Glossary of Mathematical Notation

In this appendix, we provide a glossary of mathematical notation not defined in the text.

Notation	Definition and Usage
$\in$	*Set membership.* $s \in A$ (read "s is an element of A" or "s is in A") denotes that the object s (which could be a number, a vector, a set, etc.) is an element of the set A. For example, $2 \in \{1,2,4\}$. The negation of $\in$ is denoted by $\notin$; $s \notin A$ (read "s is not an element of A" or "s is not in A") denotes that the object s is *not* an element of the set A. For example, $3 \notin \{1,2,4\}$.
$\forall$	*For all.* Used to state that all elements of a particular set satisfy a given expression. For example, $\forall x \in \{1,2,3,4,5\}$, $2x \leq 10$. The set may be omitted if it is clear from context. For example, $\forall x$, $x+1 > x$ might mean that $x+1 > x$ is true for any real number x. Similarly, $\forall k > 1, f(k) = 0$ might mean that $f(k) = 0$ is true for any integer k that is greater than one. Usually, when defining a function, we will include a "for all" statement after the equation to denote the domain of the function. For example, $g(x) = \sqrt{x}, \forall x \geq 0$.
$\exists$	*There exists.* Used to state that there is *at least one* element of a particular set that satisfies a given expression. For example, $\exists x \in \{1,2,3,4,5\}$ such that $x^2 = 16$. As with $\forall$, the set may be omitted if it is clear from context. For example, $\exists x > 0$ such that $x^2 - 3x - 1 = 0$.

$\iff$	*If and only if*. Denotes that two statements are logically equivalent; each implies the other. Sometimes written as "iff." For example, $2x = 10 \iff x = 5$.
$\square$	*Quod erat demonstrandum (QED)*. Latin for "that which was to be demonstrated." This denotes that a proof is complete.
$\triangle$	*End of example*. This nonstandard notation allows us to clearly delineate where an example ends.
$\subseteq$	*Subset*. $A \subseteq B$ (read "A is a subset of B") means that the set B contains every element in the set A. Formally, $A \subseteq B \iff \forall s \in A,\ s \in B$. For example, $\{2,3,5\} \subseteq \{1,2,3,4,5\}$. Note that $A = B \iff A \subseteq B$ and $B \subseteq A$.
$\emptyset$	The *empty set*. The set containing no elements. Sometimes written as $\{\}$.
A^C	The *complement* of a set. When all sets under consideration are subsets of some *universal set* U, the complement of a set A is the set of all elements in U that are not in A. Formally, $A^C = \{s \in U : s \notin A\}$.
$\cup$	The *union* of sets. $A \cup B$ (read "A union B") denotes the set containing all elements that are in *either* A *or* B (or both). Formally, $A \cup B = \{s : s \in A \text{ or } s \in B\}$. For example, $\{1,2,5\} \cup \{2,3\} = \{1,2,3,5\}$. Union is associative, so for multiple unions parentheses can be omitted without ambiguity. For example, $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$.
$\cap$	The *intersection* of sets. $A \cap B$ (read "A intersect B") denotes the set containing all elements that are in *both* A *and* B. Formally, $A \cap B = \{s : s \in A \text{ and } s \in B\}$. For example, $\{1,2,5,6\} \cap \{2,3,6\} = \{2,6\}$. Intersection is associative, so for multiple intersections parentheses can be omitted without ambiguity. For example, $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$.
$\setminus$	*Set subtraction*. $A \setminus B$ (read "A set-minus B") denotes the set that contains all the elements in A *except* any that are also in B. Formally, $A \setminus B = \{s \in A : s \notin B\}$. $A \setminus B$ is called the *relative complement* of B in A, since $A \setminus B = A \cup B^C$.
$\mathcal{P}(A)$	The *power set* of A, that is, the set of all subsets of A. For example, $$\mathcal{P}\big(\{1,2,3\}\big) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\big\}.$$

