) q_{n-1} = 0.$$

The left hand side of the above equality equals $-E_{n-1}^{n-1}(\iota^* x_1, \cdots, \iota^* x_{n-1}, q_1, \cdots, q_{n-2})$ and hence equals $-\frac{\chi_n(\iota^* x, q)}{\iota^* x_n - q_{n-1}}$ by Laplace expansion of matrices. Then we are done by using Proposition 2.2 and [ST97, Proposition 2.2]. □

Now we are ready to show the ring presentation of the integral quantum cohomology $QH^*(X, \mathbb{Z})$ in **Theorem 1.2**, as well as the ring homomorphism in **Theorem 1.7**.

Proof of Theorem 1.2. Abusing the notation for $\iota^* x_i$ and x_i, we notice $M_X = A \tilde{M}_X A^{-1}$ with $A = I_n + q_{n-1} B_{n-1,n}$, where $B_{n-1,n}$ is the matrix with 1 in the $(n-1, n)$-entry and zeros elsewhere. Hence, the matrices M_X and $\tilde{M}_X$ have the same characteristic polynomial, implying $\hat{E}_i^n = \chi_i$ for all $1 \le i \le n$. Again we note $\frac{\chi_n}{x_n - q_{n-1}} = E_{n-1}^{n-1}$. Thus all $\hat{E}_1^n, \cdots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}$ vanish in $QH^*(X) = QH^*(X, \mathbb{Z}) \otimes \mathbb{C}$, so do in $QH^*(X, \mathbb{Z})$. Their evaluations at $q = 0$ give the ideal $I = (e_1^n, \cdots, e_{n-1}^n, e_{n-1}^{n-1})$, providing the ring presentation $H^*(X, \mathbb{Z}) = \mathbb{Z}[x_1, \cdots, x_n]/I$ by Proposition 2.2. Therefore the statement follows from [FP97, Proposition 11]. □

Proof of Theorem 1.7. Define $\iota_q^*(x_i)=x_i$ and $\iota_q^*(q_i)=q_i$ for $1\le i\le n-2$. Define

$$\iota_q^*(x_{n-1})=x_{n-1}+q_{n-1},\qquad \iota_q^*(x_n)=x_n-q_{n-1},\quad \iota_q^*(q_{n-1})=-q_{n-1}x_n+q_{n-1}^2.$$

This induces a ring homomorphism $\iota_q^*:\mathbb{Z}[x,q]\to\mathbb{Z}[x,q]$ with

$$\iota_q^*(E_i^n(x,q))=E_i^n(x_1,\cdots,x_{n-2},\iota_q^*(x_{n-1}),\iota_q^*(x_n),q_1,\cdots,q_{n-2},\iota_q^*(q_{n-1}))=\chi_i(x,q)$$

for all $1\le i\le n$. That is, $\iota_q^*(E_i^n(x,q))=\hat{E}_i^n(x,q)$ for $1\le i\le n-1$, and $\iota_q^*(E_n^n(x,q))=(x_n-q_{n-1})E_{n-1}^{n-1}(x,q)$ by the proof of Theorem 1.2. Hence it further induces a ring homomorphism

$$\iota_q^*:QH^*(F\ell_n,\mathbb{Z})=\frac{\mathbb{Z}[x,q]}{(E_1^n,\cdots,E_n^n)}\longrightarrow\frac{\mathbb{Z}[x,q]}{(\hat{E}_1^n,\cdots,\hat{E}_{n-1}^n,E_{n-1}^{n-1})}=QH^*(X,\mathbb{Z}).$$

Hence, we are done, by noting $\sigma^i=[x_1+\cdots+x_i]$ on the left hand side, and $\xi^i=[x_1+\cdots+x_i]$ on the right hand side. (In particular, $\xi^{s_{n-1}}=[-x_n]$.) $\square$

4. Quantum Monk-Chevalley formula for X

Let $Y\in\{F\ell_n,X\}$. By d we always mean $d=(d_1,\cdots,d_{n-1})=\sum_{i=1}^{n-1}d_i[X_{s_i}]\in H_2(Y,\mathbb{Z})$. For $1\le a<b\le n$, we denote

$$\alpha_{ab}:=\sum_{i=a}^{b-1}d_i[X_{s_i}],\qquad q_{ab}:=q_aq_{a+1}\cdots q_{b-1}=q^{\alpha_{ab}}. \tag{4.1}$$

The following quantum Monk's formula for $F\ell_n$ was proved in [FGP97, Theorem 1.3] (see [FW04] for the quantum Chevalley formula for general G/P), where $\lessdot_r$ (resp. $\lessdot_r^q$) denotes the (quantum) r-Bruhat order defined in Equation (2.2) (resp. Equation (2.3)).

Proposition 4.1 (Quantum Monk's formula)**.** *For $u\in S_n$ and $1\le r<n$, in $QH^*(F\ell_n,\mathbb{Z})$ we have*

$$\sigma^{s_r}\star\sigma^u=\sum_{u\lessdot_r ut_{ab}}\sigma^{ut_{ab}}+\sum_{u\lessdot_r^q ut_{ck}}q_{ck}\sigma^{ut_{ck}}.$$

Since $\{\sigma^u\}_u$ form an $\mathbb{Z}[q]$-basis of $QH^*(F\ell_n,\mathbb{Z})$, we can write

$$\sigma^v\star\sigma^u=\sum_{w,d}N_{v,u}^{w,d}q^d\sigma^w.$$

We further denote by $N_{v,u\cup u'}^{w,d}$ the coefficient of $q^d\sigma^w$ in the product $\sigma^v\star(\sigma^u\cup\sigma^{u'})$. In particular if $v=s_r$ and $d\neq0$, then by the quantum Monk's formula, $\sigma^u\cup\sigma^{u'}$ is a sum of distinct $\sigma^{\tilde{u}}$, which contribute nonzero quantum Schubert structure constants (equal to 1) only if $d=\alpha_{ab}$ and $\tilde{u}=wt_{ab}$. Therefore, the quantum Monk-Chevalley formula in the form of Theorem 1.3 is equivalent to the following description.

Theorem 4.2. *Let $1\le r\le n-1$ and $u\in S_n$ with $u\le w_0s_{n-1}$. In $QH^*(X)$, we have*

$$\xi^{s_r}\star\xi^u=\sum_{w\le w_0s_{n-1}}N_{s_r,u}^{w,0}\xi^w+\sum_{d_{n-1}=0}N_{s_r,u}^{w,d}\xi^wq^d+\sum_{d_{n-1}=1}N_{s_r,(u\cup s_{n-1})}^{w,d}\xi^wq^d-\delta_{r,n-1}q_{n-1}\xi^u.$$

The constraint $w \le w_0 s_{n-1}$ is a prior required in the second and third sum of the formula $\xi^{s_r} \star \xi^u$, but turns out to be redundant (see Remark 4.12).

This section is devoted to a proof of the above theorem. We use the current form in Theorem 4.2, to indicate our approach that degree-d Gromov-Witten invariants of X with $d_{n-1} \le 1$ can be reduced that of $F\ell_{n-1}$, and that the vanishing of those with $d_{n-1} \ge 2$ can be confirmed.

4.1. **Degree-d Gromov-Witten invariants with $d_{n-1} \le 1$.** In this subsection, we compute Gromov-Witten invariants $\langle \beta, \gamma \rangle_{0,2,d}$ of X with $d_{n-1} \le 1$.

