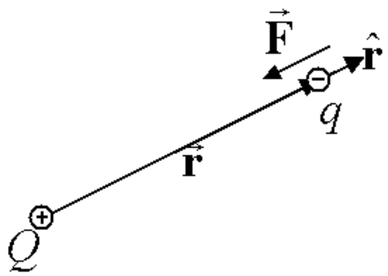
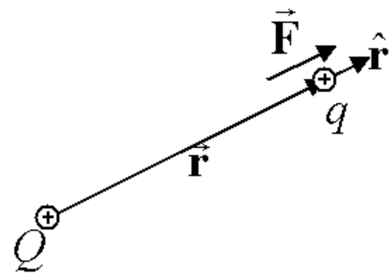
Name: _____[ANSWERS]_____

Unit C07 – Electrostatics and Gauss Law
Lecture Notes

The Coulomb Force and Coulomb's Law (22-1 and 22-3)





The coulomb force is an *attractive* force if the two charges have *opposite* sign.

The coulomb force is a *repulsive* force if the two charges have *like* sign.

Write two *vector* expressions for the force between two charges Q and q separated by a distance vector **r**.

$$\vec{\mathbf{F}} = \frac{kQq}{r^2}\hat{\mathbf{r}} \qquad \vec{\mathbf{F}} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}}$$

The value of k is $k = 9 \times 10^9$ N•m^2/C^2 and it is called the Coulomb's Law constant.

The value of ε_0 is $\varepsilon_0 = 8.85 \times 10^{-12}$ N•m^2/C^2 and it is called the permittivity of free space.

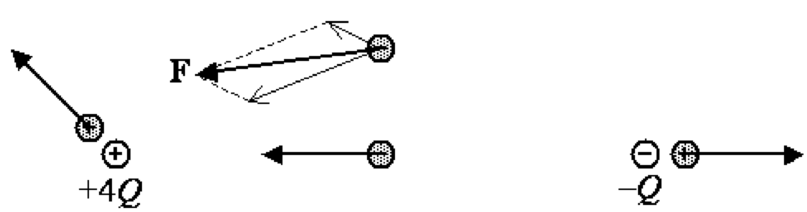
In the diagrams above, which charge has this force acting on it? Both, with equal force.

The elementary charge e is the charge carried by a proton or an electron. Its value is e = 1.9×10^{-19} coulombs.

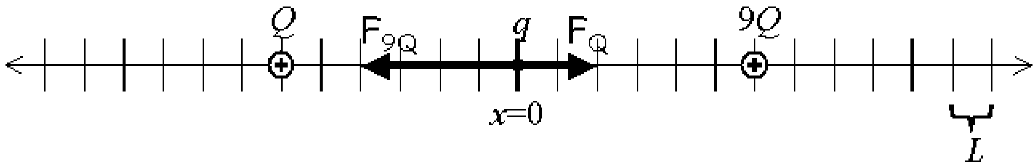
Coulomb's Law looks very much like Newton's Law of Universal Gravitation except that Coulomb's Law lacks a negative sign in front, electric charges can be negative, and the coulomb force can be attractive.

Forces from Multiple Charges (22-3)

The two white charges shown below are fixed in place. A third charge can be located at any of the gray circles. Draw an arrow representing the force acting on a charge (of the indicated sign) at each gray point.



The charge q shown is placed between two other charges Q and $9Q$, on the x-axis. Each tick mark is a distance L apart. Write an expression for the net force on charge q in terms of q, Q, L, and fundamental constants.



$F_Q = \frac{kq(Q)}{(6L)^2} = \frac{1}{36}\frac{kqQ}{L^2}$ and $F_{9Q} = \frac{kq(9Q)}{(6L)^2} = \frac{1}{4}\frac{kqQ}{L^2}$

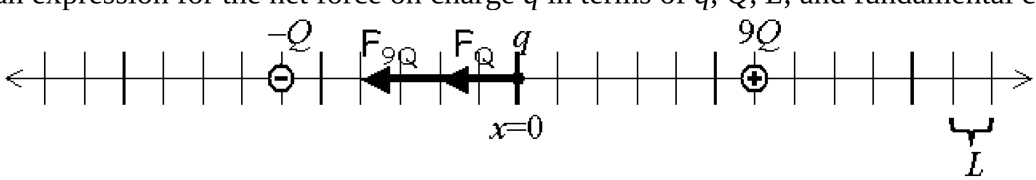
The two forces point in opposite directions (no matter the sign of q or Q) so subtract:

$$F_{net} = F_{9Q} - F_Q = \frac{2}{9}\frac{kqQ}{L^2}$$

The original location of charge q is $x = 0$. The charge q is now moved to a location on the x-axis where it experiences no net force. What is the x-value of this location?

This location must be between Q and 9Q for the forces to be in opposite directions and cancel. For the two forces to have equal magnitudes (and cancel), the 9Q force must be 3 times farther away than the 1Q. This position is at x = –3L.

The charge q shown is placed between two other charges $-Q$ and $9Q$, on the x-axis. Each tick mark is a distance L apart. Write an expression for the net force on charge q in terms of q, Q, L, and fundamental constants.



Merely changing the sign will make $F_Q = \frac{1}{36}\frac{kqQ}{L^2}$ and $F_{9Q} = \frac{1}{4}\frac{kqQ}{L^2}$ still, but the directions are now the same:

$$F_{net} = F_{9Q} + F_Q = \frac{5}{18}\frac{kqQ}{L^2}$$

The original location of charge q is $x = 0$. The charge q is now moved to a location on the x-axis where it experiences no net force. What is the x-value of this location?

Now this location must be to the left of Q for the push away/pull towards forces to be in opposite directions and cancel, and because we need the lesser charge to be closer. Again, for the two forces to have equal magnitudes, the 9Q force must be 3 times farther away than the 1Q. This position is at x = –12L.

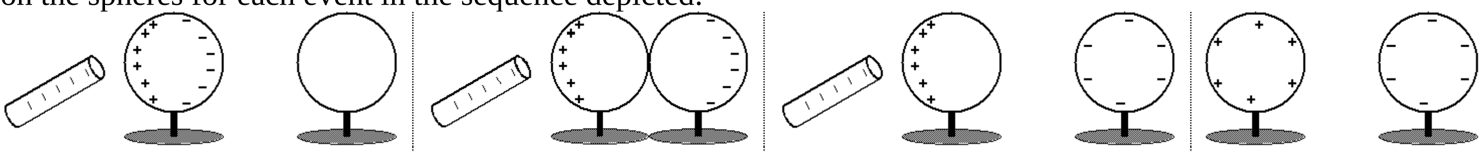
Conductors and Charging By Induction (22-2)

An electrical conductor is a material in which charges are free to move around in the material.

Conversely, an insulator is a material where charges are fixed in place and cannot move around.

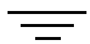
Excess charge on a conductor always resides where on the conductor? The outside surface.

Two neutral metal spheres on insulating stands are initially separated. Draw + and – signs for charge induced on the spheres for each event in the sequence depicted:

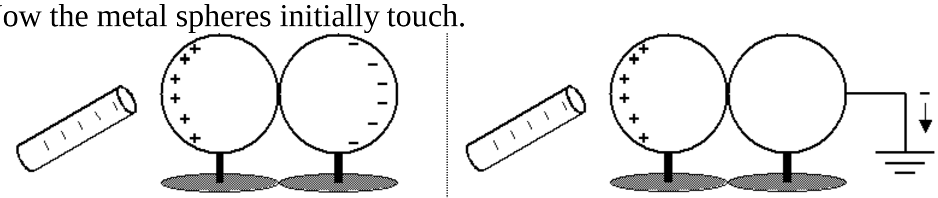


1. A negatively charged rod is brought near the left sphere.
2. With the rod in place, the spheres are made to touch.
3. The spheres are separated with the rod still in place.
4. The rod is now removed.

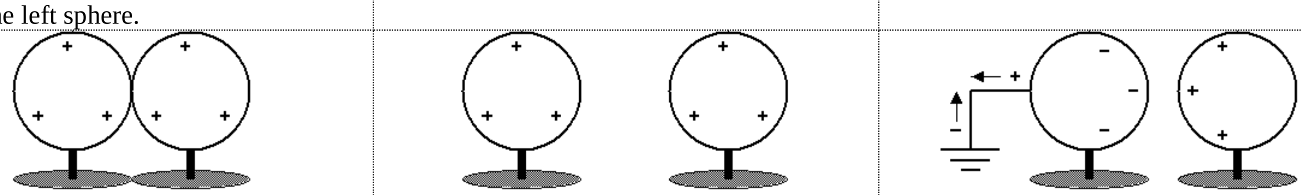
What is "ground"? any location where charges can go if they want to escape, or where charges can come in from if they want to enter the situation.

Draw the symbol for ground: 

Now the metal spheres initially touch.



1. A negatively charged rod is brought near the left sphere.
2. The right sphere is connected to ground.
3. The connection to ground is removed.



4. The negatively charged rod is removed.
5. The spheres are now separated.
6. The spheres are brought close, and the left sphere is connected to ground.

