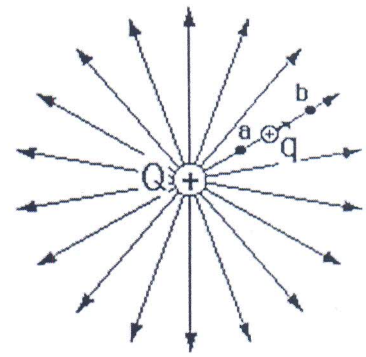
Ms. Eibling

Electrical Potential Energy and Voltage – AP Physics C

Electrostatic ~~Electrical~~ *Potential Energy* Unit (J) Not a vector



Place a small positive charge in an electric Field. The charge field a force, calculated by F= $\frac{kQq}{r^2}$

Since the electrical force is a conservative force, a electrostatic potential energy function can be derived by calculating the work done by the force on the charge as it moves from point a to point b. We will have to use what is called a path intergral here, represented by *ds*.

A path integral is an integration of the parts of the path that are parallel to the electrostatic force.

$d\vec{s}$ is tangent to the Path

The integral for work is:

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{s} = \int_a^b q_0\vec{E}\cdot d\vec{s} = q_0\int_a^b \vec{E}\cdot d\vec{s}$$

The integral for the particle's change in potential energy is:

$$\Delta U = -W_{a\to b} = -q_0\int_a^b \vec{E}\cdot d\vec{s}$$

if $a=\infty$ and $b=R$ $U = \frac{k q_1 q_2}{R}$

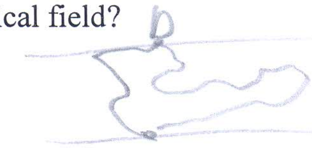
Does the path that the particle takes matter? Why or why not?

F_q is a conservative force analagous to gravitational force

Path does not matter

Would the path matter in a uniform electrical field?

No



Electrical Potential

Work per unit charge to go from ∞ to P

Electrical Potential, or ////, is a way of calculating the amount of energy per charge that a charge can carry. We calculate this as potential ENERGY divided by charge, and say that the change in Electric Potential is the change in electrostatic Potential energy divided by the charge of the test charge.

This can be derived from electrical potential energy:

$V_p = -\int_\infty^p \frac{\vec{F}_e}{q_0} \cdot d\vec{r} = -\int_\infty^p \vec{E} \cdot d\vec{s} = \frac{kQ}{R}$

$V_{\infty \text{ to } R} = \frac{U}{q_0} = \frac{kQ}{R}$

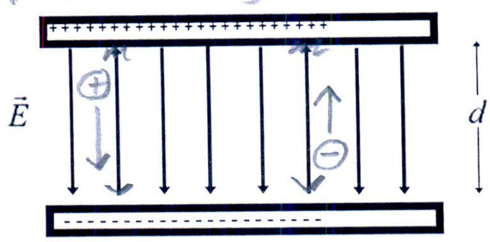
(+) $+V$ (−) $-V$

$$\Delta V = \frac{\Delta U}{q_0} = -\int_a \vec{E} \cdot d\vec{s}$$

also called V in circuits.

Electrical Potential in a Uniform Electric Field

Ex: Decreasing Electric Potential



Which are the two cases in which potential energy INCREASES in a uniform electrical field?

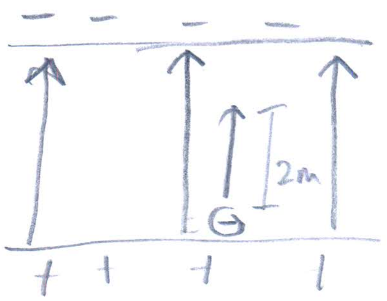
The electrical potential change between the plates can be integrated:

$$\Delta V = \frac{\Delta U}{q_0} = -\int_a^b \vec{E} \cdot d\vec{s} = -E\int_0^d d\vec{s} \Rightarrow \Delta V = -Ed$$

Thus, the electrostatic potential ENERGY between the plates becomes:

$$\Delta U = q_0 V = -q_0 E d$$

A charge of -6.0 x 10-9 C is placed in a uniform Electric Field of magnitude 1.5 x 10-2 N/C directed straight up. If the charge is moved 2.0 m upward by an external force, the work done by the electric field on the charge is:

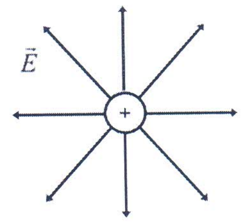


$E = 1.5 \times 10^{-2}$ N/C

$$W = \Delta U = -qEd = -(-6.0 \times 10^{-9})(1.5 \times 10^{-2} \tfrac{N}{C})(+2m)$$

$$= 1.8 \times 10^{-10}\ J$$

Electrical Potential of a Point Charge



The electrical field of a positive point charge extends outward from the charge and is equal to:

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

To find the electrical potential at a distance r in any direction from the charge, we perform the line integral from point a to point b as follows:

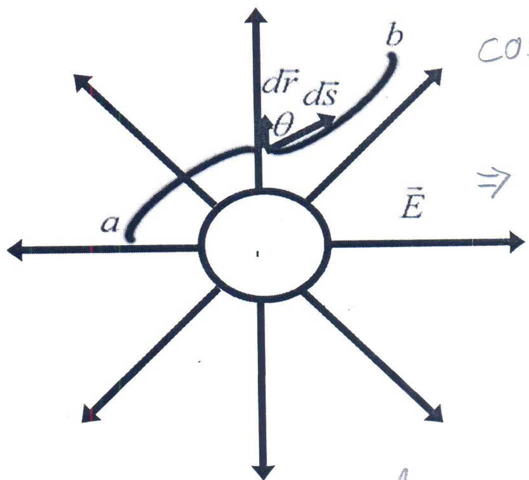
Path parallel to E-field,

$\hat{r}\cdot d\vec{s} = ds\cos\theta$

$\cos\theta = \frac{dr}{ds}$

$\Rightarrow \hat{r}\cdot d\vec{s} = ds\frac{dr}{d\theta} = dr$

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{s}$$



$$V_b - V_a = -k\int_a^b \frac{q}{r^2}\hat{r}\cdot d\vec{s}$$

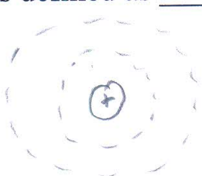
Substitute $\hat{r}\cdot d\vec{s} = dr$

$$= -\int_a^b k\frac{q}{r^2}\,dr = k\frac{q}{r}\Big|_{r_a}^{r_b}$$

$$V_b - V_a = kq\left(\frac{1}{r_b} - \frac{1}{r_a}\right) \quad \text{if } r_b \text{ or } r_a = \infty \quad V = \frac{kq}{r}$$

By convention, the electric potential assumes that point a is at ∞ when the potential is defined as 0 so that the potential at any point that is a distance r from the charge is:

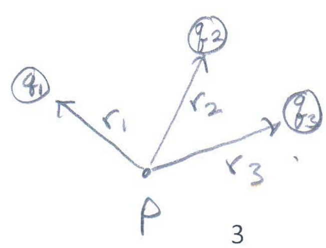
Equipotential surfaces.



$$V = \frac{kq}{r}$$

Electric potential at a point from multiple point charges is determined using superposition

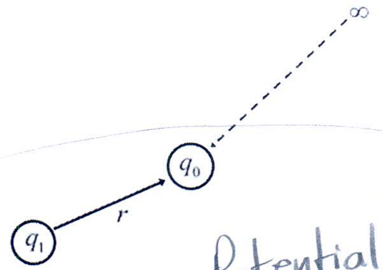
$$V_p = k\sum\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \dots\right]$$



Electrostatic ~~Electric~~ potential energy of multiple charges

Start with a positive charge q_1, which generates an electrical potential of: $V = \frac{kq_1}{r}$

From infinity, bring in another charge q_0 a distance r from the first. Calculate the electrical potential energy.



$$U = q_0 V = k\frac{q_1 q_2}{r}$$

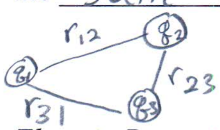
Potential between 2 q's.
Analogous to potential E btw earth & obj.

Since both charges are positive, the electric field of q_0 is doing ___−___ work on q_1, and the external force is doing positive work on q_1, increasing potential energy.

The same is true if both charges are ___−___.

If the charges are of opposite sign, the work done in bringing them closer together is ___−___, and this results in a decrease of potential energy.

If there are three or more charges, then the total Electric Potential Energy of the configuration is the sum of the pir wise potential energies.



$$U = k\sum_{i<j}\frac{q_i q_j}{r_{ij}} = k\left[\frac{q_1 q_2}{r_{12}} + \frac{q_{13}}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Electric Potential of Continuous Charge Distributions

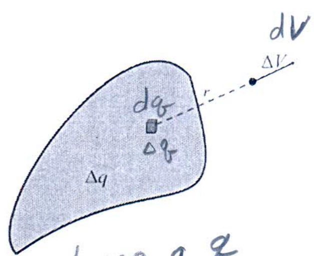
Consider a shape with a uniform distribution of charge, and find the Electric Potential at a distance r, from the shape. This is easier than calculating the Electric Field, as we don't have to worry about vectors.