4.1.1. *Unobstructedness of moduli spaces.* Recall the line bundle $\mathcal{L}_{\omega_{n-1}}$ over $F\ell_n$ defined in (2.6), the zero locus of a section of which defines the smooth Schubert divisor X.

Lemma 4.3. *Let $f : \mathbb{P}^1 \to X$ be a morphism satisfying $f_*[\mathbb{P}^1] = d$ with $d_{n-1} \le 1$. Then we have $H^1(\mathbb{P}^1, f^*T_X) = 0$.*

Proof. The exact sequence

$$0 \to T_X \to T_{F\ell_n}|_X \to E \to 0,$$

pulling back to $\mathbb{P}^1$, and then tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1)$, induces a long exact sequence:

$$\cdots \to H^0(\mathbb{P}^1, (f^*E)(-1)) \to H^1(\mathbb{P}^1, (f^*T_X)(-1)) \to H^1(\mathbb{P}^1, (f^*T_{F\ell_n}|_X)(-1)) \to \cdots .$$

Since $T_{F\ell_n}$ is globally generated and vector bundles over $\mathbb{P}^1$ are splitting, we have

$$(f^*T_{F\ell_n}|_X)(-1) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i - 1)$$

with $a_i \ge 0$ for all i. Thus $H^1(\mathbb{P}^1, (f^*T_{F\ell_n}|_X)(-1)) = 0$. On the other hand, $(f^*E)(-1) = \mathcal{O}_{\mathbb{P}^1}(d_{n-1} - 1)$ with $d_{n-1} \le 1$, which implies

$$\dim H^1(\mathbb{P}^1, (f^*T_X)(-1)) \le 1. \tag{4.2}$$

Suppose $(f^*T_X)(-1) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(b_i - 1)$, Equation (4.2) implies all $b_i \ge 0$ except at most one $b_{i_0} = -1$. In other words, $f^*T_X = \bigoplus_{i \ne i_0} \mathcal{O}_{\mathbb{P}^1}(b_i) \oplus \mathcal{O}(-1)$ with $b_i \ge 0$ or $f^*T_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(b_i)$ with $b_i \ge 0$, which implies $H^1(\mathbb{P}^1, f^*T_X) = 0$. □

Proposition 4.4. *Let $d \ge 0$ with $d_{n-1} \le 1$. Then the virtual fundamental class $[\overline{\mathcal{M}}_{0,k}(X,d)]^{\mathrm{vir}}$ coincides with the usual fundamental class $[\overline{\mathcal{M}}_{0,k}(X,d)]$.*

Proof. Let $ft : \overline{\mathcal{M}}_{0,k+1}(X,d) \longrightarrow \overline{\mathcal{M}}_{0,k}(X,d)$ be the forgetful morphism by forgetting the last marked point (which is the universal curve over $\overline{\mathcal{M}}_{0,k}(X,d)$ [BM96]). For any $(f : C \to X; pt_1, \cdots, pt_k) \in \overline{\mathcal{M}}_{0,k}(X,d)$, by using Lemma 4.3 and following (the proof of) [FP97, Lemma 10], we have $H^1(C, f^*T_X) = 0$. Therefore we have $R^1 ft_* ev_{k+1}^* T_X = 0$. Then by [BeFa97, Proposition 7.3], $[\overline{\mathcal{M}}_{0,k}(X,d)]^{\mathrm{vir}}$ is the usual fundamental class $[\overline{\mathcal{M}}_{0,k}(X,d)]$. □

4.1.2. *Computation of Gromov-Witten invariants with* $d_{n-1}=1$. In this subsection, we always assume $d\geq 0$ with $d_{n-1}=1$. For a decomposition $d=d'+d''$, we always require both $d'\geq 0$ and $d''\geq 0$. Recall the natural projection map $pr: F\ell_n\to Gr(n-1,n)=\mathbb{P}^{n-1}$. Note that $H:=pr(X)$ is a hyperplane in $\mathbb{P}^{n-1}$. The inclusion $\iota: X\hookrightarrow F\ell_n$ induces a natural inclusion $\overline{\mathcal{M}}_{0,2}(X,d)\hookrightarrow \overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)$ denoted as ι by abuse of notation. Denote

$$\begin{aligned}\mathcal{A}^\circ &:= \big\{(f:C\to F\ell_n;\ pt_1,pt_2)\in \overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)\mid f(pt_i)\in X, pr(f(pt_1))\neq pr(f(pt_2))\big\};\\ \mathcal{B} &:= \big\{(f:C\to F\ell_n;\ pt_1,pt_2)\in \overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)\mid pr(f(pt_1))= pr(f(pt_2))\in H\big\}.\end{aligned}$$

Since $d_{n-1}=1$, $pr\circ f(C)$ is a line in $\mathbb{P}^{n-1}$, containing $pr(f(pt_1)),pr(f(pt_2))$. It follows that $f(C)\subset X$ for any stable map f in $\mathcal{A}^\circ$, as the line contains the two distinct points $pr(f(pt_1)),pr(f(pt_2))$ in H has to lie in H. Note that $\mathcal{A}^\circ$ is a Zariski open dense subset of $\iota(\overline{\mathcal{M}}_{0,2}(X,d))$. Thus for the evaluation maps $ev_i:\overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)\longrightarrow F\ell_n$, we have

$$ev_1^{-1}(X)\cap ev_2^{-1}(X)=\mathcal{A}\cup\mathcal{B},\quad \text{with}\quad \mathcal{A}:=\overline{\mathcal{A}^\circ}=\iota(\overline{\mathcal{M}}_{0,2}(X,d)). \tag{4.3}$$

Here we note that $\overline{\mathcal{M}}_{0,2}(X,d)$ is proper, so $\iota(\overline{\mathcal{M}}_{0,2}(X,d))$ is closed in $\overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)$.

Denote by $X_{k,d}=ev_k^{-1}(X)\subset \overline{\mathcal{M}}_{0,k}(F\ell_n,\iota_*d)$ the space of stable maps whose last marked point is mapped to X. A stable map $(f:C\to F\ell_n,pt_1,pt_2)$ in $\mathcal{B}$ is of the form

(1) $C=C_1\cup C_2$, where the marked points x_1,x_2 are contained in C_1;
(2) $f_1=f|_{C_1}$ is a stable map to $pr^{-1}(p)$ for some $p\in H$;
(3) $f_2=f|_{C_2}$ is a stable map to Fl_n.

Actually, we can take C_1 to be the union of components of C which is maximal with respect to (1) and (2). The the union of the remaining components C_2 is connected since the image $pr(f(C))$ is a line. As a result, $\mathcal{B}$ can be written as the following union, in analogy with the boundary divisors of moduli space of stable maps (see e.g. [FP97, Section 6.2]).

$$\mathcal{B}=\bigcup_{\substack{d'+d''=d\\ d'_{n-1}=0}}\mathcal{B}_{d',d''},\quad\text{with}\quad \mathcal{B}_{d',d''}:=X_{3,d'}\times_X X_{1,d''}. \tag{4.4}$$

Here the the fiber product over X stands for the constraint $f_1(pt_3)=f_2(pt)\in X$ for the nodal point of C. By direct calculations, we have

$$\dim\mathcal{B}_{d',d''}=\langle \iota_*d',c_1(F\ell_n)\rangle+\dim X+\langle\iota_*d'',c_1(F\ell_n)\rangle-2=\langle\iota_*d,c_1(F\ell_n)\rangle+\dim X-2.$$

That is, $\mathcal{B}$ is of pure dimension. Note $\mathcal{A}$ and $\mathcal{B}$ are both of codimension 2 in $\overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)$.

