Tensor Products and Universal Properties

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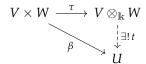
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Summary and Reflections on Exercise 5.1

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This is my (presentation) summary of Exercise 5.1 on Tensor Products.

Exercise 5.1 Statement



commutes. We write $v \otimes w := \tau(v,w)$ for all $v \in V$ and $w \in W$, and refer to $V \otimes_{\mathbb{k}} W$ as the tensor product of V and W.

(i) Uniqueness up to Unique Isomorphism

Show that if $V \otimes_{\mathbb{k}} W$ exists, then it is unique up to unique isomorphism.

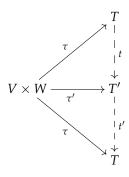
Let us assume we have two 'tensor' (or universal) spaces (T,τ) and (T',τ') , both satisfying the universal property with respect to $V\times W$. Then for the bilinear map $\tau':V\times W\to T'$, the universal property of (T,τ) gives a unique linear map $t:T\to T'$ such that $t\circ\tau=\tau'$. Similarly, the universal property of (T',τ') applied to τ yields a unique linear map $t':T'\to T$ such that $t'\circ\tau'=\tau$.

We represent a bilinear map with β

We use the dashed line to represent a unique morphism

We only use the universal property, so this applies to any object defined by such a property.

We can represent this situation with the following diagram:



This is essentially Leinster's diagram from Chapter o (see p. 5).

The same argument applied symmetri-

cally to T' gives $t \circ t' = id_{T'}$.

Now observe:

$$t' \circ t \circ \tau = t' \circ \tau' = \tau,$$

 $\Rightarrow t' \circ t = id_T \text{ by uniqueness.}$

This shows that t and t' are inverse isomorphisms, so $T \cong T'$.

Therefore, t is an isomorphism between T and T', proving uniqueness up to unique isomorphism.

(ii) Construction Using Bases

Let *B* and *C* be bases of *V* and *W*. Consider the free vector space $\mathbb{k}\langle B \times C \rangle$ and define:

$$\tau(b,c) = \delta_{(b,c)}$$

where $\delta_{(b,c)}$ denotes the formal generator for the pair $(b,c) \in B \times C$. We extend τ bilinearly. For any bilinear map $\beta: V \times W \rightarrow U$, define a linear map $t : \mathbb{k}\langle B \times C \rangle \to U$ on basis elements by $t(\delta_{(b,c)}) :=$ $\beta(b,c)$. This t satisfies $t \circ \tau = \beta$ and is unique by linearity.

Some write (b, c) directly; we use $\delta_{(b,c)}$ to make the vector space structure explicit.

$$\Bbbk\langle B\times C\rangle:=\left\{\sum_{(b,c)\in F}\lambda_{b,c}\cdot\delta_{(b,c)}\,\middle|\, F\subseteq B\times C \text{ finite, } \lambda_{b,c}\in \Bbbk\right\}$$

We have τ defined on $B \times C$ by the map of the basis elements. Now we extend it bilinearly by the fact that:

$$(v,w) \in V \times W = \sum_{i,j} \lambda_{ij} \delta_{(b_i,c_j)}$$

with $b_i \in B$ and $c_i \in C$.

Thus any (v, w) will be defined via their base elements and any linear combination of any v or w by bilinear extension:

$$(rv + sv', w) \mapsto r\delta_{(v,w)} + s\delta_{(v',w)}$$

$$(v, rw + sw') \mapsto r\delta_{(v,w)} + s\delta_{(v,w')}$$

with $r, s \in \mathbb{R}$ and $v, v' \in V$ and $w, w' \in W$ which fully describes bilinearity.

We now need to show the universal property, thus existence and uniqueness of a linear t for each bilinear β .

Existence: Let:

$$t(\tau((b,c))) = t(\delta_{(b,c)}) := \beta(b,c)$$

for each $b \in B$ and $c \in C$. Thus we defined it on the base elements of V and W and thus also of T (as we already know τ maps bases to base).

All we know about β is its bilinearity and the same is true for τ :

$$\beta(v, w) = \beta\left(\sum_{i} \lambda_{i} b_{i}, \sum_{j} \lambda_{j} c_{j}\right) = \sum_{ij} \lambda_{ij} \beta(b_{i}, c_{j})$$
$$= \sum_{ij} \lambda_{ij} t(\tau(b_{i}, c_{j})) = \sum_{ij} \lambda_{ij} t(\delta_{(b_{i}, c_{j})})$$

Now t is defined on our bases $\delta_{(b_i,c_i)}$ and thus a linear function. So we have shown existence.

Uniqueness follows by linear algebra: we have defined our linear map on a basis, and thus it is uniquely defined on the vector space (up to isomorphism).

