Reconsidered error analysis in the finite element methods

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Abstract

This article presents novel proof methods for estimating interpolation errors, predicated on the understanding that one has already studied foundational error analysis using the finite element method.

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Part I

Interpolation Error Analysis using a New Geometric Parameter

1 Preliminalies

1.1 General Convention

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants c are bounded if the maximum angle is bounded. These values vary across different contexts.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert \cdot \rvert_d$	d-dimensional Hausdorff measure
$v\vert_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x \rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x \rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
A, B	Matrices
A_{ij} or $[A]_{ij}$	Entry of A in the ith and the jth column
$A^\top$	Transpose of the matrix A
$\mathrm{Tr}(A)$	Trace of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A_{ii}$
$\det(A)$	Determinant of A
$\mathrm{diag}(A)$	Diagonal of A:
	For $A \in \mathbb{R}^{m \times n}$, $\mathrm{diag}(A)_{ij} := \delta_{ij} A_{ij}$, $1 \leq i, j \leq d$
Ax	Matrix-vector product:
	For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, $(Ax)_i := \sum_{j=1}^{d} A_{ij} x_j$ for $1 \leq i \leq d$
$A : B$	Double contraction:

	For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, $A : B := \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$
$\|A\|_2$	Operator norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_2 := \sup_{0\neq x\in\mathbb{R}^d} \frac{\|Ax\|_E}{\|x\|_E}$
$\|A\|_{\max}$	Max norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_{\max} := \max_{1\le i,j\le d} \|A_{ij}\|$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this article, we use the following facts.

For $A \in \mathbb{R}^{m\times n}$, it holds that

$$\|A\|_{\max} \le \|A\|_2 \le \sqrt{mn}\|A\|_{\max}, \tag{1.1}$$

e.g., see [19, p. 56]. For $A, B \in \mathbb{R}^{m\times m}$, it holds that

$$\|AB\|_2 \le \|A\|_2\|B\|_2. \tag{1.2}$$

If $A^\top A$ is a positive definite matrix in $\mathbb{R}^{d\times d}$, the spectral norm of the matrix $A^\top A$ is the largest eigenvalue of $A^\top A$; i.e.,

$$\|A\|_2 = \left(\lambda_{\max}(A^\top A)\right)^{1/2} = \sigma_{\max}(A), \tag{1.3}$$

where $\lambda_{\max}(A)$ and $\sigma_{\max}(A)$ are respectively the largest eigenvalues and singular values of A.

If $A \in O(d)$, because $A^\top = A^{-1}$ and

$$|Ax|_E^2 = (Ax)^\top(Ax) = x^\top A^\top Ax = x^\top A^{-1}Ax = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0\neq x\in\mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0\neq x\in\mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

1.4 Function Spaces

This article uses standard Sobolev spaces with associated norms (e.g., see [6, 14, 15]).

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \ldots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T)$, $\mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$) is spanned by the restriction to T of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$)

1.5.2 Meshes

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \operatorname{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $p_1, \dots, p_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex p_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^\partial$ be the set of faces on boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1,\infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h;\mathbb{R}^q) := \{v \in L^p(\Omega;\mathbb{R}^q) : \ v|_T \in W^{s,p}(T;\mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} := \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \quad \text{if } p \in [1,\infty),$$
$$\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} := \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h;\mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h;\mathbb{R}^q) := W^{s,2}(\mathbb{T}_h;\mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h;\mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla\varphi\|^2_{L^2(T)^d} \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.$$

For a boundary face $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$\begin{aligned}[\![v \cdot n]\!] := [\![v \cdot n]\!]_F &:= v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp, \\ [\![v]\!] := [\![v]\!]_F &:= v_\natural - v_\sharp, \quad \natural > \sharp,\end{aligned}$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_{\omega} := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F}v_\natural + \omega_{T_\sharp,F}v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi) \cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1, \infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div}; T)$ space by

$$H(\mathrm{div}; \mathbb{T}_h) := \left\{ v \in L^2(\Omega)^d;\ v|_T \in H(\mathrm{div}; T)\ \forall T \in \mathbb{T}_h \right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div}; \mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div}; \mathbb{T}_h)$,

$$(\mathrm{div}_h\, v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.5.4 Barycentric Coodinates

For a simplex $T \subset \mathbb{R}^d$, let $\{p_i\}_{i=1}^{d+1}$ be vertices of T and $(x_1^{(i)}, \dots, x_d^{(i)})^T$ coordinates of p_i. We set

$$\Delta := \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ of the point $p(x_1, \dots, x_d)$ with respect to $\{p_i\}_{i=1}^{d+1}$ are then defined as

$$\lambda_i(x) := \frac{1}{\Delta} \det \begin{array}{c} \begin{array}{ccccc} & & i & & \\ & & \smile & & \end{array} \\ \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix} \end{array}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(p_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

Let r, s be two nonnegative real numbers and $\{x_i\}_{i\in I}$ be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r \leq s, \\ \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \operatorname{card}(I)^{\frac{r-s}{rs}} \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \tag{1.4}$$

see [15, Exercise 12.1].

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.1. Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) & \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.5}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.6}$$

Proof. See, for example, [14, Corollary B.43, Theorem B.40] and [15, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.2. Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.7}$$

Proof. See, for example, [36, Theorem 2.1 (p. 61)] and the references therein. □

1.6.3 Trace Theorem

Theorem 1.3 (Trace on low-dimensional manifolds)**.** Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0, \dots, d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.

Proof. See [15, Theorem 3.15]. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [13, 8]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [2, Lemma 2.1].

Lemma 1.4. Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped concerning balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le \ell$, $p \in [1, \infty]$. It then holds that

$$\|\partial^\gamma (\varphi - Q^{(\ell)} \varphi)\|_{W^{\ell-m,p}(D)} \le C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.8}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)} \varphi)(x) := \sum_{|\delta| \le \ell - 1} \int_B \eta(y) (\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \tag{1.9}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [12, Theorem 1.1] and [13, 8, 41] which are variants of the Bramble–Hilbert lemma.

Lemma 1.5. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.10}$$

Proof. The proof is found in [12, Theorem 1.1]. $\square$

Remark 1.6. In [8, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m,\gamma) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.11}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\operatorname{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d,m,\gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d,m)$ of the estimate (1.10) does not depend on the geometric parameter γ.

Remark 1.7. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d,m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf*ü*rth [41].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \le C^{BH}(d,k,m) \operatorname{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0, 1, \dots, m-1. \tag{1.12}$$

Verf*ü*rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d,k,m) \le \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3, 1, 2) \le \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \le \frac{\sqrt{6}}{\pi} |\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

because $\operatorname{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.8 (Poincaré inequality)**.** Let $D \subset \mathbb{R}^d$ be a convex domain with diameter $\operatorname{diam}(D)$. It then holds that, for $\varphi \in H^1(D)$ with $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \le \frac{\operatorname{diam}(D)}{\pi} |\varphi|_{H^1(D)}. \tag{1.13}$$

Proof. The proof is found in [37, Theorem 3.2], also see [38]. □

Remark 1.9. The coefficient $\frac{1}{\pi}$ of (1.13) may be improved.

1.7 Abbreviated expression

FE	Finite Element
FEMs	Finite Element Methods

2 Isotropic and Anisotropic Mesh Elements

In the context of FEMs, mesh elements can be classified based on their geometric properties. *An isotropic mesh element* has equal or nearly equal edge lengths and angles, resulting in a balanced shape. In contrast, *an anisotropic mesh element* features significant variation in edge lengths and angles.

Consider the following examples: Let $s, \delta \in \mathbb{R}_+$, and $\varepsilon \ge 1$, $\varepsilon \in \mathbb{R}$.

Example 2.1. In the case of the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$, the triangle is classified as follows:

- If $\delta \approx 1$, the triangle T is considered an isotropic mesh element.
- Conversely, if δ is much less than 1, i.e., $\delta \ll 1$, the triangle T becomes an anisotropic mesh element.

Example 2.2. In this case, consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Here, the vertex p_3 introduces a parameter ε that can influence the shape of the simplex. The classification of this simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle maintains a balanced shape, making it isotropic.

- If $\varepsilon > 1$, the triangle becomes flat when $s \ll 1$, resulting in an anisotropic mesh element.

Example 2.3. Consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0,\delta s)^\top$. In this configuration, the classification of the simplex as isotropic or anisotropic depends on the value of δ:

- If $\delta \approx 1$, the triangle is an isotropic mesh element.
- If $\delta \ll 1$, the triangle becomes an anisotropic mesh element.

Example 2.4. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0,s^\varepsilon)^\top$. In this case, the classification of the simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle is isotropic because the height from p_3 is equal to the base length.
- If $\varepsilon > 1$, the triangle will be classified as anisotropic, as the edge lengths will differ significantly when $s \ll 1$.

Example 2.5. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$. The classification of the simplex into two types of anisotropic structures is determined by the values of δ and ε:

- If $1 < \varepsilon < \delta$, the triangle is flattened so that the point p_3 approaches the point p_1, i.e. the origin as $s \to 0$.
- If $1 < \delta < \varepsilon$, the triangle is flattened so that point p_3 approaches a point on the straight line $\overline{p_1 p_2}$ that does not include points p_1 and p_2 as $s \to 0$.

3 Classical Geometric Conditions

3.1 Classical Interpolation Error Estimate

Let $\widehat{T} \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^d$ be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ and $\{T, P, \Sigma\}$ with associated normed vector spaces $V(\widehat{T})$ and $V(T)$. The transformation Φ_T takes the form

$$\Phi_T : \widehat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := B_T \hat{x} + b_T \in T,$$

where $B_T \in \mathbb{R}^{d\times d}$ is an invertible matrix and $b_T \in \mathbb{R}^d$. Let $I_T : V(T) := W^{2,p}(T) \to P := \mathbb{P}^1(T)$ with $p \in [1,\infty]$ be an interpolation on T with $I_T p = p$ for any $p \in \mathcal{P}^1(T)$. According to the classical theory (e.g., see [11, 14]), there exists a positive constant c, independent of h_T, such that

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \le c \left(\|B_T\|_2 \|B_T^{-1}\|_2 \right) \|B_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity $\|B_T\|_2 \|B_T^{-1}\|_2$ is called the *Euclidean condition number* of B_T. By standard estimates (e.g., see [14, Lemma 1.100]), we have

$$\|B_T\|_2 \|B_T^{-1}\|_2 \le c\frac{h_T}{\rho_T}, \quad \|B_T\|_2 \le c h_T.$$

It thus holds that

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \le c\frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.1}$$

As a geometric condition, the *shape-regularity condition* is well known to obtain global interpolation error estimates. This condition is stated as follows.

Condition 3.1 (Shape-regularity condition)**.** There exists a constant $\gamma_1 > 0$ such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{3.2}$$

Under Condition 3.1, that is, when the quantity $\frac{h_T}{\rho_T}$ is bounded on each T, it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)},$$

where $I_h \varphi$ is the standard global linear interpolation of φ on $\mathbb{T}_h$.

3.2 Regular Mesh Conditions

Geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity condition (3.2). A proof can be found in [7, Theorem 1].

Condition 3.2 (Zlámal's condition)**.** There exists a constant $\gamma_2 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$, any simplex $T \in \mathbb{T}_h$ and any dihedral angle ψ and for $d = 3$, also any solid angle θ of T, we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \tag{3.3}$$

Condition 3.3. There exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_3 h_T^d. \tag{3.4}$$

Condition 3.4. There exists a constant $\gamma_4 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_4 |B_d^T|, \tag{3.5}$$

where $B^T \supset T$ is the circumscribed ball of T.

Note 3.5. If Condition 3.1 or 3.2 or 3.3 or 3.4 holds, a family of simplicial partitions is called *regular*.

