Reconsidered error analysis in the finite element methods

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Abstract

We introduce new proof methods for interpolation error estimates. This article is written on the assumption that standard error analysis using the finite element method is studied.

Contents

1 Preliminalies

1.1 General Convention

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants c are bounded if the maximum angle is bounded. These values vary across different contexts.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert \cdot \rvert_d$	d-dimensional Hausdorff measure
$v\vert_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x\rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x\rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
A, B	Matrices
A_{ij} or $[A]_{ij}$	Entry of A in the ith and the jth column
$A^\top$	Transpose of the matrix A
$\mathrm{Tr}(A)$	Trace of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A_{ii}$
$\det(A)$	Determinant of A
$\mathrm{diag}(A)$	Diagonal of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{diag}(A)_{ij} := \delta_{ij} A_{ij}$, $1 \leq i, j \leq d$
Ax	Matrix-vector product: For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, $(Ax)_i := \sum_{j=1}^{d} A_{ij} x_j$ for $1 \leq i \leq d$
$A : B$	Double contraction: For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$, $A : B := \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$
$\lVert A\rVert_2$	Operator norm of A: For $A \in \mathbb{R}^{d \times d}$, $\lVert A\rVert_2 := \sup_{0 \neq x \in \mathbb{R}^d} \frac{\lvert Ax\rvert_E}{\lvert x\rvert_E}$
$\lVert A\rVert_{\max}$	Max norm of A: For $A \in \mathbb{R}^{d \times d}$, $\lVert A\rVert_{\max} := \max_{1 \leq i,j \leq d} \lvert A_{ij}\rvert$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this article, we use the following facts.

For $A \in \mathbb{R}^{m\times n}$, it holds that

$$\|A\|_{\max} \le \|A\|_2 \le \sqrt{mn}\|A\|_{\max}, \tag{1.1}$$

e.g., see [12, p. 56]. For $A, B \in \mathbb{R}^{m\times m}$, it holds that

$$\|AB\|_2 \le \|A\|_2\|B\|_2. \tag{1.2}$$

If $A^\top A$ is a positive definite matrix in $\mathbb{R}^{d\times d}$, the spectral norm of the matrix $A^\top A$ is the largest eigenvalue of $A^\top A$; i.e.,

$$\|A\|_2 = \left(\lambda_{\max}(A^\top A)\right)^{1/2} = \sigma_{\max}(A), \tag{1.3}$$

where $\lambda_{\max}(A)$ and $\sigma_{\max}(A)$ are respectively the largest eigenvalues and singular values of A.

If $A \in O(d)$, because $A^\top = A^{-1}$ and

$$|Ax|_E^2 = (Ax)^\top(Ax) = x^\top A^\top Ax = x^\top A^{-1}Ax = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0\neq x\in\mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0\neq x\in\mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

1.4 Function Spaces

In this article, we use standard Sobolev spaces with associated norms (e.g., see [2, 8, 9]).

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \dots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T)$, $\mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$) is spanned by the restriction to T of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$)

1.5.2 Meshes

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T\in\mathbb{T}_h} T,$$

with $h := \max_{T\in\mathbb{T}_h} h_T$, where $h_T := \mathrm{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $p_1, \dots, p_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex p_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^\partial$ be the set of faces on boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$
\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.
$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1, \infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$
W^{s,p}(\mathbb{T}_h; \mathbb{R}^q) := \{v \in L^p(\Omega; \mathbb{R}^q) : \ v|_T \in W^{s,p}(T; \mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}
$$

with the norms

$$
\begin{aligned}
\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} &:= \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \quad \text{if } p \in [1, \infty), \\
\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} &:= \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.
\end{aligned}
$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h; \mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h; \mathbb{R}^q) := W^{s,2}(\mathbb{T}_h; \mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h; \mathbb{R})$. We use the norm

$$
|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|^2_{L^2(T)^d} \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).
$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$
[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.
$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$
\begin{aligned}
[\![v \cdot n]\!] &:= [\![v \cdot n]\!]_F := v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp, \\
[\![v]\!] &:= [\![v]\!]_F := v_\natural - v_\sharp, \quad \natural > \sharp,
\end{aligned}
$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$
\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.
$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_\omega := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F}v_\natural + \omega_{T_\sharp,F}v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi)\cdot n]\!]_F = \{\{v\}\}_{\omega,F}\cdot n_F[\![\varphi]\!]_F + [\![v\cdot n]\!]_F\{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1,\infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h\varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div};T)$ space by

$$H(\mathrm{div};\mathbb{T}_h) := \left\{v \in L^2(\Omega)^d;\ v|_T \in H(\mathrm{div};T)\ \forall T \in \mathbb{T}_h\right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div};\mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div};\mathbb{T}_h)$,

$$(\mathrm{div}_h v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.5.4 Barycentric Coodinates