The Electric Field (22-4)

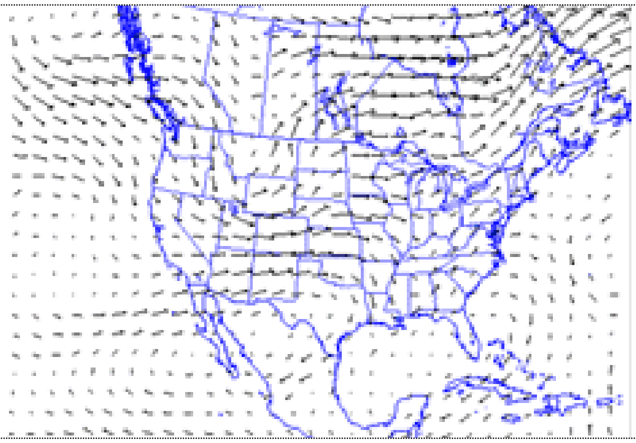
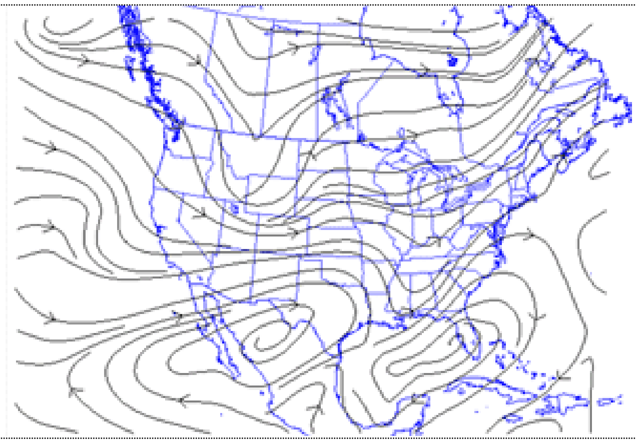
"Action at a distance" is when two objects exert forces on each other even though they are separated by a significant distance (i.e. they are not touching).

"Action at a distance" is explained by the idea that certain objects create a field in the space around them.

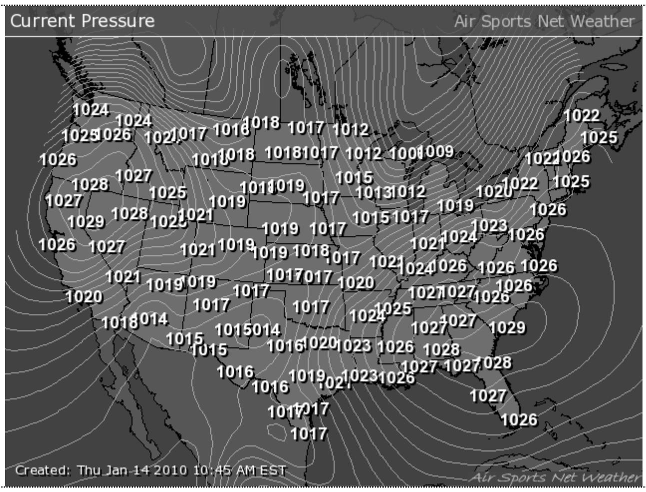
An object with mass creates a gravitational field that affects other masses by causing them to experience a gravitational force.

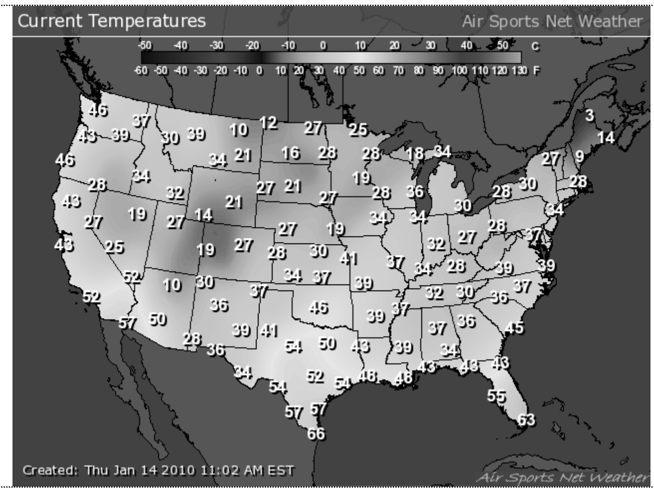
An object with charge creates an electric field that affects other charged objects by causing them to experience an electrostatic force.

Both electric and gravitational fields are called vector fields since there is both a field strength and a field direction associated with the field at every point. In meteorology, wind velocity is a vector field.

There are also such things as scalar fields, which only have a magnitude at each point. In meteorology, the pressure and temperature of the air are scalar fields.





The strength of the gravitational field at a point is defined as the ratio of the gravitational force on an object at that point to the mass of the object at that point.

The direction of the gravitational field at a point is defined as the direction that a (positive) mass would experience the gravitational force.

The basic equation for the force on a mass in a gravitational field: F = mg

The strength of the electric field at a point is defined as the ratio of the electric force on a charged object at that point to the charge of the object at that point.

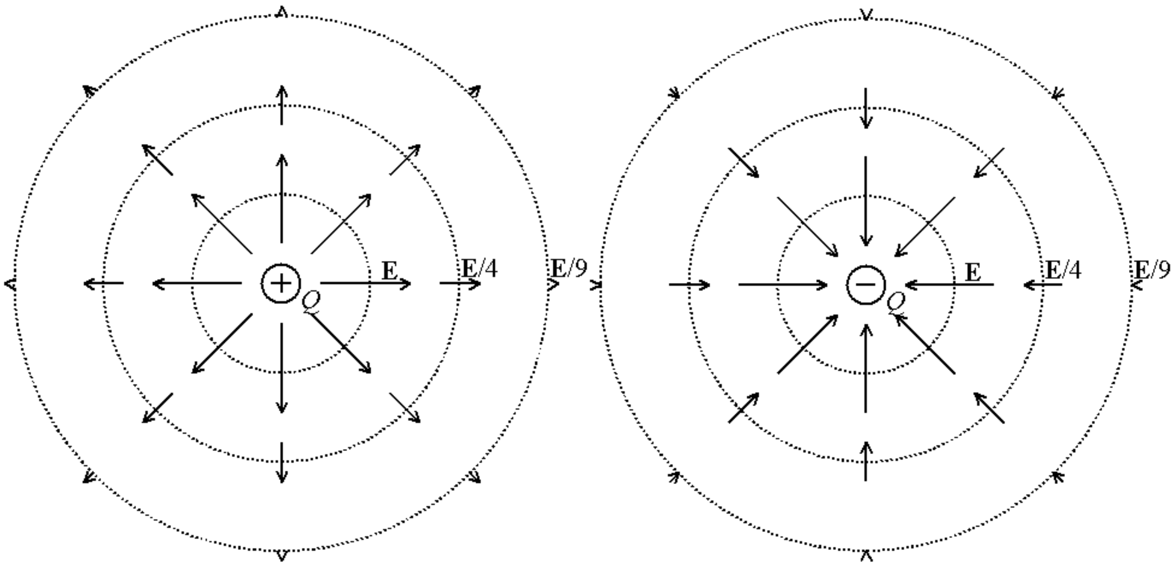
The direction of the electric field at a point is defined as the direction that a positive charge would experience the electric force.

The basic equation for the force on a charge in an electric field: F = qE

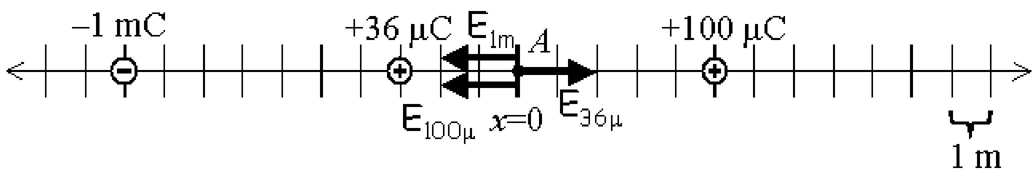
Using the above definition of electric field, write an expression for the electric field strength a distance r from a charge Q.

$$F = qE = \frac{kqQ}{r^2}$$

$$E = \frac{kQ}{r^2}$$



Determine the electric field (magnitude and direction) at point A in the diagram shown below.



$$E_{1m} = \frac{(9\times10^9)(1\times10^{-3})}{(10)^2}$$ = 90,000 N/C left

$$E_{36\mu} = \frac{(9\times10^9)(36\times10^{-6})}{(3)^2}$$ = 36,000 N/C right

$$E_{100\mu} = \frac{(9\times10^9)(100\times10^{-6})}{(5)^2}$$ = 36,000 N/C left

The total field is 90,000 N/C left

If a charge of –1 mC charge were placed at point A, what would be the force (magnitude and direction) on it?

F = qE
F = (–1 × 10^{-3} C)(90,000 N/C left)
F = 90 N right

The net charge shown on the diagram is –864 μC. Estimate the electric field 30 km from point A.

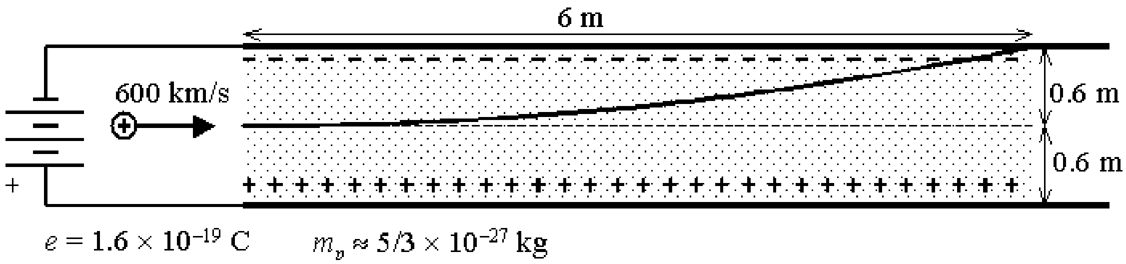
Far away from the arrangement of charges, E = kQ_{net}/r^2 like the whole thing is a point charge.
E = $(9 \times 10^9)(864 \times 10^{-6})/(30{,}000)^2$
E = (8.64 × 10^{-3} N/C

In general, when far away from an arrangement of charges, the electric field is approximated as:

A point charge that has the net charge of the entire arrangement of charges.

Motion of a Single Charge in a Uniform Electric Field (22-6)

When a charge travels through a uniform electric field, it has a constant force and acceleration.



It takes 3×10^{-16} J of energy to accelerate a proton to 600 km/s. Suppose such a proton was projected horizontally into a uniform electric field that exists between two plates (as indicated by the shaded region). The proton moves 6 m horizontally before striking one of the plates. Ignore gravity.

