Robotik - Blatt 5

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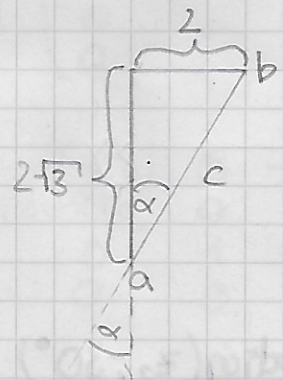
1	2	3	Σ
4,5	6,5	6	17

Aufgabe 1



$$\alpha = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° \Rightarrow -30°$$

$$c = \sqrt{(2\sqrt{3})^2+4} = \sqrt{4\cdot 3+4} = \sqrt{16} = 4$$

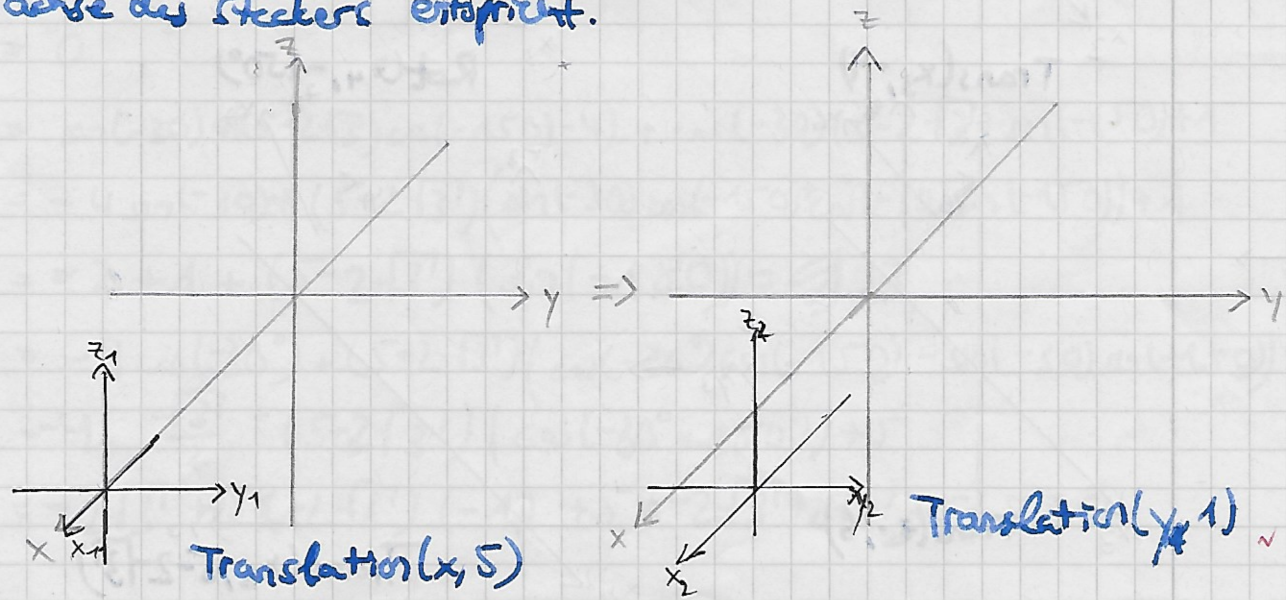
a) Transformationen:

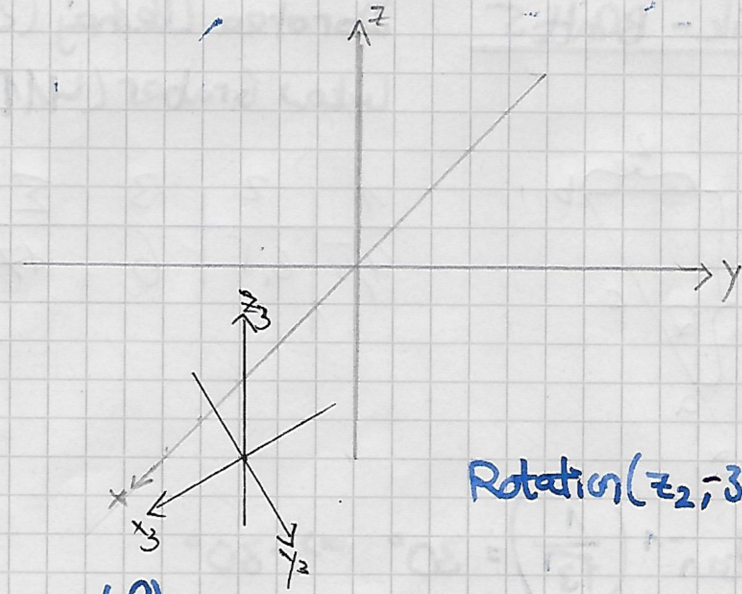
$${}^{W}T_a = \text{Translation}(x, 5)\ \text{Translation}(y, 1)\ \text{Rotation}(z, -30°)$$

