Interpreting Coefficients Under Various Combinations Of Log-Linear Models

Non transformation Interpretation

$$E[Y(1)|X] = \mathrm{beta}(0) + \mathrm{beta}(1) * X$$

$$E[Y(2)|X] = \mathrm{beta}(0) + \mathrm{beta}(1) * (X+1)$$

$$E[Y(2)|X] - E[Y(1)|X] = \mathrm{beta}(1)$$

==> Increasing 1 in X changes
y by an additive factor of beta(1)

Logging Response (Base e)
Leaving Aside Notation

$\log_e(y_1) = \beta_0 + \beta_1 X$ (Starting Specification)

$\Rightarrow y_1 = e^{\beta_0 + \beta_1 X}$ (Apply Expoentiation to both sides)

$= e^{\beta_0} * e^{\beta_1 X}$ (By $e^{a+b} = e^a * e^b$)

$\log_e(y_2) = \beta_0 + \beta_1(X+1)$ (Second Specification)

$y_2 = \underbrace{e^{\beta_0} * e^{\beta_1 X}} * e^{\beta_1}$ (Expoentiation and expanding)

$= y_1 * e^{\beta_1}$ (Simplifying)

$\Rightarrow$ increase of 1 unit changes
y_1 by a multiplicative
factor e^{β_1}

Further,

So we have y(1) being change by a 1 unit increase in x(1) as the the product of y(1) times e^beta(1), ceteris paribus which if we subtract 1 from e^beta(1), will provide the percentage change in y(1) from a 1 unit increase in x(1).

This is true because a 1 unit in x is (comparing specifications)
from above $\frac{y_2}{y_1} = \exp(\Delta \log(y)) = \exp(\beta_1)$ or
multiplicative change in y_1
Def percentage change $= \frac{y_2 - y_1}{y_1} * 100$ from
above we have this $= \frac{y_1 * \exp(\beta_1) - y_1}{y_1} * 100$

$\Rightarrow (\exp(\beta_1) - 1) * 100$ = percentage change as desired

Logging Dependent Variable

$$y_1 = \beta_0 + \beta_1 \log(x)$$

$$y_2 = \beta_0 + \beta_1 \log(x+1)$$

$$y_2 - y_1 = \beta_0 + \beta_1 \log(x+1) - \beta_0 - \beta_1 \log(x)$$

$$= \beta_1 (\log(x+1) - \log(x))$$

$$= \beta_1 \log\left(\frac{x+1}{x}\right)$$

y_1 is adjusted by $\beta_1 * \log$ of proportional change in x

$$y_2 = y_1 + \log\left\{\left(\frac{x+1}{x}\right)^{\beta_1}\right\}$$

Log Both Response and Dependent Variable

$$\log(y_1) = \beta_0 + \beta_1 \log(x_1) \quad \text{(Specif 1)}$$

$$\log(y_2) = \beta_0 + \beta_1 \log(x_2) \quad \text{(Specif 2)}$$

$$\log(y_2) - \log(y_1) = \beta_1 (\log(x_2) - \log(x_1))$$

$$= \log(y_2/y_1) = \beta_1 \left(\log\left(\frac{x_2}{x_1}\right)\right)$$

$$= \log\left((x_2/x_1)^{\beta_1}\right)$$

if $x_2 = x_1 + 1$

$$\Rightarrow \log(y_2/y_1) = \log\left(\left(\frac{x_1+1}{x_1}\right)^{\beta_1}\right) = \log\left(\left(1 + \frac{1}{x_1}\right)^{\beta_1}\right)$$

if $x_2 = x_1 * 1.1$ or a 10% increase in x_1

$$\log(y_2/y_1) = \log\left(\frac{x_1 + 0.1x_1}{x_1}\right)^{\beta_1}$$

$$= \log(1.1)^{\beta_1} = \beta_1 \log(1.1)$$

$$\cong \beta_1 * 0.09531$$

Note the approximation by first term of Taylor expansion $\log(1 + 0.1) \approx 0.1$

In summary β_1 quantifies the elasticity type relationship between relative in x and y, or β_1 describes change in y_2/y_i given proportional change in x_2/x_1