Draw + and – signs to indicate the charge on each plate, and arrows in between the plates to indicate the direction of the electric field in the shaded region.

Draw the path the proton takes and describe the path: parabolic

How much time elapses after the proton enters the field but before the proton strikes a plate?

$$x = vt$$
$$6 = (600{,}000)t$$
$$t = 10^{-5}\ \text{s}$$

What is the acceleration of the proton while it is in the electric field?

$$y = \tfrac{1}{2}at^2$$
$$0.6 = \tfrac{1}{2}a\left(10^{-5}\right)^2$$
$$a = 1.2\times10^{10}\ \text{m/s}^2$$

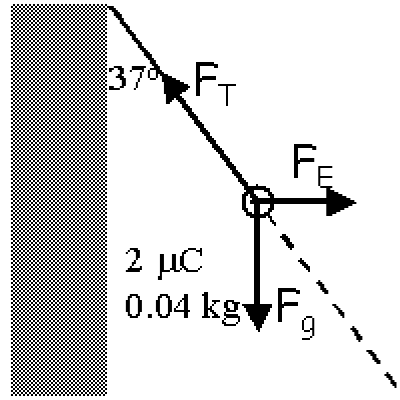
Using the value of acceleration found above, explain (1) why gravity can be safely ignored, and (2) why the acceleration can be greater than the speed of light.

Gravity is one billionth the amount of this acceleration. Acceleration can be greater than the speed of light if the time is small enough to make v = at less than the speed of light.

What is the strength of the electric field between the plates?

$$F = qE = ma$$
$$\left(1.6\times10^{-19}\right)E = \left(\tfrac{5}{3}\times10^{-27}\right)\left(1.2\times10^{10}\right)$$
$$E = 125\ \text{N/C}$$

A 0.04 kg sphere is given a charge of –2 μC. It is attached by a string to a charged wall that creates a uniform, horizontal electric field. The string makes an angle of 37° with the wall.



Draw the forces acting on the negatively-charged sphere.

What is the magnitude and direction of the uniform electric field?

The electric force on the charge is to the right, so the field must be to the left since the charge is negative.

$$F_{Tx} = F_E = qE \qquad F_{Ty} = F_g = mg$$

$$\frac{F_{Tx}}{F_{Ty}} = \tan(37) = \frac{qE}{mg} = \frac{3}{4} \rightarrow E = \frac{3mg}{4q} = \frac{3(0.04)(10)}{4(2\times10^{-6})} = 1.5\times10^{5}\ \text{N/C}$$

The string is cut at time $t = 0$. Write expressions for x and y as functions of time, and draw a dotted path on the diagram representing how the charged sphere will move after the string is cut.

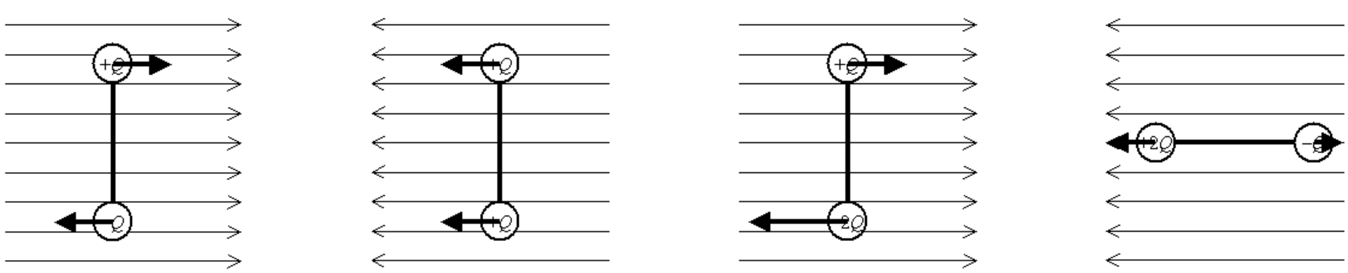
$$x = \tfrac{1}{2}a_x t^2 + v_{0x}t \rightarrow x = \tfrac{1}{2}\left(\frac{F_x}{m}\right)t^2 + (0)t \rightarrow x = \tfrac{1}{2}\left(\frac{qE}{m}\right)t^2 \rightarrow x = 3.75t^2$$

$$y = \tfrac{1}{2}a_y t^2 + v_{0y}t \rightarrow y = \tfrac{1}{2}\left(\frac{F_y}{m}\right)t^2 + (0)t \rightarrow y = \tfrac{1}{2}\left(\frac{-mg}{m}\right)t^2 \rightarrow y = -5t^2$$

<u>The Electric Dipole in a Uniform Electric Field (22-4 and 22-7)</u>

In each diagram, two metal spheres of equal mass are connected by a light, insulating rod, given the charges shown, and are initially at rest. The assembly exists in an electric field whose direction is indicated by the arrows (ignore gravity). For each diagram:

- Draw arrows representing the forces acting on the assembly.
- State the direction of the center-of-mass acceleration of the assembly (up, down, left, or right).
- State the direction of the angular acceleration of the assembly (clockwise or counterclockwise)



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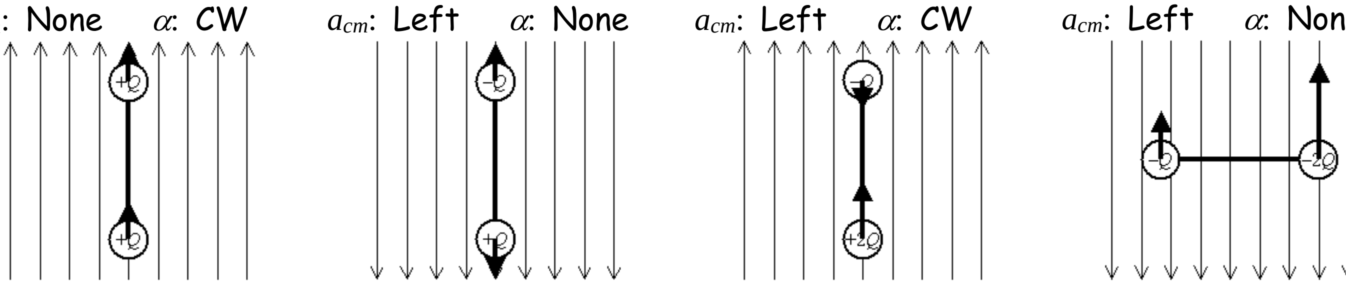

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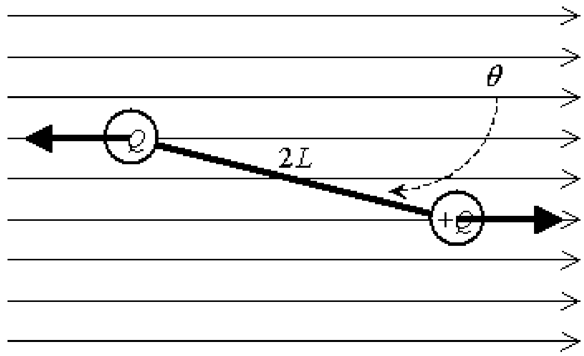
The two spheres shown each have masses *m*, opposite charges $\pm Q$, and are attached by a rod of length $2L$. This assembly is set in an electric field **E** such that the rod makes an angle θ with the field direction as shown.



Determine the moment of inertia of the assembly.

$$I = mL^2 + mL^2 = 2mL^2$$

Determine the net torque on the system.

F = qE
$\tau = r \times F = 2qEL \sin\theta$ (two forces cause two torques)

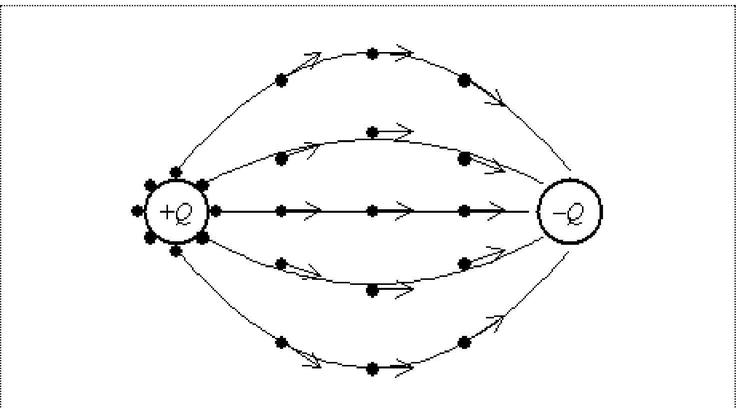
If the angle θ is sufficiently small, then the dipole will oscillate with simple harmonic motion when released from rest. Determine the period of these oscillations.

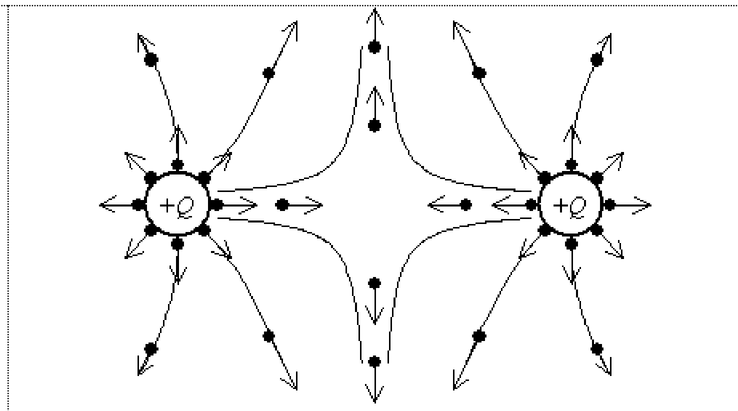
$\tau = I\alpha = (2mL^2)(d^2\theta/dt^2) = -2qEL \sin\theta$ (get $d^2\theta/dt^2$ by itself and let $\sin\theta = \theta$)
$d^2\theta/dt^2 = -(qE/mL)\theta$ (so $\omega^2 = qE/mL$)

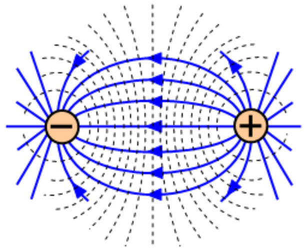
$$T = \frac{2\pi}{\omega} = \sqrt{\frac{mL}{qE}}$$

Electric Field Lines and Diagrams (22-5)

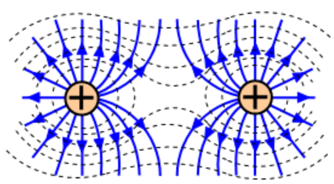
Each box contains two charges and several dots. Draw an arrow at each dot to represent the direction of the electric field at that dot. Then "connect the arrows" with curves that pass each arrow tangent to the arrow.



