Biasness

If $E[\hat{\theta}] = \theta$ or $E[\hat{\theta} - \theta] = 0$

we say $\hat{\theta}$ is an unbiased estimated of θ or $\hat{\Theta} \rightsquigarrow \Theta$

If $E[\hat{\theta} - \theta] \neq 0$ we say $\hat{\theta}$ is a biased estimate of θ

Important Properties And Def'ns of Estimators

<u>Efficency</u> - property of estimator $\ni$ as the efficiency of an estimator $\uparrow$ the number obs needed to reduce estimator variability $\downarrow$.

In the comparison of two estimator the more efficient one has a smaller variance or MSE for equivalent sample size. As we have seen before

$$mse(\hat{T}) = E[(\hat{T}-T)^2] = E[(\hat{T}-E[\hat{T}]+E[\hat{T}]-T)^2]$$

$$= Var(\hat{T}) + (E[\hat{T}]-T)^2$$

IF $\hat{T}_1$ more efficient than $\hat{T}_2 \iff MSE(\hat{T}_1) < MSE(\hat{T}_2)$

For unbiased estimators $\exists$ a lower bound on variance (called Rao-Cramer Lower Bound)

Consistency Defn

$$\lim_{n\to\infty} P((\hat{T}_n - T) > \varepsilon) = 0 \quad \text{for any } \varepsilon > 0$$

i.e. $\exists$ an "n" $\ni$ for $N > n$ this is true.

Note does not say $E[\hat{T}] = T$ (unbiased)

Ex1. Can be consistent but biased

As $n \to \infty$ $\frac{1}{n} \sum x_i + \frac{1}{n}$ gets closer to μ but is biased by the $\frac{1}{n}$ term

Ex2. Can be unbiased but not consistent

Select an random sample $\{X_1, \ldots, X_n\}$ (IID)

Let $\hat{T}_X(x) = X_n$, X_n is unbiased

$E[X_n] = \mu \ \forall n$ but for $\varepsilon > 0$

there $\nexists$ an $N > n$ $\ni$ the consistency

will hold for $\forall N_i > N$

Note if estimator is unbiased and convergens to parameter $\Longrightarrow$ it is consistent

Convergence

The property wherein certain infinite series and functions approach a limit more and more closely as an argument of the function increases or decreases or as the number of terms of the series increases.

www.britannica.com

Con vergence in Dist'n

A random sequence denoted X_n gets closer to the distribution of R.V. X as $n \uparrow$

Denote $X_n \xrightarrow{d} X$

More formally the seq X_1, X_2, X_3 converges in distribution to an RV, X

if $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$

CDF of RVs in sequence has as a lim the CDF of an non index RV

Ex

Let $X_1\ X_2,\ X_3,\ X_4, \dots$ be a seq of RVs $\ni$

$$F_{X_n}(x) = \begin{cases} 1-(1-\frac{1}{n})^{nx} & x>0 \\ \text{W} & \end{cases}$$

for $x>0$ $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \left(1-(1-\frac{1}{n})^{nx}\right)$

$$= 1 - \lim_{n\to\infty} (1-\tfrac{1}{n})^{nx}$$

$$= 1 - e^{-x} \quad \text{CDF for } \exp(1)$$

$$= F_X(x)$$

$$X_n \xrightarrow{d} X \sim \exp(1)$$

Convergence In Probability

A seq $X_1, X_2, X_3, \dots$ of R.V. converges to a number y (or RV) in probability if

$n \to \infty$, $P(|X_n - y| \leq \varepsilon) \longrightarrow 1$

for $\varepsilon > 0$ or $P(|X_n - y| > \varepsilon) \to 0$

Example Let $X_n \sim \exp(n)$ show $X_n \xrightarrow{P} 0$

$X_1, X_2, X_3, \dots$ IID converges in prob to zero with RV $X_1, X_2, \dots, X_n \geq 0$ define Supp

$$\lim_{n \to \infty} P(|X_n - 0| \geq \varepsilon) = \lim_{n \to \infty} P(X_n \geq \varepsilon), \quad X_n \geq 0$$

$$= \lim_{n \to \infty} e^{-n\varepsilon} \qquad X_n \sim \exp(n)$$

$$= 0 \quad \forall \varepsilon > 0$$

- Converging in dist'n $\Rightarrow$ dist'n $X_n \downarrow$ to dist'n X as $n \uparrow$

Whereas $X_n \Rightarrow X \Rightarrow$ RV $X_n \downarrow X$ w/ $P(X) \uparrow$ as $n \uparrow$

Convergence in prob $\Rightarrow$ Convergence in dist'n

Not the Converse.