$$= \begin{pmatrix} 1&0&0&5\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} \cos(-30°)&-\sin(-30°)&0&0\\ \sin(-30°)&\cos(-30°)&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-30°)&-\sin(-30°)&0&5\\ \sin(-30°)&\cos(-30°)&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}&\frac{1}{2}&0&5\\ -\frac{1}{2}&\frac{\sqrt{3}}{2}&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

Sei a der Ursprung des Frames a. Man erhält die Koordinaten a_1, indem man 5 Einheiten in x-Richtung, dann 1 Einheit in y-Richtung geht. Anschließend richtet man noch die Achsen aus damit die x-Achse des Frames a der Symmetrieachse des Steckers entspricht.





Rotation(z_2;30°)

$$a_a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 5 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow a_1 = (5,1,0)^T$$

$$b_a = \begin{pmatrix} -c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

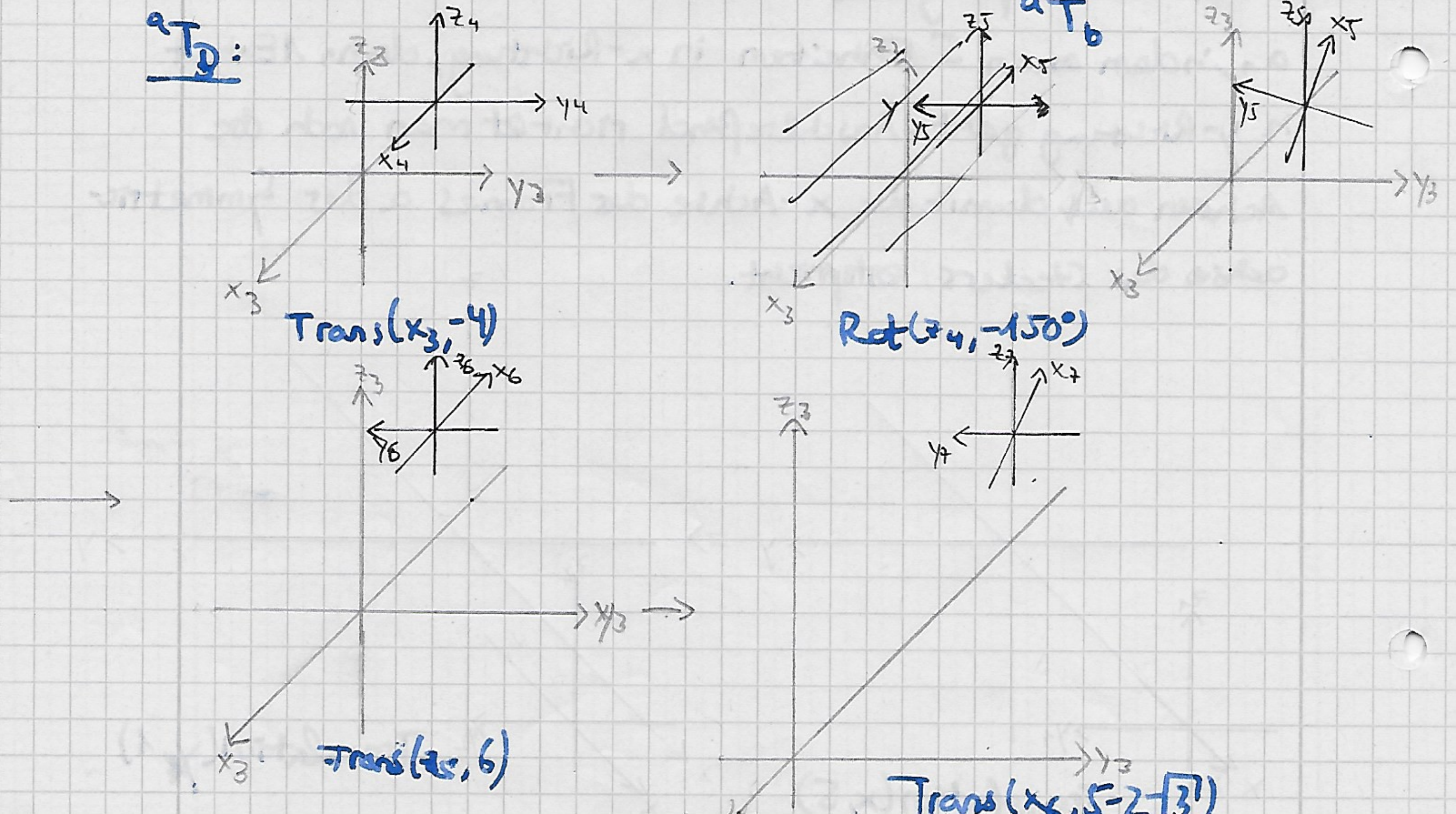
$$\Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 5 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5-2\sqrt{3} \\ 2+1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow (5-2\sqrt{3},3,0)^T = b_1$$

