Tensor Products and Universal Properties

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Summary and Reflections on Exercise 5.1
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This is my (presentation) summary of Exercise 5.1 on Tensor Products.

Exercise 5.1 Statement

Let $\Bbbk$ be a field and let $V, W \in \mathtt{Vect}_{\Bbbk}$. The tensor product of V and W is a $\Bbbk$-vector space $V \otimes_{\Bbbk} W$ together with a bilinear map $\tau : V \times W \to V \otimes_{\Bbbk} W$ satisfying the following universal property: for every $U \in \mathtt{Vect}_{\Bbbk}$ and every bilinear map $b : V \times W \to U$, there exists a unique linear map $t : V \otimes_{\Bbbk} W \to U$ such that the diagram

We represent a bilinear map with β

$$\begin{tikzcd} V \times W \arrow[r, "\tau"] \arrow[dr, "\beta"'] & V \otimes_{\Bbbk} W \arrow[d, dashed, "\exists!\, t"] \\ & U \end{tikzcd}$$

commutes. We write $v \otimes w := \tau(v, w)$ for all $v \in V$ and $w \in W$, and refer to $V \otimes_{\Bbbk} W$ as the tensor product of V and W.

We use the dashed line to represent a unique morphism

(i) Uniqueness up to Unique Isomorphism

Show that if $V \otimes_{\Bbbk} W$ exists, then it is unique up to unique isomorphism.

We only use the universal property, so this applies to any object defined by such a property.

Let us assume we have two 'tensor' (or universal) spaces (T, τ) and (T', τ'), both satisfying the universal property with respect to $V \times W$. Then for the bilinear map $\tau' : V \times W \to T'$, the universal property of (T, τ) gives a unique linear map $t : T \to T'$ such that $t \circ \tau = \tau'$. Similarly, the universal property of (T', τ') applied to τ yields a unique linear map $t' : T' \to T$ such that $t' \circ \tau' = \tau$.

We can represent this situation with the following diagram:

This is essentially Leinster's diagram from Chapter 0 (see p. 5).

$$
\begin{tikzcd}
 & T \arrow[d, dashed, "t"] \\
V \times W \arrow[ur, "\tau"] \arrow[r, "\tau'"'] \arrow[dr, "\tau"'] & T' \arrow[d, dashed, "t'"] \\
 & T
\end{tikzcd}
$$

Now observe:

The same argument applied symmetrically to T' gives $t \circ t' = \mathrm{id}_{T'}$.

$$
\begin{aligned}
& t' \circ t \circ \tau = t' \circ \tau' = \tau, \\
\Rightarrow \quad & t' \circ t = \mathrm{id}_T \quad \text{by uniqueness.}
\end{aligned}
$$

This shows that t and t' are inverse isomorphisms, so $T \cong T'$.

Therefore, t is an isomorphism between T and T', proving uniqueness up to unique isomorphism.

(ii) Construction Using Bases

Let B and C be bases of V and W. Consider the free vector space $\Bbbk\langle B \times C\rangle$ and define:

$$\tau(b,c) = \delta_{(b,c)}$$

where $\delta_{(b,c)}$ denotes the formal generator for the pair $(b,c) \in B \times C$.

Some write (b,c) directly; we use $\delta_{(b,c)}$ to make the vector space structure explicit.

We extend τ bilinearly. For any bilinear map $\beta : V \times W \to U$, define a linear map $t : \Bbbk\langle B \times C\rangle \to U$ on basis elements by $t(\delta_{(b,c)}) := \beta(b,c)$. This t satisfies $t \circ \tau = \beta$ and is unique by linearity.

$$\Bbbk\langle B \times C\rangle := \left\{ \sum_{(b,c)\in F} \lambda_{b,c} \cdot \delta_{(b,c)} \,\middle|\, F \subseteq B \times C \text{ finite, } \lambda_{b,c} \in \Bbbk \right\}$$

We have τ defined on $B \times C$ by the map of the basis elements. Now we extend it bilinearly by the fact that:

$$(v,w) \in V \times W = \sum_{i,j} \lambda_{ij} \delta_{(b_i,c_j)}$$

with $b_i \in B$ and $c_j \in C$.

Thus any (v,w) will be defined via their base elements and any linear combination of any v or w by bilinear extension:

$$(rv + sv', w) \mapsto r\delta_{(v,w)} + s\delta_{(v',w)}$$

$$(v, rw + sw') \mapsto r\delta_{(v,w)} + s\delta_{(v,w')}$$

with $r, s \in \Bbbk$ and $v, v' \in V$ and $w, w' \in W$ which fully describes bilinearity.

We now need to show the universal property, thus existence and uniqueness of a linear t for each bilinear β.

Existence: Let:

$$t(\tau((b,c))) = t(\delta_{(b,c)}) := \beta(b,c)$$

for each $b \in B$ and $c \in C$. Thus we defined it on the base elements of V and W and thus also of T (as we already know τ maps bases to base).

All we know about β is its bilinearity and the same is true for τ:

We do the double sum in one step for brevity

$$\begin{aligned}\beta(v,w) = \beta\left(\sum_i \lambda_i b_i, \sum_j \lambda_j c_j\right) &= \sum_{ij} \lambda_{ij}\beta(b_i, c_j) \\ = \sum_{ij} \lambda_{ij} t(\tau(b_i, c_j)) &= \sum_{ij} \lambda_{ij} t(\delta_{(b_i,c_j)})\end{aligned}$$

Now t is defined on our bases $\delta_{(b_i,c_j)}$ and thus a linear function. So we have shown existence.

