Tensor Products and Universal Properties

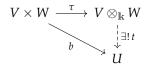
Bertold Sedlak

May 1, 2025

Summary and Reflections on Exercise 5.1 *Introduction to Category Theory — University of Vienna, ST* 2025

This is my (presentation) summary of Exercise 5.1 on Tensor Products.

Exercise 5.1 Statement



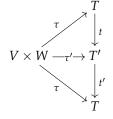
commutes. We write $v\otimes w:=\tau(v,w)$ for all $v\in V$ and $w\in W$, and refer to $V\otimes_{\Bbbk}W$ as the tensor product of V and W.

(i) Uniqueness up to Unique Isomorphism

Show that if $V \otimes_{\mathbb{k}} W$ exists, then it is unique up to unique isomorphism.

Let us assume we have two 'tensor' (or universal) spaces (T,τ) and (T',τ') , both satisfying the universal property with respect to $V\times W$. Then for the bilinear map $\tau':V\times W\to T'$, the universal property of (T,τ) gives a unique linear map $t:T\to T'$ such that $t\circ\tau=\tau'$. Similarly, the universal property of (T',τ') applied to τ yields a unique linear map $t':T'\to T$ such that $t'\circ\tau'=\tau$.

We can represent this situation with the following diagram:



We use the dashed line to represent a unique morphism

We only use the universal property, so this applies to any object defined by such a property.

This is essentially Leinster's diagram from Chapter o (see p.5).

Now observe:

$$t' \circ t \circ \tau = t' \circ \tau' = \tau,$$

 $\Rightarrow t' \circ t = id_T \text{ by uniqueness.}$

This shows that t and t' are inverse isomorphisms, so $T \cong T'$.

Therefore, t is an isomorphism between T and T', proving uniqueness up to unique isomorphism.

(ii) Construction Using Bases

Let *B* and *C* be bases of *V* and *W*. Consider the free vector space $\mathbb{k}\langle B \times C \rangle$ and define:

$$\tau(b,c) = \delta_{(b,c)}$$

where $\delta_{(b,c)}$ denotes the formal generator for the pair $(b,c) \in B \times C$. We define this space more precisely as:

$$\Bbbk \langle B \times C \rangle := \left\{ \sum_{(b,c) \in F} \lambda_{b,c} \cdot (b,c) \mid F \subseteq B \times C \text{ finite, } \lambda_{b,c} \in \Bbbk \right\}$$

We extend τ bilinearly. Given $(v, w) \in V \times W$ with

$$v = \sum_{i} \lambda_i b_i, \quad w = \sum_{j} \mu_j c_j,$$

we define:

$$au(v,w) := \sum_{i,j} \lambda_i \mu_j \cdot \delta_{(b_i,c_j)}.$$

This ensures bilinearity because:

$$\tau(rv + sv', w) = r\tau(v, w) + s\tau(v', w),$$

$$\tau(v, rw + sw') = r\tau(v, w) + s\tau(v, w')$$

for all $r, s \in \mathbb{k}$ and $v, v' \in V$, $w, w' \in W$.

Universal Property. Let $b: V \times W \rightarrow U$ be any bilinear map. Define a linear map $t : \mathbb{k}\langle B \times C \rangle \to U$ by:

$$t(\delta_{(b,c)}) := b(b,c).$$

Then for any (v, w) as above,

$$b(v,w) = \sum_{i,j} \lambda_i \mu_j \cdot b(b_i,c_j) = \sum_{i,j} \lambda_i \mu_j \cdot t(\delta_{(b_i,c_j)}) = t(\tau(v,w)).$$

Thus $t \circ \tau = b$, and t is linear by construction. Uniqueness follows from the fact that t is defined on the basis elements $\delta_{(b,c)}$.

The same argument applied symmetrically to T' gives $t \circ t' = \mathrm{id}_{T'}$.

We do the double sum in one step for brevity.

(iii) Base-Free Construction

Define $\tilde{\tau}(v, w) := \delta_{(v,w)}$ into the free vector space $\mathbb{k}\langle V \times W \rangle$, and define *R* as the subspace generated by the following relations:

$$\delta(v + v', w) - \delta(v, w) - \delta(v', w),$$

$$\delta(v, w + w') - \delta(v, w) - \delta(v, w'),$$

$$\delta(\lambda v, w) - \lambda \delta(v, w),$$

$$\delta(v, \lambda w) - \lambda \delta(v, w)$$

Then define $T := \mathbb{k}\langle V \times W \rangle / R$ show that $\tilde{\tau} : V \times W \to \mathbb{k}\langle V \times W \rangle / R$ satisfies the universal property.