3/2

falsch −1,5P, ??

b) $${}^{W}T_b = {}^{W}T_a \underbrace{\text{Trans}(x_3;4)\,\text{Rot}(z_4,-150°)\,\text{Trans}(z_5,6)\,\text{Trans}(x_6,5-2\sqrt{3})}_{{}^{a}T_b}$$

${}^{a}T_b$:



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$${}^{a}T_{b} = \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-150) & -\sin(-150) & 0 & 0 \\ \sin(-150) & \cos(150) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5-2\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-150) & -\sin(-150) & 0 & -4 \\ \sin(-150) & \cos(150) & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5-2\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-150) & -\sin(-150) & 0 & (5-2\sqrt{3})\cos(-150)-4 \\ \sin(-150) & \cos(-150) & 0 & (5-2\sqrt{3})\sin(-150) \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow {}^{w}T_{b} = {}^{w}T_{a}\,{}^{a}T_{b}$$

$$= \begin{pmatrix} \cos(-30^\circ) & -\sin(-30^\circ) & 0 & 5 \\ \sin(-30^\circ) & \cos(-30^\circ) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-150) & -\sin(-150) & 0 & (5-2\sqrt{3})\cos(-150)-4 \\ \sin(-150) & \cos(-150) & 0 & (5-2\sqrt{3})\sin(-150) \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$a_{11} = \cos(-30)\cos(-150) - \sin(-30)\sin(-150) = \cos(-30-150) = \cos(-180) = -1$

$a_{12} = -\cos(-30)\sin(150) - \sin(-30)\cos(-150) = -(\cos(-30)\sin(150) + \sin(-30)\cos(-150))$

$= -\sin(-30-150) = -\sin(-180) = 0$

$a_{13} = 0$

$a_{14} = \cos(-30)((5-2\sqrt{3})\cos(-150)-4) - \sin(-30)(5-2\sqrt{3})\sin(-150)+5$

$a_{21} = \sin(-30)\cos(-150) + \cos(-30)\sin(-150) = \sin(-30-150) = \sin(-180) = 0$

$a_{22} = -\sin(-30)\sin(-150) + \cos(-30)\cos(-150) = \cos(-30-150) = \cos(-180) = -1$

$a_{23} = 0$

$a_{24} = \sin(-30)((5-2\sqrt{3})\cos(-150)-4) + \cos(-30)(5-2\sqrt{3})\sin(-150)+1$

$= -4\sin(-30) + (5-2\sqrt{3})(\sin(-30)\cos(-150)+\cos(-30)\sin(-150))+1$

$= 2+1+(5-2\sqrt{3})(\sin(-180)) = 3$

$a_{14} = -4\cos(-30^\circ) + (5-2\sqrt{3})(\cos(-30^\circ)\cos(-150) - \sin(-30)\sin(-150)) + 5$