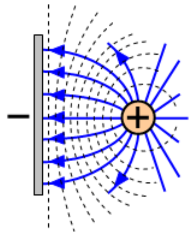




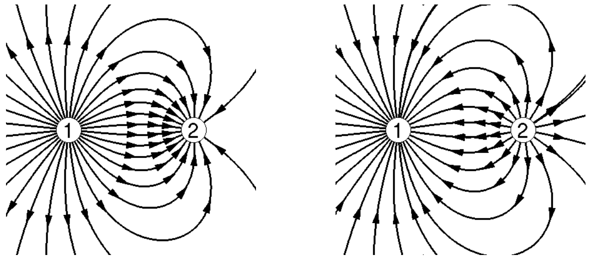
Field lines between opposite charges "connect", creating a "melon" shape.



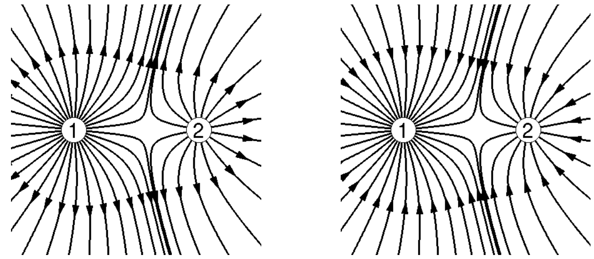
Field lines between like charges "repel", creating a "diamond" shape. The field is zero at the center of the "diamond".



Field lines always intersect conductors perpendicular to the surface of the conductor.



In these two diagrams, the left charge is stronger than the right. The charges have *opposite* sign.



In these two diagrams, the left charge is stronger than the right. The charges have *the same* sign.

Three charges create the electric field pattern shown. Some of the lines are labeled with arrows.

What is the sign of…
Charge 1? (+) (–)
Charge 2? (+) (–)
Charge 3? (+) (–)

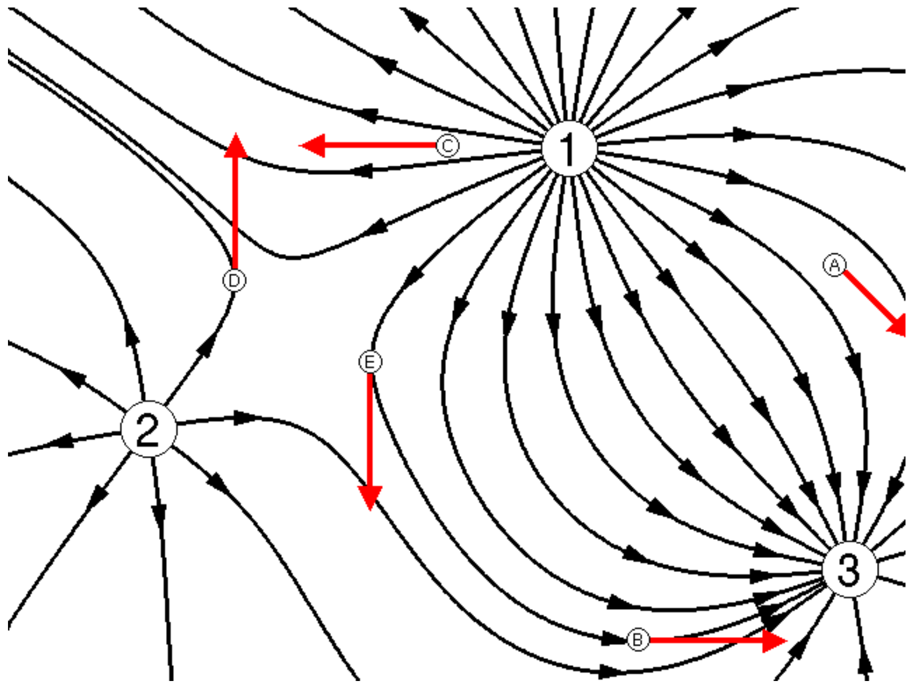
Draw vectors at each point to represent the direction of **E** at that point.

Mark with an × the position where the electric field is zero.

Which charge has the greatest magnitude? 1

Least magnitude? 2

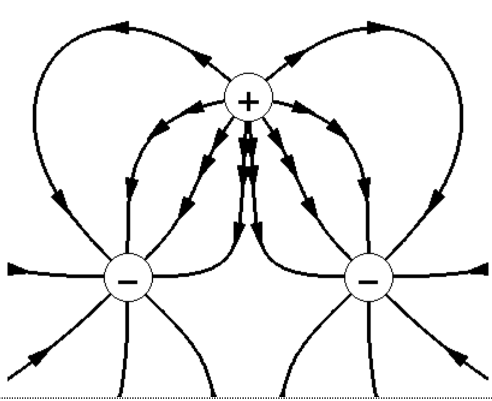
At what lettered point is the field strongest? C

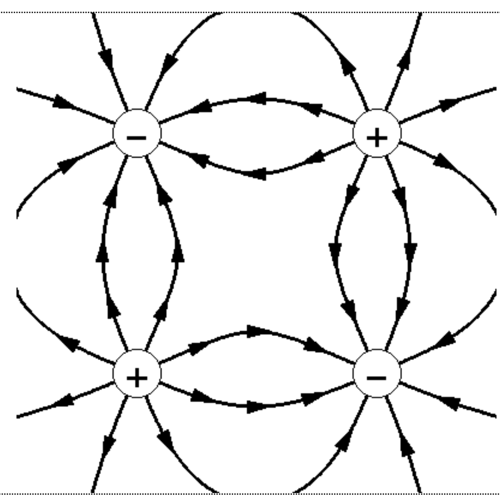


In what direction would the force on a proton be if it were placed at *E*? Down

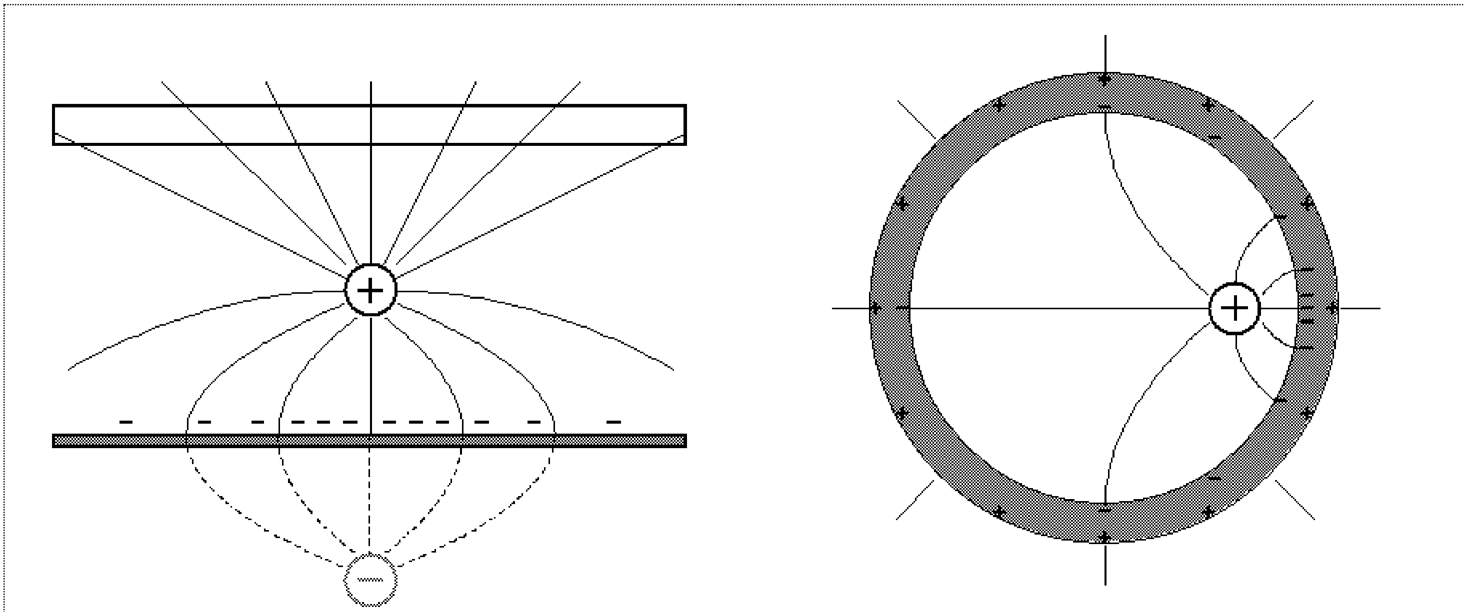
In what direction would the force on an electron be if it were placed at *D*? Down

For each set of charges shown in each box, draw several electric field lines surrounding the arrangement.



