

# Interpolation and Motion



University of Calgary **GraphicsJungle** Project

CPSC 587 2005



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- Abstract out the important features of the animation ?
- A very subjective thing to do → the appropriate feature for one kind of animation will not be the appropriate feature for another kind of animation.
- Our focus → various techniques in which the animator is responsible for specifying most of the information (low level of abstraction).



## 1. Interpolation

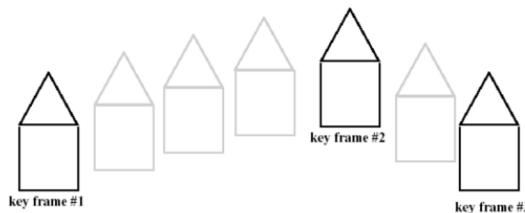
2. Motion along a curve (arc length)
3. Interpolation of rotations (quaternions)
4. Path following



- At the foundation of almost all animation is the interpolation of values.
- The simplest case is interpolating the position of a point in space.
- Even this is non-trivial to do correctly and requires some discussion of several issues:
  - the appropriate parameterization of position,
  - the appropriate interpolating function,
  - and maintaining the desired control of the interpolation over time.



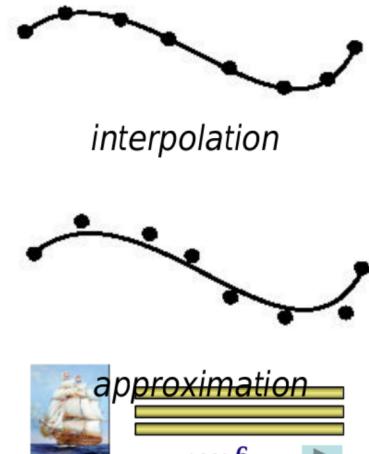
- Often, an animator has a list of values associated with a given parameter at specific frames (called *key frames* or *keys*) of the animation.



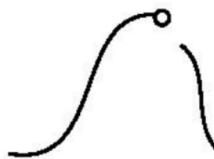
- The question to be answered is how best to generate the values of the parameter for the frames between the key frames.
- The parameter to be interpolated may be
  - a coordinate of the position of an object,
  - a joint angle of an appendage of a robot,
  - the transparency attribute of an object,
  - any other parameter



- Our focus → How to choose the most appropriate interpolation technique and apply it in the production of an animated sequence.
- One of the first decisions to make is whether the given values represent actual values that the parameter should have at the key frames (*interpolation*), or whether they are meant merely to control the interpolating function and do not represent actual values the parameter will assume (*approximation*).



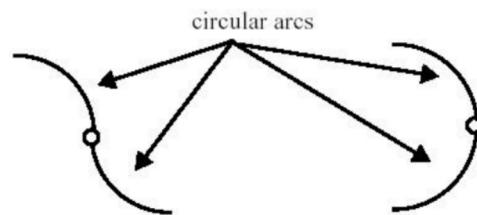
- Other issues that influence which interpolation technique to use include how smooth the resulting function needs to be (i.e. *continuity*),



a) positional discontinuity  
at the point



b) positional continuity  
but not tangential continuity  
at the point



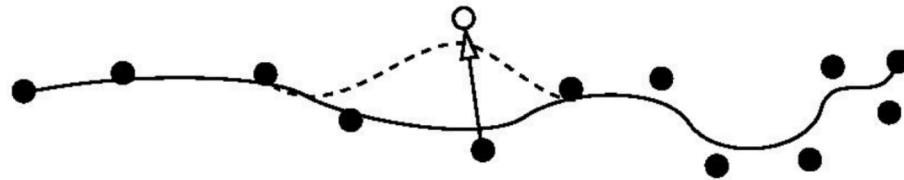
c) positional and tangential  
continuity but not curvature  
continuity at the point



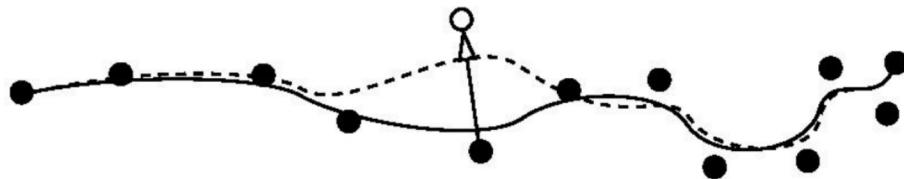
d) positional, tangential,  
and curvature continuity  
at the point



- how much computation you can afford to do (*order of interpolating polynomial*, and whether *local* or *global control* of the interpolating function is required.



**Local Control:** moving one control point only changes the curve over a finite bounded region



**Global Control:** moving one control point changes the entire curve;  
distant sections may change only slightly.



## - Spline

- Created by mathematical technique that fits a curve to a set of given points called Control Points: Curve fitting
- Artist places the points, and program creates a curve based on the points



## - Polyline

- series of straight lines through the points

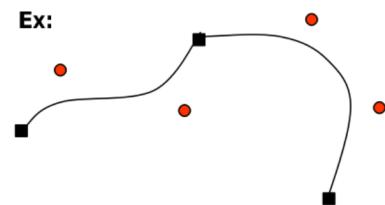


## - B-splines

- blend position of control points without passing through any of them

## - Non-uniform rational B-splines (NURB)

- Named after French engineer Pierre Bezier
- Direct control over the location of the curve with anchor points
- Over the curvature between the points with two off-the curve control points
- Difficult: off-path control points



### - Catmull-Rom Spline

- Interpolating spline
- Restricts the curvature choice but places all points on the curve
- Make drawing and editing more straightforward