$= -4\cdot\frac{\sqrt{3}}{2} + (5-2\sqrt{3})(\cos(-30^\circ-150^\circ))+5$

$= -2\sqrt{3} + (5-2\sqrt{3})(-1) + 5 = -2\sqrt{3}+2\sqrt{3}-5+5 = 0$

$a_{31} = 0$

$a_{32} = 0$

$a_{33} = 1$

$a_{34} = 6$

~~a_{44}~~

Montagebewegung nicht Pose von der Steckdose

$$^{W}T_b = \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{} = \begin{pmatrix} \cos(-180^\circ) & -\sin(-180^\circ) & 0 & 0 \\ \sin(-180^\circ) & \cos(-180^\circ) & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

0.5/2

$\Rightarrow$ OK, das war irgendwie abzusehen.

C) Man muss den Stecker so ausrichten, dass man nur noch entlang einer Achse in Richtung der Steckdose translatieren muss. Dann kann es keine Kollisionen geben.

$\Rightarrow$ Führe die Transformationen bis auf die letzte Translation auf einmal durch. Anschließend kann dann noch translatiert werden.

$$^{W}T_b = \underbrace{\left({}^{W}T_a \, Trans(x_3, -4) Rot(z_4, -150^\circ) Trans(z_5, 6) \right)}_{T_1} \; \& \; \underbrace{Trans(x_6, 5-2\sqrt{3})}_{T_2}$$

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 5-2\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(✓)

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$$T_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 5 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -4 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \left(-\frac{3}{4} - \frac{1}{4}\right) & \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) & 0 & (-2\sqrt{3}+5) \\ \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) & \left(-\frac{1}{4} - \frac{3}{4}\right) & 0 & 2+1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 5-2\sqrt{3} \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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A2

a) $$R_x(\alpha)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\alpha) & -\sin(\alpha)\\ 0 & \sin(\alpha) & \cos(\alpha)\end{pmatrix}$$

$$R_z(\beta)=\begin{pmatrix}\cos(\beta) & -\sin(\beta) & 0\\ \sin(\beta) & \cos(\beta) & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$R_x(\gamma)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\gamma) & -\sin(\gamma)\\ 0 & \sin(\gamma) & \cos(\gamma)\end{pmatrix}$$

$$XZX(\alpha,\beta,\gamma)=R_x(\alpha)\cdot R_z(\beta)\cdot R_x(\gamma)$$

$$=\begin{pmatrix}\cos(\beta) & -\sin(\beta) & 0\\ \cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) & -\sin(\alpha)\\ \sin(\alpha)\sin(\beta) & \sin(\alpha)\cos(\beta) & \cos(\alpha)\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\gamma) & -\sin(\gamma)\\ 0 & \sin(\gamma) & \cos(\gamma)\end{pmatrix}$$

$$=\begin{pmatrix}\cos(\beta) & -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma)\\ \cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta)\cos(\gamma)-\sin(\alpha)\sin(\gamma) & -\cos(\alpha)\cos(\beta)\sin(\gamma)-\sin(\alpha)\cos(\gamma)\\ \sin(\alpha)\sin(\beta) & \sin(\alpha)\cos(\beta)\cos(\gamma)+\cos(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma)+\cos(\alpha)\cos(\gamma)\end{pmatrix}$$

1/1

b)

$$XZX(0,\tfrac{\pi}{2},\tfrac{\pi}{2})=$$

$$=\begin{pmatrix}\cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2})\cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2})\sin(\frac{\pi}{2})\\ \cos(0)\sin(\frac{\pi}{2}) & \cos(0)\cos(\frac{\pi}{2})\cos(\frac{\pi}{2})-\sin(0)\sin(\frac{\pi}{2}) & -\cos(0)\cos(\frac{\pi}{2})\sin(\frac{\pi}{2})-\sin(0)\cos(\frac{\pi}{2})\\ \sin(0)\sin(\frac{\pi}{2}) & \sin(0)\cos(\frac{\pi}{2})\cos(\frac{\pi}{2})+\cos(0)\sin(\frac{\pi}{2}) & -\sin(0)\cos(\frac{\pi}{2})\sin(\frac{\pi}{2})+\cos(0)\cos(\frac{\pi}{2})\end{pmatrix}$$

$$=\begin{pmatrix}0 & -1\cdot 0 & 1\cdot 1\\ 1\cdot 1 & 1\cdot 0\cdot 0-0\cdot 1 & -1\cdot 0\cdot 1-0\cdot 0\\ 0\cdot 1 & 0\cdot 0\cdot 0+1\cdot 1 & -0\cdot 1\cdot 0+1\cdot 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

1/1

c) Jeweils die Einträge, die nicht abgeleitet werden, können als Faktoren für den Teil angesehen werden, der abgeleitet wird. Daher können die Faktoren bereits vor der Ableitung ausgewertet werden.

