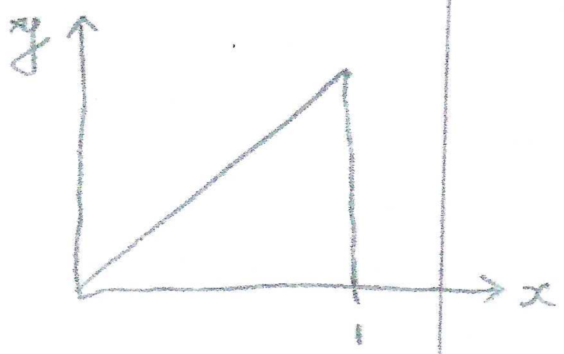
LS 4
Additional
Mat'l



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le 1,\ 0 \le y < x \\ ⓦ \end{cases}$$

want $f_{Y|X}(y|x)$

know that $f_{Y|X}(y|x) =$

$f_{XY}(x,y) / f_X(x)$

want to integrate out

$$f_X(x) = \int_0^x 2\,dy = 2y \Big|_0^x = 2x - 0$$

$$= \begin{cases} 2x & 0 \le x \le 1 \\ ⓦ \end{cases}$$

$$\Rightarrow f_{Y|X}(y|x) = \begin{cases} \frac{2}{2x} = \frac{1}{x} & 0 \le y \le x \\ ⓦ \end{cases}$$

So $E[Y|X]$ ~ $f_{Y|X}(x,y)$

$$= \int_0^x y \frac{1}{x}\,dy$$

$$= \frac{y^2}{2x} \Big|_0^x = \frac{x^2}{2x} = \frac{x}{2}$$

? rv or not

Condition Expectation

We know from previous def'n and results

$$V(Y|x) = E[Y^2|X] - E^2[Y|X]$$

we have $E[Y|X]$

So $E[Y^2|X] = \int_0^x y^2 f_{Y|X}(x,y)\,dy$

$$= \int_0^x y^2 \tfrac{1}{x}\,dy = \frac{y^3}{3x}\Big|_0^x$$

$$= \frac{x^2}{3}$$

$$\Rightarrow V(Y|X) = \frac{x^2}{3} - \left(\frac{x}{2}\right)^2 = \frac{4x^2}{12} - \frac{3x^2}{12} = \frac{x^2}{12}$$

$$Var(Y) = E[Y^2] - E^2[Y]$$

$$f_Y(y) \int_{x=y}^1 2\,dx = 2x\Big|_{x=y}^1 = 2-2y$$

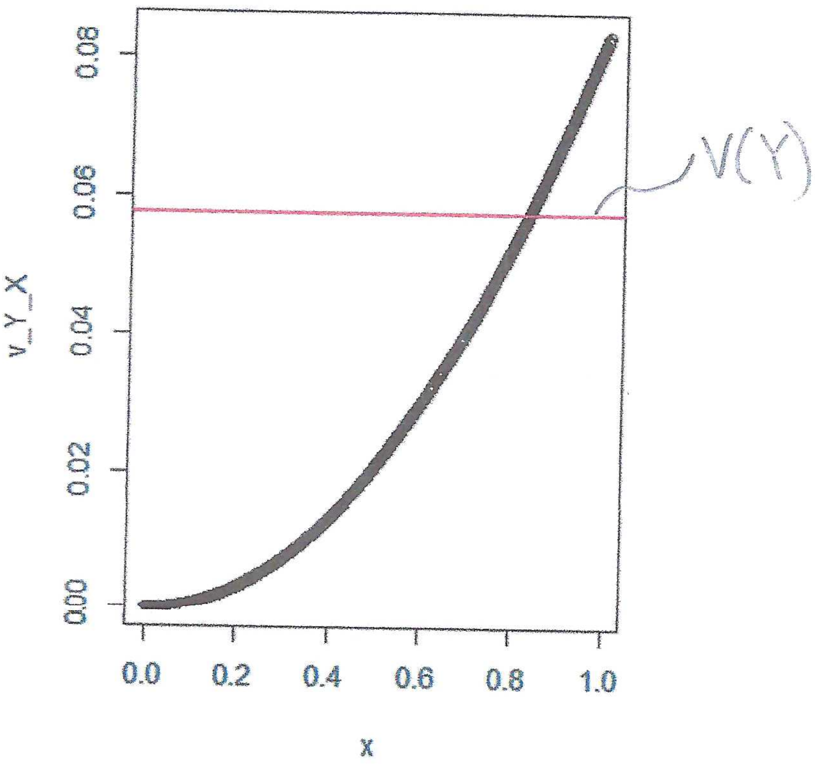
$$1 = \int_0^1 2-2y\,dy = 2y - y^2\Big|_0^1 = 2-1=1$$