*Often used in animation path movement*



Hermite

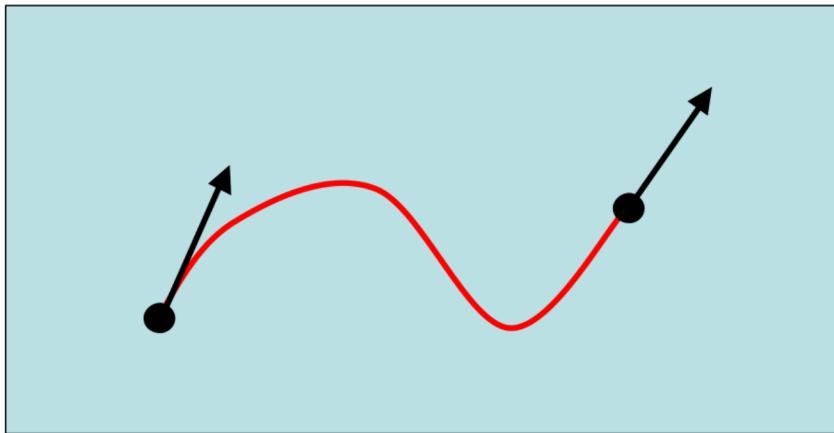
Bezier

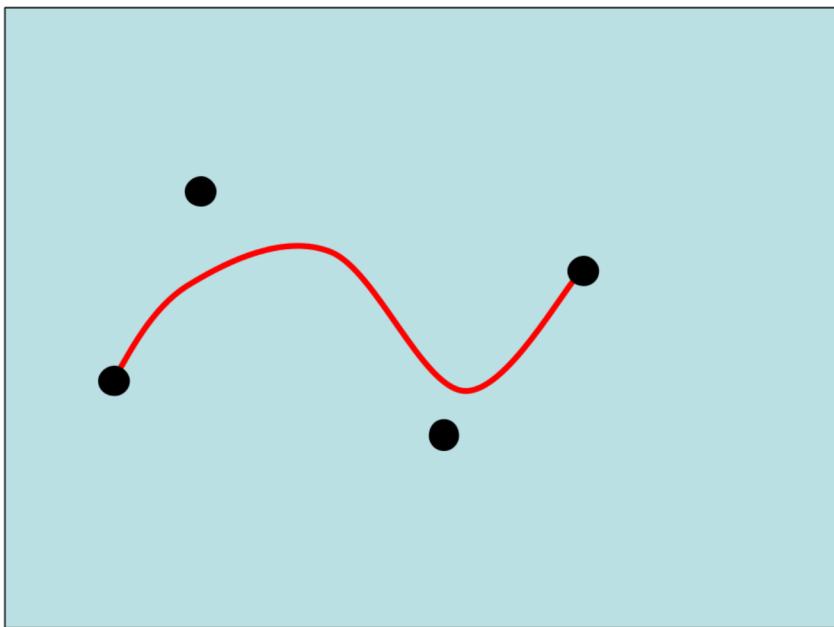
Catmull-Rom

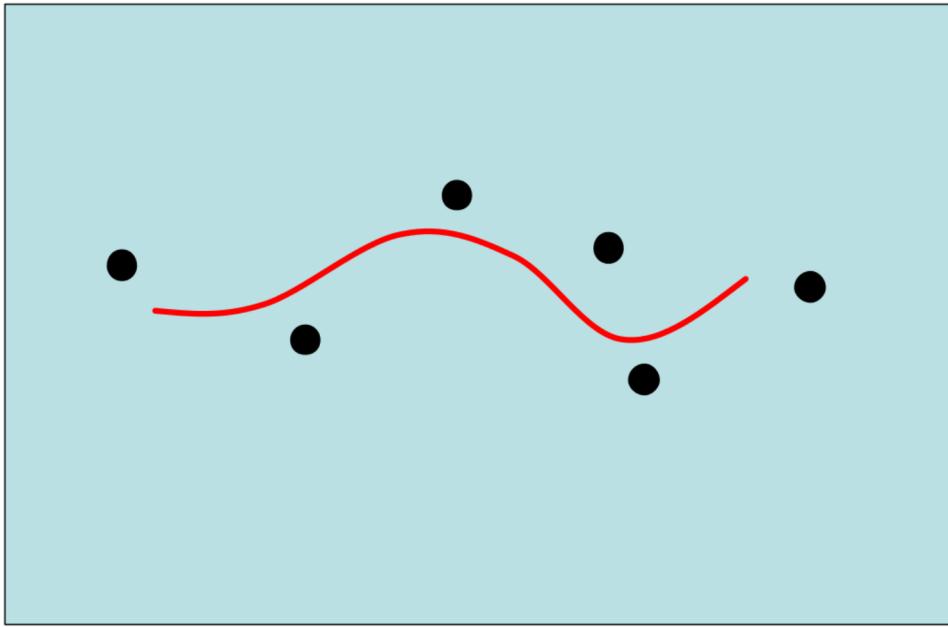
Blended parabolas

B-splines, NURBS

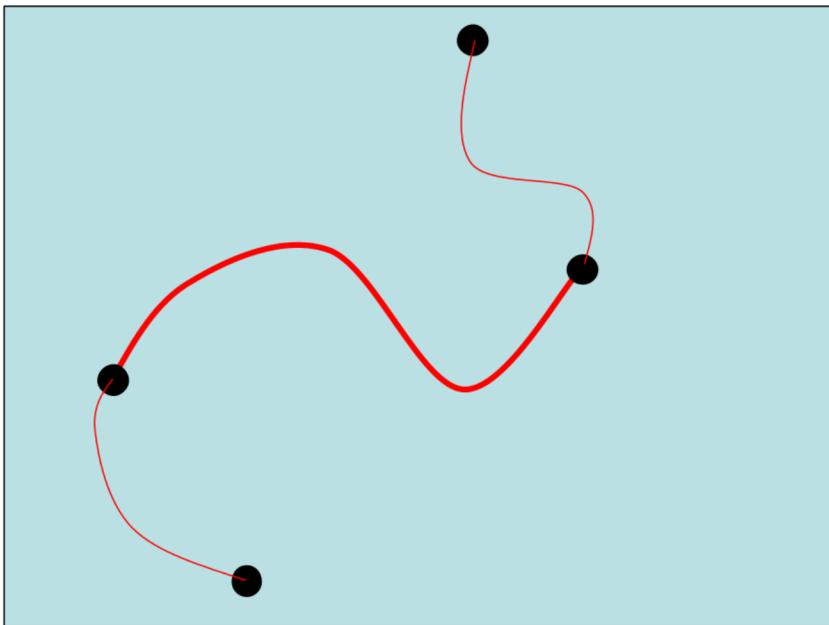




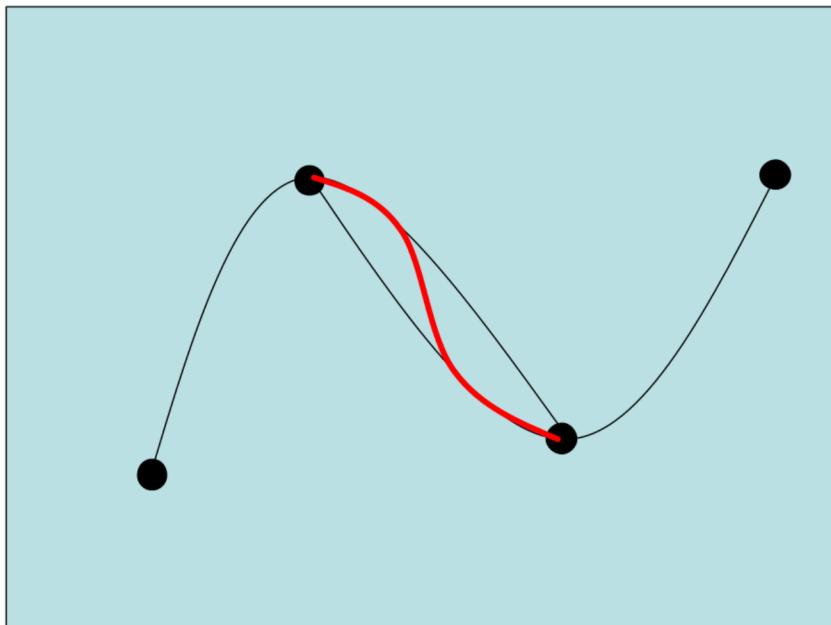




*Often used in animation path movement*



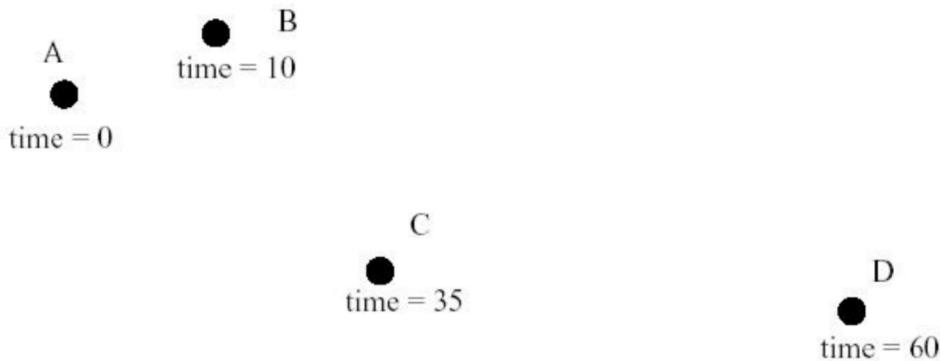
*Often used in animation path movement*



- Appendix B.4
- Michael E. Mortenson, "Geometric Modeling", John Wiley & Sons, New York, 1985.
- David F. Rogers, J. Alan Adams, "Mathematical Elements for Computer Graphics", McGraw Hill, New York, 1976.



- Let's assume an interpolating technique has been chosen and that a function  $P(t)$  has been selected which, for a given value of  $t$ , will produce a value, i.e.,  $p = P(t)$ .
- If position is being interpolated then three functions are used in the following manner. The  $x$ ,  $y$  and  $z$  coordinates for the positions at the key frames are specified.