Note 3.6. Condition 3.2 was presented by Zlámal [42] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on $\mathbb{R}^2$. Zlámal's condition can be generalised into $\mathbb{R}^n$ for any $n \in \{2, 3, \ldots\}$. Later, the shape-regularity condition (the inscribed ball condition) was introduced; see [11]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

Note 3.7. Condition 3.3 seems to be simpler than Condition 3.1, Condition 3.2 and Condition 3.4. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

3.3 What happens when anisotropic meshes are used?

Using the equivalence conditions in Section 3.2, the error estimate (3.1) is rewritten as

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T^2}{|T|_2} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.6}$$

We considered the following five anisotropic elements as in Section 2: Let $0 < s, \delta \ll 1$, $s, \delta \in \mathbb{R}$, and $\varepsilon > 1$, $\varepsilon \in \mathbb{R}$.

Example 3.8. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, \delta s)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = \delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{4}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}.$$

When $\delta \ll 1$, the interpolation error (3.6) may be large.

Example 3.9. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = 4 s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity is not satisfied. In this case, when $\varepsilon > 2$, the estimate (3.6) diverges as $s \to 0$.

Example 3.10. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. Then, we have that $h_T = s\sqrt{1+\delta^2} \approx s$, $|T|_2 = \frac{1}{2}\delta s^2$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(1+\delta^2)}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.7}$$

It is implied that the interpolation error (3.7) may be large when $\delta \ll 1$.

Example 3.11. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. Subsequently, we obtain $h_T = \sqrt{s^2 + s^{2\varepsilon}} \approx s$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s^2 + s^{2\varepsilon})}{s^{1+\varepsilon}} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$.

Example 3.12. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$. If $1 < \varepsilon < \delta$, we have $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \leq c s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$. If $1 < \delta < \varepsilon$, we have $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \leq c s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$.

Remark 3.13. As will be explained later, the factor $\frac{1}{\delta}$ in Example 3.10 is violated. The interpolation error estimate converges in the cases of Example 3.11 and Example 3.12 with $1 < \varepsilon < \delta$ using new precise interpolation error estimates under more relaxed geometric conditions.

4 Classical Relaxed Geometric Conditions

4.1 Semi-regular Mesh Conditions for $d = 2$

In 1957, Synge [39, Section 3.8] proposed the following condition.

Condition 4.1 (Synge's condition)**.** There exists $\frac{\pi}{3} \leq \gamma_5 < \pi$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_5, \tag{4.1}$$

where $\theta_{T,\max}$ is the maximal angle of T.

Under Condition 4.1, Synge proved an optimal interpolation error estimate as follows.

$$\|\varphi - I_h\varphi\|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)} \quad \text{for } p = \infty.$$

The inequality (4.1) is called *Synge's condition* or the *maximum-angle condition*. In 1976, several author's [4, 5, 20, 33] independently proved the convergence of finite element for $p < \infty$. It ensures that finite elements converge effectively when the minimum angle approaches zero as the mesh size decreases. If this condition is not met, the accuracy of interpolation for linear triangular elements can suffer, similar to the absence of Zlámal's condition, see e.g. [4, p. 223]. This underscores the importance of keeping proper geometric constraints to ensure reliable outcomes in numerical methods. Synge's condition is essential in finite element analysis.

In [34], Křížek proposed the following circumscribed ball condition for $d = 2$ which is equivalent to Synge's condition.

Condition 4.2. There exists $\gamma_6 > 0$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\frac{R_2}{h_T} \leq \gamma_6, \tag{4.2}$$

where R_2 is the radius of the circumscribed ball of $T \subset \mathbb{R}^2$.

Note 4.3. If Condition 4.1 or 4.2 holds, the associated families of partitions are called *semi-regular*.

Remark 4.4. Assume that Condition 3.3 holds, that is, there exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_3 h_T^2.$$

Let $T \subset \mathbb{R}^2$ be the triangle with vertices P_1, P_2 and P_3 such that the maximum angle $\theta_{T,\max}$ of T is $\angle P_2P_1P_3$. We then have $h_T = |P_2P_3|$ and

$$\frac{R_2}{h_T} = \frac{|P_2P_3|}{2h_T \sin\theta_{T,\max}} = \frac{|P_1P_2||P_1P_3|}{2|P_1P_2||P_1P_3| \sin\theta_{T,\max}} \leq c\frac{h_T^2}{|T|} \leq \frac{c}{\gamma_3} =: \gamma_6.$$

This implies that each regular family is semi-regular. However, the converse implication does not hold.

4.2 Semi-regular Mesh Conditions for $d = 3$

Synge's condition (4.1) is extended to the case of tetrahedra in [35].

Condition 4.5. There exists a constant $0 < \gamma_7 < \pi$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \le \gamma_7, \tag{4.3a}$$

$$\psi_{T,\max} \le \gamma_7, \tag{4.3b}$$

where $\theta_{T,\max}$ is the maximum angle of all triangular faces of the tetrahedron T and $\psi_{T,\max}$ is the maximum dihedral angle of T.

Remark 4.6. The theory of anisotropic interpolation has been advanced through extensive research ([3, 2, 9]).

Question 4.7. Is there a semi-regularity condition which equivalent to Synge's condition (4.3) for $d = 3$?

Remark 4.8. This article introduces a novel geometric condition intended to serve as an alternative to Synge's condition specifically for three-dimensional cases.

5 Settings for New Interpolation Theory

5.1 Reference Elements

We first define the reference elements $\widehat{T} \subset \mathbb{R}^d$.

Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be a reference triangle with vertices $\hat{p}_1 := (0,0)^\top$, $\hat{p}_2 := (1,0)^\top$, and $\hat{p}_3 := (0,1)^\top$.

Three-dimensional case

In the three-dimensional case, we consider the following two cases: (i) and (ii); see Condition 5.2.

Let $\widehat{T}_1$ and $\widehat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\widehat{T}_1$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (0,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$;

(ii) $\widehat{T}_2$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (1,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$.

Therefore, we set $\widehat{T} \in \{\widehat{T}_1, \widehat{T}_2\}$. Note that the case (i) is called *the regular vertex property*, see [1].

5.2 Two-step Affine Mapping

To an affine simplex $T \subset \mathbb{R}^d$, we construct two affine mappings $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$. First, we define the affine mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ as

$$\Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto \tilde{x} := \Phi_{\widetilde{T}}(\hat{x}) := A_{\widetilde{T}} \hat{x} \in \widetilde{T}, \tag{5.1}$$

where $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ is an invertible matrix. We then define the affine mapping $\Phi_T : \widetilde{T} \to T$ as follows:

$$\Phi_T : \widetilde{T} \ni \tilde{x} \mapsto x := \Phi_T(\tilde{x}) := A_T \tilde{x} + b_T \in T, \tag{5.2}$$

where $b_T \in \mathbb{R}^d$ is a vector and $A_T \in O(d)$ denotes the rotation and mirror-imaging matrix. We define the affine mapping $\Phi : \widehat{T} \to T$ as

$$\Phi := \Phi_T \circ \Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto x := \Phi(\hat{x}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\hat{x}) = A\hat{x} + b_T \in T,$$

where $A := A_T A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$.

Construct mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$

We consider the affine mapping (5.1). We define the matrix $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ as follows. We first define the diagonal matrix as

$$\widehat{A} := \operatorname{diag}(h_1, \ldots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i, \tag{5.3}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

For $d = 2$, we define the regular matrix $\widetilde{A} \in \mathbb{R}^{2 \times 2}$ as

$$\widetilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \tag{5.4}$$

with the parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For the reference element $\widehat{T}$, let $\mathfrak{T}^{(2)}$ be a family of triangles.

$$\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}) = A_{\widetilde{T}}(\widehat{T}), \quad A_{\widetilde{T}} := \widetilde{A}\widehat{A}$$

with the vertices $\tilde{p}_1 := (0,0)^\top$, $\tilde{p}_2 := (h_1, 0)^\top$ and $\tilde{p}_3 := (h_2 s, h_2 t)^\top$. Then, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$ and $h_2 = |\tilde{p}_1 - \tilde{p}_3| > 0$.

For $d = 3$, we define the regular matrices $\widetilde{A}_1, \widetilde{A}_2 \in \mathbb{R}^{3 \times 3}$ as follows:

$$\widetilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{A}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \tag{5.5}$$

with the parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ h_2 s_1 \leq h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0, \ h_3 s_{21} \leq h_1/2. \end{cases}$$

Therefore, we set $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$. For the reference elements $\widehat{T}_i$, $i = 1, 2$, let $\mathfrak{T}_i^{(3)}$, $i = 1, 2$, be a family of tetrahedra.

$$\widetilde{T}_i = \Phi_{\widetilde{T}_i}(\widehat{T}_i) = A_{\widetilde{T}_i}(\widehat{T}_i), \quad A_{\widetilde{T}_i} := \widetilde{A}_i \widehat{A}, \quad i = 1, 2,$$

with the vertices

$$\tilde{p}_1 := (0,0,0)^\top, \ \tilde{p}_2 := (h_1, 0, 0)^\top, \ \tilde{p}_4 := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^\top,$$
$$\begin{cases} \tilde{p}_3 := (h_2 s_1, h_2 t_1, 0)^\top & \text{for case (i)}, \\ \tilde{p}_3 := (h_1 - h_2 s_1, h_2 t_1, 0)^\top & \text{for case (ii)}. \end{cases}$$

Subsequently, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$, $h_3 = |\tilde{p}_1 - \tilde{p}_4| > 0$, and

$$h_2 = \begin{cases} |\tilde{p}_1 - \tilde{p}_3| > 0 & \text{for case (i)}, \\ |\tilde{p}_2 - \tilde{p}_3| > 0 & \text{for case (ii)}. \end{cases}$$

Construct mapping $\Phi_T : \widetilde{T} \to T$

We determine the affine mapping (5.2) as follows. Let $T \in \mathbb{T}_h$ have vertices p_i ($i = 1, \ldots, d+1$). Let $b_T \in \mathbb{R}^d$ be the vector and $A_T \in O(d)$ be the rotation and mirror imaging matrix such that

$$p_i = \Phi_T(\tilde{p}_i) = A_T \tilde{p}_i + b_T, \quad i \in \{1, \ldots, d+1\},$$

where vertices p_i ($i = 1, \ldots, d+1$) satisfy the following conditions:

Condition 5.1 (Case in which $d = 2$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i ($i = 1, \ldots, 3$). We assume that $\overline{p_2 p_3}$ is the longest edge of T, that is, $h_T := |p_2 - p_3|$. We set $h_1 = |p_1 - p_2|$ and $h_2 = |p_1 - p_3|$. We then assume that $h_2 \leq h_1$. Because $\frac{1}{2} h_T < h_1 \leq h_T$, $h_1 \approx h_T$.

Condition 5.2 (Case in which $d = 3$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i ($i = 1, \ldots, 4$). Let L_i ($1 \leq i \leq 6$) be the edges of T. We denote by $L_{\min}$ the edge of T with the minimum length; that is, $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$. We set $h_2 := |L_{\min}|$ and assume that

$$\text{the endpoints of } L_{\min} \text{ are either } \{p_1, p_3\} \text{ or } \{p_2, p_3\}.$$

Among the four edges sharing an endpoint with $L_{\min}$, we consider the longest edge $L_{\max}^{(\min)}$. Let p_1 and p_2 be the endpoints of edge $L_{\max}^{(\min)}$. Thus, we have

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting $\mathbb{R}^3$ with a plane that contains the midpoint of the edge $L_{\max}^{(\min)}$ and is perpendicular to the vector $p_1 - p_2$. Thus, there are two cases.

(Type i) p_3 and p_4 belong to the same half-space;

(Type ii) p_3 and p_4 belong to different half-spaces.

In each case, we set

(Type i) p_1 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_1 - p_3|$;

(Type ii) p_2 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_2 - p_3|$.

Finally, we set $h_3 = |p_1 - p_4|$. We implicitly assume that p_1 and p_4 belong to the same half-space. Additionally, note that $h_1 \approx h_T$.

