The Discrete-Time Fourier Transform and Convolution Theorems: A Brief Tutorial

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1 Definitions and interpretation

1.1 Units

Throughout this semester, we will use the integer-valued variable n as the time variable for discrete-time signal processing; that is, $n = -\infty, ..., -1, 0, 1, ..., \infty$. In this convention, the unit of n is dimensionless, but be aware that in reality the sampling period is $1/f_s$, where f_s denotes the sampling rate. In some text books, $1/f_s$ is denoted as T, and its unit is second. In this course we do not do this, thereby favoring conciseness of notations.

1.2 Definition of DTFT

The *discrete-time Fourier transform* (DTFT) of a discrete-time signal $x[n]$ is a function of frequency ω defined as follows:

$$X(\omega) \triangleq \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}. \tag{1}$$

Conceptually, the DTFT allows us to check how much of a tonal component at frequency ω is in $x[n]$. The DTFT of a signal is often also called a *spectrum*. Note that $X(\omega)$ is complex-valued. So, the absolute value $|X(\omega)|$ represents the tonal component's magnitude, and the angle $\angle X(\omega)$ tells us the phase of the tonal component.

Also note that Eq. (1) defines the DTFT as the inner product between the signal and the complex exponential tone $e^{-j\omega n}$. Because complex exponentials $\{e^{-j\omega n}, \omega \in [-\pi, \pi]\}$ form a *complete* family of orthogonal bases for (practically) all signals of interest, what Eq. (1) essentially does is to project $x[n]$ onto the space spanned by all complex exponentials.

Remarks:
A little bit about notations — unless mentioned otherwise, we will use lowercase variables (e.g., x and y) to denote signals in the time domain, and their uppercase counterparts (i.e., X and Y) to denote their spectra.

1.3 Inverse transform

Sometimes we are given the Fourier transform $X(\omega)$ and need to reconstruct the signal in the time domain. This can be done by calculating the following integral:

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega. \tag{2}$$

Intuitively, Eq. (2) does three things:

- take $X(\omega)$ as the coefficient along the dimension ω;
- multiply $X(\omega)$ with the basis function $e^{j\omega n}$;
- gathering all terms by integrating with respect to ω.

The statements above can be understood intuitively, but we must prove that Eq. (2) is indeed the inverse of Eq. (1). A sketch of the proof is given as follows,

$$\begin{aligned}
\hat{x}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega && (3)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}\right)e^{j\omega n}d\omega && (4)\\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x(m)\int_{-\pi}^{\pi} e^{j(n-m)\omega}d\omega && (5)
\end{aligned}$$

Because the integral on the right-hand side of Eq. (5) vanishes except when $m = n$, the equation can be simplified as

$$\begin{aligned}
\hat{x}[n] &= \frac{1}{2\pi}x[n]\int_{-\pi}^{\pi} e^{j\cdot 0\cdot\omega}d\omega\\
&= \frac{1}{2\pi}x[n]\cdot 2\pi\\
&= x[n].
\end{aligned}$$

Remarks:
Note that the range of integral is from $-\pi$ to π — because we follow a convention which defines time as a dimensionless quantity, the frequency becomes dimensionless, too.

2 Properties of discrete-time Fourier transform

2.1 Periodicity in frequency

Note that if ω is replaced by $\omega + 2\pi$ in Eq. (1), the DTFT of a signal remains the same; that is, for any signal $x[n]$, its DTFT $X(\omega)$ has the following property:

$$X(\omega + 2\pi) = X(\omega) \tag{6}$$

at any ω. In other words, the spectrum is periodic. If we convert back to the time-frequency units in reality, this means that a signal DTFT repeats at a period of f_s in the frequency domain.

$\boxed{\text{What does this mean?}}$

Recall that DTFT is a projection onto the space spanned by complex exponentials. Equation (6) tells us that frequency $\omega + 2\pi$ is indistinguishable to the frequency ω (assuming $0 < \omega < \pi$). So, if an analog signal is sampled at a rate of f_s, that any spectral component at a frequency f higher than f_s would be considered by the system as if its frequency were $f' = \mod(f, f_s)$. This is called *aliasing* and is undesired for most applications — think of a system that converts ultrasounds to low frequency sounds, for instance. So, in the design of an analog-to-digital (A/D) converter, one must prevent aliasing by introducing a low pass analog filter before sampling the analog signal in time. Such a filter is called an *anti-aliasing filter*.

