Tensor Products and Universal Properties

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Summary and Reflections on Exercise 5.1

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This is my (presentation) summary of Exercise 5.1 on Tensor Products.

Exercise 5.1 Statement

Let $\Bbbk$ be a field and let $V, W \in \mathtt{Vect}_{\Bbbk}$. The tensor product of V and W is a $\Bbbk$-vector space $V \otimes_{\Bbbk} W$ together with a bilinear map $\tau : V \times W \to V \otimes_{\Bbbk} W$ satisfying the following universal property: for every $U \in \mathtt{Vect}_{\Bbbk}$ and every bilinear map $b : V \times W \to U$, there exists a unique linear map $t : V \otimes_{\Bbbk} W \to U$ such that the diagram

We use the dashed line to represent a unique morphism

$$\begin{tikzcd}
V \times W \arrow[r, "\tau"] \arrow[dr, "b"'] & V \otimes_{\Bbbk} W \arrow[d, dashed, "\exists!\, t"] \\
& U
\end{tikzcd}$$

commutes. We write $v \otimes w := \tau(v, w)$ for all $v \in V$ and $w \in W$, and refer to $V \otimes_{\Bbbk} W$ as the tensor product of V and W.

(i) Uniqueness up to Unique Isomorphism

Show that if $V \otimes_{\Bbbk} W$ exists, then it is unique up to unique isomorphism.

We only use the universal property, so this applies to any object defined by such a property.

Let us assume we have two 'tensor' (or universal) spaces (T, τ) and (T', τ'), both satisfying the universal property with respect to $V \times W$. Then for the bilinear map $\tau' : V \times W \to T'$, the universal property of (T, τ) gives a unique linear map $t : T \to T'$ such that $t \circ \tau = \tau'$. Similarly, the universal property of (T', τ') applied to τ yields a unique linear map $t' : T' \to T$ such that $t' \circ \tau' = \tau$.

We can represent this situation with the following diagram:

This is essentially Leinster's diagram from Chapter 0 (see p.5).

$$\begin{tikzcd}
& T \arrow[d, "t"] \\
V \times W \arrow[ur, "\tau"] \arrow[r, "\tau'"] \arrow[dr, "\tau"'] & T' \arrow[d, "t'"] \\
& T
\end{tikzcd}$$

Now observe:

$$
\begin{aligned}
& t' \circ t \circ \tau = t' \circ \tau' = \tau, \\
\Rightarrow \quad & t' \circ t = \mathrm{id}_T \quad \text{by uniqueness.}
\end{aligned}
$$

The same argument applied symmetrically to T' gives $t \circ t' = \mathrm{id}_{T'}$.

This shows that t and t' are inverse isomorphisms, so $T \cong T'$.

Therefore, t is an isomorphism between T and T', proving uniqueness up to unique isomorphism.

(ii) Construction Using Bases

Let B and C be bases of V and W. Consider the free vector space $\Bbbk\langle B \times C\rangle$ and define:

$$\tau(b,c) = \delta_{(b,c)}$$

where $\delta_{(b,c)}$ denotes the formal generator for the pair $(b,c) \in B \times C$.

We define this space more precisely as:

$$\Bbbk\langle B \times C\rangle := \left\{ \sum_{(b,c)\in F} \lambda_{b,c} \cdot (b,c) \;\middle|\; F \subseteq B \times C \text{ finite, } \lambda_{b,c} \in \Bbbk \right\}$$

We extend τ bilinearly. Given $(v,w) \in V \times W$ with

$$v = \sum_i \lambda_i b_i, \quad w = \sum_j \mu_j c_j,$$

we define:

$$\tau(v,w) := \sum_{i,j} \lambda_i \mu_j \cdot \delta_{(b_i,c_j)}.$$

This ensures bilinearity because:

$$
\begin{aligned}
\tau(rv + sv', w) &= r\tau(v,w) + s\tau(v',w), \\
\tau(v, rw + sw') &= r\tau(v,w) + s\tau(v,w')
\end{aligned}
$$

for all $r, s \in \Bbbk$ and $v, v' \in V$, $w, w' \in W$.

Universal Property. Let $b : V \times W \to U$ be any bilinear map. Define a linear map $t : \Bbbk\langle B \times C\rangle \to U$ by:

$$t(\delta_{(b,c)}) := b(b,c).$$

Then for any (v,w) as above,

We do the double sum in one step for brevity.

$$b(v,w) = \sum_{i,j} \lambda_i \mu_j \cdot b(b_i, c_j) = \sum_{i,j} \lambda_i \mu_j \cdot t(\delta_{(b_i,c_j)}) = t(\tau(v,w)).$$

Thus $t \circ \tau = b$, and t is linear by construction. Uniqueness follows from the fact that t is defined on the basis elements $\delta_{(b,c)}$.