Note 5.3. As an example, we define the matrices A_T as

$$A_T := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad A_T := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ denotes the angle.

Note 5.4. None of the lengths of the edges of a simplex or the measures of the simplex are changed by the transformation, i.e.,

$$h_i \leq h_T, \quad i = 1, \ldots, d. \tag{5.6}$$

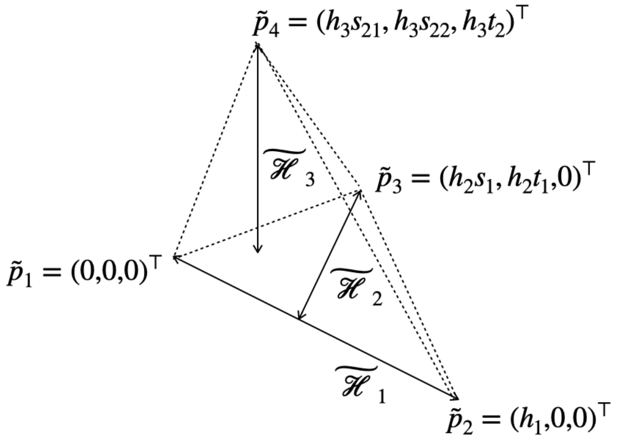

Fig. 1: New parameters $\widetilde{\mathscr{H}_i}$, $i = 1, 2, 3$

5.3 Additional Notations and Assumptions

For convenience, we introduce the following additional notation. We define a parameter $\widetilde{\mathscr{H}_i}$, $i = 1, \ldots, d$, as

$$\begin{cases} \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t \quad \text{if } d = 2, \\ \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t_1, \quad \widetilde{\mathscr{H}_3} := h_3 t_2 \quad \text{if } d = 3, \end{cases}$$

see Fig. 1.

Assumption 5.5. In an anisotropic interpolation error analysis, we impose a geometric condition for the simplex $\widetilde{T}$:

1. If $d = 2$, there are no additional conditions;
2. If $d = 3$, there exists a positive constant M independent of $h_{\widetilde{T}}$ such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$. Note that if $s_{22} \neq 0$, this condition means that the order concerning h_T of h_3 coincides with the order of h_2, and if $s_{22} = 0$, the order of h_3 may be different from that of h_2.

We define the vectors $r_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows: If $d = 2$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

see Fig. 2, and if $d = 3$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \dfrac{p_3 - p_1}{|p_3 - p_1|}, \quad \text{for case (i)}, \\ r_2 := \dfrac{p_3 - p_2}{|p_3 - p_2|} \quad \text{for case (ii)}, \end{cases}$$

see Fig 3 for (Type i) and Fig 4 for (Type ii). Furthermore, we define the vectors $\tilde{r}_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows. If $d = 2$,

$$\tilde{r}_1 := (1, 0)^\top, \quad \tilde{r}_2 := (s, t)^\top,$$

and if $d = 3$,

$$\tilde{r}_1 := (1, 0, 0)^\top, \quad \tilde{r}_3 := (s_{21}, s_{22}, t_2)^\top, \quad \begin{cases} \tilde{r}_2 := (s_1, t_1, 0)^\top \quad \text{for case (i)}, \\ \tilde{r}_2 := (-s_1, t_1, 0)^\top \quad \text{for case (ii)}. \end{cases}$$

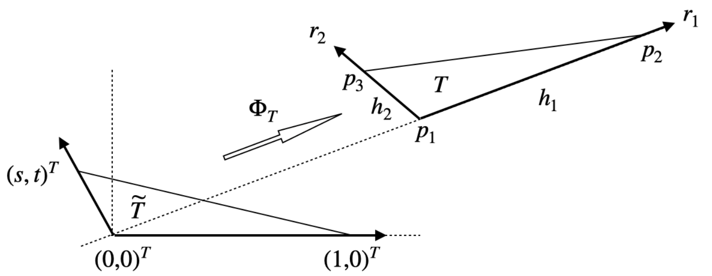

Fig. 2: Affine mapping Φ_T and vectors r_i, $i = 1, 2$

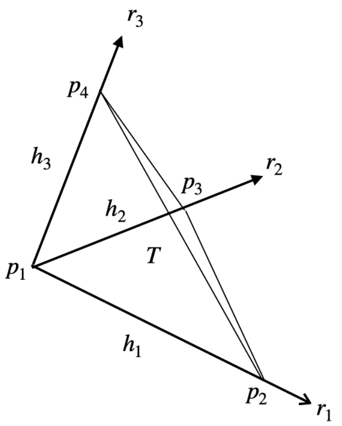

Fig. 3: (Type i) Vectors r_i, $i = 1, 2, 3$

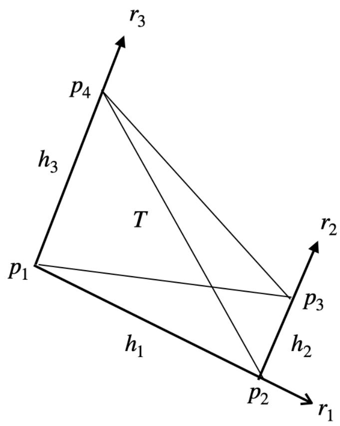

Fig. 4: (Type ii) Vectors r_i, $i = 1, 2, 3$

Remark 5.6. The vectors $\tilde{r}_i$, $i \in \{1, \ldots, d\}$ are unit vectors. Indeed, if $d = 2$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s^2 + t^2} = 1,$$

if $d = 3$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s_1^2 + t_1^2} = 1, \quad |\tilde{r}_3|_E = \sqrt{s_{21}^2 + s_{22}^2 + t_2^2} = 1.$$

For a sufficiently smooth function φ and vector function $v := (v_1, \ldots, v_d)^\top$, we define the directional derivative of $i \in \{1, \ldots, d\}$ as:

$$\frac{\partial \varphi}{\partial r_i} := (r_i \cdot \nabla_x)\varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}},$$
$$\frac{\partial v}{\partial r_i} := \left(\frac{\partial v_1}{\partial r_i}, \ldots, \frac{\partial v_d}{\partial r_i} \right)^\top = ((r_i \cdot \nabla_x) v_1, \ldots, (r_i \cdot \nabla_x) v_d)^\top.$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we use the following notation.

$$\partial^\beta \varphi := \frac{\partial^{|\beta|} \varphi}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|} \varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}}.$$

Note that $\partial^\beta \varphi \neq \partial_r^\beta \varphi$.

6 New Semi-regularity Condition

6.1 New Geometric Parameter and Condition

We proposed a new geometric parameter H_T in [30].

Definition 6.1. Parameter H_T is defined as follows:

$$H_T := \frac{\prod_{i=1}^{d} h_i}{|T|_d} h_T.$$

We introduce geometric conditions to obtain the optimal convergence rate of the anisotropic error estimates.

Condition 6.2. A family of meshes $\{\mathbb{T}_h\}$ is semi-regular if there exists $\gamma_0 > 0$ such that

$$\frac{H_T}{h_T} \le \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{6.1}$$

Remark 6.3. The geometric condition in (6.1) is equivalent to the maximum angle condition (Section 7).

Remark 6.4. We consider the good elements on the meshes in Section 8. On anisotropic meshes, good elements may satisfy the following conditions:

$(d = 2)$ $h_2 \approx h_2 t$;

$(d = 3)$ $h_2 \approx h_2 t_1$ and $h_3 \approx h_3 t_2$.

6.2 Properties of the New Geometric Parameter

We first show the relation between h_T and H_T.

Lemma 6.5. It holds that

$$h_T \le \frac{1}{2} H_T \quad \text{if } d = 2, \tag{6.2}$$

$$h_T < \frac{1}{6} H_T \quad \text{if } d = 3. \tag{6.3}$$

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. By constructing the standard element in the two-dimensional case, the angle $\theta_{\max} := \angle p_2 p_1 p_3$ is the maximum angle of T. We then have $\frac{\pi}{3} < \theta_{\max} < \pi$, that is, $0 < \sin\theta_{\max} \le 1$. Therefore, it holds that

$$H_T = \frac{h_1 h_2}{|T|_2} h_T = \frac{2}{\sin\theta_{\max}} h_T \ge 2 h_T.$$

We here used the fact that $|T|_2 = \frac{1}{2} h_1 h_2 \sin\theta_{\max}$.

Three-dimensional case. We denote by ϕ_T the angle between the base $\triangle p_1 p_2 p_3$ of T and the segment $\overline{p_1 p_4}$. Recall that there are two types of standard elements, (Type i) or (Type ii). We denote by θ_T

(Type i) the angle between the segments $\overline{p_1 p_2}$ and $\overline{p_1 p_3}$, that is, $\theta_T := \angle p_2 p_1 p_3$, or

(Type ii) the angle between the segments $\overline{p_2 p_1}$ and $\overline{p_2 p_3}$, that is, $\theta_T := \angle p_1 p_2 p_3$.

We set $t_1 := \sin\theta_T$ and $t_2 := \sin\phi_T$. By constructing the standard element in the three-dimensional case, the angle $\angle p_1p_3p_2$ is the maximum angle of the base $\triangle p_1p_2p_3$ of T. Therefore, we have $0 < \theta_T < \frac{\pi}{2}$. Because $0 < \phi_T < \pi$, it holds that

$$H_T = \frac{h_1h_2h_3}{|T|_3} h_T = \frac{6}{\sin\theta_T \sin\phi_T} h_T > 6h_T.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1h_2h_3 \sin\theta_T \sin\phi_T$. $\square$

We introduce another geometric parameter regarding Definition 6.1.

Definition 6.6 (Another parameter H_T^*)**.** For $T \in \mathbb{T}_h$, we denote by L_i edges of the simplex T. We define the new parameter H_T^* as

$$H_T^* := \frac{h_T^2}{|T|_2} \min_{1\le i\le 3} |L_i| \quad \text{if } d = 2, \quad H_T^* := \frac{h_T^2}{|T|_3} \min_{1\le i,j\le 6, i\neq j} |L_i||L_j| \quad \text{if } d = 3. \tag{6.4}$$

The parameters H_T^* and H_T are equivalent.

Lemma 6.7. It holds that

$$\frac{1}{2} H_T^* < H_T < 2H_T^*. \tag{6.5}$$

Furthermore, H_T^* is equivalent to the circumradius R_2 of T in the two-dimensional case.

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Let L_i $(i = 1, 2, 3)$ denote edges of the triangle T with $|L_1| \le |L_2| \le |L_3|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_3| = h_T$. Because $h_2 \le h_1 < 2h_T$ and $h_T < h_1 + h_2 \le 2h_1$ for the triangle $\triangle p_1p_2p_3$, it holds that

$$\frac{1}{2} h_T < h_1 = |L_2| < 2h_T = 2h_T.$$

We thus have

$$\frac{1}{2} H_T^* = \frac{1}{2} \frac{|L_1|}{|T|_2} h_T^2 < H_T = \frac{h_1h_2}{|T|_2} h_T < 2 \frac{|L_1|}{|T|_2} h_T^2 = 2H_T^*.$$

Furthermore, it holds that

$$2R_2 = 2\frac{|L_1||L_2||L_3|}{4|T|_2} < H_T^* = \frac{|L_1|}{|T|_2} h_T^2 < 8 \frac{|L_1||L_2||L_3|}{4|T|_2} = 8R_2.$$

Three-dimensional case. Let L_i $(i = 1, \ldots, 6)$ denote edges of the triangle T with $|L_1| \le |L_2| \le \cdots \le |L_6|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_6|$. Recall that there are two types of standard elements, (Type i) or (Type ii).