Proposition 4.5. *Let $d\geq 0$ with $d_{n-1}=1$. For $u,w\in S_n$ with $u,w\leq w_0s_{n-1}$, we have*

$$\langle\xi^u,\xi^w\rangle^X_{0,2,d}=\langle\sigma^u\cup\sigma^{s_{n-1}},\sigma^w\cup\sigma^{s_{n-1}}\rangle^{F\ell_n}_{0,2,\iota_*d}-\delta_{q^d,q_{n-1}}\langle\sigma^u,\sigma^w,\sigma^{s_{n-1}}\rangle^{F\ell_n}_{0,3,0}.$$

Proof. Note that the Schubert divisor X defines the Schubert class $\sigma^{s_{n-1}}=PD([X])$, and $ev_i^{-1}(X)$ are divisors of $\overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_*d)$. As discussed above, we have the decomposition $ev_1^{-1}(X)\cap ev_2^{-1}(X)=\overline{\mathcal{M}}_{0,2}(X,d)\cup\bigcup_{(d',d'')}X_{3,d'}\times_X X_{1,d''}$ with each irreducible component of codimension 2 in $\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$. Hence, we have

$$\begin{aligned}
&\langle \sigma^u \cup \sigma^{s_{n-1}}, \sigma^w \cup \sigma^{s_{n-1}} \rangle_{0,2,\iota_* d}^{F\ell_n} \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_1^* \sigma^{s_{n-1}} \cup ev_2{}^* \sigma^{s_{n-1}} \\
&= \int_{[ev_1^{-1}(X) \cap ev_2^{-1}(X)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w + \sum_{(d',d'')} \int_{[X_{3,d'} \times_X X_{2,d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w.
\end{aligned}$$

The first term in the last equality is equal to

$$\int_{[\overline{\mathcal{M}}_{0,2}(X,d))]} \iota^*(ev_1^* \sigma^u \cup ev_2^* \sigma^w) = \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]} ev_1^* \iota^* \sigma^u \cup ev_2^* \iota * \sigma^w = \langle \xi^u, \xi^w \rangle_{0,2,d}^X.$$

Note $ev_1 \times ev_2 : X_{3,d'} \times_X X_{1,d''} \to X \times X$ factors through

$$X_{3,d'} \times_X X_{1,d''} \xrightarrow{\phi} X_{3,d'} \xrightarrow{ev_1 \times ev_2} X \times X,$$

where ϕ is a fibration with generic fibers of dimension the same as that of the generic fiber of the evaluation map $\overline{\mathcal{M}}_{0,1}(F\ell_n, \iota_* d'') \to F\ell_n$, namely of dimension $\langle \iota_* d'', c_1(T_{F\ell_n}) \rangle + 1 - 3$. Hence, $\phi_*[X_{3,d'} \times_X X_{1,d''}] = 0$ unless $\langle \iota_* d'', c_1(T_{F\ell_n}) \rangle = 2$, i.e. $d'' = [X_{s_{n-1}}] = \alpha_{n-1,n}$. In this case, $\overline{\mathcal{M}}_{0,1}(F\ell_n, d'') \cong F\ell_n$, thus $X_{3,d'} \times_X X_{1,d''} \to X_{3,d'}$ is of degree 1. Hence, we have

$$\begin{aligned}
\sum_{(d',d'')} \int_{[X_{3,d'} \times_X X_{1,d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w &= \sum_{(d',d'')} \int_{[X_{3,d'} \times_X X_{1,d''}]} \phi^* \circ (ev_1 \times ev_2)^* (\sigma^u \boxtimes \sigma^w) \\
&= \sum_{(d',\alpha_{n-1,n})} \int_{[X_{3,d'}]} (ev_1 \times ev_2)^* (\sigma^u \boxtimes \sigma^w) \\
&= \sum_{(d',\alpha_{n-1,n})} \int_{[\overline{\mathcal{M}}_{0,3}(F\ell_3, \iota_* d')]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}}.
\end{aligned}$$

Recall that $d'_{n-1} = 0$. If $d' \neq 0$, then by the divisor axiom, we have $\int_{[\overline{\mathcal{M}}_{0,3}(F\ell_3, \iota_* d')]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}} = \int_{\iota_* d'} \sigma^{s_{n-1}} \int_{\overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d')} ev_1^* \sigma^u \cup ev_2^* \sigma^w = 0$. Hence, the above sum is nonzero only if $d = \alpha_{n-1,n} = d''$, in which case we have

$$\sum_{(d',d'')} \int_{[X_{3,d'} \times_X X_{1,d''}]} ev_1^* \sigma^u \cup ev_2^* \sigma^w = \int_{[\overline{\mathcal{M}}_{0,3}(F\ell_3, 0)]} ev_1^* \sigma^u \cup ev_2^* \sigma^w \cup ev_3^* \sigma^{s_{n-1}} = \langle \sigma^u, \sigma^w, \sigma^{s_{n-1}} \rangle_{0,3,0}^{F\ell_n}.$$

Hence, the statement follows. □

4.1.3. *Computation of Gromov-Witten invariants with $d_{n-1} = 0$.* The analysis for $d_{n-1} = 0$ is similar to but much simpler than that for $d_{n-1} = 1$.

Proposition 4.6. *Let $d \geq 0$ with $d_{n-1} = 0$. For $u, w \in S_n$ with $u, w \leq w_0 s_{n-1}$, we have*

$$\langle \xi^u, \xi^w \rangle_{0,2,d}^X = \langle \sigma^u, \sigma^w \cup \sigma^{s_{n-1}} \rangle_{0,2,\iota_* d}^{F\ell_n}.$$

Proof. For $i \in \{1,2\}$ and $(f : C \to F\ell_n; pt_1, pt_2) \in ev_i^{-1}(X) \subset \overline{\mathcal{M}}_{0,2}(F\ell_n, \iota_* d)$, we have $pr_*([f(C)]) = 0$, so that $pr(f(C))$ consists of a point in $\mathbb{P}^{n-1}$.Moreover, $f(pt_1) = f(pt_2) = ev_i(f) \in X$. It follows that $f(C) \subset X$. Hence, $ev_i^{-1}(X) = \iota(\overline{\mathcal{M}}_{0,2}(X,d))$ for any $i \in \{1,2\}$. Recall $\xi^u = \iota^*\sigma^u$ and note $PD([ev_i^{-1}(X)]) = ev_i^*\sigma^{s_{n-1}}$. Hence, we have

$$\begin{aligned}
\langle \xi^u, \xi^w \rangle_{0,2,d}^X &= \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]} ev_1^*\iota^*\sigma^u \cup ev_2^*\iota^*\sigma^w \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]} \iota^*(ev_1^*\sigma^u \cup ev_2^*\sigma^w) \\
&= \int_{[\iota(\overline{\mathcal{M}}_{0,2}(X,d))]} ev_1^*\sigma^u \cup ev_2^*\sigma^w \\
&= \int_{[\overline{\mathcal{M}}_{0,2}(F\ell_n,\iota_* d)]} ev_1^*\sigma^u \cup ev_2^*\sigma^w \cup ev_2^*\sigma^{s_{n-1}}. \qquad \square
\end{aligned}$$

4.2. Vanishing of Gromov-Witten invariants with $d_{n-1} \geq 2$.

4.2.1. *Vanishing by curve neighborhood technique.* We use the curve neighborhood technique developed by Buch and Mihalcea [BM15], to show the vanishing of Gromov-Witten invariants of degree d with $d_{n-1} \geq 2$ and $d \neq \alpha_{in} + \alpha_{n-1,n}$.