Universal Property. Let $b: V \times W \to U$ be any bilinear map. Define $t: T \to U$ by:

$$t(\delta(v, w) + R) := b(v, w).$$

This is well-defined because b is bilinear and R captures exactly the relations needed for bilinearity.

We now have:

$$t(\tau(v,w)) = t(\pi(\delta(v,w))) = b(v,w),$$

so $t \circ \widetilde{\tau} = b$ as desired.

Uniqueness follows because $\tilde{\tau}$ maps all of $V \times W$ (not just a basis), and any linear t' such that $t' \circ \tilde{\tau} = b$ must coincide with t on all elements, hence be equal.

(iv) Computational Rules

We now explain and justify the familiar tensor identities:

$$(v + v') \otimes w = v \otimes w + v' \otimes w,$$

$$v \otimes (w + w') = v \otimes w + v \otimes w',$$

$$(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$$

These follow directly from the relations used to define the quotient space $T := \mathbb{k}\langle V \times W \rangle / R$, and thus hold by construction.

They express bilinearity of the map $\tau: V \times W \to T$ in a concise in-line notation:

$$\tau(v+v',w) = \tau(v,w) + \tau(v',w),$$

$$\tau(v,w+w') = \tau(v,w) + \tau(v,w'),$$

$$\tau(\lambda v,w) = \lambda \tau(v,w) = \tau(v,\lambda w).$$

We define *R* (kernel/nullspace) to enforce bilinear identities. E.g., $\delta(v +$ $v', w) - \delta(v, w) - \delta(v', w) \in R$ implies we identify $\delta(v + v', w) \sim \delta(v, w) +$ $\delta(v', w)$. as formally there is no reason to identify them!

Remark: The free vector space $\mathbb{k}\langle V \times W \rangle$ consists of *all* finite linear combinations of pairs (v, w) with $v \in V$ and $w \in W$. These are not linearly independent in general - e.g.,

$$\lambda_1(1,0) + \lambda_2(0,1) + \lambda_3(1,1)$$

may exhibit linear dependence depending on the imposed relations. Thus, R is necessary to reflect bilinearity.

This bilinearity is exactly what ensures that any bilinear map $b: V \times W \rightarrow U$ factors uniquely through τ , via a linear map t: $V \otimes W \rightarrow U$.

(v) The Tensor Product as a Functor

Lift the operation $(-) \otimes_{\mathbb{k}} (-)$ to a functor.

$$T: \mathsf{Vect}_{\Bbbk} imes \mathsf{Vect}_{\Bbbk} o \mathsf{Vect}_{\Bbbk}$$

The essential structure is already in place:

- On objects: for $(V, W) \in \mathsf{Vect}_{\mathbb{k}} \times \mathsf{Vect}_{\mathbb{k}}$, define $T(V, W) := V \otimes W$.
- On morphisms: a morphism $(\varphi, \psi) : (V, W) \to (V', W')$ consists of linear maps $\varphi: V \to V'$, $\psi: W \to W'$. Define:

$$T(\varphi,\psi)(v\otimes w):=\varphi(v)\otimes\psi(w)$$

This is well-defined and respects bilinearity and linearity, hence it is a morphism in $Vect_k$.

Identity and composition are preserved:

$$T(\mathrm{id}_V,\mathrm{id}_W)(v\otimes w)=v\otimes w,$$
 $T(\varphi_2,\psi_2)\circ T(\varphi_1,\psi_1)(v\otimes w)=(\varphi_2\circ\varphi_1)(v)\otimes (\psi_2\circ\psi_1)(w).$

Thus *T* is indeed a functor.

References

This summary is based on the material and exercises from:

- Tom Leinster, Basic Category Theory (Cambridge University Press)
- Steven Roman, Advanced Linear Algebra (Springer)

This is reminiscent of multiplication:

$$m(a,b) := a \cdot b$$

These rules are defining properties of the tensor symbol \otimes .

We now use *T* for the functor, not the tensor space itself.

We omit $\otimes_{\mathbb{k}}$ subscripts when the field is fixed contextually.

Tensoring commutes with morphisms: the diagram illustrates how morphisms are "distributed" by T.