(Type i) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$ and $h_6 := |p_2 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \le \min\{h_3, h_4, h_6\} \le \max\{h_3, h_4, h_6\} \le h_1. \tag{6.6}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1p_2p_4$, it holds that

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T & \text{or} \\ h_3 \le h_1 \le h_5 = h_T. \end{cases}$$

We thus have

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T \quad \text{or} \\ h_3 \le h_1 \le h_T < 2h_1, \quad \frac{1}{2}h_T < h_1 \le h_T. \end{cases}$$

Because $h_3 \le h_5$, the length of the edge L_2 is equal to the one of h_3, h_4 or h_6.

Assume that $|L_2| = h_3$. We then have

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_T \le \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T^*(< 2H_T^*).$$

Assume that $|L_2| = h_4$. We consider the triangle $\triangle p_1p_3p_4$. From the assumption, we have $h_2 \le h_4 \le h_3$ and $\frac{1}{2}h_3 < h_4 \le h_3$. We then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_{T^s} < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

Assume that $|L_2| = h_6$. We consider the triangle $\triangle p_1p_2p_3$. Because p_1 and p_3 belong to the same half-space for the triangle $\triangle p_1p_2p_3$, it holds that $h_2 \le h_6 \le h_1$ and $\frac{1}{2}h_1 < h_6 \le h_1$. From (6.6), we have

$$\frac{1}{2}h_3 \le \frac{1}{2}h_1 < h_6 \le h_1.$$

Because $h_6 \le h_3$, we then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_T < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

(Type ii) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$, and $h_6 := |p_1 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \le \min\{h_4, h_5, h_6\} \le \max\{h_4, h_5, h_6\} \le h_1. \tag{6.7}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1p_2p_4$ and (6.7), it holds that

$$h_3 \le h_5 \le h_1.$$

This implies that $h_1 = h_T$. Therefore, the length of the edge L_2 is equal to the one of h_3, h_4, or h_6.

Assume that $|L_2| = h_3$. We then have

$$\begin{aligned}\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T &= \frac{h_1h_2h_3}{|T|_3}h_T \\ &= \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T^*(< 2H_T).\end{aligned}$$

Assume that $|L_2| = h_4$. For the triangle $\triangle p_2p_3p_4$, we have

$$h_2 \le h_4 \le h_5 < 2h_4.$$

Because $h_3 \leq h_5$ and $h_4 \leq h_3$, it holds that

$$\left(\frac{1}{2} H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2 H_T^*.$$

Assume that $|L_2| = h_6$. We have $h_1 < h_2 + h_6 < 2h_6$ for the triangle $\triangle p_1 p_2 p_3$. Therefore, since $h_6 \leq h_3 \leq h_1$, we obtain

$$\left(\frac{1}{2} H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2 H_T^*.$$

□

6.3 Euclidean Condition Number

Examining the Euclidean condition number is useful for deriving appropriate interpolation error estimates.

Lemma 6.8. It holds that

$$\|\widehat{A}\|_2 \leq h_T, \quad \|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}, \tag{6.8a}$$

$$\|\widetilde{A}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3, \end{cases} \qquad \|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 \leq \begin{cases} \frac{h_1 h_2}{|T|_2} = \frac{H_T}{h_T} & \text{if } d = 2, \\ \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_3} = \frac{2}{3} \frac{H_T}{h_T} & \text{if } d = 3, \end{cases} \tag{6.8b}$$

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1. \tag{6.8c}$$

where a parameter H_T is defined in Definition 6.1. Furthermore, we have

$$|\det(A_{\widetilde{T}})| = |\det(\widetilde{A})||\det(\widehat{A})| = \frac{|T|_d}{|\widetilde{T}|_d} \frac{|\widetilde{T}|_d}{|\widehat{T}|_d} = d!|T|_d, \quad |\det(A_T)| = 1. \tag{6.9}$$

Proof. We first show the equality (6.9). Because

$$\int_T dx = \int_{\widetilde{T}} |\det(A_T)| d\tilde{x}, \quad \int_{\widetilde{T}} d\tilde{x} = \int_{\widehat{T}} |\det(A_{\widetilde{T}})| d\hat{x},$$

and $|T|_d = |\widetilde{T}|_d$, we conclude (6.9).

We show the equality (6.8a). From

$$(\widehat{A})^\top \widehat{A} = \mathrm{diag}(h_1^2, \ldots, h_d^2), \quad \widehat{A}^{-1} \widehat{A}^{-\top} = \mathrm{diag}(h_1^{-2}, \ldots, h_d^{-2}),$$

we have

$$\|\widehat{A}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} = \max\{h_1, \cdots, h_d\} \leq h_T,$$

and

$$\|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} \lambda_{\max}(\widehat{A}^{-1} \widehat{A}^{-\top})^{\frac{1}{2}} = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}},$$

which leads to (6.8a).

We next show the equality (6.8b). We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Because

$$\widetilde{A}^\top \widetilde{A} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad \widetilde{A}^{-1} \widetilde{A}^{-\top} = \frac{1}{t^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}, \quad |s| \leq 1,$$

we have

$$\|\widetilde{A}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \leq (1 + |s|)^{\frac{1}{2}} \leq \sqrt{2},$$

and

$$\|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}^{-1} \widetilde{A}^{-\top})^{\frac{1}{2}} \leq \frac{2}{t} = \frac{h_1 h_2}{|T|_d},$$

which leads to (6.8b) for $d = 2$. Here, we used the fact that $|\widetilde{T}|_d = \frac{1}{2} h_1 h_2 t$ and $|T|_d = |\widetilde{T}|_d$.

Three-dimensional case. The matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ introduced in (5.5) can be decomposed as $\widetilde{A}_1 = \widetilde{M}_0 \widetilde{M}_1$ and $\widetilde{A}_2 = \widetilde{M}_0 \widetilde{M}_2$ with

$$\widetilde{M}_0 := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{M}_1 := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{M}_2 := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\widetilde{M}_2^\top \widetilde{M}_2$ coincide with those of $\widetilde{M}_1^\top \widetilde{M}_1$, and only Case (i) is shown.

We have the inequalities

$$\begin{aligned} \|\widetilde{A}_1\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \leq \lambda_{\max}(\widetilde{M}_0^\top \widetilde{M}_0)^{\frac{1}{2}} \lambda_{\max}(\widetilde{M}_1^\top \widetilde{M}_1)^{\frac{1}{2}} \\ &\leq \left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)^{\frac{1}{2}} (1 + |s_1|)^{\frac{1}{2}} \leq 2, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{A}_1\|_2 \|\widetilde{A}_1^{-1}\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}_1^{-1} \widetilde{A}_1^{-\top})^{\frac{1}{2}} \\ &\leq \frac{\left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)(1 + |s_1|)}{t_1 t_2} \leq \frac{4}{t_1 t_2} = \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_d}, \end{aligned}$$

where we used the fact that $|\widetilde{T}|_d = \frac{1}{6} h_1 h_2 h_3 t_1 t_2$ and $|T|_d = |\widetilde{T}|_d$.

Because the length of all edges of a simplex and measure of the simplex is not changed by a rotation and mirror imaging matrix and $A_T, A_T^{-1} \in O(d)$,

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1,$$

which is (6.8c). □

7 New Geometric Mesh Condition equivalent to the Maximum-angle Condition

7.1 Statements

We state the following theorems concerning the new condition.

Theorem 7.1. Condition 6.2 holds if and only if Condition 4.1 holds when $d = 2$.

Proof. In the case of $d = 2$, we use the previous result presented in [34]; i.e., there exists a constant $\gamma_6 > 0$ such that

$$\frac{R_2}{h_T} \leq \gamma_6 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h,$$

if and only if Condition 4.1 is satisfied. Combining this result with H_T being equivalent to the circumradius R_2 of T (Lemma 6.7), we have the desired conclusion. □

Theorem 7.2. Condition 6.2 holds if and only if Condition 4.5 holds when $d = 3$.

The proof can be found in [29]. Preparation is needed to prove the three-dimensional case. The following subsection shows the symbols used only in this section.

7.2 Notation

Let $T \in \mathbb{T}_h$ be the standard element in $\mathbb{R}^3$ with vertices, P_1, P_2, P_3 and P_4. Let F_i be the face of a simplex T opposite to the vertex P_i. We denote by $\psi^{i,j}$ (Table 5) the angle between the face F_i and the face F_j, see Figure 5. Note that $\psi^{i,j} = \psi^{j,i}$. Furthermore, we denote by θ^i_j (Table 6) the internal angle at the vertex P_j on the face F_i and by ϕ^i_j (Table 7) the angle between the face F_i and the segment $\overline{P_j P_i}$.

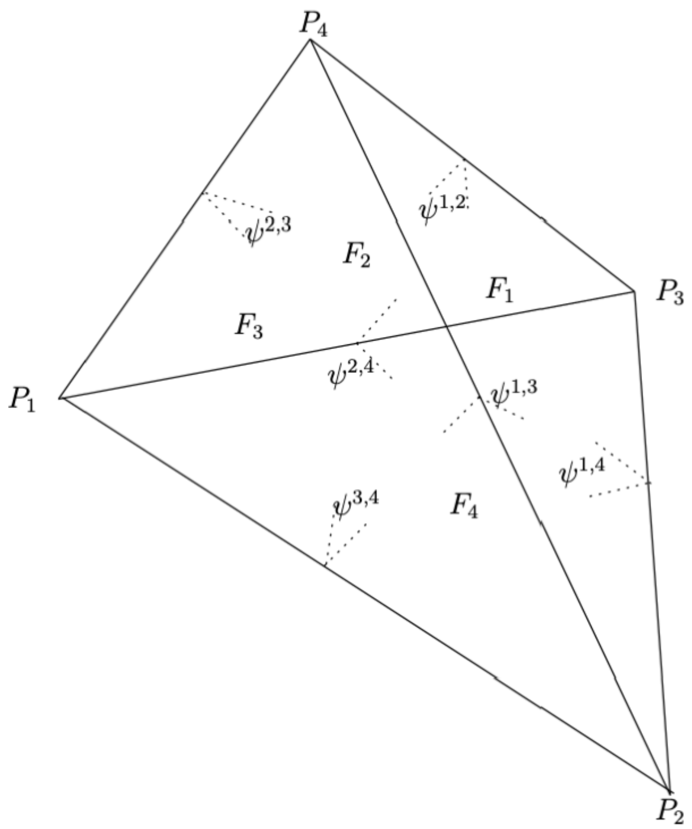

Fig. 5: Tetrahedra

7.3 Preliminaries: Part 1

We introduce three lemmata.