(iii) Base-Free Construction

Define $\widetilde{\tau}(v, w) := \delta_{(v,w)}$ into the free vector space $\Bbbk\langle V \times W\rangle$, and define R as the subspace generated by the following relations:

$$\begin{aligned} \delta(v+v', w) - \delta(v, w) - \delta(v', w), \\ \delta(v, w+w') - \delta(v, w) - \delta(v, w'), \\ \delta(\lambda v, w) - \lambda\delta(v, w), \\ \delta(v, \lambda w) - \lambda\delta(v, w) \end{aligned}$$

Then define $T := \Bbbk\langle V \times W\rangle / R$ show that $\widetilde{\tau} : V \times W \to \Bbbk\langle V \times W\rangle / R$ satisfies the universal property.

We define R (kernel/nullspace) to enforce bilinear identities. E.g., $\delta(v + v', w) - \delta(v, w) - \delta(v', w) \in R$ implies we identify $\delta(v + v', w) \sim \delta(v, w) + \delta(v', w)$. as formally there is no reason to identify them!

Remark: The free vector space $\Bbbk\langle V \times W\rangle$ consists of *all* finite linear combinations of pairs (v, w) with $v \in V$ and $w \in W$. These are not linearly independent in general – e.g.,

$$\lambda_1(1, 0) + \lambda_2(0, 1) + \lambda_3(1, 1)$$

may exhibit linear dependence depending on the imposed relations. Thus, R is necessary to reflect bilinearity.

Universal Property. Let $b : V \times W \to U$ be any bilinear map. Define $t : T \to U$ by:

$$t(\delta(v, w) + R) := b(v, w).$$

This is well-defined because b is bilinear and R captures exactly the relations needed for bilinearity.

We now have:

$$t(\tau(v, w)) = t(\pi(\delta(v, w))) = b(v, w),$$

so $t \circ \widetilde{\tau} = b$ as desired.

Uniqueness follows because $\widetilde{\tau}$ maps all of $V \times W$ (not just a basis), and any linear t' such that $t' \circ \widetilde{\tau} = b$ must coincide with t on all elements, hence be equal.

(iv) Computational Rules

We now explain and justify the familiar tensor identities:

$$\begin{aligned} (v + v') \otimes w &= v \otimes w + v' \otimes w, \\ v \otimes (w + w') &= v \otimes w + v \otimes w', \\ (\lambda v) \otimes w &= \lambda(v \otimes w) = v \otimes (\lambda w) \end{aligned}$$

These follow directly from the relations used to define the quotient space $T := \Bbbk\langle V \times W\rangle / R$, and thus hold by construction.

They express bilinearity of the map $\tau : V \times W \to T$ in a concise in-line notation:

$$\begin{aligned} \tau(v + v', w) &= \tau(v, w) + \tau(v', w), \\ \tau(v, w + w') &= \tau(v, w) + \tau(v, w'), \\ \tau(\lambda v, w) &= \lambda\tau(v, w) = \tau(v, \lambda w). \end{aligned}$$

This bilinearity is exactly what ensures that any bilinear map $b : V \times W \to U$ factors uniquely through τ, via a linear map $t : V \otimes W \to U$.

This is reminiscent of multiplication:

$$m(a, b) := a \cdot b$$

These rules are defining properties of the tensor symbol $\otimes$.

(v) The Tensor Product as a Functor

Lift the operation $(-) \otimes_{\Bbbk} (-)$ to a functor.

We now use T for the functor, not the tensor space itself.

$$T : \mathtt{Vect}_{\Bbbk} \times \mathtt{Vect}_{\Bbbk} \to \mathtt{Vect}_{\Bbbk}$$

The essential structure is already in place:

- **On objects:** for $(V, W) \in \mathtt{Vect}_{\Bbbk} \times \mathtt{Vect}_{\Bbbk}$, define $T(V, W) := V \otimes W$.
- **On morphisms:** a morphism $(\varphi, \psi) : (V, W) \to (V', W')$ consists of linear maps $\varphi : V \to V'$, $\psi : W \to W'$. Define:

$$T(\varphi, \psi)(v \otimes w) := \varphi(v) \otimes \psi(w)$$

This is well-defined and respects bilinearity and linearity, hence it is a morphism in $\mathtt{Vect}_{\Bbbk}$.

Identity and composition are preserved:

We omit $\otimes_{\Bbbk}$ subscripts when the field is fixed contextually.

$$\begin{aligned} T(\mathrm{id}_V, \mathrm{id}_W)(v \otimes w) &= v \otimes w, \\ T(\varphi_2, \psi_2) \circ T(\varphi_1, \psi_1)(v \otimes w) &= (\varphi_2 \circ \varphi_1)(v) \otimes (\psi_2 \circ \psi_1)(w). \end{aligned}$$

Thus T is indeed a functor.

Tensoring commutes with morphisms: the diagram illustrates how morphisms are "distributed" by T.

$$\begin{tikzcd} (V, W) \arrow[r, mapsto] \arrow[d, mapsto, "{(\varphi,\psi)}"'] & V \otimes W \arrow[d, mapsto, "\varphi \otimes \psi"] \\ (V', W') \arrow[r, mapsto] & V' \otimes W' \end{tikzcd}$$

References

This summary is based on the material and exercises from:

- Tom Leinster, *Basic Category Theory* (Cambridge University Press)
- Steven Roman, *Advanced Linear Algebra* (Springer)