$$\Rightarrow f_Y(y) = \begin{cases} 2-2y & 0\le y\le 1 \\ 0 \end{cases}$$

$$E[Y] = \int_0^1 y(2-2y)\,dy = y^2 - \frac{2y^3}{3}\Big|_0^1$$
$$= 0.33$$

$$E[Y^2] = \int_0^1 (2y^2-2y^3)\,dy = (2/3)y^3 - \left(\tfrac{1}{2}\right)y^4\Big|_0^1$$
$$= 0.166$$

$$Var(Y) = 0.0578$$


0.00
0.02
0.04
0.06
0.08
v_Y_X
0.0
0.2
0.4
0.6
0.8
1.0
x
V(Y)

Law of Iterated Expectation

(a.k.a. LIE, Law of total expectations, Adam's Law)

$$E[Y] = E_x[E_Y[Y|X]]$$

Intuition

Let X be a RV $\triangleq$ wtg of

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adults in U.S. population

$$E[X] = \sum_{x_i = \text{wtgs}} x_i \, p(X = x_i) \quad (1)$$

But alternatively we could break-up U.S. pop by gender, lets define X|Y as wtg of adults in U.S. population given their gender

So we can compute $E_x[X|Y=♂]$ & $E_x[X|Y=♀]$

Now we divide the pop and perform iteratively for ♀, ♂.

What is left is to bring the two expected values together by wgting by proportion of ♀ and ♂ in the pop

$$E_Y[X|Y=\text{gender}] \quad E_x[X|Y=♀] \cdot P(Y=♀) + E_x[X|Y=♂] \cdot P(Y=♂)$$

$= E[X]$

What is the value? If know $P(Y=y_i)$ & $E[X|Y=y]$ can compute $E[X]$

Restatement of Proof of L.I.E. Discrete R.V.

$E[X|Y]$

$$= \sum_x x \cdot P(X=x|Y) \quad (1)$$

$\triangleq g(Y)$ is RV fcn of Y if y is abstract

So then $E[g(Y)]$

$$= \sum_y g(Y)\, P(Y=y) \quad (2)$$

$$= \sum_y \sum_x x\, P(X=x|Y) \cdot P(Y=y)$$

by substituting (1) in (2)

$P(x,Y) = P(X|Y) \cdot P(Y)$

$$= \sum_y \sum_x x\, P(X=x, Y=y)$$

$$= \sum_x x \sum_y P(X=x, Y=y)$$

$$= \sum_x x\, P(X=x) = E[X]$$

Result from LS4

Proof Continuous form of LIE

We know $E[Y|x]$ is a fcn of x

$$E[E[Y|X]] = \int E[Y|x] f_X(x)\,dx$$

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy \right) f_X(x)\,dx$$

$$= \iint y \frac{f_{X,Y}(x,y)}{f_X(x)} f_X(x)\,dy\,dx$$

$$= \iint y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= E[Y]$$

Eve's Law (EVVE's Law)

Law of Total Variance (w/o proof)

$$\left.\begin{aligned} Var(X) = \; & E[Var(X|Y)] \\ & + Var(E[X|Y]) \end{aligned}\right\} EV + VE$$

But the proof quite nicely

Use the L.I.E.

Apply definitions!!

Proof Law of Total Variance

$$E[Var(X|Y)] + Var(E[X|Y])$$

First term

def $Var(X|Y=y) = E[X^2|Y=y] - E^2[X|Y=y]$

$$E[Var(X|Y)] = E[X^2] - E[(E[X|Y])^2]$$

(By LIE and linear operator)

Second term

$$Var(E[X|Y]) = E[(E[X|Y])^2] - \underbrace{(E[E[X|Y]])^2}_{\text{By LIE}}$$

$$= (E[X])^2$$

Adding them together

$$E[Var(X|Y)] + Var(E[X|Y])$$

$$= E[X^2] - E[(E[X|Y])^2] + E[(E[X|Y])^2] - E^2[X]$$

$$= Var(X)$$

Professorial Mistake

Let X = number of questions asked by a student

Let X take on the values $\{1, 2, 3\}$

$$P(X = x) = \begin{cases} 1/3 & \text{if } x = 1 \\ 1/3 & \text{if } x = 2 \\ 1/3 & \text{if } x = 3 \\ \text{W} & \end{cases}$$

Let Y be the number of incorrect answers by instructor.