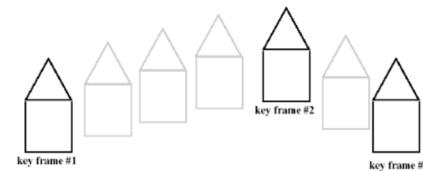
- Key frames are associated with specific values of the time ( $t$ ) parameter.
- The  $x$ ,  $y$  and  $z$  coordinates are considered independently.
- For example, the points  $(x,t)$  are used as control points for the interpolating curve so that  $X=P_x(t)$
- Similarly,  $Y=P_y(t)$  and  $Z=P_z(t)$  are formed so that at any time,  $t$ , a position  $(x,y,z)$  can be produced.

A  
time = 0

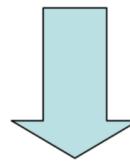
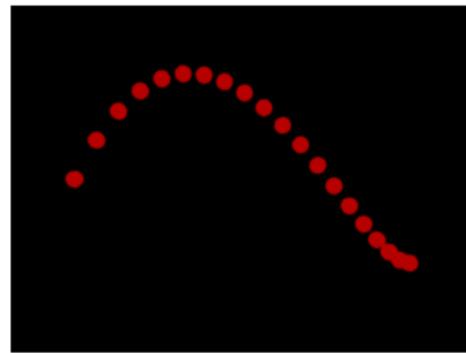
B  
time = 10

C  
time = 35

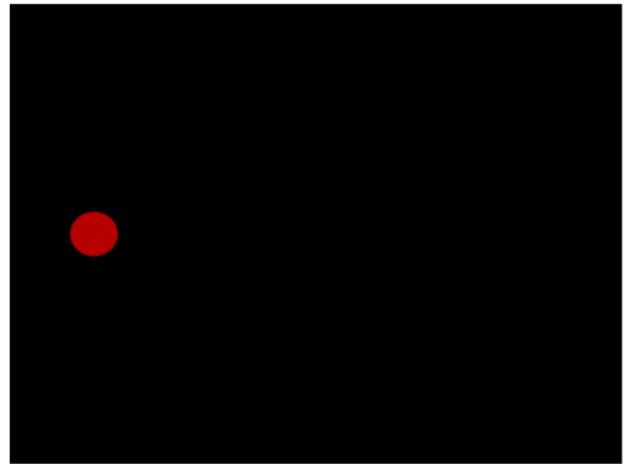
D  
time = 60



- Varying the parameter of interpolation, in this case  $t$ , by a constant amount, does not mean that the resulting values, in this case Euclidean position, will vary by a constant amount.
- This means that just because  $t$  changes at a constant rate, the interpolated value will not necessarily, in fact seldom, have a constant speed.



- This animation shows a ball moving along a parametrically defined cubic curve:  
 $P(t) = a*t**e + b*t**2 + c*t + d$   
by uniformly varying the parameter (i.e.  $t = 0.0, 0.1, 0.2, \dots$ ).
- Note that it travels further between frames at the start of the curve than at the end of the curve: **Equi-distant values in parametric space do not result in equi-distant points in Euclidean space.**



- In order to ensure a constant speed for the interpolated value, the interpolating function has to be parameterized by **arc length**, i.e., distance along the curve of interpolation.
- Some type of reparameterization by arc length should be performed for most applications.
- Usually this reparameterization can be approximated without adding undue overhead or complexity to the interpolating function.



1. Interpolation
2. **Motion along a curve (arc length)**
3. Interpolation of rotations (quaternions)
4. Path following

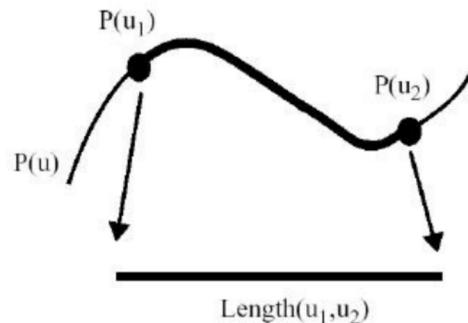


- Assume that we have a spline defined by  $P(t)$ .
- No guarantee that the variance in  $t$  is directly related to the distance traveled along the curve.
  - That is to say, that the distance traveled from  $P(0)$  to  $P(.2)$  is not necessarily twice as far as the distance traveled along the curve from  $P(.2)$  to  $P(.3)$ .
- In order to establish this relationship we have to reparameterize the interpolating spline by arc length or some scalar of arc length.
- There are various ways to approach this.



Analytically, arc length is defined as:

$$L = \int_{u1}^{u2} |dP/du| du$$



We use **Gaussian quadrature**  
(see "Numerical Recipes")  
to reduce the integral to

$$L = \sum_{i=1}^n w_i f(u_i)$$

- n is the number of sample points,
- wi are the weight values,
- ui are the sampled values.

See Mortenson, "Geometric Modeling", pp. 299-300

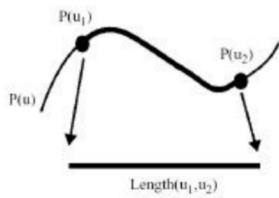
The u's can be normalized to the range zero to one and tables of weights and sample values can be found in tables (see Appendix B).



- Most of the time, the curves that arise in computer animation applications are not analytically reparameterizable by arc length.
- They must be reparameterized numerically.
- A simple, but somewhat inaccurate, approach to this reparameterization is to precompute a table of values which relates the original parameter with an arc length parameter.



- The number of entries in the table depends on the accuracy with which the arc length must be computed. This is determined by the application.
- The function is evaluated at  $n$  equidistant parameter values (equidistant in parameter space, e.g.,  $t = 0.0, 0.01, 0.02, 0.03$ , etc.).  $N$  should be sufficiently large to ensure that the resulting arc lengths are within tolerance; this will become clear as the technique is described.



Index	Parametric Entry	Arc Length
0	0.00	0.000
1	0.05	0.080
2	0.10	0.150
3	0.15	0.230
4	0.20	0.320
5	0.25	0.400
6	0.30	0.500
7	0.35	0.600
8	0.40	0.720
9	0.45	0.800
10	0.50	0.860
11	0.55	0.900
12	0.60	0.920
13	0.65	0.932
14	0.70	0.944
15	0.75	0.959
16	0.80	0.972
17	0.85	0.984
18	0.90	0.994
19	0.95	0.998
20	1.00	1.000

How to build this table?



$$u = 0.00, 0.05, 0.10, 0.15, \dots, 1.0$$

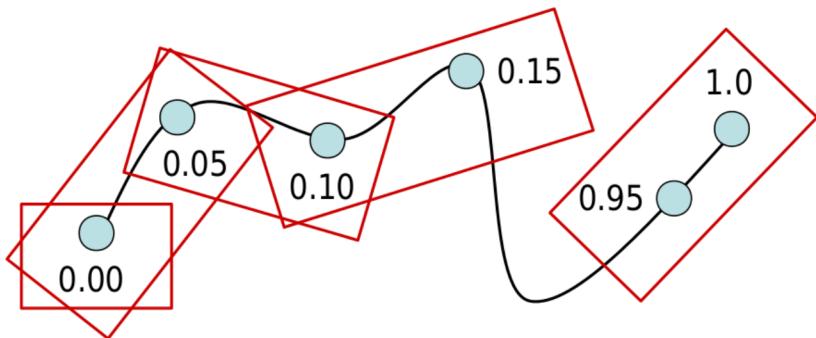


Table G( $u$ )



Index	Parametric Entry	Arc Length
0	0.00	0.000
1	0.05	0.080
2	0.10	0.150
3	0.15	0.230
4	0.20	0.320
5	0.25	0.400
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7	0.35	0.600
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16	0.80	0.972
17	0.85	0.984
18	0.90	0.994
19	0.95	0.998
20	1.00	1.000

$$G(0.0) = 0$$

$$G(0.05) = \text{dist. between } P(0.00) \text{ and } P(0.05)$$

$$G(0.10) = G(0.05) + \text{dist. between } P(0.05) \text{ and } P(0.10)$$

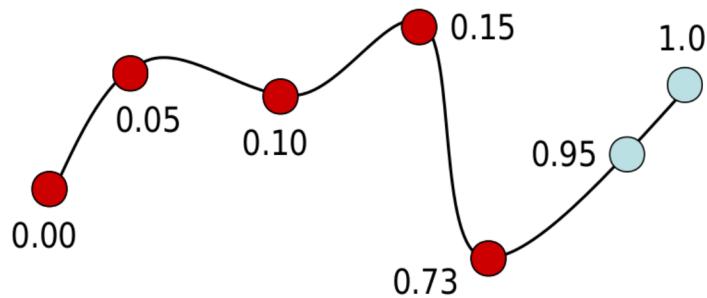
$$G(0.15) = G(0.10) + \text{dist. between } P(0.10) \text{ and } P(0.15)$$

...

$$G(1.00) = G(0.95) + \text{dist. between } P(0.95) \text{ and } P(1.00)$$

Arc Length in the table is dist/max. dist.