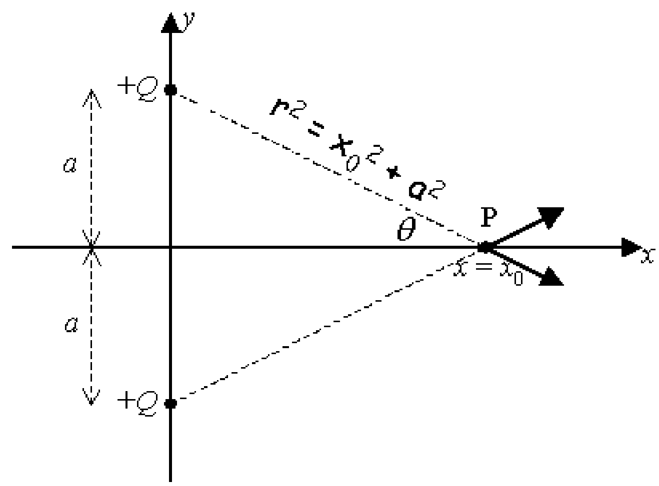


The shaded regions are conductors. The clear regions are insulators.



Calculating Forces and Electric Field in Two Dimensions (22-3 and 22-4)

In the diagram shown to the right, two equal charges $+Q$ are placed on the y-axis at distances a on opposite sides of the origin. Point P is a location on the x-axis at $x = x_0$.



Determine the magnitude and direction of the electric field at point P in terms of Q, a, x_0, and k.

X-components don't cancel:

$$E_x = \frac{kQ}{r^2}\cos\theta$$

But note that $r^2 = x_0^2 + a^2$ and $\cos\theta = x_0/r$

Also note that there are two charges.

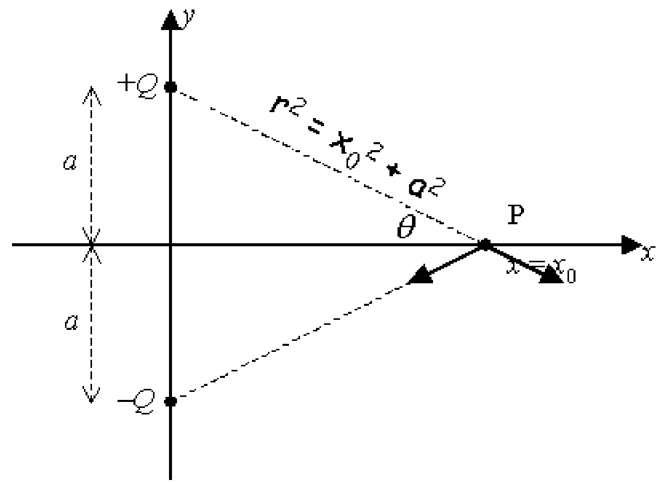
$$E = \frac{2kQ}{r^2}\frac{x_0}{r} \quad \rightarrow \quad E = \frac{2kQx_0}{\left(x_0^2 + a^2\right)^{3/2}}$$

Show that, at sufficiently large distances from the origin, the electric field can be approximated by the equation $E \approx kQ_{net}/r^2$.

Far away, $r^2 = x_0^2 + a^2$ becomes $r^2 = x_0^2$ becomes $r = x_0$ (because a is small compared to x_0)

$$E = \frac{2kQx_0}{\left(x_0^2 + a^2\right)^{3/2}} \quad \rightarrow \quad E = \frac{2kQr}{\left(r^2\right)^{3/2}} = \frac{2kQ}{r^2}$$

Now consider the same situation, but where the bottom charge is replaced with a charge $-Q$. Determine the magnitude and direction of the electric field at point P in terms of Q, a, x_0, and k.



Now the y-components add up:

$$E_y = \frac{kQ}{r^2}\sin\theta$$

$$E = \frac{2kQ}{r^2}\frac{a}{r}$$

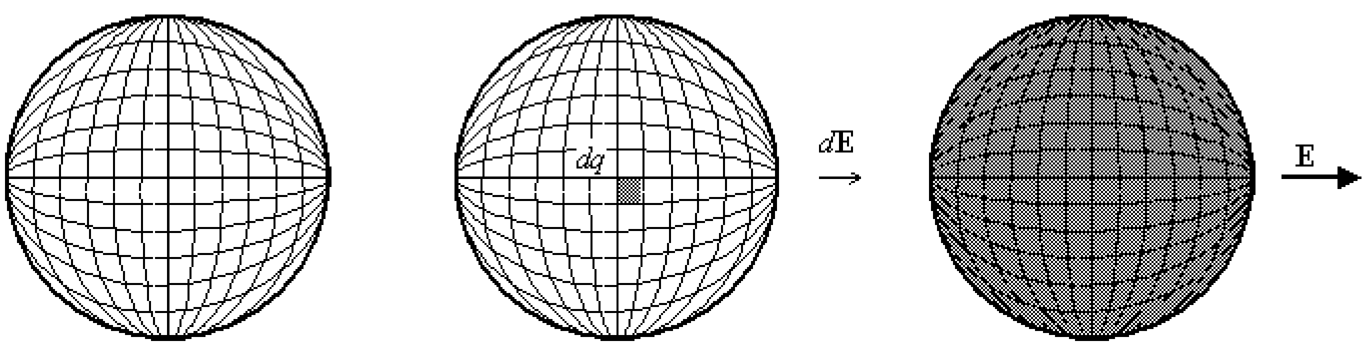
$$E = \frac{2kQa}{\left(x_0^{\ 2} + a^2\right)^{3/2}}$$

Give two reasons why $E \approx kQ_{net}/r^2$ does not adequately describe the electric field at long distances.

First, the net charge is zero. Second, for large x, $E = 2kQa/r^3$, so the field varies as $1/r^3$ and not as $1/r^2$.

Electric Field due to Continuous Charge Distributions (23-1)

A "continuous charge distribution" means a charged object or arrangement of objects that is not shaped like a point. We only know how to find the electric field of a point-charge using $E = kQ/r^2$, so in order to find the electric field due to a continuous charge distribution, we need to:



Split the object into tiny pieces dq. Find the tiny electric field $d\mathbf{E}$ from each single piece dq. Add up all the $d\mathbf{E}$'s from all of the dq's to get the total $\mathbf{E}$.

Write an expression for the tiny electric field $d\mathbf{E}$ created by a tiny charge dq that is a distance r away.

$$dE = \frac{k \cdot dq}{r^2}$$

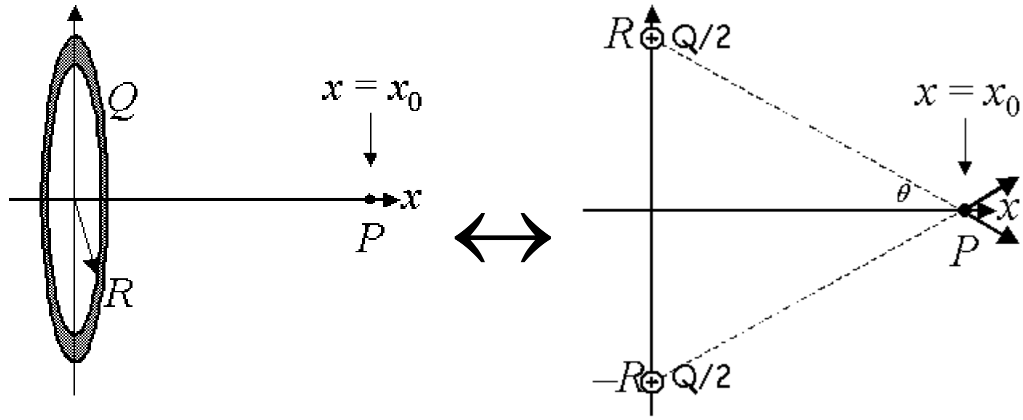
Write an expression for the total electric field $\mathbf{E}$ created by all tiny charges dq.

$$E = \int \frac{k \cdot dq}{r^2}$$

[*Ring of Charge*]

A circular ring is centered at the origin and oriented in the *y*-*z* plane. The ring carries a uniformly distributed total charge *Q*, as shown in the left diagram.

On the right diagram, draw another arrangement of charge that results in the same electric field strength at point *P* as the ring of charge.



Using the simpler arrangement, determine the electric field at point *P* for the ring of charge.

$E_x = \frac{2k(Q/2)}{r^2}\cos\theta$ (where cosθ = x_0/r and there are two x-comp's of field from Q/2 charges)

$$E = \frac{kQ}{r^2}\frac{x_0}{r} = \frac{kQx_0}{\left(x_0{}^2 + R^2\right)^{3/2}}$$

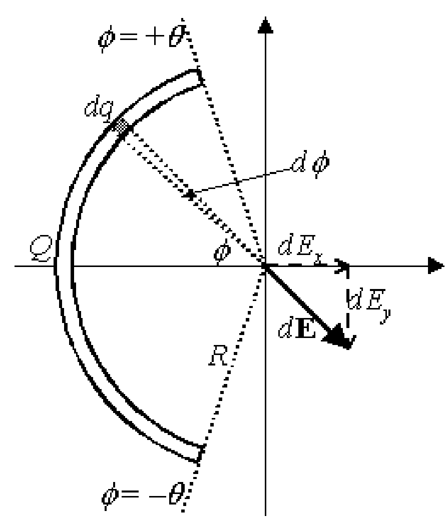
The symmetry of the ring of charge (that all components other than the x-components cancel) is also obeyed by the two half-charges, and the two half-charges are the same distance from P as all of the charges on the ring.

[*Arc of Charge*]

Consider an arc with a curvature center at the origin. The arc holds a total charge *Q*, uniformly distributed over its length.

What is the linear charge density λ of the arc?

The length of the arc is L = (R)(2θ) = 2Rθ
(Length is radius•radians)



Determine the electric field strength at the origin.