Table 5: $\psi^{i,j}$

	F_1	F_2	F_3	F_4
F_1	-	$\psi^{1,2}$	$\psi^{1,3}$	$\psi^{1,4}$
F_2	$\psi^{2,1}$	-	$\psi^{2,3}$	$\psi^{2,4}$
F_3	$\psi^{3,1}$	$\psi^{3,2}$	-	$\psi^{3,4}$
F_4	$\psi^{4,1}$	$\psi^{4,2}$	$\psi^{4,3}$	-

Table 6: θ^i_j

	F_1	F_2	F_3	F_4
P_1	-	θ^2_1	θ^3_1	θ^4_1
P_2	θ^1_2	-	θ^3_2	θ^4_2
P_3	θ^1_3	θ^2_3	-	θ^4_3
P_4	θ^1_4	θ^2_4	θ^3_4	-

Table 7: ϕ^i_j

	F_1	F_2	F_3	F_4
P_1	-	ϕ^2_1	ϕ^3_1	ϕ^4_1
P_2	ϕ^1_2	-	ϕ^3_2	ϕ^4_2
P_3	ϕ^1_3	ϕ^2_3	-	ϕ^4_3
P_4	ϕ^1_4	ϕ^2_4	ϕ^3_4	-

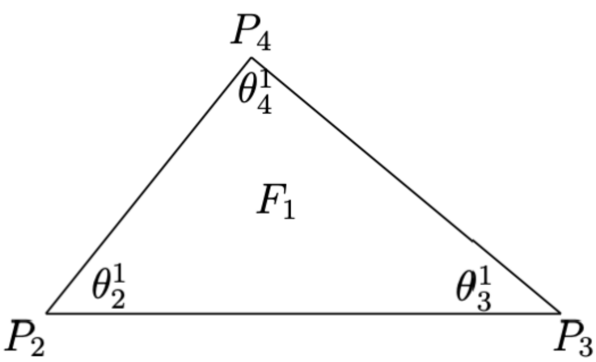

Fig. 6: Face 1

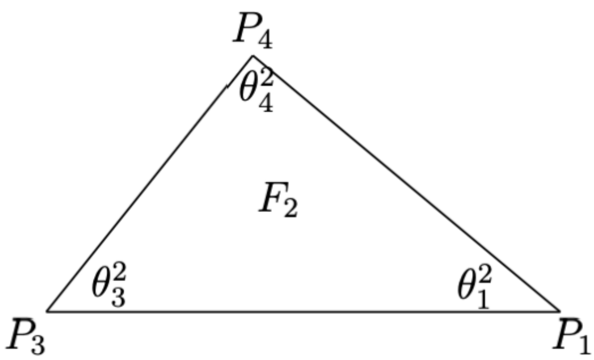

Fig. 7: Face 2

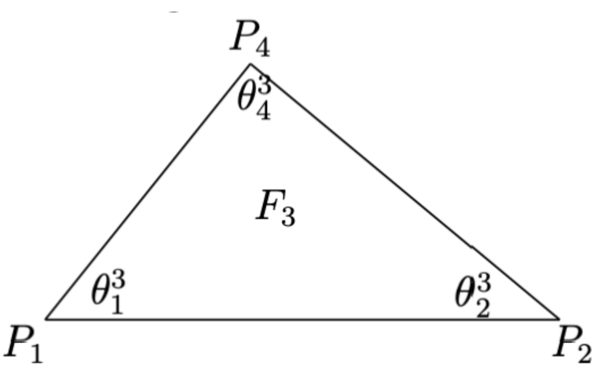

Fig. 8: Face 3

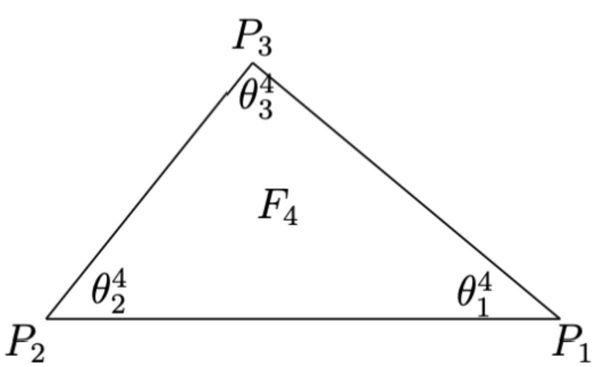

Fig. 9: Face 4

Lemma 7.3. Let $K \subset \mathbb{R}^2$ be a simplex and let θ_1, θ_2 and θ_3 be internal angles of K with $\theta_1 \le \theta_2 \le \theta_3$. If there exists $0 < \theta_0 < \pi$, $\theta_0 \in \mathbb{R}$, such that $\theta_3 \le \theta_0$, we then have

$$\sin\theta_2,\ \sin\theta_3 \ge \min\left\{\sin\frac{\pi-\theta_0}{2}, \sin\theta_0\right\}.$$

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_1 \le \theta_2 \le \theta_3$, we have

$$\theta_0 \ge \theta_3 \ge \theta_2 \ge \frac{\theta_1+\theta_2}{2} \ge \frac{\pi-\theta_3}{2} \ge \frac{\pi-\theta_0}{2},$$

which leads to the target inequality. □

Lemma 7.4. Let $K \subset \mathbb{R}^2$ be a simplex with internal angles θ_1, θ_2 and θ_3. For any fixed $\gamma \in \mathbb{R}$ with $0 < \gamma < \pi$, we assume that $\pi - \gamma \le \theta_i$, $i \in \{1,2,3\}$. We then have $\theta_{i+1},\ \theta_{i+2} \le \gamma$, where the indices i, $i+1$ and $i+2$ have to be understood "mod 3".

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$, we have

$$\theta_{i+1} = \pi - \theta_i - \theta_{i+2} < \pi - \theta_i \le \pi - (\pi - \gamma) = \gamma.$$

□

Lemma 7.5. Let $\gamma \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma < \pi$. It then holds that

$$0 < \frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} \le 1.$$

Proof. Because $\cos\gamma = 1 - 2\sin^2\frac{\gamma}{2}$, we have

$$\frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} = \frac{2 - 2\sin^2\frac{\gamma}{2}}{\sin\frac{\gamma}{2} + 1} = 2\left(1 - \sin\frac{\gamma}{2}\right).$$

Therefore, for $\frac{\pi}{3} \le \gamma < \pi$, the target inequality holds. □

7.4 Preliminaries: Part 2

Lemma 7.6 (Cosine rules for the sides and for the angles)**.** It holds that

$$\begin{aligned}
\cos\theta_j^{j+3} &= \cos\theta_j^{j+1}\cos\theta_j^{j+2} + \sin\theta_j^{j+1}\sin\theta_j^{j+2}\cos\psi^{j+1,j+2}, && (7.1a)\\
\cos\theta_j^{j+1} &= \cos\theta_j^{j+2}\cos\theta_j^{j+3} + \sin\theta_j^{j+2}\sin\theta_j^{j+3}\cos\psi^{j+2,j+3}, && (7.1b)\\
\cos\theta_j^{j+2} &= \cos\theta_j^{j+3}\cos\theta_j^{j+1} + \sin\theta_j^{j+3}\sin\theta_j^{j+1}\cos\psi^{j+3,j+1}, && (7.1c)\\
\cos\psi^{j+1,j+2} &= \sin\psi^{j+2,j+3}\sin\psi^{j+3,j+1}\cos\theta_j^{j+3} - \cos\psi^{j+2,j+3}\cos\psi^{j+3,j+1}, && (7.1d)\\
\cos\psi^{j+2,j+3} &= \sin\psi^{j+3,j+1}\sin\psi^{j+1,j+2}\cos\theta_j^{j+1} - \cos\psi^{j+3,j+1}\cos\psi^{j+1,j+2}, && (7.1e)\\
\cos\psi^{j+3,j+1} &= \sin\psi^{j+1,j+2}\sin\psi^{j+2,j+3}\cos\theta_j^{j+2} - \cos\psi^{j+1,j+2}\cos\psi^{j+2,j+3}, && (7.1f)
\end{aligned}$$

where the indices j, $j+1$, $j+2$ and $j+3$ have to be understood "mod 4".

Proof. A proof can be found in [18, 40]. □

Lemma 7.7. Let $\gamma_{\max} \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma_{\max} < \pi$ satisfy Condition 4.5 for the maximum solid $\theta_{T,\max}$ and the maximum dihedral $\psi_{T,\max}$ of T. Assume that for each $j = 1, 2$, θ_j^4 is not the minimum angle of $\triangle P_1 P_2 P_3$ and $\theta_j^4 < \frac{\pi}{2}$. Then, setting $\delta := \delta(\gamma_{\max})$, $0 < \delta \le \frac{\pi}{2}$ such that

$$\sin \delta = \left(\frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{j+1,4} \ge \delta, \quad \text{or} \quad \psi^{3,4} \ge \delta, \tag{7.2}$$

where the indices j and $j+1$ have to be understood "mod 2".

Proof. From Lemma 7.5, we have

$$0 < \frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \le 1,$$

because $\frac{\pi}{3} \le \gamma_{\max} < \pi$. Therefore, δ is well-defined.

We use proof by contradiction. Suppose that

$$0 < \psi^{j+1,4} < \delta, \quad 0 < \psi^{3,4} < \delta,$$

that is,

$$0 < \sin \psi^{j+1,4} \sin \psi^{3,4} < \sin^2 \delta, \text{ and } 1 > \cos \psi^{j+1,4} \cos \psi^{3,4} > \cos^2 \delta \ge 0.$$

From Lemma 7.3 and assumption, we have

$$\frac{\pi - \gamma_{\max}}{2} \le \theta_j^4 < \frac{\pi}{2},$$

which implies

$$0 < \cos \theta_j^4 \le \cos \left(\frac{\pi - \gamma_{\max}}{2} \right) = \sin \frac{\gamma_{\max}}{2}.$$

We thus obtain

$$\sin \psi^{j+1,4} \sin \psi^{3,4} \cos \theta_j^4 < \sin^2 \delta \sin \frac{\gamma_{\max}}{2}.$$

Using the cosine rule (7.1d) with $j = 1$ and the above inequalities yield

$$\begin{aligned}
\cos \psi_{2,3} &= \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2} \\
&< \sin^2 \delta \sin \frac{\gamma_{\max}}{2} - (1 - \sin^2 \delta) \\
&= \frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \left(\sin \frac{\gamma_{\max}}{2} + 1 \right) - 1 = \cos \gamma_{\max}.
\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{2,3} \le \gamma_{\max} < \pi$, that is, $\cos \psi^{2,3} \ge \cos \gamma_{\max}$.

Analogously, using the cosine rule (7.1f) with $j = 2$ and the above inequalities yield

$$\begin{aligned}
\cos \psi^{1,3} &= \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 - \cos \psi^{3,4} \cos \psi^{4,1} \\
&< \cos \gamma_{\max}.
\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{1,3} \le \gamma_{\max} < \pi$, that is, $\cos \psi^{1,3} \ge \cos \gamma_{\max}$. □

Corollary 7.8. For each $j = 1, 2$, under assumptions in Lemma 7.7, it holds that setting $C_0 := \min\{\delta, \gamma_{\max}\}$,

$$\sin \psi^{j+1,4} \geq C_0, \quad \text{or} \quad \sin \psi^{3,4} \geq C_0$$

where the indices j and $j+1$ have to be understood "mod 2".

Lemma 7.9. For any $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ and $k \in \{1, 2, 3, 4\}$, $k \neq i, j$, it holds that

$$\sin \phi_j^i = \sin \theta_j^k \sin \psi^{k,i}.$$

Proof. We only show the case $i = 4$, $j = 1$ and $k = 2$. We then have

$$\sin \phi_1^4 = |\overline{P_1 P_4}| \sin \theta_1^2 \times \frac{1}{|\overline{P_1 P_4}|} \sin \psi^{2,4} = \sin \theta_1^2 \sin \psi^{2,4}.$$

□

Lemma 7.10. Assume that there exists a positive constant M_j ($j = 1, 2$) with $0 < M_j < 1$ such that

$$\sin \theta_j^4 \sin \phi_1^4 > M_j, \quad j = 1, 2.$$

Setting $\gamma(M_j) := \pi - \sin^{-1} M_j$ ($j = 1, 2$), we have $\frac{\pi}{2} < \gamma(M_j) < \pi$ and it holds that for each $j = 1, 2$,

$$\begin{aligned}
&\theta_1^4, \ \theta_2^4, \ \theta_3^4 < \gamma(M_j), \\
&\theta_2^3, \ \theta_4^3, \ \theta_3^2, \ \theta_4^2, \ \theta_1^2, \ \theta_1^3, \ \psi^{2,4}, \ \psi^{3,4} < \gamma(M_j).
\end{aligned}$$

Proof. From assumption, we have, for each $j = 1, 2$,

$$\begin{aligned}
&\sin \theta_j^4 \geq \sin \theta_j^4 \sin \phi_1^4 > M_j, \\
&\sin \phi_1^4 > M_j.
\end{aligned}$$

The definition of $\gamma(M_j)$ and Lemma 7.4 yield, for each $j = 1, 2$,

$$\begin{aligned}
&\pi - \gamma < \theta_j^4 < \gamma(M_j), \quad \theta_{j+1}^4 < \gamma(M_j), \quad \theta_{j+2}^4 < \gamma(M_j), \\
&\pi - \gamma < \phi_1^4 < \gamma(M_j),
\end{aligned}$$

where the indices j, $j+1$ and $j+2$ have to be understood "mod 3".