$$\boxed{V = \frac{kQ}{r}} \Rightarrow dV = \frac{k\,dq}{r}$$

$$\Delta V = k\frac{\Delta q}{r}$$

$$V = k\lim_{\Delta q_i \to 0}\frac{\Delta q_i}{r_i}$$

$$\boxed{V = k\int\frac{dq}{r}}$$



loose a q →

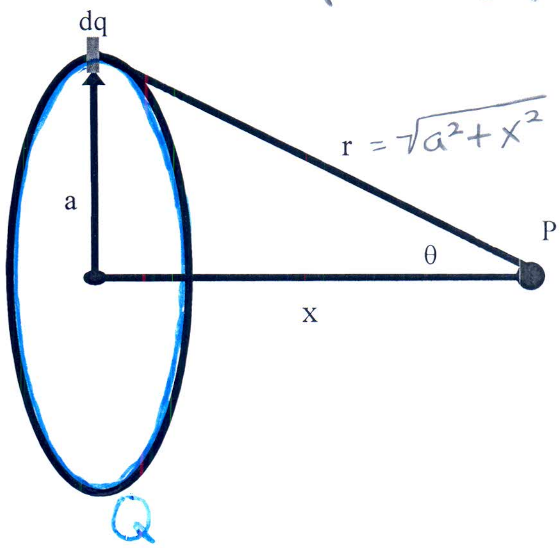
Loose an r ↓

$\vec{F} = k\frac{\lvert q_1\rvert\lvert q_2\rvert}{r^2}\hat{r}$	$\vec{E} = k\frac{\lvert q_1\rvert}{r^2}\hat{r}$	Vector
$U = k\frac{q_1 q_2}{r}$	$V = k\frac{q_1}{r}$	Scalar

Uniform Ring of Charge

$$V = \frac{kQ}{r} = k\int \frac{dq}{r}$$

Consider a uniformly positively charged ring of radius a, with total charge Q. Find the Electric Potential at point P, a distance x away from the center of the ring:



Derive an expression for r:

$$r = \sqrt{a^2+x^2}$$

Derive an expression for the charge on the ring and integrate in terms of dq.

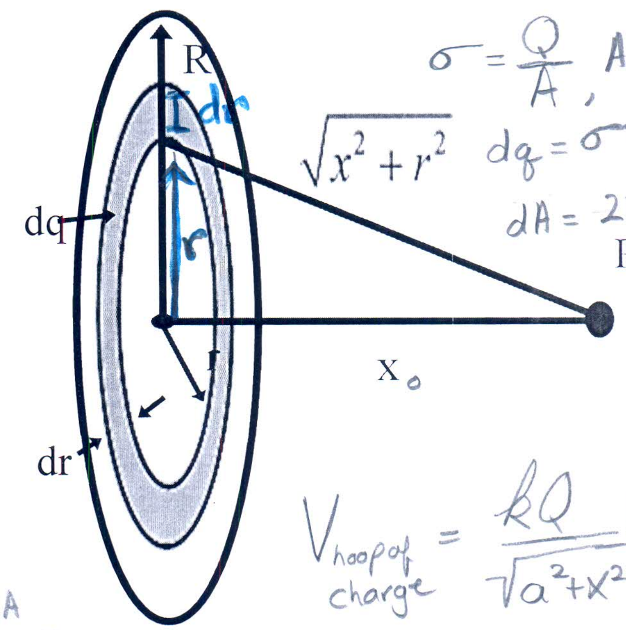
$$V = k\int \frac{dq}{r} = k\int \frac{dq}{\sqrt{a^2+x^2}}$$

$$= \frac{k}{\sqrt{a^2+x^2}}\int dq = \frac{kQ}{\sqrt{a^2+x^2}}$$

as $a \to 0$, $x \simeq r$ and $V = \frac{kQ}{\sqrt{r^2}} = \frac{kQ}{r}$

Uniform Disc of Charge

Calculate the Electric Field at point P, along the center axis of a uniform disc of radius R and having a uniform positive charge. Evaluate the disc as a series of concentric rings of charge, dq, centered on its center axis (along the x axis). Remember to use surface charge density.



$\sigma = \frac{Q}{A}$, $A = \pi R^2$

$dq = \sigma\, dA$

$dA = 2\pi r\, dr$

$$V_{\text{hoop of charge}} = \frac{kQ}{\sqrt{a^2+x^2}}$$

$$dV = \frac{k}{\sqrt{r^2+x_0^2}}\, dq$$

$$V = k\int \frac{\overbrace{\sigma\, dA}^{dq}}{\sqrt{r^2+x_0^2}} = k\int \frac{\sigma \overbrace{[2\pi r\, dr]}^{dA}}{\sqrt{r^2+x_0^2}}$$

$$= \pi\sigma k \int_0^R \frac{2r\, dr}{\sqrt{r^2+x_0^2}} = \pi\sigma k \int u^{-1/2} du = \pi\sigma k \left| \frac{u^{1/2}}{1/2} \right. = 2\pi\sigma k \left| \sqrt{r^2+x_0^2} \right|_0^R$$

$u = r^2 + x_0^2$

$du = 2r\, dr$

$$V = \boxed{2\pi\sigma k\left[\sqrt{R^2+x_0^2} - x_0\right]}$$

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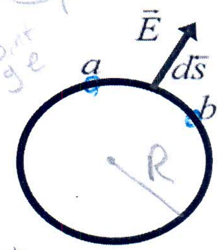
Spherical Conductor