$\boxed{\text{What is the bandwidth of an ideal anti-aliasing filter?}}$

2.2 Symmetry

Below are two theorems that reflect the time-frequency symmetry of DTFT. The first theorem is good to know because we mostly deal with real-valued signals. The second theorem is the first theorem in its time-frequency dual form.

THEOREM (Symmetry in frequency):

If $x[n]$ is real, then $X(-\omega) = X^*(\omega)$ for all ω.

Proof:

$$X^*(\omega) = \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n] e^{j\omega n} \tag{7}$$

Because $x[n]$ is real, i.e., $x^*[n] = x[n]$, it is straightforward to check that the right-most term in the equation above is indeed equal to $X(-\omega)$.

Remarks:
This theorem enables us to reconstruct a real signal if we only know its DTFT from 0 to π.

THEOREM (Symmetry in time):

If $x[n]$ is real and $x[-n] = x[n]$ for all n, then $X(\omega)$ is real.

Proof:

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= x[0] + \sum_{n=1}^{\infty} \left(x[n]e^{-j\omega n} + x[-n]e^{j\omega n}\right) \\
&= x[0] + \sum_{n=1}^{\infty} x[n] \left(e^{-j\omega n} + e^{j\omega n}\right) \\
&= x[0] + \sum_{n=1}^{\infty} x[n] \cdot 2\cos(\omega n) \\
&\in \mathcal{R}
\end{aligned}
$$

2.3 Shift in time

The shift theorem states that a delay in the time domain is equivalent to a linear phase shift factor in the frequency domain.

THEOREM Shift:

Let $y[n]$ be the delay of $x[n]$ by m samples, i.e.,

$$y[n] = x[n-m]$$

for all n. Then,

$$Y(\omega) = e^{-jm\omega} \cdot X(\omega).$$

The proof is left as an exercise.

3 Properties of Linear Time-Invariant Systems

3.1 Definition

A system is said to be *Linear time-invariant* (LTI) if the relation of its input and output satisfies the following criterion,

- **Linearity**: if a stimulus $x_1[n], n = 0, 1, 2, ...$ causes response $y_1[n]$, and $x_2[n]$ causes $y_2[n]$, then $ax_1[n] + bx_2[n]$ causes $ay_1[n] + by_2[n]$.
- **Time-invariance**: if $x[n], n = 0, 1, 2, ...$ causes a response $y[n]$, then $x[n-m]$, where m is a fixed integer, causes $y[n-m]$.

The *impulse response* is simply an LTI system's response to the Dirac impulse function $\delta[n]$.[1] An LTI system is completely characterized by its impulse response $h[n]$ in the sense that, given any arbitrary input $x[n]$, the system's response $y[n]$ can be calculated via convolution of $x[n]$ and $h[n]$:

$$y[n] = x[n] * h[n] \triangleq \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]. \tag{8}$$

Remarks:
Note that convolution is *commutative*, e.g., $x * h = h * x$. (How?)

3.2 Causality

Usually (I bet 99% of time in my career) we only deal with LTI systems that are *causal*, i.e., we assume that $h[n] = 0$ for all $n < 0$. In this case, Eq. (8) can be re-written as

$$y[n] = \sum_{m=-\infty}^{n} x[m] \cdot h[n-m]. \tag{9}$$

Remarks:
How do we interpret the equation above?

3.3 The convolution theorem

The convolution theorem states that convolution in the time domain is equivalent to multiplication in the frequency domain. Conceptually, we can regard one signal as the input to an LTI system and the other signal as the impulse response of the LTI system. Then, the output spectrum can be calculated by multiplying the input spectrum with the *transfer function* of the LTI system.

THEOREM (Convolution):

Let $x_1[n]$ and $x_2[n]$ be two discrete-time signals. If

$$y[n] = x_1[n] * x_2[n]$$

then

$$Y(\omega) = X_1(\omega) \cdot X_2(\omega).$$

[1] The Dirac impulse function $\delta[n]$ in discrete-time is defined as follows: $\delta[n] = 1$ for $n = 0$ and $\delta[n] = 0$ otherwise.