We obtain, from Lemma 7.9,

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} = \sin \theta_1^3 \sin \psi^{3,4} > M_j, \quad j = 1, 2.$$

We then have, for each $j = 1, 2$,

$$\sin \theta_1^2, \ \sin \psi^{2,4}, \ \sin \theta_1^3, \ \sin \psi^{3,4} > M_j,$$

that is,

$$\pi - \gamma(M_j) < \theta_1^2, \ \theta_1^3, \ \psi^{2,4}, \ \psi^{3,4} < \gamma(M_j).$$

On $\triangle P_1 P_2 P_4$ and $\triangle P_1 P_3 P_4$, using Lemma 7.4 yields

$$\theta_2^3, \ \theta_4^3, \ \theta_3^2, \ \theta_4^2 < \gamma(M_j), \quad j = 1, 2.$$

□

By analogous argument with Lemma 7.10, we get the following two lemmata.

Lemma 7.11. Assume that there exists M_3 with $0 < M_3 < 1$ such that

$$\sin\theta_3^1 \sin\phi_3^1 > M_3.$$

Setting $\gamma(M_3) := \pi - \sin^{-1} M_3$, we have $\frac{\pi}{2} < \gamma(M_3) < \pi$ and it holds that

$$\theta_3^2,\ \theta_3^4,\ \theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Proof. From assumption, we have

$$\sin\theta_3^1 \geq \sin\theta_3^1 \sin\phi_3^1 > M_3, \quad \sin\phi_3^1 > M_3.$$

Using the definition of $\gamma(M_3)$ yields

$$\pi - \gamma < \theta_1^3 < \gamma(M_3), \quad \pi - \gamma < \phi_1^3 < \gamma(M_3).$$

We obtain, from Lemma 7.9,

$$\sin\phi_3^1 = \sin\theta_3^2 \sin\psi^{2,1} = \sin\theta_3^4 \sin\psi^{4,1} > M_3.$$

We then have

$$\sin\theta_3^2,\ \sin\psi^{2,1},\ \sin\theta_3^4,\ \sin\psi^{4,1} > M_3,$$

that is,

$$\pi - \gamma(M_3) < \theta_3^2,\ \theta_3^4,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Meanwhile, on $\triangle P_2P_3P_4$, using Lemma 7.4, we have

$$\theta_2^1,\ \theta_4^1 < \gamma(M_3).$$

□

Lemma 7.12. Assume that there exists M_4 with $0 < M_4 < 1$ such that

$$\sin\theta_2^1 \sin\phi_4^1 > M_4.$$

Setting $\gamma(M_4) := \pi - \sin^{-1} M_4$, we have $\frac{\pi}{2} < \gamma(M_4) < \pi$ and it holds that

$$\theta_4^2,\ \theta_4^3,\ \theta_2^1,\ \theta_3^1,\ \theta_4^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(M_4).$$

Proof. The proof is obtained by using an analogous argument with Lemma 7.11. □

7.5 Proof of Theorem 7.2 in (Type i)

7.5.1 Condition 4.5 ⇒ Condition 6.2

We set $t_1 := \sin\theta_1^4$ and $t_2 := \sin\phi_1^4$. We then have

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4}.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin\theta_1^4 \sin\phi_1^4$. By construct of the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1 P_2 P_3$ of T. We hence have $\theta_1^4 < \frac{\pi}{2}$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^4 \leq \gamma_{11}, \quad \sin\theta_1^4 \geq \min\left\{ \sin\frac{\pi - \gamma_{11}}{2}, \sin\gamma_{11} \right\} =: C_1.$$

Due to Lemma 7.7, setting $\delta := \delta(\gamma_{11})$, $0 < \delta \leq \frac{\pi}{2}$ such that

$$\sin\delta = \left(\frac{\cos\gamma_{11} + 1}{\sin\frac{\gamma_{11}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that $\psi^{2,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_1^4 = \sin\theta_1^2 \sin\psi^{2,4} \geq C_0 \sin\theta_1^2.$$

By construct of the standard element (Type i), the angle θ_1^2 is not the minimum angle of $\triangle P_1 P_3 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^2 \leq \gamma_{11}, \quad \sin\theta_1^2 \geq C_1.$$

We thus obtain

$$\sin\phi_1^4 \geq C_0 C_1.$$

Suppose that $\psi^{3,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_1^4 = \sin\theta_1^3 \sin\psi^{3,4} \geq C_0 \sin\theta_1^3.$$

By construct of the standard element (Type i), the angle θ_1^3 is not the minimum angle of $\triangle P_1 P_2 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin\theta_1^3 \geq C_1.$$

We thus obtain

$$\sin\phi_1^4 \geq C_0 C_1.$$

In both cases

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yield

$$\frac{H_T}{h_T} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \leq \frac{6}{C_0 C_1^2} =: D_1 > 0,$$

that is, Condition 6.2 holds. □

7.5.2 Condition 6.2 $\Rightarrow$ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \leq \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_1^4 < \frac{\pi}{2}$ and $\sin\theta_1^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_1^4 \sin\phi_1^4 \geq \frac{6}{\gamma_9} =: C_2.$$

From Lemma 7.10 with $j = 1$, setting $\gamma(C_2) := \pi - \sin^{-1} C_2$, we have $\frac{\pi}{2} < \gamma(C_2) < \pi$ and it holds that

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2),\\
&\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
|T|_3 &= \frac{1}{3} \times \frac{1}{2} |\overline{P_2P_3}||\overline{P_3P_4}| \sin\theta_3^1 \times h_2 \sin\phi_3^1 = \frac{1}{6} h_2 |\overline{P_2P_3}||\overline{P_3P_4}| \sin\theta_3^1 \sin\phi_3^1\\
&< \frac{1}{3} h_1 h_2 h_3 \sin\theta_3^1 \sin\phi_3^1,
\end{aligned}$$

where we used the fact that $|\overline{P_3P_4}| < |\overline{P_1P_4}| + |\overline{P_1P_3}| \leq 2h_3$ on $\triangle P_1P_3P_4$ and $|\overline{P_2P_3}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_T}{h_T} > \frac{3}{\sin\theta_3^1 \sin\phi_3^1},$$

that is,

$$\sin\theta_3^1 \sin\phi_3^1 > \frac{3}{\gamma_9} =: C_3.$$

From Lemma 7.11, setting $\gamma(C_3) := \pi - \sin^{-1} C_3$, we have $\frac{\pi}{2} < \gamma(C_3) < \pi$ and it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(C_3).$$

Due to the cosine rule (7.1f) with $j = 2$, we get

$$\cos\psi^{1,3} = \sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 - \cos\psi^{3,4}\cos\psi^{4,1}.$$

By constructing the standard element (Type i), the angle θ_2^4 is the minimum angle of $\triangle P_1P_2P_3$. Therefore, we have

$$\begin{aligned}
&\cos\theta_2^4 \geq \frac{1}{2} \quad \text{because } \theta_2^4 \leq \frac{\pi}{3},\\
&\sin\psi^{3,4}\sin\psi^{4,1}\cos\theta_2^4 > 0, \quad \text{because } \sin\psi^{3,4}\sin\psi^{4,1} > 0,
\end{aligned}$$

and thus

$$\cos\psi^{1,3} > -\cos\psi^{3,4}\cos\psi^{4,1}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,1} > C_3$ yields

$$\begin{aligned}
\cos \psi^{1,3} &> -\cos \psi^{3,4} \cos \psi^{4,1} \\
&\geq -|\cos \psi^{3,4}||\cos \psi^{4,1}| = -\sqrt{1-\sin^2 \psi^{3,4}}\sqrt{1-\sin^2 \psi^{4,1}} \\
&> -\sqrt{1-C_2^2}\sqrt{1-C_3^2} =: C_4 > -1.
\end{aligned}$$

Setting $\gamma(C_4) := \cos^{-1} C_4$, it holds that

$$\psi^{1,3} < \gamma(C_4) < \pi.$$

Due to the cosine rule (7.1d) with $j = 1$, we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1 P_2 P_3$ of T^s. We hence have $\theta_1^4 < \frac{\pi}{2}$. Therefore, we have

$$\begin{aligned}
&\cos \theta_1^4 > 0 \quad \text{because } \theta_1^4 \leq \frac{\pi}{2}, \\
&\sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 > 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0,
\end{aligned}$$

and thus

$$\cos \psi^{2,3} > -\cos \psi^{3,4} \cos \psi^{4,2}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,2} > C_2$ yield

$$\begin{aligned}
\cos \psi^{2,3} &> -\cos \psi^{3,4} \cos \psi^{4,2} \\
&\geq -|\cos \psi^{3,4}||\cos \psi^{4,2}| = -\sqrt{1-\sin^2 \psi^{3,4}}\sqrt{1-\sin^2 \psi^{4,2}} \\
&> -(1-C_2^2) =: C_5 > -1.
\end{aligned}$$

Setting $\gamma(C_5) := \cos^{-1} C_5$, it holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_3), \gamma(C_4), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, Condition 4.5 holds. $\square$

7.6 Proof of Theorem 7.2 in (Type ii)

7.6.1 Condition 4.5 $\Rightarrow$ Condition 6.2

We set $t_1 := \sin \theta_2^4$ and $t_2 := \sin \phi_1^4$. We then have

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4}.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_2^4 \sin \phi_1^4$. By construct of the standard element (Type ii), the angle θ_3^4 and θ_1^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1 P_2 P_3$ of T^s. We hence have $\theta_2^4 < \frac{\pi}{2}$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^4 \leq \gamma_{11}, \quad \sin \theta_2^4 \geq C_1.$$

Due to Lemma 7.7, it holds that

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that $\psi^{1,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_2^4 = \sin\theta_2^1 \sin\psi^{1,4} \geq C_0 \sin\theta_2^1.$$

Furthermore, it holds that

$$\sin\phi_1^4 = \frac{|\overline{P_2P_4}| \sin\phi_2^4}{h_3}.$$

By construct of the standard element (Type ii), the angle θ_2^1 is not the minimum angle of $\triangle P_2P_3P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^1 \leq \gamma_{11}, \quad \sin\theta_2^1 \geq C_1.$$

Because $h_3 = |\overline{P_1P_4}| < |\overline{P_2P_4}|$ on $\triangle P_1P_2P_4$, we thus obtain

$$\sin\phi_1^4 = \frac{|\overline{P_2P_4}|}{h_3} \sin\phi_2^4 > C_0C_1.$$

Suppose that $\psi^{3,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_1^4 = \sin\theta_1^3 \sin\psi^{3,4} \geq C_0 \sin\theta_1^3.$$

By constructing the standard element (Type ii), the angle θ_1^3 is not the minimum angle of $\triangle P_1P_2P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin\theta_1^3 \geq C_1.$$

We thus obtain

$$\sin\phi_1^4 > C_0C_1.$$

In both cases

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yields

$$\frac{H_T}{h_T} = \frac{6}{\sin\theta_2^4 \sin\phi_1^4} \leq \frac{6}{C_0C_1^2} = D_1 > 0,$$

that is, Condition 6.2 holds. $\square$

7.6.2 Condition 6.2 $\Rightarrow$ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1h_2h_3}{|T|_3} = \frac{6}{\sin\theta_2^4 \sin\phi_1^4} \leq \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_2^4 < \frac{\pi}{2}$ and $\sin\theta_2^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_2^4 \sin\phi_1^4 \geq \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.10 with $j = 2$, it holds that

$$\begin{aligned}&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2),\\ &\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).\end{aligned}$$

