The completion of a normed space

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Let X be a normed space (not complete). Consider the vector space c of Cauchy sequences in X and its subspace c_0 of sequences that converge to zero. Consider finally the space

$$Z := c/c_0 = \{\mathbf{x} + c_0 \,:\, \mathbf{x} \in c\}.$$

Note that this is the space of cosets defined by the equivalence relation $(x_n)_{n\ge1} \equiv (y_n)_{n\ge1}$ when $\|x_n - y_n\| \to 0$, that is, we identify Cauchy sequences whose difference converges to zero. We then define

$$|\!|\!|\mathbf{x} + c_0|\!|\!| := \lim_{n\to\infty} \|x_n\|.$$

Z is a normed space. Some simple facts:

(a) If $(x_n)_{n\ge1}$ is Cauchy, then, since

$$|\|x_n\| - \|x_m\|| \le \|x_n - x_m\|,$$

it follows that $\lim_n \|x_n\|$ is well defined.

(b) If $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ are Cauchy sequences such that $(x_n - y_n)_{n\ge1} \in c_0$, then, using the inequalities

$$\|x_n\| \le \|x_n - y_n\| + \|y_n\|, \qquad \|y_n\| \le \|x_n - y_n\| + \|x_n\|,$$

it follows that $\lim_n \|x_n\| = \lim_n \|y_n\|$. This proves that the definition of $|\!|\!| \cdot |\!|\!| : Z \to [0,\infty)$ is correct, that is, the definition does not depend on the particular representative of the coset.

(c) If $|\!|\!|\mathbf{x} + c_0|\!|\!| = 0$, then $\lim_n \|x_n\| = 0$ and therefore $\mathbf{x} \in c_0$, or, equivalently $\mathbf{x} + c_0 = \mathbf{0} + c_0$, which is the zero element of the quotien space Z.

(d) Since $\lambda(\mathbf{x} + c_0) = \lambda\mathbf{x} + c_0$, it is easy to prove that

$$|\!|\!|\lambda(\mathbf{x} + c_0)|\!|\!| = \lim_{n\to\infty} |\lambda x_n\| = |\lambda| \lim_{n\to\infty} \|x_n\| = |\lambda|\, |\!|\!|\mathbf{x} + c_0|\!|\!| \qquad \forall \lambda \in \mathbb{K}.$$

(e) The triangle inequality for $|\!|\!| \cdot |\!|\!|$ follows from the fact that $(\mathbf{x} + c_0) + (\mathbf{y} + c_0) = (\mathbf{x} + \mathbf{y}) + c_0$ and the triangle inequality for the norm of X.

We have thus proved that Z is a normed space.

The canonical inclusion of X into Z. Consider now the operator $\boldsymbol{i} : X \to Z$ given by $\boldsymbol{i}x = (x, x, \ldots, x, \ldots) + c_0$, that is, to every $x \in X$ we associate the (coset containing the) constant sequence where all the elements are equal to x. Note that

$$|||\boldsymbol{i}x||| = \|x\|.$$

This means that $\boldsymbol{i}$ is an isometry and therefore injective.

$\boldsymbol{i}(X)$ is dense in Z. Fix $\mathbf{x} \in c$. Let $\varepsilon > 0$ and take N such that

$$\|x_n - x_m\| < \varepsilon \qquad \forall n, m \geq N.$$

Then

$$\|x_n - x_N\| < \varepsilon \qquad \forall n \geq N,$$

and from there

$$\lim_{n\to\infty} \|x_n - x_N\| \leq \varepsilon.$$

(Note that the limit does actually exist. Why?) Therefore

$$|||\mathbf{x} + c_0 - \boldsymbol{i}x_N||| \leq \varepsilon.$$

This proves that the set $\boldsymbol{i}(X) = \{\boldsymbol{i}x \,:\, x \in X\}$ is dense in Z.

Z is a Banach space. To keep notation more or less clean, we will use superscripts for elements of a sequence of sequences. Let thus $(\mathbf{x}_N)_{N\geq 1}$ be a sequence of elements of c such that $(\mathbf{x}^N + c_0)_{N\geq 1}$ is a Cauchy sequence in Z. Since $\mathbf{x}^N \in c$, there exists n_N such that

$$\|x_n^N - x_m^N\| \leq \tfrac{1}{N} \quad n, m \geq n_N.$$

We then choose $y_N := x_{n_N}^N$ and construct a sequence $\mathbf{y} := (y_N)_{N\geq 1}$. Let us first show that $\mathbf{y} \in c$. For arbitrary $\varepsilon > 0$ there exists N_0 such that

$$|||\mathbf{x}^N - \mathbf{x}^M + c_0||| \leq \varepsilon \qquad \forall N, M \geq N_0 \geq \tfrac{1}{\varepsilon}.$$

We momentarily fix $N, M \geq N_0$ and choose n_ε such that

$$\|x_n^N - x_n^M\| \leq 2\varepsilon \qquad n \geq n_\varepsilon.$$

(Note that we can do this because $|||\mathbf{x}^N - \mathbf{x}^M + c_0||| = \lim_n \|x_n^N - x_n^M\| \leq \varepsilon$. Taking $n := \max\{n_\varepsilon, n_N, n_M\}$, we have the bound

$$\|y_N - y_M\| \leq \|y_N - x_n^N\| + \|x_n^N - x_n^M\| + \|x_n^M - y_M\| \leq \tfrac{1}{N} + 2\varepsilon + \tfrac{1}{M} \leq 4\varepsilon.$$

We have thus shown that for all $\varepsilon > 0$ there exists N_0 such that $\|y_N - y_M\| \leq 4\varepsilon$ if $N, M \geq N_0$. Let us finally prove that $(\mathbf{x}^N + c_0)_{N\geq 1}$ converges to $\mathbf{y} + c_0$. Given $\varepsilon > 0$ we first choose n_ε such that

$$\|y_n - y_m\| < \varepsilon \qquad \forall n, m \geq n_\varepsilon$$

and then pick $N_0 \geq \max\{1/\varepsilon, n_\varepsilon\}$. Then for $N \geq N_0$

$$\|x_n^N - y_N\| = \|x_n^N - x_{n_N}^N\| \leq \tfrac{1}{N} \leq \tfrac{1}{N_0} \leq \varepsilon \qquad \forall n \geq n_N.$$

Fix momentarily $N \geq N_0$. Then

$$\|x_n^N - y_n\| \leq \|x_n^N - y_N\| + \|y_N - y_n\| \leq 2\varepsilon \qquad \forall n \geq \max\{n_N, n_\varepsilon\}$$

and therefore

$$|\!|\!|\mathbf{x}^N - \mathbf{y} + c_0|\!|\!| = \lim_{n\to\infty} \|x_N^N - y_n\| \leq 2\varepsilon.$$

In summary, for all $\varepsilon > 0$ there exists N_0 such that

$$|\!|\!|(\mathbf{x}^N + c_0) - (\mathbf{y} + c_0)|\!|\!| \leq 2\varepsilon \qquad \forall N \geq N_0,$$

that is, $\lim_N(\mathbf{x}^N + c_0) = \mathbf{y} + c_0$ in Z.

Final words. It can be proved that the space Z is essentially unique. If there is another space with the same properties (Banach and such that there is an isometry from X to the space in such a way that the image of X is dense), then this space is isometrically isomorphic to Z. The space Z is called the **completion of** X**.**