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$\beta = \pi, \gamma = 0$:

$XZX(\alpha, \pi, 0) =$

$$= \begin{pmatrix} \cos(\pi) & -\sin(\pi)\cos(0) & \sin(\pi)\sin(0) \\ \cos(\alpha)\sin(\pi) & \cos(\alpha)\cos(\pi)\cos(0) - \sin(\alpha)\sin(0) & -\cos(\alpha)\cos(\pi)\sin(0) - \sin(\alpha)\cos(0) \\ \sin(\alpha)\sin(\pi) & \sin(\alpha)\cos(\pi)\cos(0) + \cos(\alpha)\sin(0) & -\sin(\alpha)\cos(\pi)\sin(0) + \cos(\alpha)\cos(0) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -0 \cdot 1 & 0 \cdot 0 \\ \cos(\alpha) \cdot 0 & -\cos(\alpha) \cdot 1 \cdot 1 - \sin(\alpha) \cdot 0 & \cos(\alpha) \cdot 1 \cdot 0 - \sin(\alpha) \cdot 1 \\ \sin(\alpha) \cdot 0 & -\sin(\alpha) \cdot 1 \cdot 1 + \cos(\alpha) \cdot 0 & \sin(\alpha) \cdot 1 \cdot 0 + \cos(\alpha) \cdot 1 \end{pmatrix}$$ $-0{,}5$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos(\alpha) & -\sin(\alpha) \\ 0 & -\sin(\alpha) & +\cos(\alpha) \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial \alpha} XZX(\alpha, \pi, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin(\alpha) & -\cos(\alpha) \\ 0 & -\cos(\alpha) & \sin(\alpha) \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial \alpha} XZX\left(\frac{\pi}{2}, \pi, 0\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin(\frac{\pi}{2}) & -\cos(\frac{\pi}{2}) \\ 0 & -\cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\alpha = \frac{\pi}{2}, \beta = \pi$:

$XZX(\frac{\pi}{2}, \pi, \gamma)$

$$= \begin{pmatrix} \cos(\pi) & -\sin(\pi)\cos(\gamma) & \sin(\pi)\sin(\gamma) \\ \cos(\frac{\pi}{2})\sin(\pi) & \cos(\frac{\pi}{2})\cos(\pi)\cos(\gamma) - \sin(\frac{\pi}{2})\sin(\gamma) & -\cos(\frac{\pi}{2})\cos(\pi)\sin(\gamma) - \sin(\frac{\pi}{2})\cos(\gamma) \\ \sin(\frac{\pi}{2})\sin(\pi) & \sin(\frac{\pi}{2})\cos(\pi)\cos(\gamma) + \cos(\frac{\pi}{2})\sin(\gamma) & -\sin(\frac{\pi}{2})\cos(\pi)\sin(\gamma) + \cos(\frac{\pi}{2})\cos(\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \cdot \cos(\gamma) & 0 \cdot \sin(\gamma) \\ 0 \cdot 0 & 0 \cdot (-1)\cos(\gamma) - 1 \cdot \sin(\gamma) & -0 \cdot (-1)\sin(\gamma) - 1\cos(\gamma) \\ 1 \cdot 0 & 1 \cdot (-1)\cos(\gamma) + 0 \cdot \sin(\gamma) & -1 \cdot (-1)\sin(\gamma) + 0 \cdot \cos(\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sin(\gamma) & -\cos(\gamma) \\ 0 & -\cos(\gamma) & \sin(\gamma) \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial \gamma} XZX(\tfrac{\pi}{2}, \pi, \gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos(\gamma) & \sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial \gamma} XZX(\tfrac{\pi}{2}, \pi, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2,5 / 3

d) Es geht durch die Drehungen eine Achse verloren, genauer gesagt die x-Achse / Einheitsvektor für x-Achse: Der erste Spaltenvektor für beide partiellen Ableitungen beträgt $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Für $\beta = 3\pi$ erhält für $\sin(\beta)$ und $\cos(\beta)$ den jeweils gleichen Betrag wie in den vorigen Beispielen.