Furthermore, we write

$$\begin{aligned}|T|_3 &= \frac{1}{3} \times \frac{1}{2}|\overline{P_2P_4}||\overline{P_2P_3}| \sin\theta_2^1 \times h_3 \sin\phi_4^1\\ &< \frac{1}{6} h_1 h_2 h_3 \sin\theta_2^1 \sin\phi_4^1,\end{aligned}$$

where we used the fact that $|\overline{P_3P_2}| = h_2$ and $|\overline{P_2P_4}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_{T^s}}{h_{T^s}} > \frac{6}{\sin\theta_2^1 \sin\phi_4^1},$$

that is,

$$\sin\theta_2^1 \sin\phi_4^1 > \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.12, it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(C_2).$$

Due to the cosine rule (7.1e) with $j = 2$, we get

$$\cos\psi^{4,1} = \sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 - \cos\psi^{1,3} \cos\psi^{3,4}.$$

By constructing the standard element (Type ii), the angle θ_2^3 is the minimum angle of $\triangle P_1P_2P_4$. Therefore, we have

$$\begin{aligned}&\cos\theta_2^3 \geq \frac{1}{2} \quad \text{because } \theta_2^3 \leq \frac{\pi}{3},\\ &\sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 > 0, \quad \text{because } \sin\psi^{1,3} \sin\psi^{3,4} > 0,\end{aligned}$$

and thus

$$\cos\psi^{4,1} > -\cos\psi^{1,3} \cos\psi^{3,4}.$$

Using $\sin\psi^{1,3} > C_2$ and $\sin\psi^{3,4} > C_2$ yield

$$\begin{aligned}\cos\psi^{4,1} &> -\cos\psi^{1,3} \cos\psi^{3,4}\\ &\geq -\sqrt{1 - \sin^2\psi^{1,3}}\sqrt{1 - \sin^2\psi^{3,4}}\\ &> -(1 - C_2^2) = C_5 > -1.\end{aligned}$$

It then holds that

$$\psi^{4,1} < \gamma(C_5) < \pi.$$

Due to the cosine rule (7.1d) with $j = 1$, we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type ii), the angle θ_1^4 is the minimum angle of $\triangle P_1 P_2 P_3$. We hence have $\theta_1^4 < \frac{\pi}{3}$. Therefore, we have

$$\begin{aligned}
&\cos \theta_1^4 \geq \frac{1}{2} \quad \text{because } \theta_1^4 \leq \frac{\pi}{3}, \\
&\sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 > 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0,
\end{aligned}$$

and thus

$$\cos \psi^{2,3} > -\cos \psi^{3,4} \cos \psi^{4,2}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,2} > C_2$ yield

$$\begin{aligned}
\cos \psi^{2,3} &> -\cos \psi^{3,4} \cos \psi^{4,2} \\
&\geq -\sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,2}} \\
&> -(1 - C_2^2) = C_5 > -1.
\end{aligned}$$

It then holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_2), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, Condition 4.5 holds. $\square$

8 Good Elements or not for $d = 2, 3$?

In this subsection, we consider good elements on meshes. In this paper, we define 'good elements' on meshes as the existence of a positive constant $\gamma_0 > 0$ satisfying (6.1). We treat a "Right-angled triangle", "Blade" and "Dagger" for $d = 2$, and "Spire", "Spear", "Spindle", "Spike", "Splinter" and "Sliver" for $d = 3$, which are introduced in [10]. We give the quantities $h_{\max}/h_{\min}$ and H_T/h_T for those elements. The parameters $h_{\max}$ and $h_{\min}$ are defined as

$$h_{\max} := \max\{h_1, \ldots, h_d\}, \quad h_{\min} := \min\{h_1, \ldots, h_d\}. \tag{8.1}$$

8.1 Isotropic Mesh Elements

Recall that an isotropic mesh element has equal or nearly equal edge lengths and angles, resulting in a balanced shape. Then, the geometric condition (3.4) is satisfied. Therefore, it holds that

$$\frac{H_T}{h_T} \leq \frac{h_T^d}{|T|_d} \leq \frac{1}{\gamma_3}, \quad \frac{h_{\max}}{h_{\min}} \leq c \frac{h_T^d}{|T|_d} \leq \frac{c}{\gamma_3}.$$

In this case, elements satisfying the geometric condition (3.4) are "good."

8.2 Anisotropic mesh: two-dimensional case

Let $S \subset \mathbb{R}^2$ be a triangle. Let $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon, \delta, \gamma \in \mathbb{R}$

Example 8.1 (Right-angled triangle)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = s$ and $h_2 = s^\varepsilon$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \le s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element S is "good."

Example 8.2 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ and $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \le c s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \le c. \end{aligned}$$

In this case, the element S is "good."

Remark 8.3. In the above examples, $h_2 \approx \widetilde{\mathscr{H}_2}$ holds. That is, the good element $S \subset \mathbb{R}^2$ may satisfy conditions such as $h_2 \approx \widetilde{\mathscr{H}_2}$.

Example 8.4 (Blade)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = h_2 = \sqrt{s^2 + s^{2\varepsilon}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \to \infty \quad \text{as } s \to 0.$$

In this case, the element S is "not good."

Example 8.5 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ and $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \le c s^{1-\delta} \to \infty \quad \text{as } s \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \le c s^{\delta-\varepsilon} \to \infty \quad \text{as } s \to 0. \end{aligned}$$

In this case, the element S is "not good."

Anisotropic elements in the next two examples are also "good." However, these examples differ slightly from Examples 8.1 and 8.4.

Example 8.6 (Right-angled triangle)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = s$ and $h_2 = \delta s$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{1}{\delta}, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element S is "good." However, the factor $\frac{1}{\delta}$ is very large.

Example 8.7 (Blade)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = h_2 = s\sqrt{1 + \delta^2}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2(1+\delta^2)}{\delta s^2} \le \frac{c}{\delta},$$

In this case, the element S is "good." However, the factor $\frac{1}{\delta}$ is very large.

8.3 Anisotropic mesh: three-dimensional case

Example 8.8. Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let S be the base of T; i.e., $S = \triangle p_1 p_2 p_3$. Recall that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{h_1 h_2}{\frac{1}{2} h_1 h_2 t_1} \frac{h_3}{\frac{1}{3} h_3 t_2} \leq \frac{H_S}{h_S} \frac{h_3}{\frac{1}{3} \widetilde{\mathscr{H}_3}}. \tag{8.2}$$

If the triangle S is "not good" such as in Examples 8.4 and 8.5, the quantity (8.2) may diverge. In the following, we consider the case that the triangle S is "good".

Assume that there exists a positive constant M such that $\frac{H_S}{h_S} \leq M$. For simplicity, we set $p_1 := (0,0,0)^\top$, $p_2 := (2s,0,0)^\top$, and $p_3 := (2s - \sqrt{4s^2 - s^{2\gamma}}, s^\gamma, 0)^\top$ with $1 < \gamma$. Then,

$$h_1 = 2s, \quad h_2 = \sqrt{\frac{4s^{2\gamma}}{2 + \sqrt{4 - s^{2\gamma - 2}}}},$$

and because $h_{\max} \approx cs$,

$$\frac{h_{\max}}{h_{\min}} \leq \frac{cs}{h_2} \leq cs^{1-\gamma} \to \infty \quad \text{as } s \to 0.$$

If we set $p_4 := (s, 0, s^\varepsilon)^\top$ with $1 < \varepsilon$, the triangle $\triangle p_1 p_2 p_4$ is the blade (Example 8.4). Then,

$$h_3 = \sqrt{s^2 + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{2+\gamma}}{s^{1+\gamma+\varepsilon}} \leq cs^{1-\varepsilon} \to \infty \quad \text{as } s \to 0.$$

In this case, the element T is "not good."

If we set $p_4 := (s^\delta, 0, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon < \gamma$, the triangle $\triangle p_1 p_2 p_4$ is the dagger (Example 8.5, Fig. 10). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\delta}}{s^{1+\gamma+\varepsilon}} \leq cs^{\delta-\varepsilon} \to \infty \quad \text{as } s \to 0.$$

In this case, the element T is "not good."

If we set $p_4 := (s^\delta, 0, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta < \gamma$, the triangle $\triangle p_1 p_2 p_4$ is the dagger (Example 8.2). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\varepsilon}}{s^{1+\gamma+\varepsilon}} \leq c.$$

In this case, the element T is "good" and $h_3 \approx h_3 t_2 = \widetilde{\mathscr{H}_3}$ holds.

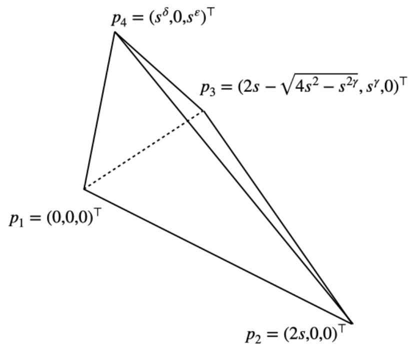

Fig. 10: Example 8.8

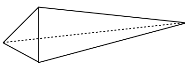

Fig. 11: Spire

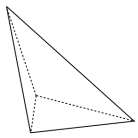

Fig. 12: Spear

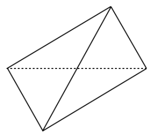

Fig. 13: Spindle

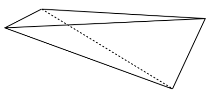

Fig. 14: Spike

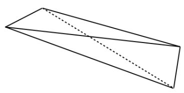

Fig. 15: Splinter

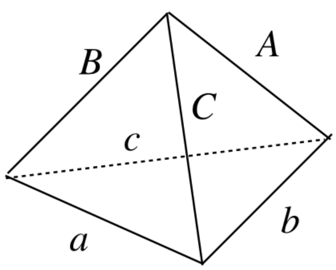

Fig. 16: R_3

Example 8.9. In [10], the spire has a cycle of three daggers among its four triangles; see Figure 11. The splinter has four daggers; see Figure 15. The spear and spike have two daggers and two blades as triangles; see Figures 12, 14. The spindle has four blades as triangles; see Figure 13.

Remark 8.10. The above examples reveal that the good element $T \subset \mathbb{R}^3$ may satisfy conditions such as $h_2 \approx \tilde{\mathscr{H}_2}$ and $h_3 \approx \tilde{\mathscr{H}_3}$.

Example 8.11. Using an element T called *Sliver*, we compare the three quantities $\frac{h_T^3}{|T|_3}$, $\frac{H_T}{h_T}$, and $\frac{R_3}{h_T}$, where the formulation of the circumradius R_3 of a tetrahedron T is as follows, e.g., see [21]. Let a, b and c be the lengths of the three edges of T and A, B, C the length of the opposite edges of a, b, c, respectively. Then,

$$R_3 = \frac{\sqrt{(aA + bB + cC)(aA + bB - cC)(aA - bB + cC)(-aA + bB + cC)}}{24|T|_3},$$

see Fig. 16.