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$\lvert A \rvert$	The *cardinality* (or *size*) of a set. For finite sets, the cardinality is the number of elements in the set. For example, $\lvert\{1,2,3,5,6,8\}\rvert = 6$.
$\mathbb{N}$	The set of all *natural numbers*, that is, positive integers: $\mathbb{N} = \{1,2,3,\ldots\}$. Note that while some texts include zero in $\mathbb{N}$, we do not.
$\mathbb{Z}$	The set of all *integers*: $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. (The letter "Z" here stands for *Zahlen*, German for "numbers.")
$\mathbb{R}$	The set of all *real numbers*. In classical mathematics, there are many ways of rigorously defining the real numbers, but for our purposes it suffices to say that the real numbers are all the points on a continuous number line.
$\mathbb{R}^n$	The *real coordinate space* of dimension n, that is, the set of all *vectors* of length n with real entries. Formally, $\mathbb{R}^n = \{(x_1,x_2,\ldots,x_n) : x_i \in \mathbb{R}, \forall i \in \{1,2,\ldots,n\}\}$. $\mathbb{N}^n$, $\mathbb{Z}^n$, etc. are defined analogously.
$f : S \to T$	A *function* f from S to T. The set S is the *domain* of f, the set of all values s for which $f(s)$ is defined. The set T is the *codomain* of f, a set that contains all possible values of $f(s)$. Formally, $\forall s \in S$, $f(s) \in T$.
$f(A)$	The *image* of the set A under the function f, that is, the set of all values the function f can take on when applied to an element of A. Formally, for a function $f : S \to T$ and a set $A \subseteq S$, $f(A) = \{t \in T : \exists a \in A \text{ such that } f(a) = t\}$. Note that $f(S)$ is the *range* of f.
$g \circ f$	The *composition* of two functions. For functions $f : S \to T$ and $g : U \to V$, where $f(S) \subseteq U \subseteq T$, $g \circ f : S \to V$ is the function that first applies f and then applies g to its output: $\forall s \in S$, $(g \circ f)(s) = g(f(s))$.
$\sum$	*Summation* of a sequence. For example, $$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$
$\prod$	*Product* of a sequence. For example, $$\prod_{i=1}^{6}(i+1) = (1+1)(2+1)(3+1)(4+1)(5+1)(6+1) = 5{,}040.$$

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$\arg\max$	The value, among all values in a particular set, that maximizes the given expression. For example, $\arg\max_{x\in\mathbb{R}}(-x^2-2x+3) = -1$, as $-x^2-2x+3$ attains its maximum value at $x=-1$. Our use of $\arg\max$ assumes that there exists a unique maximum. More generally, $\arg\max$ denotes the set of values at which the maximum is attained.
$\arg\min$	The value, among all values in a particular set, that minimizes the given expression. For example, $\arg\min_{x\in\mathbb{R}}(x^2-4x+1) = 2$, as x^2-4x+1 attains its minimum value at $x=2$. Our use of $\arg\min$ assumes that there exists a unique minimum. More generally, $\arg\min$ refers to the set of values at which the minimum is attained.
$\mathbb{A}^T$	The *transpose* of a matrix. $\mathbb{A}^T$ denotes the matrix whose columns are the rows of $\mathbb{A}$. Sometimes written as $\mathbb{A}'$. For example, if $$\mathbb{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \text{then} \quad \mathbb{A}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$
$\mathbb{A}^{-1}$	The *inverse* of a (square) matrix. If $\mathbb{A}$ is *invertible*, then $\mathbb{A}^{-1}$ denotes the matrix that, when multiplied by $\mathbb{A}$, yields the identity matrix of the appropriate dimensions: $$\mathbb{A}\mathbb{A}^{-1} = \mathbb{A}^{-1}\mathbb{A} = \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$ For example, if $$\mathbb{A} = \begin{pmatrix} 2 & 1 & 1 \\ -5 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad \text{then} \quad \mathbb{A}^{-1} = \begin{pmatrix} -3 & -2 & -3 \\ 5 & 3 & 5 \\ 2 & 1 & 1 \end{pmatrix}.$$

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