## How to use this table? (1)



User wants to know the distance (arc length) from the beginning of the curve ( $u=0.00$ ) to the point ( $u=0.73$ )

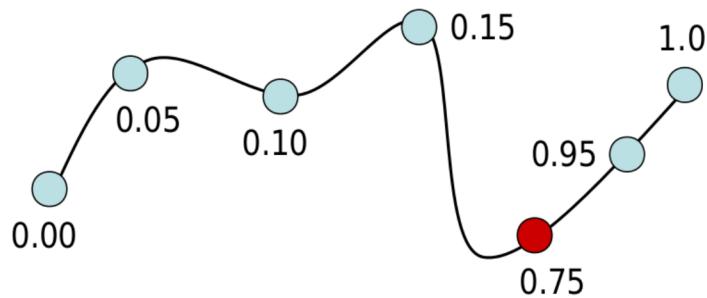
$$i = (\text{int})\left(\frac{\text{given parametric value}}{\text{distance between entries}}\right) = (\text{int})\left(\frac{0.73}{0.05}\right) = 14$$

Index	Parametric Entry	Arc Length
0	0.00	0.000
1	0.05	0.080
2	0.10	0.150
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4	0.20	0.320
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15	0.75	0.959
16	0.80	0.972
17	0.85	0.984
18	0.90	0.994
19	0.95	0.998
20	1.00	1.000

$$\begin{aligned}
 L &= \text{ArcLength}[i] + \frac{(\text{Given Value} - \text{Value}[i])}{(\text{Value}[i+1] - \text{Value}[i])} \cdot (\text{ArcLength}[i+1] - \text{ArcLength}[i]) \\
 &= 0.944 + \frac{0.73 - 0.70}{0.75 - 0.70} \cdot (0.959 - 0.944) \\
 &= 0.953
 \end{aligned}$$

## How to use this table? (2)

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User wants to know the value of  $u$  given the arc length;  
For example,  $L = 0.75$ ,  $u = ?$

$$0.75 = 0.72 + t(0.8 - 0.72)$$

$$t = \frac{0.75 - 0.72}{0.8 - 0.72} = \frac{0.03}{0.08} = 0.375$$

$$u = 0.4 + t(0.45 - 0.4)$$

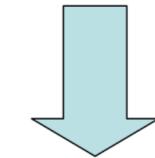
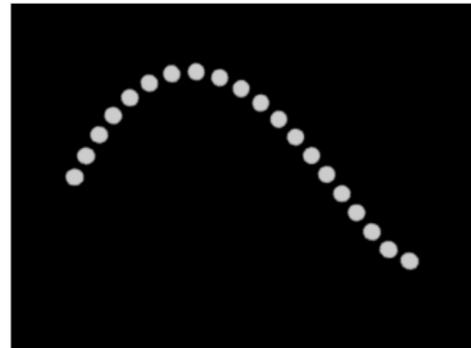
$$u = 0.4 + 0.375(0.45 - 0.4) = 0.41875$$

Index	Parametric Entry	Arc Length
0	0.00	0.000
1	0.05	0.080
2	0.10	0.150
3	0.15	0.230
4	0.20	0.320
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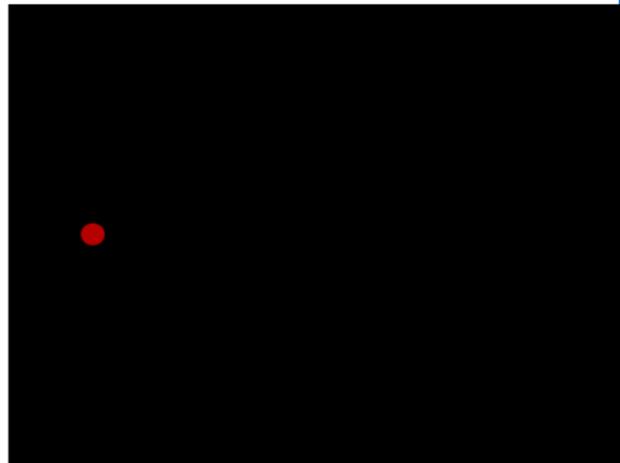


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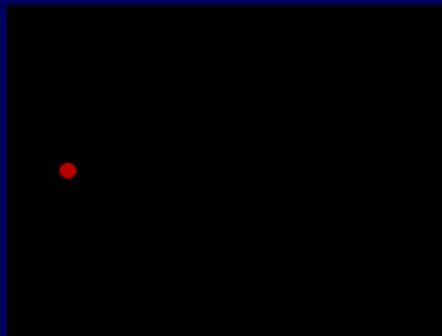
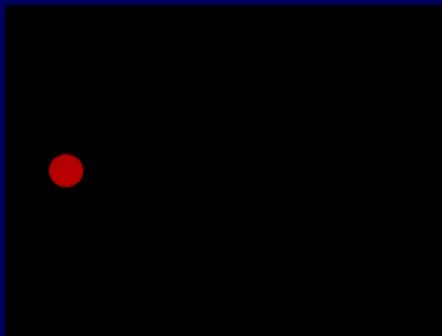
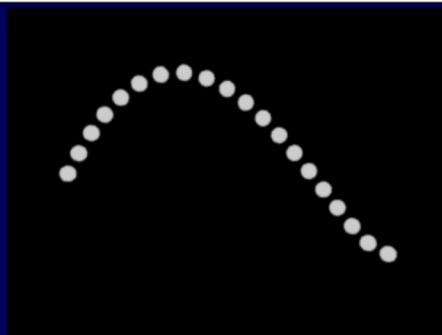
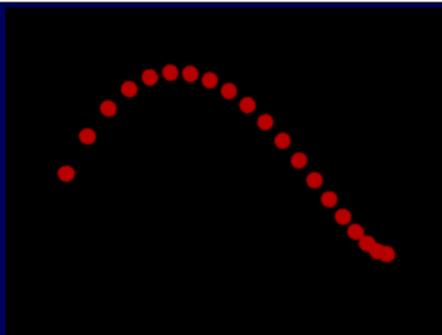
- Using the parameter values and corresponding values of the function, a table is built which records accumulated linear distances between the computed points.
- Entries are then made in the table which record normalized distances along the curve.
- These entries are monotonic, increasing from zero to one, and represent the arc length parameterization (a scaled version of the arc length).
- This table can be precomputed before being used in the animation.
- This table can be used to determine the functional parameter needed to produce a point along the curve that corresponds to arc length and effectively parameterizes the function by arc length.



- This animation shows a ball moving along a parametrically defined cubic curve:  $P(t) = a*t^{**3} + b*t^{**2} + c*t + d$
- The parameter,  $t$ , was uniformly varied from 0 to 1 generating 1000 samples.
- A table of linearly approximated normalized distances at those points was constructed.
- Then a final table of 20 points was extracted from the list to be as equally spaced as possible.
- No interpolation between samples was performed, although that would have made the final points more equally spaced. Note the ball travels pretty much the same amount between frames throughout the curve.