Definition 4.7. *Let $d \geq 0$ and $u \in S_n$, which further satisfies $u(n) \neq 1$ for $Y = X$. The curve neighborhood $\Gamma_d^Y(X_u)$ of X_u of degree d is a reduced subscheme of Y defined by*

$$\Gamma_d^Y(X_u) = ev_2(ev_1^{-1}(X_u)).$$

The permutation $z_d \in S_n$ associated with $d \geq 0$ is defined by using the Heck product $\bullet$ on S_n as follows. Note

$$w \bullet s_i = \begin{cases} ws_i, & \text{if } \ell(ws_i) > \ell(w), \\ w, & \text{otherwise.} \end{cases}$$

Take a sequence $(\alpha_{i_1j_1}, \alpha_{i_2j_2}, \cdots, \alpha_{i_kj_k})$ of maximal elements $\alpha_{i_rj_r} = \sum_{m=i_r}^{j_r-1}[X_{s_m}]$ with respect to d; that is, each $\alpha_{i_rj_r}$ is maximal in the sense $\alpha_{i_rj_r} \in A_r := \{\alpha_{ab} \mid d - \alpha_{ab} - \sum_{m=1}^{r-1}\alpha_{i_mj_m} \geq 0\}$ with $j_r - i_r = \max\{b - a \mid \alpha_{ab} \in A_r\}$. Then k depends only on d, and

$$z_d := t_{i_1j_1} \bullet t_{i_2j_2} \bullet \cdots \bullet t_{i_kj_k} \in S_n$$

is also independent of choices of the sequences of maximal elements with respect to d.

Proposition 4.8 ([BM15, Theorem 5.1])**.** $\Gamma_d^{F\ell_n}(X_u) = X_{u \bullet z_d}$, *for $u \in S_n$ and $d \geq 0$.*

Lemma 4.9. *Let $d \geq 0$ with $d_{n-1} \geq 2$. Then we have $\ell(z_d) \leq \langle d, c_1(T_X)\rangle - 1$, with equality holding only if $d = \alpha_{in} + \alpha_{n-1,n}$ for some i.*

Proof. Take a sequence $(\alpha_{i_1j_1}, \alpha_{i_2j_2}, \cdots, \alpha_{i_kj_k})$ of maximal elements with respect to d. Without loss of generality, we can assume $j_r = n$ for $1 \leq r \leq d_{n-1}$ (by noting that the corresponding transpositions of the form t_{cn} are disjoint with that of the form t_{ab} with $b < n$; otherwise, $b = c$, and α_{an} would be a bigger element than α_{cn}). Note $\ell(t_{ab}) = 2b - 2a - 1 = 2|\alpha_{ab}| - 1$,

$\ell(t_{an}\bullet t_{bn})\le \ell(t_{an})+\ell(t_{bn})-1$ (since t_{an} (resp. t_{bn}) has a reduced expression ending (resp. starting) with s_{n-1}). Hence, $k\ge d_{n-1}\ge 2$, and we have

$$\begin{aligned}\ell(z_d) &\le \ell(t_{i_1n}\bullet\cdots\bullet t_{i_{d_{n-1}}n})+\ell(t_{i_{d_{n-1}+1}j_{d_{n-1}+1}}\bullet\cdots\bullet t_{i_kj_k})\\ &\le \sum_{r=1}^{d_{n-1}}\ell(t_{i_rn})-(d_{n-1}-1)+\sum_{r=d_{n-1}+1}^{k}\ell(t_{i_rj_r})\\ &=2|d|-k-(d_{n-1}-1)\\ &\le 2|d|-d_{n-1}-(d_{n-1}-1)=\langle d,c_1(T_X)\rangle-d_{n-1}+1.\end{aligned}$$

Hence, $\ell(z_d)\le\langle d,c_1(T_X)\rangle-1$, with equality holding only if $k=d_{n-1}=2$. When the equality holds, all the above inequalities are equalities. In particular, we have $d=\alpha_{in}+\alpha_{jn}$ with $i\le j$, and $\ell(t_{in}\cdot(s_{n-1}t_{jn})=\ell(t_{in}\bullet t_{jn})=\ell(t_{in})+\ell(t_{jn})-1=\ell(t_{in}s_{n-1}t_{jn})$. It follows that $s_js_{j+1}\cdots s_{n-2}t_{jn}$ is a reduced expression, where $t_{jn}=s_{n-1}s_{n-2}\cdots s_j\cdots s_{n-2}s_{n-1}$ is reduced. Then $s_js_{j+1}\cdots s_{n-2}t_{jn}(n-1)>s_js_{j+1}\cdots s_{n-2}t_{jn}(n)$, resulting in a contraction $j>j+1$ if $j<n-1$. □

Proposition 4.10. *Let $d\ge 0$ satisfy $d_{n-1}\ge 2$ and $d\ne\alpha_{in}+\alpha_{n-1,n}$ for $1\le i\le n-1$. Then for any $\beta,\gamma\in H^*(X)$, we have $\langle\beta,\gamma\rangle_{0,2,d}=0$.*

Proof. Take any $u,v\in S_n$ with $u,v\le w_0s_{n-1}$. Note $\Gamma_d^X(X_u)\subseteq\Gamma_d^{F\ell_n}(X_u)=X_{u\bullet z_d}$ by Proposition 4.8. Since $d_{n-1}\ge 2$ and $d\ne\alpha_{in}+\alpha_{n-1,n}$, by Lemma 4.9, we have

$$\dim\Gamma_d^X(X_u)\le\dim(X_{u\bullet z_d})=\ell(u\bullet z_d)\le\ell(u)+\ell(z_d)<\ell(u)+\langle d,c_1(T_X)\rangle-1.$$

Denote $PD([X_u])$ as $[X_u]$ by abuse of notation. Using projection formula, we have

$$\int_{[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}}}ev_1^*([X_u])\cup ev_2^*([X_v])=\int_X(ev_2)_*(ev_1^*([X_u])\cap[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}})\cup[X_v].$$

The cycle $(ev_2)(ev_1^{-1}([X_u]))$ is supported on the curve neighborhood $\Gamma_d^X(X_u)$, and the pushforward $(ev_2)_*(ev_1^*([X_u])\cap[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}})$ is non-zero only if the curve neighborhood $\Gamma_d^X(X_u)$ has components of dimension

$$\operatorname{expdim}\overline{\mathcal{M}}_{0,2}(X,d)-(\dim X-\ell(u))=\langle d,c_1(T_X)\rangle-1+\ell(u).$$

However, such components do not exist by the above estimation of $\dim\Gamma_d^X(X_u)$.

Hence, $\langle[X_u],[X_v]\rangle_{0,2,d}=\int_{[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}}}ev_1^*([X_u])\cup ev_2^*([X_v])=0$. Since $\{[X_u]\}_{u\le w_0s_{n-1}}$ is a basis of $H^*(X,\mathbb{Z})$, the statement follows. □

4.2.2. *Vanishing for specific degrees.* It remains to show the vanishing of Gromov-Witten invariants for $d=\alpha_{in}+\alpha_{n-1,n}$. We follow a similar idea to the approach to proving [HKLS24, Lemma 3.7] and [HKLS25, Theorem 4.1 (b)].