Because of symmetry, only x-components will add up:

$$dE_x = \frac{k \cdot dq}{r^2}\cos\phi$$

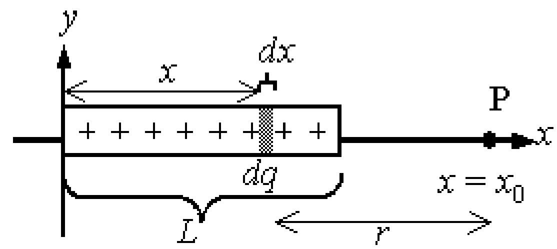
But note that r = R (a constant), dq = λ dL = λR dφ (tiny length is radius•tiny radians)

$$dE_x = \frac{k\lambda R}{R^2}\cos\phi \cdot d\phi \quad \rightarrow \quad E_x = \frac{k\lambda}{R}\int_{\phi=-\theta}^{\phi=+\theta}\cos\phi \cdot d\phi$$

$$E_x = \frac{2k\lambda}{R}\sin\theta$$

[*Line of Charge*]

A rod of length L and uniform linear charge density λ is set along the x-axis with its left end at the origin.



Write an expression for the total charge on the rod.

Q = λL

Determine the electric field strength at a point $x = x_0$ on the x-axis.

Let x represent the position of one "piece" of the charge on the rod. Then r = x_0 – x.
The charge on the rod is positioned between x = 0 and x = L:

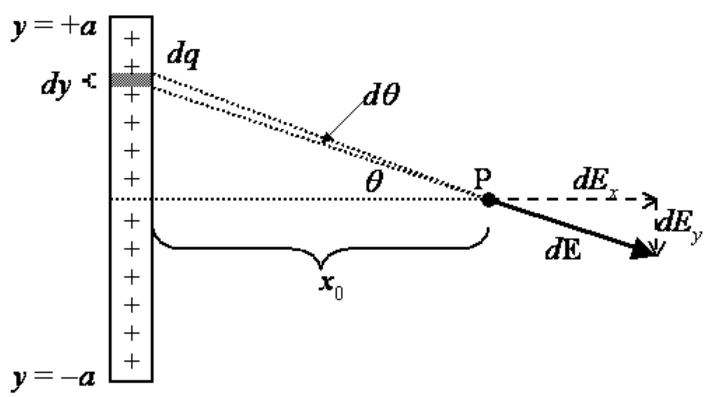
$$dE = \frac{k \cdot dq}{r^2}$$

Use the charge density to express dq in terms of dx: dq = λ dx

$$dE = \frac{k\lambda}{(x_0 - x)^2} dx \quad \rightarrow \quad E = \int_{x=0}^{x=L} \frac{k\lambda}{(x_0 - x)^2} dx$$

$$E = \left[\frac{k\lambda}{(x_0 - x)} \right]_{x=0}^{x=L} = k\lambda \left[\frac{1}{x_0 - L} - \frac{1}{x_0} \right]$$

A different rod of length 2a and uniform linear charge density λ is set along the y-axis with its center at the origin.



Determine the electric field strength at a point $x = x_0$ on the x-axis.

Let θ be our variable of integration. Set up the following relationships:
λ = dq/dy, so dq = λ dy
y = $x_0\tan\theta$, so dy = $x_0\sec^2\theta\, d\theta$ [also note that theta's bounds are $-\tan^{-1}(a/x_0) < \theta < \tan^{-1}(a/x_0)$]
r = $x_0\sec\theta$

$$dE_x = \frac{k \cdot dq}{r^2}\cos\theta = \frac{k\lambda x_0 \sec^2\theta}{x_0{}^2 \sec^2\theta}\cos\theta \cdot d\theta = \frac{k\lambda}{x_0}\cos\theta \cdot d\theta$$

$$E_x = \int_{\theta=-\arctan(a/x_0)}^{\theta=-\arctan(a/x_0)} \frac{k\lambda}{x_0}\cos\theta \cdot d\theta = \frac{2k\lambda}{x_0}\sin(\arctan(a/x_0))$$

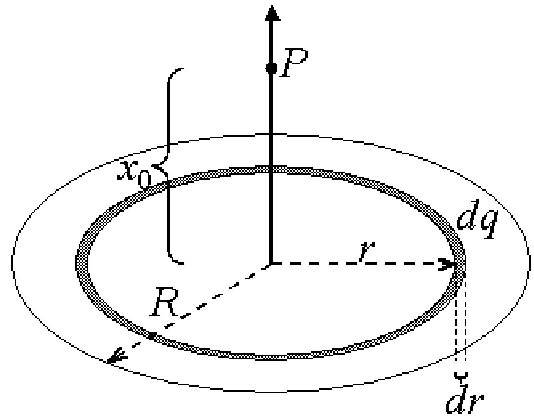
$$E = \frac{2k\lambda a}{x_0\sqrt{x_0{}^2 + a^2}}$$

What would the electric field be at x_0 if the rod were made infinitely long without changing its charge density?

If a >> x_0, then $a/\sqrt{x_0{}^2 + a^2} \approx 1$, so $E = \dfrac{2k\lambda}{x_0}$

[*Disk of Charge*]

Finally, consider a flat disk centered at the origin and a point P on the disk's axis a distance x_0 from the origin. The disk has a radius R and an area charge density σ.



Write an expression for the total charge of the disk.

$Q = \sigma A = \sigma \pi R^2$

The disk can be treated as many thin rings of varying radius r and thickness dr. What is the charge dq contained in the ring of radius r shown in the diagram?

$dq = \sigma \cdot dA$ and $A = \pi r^2$, so $dA = 2\pi r\, dr$
$dq = 2\pi\sigma r \cdot dr$

What is the electric field dE created only by the thin ring of charge dq shown in the diagram?

Ring of charge has electric field $E = \dfrac{kQx_0}{\left(x_0{}^2 + r^2\right)^{3/2}}$, so ring of tiny charge has electric field

$$dE = \frac{kx_0}{\left(x_0{}^2 + r^2\right)^{3/2}} dq \quad \rightarrow \quad dE = \frac{2\pi\sigma k x_0 r}{\left(x_0{}^2 + r^2\right)^{3/2}} dr$$

What is the total electric field created at point P by all of the charge on the disk?

$$E = \pi\sigma k x_0 \int_{r=0}^{r=R} \frac{2r}{\left(x_0{}^2 + r^2\right)^{3/2}} dr$$ (if $u = x_0{}^2 + r^2$, and $du = 2r\, dr$, then we have $u^{-3/2}du$)

$$E = 2\pi\sigma k x_0 \left[\frac{-1}{\sqrt{x_0{}^2 + r^2}}\right]_{r=0}^{r=R} \quad \rightarrow \quad E = 2\pi\sigma k x_0 \left[\frac{1}{x_0} - \frac{1}{\sqrt{x_0{}^2 + R^2}}\right]$$

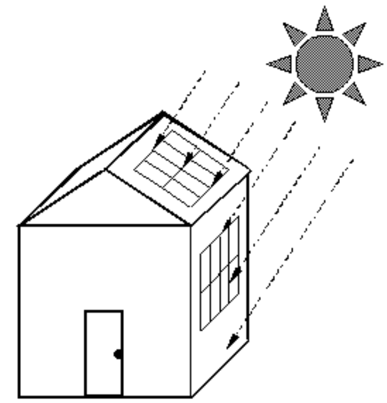
If the radius of the disk were increased to infinity, but the charge density remained the same, then what would the electric field created by this infinite plane of charge be?

$\dfrac{1}{\sqrt{x_0{}^2 + R^2}} \rightarrow 0$ as R gets huge, so $E = 2\pi\sigma k$ (note that this is a constant value!!!)

Introduction to Gauss' Law (23-2)

[*Electric Flux*]

Consider a house with two windows. The windows have equal size, but one is on the roof and the other is on a wall as shown. Which window has more light passing through it?



The window on the roof

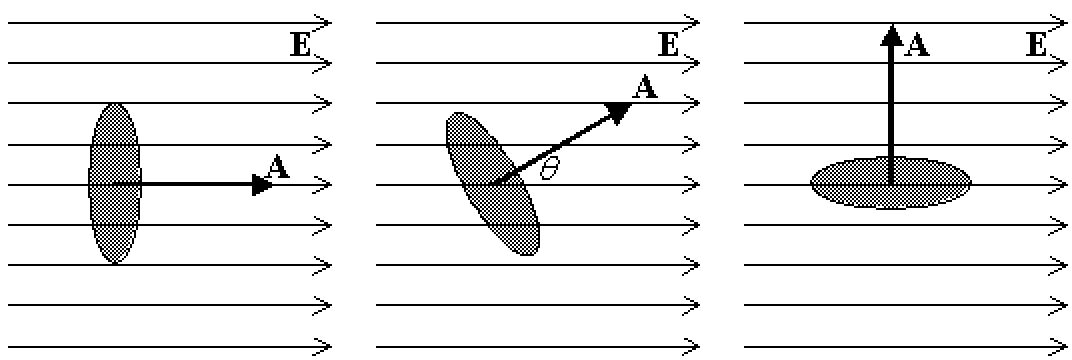
Explain what "flux" is.

The "flow" of some physical quantity (that is defined by a vector field) through a surface area.

From the example of the sunlight and the house, what three things affect how much light passes through the window?

The intensity of the light, the area of the window, and the orientation of the window

Gauss' Law deals with the flux of electric field, rather than the flux of light energy or water flow. In order to discuss the flux through a surface, we first must define an **area vector** for the surface.



Which diagram shows the greatest electric field flux through the surface? Left Least flux? Right

Write the basic equation for the flux of a uniform electric field **E** through a flat surface of area **A**:

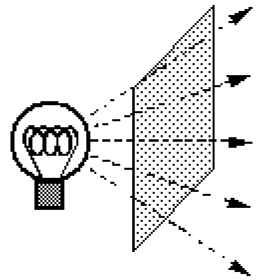
$\Phi = E \bullet A$

Explain what the sign (+ or –) of flux represents about "flow" and the surface involved.