Let $T \subset \mathbb{R}^3$ be the simplex with vertices $p_1 := (s^{\varepsilon_2}, 0, 0)^\top$, $p_2 := (-s^{\varepsilon_2}, 0, 0)^\top$, $p_3 := (0, -s, s^{\varepsilon_1})^\top$, and $p_4 := (0, s, s^{\varepsilon_1})^\top$ $(\varepsilon_1, \varepsilon_2 > 1)$, where $s := \frac{1}{N}$, $N \in \mathbb{N}$, see Fig. 17. Let L_i $(1 \leq i \leq 6)$ be the edges of T with $h_{\min} = L_1 \leq L_2 \leq \cdots \leq L_6 = h_T$. Recall that $h_{\max} \approx h_T$ and

$$\frac{h_{\max}}{h_{\min}} \leq c \frac{L_6}{L_1}, \quad \frac{H_T}{h_T} = \frac{L_1 L_2}{|T|_3} h_T.$$

Table 8: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T $(\varepsilon_1 = 1.5, \varepsilon_2 = 1.0)$

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	1.4033	6.7882e+01	3.4471e+01	5.0195e-01
64	1.5625e-02	1.4087	9.6000e+01	4.8375e+01	5.0098e-01
128	7.8125e-03	1.4115	1.3576e+02	6.8147e+01	5.0049e-01

Table 9: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T $(\varepsilon_1 = 1.0, \varepsilon_2 = 1.5)$

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	6.7882e+01	8.5513	5.0006e-01
64	1.5625e-02	8.0000	9.6000e+01	8.5184	5.0002e-01
128	7.8125e-03	1.1314e+01	1.3576e+02	8.5018	5.0000e-01

Table 10: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T $(\varepsilon_1 = 1.5, \varepsilon_2 = 1.5)$

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	3.8400e+02	3.4986e+01	1.4170
64	1.5625e-02	8.0000	7.6800e+02	4.8744e+01	2.0010
128	7.8125e-03	1.1314e+01	1.5360e+03	6.8411e+01	2.8288

In Table 8, the angle between $\triangle p_1p_2p_3$ and $\triangle p_1p_2p_4$ tends to π as $s \to 0$, and the simplex T is "not good." In Table 9, the angle between $\triangle p_1p_3p_4$ and $\triangle p_2p_3p_4$ tends to 0 as $s \to 0$, the simplex T is "good." In Table 10, from the numerical result, the simplex T is "not good."

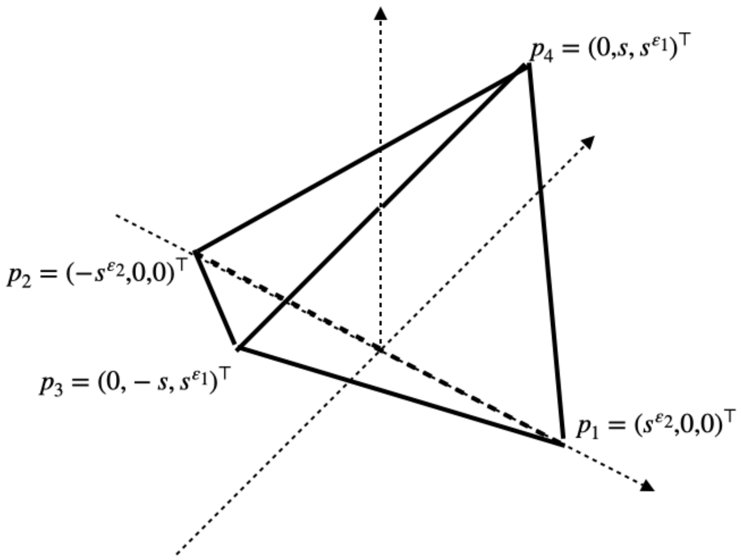

Fig. 17: Sliver

9 Interpolation of Smooth Function

9.1 Finite Element Generation

We follow the procedure described in [15, Chapter 9] and [14, Section 1.4.1 and 1.2.1]; also see [30, Section 3.5]. The definition of a *finite element* can be found in [11, p. 78] and [15, Definition 5.2].

For the reference element $\widehat{T}$ defined in Sections 5.1, let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a fixed reference finite element, where $\widehat{P}$ is a vector space of functions $\hat{q} : \widehat{T} \to \mathbb{R}^n$ for some positive integer n (typically $n = 1$ or $n = d$) and $\widehat{\Sigma}$ is a set of n_0 linear forms $\{\hat{\chi}_1, \dots, \hat{\chi}_{n_0}\}$ such that

$$\widehat{P} \ni \hat{q} \mapsto (\hat{\chi}_1(\hat{q}), \dots, \hat{\chi}_{n_0}(\hat{q}))^T \in \mathbb{R}^{n_0}$$

is bijective; i.e., $\widehat{\Sigma}$ is a basis for $\mathcal{L}(\widehat{P}; \mathbb{R})$. Further, we denote by $\{\hat{\theta}_1, \dots, \hat{\theta}_{n_0}\}$ in $\widehat{P}$ the local ($\mathbb{R}^n$-valued) shape functions such that

$$\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}, \quad 1 \le i, j \le n_0.$$

Let $V(\widehat{T})$ be a normed vector space of functions $\hat{\varphi} : \widehat{T} \to \mathbb{R}^n$ such that $\widehat{P} \subset V(\widehat{T})$ and the linear forms $\{\hat{\chi}_1, \dots, \hat{\chi}_{n_0}\}$ can be extended to $V(\widehat{T})'$, i.e., there exist $\{\bar{\chi}_1, \dots, \bar{\chi}_{n_0}\}$ and c_χ such that $\hat{\chi}_i(\hat{q}) = \bar{\chi}_i(\hat{q})$ for any $\hat{q} \in \widehat{P}$, and $|\bar{\chi}_i(\hat{v})| \le c_\chi \|\hat{v}\|_{V(\widehat{T})}$ and for $i \in \{1, \dots, n_0\}$. We use the same symbol $\hat{\chi}_i$ instead of $\bar{\chi}_i$. The local interpolation operator $I_{\widehat{T}}$ is then defined by

$$I_{\widehat{T}} : V(\widehat{T}) \ni \hat{\varphi} \mapsto \sum_{i=1}^{n_0} \hat{\chi}_i(\hat{\varphi})\hat{\theta}_i \in \widehat{P}. \tag{9.1}$$

It obviously holds that, for any $\hat{\varphi} \in V(\widehat{T})$,

$$\hat{\chi}_i(I_{\widehat{T}}\hat{\varphi}) = \hat{\chi}_i(\hat{\varphi}) \quad i = 1, \dots, n_0. \tag{9.2}$$

Proposition 9.1. $\widehat{P}$ is invariant under $I_{\widehat{T}}$, that is,

$$I_{\widehat{T}}\hat{q} = \hat{q} \quad \forall \hat{q} \in \widehat{P}. \tag{9.3}$$

Proof. Let $\hat{q} = \sum_{j=1}^{n_0} \alpha_j \hat{\theta}_j$ for $\alpha_j \in \mathbb{R}$, $1 \leq j \leq n_0$. Then,

$$I_{\widehat{T}}\hat{q} = \sum_{i,j=1}^{n_0} \alpha_j \hat{\chi}_i(\hat{\theta}_j)\hat{\theta}_i = \hat{q}.$$

□

Let $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$ be the two affine mappings defined in Section 5.2. For any $T \in \mathbb{T}_h$ with $T = \Phi(\widehat{T}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\widehat{T})$, we fdefine a Banach space $V(T)$ of $\mathbb{R}^n$-valued functions that is the counterpart of $V(\widehat{T})$ and define a linear bijection mapping by

$$\psi := \psi_{\widehat{T}} \circ \psi_{\widetilde{T}} : V(T) \ni \varphi \mapsto \hat{\varphi} := \psi(\varphi) := \varphi \circ \Phi \in V(\widehat{T}),$$

with two linear bijection mappings:

$$\begin{aligned} \psi_{\widetilde{T}} &: V(T) \ni \varphi \mapsto \tilde{\varphi} := \psi_{\widetilde{T}}(\varphi) := \varphi \circ \Phi_T \in V(\widetilde{T}), \\ \psi_{\widehat{T}} &: V(\widetilde{T}) \ni \tilde{\varphi} \mapsto \hat{\varphi} := \psi_{\widehat{T}}(\tilde{\varphi}) := \tilde{\varphi} \circ \Phi_{\widetilde{T}} \in V(\widehat{T}). \end{aligned}$$

Triples $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T, P, \Sigma\}$ are defined as follows:

$$\begin{cases} \widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}); \\ \widetilde{P} = \{\psi_{\widehat{T}}^{-1}(\hat{q});\ \hat{q} \in \widehat{P}\}; \\ \widetilde{\Sigma} = \{\{\tilde{\chi}_i\}_{1 \leq i \leq n_0};\ \tilde{\chi}_i = \hat{\chi}_i(\psi_{\widehat{T}}(\tilde{q})), \forall \tilde{q} \in \widetilde{P}, \hat{\chi}_i \in \widehat{\Sigma}\}, \end{cases}$$

and

$$\begin{cases} T = \Phi_T(\widetilde{T}); \\ P = \{\psi_{\widetilde{T}}^{-1}(\tilde{q});\ \tilde{q} \in \widetilde{P}\}; \\ \Sigma = \{\{\chi_i\}_{1 \leq i \leq n_0};\ \chi_i = \tilde{\chi}_i(\psi_{\widetilde{T}}(q)), \forall q \in P, \tilde{\chi}_i \in \widetilde{\Sigma}\}. \end{cases}$$

Proposition 9.2. The triples $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T, P, \Sigma\}$ are finite elements.

Proof. A proof can be obtained similarly for [15, Proposition 9.2].

□

The local shape functions are $\tilde{\theta}_i = \psi_{\widehat{T}}^{-1}(\hat{\theta}_i)$ and $\theta_i = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$, $1 \leq i \leq n_0$, and the associated local interpolation operators are respectively defined by

$$I_{\widetilde{T}} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}\tilde{\varphi} := \sum_{i=1}^{n_0} \tilde{\chi}_i(\tilde{\varphi})\tilde{\theta}_i \in \widetilde{P}, \tag{9.4}$$

$$I_T : V(T) \ni \varphi \mapsto I_T\varphi := \sum_{i=1}^{n_0} \chi_i(\varphi)\theta_i \in P. \tag{9.5}$$

The following diagrams play an important role in analysing the interpolation error.

Proposition 9.3 (Commuting diagrams)**.** The diagrams

$$\begin{tikzcd} V(T) \arrow[r, "\psi_{\widetilde{T}}"] \arrow[d, "I_T"'] & V(\widetilde{T}) \arrow[r, "\psi_{\widehat{T}}"] \arrow[d, "I_{\widetilde{T}}"'] & V(\widehat{T}) \arrow[d, "I_{\widehat{T}}"] \\ P \arrow[r, "\psi_{\widetilde{T}}"'] & \widetilde{P} \arrow[r, "\psi_{\widehat{T}}"'] & \widehat{P} \end{tikzcd}$$

commute. Furthermore, $\widetilde{P}$ and P are respectively invariant under $I_{\widetilde{T}}$ and I_T.

Proof. A proof can be obtained similarly for [15, Proposition 9.3].

Let $\tilde{\varphi} \in V(\widetilde{T})$. The definition of $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ implies that

$$I_{\widehat{T}}(\psi_{\widehat{T}}(\tilde{\varphi})) = \sum_{i=1}^{n_0} \hat{\chi}_i(\psi_{\widehat{T}}(\tilde{\varphi}))\hat{\theta}_i = \sum_{i=1}^{n_0} \tilde{\chi}_i(\tilde{\varphi})\psi_{\widehat{T}}(\tilde{\theta}_i) = \psi_{\widehat{T}}(I_{\widetilde{T}}\tilde{\varphi}).$$

Here, we used the linearity of $\psi_{\widehat{T}}$. Therefore, the right diagram commutes.

Let $\tilde{q} \in \widetilde{P}$. Because $\psi_{\widehat{T}}(\tilde{q}) \in \widehat{P}$ and $\widehat{P}$ is invariant under $I_{\widehat{T}}$,

$$I_{\widetilde{T}}(\tilde{q}) = \psi_{\widehat{T}}^{-1}(I_{\widehat{T}}(\psi_{\widehat{T}}(\tilde{q}))) = \psi_{\widehat{T}}^{-1}(\psi_{\widehat{T}}(\tilde{q})) = \tilde{q}.$$

Another diagram can be proved in the same way.

$\square$

Example 9.4. Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element.

1. For the Lagrange finite element of degree k, we set $V(\widehat{T}) := \mathcal{C}^0(\widehat{T})$.
2. For the Hermite finite element, we set $V(\widehat{T}) := \mathcal{C}^1(\widehat{T})$.
3. For the Crouzeix–Raviart finite element with $k = 1$, we set $V(\widehat{T}) := W^{1,1}(\widehat{T})$.

9.2 Remarks on the Anisotropic Interpolation Error

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