Let us consider

$$\mathcal{P}=F\ell_{1,\dots,n-2;n},\qquad Y=\{V_\bullet\in\mathcal{P}\mid F_1\subseteq V_{n-2}\}.$$

We have a natural projection $\pi:X\to\mathcal{P}$ by forgetting V_{n-1}. The fiber at a point of Y is $\mathbb{P}^1$, while the fiber is a point at a point $\mathcal{P}\setminus Y$. Note

$$\dim\mathcal{P}=\dim X,\qquad\dim Y=\dim\mathcal{P}-2.$$

Proposition 4.11. *Let $d = \alpha_{in} + \alpha_{n-1,n}$ where $1 \le i \le n-1$. Then for any $\beta_1, \beta_2 \in H^*(X)$, we have $\langle \beta_1, \beta_2 \rangle_{0,2,d} = 0$.*

Proof. We discuss the degrees in two cases.

Case $i = n-1$. In this case, $\pi_* d = 0$. Therefore any stable map of degree d is contained in a fiber of π at some point of Y. This defines a morphism from the moduli space to $\mathcal{P}$ whose image is included in Y. Denoting $\mathrm{ev} = ev_1 \times ev_2$ and by Δ the diagonal map, we have

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,2}(X,d) \arrow[r, "\mathrm{ev}"] \arrow[d] & X \times X \arrow[d] \\ \mathcal{P} \arrow[r, "\Delta"] & \mathcal{P} \times \mathcal{P} \end{tikzcd}$$

The space $\overline{\mathcal{M}}_{0,2}(X,d)$ has expected dimension

$$\dim X + \deg_X q_{n-1}^2 + 2 - 3 = \dim X + 1.$$

While the image of ev lies in the preimage of Y of $X \times_{\mathcal{P}} X$, which is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over Y. Thus its dimension is

$$\dim Y + 2 = \dim X - 2 + 2 = \dim X.$$

Thus $\mathrm{ev}_*[\overline{\mathcal{M}}_2(M,d)]^{\mathrm{vir}} = 0$.

Case $i < n-1$. We have a similar commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,2}(X,d) \arrow[r, "\mathrm{ev}"] \arrow[d, "\hat{\pi}"'] & X \times X \arrow[d, "\pi\times\pi"] \\ \overline{\mathcal{M}}_{0,2}(\mathcal{P},\pi_* d) \arrow[r, "\mathrm{ev}_{\mathcal{P}}"] & \mathcal{P} \times \mathcal{P}. \end{tikzcd}$$

Note that $\deg_X \alpha_{j,j+1} = 2 = \deg_{\mathcal{P}} \pi_* \alpha_{j,j+1}$ for $j < n-2$, $\deg_X \alpha_{n-2,n-1} = 2 = \deg_{\mathcal{P}} \pi_* \alpha_{n-2,n-1} - 1$ and $\deg_X \alpha_{n-1,n} = 1$. Hence, we have

$$\begin{aligned} \exp\dim \overline{\mathcal{M}}_{0,2}(X,d) &= \dim X + \deg_X q^d + 2 - 3, \\ \exp\dim \overline{\mathcal{M}}_{0,2}(\mathcal{P},\pi_* d) &= \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P},\pi_* d) = \exp\dim \overline{\mathcal{M}}_{0,2}(X,d) - 1. \end{aligned}$$

Moreover the map $\hat{\pi}$ cannot be surjective, since any stable map in the image has an extra constrain that C intersects with Y. This reduces one more dimension:

$$\dim(\mathrm{im}(\hat{\pi})) \le \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P},\pi_* d) - 1.$$

As a result, $\mathrm{ev}_*[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}}$ is supported over

$$(\pi \times \pi)^{-1}(\mathrm{im}(\mathrm{ev}_{\mathcal{P}} \circ \hat{\pi})).$$

Since the fibers of $\pi \times \pi$ have dimension at most 2, it has dimension at most

$$\begin{aligned} \dim(\mathrm{im}(\mathrm{ev}_{\mathcal{P}} \circ \hat{\pi})) + 2 &\le \dim(\mathrm{im}\,\hat{\pi}) + 2 \\ &\le \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P},\pi_* d) - 1 + 2 = \exp\dim \overline{\mathcal{M}}_{0,2}(X,d). \end{aligned}$$

If $\mathrm{ev}_*[\overline{\mathcal{M}}_{0,2}(X,d)]^{\mathrm{vir}} \ne 0$, then the equalities must be achieved. Note that $\pi \times \pi$ has two-dimensional fibers only at points of $Y \times Y$, the equality holds only if $\mathrm{ev}_{\mathcal{P}}$ restricts to a morphism $Z_1 \to Z_2$ of finite degree, where Z_1 is a component of $\mathrm{im}\,\hat{\pi}$ of codimension 1 in $\overline{\mathcal{M}}_{0,2}(\mathcal{P}, \pi_* d)$ and Z_2 is a component of $\mathrm{im}(\mathrm{ev}_{\mathcal{P}})$ contained in $Y \times Y$. By (the proof of) [BCMP13, Lemma 3.8], Z_2 is contained in a locally trivial fibration over $Y_1 \subset Y$ with fiber $\Gamma^{\mathcal{P}}_{\pi_* d}(y)$, where Y_1 denotes the natural projection of $Z_2 \subset Y \times Y$ to the first factor. By [BM15, Theorem 5.1], the fiber $\Gamma^{\mathcal{P}}_{\pi_* d}(y)$ is a Schubert variety of $\mathcal{P}$, indexed by the permutation $z^{\mathcal{P}}_{\pi_* d} = z_{\alpha_{in}} s_{n-1} = t_{in} s_{n-1}$. Thus is of dimension $\ell(z^{\mathcal{P}}_{\pi_* d}) = \ell(t_{in}) - 1$.

$$\dim Z_2 = \dim Y_1 + \dim \Gamma^{\mathcal{P}}_{\pi_* d}(y) \le \dim Y + \ell(t_{in}) - 1 = \dim X - 2 + 2(n-i) - 1 - 1,$$
$$\dim Z_1 = \dim \overline{\mathcal{M}}_{0,2}(\mathcal{P}, d) - 1 = \exp\dim \overline{\mathcal{M}}_{0,2}(X, d) - 2 = \dim X + 2(n-i) - 1 - 2,$$

resulting in a contradiction $0 = \dim Z_2 - \dim Z_1 \le -1$. □

4.3. **Proof of Theorem 4.2.** Note that $\{PD([X_u])\}_u$ is the dual basis of $\{\xi^u\}_u$ with respect to the Poincaré pairing. Write $PD([X_u]) = \sum_\gamma a^u_\gamma \xi^\gamma$. Using the projection formula,

$$\delta_{u,w} = \int_{[X]} PD([X_u]) \cup \xi^w = \int_{[X]} \iota^*(\sum_\gamma a^u_\gamma \sigma^\gamma) \cup \iota^* \sigma^w = \int_{[F\ell_n]} \sum_\gamma a^u_\gamma \sigma^\gamma \cup \iota_*(\iota^* \sigma^w).$$

Note $\iota_*(\iota^*\sigma^w) = \iota_*(\iota^*\sigma^w \cup \xi^{\mathrm{id}}) = \sigma^w \cup PD([X]) = \sigma^w \cup \sigma^{s_{n-1}}$. The permutation u varies in S_n with $u(n) \ne 1$. Hence,

$$\sum_\gamma a^w_\gamma \sigma^\gamma \cup \sigma^{s_{n-1}} = (\sigma^w)^\vee + \sum_{\eta(n)=1} b_\eta (\sigma^\eta)^\vee.$$

We have

$$\begin{aligned}\xi^{s_r} \star \xi^u &= \sum_{w,d} \langle \xi^{s_r}, \xi^u, PD([X_w])\rangle^X_{0,3,d} \xi^w q^d \\ &= \sum_{v \le w_0 s_{n-1}} \langle \xi^{s_r}, \xi^u, PD([X_w])\rangle^X_{0,3,0} \xi^w + \sum_{w \le w_0 s_{n-1}, d \ne 0} \langle \xi^{s_r}, \xi^u, PD([X_w])\rangle^X_{0,3,d} \xi^w q^d.\end{aligned}$$

By Proposition 4.10 and Proposition 4.11, there are no q^d-terms in the second sum whenever $d_{n-1} \ge 2$. By the divisor axiom in Gromov-Witten theory, for $d \ne 0$ we have $\langle \xi^{s_r}, \xi^u, [X_v]\rangle^X_{0,3,d} = \int_d \xi^{s_r} \langle \xi^u, \sum_\gamma a^v_\gamma \xi^\gamma\rangle_{0,2,d}$.