Recall that the Electric Field due to a conducting sphere is as shown below. All of the charge resides on the surface, and the Electric Field is perpendicular to the surface at every point and is zero within the sphere. The Electric Potential difference between any two points a and b on the surface of the conductor is then:

Ex Vanda Graph $R = .3$ $+10\mu C$ behaves like point charge

$$V_p = -\int_{\infty}^{r} \vec{F}_E \cdot d\vec{r} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = -\int_{\infty}^{r} \frac{kQ}{r^2} \cdot dr = \frac{kQ}{r}$$



$$\Delta V = -\int_{0}^{b} \vec{E} \cdot d\vec{s} = 0$$

$\vec{E}$ is $\perp$ to $d\vec{s}$.

The electric potential is constant everywhere on the surface, it is an equipotential surface.

$$V = \frac{kQ}{R}$$

The electrical field within a conductor is 0. Choose one point inside the sphere and one on the surface, and integrate.

$$\Delta V = -\int_{0}^{R} \vec{E} \cdot d\vec{s} = 0$$

The electrical potential within the conductor is equal to that on the surface of the conductor.

Deriving electrical field from electric potential

Supposing that you have the equation $\Delta V = -\int_a^b \vec{E} \cdot d\vec{s}$ and we want to find the electrical field by measuring voltages.

Take the derivative of both sides and switch to Cartesian coordinates to find the electric field along the x. You can do similar things for the y and z axes. You get:

$$E_x = \frac{dV}{dx}$$

If you want to do this in 3 dimensions, you need a partial derivative and some vector calculus, which looks like this:

Gradient

$$\vec{E} = -\vec{\nabla} V = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V \qquad \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ Del operator

P2

$\vec{\nabla} V$ Gradient of V

We're not doing this in this class, but if you're doing any engineering at all, you will do this later, and this will help you connect what we do here to what you will do there.

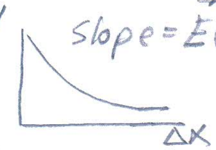
Let's just deal with 2 dimensions:

For a given Electric Potential (Voltage) map, the Electric Field is found by:

Connecting all points of equal potential - these are called Equipotential Lines.

The Electric Field lines are perpendicular to the equipotential lines and point in the direction of decreasing potential (the negative sign).

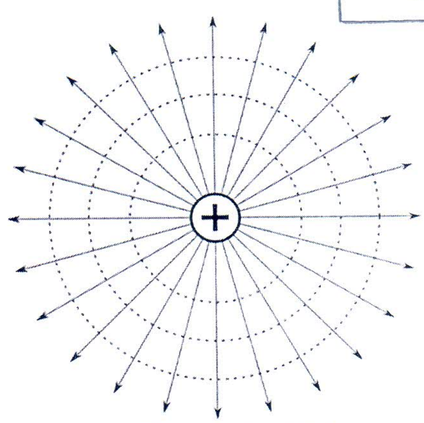
The slope of the Voltage vs. position graph is the Magnitude of the Electric Field.



Slope = $\vec{E}$ field.

The Electric Field is perpendicular to the Equipotential lines because ΔV = 0 along the Equipotential lines. This is written mathematically:

$$E_x = -\frac{dV}{dx}$$



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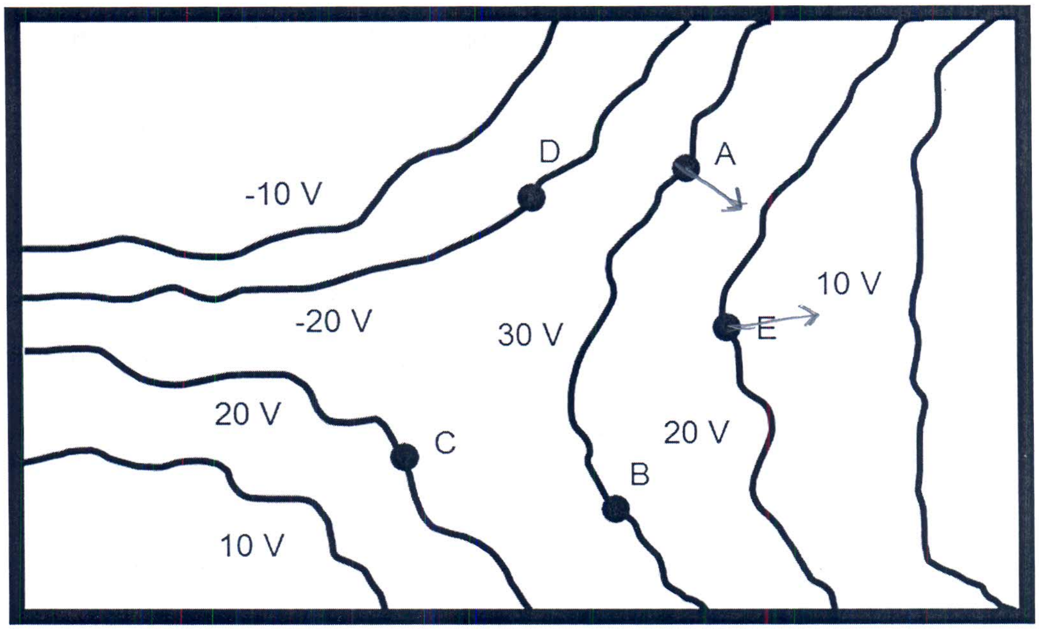
$V = \frac{kQ}{r}$