For a simplex $T \subset \mathbb{R}^d$, let $\{p_i\}_{i=1}^{d+1}$ be vertices of T and $(x_1^{(i)},\ldots,x_d^{(i)})^T$ coordinates of p_i. We set

$$\Delta := \det\begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ of the point $p(x_1,\ldots,x_d)$ with respect to $\{p_i\}_{i=1}^{d+1}$ are then defined as

$$\lambda_i(x) := \frac{1}{\Delta}\det\begin{array}{c} \begin{array}{ccccc} & & i & & \\ & & \smile & & \end{array} \\ \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix} \end{array}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(p_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1}\lambda_i(x) = 1.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

Let r, s be two nonnegative real numbers and $\{x_i\}_{i\in I}$ be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r \leq s, \\ \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \operatorname{card}(I)^{\frac{r-s}{rs}} \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \tag{1.4}$$

see [9, Exercise 12.1].

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.1. Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) & \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.5}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.6}$$

Proof. See, for example, [8, Corollary B.43, Theorem B.40] and [9, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.2. Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.7}$$

Proof. See, for example, [24, Theorem 2.1 (p. 61)] and the references therein. □

1.6.3 Trace Theorem

Theorem 1.3 (Trace on low-dimensional manifolds)**.** Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0, \ldots, d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.

Proof. See [9, Theorem 3.15]. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [7, 4]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [1, Lemma 2.1].

Lemma 1.4. Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped with respect to balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leq m \leq \ell$, $p \in [1, \infty]$. It then holds that

$$\|\partial^\gamma(\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \leq C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.8}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \leq \ell-1} \int_B \eta(y)(\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \tag{1.9}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [6, Theorem 1.1] and [7, 4, 27] which are variants of the Bramble–Hilbert lemma.

Lemma 1.5. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d,m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \ldots, m. \tag{1.10}$$

Proof. The proof is found in [6, Theorem 1.1]. □

Remark 1.6. In [4, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d,m,\gamma) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \ldots, m. \tag{1.11}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\operatorname{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d,m,\gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d,m)$ of the estimate (1.10) does not depend on the geometric parameter γ.

Remark 1.7. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d,m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf*ü*rth [27].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \leq C^{BH}(d,k,m) \operatorname{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0, 1, \ldots, m-1. \tag{1.12}$$

Verf*ü*rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d,k,m) \leq \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3, 1, 2) \le \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \le \frac{\sqrt{6}}{\pi}|\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

because $\mathrm{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.8 (Poincaré inequality)**.** Let $D \subset \mathbb{R}^d$ be a convex domain with diameter $\mathrm{diam}(D)$. It then holds that, for $\varphi \in H^1(D)$ with $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \le \frac{\mathrm{diam}(D)}{\pi}|\varphi|_{H^1(D)}. \tag{1.13}$$

Proof. The proof is found in [25, Theorem 3.2], also see [26]. □

Remark 1.9. The coefficient $\frac{1}{\pi}$ of (1.13) may be improved.

1.7 Abbreviated expression

FE	Finite Element
FEMs	Finite Element Methods

2 Isotropic and Anisotropic Mesh Elements

In the context of FEMs, mesh elements can be classified based on their geometric properties. *An isotropic mesh element* has equal or nearly equal edge lengths and angles, resulting in a balanced shape. In contrast, *an anisotropic mesh element* features significant variation in edge lengths and angles.

Consider the following examples: Let $s, \delta \in \mathbb{R}_+$, and $\varepsilon \ge 1$, $\varepsilon \in \mathbb{R}$.

Example 2.1. In the case of the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$, the triangle is classified as follows:

- If $\delta \approx 1$, the triangle T is considered an isotropic mesh element.
- Conversely, if $\delta \ll 1$, the triangle T becomes an anisotropic mesh element.

Example 2.2. In this case, consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Here, the vertex p_3 introduces a parameter ε that can influence the shape of the simplex. The classification of this simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle maintains a balanced shape, making it isotropic.
- If $\varepsilon \ge 1$, the triangle becomes flat when $s \ll 1$, resulting in an anisotropic mesh element.

Example 2.3. Consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. In this configuration, the classification of the simplex as isotropic or anisotropic depends on the value of δ:

- If $\delta \approx 1$, the triangle is an isotropic mesh element.
- If δ is much less than 1, i.e., $\delta \ll 1$, the triangle becomes an anisotropic mesh element.

Example 2.4. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. In this case, the classification of the simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle is isotropic because the height from p_3 is equal to the base length.
- If $\varepsilon \geq 1$, the triangle will be classified as anisotropic, as the edge lengths will differ significantly when $s \ll 1$.