For $d_{n-1} = 1$, by Proposition 4.5, we have

$$\begin{aligned}&\int_d \xi^{s_r} \langle \xi^u, \sum_\gamma a^w_\gamma \xi^\gamma \rangle^X_{0,2,d} q^d \\ =&\int_d \sigma^{s_r} \langle \sigma^u \cup \sigma^{s_{n-1}}, \sum_\gamma a^w_\gamma \sigma^\gamma \cup \sigma^{s_{n-1}} \rangle^{F\ell_n}_{0,2,d} q^d - \int_d \xi^{s_r} \delta_{q^d, q_n - 1} \langle \sigma^u, \sum_\gamma a^w_\gamma \sigma^\gamma, \sigma^{s_{n-1}}\rangle^{F\ell_n}_{0,3,0} q_{n-1} \\ =&\int_d \sigma^{s_r} \langle \sigma^u \cup \sigma^{s_{n-1}}, \sum_\gamma a^w_\gamma \sigma^\gamma \cup \sigma^{s_{n-1}} \rangle^{F\ell_n}_{0,2,d} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \int_{[F\ell_n]} \sigma^u \cup (\sum_\gamma a^w_\gamma \sigma^\gamma) \cup \sigma^{s_{n-1}} \\ =&\langle \sigma^{s_r}, \sigma^u \cup \sigma^{s_{n-1}}, (\sigma^w)^\vee + \sum_{\eta(n)=1} b_\eta (\sigma^\eta)^\vee \rangle^{F\ell_n}_{0,3,d} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \int_{[F\ell_n]} \sigma^u \cup ((\sigma^w)^\vee + \sum_{\eta(n)=1} b_\eta (\sigma^\eta)^\vee) \\ =&\langle \sigma^{s_r}, \sigma^u \cup \sigma^{s_{n-1}}, ((\sigma^w)^\vee)^\vee \rangle^{F\ell_n}_{0,3,d} q^d - \int_d \xi^{s_r} \delta_{q^d, q_{n-1}} q_{n-1} \delta_{u,w}\end{aligned}$$

The last equality holds by noting $\int_{[F\ell_n]}\sigma^u\cup(\sigma^\eta)^\vee=0$ (since $u(n)\neq 1$)) and

$$\langle\sigma^{s_r},\sigma^u\cup\sigma^{s_{n-1}},(\sigma^\eta)^\vee\rangle_{0,3,d}^{F\ell_n}=\sum_{\tilde u}\langle\sigma^{s_r},\sigma^{\tilde u},(\sigma^\eta)^\vee\rangle_{0,3,d}^{F\ell_n}=\sum_{\tilde u}N_{s_r,\tilde u}^{\eta,d}=0$$

for any permutation η with $\eta(n)=1$. Indeed, by the quantum Monk's formula for $F\ell_n$, $u\lessdot_{n-1}\tilde u$ and for $\hat w\in S_n$, $N_{s_r,\tilde u}^{\hat w,d}\neq 0$ only if $\tilde u\lessdot^q\hat w=\tilde u t_{an}$ for some a (since $d_{n-1}=1$).

i) If $\tilde u\leq w_0s_{n-1}$, then $\hat w\leq\tilde u\leq w_0s_{n-1}$, i.e. $\hat w(n)\neq 1$;
ii) If $\tilde u\not\leq w_0s_{n-1}$, i.e. $\tilde u(n)=1$, then $\hat w(n)=\tilde u t_{an}(n)=\tilde u(a)\neq\tilde u(n)=1$.

Thus the sum is vanishing for any η with $\eta(n)=1$.

For $d\neq 0$ with $d_{n-1}=0$, by Proposition 4.6, we have

$$\begin{aligned}\int_d\xi^{s_r}\langle\xi^u,\sum_\gamma a_\gamma^w\xi^\gamma\rangle_{0,2,d}^Xq^d&=\int_d\sigma^{s_r}\langle\sigma^u,\sum_\gamma a_\gamma^w\sigma^\gamma\cup\sigma^{s_{n-1}}\rangle_{0,2,d}^{F\ell_n}q^d\\&=\langle\sigma^{s_r},\sigma^u,(\sigma^w)^\vee+\sum_{\eta(n)=1}b_\eta(\sigma^\eta)^\vee\rangle_{0,3,d}^{F\ell_n}q^d\\&=\langle\sigma^{s_r},\sigma^u,(\sigma^w)^\vee\rangle_{0,3,d}^{F\ell_n}q^d.\end{aligned}$$

The last equality holds again by noting $\langle\sigma^{s_r},\sigma^u,(\sigma^{\hat w})^\vee\rangle_{0,3,d}^{F\ell_n}q^d=0$ unless $u\lessdot_r^q\hat w$, implying $\hat w(n)\neq 1$. Hence, we are done. □

Remark 4.12. *The arguments i) and ii) in the above proof say that for any $u,w\in S_n$ and any a,k, the hypothesis $u\lessdot_{n-1}wt_{an}\lessdot_k^q w$ implies $w(n)\neq 1$, i.e. $w\leq w_0s_{n-1}$.*

5. Quantum Schubert polynomials for X

This section is devoted to a proof of Theorem 1.6, namely for any w, the quantum Schubert polynomial $\mathfrak{S}_w^q$ of Fomin, Gelfand and Postnikov represents the pullback Schubert class ξ^w, under the canonical ring isomorphism in Theorem 1.2.

Recall

$$q_{ab}:=q_aq_{a+1}\cdots q_{b-1}\quad\text{for}\quad 1\leq a<b\leq n.$$

5.1. **Quantum Schubert polynomials.** The classical Schubert polynomials were introduced by Lascoux and Schützenberger [LS82] by using the divided difference operators ∂_i's of Bernstein, Gelfand and Gelfand [BGG73]. Precisely, for $f=f(x_1,\cdots,x_n)\in\mathbb{Z}[x]$ and $w\in S_n$, denote $wf=f(x_{w^{-1}(1)},\cdots,x_{w^{-1}(n)})$. Then $\partial_if:=\frac{f-s_if}{x_i-x_{i+1}}\in\mathbb{Z}[x]$, and the classcial Schubert polynomials $\mathfrak{S}_w(x)$ is recursively defined by

$$\mathfrak{S}_{w_0}=x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\qquad\text{and}\quad\mathfrak{S}_{ws_i}=\partial_i\mathfrak{S}_w\quad\text{whenver}\quad\ell(ws_i)=\ell(w)-1.\tag{5.1}$$

The following were shown in [LS82].