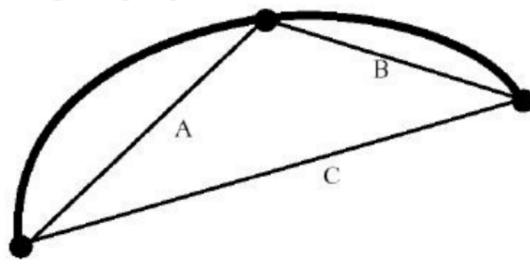
# Parametric Motion X Arc Length Motion



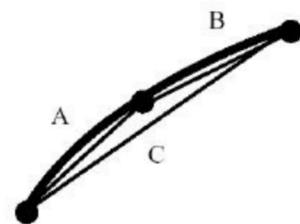
- An alternative technique is computationally more efficient but programmatically more difficult. This technique uses adaptive Gaussian quadrature to determine the value of the function at the point along the curve that corresponds to the given arc length.
- Guenter, B., and Parent, R.,  
"Computing the Arc Length of Parametric Curves,"  
IEEE Computer Graphics & Applications,  
May 1990, Vol. 10, No. 3., pp. 72-78.



## Subdivision Criteria

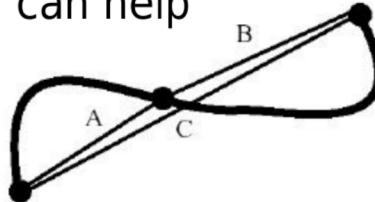


Lengths  $A+B-C$  above error tolerance



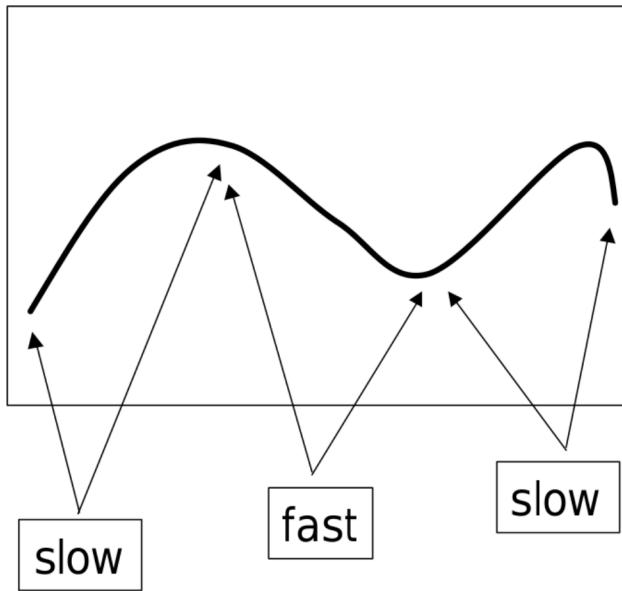
Lengths  $A+B-C$  within error tolerance

One extra subdivision can help



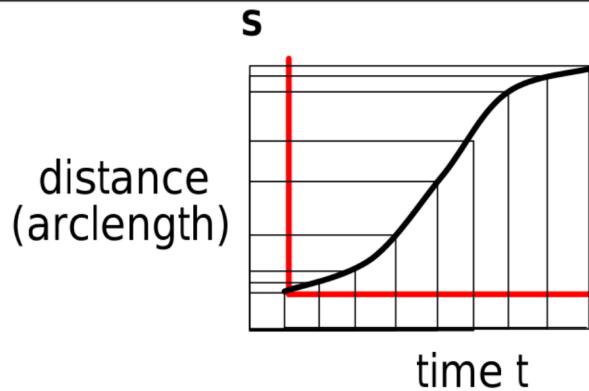
Lengths  $A+B-C$  erroneously reports that the error is within tolerance





Note the difference between interpolating position along a curve and the speed along which the interpolation proceeds (consider the interpolating parameter to be 'time').

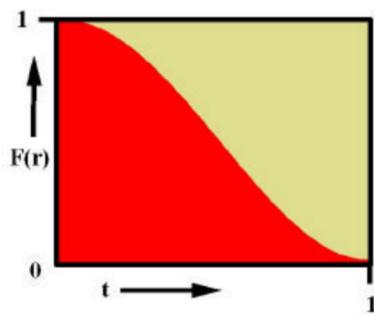
For example you can do linear interpolation in space but cubic interpolation of the distance with respect to the time parameter.



At a given time,  $t$ ,  $s(t)$  is the desired distance to have travelled.

Arc-length table gives corresponding parameter  $u$  for that distance.

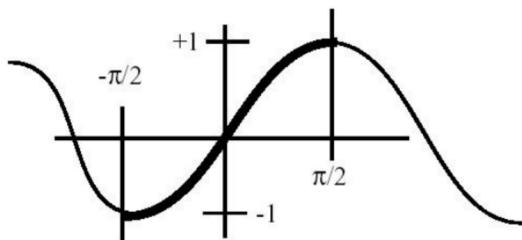




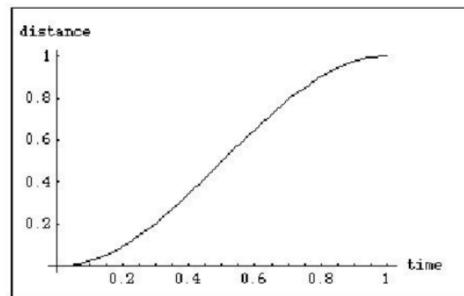
1.  $s(t)$  should be monotonic in  $t$   
i.e. traversed without going backwards in  $t$
2.  $s(t)$  should be continuous.  
No jumps from one point to the next on the curve.

Normalizing (0-1 range) makes it easier to use in conjunction  
with arc length and other functions.





a) Sine curve segment to use as ease-in/ease-out control



b) Sine curve segment mapped to useful values

map parameter values  $[0, +1]$  into domain of section of curve.  $[-\pi/, +\pi/2]$

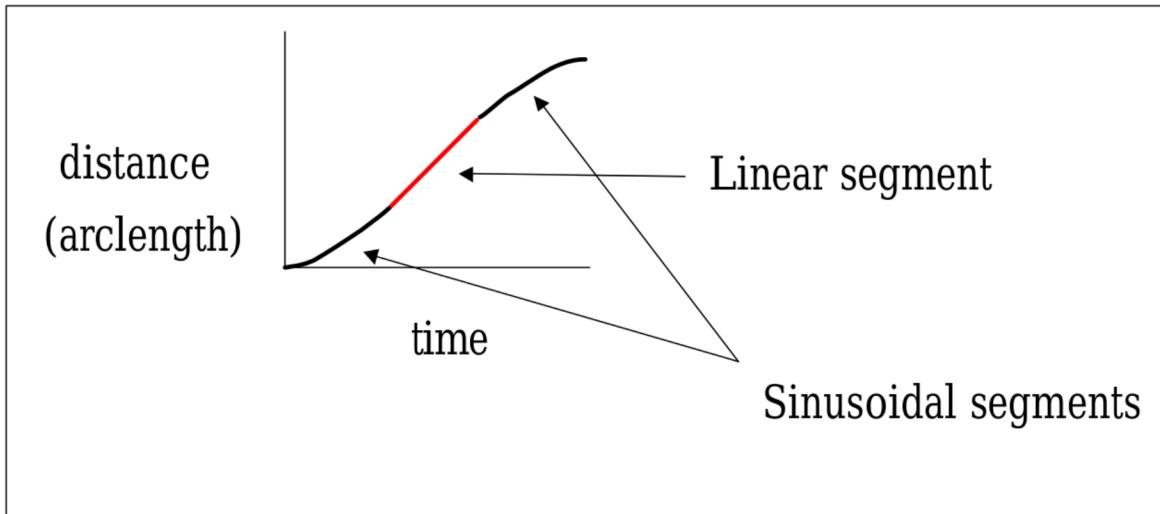
$$s(t) = \text{ease}(t) = \frac{\sin\left(t\pi - \frac{\pi}{2}\right) + 1}{2}$$

$$\begin{aligned} t = 0.25 &\rightarrow s(t) = 0.1465 \\ t = 0.75 &\rightarrow s(t) = 0.8535 \end{aligned}$$