Proof:

$$
\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x_1[m] \cdot x_2[n-m] \right) e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} x_1[m] \left(\sum_{n=-\infty}^{\infty} x_2[n-m]e^{-j\omega n} \right) \\
&= \sum_{m=-\infty}^{\infty} x_1[m] \left(e^{-j\omega m} X_2(\omega) \right) \\
&= X_2(\omega) \sum_{m=-\infty}^{\infty} x_1[m]e^{-j\omega m} \\
&= X_2(\omega)X_1(\omega).
\end{aligned}
$$

Remarks:
By duality, multiplication of two signals in the *time* domain is also equivalent to convolution in the frequency domain. This will be important when we discuss finite-impulse-response (FIR) filter design using the window method [1].

3.4 Energy of a deterministic signal

In engineering and physics, energy equals power integrated over time. Power is often proportional to the square of a physical quantity such as voltage, current, particle velocity, etc. For example, given a time-varying voltage $V(t)$ that is applied to a resistor R, the heat dissipation H can be calculated as follows:

$$H = \int \frac{[V(t)]^2}{R} dt. \tag{10}$$

In discrete-time signal processing, we *borrow* the energy concept from physics by defining the energy F of a signal $x[n]$ as x^2 summed over time:

$$F = \sum_{n=-\infty}^{\infty} (x[n])^2. \tag{11}$$

The following theorem tells us that the energy can be calculated either in the time domain or in the frequency domain.

THEOREM (Parseval's theorem):

Let $x[n]$ be a real-valued, discrete-time signal and $X(\omega)$ be its DTFT. Then, the following equation always holds:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2.$$

Proof:
First, construct the autocorrelation function

$$y[m] \triangleq \sum_{n=-\infty}^{\infty} x[n]\cdot x[n-m].$$

Note that $y[n] = x[n] * x[-n]$, and in particular, $y[0] = \sum_{n=-\infty}^{\infty} x^2[n]$. Applying the convolution theorem, $y[n]$'s Fourier transform of $Y(\omega)$ can be expressed in terms of $X(\omega)$ as follows,

$$Y(\omega) = X(\omega)\cdot\left(\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}\right) = X(\omega)\cdot X^*(\omega) = |X(\omega)|^2.$$

Calculating the inverse DTFT at time 0, then we reach that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^2[n] = y[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega\cdot 0}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega.\end{aligned}$$

Remarks:
Parseval's theorem tells us that the DTFT is a linear transform that *preserves the norm* of a signal (up to a factor of $\sqrt{1/2\pi}$). Therefore, we can think of Fourier transform as a rotation in the infinitely-many dimensional space.

4 Exercises

Please do as much of the following exercises as you could. Some of them might be selected as questions for future quizzes.

1. Prove the shift-theorem.

2. Describe artifacts that one might hear when playing back an audio signal that is sampled from an analog acoustic signal without anti-aliasing filtering.

3. What does it mean to have a non-causal system? Is it possible to have a non-causal system physically?

4. Argue that the output an LTI system produces indeed has to be the convolution of the impulse response and an input.

5. Calculate the DTFT of $h[n] = u[n]e^{-\beta n}\cos(\omega_0 n)$, where $\beta > 0$ and $u[n] = 0$ for $n < 0$ and $u[n] = 1$ otherwise. Write a `MATLAB` script to plot the magnitude spectrum $|H(\omega)|$ and phase spectrum $\angle H(\omega)$.

6. Continued from above, argue that $h[n]$ is the impulse response to a second-order system whose input-output relation can be defined as $y[n] = a_1 y[n-1] + a_2 y[n-2] + x[n]$. Find out coefficients a_1 and a_2 in terms of β and ω_0.

7. Continued from above, what factors determine the sharpness of resonance at ω_0 and the slope of phase accumulation at ω_0? Are they correlated?

References

[1] Alan V. Oppenheim and Ronald W. Schafer. *Discrete-Time Signal Processing, 3rd Ed.* Pearson, New York, 2010.