(iii) Base-Free Construction

Define $\tilde{\tau}(v, w) := \delta_{(v,w)}$ into the free vector space $\mathbb{k}\langle V \times W \rangle$, and define *R* as the subspace generated by the following relations:

$$\begin{split} \delta(v+v',w) - \delta(v,w) - \delta(v',w), \\ \delta(v,w+w') - \delta(v,w) - \delta(v,w'), \\ \delta(\lambda v,w) - \lambda \delta(v,w), \\ \delta(v,\lambda w) - \lambda \delta(v,w) \end{split}$$

Then define $T := \mathbb{k}\langle V \times W \rangle / R$ show that $\tilde{\tau} : V \times W \to \mathbb{k}\langle V \times W \rangle / R$ satisfies the universal property.

Universal Property. Let $\beta: V \times W \to U$ be any bilinear map. Define $t: T \to U$ by:

$$t(\delta(v, w) + R) := \beta(v, w).$$

This is well-defined because β is bilinear and R captures exactly the relations needed for bilinearity.

We do the double sum in one step for brevity

We define *R* (kernel/nullspace) to enforce bilinear identities. E.g., $\delta(v +$ $v', w) - \delta(v, w) - \delta(v', w) \in R$ implies we identify $\delta(v + v', w) \sim \delta(v, w) +$ $\delta(v', w)$. as formally there is no reason to identify them!

Remark: The free vector space $\mathbb{k}\langle V \times W \rangle$ consists of *all* finite linear combinations of pairs (v, w) with $v \in V$ and $w \in W$. These are not linearly independent in general - e.g.,

$$\lambda_1(1,0) + \lambda_2(0,1) + \lambda_3(1,1)$$

may exhibit linear dependence depending on the imposed relations. Thus, R is necessary to reflect bilinearity.

We now have:

$$t(\tau(v,w)) = t(\pi(\delta(v,w))) = \beta(v,w),$$

so $t \circ \widetilde{\tau} = b$ as desired.

Uniqueness follows because $\tilde{\tau}$ maps all of $V \times W$ (not just a basis), and any linear t' such that $t' \circ \tilde{\tau} = \beta$ must coincide with t on all elements, hence be equal.

(iv) Computational Rules

We now explain and justify the familiar tensor identities:

$$(v + v') \otimes w = v \otimes w + v' \otimes w,$$

$$v \otimes (w + w') = v \otimes w + v \otimes w',$$

$$(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$$

These follow directly from the relations used to define the quotient space $T := \mathbb{k}\langle V \times W \rangle / R$, and thus hold by construction.

They express bilinearity of the map $\tau: V \times W \to T$ in a concise in-line notation:

$$\tau(v+v',w) = \tau(v,w) + \tau(v',w),$$

$$\tau(v,w+w') = \tau(v,w) + \tau(v,w'),$$

$$\tau(\lambda v,w) = \lambda \tau(v,w) = \tau(v,\lambda w).$$

This bilinearity is exactly what ensures that any bilinear map $b: V \times W \rightarrow U$ factors uniquely through τ , via a linear map t: $V \otimes W \rightarrow U$.

(v) The Tensor Product as a Functor

Lift the operation $(-) \otimes_{\mathbb{k}} (-)$ to a functor.

$$T: \mathsf{Vect}_{\Bbbk} imes \mathsf{Vect}_{\Bbbk} o \mathsf{Vect}_{\Bbbk}$$

The essential structure is already in place:

- On objects: for $(V, W) \in Vect_{\mathbb{R}} \times Vect_{\mathbb{R}}$, define $T(V, W) := V \otimes W$.
- On morphisms: a morphism $(\varphi, \psi) : (V, W) \to (V', W')$ consists of linear maps $\varphi: V \to V'$, $\psi: W \to W'$. Define:

$$T(\varphi, \psi)(v \otimes w) := \varphi(v) \otimes \psi(w)$$

This is well-defined and respects bilinearity and linearity, hence it is a morphism in Vect_k.

This is reminiscent of multiplication:

$$m(a,b):=a\cdot b$$

These rules are defining properties of the tensor symbol \otimes .

We now use T for the functor, not the tensor space itself.

Identity and composition are preserved:

$$T(\mathrm{id}_V,\mathrm{id}_W)(v\otimes w)=v\otimes w,$$
 $T(\varphi_2,\psi_2)\circ T(\varphi_1,\psi_1)(v\otimes w)=(\varphi_2\circ\varphi_1)(v)\otimes (\psi_2\circ\psi_1)(w).$

Thus *T* is indeed a functor.

$$(V,W) \longmapsto V \otimes W$$

$$(\varphi,\psi) \downarrow \qquad \qquad \qquad \downarrow^{\varphi \otimes \psi}$$

$$(V',W') \longmapsto V' \otimes W'$$

We omit $\otimes_{\mathbb{k}}$ subscripts when the field is fixed contextually.

Tensoring commutes with morphisms: the diagram illustrates how morphisms are "distributed" by *T*.

References

This summary is based on the material and exercises from:

- Tom Leinster, Basic Category Theory (Cambridge University Press)
- Steven Roman, Advanced Linear Algebra (Springer)
- This document uses the tufte-latex class