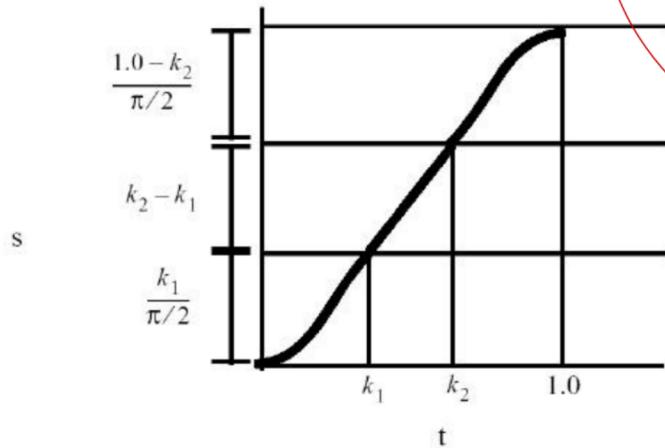
- Another method is to have the user specify times t1 and t2.
- A sinusoidal curve is used for velocity to implement an acceleration from time 0 to t1.
- A sinusoidal curve is also used for velocity to implement deceleration from time t2 to 1.
- Between times t1 and t2, a constant velocity is used.
- This is done by taking a parameter t in the range 0 to 1 and remapping it into that range according to the above velocity curves to get a new parameter rt.
- So as t varies uniformly from 0 to 1, rt will accelerate from 0, then maintain a constant parametric velocity and then decelerate back to 1.



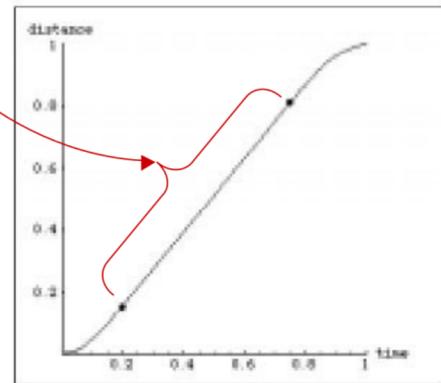


$$\frac{k_1}{\pi/2} + k_2 - k_1 + \frac{1.0 - k_2}{\pi/2} = 1.$$

**Constant Speed**



a) Ease-in/ease-out curve as it is initially pieced together

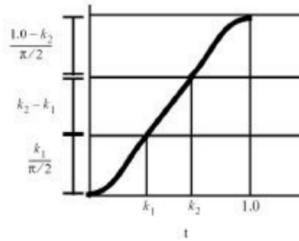


b) curve segments scaled into useful values with points marking segment junctions.

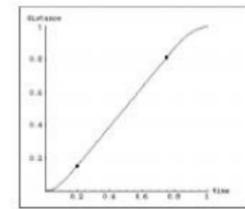


$$\frac{\hat{k}_1}{\pi/2} + k_2 - \hat{k}_1 + \frac{1.0 - \hat{k}_2}{\pi/2} =$$

$f$  is the total distance travelled  
to normalize the distance  
scale each segment by dividing by  $f$ ,



a) Ease-in/ease-out curve as it is initially pieced together.



b) curve segments scaled into useful values with points marking segment functions.

$$east(t) = \begin{cases} \left( k_1 \cdot \frac{2}{\pi} \cdot \sin\left(\left(\frac{t}{k_1} \cdot \frac{\pi}{2} - \frac{\pi}{2}\right) + 1\right)\right)/f & t \leq k_1 \\ \left(\frac{k_1}{\pi/2} + t - k_1\right)/f & k_1 \leq t \leq k_2 \\ \left(\frac{k_1}{\pi/2} + k_2 - k_1 + \left((1 - k_2) \cdot \frac{2}{\pi}\right) \sin\left(\frac{t - k_2}{1.0 - k_2} \cdot \frac{\pi}{2}\right)\right)/f & k_2 \leq t \end{cases}$$

where  $f = k_1 \cdot 2/\pi + k_2 - k_1 + (1.0 - k_2) \cdot 2/\pi$



```
float ease(float t, float k1, float k2)
```

```
{
```

```
    float t1,t2;
```

```
    float f,s;
```

$$f = k1*2/3.1415926535 + k2 - k1 + (1.0-k2)$$

```
if (t < k1) {
```

$$s = k1*(2/\pi)*(sin((t/k1)*\pi/2-\pi/2)+1);$$

```
}
```

```
else if (t < k2) {
```

$$s = (2*k1/\pi + t-k1);$$

```
}
```

```
else {
```

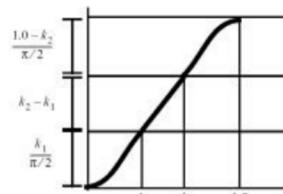
$$s = 2*k1/\pi + k2-k1 + ((1-k2)*(2/\pi))*sin(((t-k2)/(1.0-k2))*\pi/2);$$

```
}
```

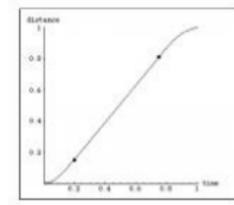
```
return (s/f);
```

```
}
```

$$\frac{k_1}{\pi/2} + k_2 - k_1 + \frac{1.0 - k_2}{\pi/2}$$



a) Ease-in/ease-out curve as it is initially pieced together



b) curve segments scaled into useful values with points marking segment junctions.



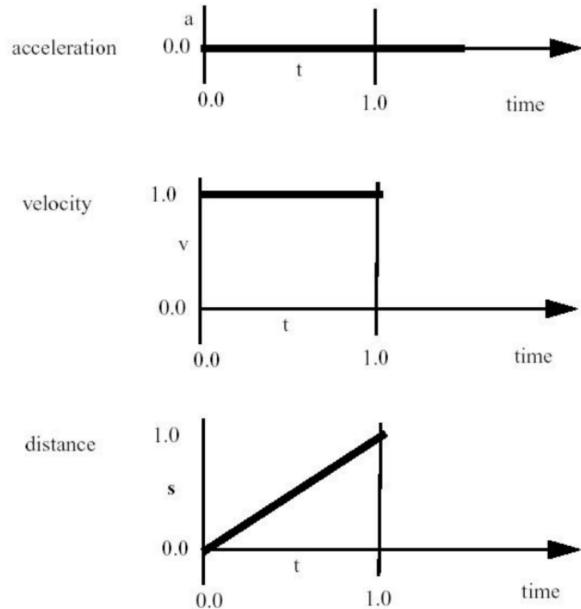
## Parabolic Ease-In/Ease-Out

- An alternative approach and one that avoids the transcendental function evaluation, or corresponding table look-up and interpolation, is to establish basic assumptions about the acceleration and, from there, integrate to get the resulting interpolation function.

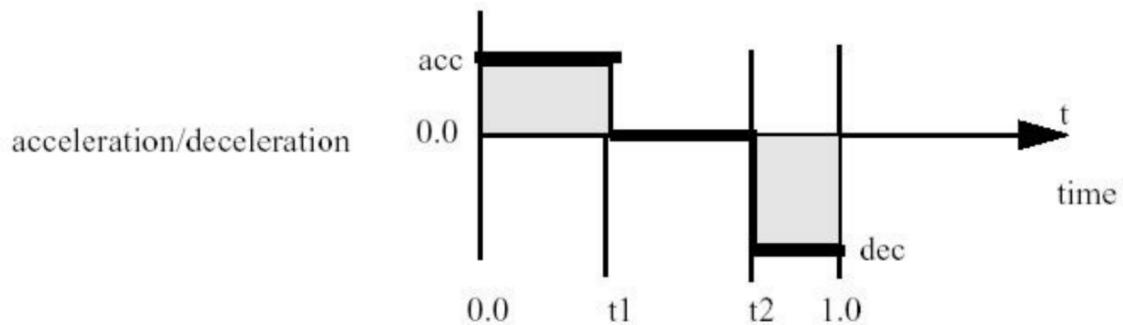


## Parabolic Ease-In/Ease-Out

- The default case of no ease-in/ease-out would produce a velocity curve that is a horizontal straight line of  $v_0$  as it goes from 0 to 1.
- The distance covered would be  $\text{ease}(1) = v_0 * 1$ .
- To implement an ease-in/ease-out function, assume constant acceleration and deceleration at the beginning and end of the motion, and zero acceleration during the middle of the motion.



