Proof of variance and bias relationship

$$
\begin{aligned}
\operatorname{MSE}(\hat{\theta}) &= \operatorname{E}_\theta\left[(\hat{\theta}-\theta)^2\right] \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]+\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2\right] \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)^2+2\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)+\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2\right] \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)^2\right]+\operatorname{E}_\theta\left[2\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)\right]+\operatorname{E}_\theta\left[\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2\right] \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)^2\right]+2\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)\operatorname{E}_\theta\left[\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right]+\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2 && \operatorname{E}_\theta[\hat{\theta}]-\theta=\text{const.} \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)^2\right]+2\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)\left(\operatorname{E}_\theta[\hat{\theta}]-\operatorname{E}_\theta[\hat{\theta}]\right)+\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2 && \operatorname{E}_\theta[\hat{\theta}]=\text{const.} \\
&= \operatorname{E}_\theta\left[\left(\hat{\theta}-\operatorname{E}_\theta[\hat{\theta}]\right)^2\right]+\left(\operatorname{E}_\theta[\hat{\theta}]-\theta\right)^2 \\
&= \operatorname{Var}_\theta(\hat{\theta})+\operatorname{Bias}_\theta(\hat{\theta},\theta)^2
\end{aligned}
$$

0

After https://en.wikipedia.org/wiki/Mean_squared_error