Intuitively, Xn converging to X in distribution means that the distribution of Xn gets very close to the distribution of X as n grows, whereas Xn converging to X in probability means that the random variable Xn gets very close to the random variable X (with very high probability) as n grows.

Implies a non-zero prob that X != Y

Suppose that Y has the same distribution as X, but P(X=Y)≠1. Then Xn converging in distribution to X implies that Xn converges in distribution to Y. But if Xn converges in probability to X, then Xn does not converge in probability to Y; after all, for n large Xn will get very close to X (with high probability), not Y. Because no matter how large n gets get P(Xn - Y) does not go to 0

Convergence in probability is stronger, in the sense that convergence in probability to X implies convergence in distribution to X. An important special case where these two forms of convergence turn out to be equivalent is when X is a constant. (After Internet Sources)

Proof Chebyshev Inequality, Cont RV

Show $P(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$

Let X be a continuous R.V.

w/ $E[X]=\mu$, $|\mu|<\infty$, $V(X)=\sigma^2<\infty$, $\varepsilon>0$

$$E[(X-E[\mu])^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\geq \int_{-\infty}^{\mu-\varepsilon} \underbrace{(x-\mu)^2 f_X(x)\,dx}_{A} + \int_{\mu+\varepsilon}^{\infty} A$$

Subject to the supp of X.

Next by $-\infty < x < \mu-\varepsilon$, $x \leq \mu-\varepsilon \Rightarrow \varepsilon \leq |x-\mu| \Rightarrow$

$\varepsilon^2 \leq (x-\mu)^2$ (recall $\varepsilon>0$)

$$\Rightarrow \geq \int_{-\infty}^{\mu-\varepsilon} \varepsilon^2 f_X(x)\,dx + \int_{\mu+\varepsilon}^{\infty} \varepsilon^2 f_X(x)\,dx$$

$$= \varepsilon^2\left(\int_{-\infty}^{\mu-\varepsilon} f_X(x)\,dx + \int_{\mu+\varepsilon}^{\infty} f_X(x)\,dx\right)$$

$$= \varepsilon^2\left(P(X \leq \mu-\varepsilon) + P(X \geq \mu+\varepsilon)\right)$$

$$= \varepsilon^2\left(P((X-\mu) \leq -\varepsilon) + P((X-\mu) \geq \varepsilon)\right)$$

$$= \varepsilon^2 P(|X-\mu| \geq \varepsilon)$$

$\Rightarrow \sigma^2 \geq \varepsilon^2 P(|X-\mu| \geq \varepsilon)$ from above

$\Rightarrow \varepsilon^2 P(|X-\mu| \geq \varepsilon) \leq \sigma^2$

$\Rightarrow P(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$ dividing through by $\varepsilon^2 > 0$

Law of Large Number, Finite Varaince

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ IID RS

Let $E[X] = \mu$ where $|\mu| < \infty$ & $Var(X) = \sigma^2 < \infty$ Then

$$\lim_{n\to\infty} \left(P(|\bar{X}_n - \mu| \geq \varepsilon)\right) = 0 \quad \forall \varepsilon > 0 \text{ or}$$

$$\bar{X}_n \xrightarrow{P} \mu$$

Comment Describes the result of repeating same experiment a large number of times

$\Rightarrow$ the average of results should be increasing close to the expected value

$\Rightarrow$ A stable long term result

$X_1, X_2, \cdots$ infinite & IID $E[X_1] = E[X_2] = \cdots \mu$

$|\mu| < \infty$

$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. or

$\bar{X}_n \xrightarrow{P} \mu$ as $n \to \infty$

$$E[\bar{X}_n] = \frac{E[X_1] + E[X_2] + \cdots + E[X_n]}{n}$$

$$= \frac{n\,E[X]}{n} = E[X] = \mu$$

$$Var(\bar{X}_n) = Var((X_1 + X_2 + \cdots + X_n)/n) \text{ by IID}$$

$$= \frac{Var(X_1) + Var(X_2) + \cdots + Var(X_n)}{n^2}$$

$$= \frac{n\,Var(X)}{n^2} = \frac{Var(X)}{n}$$

Proof Law of Large Numbers

From Chebyshev we have

$$P(|X-\mu| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2} \quad \forall \varepsilon > 0$$

$$\Longrightarrow P(|\bar{X}_n - \mu_n| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \quad \forall \varepsilon > 0$$