(1) $\Phi(\sigma^w)=[\mathfrak{S}_w(x)]$ under the canonical ring isomorphism Φ in Proposition 2.1.
(2) $\{e_{i_1}^1e_{i_2}^2\cdots e_{i_{n-1}}^{n-1}\}_{0\leq i_j\leq j}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[x]$.

Therefore, we have the linear expansion $\mathfrak{S}_w=\sum\alpha_{i_1\dots i_{n-1}}e_{i_1}^1e_{i_2}^2\cdots e_{i_{n-1}}^{n-1}$. In [FGP97], Fomin, Gelfand and Postnikov introduced the quantum Schubert polynomial

$$\mathfrak{S}_w^q:=\sum\alpha_{i_1\dots i_{n-1}}E_{i_1}^1E_{i_2}^2\cdots E_{i_{n-1}}^{n-1}.\tag{5.2}$$

They also showed $\Phi_q(\sigma^w) = [\mathfrak{S}^q_w]$, under the canonical ring isomorphism in Φ_q [GK95],

$$\Phi_q : QH^*(F\ell_n, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \cdots, x_n, q_1, \cdots, q_{n-1}]/(E^n_1, \cdots, E^n_n). \tag{5.3}$$

Recall the (quantum) r-Bruhat order $\lessdot_r$ (resp. $\lessdot^q_r$) defined in Equation (2.2) (resp. Equation (2.3)). The following is the quantum Monk's formula on the level of polynomials, proved in the first half of [FGP97, Theorem 7.1].

Proposition 5.1 (Quantum Monk's formula). *For $u \in S_n$ and $1 \le r < n$, in $\mathbb{Z}[x]$ we have*

$$\mathfrak{S}^q_{s_r}\mathfrak{S}^q_u = (x_1 + \cdots + x_r)\mathfrak{S}^q_u = \sum_{u \lessdot_r ut_{ab}} \mathfrak{S}^q_{ut_{ab}} + \sum_{u \lessdot^q_r ut_{ck}} q_{ck}\mathfrak{S}^q_{ut_{ck}}.$$

Lemma 5.2. *Let $u, w \in S_n$ and $1 \le a < b \le n$.*

1. *$u \lessdot ut_{ab}$ if and only if $u(a) < u(b)$ and for any $a < c < b$, we have $u(c) \notin [u(a), u(b)]$.*
2. *$u \lessdot^q ut_{ab}$ if and only if $u(a) > u(b)$ and for any $a < c < b$, we have $u(c) \in [u(b), u(a)]$.*

Proof. Note $\ell(t_{ab}) = 2b - 2a - 1$. The statement follows from a direct counting of the number of inversions, which defines the length of a permutation. □

The next proposition is a special case of the second half of [FGP97, Theorem 7.1], with a slightly more precise description than that in loc. cit.; see also [LOTRZ25, Theorem 4]. This special case will play a crucial role in our proof of Theorem 1.6. A permutation $w \in S_n$ is said to have a descent at the k-th position if $w(k+1) < w(k)$.

Proposition 5.3 (Transition equation). *Let $w \in S_n \setminus \{\mathrm{id}\}$. Denote by i the last descent position of w. Take the maximal j with $w(j) < w(i)$. Then $u := wt_{ij}$ satisfies $u \lessdot w$, and we have*

$$\mathfrak{S}^q_w = x_i\mathfrak{S}^q_u + \sum_{u \lessdot ut_{hi}} \mathfrak{S}^q_{ut_{hi}} + \sum_{u \lessdot^q ut_{hi}} q_{hi}\mathfrak{S}^q_{ut_{hi}} - \sum_{u \lessdot^q ut_{ik}} q_{ik}\mathfrak{S}^q_{ut_{ik}}.$$

Proof. It follows directly from the definition that $i < j$ and that for any $i < c < j$, $u(c) = w(c) < w(j) = u(i)$ (where the inequality holds since i is the last descent position). Thus $u \lessdot w$ by Lemma 5.2 (1).

By Proposition 5.1, we compare the two quantum Monk's formulas

$$(x_1 + x_2 + \cdots + x_{i-1})\mathfrak{S}^q_u = \sum_{u \lessdot_{i-1} ut_{ab}} \mathfrak{S}^q_{ut_{ab}} + \sum_{u \lessdot^q_{i-1} ut_{ck}} q_{ck}\mathfrak{S}^q_{ut_{ck}};$$

$$(x_1 + x_2 + \cdots + x_i)\mathfrak{S}^q_u = \sum_{u \lessdot_i ut_{ab}} \mathfrak{S}^q_{ut_{ab}} + \sum_{u \lessdot^q_i ut_{ck}} q_{ck}\mathfrak{S}^q_{ut_{ck}}.$$

Notice that if $c \le i-1 < i < k$, then $u \lessdot^q_{i-1} ut_{ck}$ implies $u \lessdot^q_i ut_{ck}$. Therefore the difference of the two products involving quantum parts happens exactly when either $k = i$ in the first product or $c = i$ in the second product. Hence, the quantum part of the statement follows.

For the classical part, the same argument applies once we show that $u \lessdot ut_{ib}$ implies $b = j$. Indeed, assume $u \lessdot ut_{ib}$ for some $b \ne j$. Then $w(j) = u(i) < u(b) = w(b)$ by Lemma 5.2. Since i is the last descent position of w, it follows that $j < b$. Furthermore, we have $u(j) = w(i) < w(b) = u(b)$, since j is maximal with respect to $w(j) < w(i)$. But then we would have $i < j < b$ and $u(i) < u(j) < u(b)$, contradicting with $u \lessdot ut_{ib}$ by Lemma 5.2. □

Lemma 5.4. *Let $w \in S_n \setminus \{\mathrm{id}\}$ with $w(n) \neq 1$. Then all the permutations v occurring on the right hand side of the formula of $\mathfrak{S}_w^q$ in Proposition 5.3 satisfy $v(n) \neq 1$.*

Proof. With the same notation in Proposition 5.3, if $v = u = wt_{ij}$, then $u(n) \geq u(j) > u(i) \geq 1$. If $v = ut_{hi}$, then $v(n) = u(n) \neq 1$ by noting $i < n$. It remains to discuss the case $v = ut_{ik}$. Since $u \lessdot^q ut_{ik}$, by Lemma 5.2, $u(k) < u(i) = w(j)$, so $k < j$ since i is the last descent postion. In particuar, $k \neq n$. Thus, $v(n) = u(n) \neq 1$. □

Definition 5.5. *For $u, w \in W$, we say $u \prec w$ if and only if*

$$(\ell(u), -u(n), -u(n-1), \cdots, -u(1)) < (\ell(w), -w(n), -w(n-1), \cdots, -w(1))$$

with respect to the lexicographic order. This defines a total order $\prec$ on S_n.

Lemma 5.6. *Let $w \in S_n \setminus \{\mathrm{id}\}$. Then all the permutations occurring on the right hand side of the formula of $\mathfrak{S}_w^q$ in Proposition 5.3 are strictly smaller than w with the order $\prec$.*

Proof. Let u be as in Proposition 5.3, then $u \prec w$ since $\ell(u) = \ell(w) - 1$. For $u \lessdot ut_{hi}$, we have $\ell(w) = \ell(ut_{hi}) = \ell(u) + 1$. By Lemma 5.2, we have $w(j) = u(i) < u(j) = ut_{hi}(j)$. Combining this with the property $w(a) = u(a) = ut_{hi}(a)$ for $a > j$, we obtain $ut_{hi} \prec w$. Permutations in the quantum part are all of length smaller than $\ell(w)$, and hence are strictly smaller than w with respect to the total order $\prec$. □

5.2. **Proof of Theorem 1.6.** To achieve our aim, we first show that the pullback Schubert classes ξ^w admit exactly the same transition equations in the quantum cohomology $QH^*(X)$ as that for $\mathfrak{S}_w^q$ on the level of polynomials.