3 Classical Geometric Conditions

3.1 Classical Interpolation Error Estimate

Let $\widehat{T} \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^d$ be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ and $\{T, P, \Sigma\}$ with associated normed vector spaces $V(\widehat{T})$ and $V(T)$. The transformation Φ_T takes the form

$$\Phi_T : \widehat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := B_T \hat{x} + b_T \in T,$$

where $B_T \in \mathbb{R}^{d \times d}$ is an invertible matrix and $b_T \in \mathbb{R}^d$. Let $I_T : V(T) := W^{2,p}(T) \to P := \mathbb{P}^1(T)$ with $p \in [1, \infty]$ be an interpolation on T with $I_T p = p$ for any $p \in \mathcal{P}^1(T)$. According to the classical theory (e.g., see [5, 8]), there exists a positive constant c, independent of h_T, such that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \left(\|B_T\|_2 \|B_T^{-1}\|_2 \right) \|B_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity $\|B_T\|_2 \|B_T^{-1}\|_2$ is called the *Euclidean condition number* of B_T. By standard estimates (e.g., see [8, Lemma 1.100]), we have

$$\|B_T\|_2 \|B_T^{-1}\|_2 \leq c \frac{h_T}{\rho_T}, \quad \|B_T\|_2 \leq c h_T.$$

It thus holds that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.1}$$

As a geometric condition, the *shape-regularity condition* is well known to obtain global interpolation error estimates. This condition states as follows.

Condition 3.1 (Shape-regularity condition)**.** There exists a constant $\gamma_1 > 0$ such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{3.2}$$

Under Condition 3.1, that is, when the quantity $\frac{h_T}{\rho_T}$ is bounded on each T, it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq c h |\varphi|_{W^{2,p}(\Omega)},$$

where $I_h \varphi$ is the standard global linear interpolation of φ on $\mathbb{T}_h$.

3.2 Regular Mesh Conditions

Geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity condition (3.2). A proof can be found in [3, Theorem 1].

Condition 3.2 (Zlámal's condition)**.** There exists a constant $\gamma_2 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$, any simplex $T \in \mathbb{T}_h$ and any dihedral angle ψ and for $d = 3$, also any solid angle θ of T, we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \tag{3.3}$$

Condition 3.3. There exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_3 h_T^d. \tag{3.4}$$

Condition 3.4. There exists a constant $\gamma_4 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_4 |B_d^T|, \tag{3.5}$$

where $B^T \supset T$ is the circumscribed ball of T.

Note 3.5. If Condition 3.1 or 3.2 or 3.3 or 3.4 holds, a family of simplicial partitions is called *regular*.

Note 3.6. Condition 3.2 was presented by Zlámal [28] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on $\mathbb{R}^2$. Zlámal's condition can be generalised into $\mathbb{R}^n$ for any $n \in \{2, 3, \ldots\}$. Later, the shape-regularity condition (the inscribed ball condition) was introduced; see [5]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

Note 3.7. Condition 3.3 seems to be simpler than Condition 3.1, Condition 3.2 and Condition 3.4. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

3.3 What happens when anisotropic meshes are used?

Using the equivalence conditions in Section 3.2, the error estimate (3.1) is rewritten as

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T^2}{|T|_2} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.6}$$

We considered the following four anisotropic elements as in Section 2: Let $0 < s, \delta \ll 1$, $s, \delta \in \mathbb{R}$, and $\varepsilon > 1$, $\varepsilon \in \mathbb{R}$.

Example 3.8. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = \delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{4}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}.$$

When $\delta \ll 1$, the interpolation error (3.6) may be large.

Example 3.9. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = s^{1+\varepsilon}$, and

$$\frac{h_T^2}{|T|_2} = 4s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity is not satisfied. In this case, when $\varepsilon > 2$, the estimate (3.6) diverges as $s \to 0$.

Example 3.10. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. Then, we have that $h_T = s\sqrt{1+\delta^2} \approx s$, and $|T|_2 = \frac{1}{2}\delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{2(1+\delta^2)}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.7}$$

It is implied that the interpolation error (3.7) may be large when $\delta \ll 1$.

Example 3.11. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. Subsequently, we obtain $h_T = \sqrt{s^2 + s^{2\varepsilon}} \approx s$, and $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$, and

$$\frac{h_T^2}{|T|_2} = \frac{2(s^2 + s^{2\varepsilon})}{s^{1+\varepsilon}} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$.

Remark 3.12. As will be explained later, the factor $\frac{1}{\delta}$ in Example 3.10 is violated and the interpolation error estimate converges in the case of Example 3.11 using new precise interpolation error estimates under more relaxed geometric conditions.

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