Clearly $Y = \{0, 1, 3, 3\}$

Given a question is asked the probability of an incorrect answer is 0.25 $\Longrightarrow$ $P(\text{correct ans}) = 1 - 0.25 = 0.75$

Let compute conditional Expected values

$$E[Y|X=1] = 0 \cdot P(Y=0|X=1) + 1 \cdot P(Y=1/X=1) + 2 \cdot P(Y=2|X=1) + 3 \cdot P(Y=3|X=1)$$
$$= 0 + 1 \cdot .25 + 2 \cdot 0 + 3 \cdot 0 = .25$$

$$E[Y|X=2] = 0 \cdot P(Y=0|X=2) + 1 \cdot P(Y=1|X=2) + 2\, P(Y=2|X=2) + 3 \cdot P(Y=3|X=2)$$
$$= 0 + 1 \cdot 2 \cdot .25 * .75 + 2 \cdot 1 \cdot .25^2 + 3 \cdot 0 = 0.5$$

$$E[Y|X=3] = 0 \cdot P(Y=0|X=3) + 1 \cdot P(Y=1|X=3) + 2 \cdot P(Y=2|X=3) + 3 \cdot P(Y=3|X=3)$$
$$= 0 + 1 \cdot 3 \cdot .25 \cdot .75^2 + 2 \cdot 3 \cdot .25^2 \cdot .75 + 3 \cdot 1 \cdot .25^3$$
$$= 0.[illegible]$$

$E[Y|X=3] = 0.75$

$\Rightarrow E[E[Y|X]] = E[Y]$ By LIE

$$= \sum_{x \in \Omega UT} E[Y|X] * P(X=x)$$

$$= 0.25 * \frac{1}{3} + 0.5 \cdot \frac{1}{3} + 0.75 \cdot \frac{1}{3}$$

$$= \frac{1}{12} + \frac{2}{12} + \frac{3}{12} = \frac{1}{2}$$

$$E[Y|X] = \begin{cases} 0.25 & x=1 \\ 0.5 & x=2 \\ 0.75 & x=3 \end{cases}$$

more initutively

$\Rightarrow E[Y|X=x] = 0.25 * x$

$= .25x$ if x is specified

Recall

$P(x,y) = P(y|x) \cdot P(x)$

$P(x,y)$	$x=1$	$x=2$	$x=3$	$P(Y=y)$
$y=0$	$3/4 \cdot 1/3$	$(3/4)^2\, 1/3$	$(3/4)^3 * 1/3$	
$y=1$	$1/4 \cdot 1/3$	$2 * .25 * .75 * 1/3$	$3 * .25 * .75^2 * 1/3$	$\binom{3}{1} = \frac{3!}{2! \cdot 1!}$
$y=2$	$0 \cdot 1/3$	$0.25^2 * 1/3$	$3 * .25^2 * .75 * 1/3$	$\binom{3}{2} = \frac{3!}{1! \cdot 2!}$
$y=3$	$0 * 1/3$	$0 * 1/3$	$0.25^3 * 1/3$	
$P(X=x)$	$1/3$	$1/3$	$1/3$	

$P(x,y)$	$x=1$	$x=2$	$x=3$	$P(Y=y)$
$y=0$	0.25	0.1875	0.1406	0.5781
$y=1$	0.0833	0.125	0.1406	0.3489
$y=2$	0	0.0206	0.0469	0.0677
$y=3$	0	0	0.0052	0.0052
$P(X=x)$	$1/3$	$1/3$	$1/3$	.9999

$$E[XY] = \sum_{x \in Out X} \sum_{y \in Out Y} x\, y\, P(X=x, Y=y)$$

$$1*1*0.0833 + 1*2*0.125 + 3*0.1406 + 4*0.0208 + 6*0.0469 + 9*0.0052$$
$$= 1.9153$$

$$Cov(X,Y) = E[XY] - E[X]\,E[Y]$$
$$= 1.9153 - [1*1/3 + 2*1/3 + 3*1/3]*\tfrac{1}{2}$$
$$= 1.915 - 2*\tfrac{1}{2} = 0.915$$