2/2

F.F

$$\frac{\partial}{\partial \gamma} XZX(\pi/2, \pi, 0) = -\frac{\partial}{\partial \alpha} XZX(\pi/2, \pi, 0)$$

Aufgabe 3

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a)

$45° \Leftrightarrow 90° \Rightarrow \frac{\pi}{2}$

$$Q_1 = [\cos(45°), \sin(45°)\begin{pmatrix}0\\1\\0\end{pmatrix}] = [\cos(45°), \begin{pmatrix}0\\ \sin(45°)\\0\end{pmatrix}] = [\frac{\sqrt{2}}{2}, \begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\0\end{pmatrix}]$$

$$Q_2 = [\cos(-45°), \sin(-45°)\begin{pmatrix}0\\0\\1\end{pmatrix}] = [\cos(-45), \begin{pmatrix}0\\0\\ \sin(-45)\end{pmatrix}] = [\frac{\sqrt{2}}{2}, \begin{pmatrix}0\\0\\ -\frac{\sqrt{2}}{2}\end{pmatrix}]$$

$$Q_a = Q_2 \times Q_1 = [\cos(-45)\cos(45°) - \begin{pmatrix}0\\0\\-\frac{\sqrt{2}}{2}\end{pmatrix} \cdot \begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\0\end{pmatrix},$$

$$\cos(-45)\begin{pmatrix}0\\ \sin(45)\\0\end{pmatrix} + \cos(45)\begin{pmatrix}0\\0\\ \sin(-45)\end{pmatrix} + \begin{pmatrix}0\\0\\ \sin(-45)\end{pmatrix} \times \begin{pmatrix}0\\ \sin(45)\\0\end{pmatrix}]$$

$$s_a = \cos(-45)\cos(45) - \begin{pmatrix}0\\0\\-\frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\0\end{pmatrix} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - (0 \cdot 0 + 0 \cdot \frac{\sqrt{2}}{2} + 0 \cdot (-\frac{\sqrt{2}}{2}))$$

$$= \frac{2}{4} = \frac{1}{2}$$

2/2

$$v_a = \begin{pmatrix}0\\ \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\\0\end{pmatrix} + \begin{pmatrix}0\\0\\ \frac{\sqrt{2}}{2} \cdot (-\frac{\sqrt{2}}{2})\end{pmatrix} + \begin{pmatrix}\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

$$Q_a = [\frac{1}{2}, \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}]$$

b)

Rotationen mit Quaternionen: $Q \cdot P \cdot Q^*$

$$Q = [\frac{1}{2}, \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}], \quad Q^* = [\frac{1}{2}, -\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}]$$

$$Q = (\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j - \frac{1}{2}k)$$

$$P = 0 + 1 \cdot i - 1j + 1k = i - j + k$$

$$Q^* = (\frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k)$$

$$Q * P * Q^* = \left(\tfrac{1}{2} + \tfrac{1}{2} i + \tfrac{1}{2} j - \tfrac{1}{2} k\right)(i - j + k)\left(\tfrac{1}{2} - \tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k\right)$$

$$= \left(\tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k + \tfrac{1}{2} i^2 - \tfrac{1}{2} ij + \tfrac{1}{2} ik + \tfrac{1}{2} ji - \tfrac{1}{2} j^2 + \tfrac{1}{2} jk - \tfrac{1}{2} ki + \tfrac{1}{2} kj - \tfrac{1}{2} k^2\right)\left(\tfrac{1}{2} - \tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k\right)$$

$$= \left(\tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k - \tfrac{1}{2} - \tfrac{1}{2} k - \tfrac{1}{2} j - \tfrac{1}{2} k + \tfrac{1}{2} + \tfrac{1}{2} i - \tfrac{1}{2} j - \tfrac{1}{2} i + \tfrac{1}{2}\right)\left(\tfrac{1}{2} - \tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k\right)$$