## Parabolic Ease-In/Ease-Out



$$a = acc \quad 0 < t < t1$$

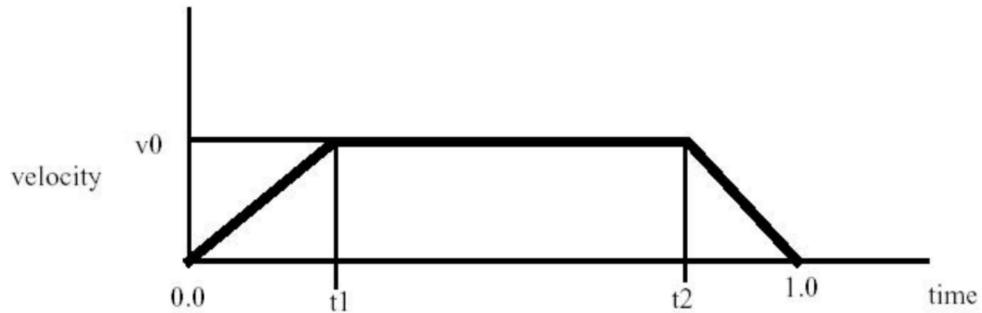
$$a = 0.0 \quad t1 < t < t2$$

$$a = dec \quad t2 < t < 1.0$$



## Parabolic Ease-In/Ease-Out

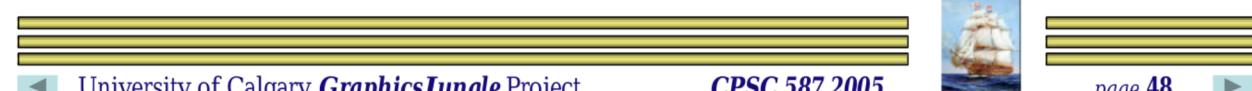
- Integrate to get velocity. This produces a linear ramp for accelerating and decelerating velocities



$$v = v_0 \cdot \frac{t}{t_1} \quad 0.0 < t < t_1$$

$$v = v_0 \quad t_1 < t < t_2$$

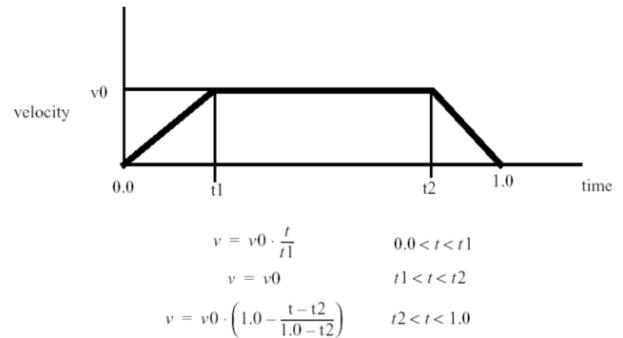
$$v = v_0 \cdot \left(1.0 - \frac{t - t_2}{1.0 - t_2}\right) \quad t_2 < t < 1.0$$



## Parabolic Ease-In/Ease-Out

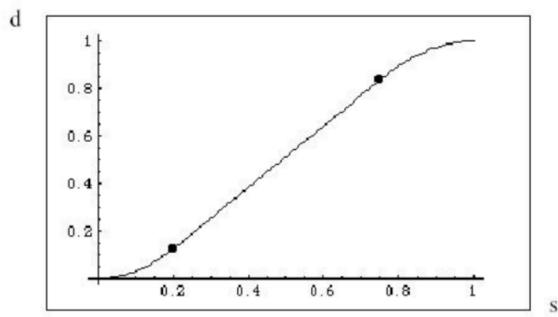
- More intuitive if user specifies  $t_1$  and  $t_2$ , and the system can solve for the maximum velocity

$$v_0 = \frac{2.0}{(t_2 - t_1 + 1.0)}$$



## Parabolic Ease-In/Ease

Distance function  
with parabolic  
sections at both ends.



$$d = v_0 \cdot \frac{t^2}{2 \cdot t_1} \quad 0.0 < t < t_1$$

$$d = v_0 \cdot \frac{t_1}{2} + v_0 \cdot (t - t_1) \quad t_1 < t < t_2$$

$$d = v_0 \cdot \frac{t_1}{2} + v_0 \cdot (t_2 - t_1) + \left( v_0 - \frac{\left( v_0 \cdot \frac{t - t_2}{1 - t_2} \right)}{2} \right) \cdot (t - t_2) \quad t_2 < t < 1.0$$



## Parabolic Ease-In/Ease-Out

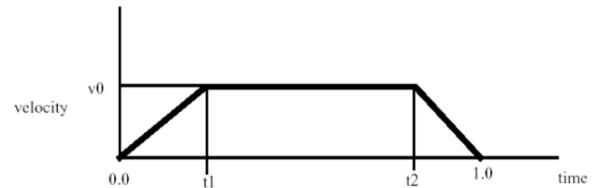
- *Distance* in this case is in parametric space, or t-space, and is the distance covered by the output value of  $t$ .
- For a given range of  $t = [0,1]$ , the user specifies times to control acceleration and deceleration:  $t_1$  and  $t_2$ .
- Acceleration occurs from time 0 to time  $t_1$ .
- Deceleration occurs from time  $t_2$  to time 1.



## Parabolic Ease-In/Ease-Out

```
double ease(double t, double t1, double t2)
{
    double v0,a1,a2;

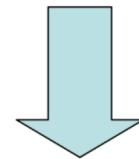
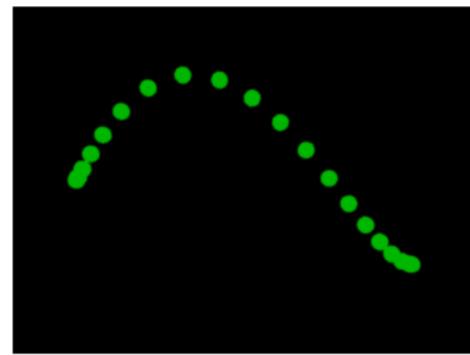
    v0 = 2/(1+t2-t1); /* constant velocity attained */
    if (t<t1)
        { d = v0*t*t/(2*t1); }
    else
        {
            d = v0*t1/2;
            if (t<t2)
                { d += (t-t1)*v0; }
            else
                {
                    d += (t2-t1)*v0;
                    d += (t-t*t/2-t2+t2*b/2)*v0/(1-t2);
                }
        }
    return (d);
}
```



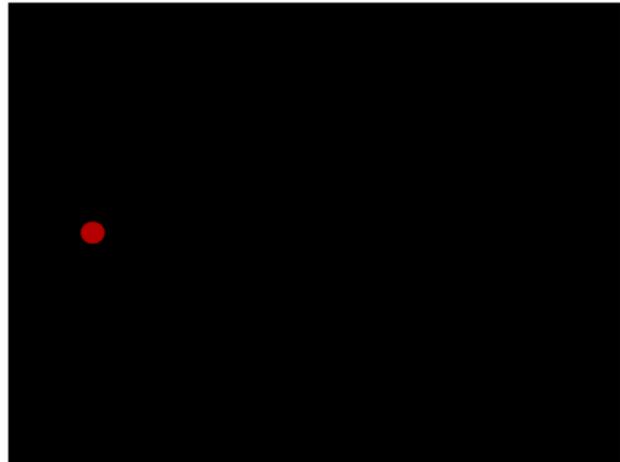
$$v = v_0 \cdot \frac{t}{t_1} \quad 0.0 < t < t_1$$