Uniqueness follows by linear algebra: we have defined our linear map on a basis, and thus it is uniquely defined on the vector space (up to isomorphism).

(iii) Base-Free Construction

Define $\widetilde{\tau}(v,w) := \delta_{(v,w)}$ into the free vector space $\Bbbk\langle V \times W\rangle$, and define R as the subspace generated by the following relations:

$$\begin{aligned}\delta(v + v', w) - \delta(v,w) - \delta(v', w), \\ \delta(v, w + w') - \delta(v,w) - \delta(v, w'), \\ \delta(\lambda v, w) - \lambda\delta(v,w), \\ \delta(v, \lambda w) - \lambda\delta(v,w)\end{aligned}$$

Then define $T := \Bbbk\langle V \times W\rangle / R$ show that $\widetilde{\tau} : V \times W \to \Bbbk\langle V \times W\rangle / R$ satisfies the universal property.

We define R (kernel/nullspace) to enforce bilinear identities. E.g., $\delta(v + v', w) - \delta(v,w) - \delta(v',w) \in R$ implies we identify $\delta(v+v',w) \sim \delta(v,w) + \delta(v',w)$. as formally there is no reason to identify them!

Remark: The free vector space $\Bbbk\langle V \times W\rangle$ consists of *all* finite linear combinations of pairs (v,w) with $v \in V$ and $w \in W$. These are not linearly independent in general – e.g.,

$$\lambda_1(1,0) + \lambda_2(0,1) + \lambda_3(1,1)$$

may exhibit linear dependence depending on the imposed relations. Thus, R is necessary to reflect bilinearity.

Universal Property. Let $\beta : V \times W \to U$ be any bilinear map. Define $t : T \to U$ by:

$$t(\delta(v,w) + R) := \beta(v,w).$$

This is well-defined because β is bilinear and R captures exactly the relations needed for bilinearity.

We now have:

$$t(\tau(v,w)) = t(\pi(\delta(v,w))) = \beta(v,w),$$

so $t \circ \widetilde{\tau} = b$ as desired.

Uniqueness follows because $\widetilde{\tau}$ maps all of $V \times W$ (not just a basis), and any linear t' such that $t' \circ \widetilde{\tau} = \beta$ must coincide with t on all elements, hence be equal.

(iv) Computational Rules

We now explain and justify the familiar tensor identities:

$$\begin{aligned}(v+v') \otimes w &= v \otimes w + v' \otimes w,\\ v \otimes (w+w') &= v \otimes w + v \otimes w',\\ (\lambda v) \otimes w &= \lambda(v \otimes w) = v \otimes (\lambda w)\end{aligned}$$

These follow directly from the relations used to define the quotient space $T := \Bbbk\langle V \times W\rangle / R$, and thus hold by construction.

They express bilinearity of the map $\tau : V \times W \to T$ in a concise in-line notation:

$$\begin{aligned}\tau(v+v',w) &= \tau(v,w) + \tau(v',w),\\ \tau(v,w+w') &= \tau(v,w) + \tau(v,w'),\\ \tau(\lambda v,w) &= \lambda\tau(v,w) = \tau(v,\lambda w).\end{aligned}$$

This bilinearity is exactly what ensures that any bilinear map $b : V \times W \to U$ factors uniquely through τ, via a linear map $t : V \otimes W \to U$.

This is reminiscent of multiplication:

$$m(a,b) := a \cdot b$$

These rules are defining properties of the tensor symbol $\otimes$.

(v) The Tensor Product as a Functor

Lift the operation $(-) \otimes_{\Bbbk} (-)$ to a functor.

We now use T for the functor, not the tensor space itself.

$$T : \mathtt{Vect}_{\Bbbk} \times \mathtt{Vect}_{\Bbbk} \to \mathtt{Vect}_{\Bbbk}$$

The essential structure is already in place:

- **On objects:** for $(V,W) \in \mathtt{Vect}_{\Bbbk} \times \mathtt{Vect}_{\Bbbk}$, define $T(V,W) := V \otimes W$.
- **On morphisms:** a morphism $(\varphi,\psi) : (V,W) \to (V',W')$ consists of linear maps $\varphi : V \to V'$, $\psi : W \to W'$. Define:

$$T(\varphi,\psi)(v \otimes w) := \varphi(v) \otimes \psi(w)$$

This is well-defined and respects bilinearity and linearity, hence it is a morphism in $\mathtt{Vect}_{\Bbbk}$.

Identity and composition are preserved:

We omit $\otimes_{\Bbbk}$ subscripts when the field is fixed contextually.

$$\begin{aligned} T(\mathrm{id}_V, \mathrm{id}_W)(v \otimes w) &= v \otimes w, \\ T(\varphi_2, \psi_2) \circ T(\varphi_1, \psi_1)(v \otimes w) &= (\varphi_2 \circ \varphi_1)(v) \otimes (\psi_2 \circ \psi_1)(w). \end{aligned}$$

Thus T is indeed a functor.

Tensoring commutes with morphisms: the diagram illustrates how morphisms are "distributed" by T.

$$\begin{tikzcd} (V, W) \arrow[r, mapsto] \arrow[d, mapsto, "(\varphi,\psi)"'] & V \otimes W \arrow[d, mapsto, "\varphi\otimes\psi"] \\ (V', W') \arrow[r, mapsto] & V' \otimes W' \end{tikzcd}$$

References

This summary is based on the material and exercises from:

- Tom Leinster, *Basic Category Theory* (Cambridge University Press)
- Steven Roman, *Advanced Linear Algebra* (Springer)
- This document uses the `tufte-latex` class