$E_x = -\frac{dV}{dx}$

Draw the electric field vector at points A-E.



A test charge of magnitude q moves from point A to point B. What is the work done on the particle by the Electric Field?

0 $\int d\vec{s}$ along field line is $\perp$ to E

$W = -\Delta E = -q_0 \Delta V$ or $W = -\Delta U = -\int \vec{F_e} \cdot d\vec{s} = q \int \vec{E} \cdot d\vec{s} = 0$

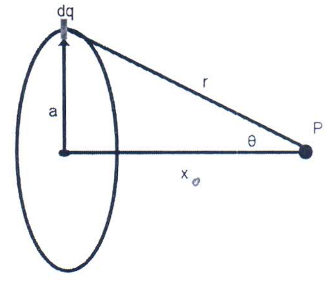
A test charge of magnitude q moves from point E to point A. What is the work done on the charge by the Electric Field?

$$W = -q\Delta V = -q[V_f - V_i] = -q[30V - 20V]$$

$$= -10q \ [J]$$

Earlier in this unit, the Electric Potential for a uniformly charged ring was calculated at point P. Calculate the Electric Field at point P using Voltage.

$$E_x = -\frac{dV}{dx}, \quad V = \frac{kQ}{\sqrt{a^2+x^2}}$$



$$E_x = -\frac{d}{dx}\left[kQ(a^2+x^2)^{-\frac{1}{2}}\right] = -\left[-\frac{1}{2}kQ(a^2+x^2)^{-3/2} \cdot 2x\right]$$

$$E_x = \frac{+kQx}{(a^2+x^2)^{3/2}}$$

Capacitance

$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Qd}{A\varepsilon_0}} =$ $d \updownarrow$

$E = \frac{Q}{A\varepsilon_0}$

$V_{ab} = -\int E\,dr = -\frac{Q}{A\varepsilon_0}d$

A capacitor can be made by putting two conductors near each other, separated by an insulator, such as air, rubber, or any other material. Each conductor has an equal magnitude, but opposite charge, so the net charge across the two conductors is 0.

P2

The amount of charge stored on each conductor has been found experimentally to be proportional to the ΔV between them. This proportionality constant is called Capacitance and is represented as follows:

$$C = \frac{Q}{\Delta V}$$

1 Farad (F) = 1 Coulomb/Volt

The unit of capacitance is named after Michael Faraday and is equal to :

1 Farad (F) = 1 Coulomb/Volt. A coulomb is a huge unit of charge, so typically smaller charges are stored, resulting in anywhere between micro and pico-Farad capacitances. $\mu F = 10^{-6} F$ $pF = \times 10^{-12} F$

Capacitors are charged by connecting opposite ends of a battery to each conductor. Once the plates have achieved a voltage (potential difference) equal and opposite to the battery, current stops and the capacitor is fully charged (Q). The charge stays on the capacitor even after the battery is disconnected. This makes them quite dangerous.

Capacitors are used to store energy in their E-field, and are drawn in circuit diagrams as:

We calculated the simplest version of a capacitor, which is the parallel plate capacitor, which has an electric field of:

$$E = \frac{\sigma}{\varepsilon_0}$$

The electric field outside the capacitor is 0.

Calculate the charge density, σ, on either plate.

Take the standard formula for voltage between parallel plates and substitute the formula derived above into it.

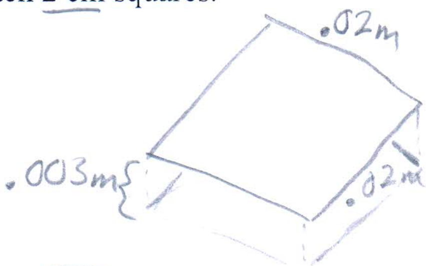
$$\boxed{\sigma = \frac{Q}{A}}$$

$$\Delta V = Ed = \left[\frac{\sigma}{\varepsilon_0}\right] d = \frac{Q}{\varepsilon_0 A}\cdot d$$

Substitute the relationship you just found for electric potential into the capacitance formula:

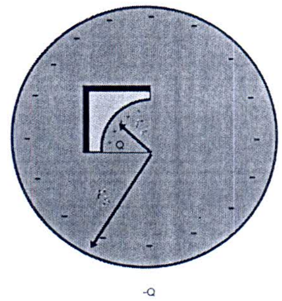
$$C = \frac{Q}{\Delta V} = \frac{Q}{\left[\frac{Q}{\varepsilon_0 A}\cdot d\right]} = \boxed{\frac{\varepsilon_0 A}{d}}$$

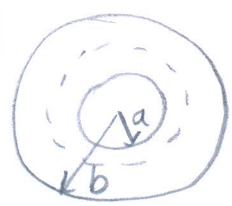
23. The plates of a parallel plate capacitor are 3.0×10^{-3} m apart and they are each 2 cm squares. What is the capacitance of the capacitor? $\varepsilon_0 = 8.85x10^{-12}\ F/m$



$$C = \frac{\varepsilon_0 (.02)^2}{.003}$$

Another common capacitor design is a positively charged sphere within a negatively charged sphere, shown on the right. The Electric Field at a distance r, where ra < r < rb was found earlier, using Gauss's Law and equals: $SA = 4\pi r^2$





$$EA = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \boxed{E = \frac{Q_{enc}}{A\varepsilon_0}} = \frac{Q_{enc}}{4\pi r^2 \varepsilon_0} = \boxed{\frac{kQ}{r^2}}$$

Note that only the enclosed +Q charge contributes to the field. -Q is outside this volume (charge resides on the surface of the larger sphere) and does not add to the internal field.

Calculate the potential difference in the region between the spheres:

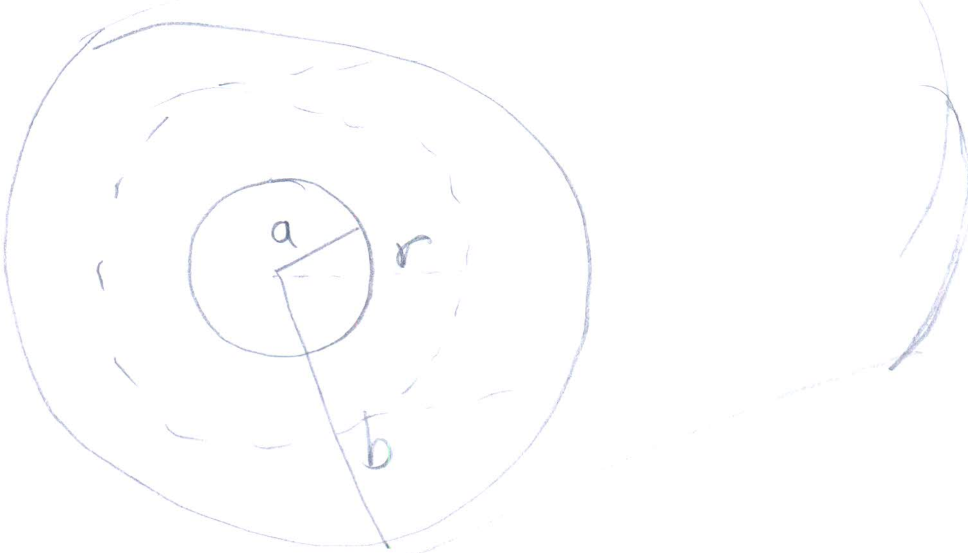
$$\boxed{V_{ab} = -\int_a^b E\,dr} = -\int_a^b \frac{kQ}{r^2}\,dr = -kQ\int r^{-2}\,dr = -kQ\left|\frac{r^{-1}}{-1}\right| = kQ\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$= \boxed{kQ\left(\frac{a-b}{ab}\right)}$$

Find the capacitance of this configuration:

$$\boxed{C = \frac{Q}{V_{ab}}} = \frac{Q}{kQ\left(\frac{a-b}{ab}\right)} = \boxed{\frac{ab}{k(a-b)}}$$

What is the capacitance for a cylindrical capacitor?



1) Find E using Gauss $\oint E\,da = \frac{Q}{\varepsilon_0}$

2) Find V_{ab} using E

3) $C = \frac{Q}{V_{ab}}$

1) $\boxed{EA = \frac{Q}{\varepsilon_0}} \Rightarrow \boxed{E} = \frac{Q}{A\varepsilon_0} = \frac{Q}{2\pi r L \varepsilon_0} = \boxed{\frac{kQ}{2rL}}$

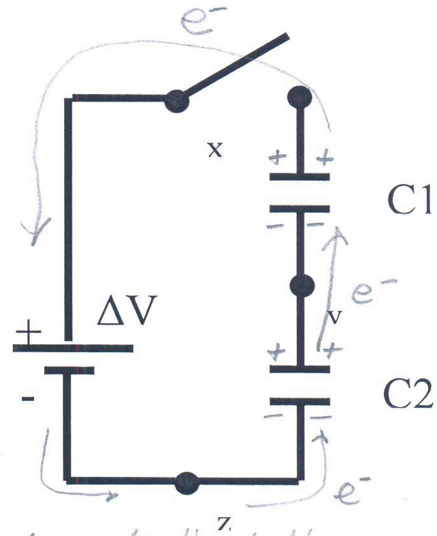
$A = 2\pi r L$ $\qquad a < r < b$

2) $\boxed{V_{ab}} = \int dV = \boxed{-\int E\,dr} = -\int_a^b \frac{kQ}{2rL} = -\frac{kQ}{2L}\int_a^b \frac{1}{r}\,dr$