$$= \left(-\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} i + \tfrac{1}{2} i - \tfrac{1}{2} i - \tfrac{1}{2} j - \tfrac{1}{2} j - \tfrac{1}{2} j + \tfrac{1}{2} k - \tfrac{1}{2} k - \tfrac{1}{2} k\right)\left(\tfrac{1}{2} - \tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k\right)$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{2} i - \tfrac{3}{2} j - \tfrac{1}{2} k\right)\left(\tfrac{1}{2} - \tfrac{1}{2} i - \tfrac{1}{2} j + \tfrac{1}{2} k\right)$$

$$= \tfrac{1}{4} - \tfrac{1}{4} i - \tfrac{1}{4} j + \tfrac{1}{4} k + \tfrac{1}{4} i - \tfrac{1}{4} i^2 - \tfrac{1}{4} ij + \tfrac{1}{4} ik - \tfrac{3}{4} j + \tfrac{3}{4} ji + \tfrac{3}{4} j^2 - \tfrac{3}{4} jk - \tfrac{1}{4} k + \tfrac{1}{4} ki + \tfrac{1}{4} kj - \tfrac{1}{4} k^2$$

$$= \tfrac{1}{4} - \tfrac{1}{4} i - \tfrac{1}{4} j + \tfrac{1}{4} k + \tfrac{1}{4} i + \tfrac{1}{4} - \tfrac{1}{4} k - \tfrac{1}{4} j - \tfrac{3}{4} j - \tfrac{3}{4} k - \tfrac{3}{4} - \tfrac{3}{4} i - \tfrac{1}{4} k + \tfrac{1}{4} j - \tfrac{1}{4} i + \tfrac{1}{4}$$

$$= \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{3}{4} + \tfrac{1}{4} - \tfrac{1}{4} i + \tfrac{1}{4} i - \tfrac{3}{4} i - \tfrac{1}{4} i - \tfrac{1}{4} j - \tfrac{1}{4} j - \tfrac{3}{4} j + \tfrac{1}{4} j + \tfrac{1}{4} k - \tfrac{1}{4} k - \tfrac{3}{4} k - \tfrac{1}{4} k$$

$$= 0 - 1i - 1j - 1k = \left[0, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right]$$

$$\Rightarrow P' = (-1, -1, -1)^T$$

1/1

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c) $Q = [\cos(\frac{\phi}{2}), w \sin(\frac{\phi}{2})]$

$Q^* = [\cos(\frac{\phi}{2}), -w \sin(\frac{\phi}{2})]$

$Q^* \times Q =$

$$s = \cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) - \begin{pmatrix} x\sin(\frac{\phi}{2}) \\ y\sin(\frac{\phi}{2}) \\ z\sin(\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} -x\sin(\frac{\phi}{2}) \\ -y\sin(\frac{\phi}{2}) \\ -z\sin(\frac{\phi}{2}) \end{pmatrix}$$

$$= \cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + x^2\sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) + y^2\sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) + z^2\sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)$$

$$= \cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(x^2+y^2+z^2)$$

$$= \cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\left(\sqrt{x^2+y^2+z^2}\right)^2$$

$$= \cos\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) = \cos\left(\frac{\phi}{2} - \frac{\phi}{2}\right) = \cos(0) = 1$$

$$v = -\cos\left(\frac{\phi}{2}\right) w\sin\left(\frac{\phi}{2}\right) + \cos\left(\frac{\phi}{2}\right) w\sin\left(\frac{\phi}{2}\right) + \begin{pmatrix} x\sin(\frac{\phi}{2}) \\ y\sin(\frac{\phi}{2}) \\ z\sin(\frac{\phi}{2}) \end{pmatrix} \times \begin{pmatrix} -x\sin(\frac{\phi}{2}) \\ -y\sin(\frac{\phi}{2}) \\ -z\sin(\frac{\phi}{2}) \end{pmatrix}$$

$$= 0 + \Big(-yz\sin^2\left(\tfrac{\phi}{2}\right) + yz\sin^2\left(\tfrac{\phi}{2}\right),\ -xz\sin^2\left(\tfrac{\phi}{2}\right) + xz\sin^2\left(\tfrac{\phi}{2}\right),$$
$$-xy\sin^2\left(\tfrac{\phi}{2}\right) + xy\sin^2\left(\tfrac{\phi}{2}\right)\Big)^T$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad Q^* \times Q = [1, 0, 0, 0]$