Proposition 5.7. *With the same notation as in Proposition 5.3, in $QH^*(X)$ we have*

$$\xi^w = x_i \xi^u + \sum_{u \lessdot ut_{hi}} \xi^{ut_{hi}} + \sum_{u \lessdot^q ut_{hi}} q_{hi} \xi^{ut_{hi}} - \sum_{u \lessdot^q ut_{ik}} q_{ik} \xi^{ut_{ik}}.$$

Proof. Recall $w = ut_{ij}$ and that i is the last descent of w, so $i \leq n-1$ and $x_i = \xi^{s_i} - \xi^{s_{i-1}}$. We compare the two quantum Monk-Chevalley formulas

$$\xi^{s_{i-1}} \star \xi^u = \sum_{u \lessdot_{i-1} ut_{ab} \leq w_0 s_{n-1}} \xi^{ut_{ab}} + \sum_{u \lessdot^q_{i-1} ut_{ck} \text{ with } k<n} \xi^{ut_{ck}} q_{ck} + \sum_{u \lessdot ut_{an} \lessdot^q_{i-1} ut_{an}t_{cn} \leq w_0 s_{n-1}} \xi^{ut_{an}t_{cn}} q_{cn} - 0,$$

$$\xi^{s_i} \star \xi^u = \sum_{u \lessdot_i ut_{ab} \leq w_0 s_{n-1}} \xi^{ut_{ab}} + \sum_{u \lessdot^q_i ut_{ck} \text{ with } k<n} \xi^{ut_{ck}} q_{ck} + \sum_{u \lessdot ut_{an} \lessdot^q_i ut_{an}t_{cn} \leq w_0 s_{n-1}} \xi^{ut_{an}t_{cn}} q_{cn} - \delta_{i,n-1} q_{n-1} \xi^u.$$

The classical part follows from the same argument as in Proposition 5.3, where the constraint $u_{hi} \leq w_0 s_{n-1}$ is redundant by Lemma 5.4. Again note that if $c \leq i-1 < i < k$, then $u \lessdot^q_{i-1} ut_{ck}$ implies $u \lessdot^q_i ut_{ck}$. Therefore the difference of the sum involving quantum parts in the two quantum products happens exactly when either $k = i$ in the first product or $c = i$ in the second product. The part of $k = i$ in the first product is exactly the second sum in the equation for ξ^w in the statement. It remains to show that the rest is given by $c = i$ part in $\xi^{s_i} \star \xi^u$ together with $\delta_{i,n-1} q_{n-1} \xi^u$, namely to show

$$\sum_{u \lessdot^q ut_{ik}} \xi^{ut_{ik}} q_{ik} = \sum_{u \lessdot^q ut_{ik} \text{ with } k<n} \xi^{ut_{ik}} q_{ik} + \sum_{u \lessdot ut_{an} \lessdot^q_i ut_{an}t_{in} \leq w_0 s_{n-1}} \xi^{ut_{an}t_{in}} q_{in} - \delta_{i,n-1} q_{n-1} \xi^u. \tag{5.4}$$

Denote by RHS (resp. LHS) the right (resp. left) hand side of the above equation to prove.

(1) Case $i = n-1$. Then $j = n$, the sum in LHS is empty (otherwise we would have $u \lessdot^q ut_{ik} = ut_{n-1,n} = w$, contradicting $u \lessdot w$), and the first sum in RHS is empty as well. The constraints
$$u \lessdot ut_{an} \lessdot^q ut_{an}t_{in}$$
imply $a = n-1$ by Lemma 5.2(Otherwise, $a < n-1 = i$, then $ut_{an}(i) = u(i) < u(a) = ut_{an}(n)$). Then we have $ut_{an}t_{cn} = u$, which automatically satisfies $u \leq w_0 s_{n-1}$ by Lemma 5.4. Hence, RHS $= 0 + \xi^u q_{n-1} - \xi^u q_{n-1} = 0 =$ LHS.

(2) Case $i < n-1$. Then $\delta_{i,n-1} = 0$.

For $u \lessdot^q ut_{ik}$ on the LHS, we have $k \neq j$ since $u \lessdot ut_{ij} = w$. By Lemma 5.2, $w(k) = u(k) < u(i) = w(j)$. Since i is the last descent, we have $k < j$. In particular, $k < n$, thus the LHS is equal to the first sum of the RHS. It remains to show that the second sum on the RHS is zero. Suppose we have $u \lessdot ut_{an} \lessdot_i^q ut_{an}t_{in}$. By the choice of i, j, u also has no descent after i. We also note that j is the minimal integer greater than i that satisfies $u(i) < u(j)$. These two properties will be used over and over again in the following argument. If $i < a < n-1$, then $u(n-1) \in [u(a), u(n)]$, contradicting with Lemma 5.2. Therefore, either $a = n-1$ or $a \leq i$. If $a = n-1$, by Lemma 5.2, we have $u(n-1) = ut_{an}(n) < ut_{an}(i) = u(i)$, so $j = n$. Then $ut_{an}(n-1) = u(n) = u(j) > u(i) = ut_{an}(i)$. Then $ut_{an}(n-1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. If $a = i$, then from $u \lessdot ut_{an}$ and Lemma 5.2, we have $j = n$. Then $ut_{an}(n-1) = u(n-1) < u(i) = ut_{an}(n)$, Then $ut_{an}(n-1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. If $a < i$, then by Lemma 5.2, we have $ut_{an}(n-1) = u(n-1) < u(a) = ut_{an}(n)$, Then $ut_{an}(n-1) \notin [ut_{an}(n), ut_{an}(i)]$, contradicting Lemma 5.2. □

Proof of Theorem 1.6. By (the proof of) Theorem 1.2, the canonical ring isomorphism
$$\Psi_q : QH^*(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[x_1, \cdots, x_n, q_1, \cdots, q_{n-1}] \Big/ \big(\hat{E}_1^n, \cdots, \hat{E}_{n-1}^n, E_{n-1}^{n-1}\big).$$
satisfies $\Psi_q(\xi^i) = [x_1 + x_2 + \cdots + x_i]$ for $1 \leq i \leq n-1$. Namely $\Psi_q(\xi^w) = [\mathfrak{S}_w^q]$ holds for all $w \in S_n$ with $\ell(w) = 1$ (which all satisfy $w \leq w_0 s_{n-1}$).

By (2.1), $w \leq w_0 s_{n-1}$ holds if and only if $w(n) \neq 1$. By Proposition 5.7, Lemma 5.4 and Lemma 5.6, every ξ^w with $w(n) \neq 1$ can be written as a $\mathbb{Z}[q]$-linear combination of classes ξ^v with $v \prec w$ and $v(n) \neq 1$. By Proposition 5.3, $\mathfrak{S}_w^q$ can also be written as exactly the same $\mathbb{Z}[q]$-linear combination of $\mathfrak{S}_v^q$ on the level of polynomials. Hence, the statement follows immediately from the mathematical induction on the totally-ordered subset $(\{w\}_{w \leq w_0 s_{n-1}}, \prec)$. □

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