$= -\frac{kQ}{2L} \Big|_a^b \ln r = \boxed{\frac{kQ}{2L}\left[\ln\frac{a}{b}\right]}$

3) $\boxed{C = \frac{Q}{V_{ab}}} = \frac{Q}{\frac{kQ}{2L}\ln\left(\frac{a}{b}\right)} = \boxed{\frac{2L}{k\ln\frac{a}{b}}}$

Capacitors in Parallel and in Series



What can we assume the charge on capacitors C1 and C2 to be at this point?

$Q = 0$

Describe what will happen when the switch is closed. e⁻s move from the - terminal to the bottom of C2. e⁻s also move from the top of C2 to the bottom of C1. e⁻s move from the top of C1 towards the battery. This stops when ΔV accross both capacitors = ΔV battery.

Magnitude of the charge on each capacitor is equal to the charge supplied by the ~~battery~~. circuit.

If the 2 real capacitors were replaced by 1 virtual capacitor of the same capacitance as both combined, derive the capacitance of this capacitor.

$$\frac{Q}{C_{equ}} = \Delta V = \Delta V_1 + \Delta V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left[\frac{1}{C_1} + \frac{1}{C_2}\right]$$

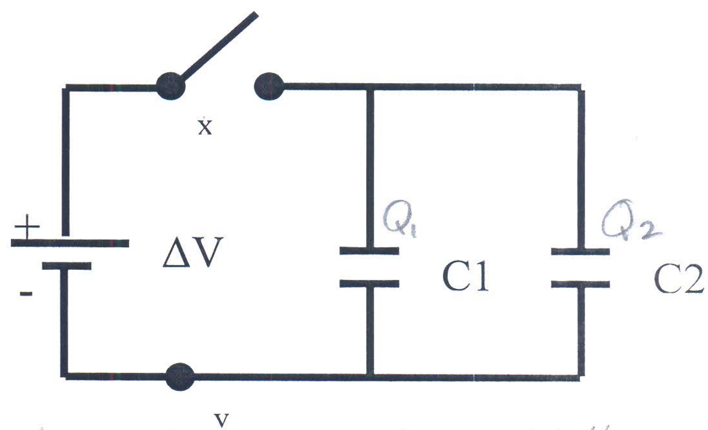
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Assume Q is the same on all capacitors.

So the general equation for capacitors in Series is written:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$$

Now let's take another situation, two capacitors in parallel.



After the switch is closed, the voltage across each capacitor will equal the battery.

What is Ceq for capacitors in parallel?

Write the equation for charge on each capacitor:

$C = \frac{Q}{\Delta V} \Rightarrow$ $Q_1 = C_1 \Delta V$ $\quad Q_2 = C_2 \Delta V$

Derive an expression for conservation of charge in this circuit. An equivalent capacitor will have to have the charge ~~of both~~ accross it.

$$Q_{net} = Q_1 + Q_2 = C_1 \Delta V + C_2 \Delta V = (C_1 + C_2)\Delta V$$

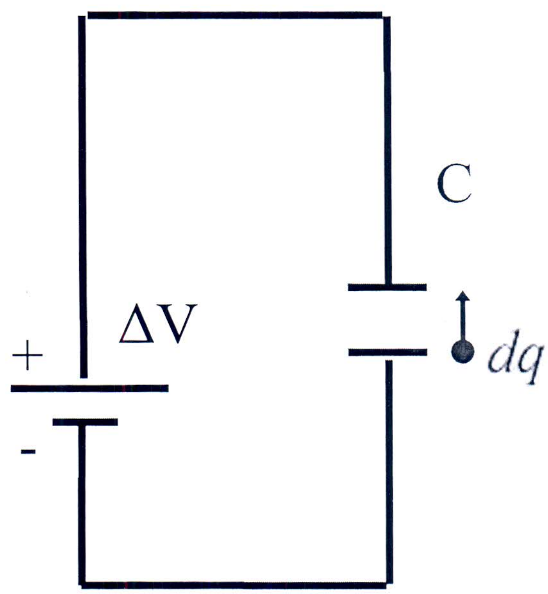
If we replaced both capacitors with 1 equivalent capacitor, derive an expression for its capacitance:

$$C_{eq} = C_1 + C_2$$

The general law for net capacitance of capacitors in parallel is:

$$C_{eq} = C_1 + C_2 + C_3 + \dots$$

Energy Stored in a Capacitor



Let us assume that we are charging this capacitor from 0 charge. Each small charge put on the capacitor is represented by *dq*.