Rückrichtung gilt ebenfalls: $Q \times Q^* = [1, 0, 0, 0]$

d) $XZX(\alpha,\beta,\gamma) = R_x(\alpha) R_z(\beta) R(\gamma) \hat{=} Q_{x\gamma} \cdot Q_{z\beta} \cdot Q_{x\alpha}$

$Q_{x\alpha} = [\cos(\frac{\alpha}{2}), \sin(\frac{\alpha}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]$

$Q_{z\beta} = [\cos(\frac{\beta}{2}), \sin(\frac{\beta}{2}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}]$

$Q_{x\gamma} = [\cos(\frac{\gamma}{2}), \sin(\frac{\gamma}{2}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]$

$Q_{xz} = Q_{z\beta} * Q_{x\alpha}$:

$$s = \cos(\frac{\beta}{2})\cos(\frac{\alpha}{2}) - \begin{pmatrix} 0 \\ 0 \\ \sin(\frac{\beta}{2}) \end{pmatrix} \cdot \begin{pmatrix} \sin(\frac{\alpha}{2}) \\ 0 \\ 0 \end{pmatrix} = \cos(\frac{\beta}{2})\cos(\frac{\alpha}{2})$$

$$v = \cos(\frac{\beta}{2}) \begin{pmatrix} \sin(\frac{\alpha}{2}) \\ 0 \\ 0 \end{pmatrix} + \cos(\frac{\alpha}{2}) \begin{pmatrix} 0 \\ 0 \\ \sin(\frac{\beta}{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin(\frac{\beta}{2}) \end{pmatrix} \times \begin{pmatrix} \sin(\frac{\alpha}{2}) \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \sin(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \end{pmatrix}$$

$$\Rightarrow Q_{xz} = [\cos(\frac{\beta}{2})\cos(\frac{\alpha}{2}), \begin{pmatrix} \cos(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \sin(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \end{pmatrix}]$$

$Q_{xzx} = Q_{x\gamma} Q_{xz}$ *

$$s = \cos(\frac{\gamma}{2})\cos(\frac{\beta}{2})\cos(\frac{\alpha}{2}) - \begin{pmatrix} \sin(\frac{\gamma}{2}) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \cos(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \sin(\frac{\beta}{2})\sin(\frac{\alpha}{2}) \\ \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2}) \end{pmatrix}$$

$$= \cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\cos(\frac{\gamma}{2}) - \sin(\frac{\alpha}{2})\cos(\frac{\beta}{2})\sin(\frac{\gamma}{2})$$

$$= \cos(\frac{\beta}{2})\left(\cos(\frac{\alpha}{2})\cos(\frac{\gamma}{2}) - \sin(\frac{\alpha}{2})\sin(\frac{\gamma}{2})\right)$$

$$= \cos(\frac{\beta}{2})\cos(\frac{\alpha}{2} + \frac{\gamma}{2})$$

$$v = \cos\left(\frac{\gamma}{2}\right)\begin{pmatrix}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\\ \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\end{pmatrix} + \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}\right)\begin{pmatrix}\sin\left(\frac{\gamma}{2}\right)\\ 0\\ 0\end{pmatrix}$$

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$$+ \begin{pmatrix}\sin\left(\frac{\gamma}{2}\right)\\ 0\\ 0\end{pmatrix} \times \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\\ \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\\ \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\\ \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\end{pmatrix} + \begin{pmatrix}\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\\ 0\\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}0\\ 0 - \sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\\ \sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\\ \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\\ \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right)\\ \sin\left(\frac{\beta}{2}\right)\left(\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\gamma}{2}\right) - \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right)\\ \sin\left(\frac{\beta}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right)\end{pmatrix}$$

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$$= \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2} + \frac{\gamma}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2} - \frac{\gamma}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2} - \frac{\gamma}{2}\right)\end{pmatrix}$$

$$\Rightarrow Q_{xzx} = \left[\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2} + \frac{\gamma}{2}\right),\ \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2} + \frac{\gamma}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2} - \frac{\gamma}{2}\right)\\ \sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2} - \frac{\gamma}{2}\right)\end{pmatrix}\right]$$