Where $E[\bar{X}_n] = \frac{nE[X]}{n} = E[X] = \mu$

$Var(\bar{X}) = Var(X)/n = \sigma^2/n$

$$\lim_{n\to\infty} (P(|\bar{X}_n - \mu| \geq \varepsilon)) \leq \lim_{n\to\infty} \left(\frac{\sigma^2}{n\varepsilon^2}\right) = 0$$

or

$$\lim_{n\to\infty} (P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$$

$$\Longrightarrow \bar{X}_n \xrightarrow{P} \mu$$

CLT $\equiv$ Central Limit Theorem
"Statistician Shim"

Lots of Def'n

If $-\infty < E[X] =: \mu < \infty$ and $Var(X) =: \sigma^2 < \infty$
and sample $\{X_i\}$ are iid RV

Then for any distribution the sums & averages
of sample realizations of size n converges
in distribution to normal distribution
w/ mean $= \mu$ and variance $= \sigma^2/n$
convergence speed and asymptotics in part
dep on dist'n skewness and size n

$X \sim B(0.7) \Rightarrow$

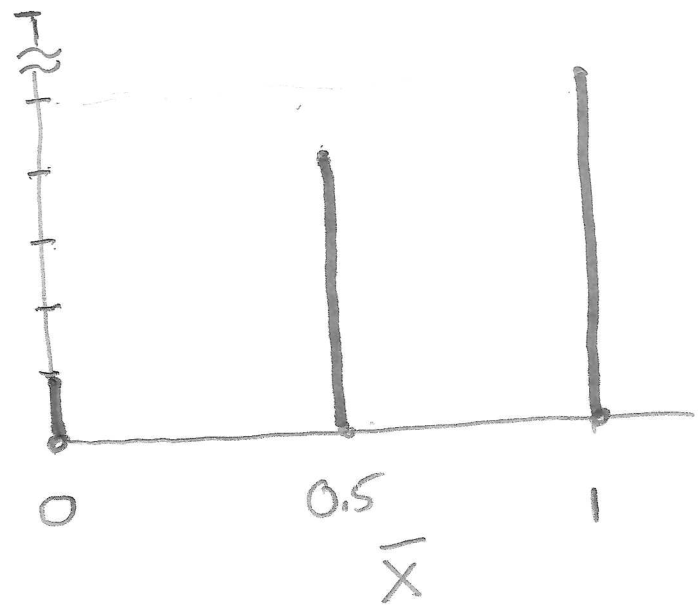
$$X = \{0,1\}$$

$$P(X=x) = \begin{cases} 0.7 & x=1 \\ 0.3 & x=0 \end{cases}$$

W

2 coins

x	$P(X=x)$	Value $\bar{X}$
2 H	$1 \cdot 0.7^2 = 0.49$	1
1 H	$2 \cdot 0.7 \cdot 0.3 = 0.42$	0.5
0 H	$1 \cdot 0.3^2 = 0.09$	0



$P(H) = 0.7 \Rightarrow P(T) = 0.3$

5 Coins

x	$P(X=x)$		Value
5 H	0.16807	$\binom{5}{5} 0.7^{5}$	1
4 H	0.36015	$\binom{5}{4} 0.7^4 .3^1$	.8
3 H	0.3087	$\binom{5}{3} 0.7^3 * 3^2$	.6
2 H	0.1323	$\binom{5}{2} 0.7^2 * 0.3^3$	.4
1 H	0.02835	$\binom{5}{1} 0.7^1 * 0.3^4$	.2
0 H	0.00243	$\binom{5}{0} 0.3^5$	0

3 Coins

x	$P(X=x)$		V
3 H	0.343	$\binom{3}{3} 0.7^3$	$(1+1+1)/3 = 1$
2 H	0.441	$\binom{3}{2} 0.7^2 * 0.3^1$	$(1+1+0)/3 = .67$
1 H	0.189	$\binom{3}{1} 0.7 * 0.3^2$	$(1+0+0)/3 = .33$
0 H	0.027	$\binom{3}{0} 0.3^3$	$(0+0+0)/3 = 0$