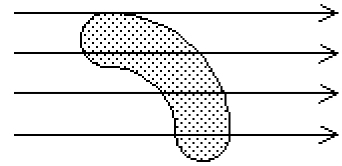
Positive flux flows "out" of a surface area, negative flux flows "into" a surface area.

[*Surface Integral*]

In the above examples, the electric field is uniform and the surface is flat. But what if the field were different in different places or the surface was not flat? How could we find the total flux through the surface?



The light rays are not uniform (going in the same direction) so *the magnitude and orientation of the light rays and the surface is different in different places.*



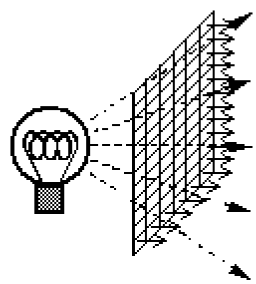
The field lines are uniform, but the surface is not flat, so that the *orientation of the surface and the field lines is different in different places.*

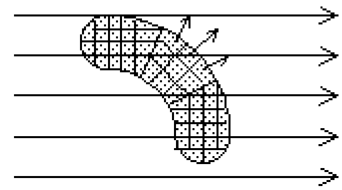
In order to handle these situations, we follow these three steps:

1. Split the surface into tiny areas dA each with its own area vector.

2. Take the tiny flux through each tiny surface: $d\Phi = E \bullet dA$

3. Add up all of the tiny fluxes to get the total flux.





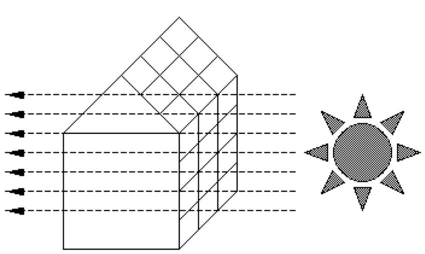
The new equation for the total flux of a non-uniform **E**-field through an irregularly-shaped surface is:

$\Phi = \int \mathbf{E} \bullet d\mathbf{A}$

A surface integral is NOT a Riemann (raise-the-power-and-divide-by-the-new-power) integral. Students should not wonder "how" they are going to actually do the math of taking this integral, but should only understand CONCEPTUALLY what they are doing at this point.

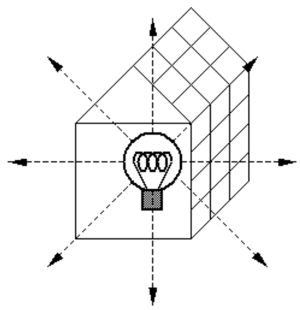
[*Enclosed Charge*]

Consider a house made entirely of glass, including the floor. State whether the net flux of light through the glass surface of the house is positive, negative, or zero.



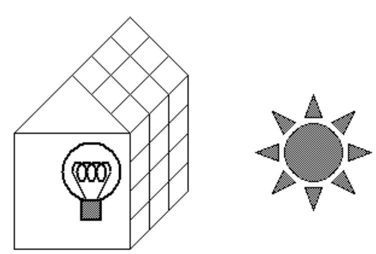
Sunlight shines in one side and out the other.

Net flux is: (+) (–) **(0)**



A lightbulb shines inside.

Net flux is: **(+)** (–) (0)



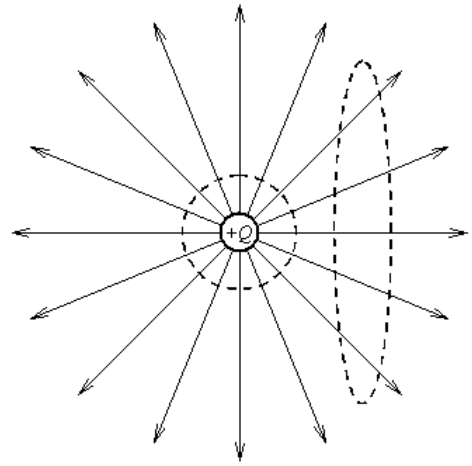
Now both the sun and the bulb are in place.

Net flux is: **(+)** (–) (0)

In the last diagram, the flux of light out of the glass house is only due to the lightbulb inside.

If there were to be negative net flux in the first diagram, what would have to be added?

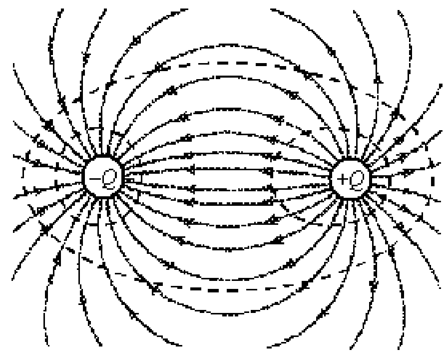
A light-absorbing object, such as a black box or black hole.



Definition: A "closed surface" is a surface fully encloses a region of space.

The point is this: the ***net flux*** through a ***closed surface*** depends only on what is inside the closed surface. Objects outside the closed surface do not contribute to the net flux, since anything emitted by an outside object will flow *into and out of* the surface, canceling out.

The surface enclosing the charge in this diagram has a net outward (positive) flux because it has 16 arrows going outward through it and no arrows going into it. The surface on the right has 5 arrows going into it and the same 5 arrows leaving it, which means it has zero net flux.



The net flux through the surface enclosing the negative charge is negative (field lines go into the surface). The net flux through the positive charge is positive (field lines go out of the surface). The net flux through the large, all-encompassing surface is zero, because for every field line that goes into the surface, there is a field line that goes out of the surface.

[*Gauss Law*]

Conceptual Statement of Gauss' Law: The **net** flux of electric field through a **closed** surface is directly proportional to the net charge enclosed within the surface.

Mathematical Statement of Gauss' Law:

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$ (both sides of this equation represent net flux through the closed surface)

The circle on the integral replaces any sort of "bounds" that would be on a Riemann integral. The circle means that the surface of which dA is a piece is a closed surface.

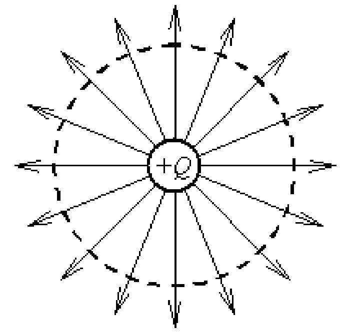
When we apply Gauss' Law to any problem, *we get to choose what shape the surface takes*. To avoid doing any real calculus, we always want to choose a surface shape such that:

- The surface obeys the same symmetry as the charge distribution in the problem.
- At every point on the surface, the electric field strength is the same.
- At every point on the surface, the electric field vector is either tangent or normal to the surface.

Using Gauss' Law to Calculate Electric Field (23-3)

[*Point Charge*]

Consider a point charge Q isolated from other charges. We wish to find the electric field a distance r from the charge.



Describe an appropriate Gaussian surface to use to find the electric field. Draw it on the diagram.

A sphere that is centered on the charge. All points on the sphere are equal distance from Q, and so must have the same field strength. Also, the field is normal to the surface at all points.

Apply Gauss' Law to find an expression for the electric field a distance r from the charge.

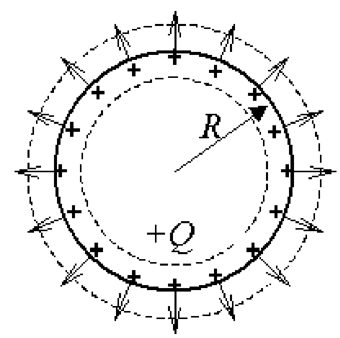
$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0}$$ (The strength of E is the same everywhere, and the • is 1 since E || dA always)

$$E\oint d\mathbf{A} = \frac{Q}{\varepsilon_0}$$ (The charge enclosed is Q, and E (same strength) can "factor out" of the integral)

$$E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$ (Integral adds up the area of the sphere)

[*Spherical Conductor*]

Now consider a spherical conductor of radius R and given a net charge Q.



Describe an appropriate Gaussian surface to use to find the electric field an arbitrary distance r from the center of the conductor. Draw a surface for $r > R$ in the diagram.

Still a sphere that is concentric now with the conductor.

Apply Gauss' Law to find an expression for the electric field a distance r from the charge, where $r > R$.

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0}$$ (Again E is the same strength everywhere on the surface, and the • is 1)

$$E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0}$$ (Under these circumstances, the integral just counts the area of the surface)

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$ (We do not have to do any real calculus if the surface obeys our 3 rules above)

What is the electric field a distance $r < R$ from the center of the sphere?

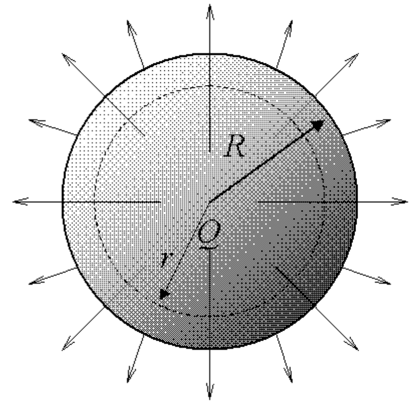
Zero, this region is inside of the conductor. Therefore, no net charge is enclosed INSIDE.

What does this illustrate about the charge given to a conductor?

All of the excess charge resides on the surface of the conductor.

[*Spherical Insulator with Uniform Charge Density*]

Now consider a spherical insulator of radius R and a net charge Q. The charge is uniformly distributed over the volume of the sphere.



Explain why the electric field *outside* the sphere obeys the same equation as the point charge and spherical conductor electric fields found above.

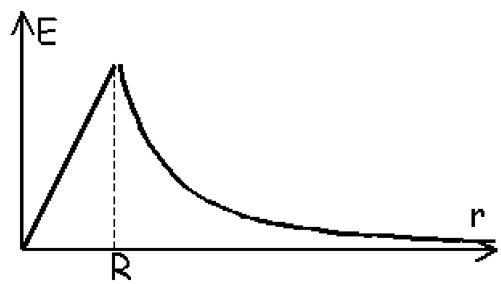
Two reasons: (1) the situation obeys the same (spherical) symmetry as before (so use a spherical surface), and (2) the charge enclosed when OUTSIDE is the entire charge Q.