How much work does it take to move the first *dq* to the other side of the capacitor?

The next *dq* now is opposed by *dq* worth of charge, and so on as we continue to charge it, so it takes more and more work to move the charges as the capacitor gets charged.

Derive an expression for the work done to move any charge *dq*. $W = \underbrace{\Delta E}_{\text{electrostatic energy}} = q\Delta V \Rightarrow dW = \Delta V \cdot dq$ for a capacitor $C = \frac{q}{\Delta V} \Rightarrow \Delta V = \frac{q}{C}$

$$\int dW = \int_0^Q \frac{q}{C}\,dq = \left.\frac{q^2}{2C}\right|_0^Q = \boxed{\frac{Q^2}{2C}} = \boxed{W}$$

Then integrate it for all charges moved from 0 to Q to find the work done by the battery.

$$U = W = \frac{Q^2}{2C}$$

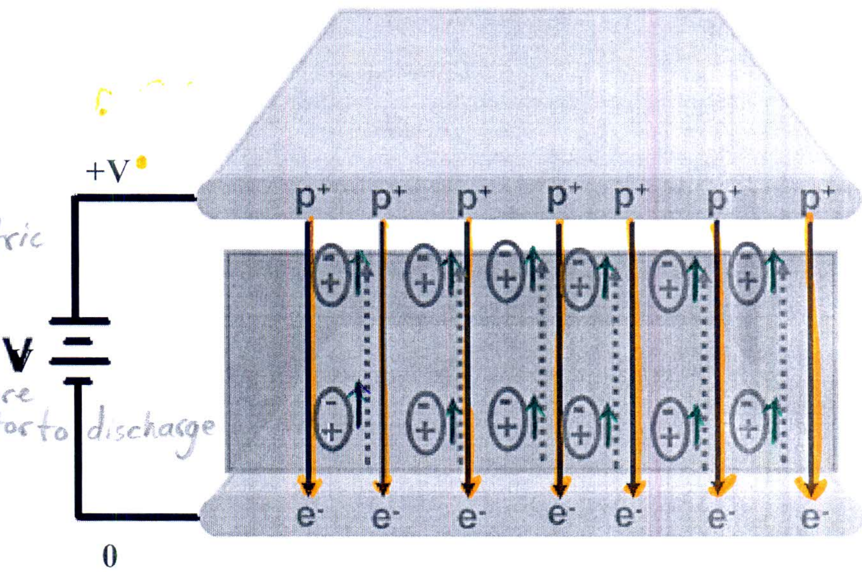
Substitute in Q=CV to find the two alternate expressions for energy in a capacitor.

$$U_{el} = \frac{(\overbrace{C\Delta V}^{Q})^2}{2C} = \boxed{\frac{C \cdot \Delta V^2}{2}} \qquad U_{el} = \frac{Q[C\Delta V]}{2C} = \boxed{\frac{Q \cdot \Delta V}{2}}$$

Dielectrics

Dielectrics are insulating materials which are placed between the positive and negative conductors of a capacitor. They help to maintain the shape of the capacitor, preventing the oppositely charged conductors from coming in contact with one another, which would result in a discharge of the stored energy. They allow the capacitor to reach a higher potential difference then it could normally before dielectric breakdown, which is the ionization of the air around the capacitor which would result in charge leaving the capacitor. They also enable the capacitor to increase its capacitance - its ability to store more charge per a given applied Electric Potential difference.

What will be the effect of the dielectric on the electric field in the diagram on the right?



E-field created by dielectric opposes E-field created by battery. ⇒ F_q is less for the same amount of charge & more F_q is required for the capacitor to discharge
⇒

This will allow the capacitor to store more energy per unit charge, thus increasing its potential difference (ΔV) and capacitance (C)

The dielectric constant is denoted by K and is defined as the ratio of the capacitance with dielectric to the capacitance w/o dielectric.

$$K = \frac{C}{C_o}, \quad C_{o\,\text{parallel plate}} = \frac{\varepsilon_o A}{d}$$

The capacitance equation with a dielectric is thus now:

$$C = K \underbrace{\frac{\varepsilon_o A}{d}}_{C_o}$$

The plates of a parallel plate capacitor are 5×10^{-4} m apart and their width and height are 2 cm and 3 cm respectively. Afterwards, plexiglas, a dielectric, is placed between the plates. What is the capacitance of the capacitor before and after the dielectric is added? $\kappa_{plexiglas} = 3.4$

$$C_o = \frac{\varepsilon_o A}{d} = \frac{(8.85\times10^{-12}\,F/m)(.02\,m)(.03\,m)}{5\times10^{-4}\,m} = 1.062\times10^{-11}\,F$$

$$C = K \cdot C_o = 3.4(1.062\times10^{-11}F) \cong 3.61\times10^{-11}F$$