$$v = v_0 \quad t_1 < t < t_2$$

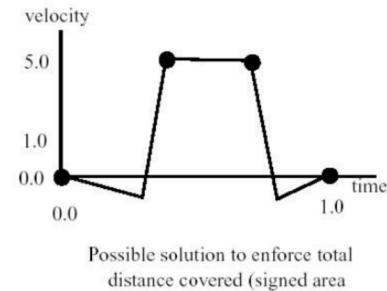
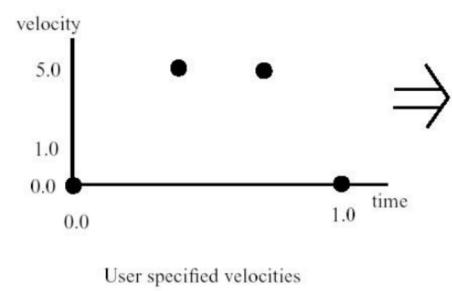
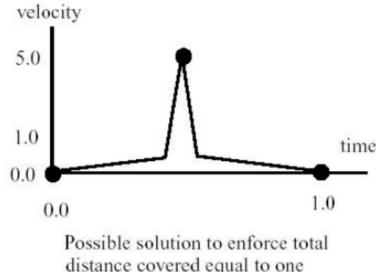
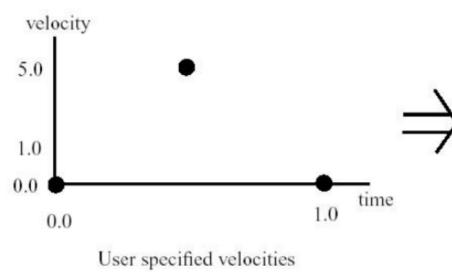
$$v = v_0 \cdot \left(1.0 - \frac{t-t_2}{1.0-t_2}\right) \quad t_2 < t < 1.0$$

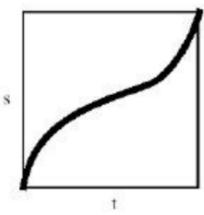


- This animation uses the same procedure as the arc length animation with one exception.
- Before searching through the table of values to find the entry with the closest normalized distance, the parameter is passed through a constant-acceleration ease function which recomputes the value of the parameter.
- Note how, as the animation starts, the ball slowly increases its speed, maintains a constant speed, and then slows down at the end of the curve.
- Acceleration takes place until  $t=0.4$ ; deceleration starts at  $t=0.7$ .



## Some non-intuitive results of user-specified values on the velocity-time curve.

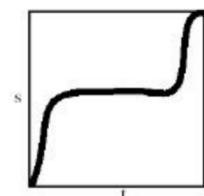




a) starts and ends abruptly



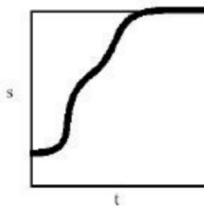
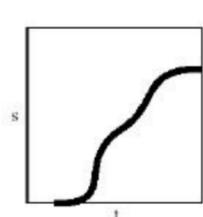
b) backs up



c) stalls



d) smoothly starts and stops

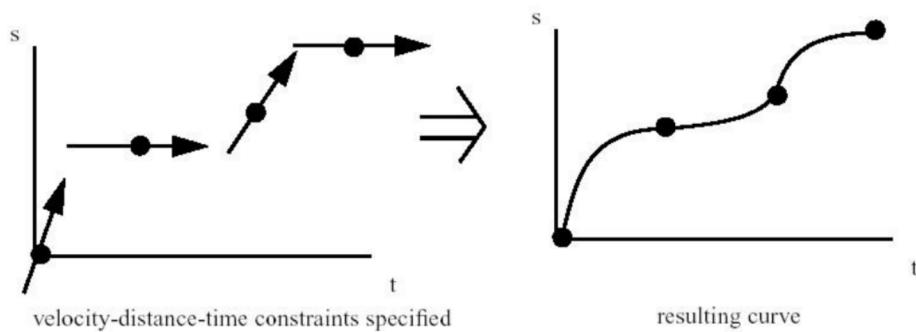
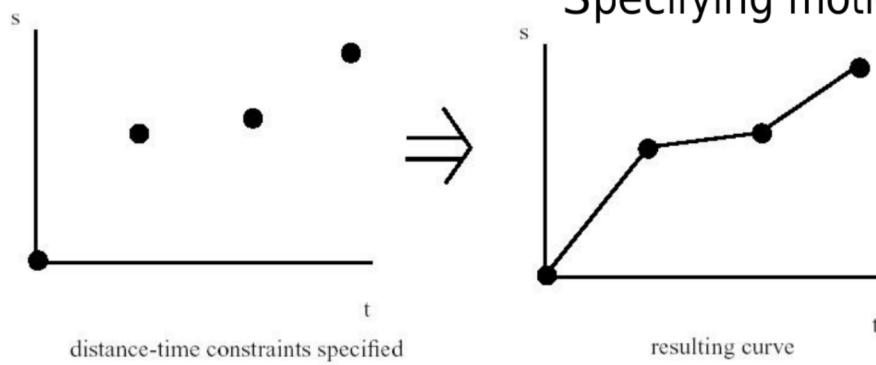
e) starts part way along the curve and gets to the end before  $t=1.0$ 

f) waits awhile before starting and doesn't get all the way to the end

## Distance Time Functions



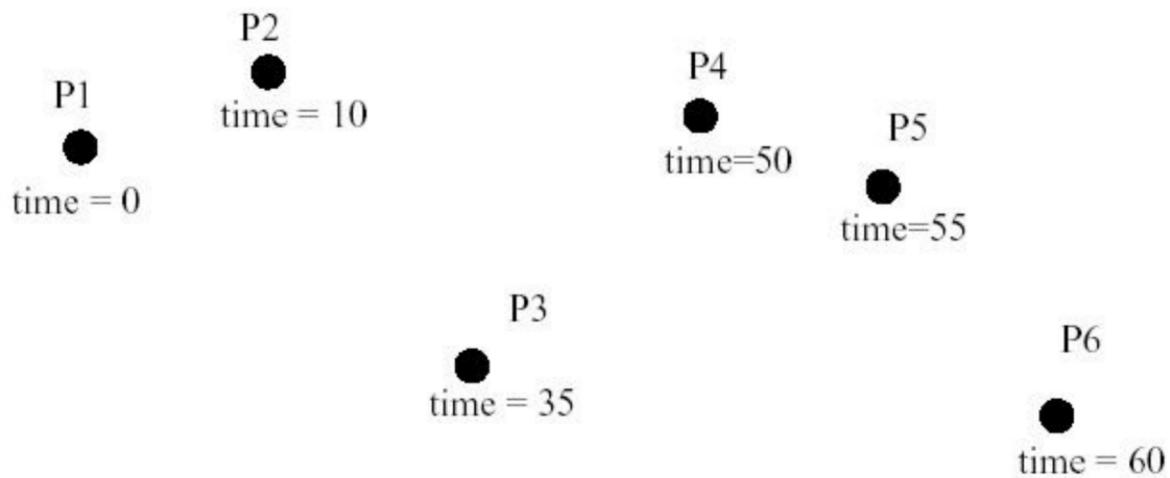
## Specifying motion constraints



resulting curve



## Position-time constraints



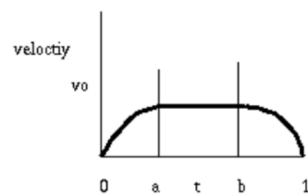
## with Intermediate Constant Speed

- Another method is to have the user specify times a and b, similar to the constant acceleration technique described above. However, in this case a segment of the sine curve is used to control the velocity acceleration and deceleration. Between times a and b, a constant velocity is used.



## with Intermediate Constant Speed

- The development of this is similar to that for constant acceleration. A velocity curve is constructed from sine segments with interior constant velocity. The constant velocity,  $v_0$ , is unknown.



- The distance covered can be computed as a function of  $v_0$ . A normalized distance of one can be used and  $v_0$  can be solved for. From this the equations are integrated and equations for the distance covered can be derived (and is left as an exercise for the reader).