How much charge is enclosed in a sphere of radius $r < R$?

If the charge density is uniform (which it is), we can set up a density ratio (charge/vol. = const)

$\frac{Q_{Total}}{V_{Total}} = \frac{Q_{enc}}{V_{enc}}$ → $\frac{Q}{R^3} = \frac{Q_{enc}}{r^3}$ (factors of $4\pi/3$ cancel) → $Q_{enc} = Q\frac{r^3}{R^3}$

What is the electric field a distance $r < R$ from the center of the sphere?



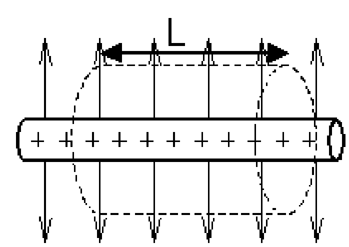
$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0}$

$E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$ (Plug in area and charge enclosed)

$E = \frac{Q}{4\pi\varepsilon_0 R^3} r$ (Field varies linearly inside uniform density)

[*Line of Charge*]

Now consider a very long, very thin linear charge distribution. The linear charge density of this distribution is λ. Determine the electric field a distance r from this line of charge. Explain each step.



Choose a cylinder (coaxial to the line) for a surface. Here, only the lateral area has electric field flux—the end caps do not!

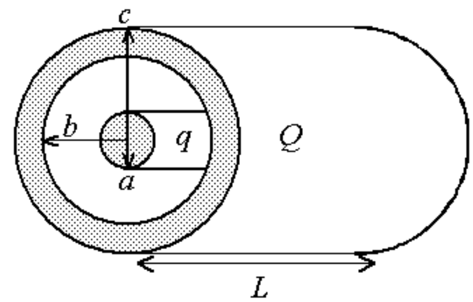
$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0}$

$E(2\pi r\ell) = \frac{\lambda\ell}{\varepsilon_0}$ (Only the lateral area has flux, and the charge enclosed is density×length)

$E = \frac{\lambda}{2\pi\varepsilon_0 r}$

[*Concentric Cylinders*]

A solid cylinder of radius a is concentric with a cylindrical shell of inner radius b and outer radius c. Both materials are conductors. The inner conductor has a net charge q and the outer conductor has a net charge Q. Both conductors have a length L. Ignore edge effects.



Determine the electric field at a distance r from the axis if:

$r < a$

Zero—inside a conductor

$a < r < b$

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \rightarrow E(2\pi r L) = \frac{q}{\varepsilon_0}$$

Let the surface be a coaxial cylinder that is exactly length L so that the entire inner charge q is enclosed.

$$E = \frac{q}{2\pi\varepsilon_0 L r}$$

$b < r < c$

Zero—inside a conductor

$r > c$

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \rightarrow E(2\pi r L) = \frac{Q+q}{\varepsilon_0}$$

Let the surface be a coaxial cylinder that is exactly length L so that the entire charge Q+q is enclosed.

$$E = \frac{Q+q}{2\pi\varepsilon_0 L r}$$

What is the surface charge density on each of the following surfaces, assuming the charge is uniformly distributed and no charge resides on the ends of the cylinders?

$r = a$

There is a net charge of +q on this surface:

σ = q/A = $\frac{q}{2\pi a L}$

$r = b$

There is a net charge of –q on this surface:

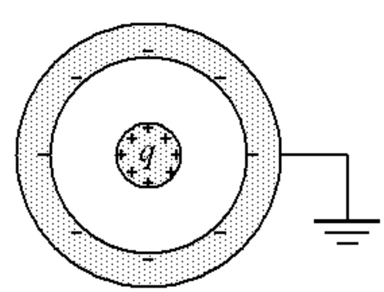
σ = q/A = $-\frac{q}{2\pi a L}$

$r = c$

There is a net charge of Q+q on this surface:

σ = Q_{net}/A = $\frac{Q+q}{2\pi a L}$

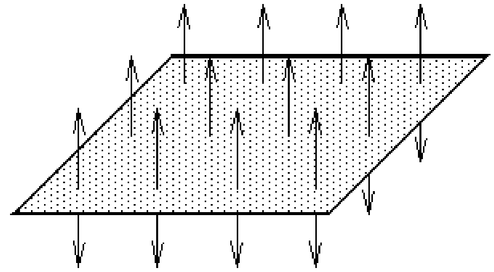
If the outer conductor were connected to ground, how much charge would reside on it, and where would this charge reside?



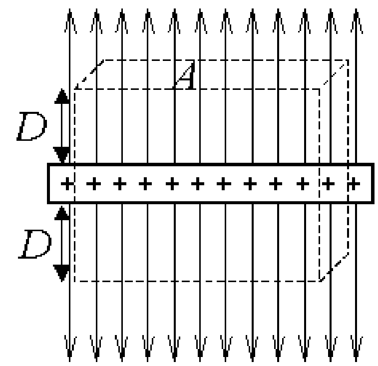
The outer charge of Q+q would escape to ground, leaving only the –q on the inner surface.

[*Plane of Charge*]

Now consider an infinitely large plane of charge with area charge density σ. Draw a Gaussian surface on the diagram and determine the electric field strength a distance D from the plane.



Choose a rectangular prism with 2 faces parallel to the plane. Let these faces have area A and be opposite distances D from the plane. Only these two areas have electric flux. The enclosed charge is (density)×(area)



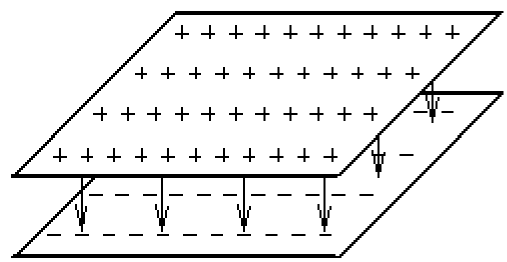
$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(2A) = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

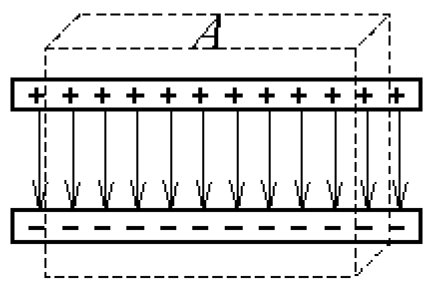
[*Parallel Plates*]

Now consider two infinitely large, parallel planes of charge, each with area charge density σ, but with opposite sign.



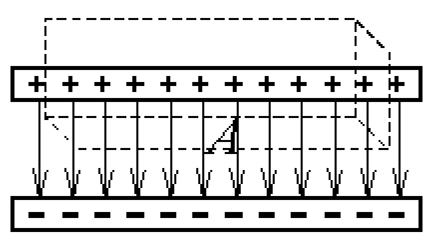
Use Gauss' Law to show that no electric field can exist outside the two plates.

Choose another rectangular prism with both parallel faces outside of the two parallel sheets. This prism encloses no net charge, and so there must be no electric flux out of the two areas A and thus no electric field.



Now determine the electric field strength between the plates.

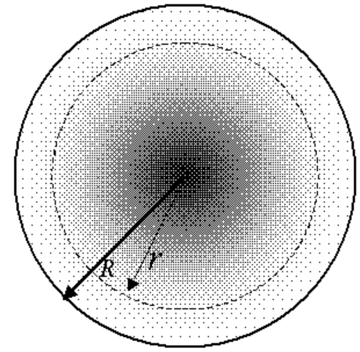
Now let one of the parallel areas be between the plates so that it has electric flux through it. Only one of the sheets is enclosed, so the charge enclosed is (density)×(area):



$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \quad \rightarrow \quad E(A) = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{\varepsilon_0}$$

Calculating Electric Field From Non-Uniform Charge Densities (23-3)



Finally, consider an insulating sphere of radius R with a positive charge distributed throughout its volume. The charge density is NOT uniform, but instead varies according to the function

$\rho = \rho_0 \frac{R}{r}$ (ρ_0 is a constant)

We wish to determine the electric field at any distance r from the center of the sphere. Answer all of the following in terms of ρ_0, R, and fundamental constants.

Determine the amount of charge enclosed within a sphere of radius r, where $r < R$.

The volumeof a sphere is $V = \frac{4}{3}\pi r^3$. Split the sphere of radius r into spherical shells of tiny thickness dr and volume $dV = 4\pi r^2 dr$. The tiny charge of each shell is $dQ = \rho \cdot dV$. Add this up:

$$Q_{enc} = \int_0^r \left(\rho_0 \frac{R}{r}\right)\left(4\pi r^2 dr\right) = 4\pi R\rho_0 \int_0^r r \cdot dr = 2\pi R\rho_0 r^2$$

Determine the total amount of charge within the sphere. Then write an expression for the electric field as a function of r if $r > R$.

$$Q_{Total} = Q_{enc}(r = R) = 2\pi R^3 \rho_0$$

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \quad \rightarrow \quad E\left(4\pi r^2\right) = \frac{2\pi R^3 \rho_0}{\varepsilon_0}$$

$$E = \frac{2\pi R^3 \rho_0}{4\pi \varepsilon_0 r^2}$$

Write an expression for the electric field as a function of r if $r < R$.

$$\oint \mathbf{E} \bullet d\mathbf{A} = \frac{Q_{enc}}{\varepsilon_0} \quad \rightarrow \quad E\left(4\pi r^2\right) = \frac{2\pi R r^2 \rho_0}{\varepsilon_0}$$

$$E = \frac{R\rho_0}{2\varepsilon_0}$$

Sketch a graph of E vs. r. Label the vertical axis with appropriate expressions.