Formulas for wave-structure interaction

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The polyhedral domain Ω is subdivided into two polyhedra: Ω_- and Ω_+. The common interface is called Γ_{int}. In this first version Ω_- has no exterior boundary, i.e., its only boundary is the interface. The exterior boundary of Ω_+ is $\Gamma_{ext} = \partial\Omega = \partial\Omega_+ \setminus \Gamma_{int}$.

The equations for the Laplace-domain wave structure interaction are:

$$\begin{aligned}
-\operatorname{div}\sigma(\mathbf{u}) + s^2\rho\mathbf{u} &= \mathbf{0} && \text{in } \Omega_-,\\
\sigma(\mathbf{u}) &= 2\mu\varepsilon(\mathbf{u}) + \lambda(\nabla\cdot\mathbf{u})\mathrm{I} &&\\
-\Delta v + s^2 v &= 0 && \text{in } \Omega_+,\\
s\gamma v\,\mathbf{n} + \sigma(\mathbf{u})\mathbf{n} &= \mathrm{G}\mathbf{n}, && \text{on } \Gamma_{int},\\
s\gamma\mathbf{u}\cdot\mathbf{n} + \partial_n v &= \mathbf{g}\cdot\mathbf{n}, && \text{on } \Gamma_{int},\\
\partial_n v + s\gamma v &= \partial_n v^{inc} + s\gamma v^{inc}, && \text{on } \Gamma_{ext}.
\end{aligned}$$

Bilinear forms:

$$\begin{aligned}
a_{el}(\mathbf{u},\mathbf{w}) :=& (\sigma(\mathbf{u}),\varepsilon(\mathbf{w}))_{\Omega_-} + s^2(\rho\mathbf{u},\mathbf{w})_{\Omega_-},\\
a_{ac}(v,w) :=& (\nabla v,\nabla w)_{\Omega_+} + s^2(v,w)_{\Omega_+} + s\langle\gamma v,\gamma w\rangle_{\Gamma_{ext}},\\
c(v,\mathbf{w}) :=& \langle\gamma v,\gamma\mathbf{w}\cdot\mathbf{n}\rangle_{\Gamma_{int}},\\
c'(\mathbf{u},w) :=& c(w,\mathbf{u}),\\
a((\mathbf{u},v),(\mathbf{w},w)) :=& a_{el}(\mathbf{u},\mathbf{w}) + sc(v,\mathbf{w})\\
& - sc'(\mathbf{u},w) + a_{ac}(v,w).
\end{aligned}$$

Right-hand sides:

$$\begin{aligned}
\ell_{ac}(w) :=& \langle\partial_n v^{inc},\gamma w\rangle_{\Gamma_{ext}} + s\,\langle\gamma v^{inc},\gamma w\rangle_{\Gamma_{ext}},\\
\ell_{c,ac}(w) :=& \langle\mathbf{g}\cdot\mathbf{n},\gamma w\rangle_{\Gamma_{int}},\\
\ell_{c,el}(\mathbf{w}) :=& \langle\mathrm{G}\mathbf{n},\gamma\mathbf{w}\rangle_{\Gamma_{int}},\\
\ell((\mathbf{w},w) :=& \ell_{c,el}(\mathbf{w}) + \ell_{ac}(w) - \ell_{c,ac}(w)
\end{aligned}$$

The function `couplingMatrices` computes the matrix corresponding to this problem using $\mathcal{P}_k$ elements on the subdomains. To be more precise: we consider a triangulation $\mathcal{T}_h$ of Ω, respecting the interface Γ_{int} and we take the subtriangulations $\mathcal{T}_h^{\pm}$ of $\Omega_\pm$; the finite element space for acoustics is

$$V_h^+ := \{v_h \in \mathcal{C}(\Omega_+) \,:\, v_j|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h^+\},$$

while the finite element space for elasticity is

$$\mathbf{V}_h^- := V_h^- \times V_h^- \times V_h^-, \qquad V_h^- := \{v_h \in \mathcal{C}(\Omega_-) \,:\, v_j|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h^-\}.$$

We will denote

$$d_+ := \dim V_h^+, \qquad d_- := \dim V_h^-.$$

From the point of view of the triangulations, we assume that Γ_{ext} is the Neumann boundary of the global domain, which does not contain any other boundary. The subdomain Ω_- does not have any tagged boundary, and the interface Γ_{int} is computed at the time of creating the coupling matrices (in `interfaceMatrices3D.m` and then discarded). The subdomain Ω_+ has a Neumann boundary equal to Γ_{ext} and the interface Γ_{int}, as part of the booundary of Ω_+ is never computed. This can happen because all computations for the interface matrices and vectors are done from the point of view of the interior domain.

The process to build the matrix and vector corresponding to the above bilinear and linear forms is organized as follows:

- We compute the variable (μ, λ) elastic stiffness matrix and variable ρ mass matrix for elasticity (a block diagonal matrix whose blocks are scalar mass matrices). With this we create the matrix
$$\mathrm{EL} := s^2 \mathrm{M}_{el} + \mathrm{S}_{el}.$$
- We then compute the acoustic stiffness matrix (corresponding to the Laplace operator), the unit mass matrix, and the Γ_{ext} boundary mass matrix. With this we create the matrix
$$\mathrm{AC} := s^2 \mathrm{M}_{ac} + \mathrm{S}_{ac} + s\mathrm{M}_{bd,ext}.$$
- The $3d_- \times d_+$ matrix C, corresponding to the bilinear form c is computed as
$$\mathrm{C} := \begin{bmatrix} \mathrm{C}_x \\ \mathrm{C}_y \\ \mathrm{C}_z \end{bmatrix}$$
where
$$C_\star := \int_{\Gamma_{int}} P_i^- P_j^+ n_\star, \qquad i = 1, \ldots, d_-, \qquad j = 1, \ldots, d_+, \quad \star \in \{x, y, z\}.$$
These blocks are the second, third, and fourth output of `interfaceMatrices3D.m`.
- We then build the global matrix
$$\begin{bmatrix} \mathrm{EL} & s\,\mathrm{C} \\ -s\mathrm{C}^\top & \mathrm{AC} \end{bmatrix}.$$
- For the right-hand side vectors, we first compute vectors on the interfaces. A $3d_-$-column vector
$$\mathbf{b}_{el} := \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$
where:
$$\mathbf{b}_\star := \int_{\Gamma_{int}} (\mathrm{G}\mathbf{n})_\star P_i^- \qquad i = 1, \ldots, d_-, \quad \star \in \{1, 2, 3\}.$$
This is the fifth output of `interfaceMatrices3D.m`. A d_+-column vector
$$\mathbf{b}_{ac} := \int_{\Gamma int} (\mathbf{g} \cdot \mathbf{n}) P_i^+ \qquad i = 1, \ldots, d_+,$$
is given as the sixth output of `interfaceMatrices3D.m`.
- We next compute the d_+-column vector
$$\mathbf{b}_{ext} := \int_{\Gamma_{ext}} (\nabla v^{inc} \cdot \mathbf{n}) P_i^+ + s \int_{\Gamma exi} v^{inc} P_i^+ \qquad i = 1, \ldots, d_+.$$
- We finally build the full vector for the coupled system
$$\begin{bmatrix} \mathbf{b}_{el} \\ \mathbf{b}_{ext} - \mathbf{b}_{ac} \end{bmatrix}.$$