Reconsidered error analysis in the finite element methods

Hiroki ISHIZAKA *

April 7, 2025

Abstract

This article presents novel proof methods for estimating interpolation errors, predicated on the understanding that one has already studied foundational error analysis using the finite element method.

Contents

*h.ishizaka005@gmail.com

Part I
Interpolation Error Analysis using a New Geometric Parameter

1 Preliminalies

1.1 General Convention

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants c are bounded if the maximum angle is bounded. These values vary across different contexts.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert \cdot \rvert_d$	d-dimensional Hausdorff measure
$v\vert_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x \rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x \rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
A, B	Matrices
A_{ij} or $[A]_{ij}$	Entry of A in the ith and the jth column
$A^\top$	Transpose of the matrix A
$\mathrm{Tr}(A)$	Trace of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A_{ii}$
$\det(A)$	Determinant of A
$\mathrm{diag}(A)$	Diagonal of A:
	For $A \in \mathbb{R}^{m \times n}$, $\mathrm{diag}(A)_{ij} := \delta_{ij} A_{ij}$, $1 \le i, j \le d$
Ax	Matrix-vector product:
	For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, $(Ax)_i := \sum_{j=1}^{d} A_{ij} x_j$ for $1 \le i \le d$
$A : B$	Double contraction:

	For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, $A : B := \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$
$\|A\|_2$	Operator norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_2 := \sup_{0\neq x\in\mathbb{R}^d} \frac{\|Ax\|_E}{\|x\|_E}$
$\|A\|_{\max}$	Max norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_{\max} := \max_{1\le i,j\le d} \|A_{ij}\|$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this article, we use the following facts.

For $A \in \mathbb{R}^{m\times n}$, it holds that

$$\|A\|_{\max} \le \|A\|_2 \le \sqrt{mn}\|A\|_{\max}, \tag{1.1}$$

e.g., see [17, p. 56]. For $A, B \in \mathbb{R}^{m\times m}$, it holds that

$$\|AB\|_2 \le \|A\|_2\|B\|_2. \tag{1.2}$$

If $A^\top A$ is a positive definite matrix in $\mathbb{R}^{d\times d}$, the spectral norm of the matrix $A^\top A$ is the largest eigenvalue of $A^\top A$; i.e.,

$$\|A\|_2 = \left(\lambda_{\max}(A^\top A)\right)^{1/2} = \sigma_{\max}(A), \tag{1.3}$$

where $\lambda_{\max}(A)$ and $\sigma_{\max}(A)$ are respectively the largest eigenvalues and singular values of A.

If $A \in O(d)$, because $A^\top = A^{-1}$ and

$$|Ax|_E^2 = (Ax)^\top(Ax) = x^\top A^\top Ax = x^\top A^{-1}Ax = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0\neq x\in\mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0\neq x\in\mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

1.4 Function Spaces

This article uses standard Sobolev spaces with associated norms (e.g., see [6, 13, 14]).

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \ldots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T)$, $\mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$) is spanned by the restriction to T of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$)

1.5.2 Meshes

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \mathrm{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $p_1, \dots, p_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex p_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^\partial$ be the set of faces on boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1, \infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h; \mathbb{R}^q) := \{v \in L^p(\Omega; \mathbb{R}^q) : \ v|_T \in W^{s,p}(T; \mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} := \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \quad \text{if } p \in [1, \infty),$$
$$\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} := \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h; \mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h; \mathbb{R}^q) := W^{s,2}(\mathbb{T}_h; \mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h; \mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|^2_{L^2(T)^d} \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$\begin{aligned}[\![v \cdot n]\!] := [\![v \cdot n]\!]_F &:= v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp, \\ [\![v]\!] := [\![v]\!]_F &:= v_\natural - v_\sharp, \quad \natural > \sharp,\end{aligned}$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_\omega := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F} v_\natural + \omega_{T_\sharp,F} v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi) \cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1, \infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div}; T)$ space by

$$H(\mathrm{div}; \mathbb{T}_h) := \left\{ v \in L^2(\Omega)^d; \ v|_T \in H(\mathrm{div}; T) \ \forall T \in \mathbb{T}_h \right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div}; \mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div}; \mathbb{T}_h)$,

$$(\mathrm{div}_h v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.5.4 Barycentric Coodinates

For a simplex $T \subset \mathbb{R}^d$, let $\{p_i\}_{i=1}^{d+1}$ be vertices of T and $(x_1^{(i)}, \dots, x_d^{(i)})^T$ coordinates of p_i. We set

$$\Delta := \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ of the point $p(x_1, \dots, x_d)$ with respect to $\{p_i\}_{i=1}^{d+1}$ are then defined as

$$\lambda_i(x) := \frac{1}{\Delta} \det \begin{pmatrix} 1 & \cdots & \overset{\overset{i}{\smile}}{1} & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(p_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

Let r, s be two nonnegative real numbers and $\{x_i\}_{i\in I}$ be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r \leq s, \\ \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \operatorname{card}(I)^{\frac{r-s}{rs}} \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \tag{1.1}$$

see [14, Exercise 12.1].

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.1. Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) \quad \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) \quad \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) \quad \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.2}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.3}$$

Proof. See, for example, [13, Corollary B.43, Theorem B.40] and [14, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.2. Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.4}$$

Proof. See, for example, [33, Theorem 2.1 (p. 61)] and the references therein. □

1.6.3 Trace Theorem

Theorem 1.3 (Trace on low-dimensional manifolds)**.** Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0, \ldots, d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.

Proof. See [14, Theorem 3.15]. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [12, 8]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [2, Lemma 2.1].

Lemma 1.4. Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped concerning balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le \ell$, $p \in [1,\infty]$. It then holds that

$$\|\partial^\gamma(\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \le C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.5}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \le \ell-1} \int_B \eta(y)(\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \tag{1.6}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [11, Theorem 1.1] and [12, 8, 37] which are variants of the Bramble–Hilbert lemma.

Lemma 1.5. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0,1,\ldots,m. \tag{1.7}$$

Proof. The proof is found in [11, Theorem 1.1]. □

Remark 1.6. In [8, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r : D$ is star-shaped with respect to a ball of radius $r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m,\gamma) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0,1,\ldots,m. \tag{1.8}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\operatorname{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d,m,\gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d,m)$ of the estimate (1.7) does not depend on the geometric parameter γ.

Remark 1.7. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d,m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf$\ddot{u}$rth [37].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \le C^{BH}(d,k,m) \operatorname{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0,1,\ldots,m-1. \tag{1.9}$$

Verf$\ddot{u}$rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d,k,m) \le \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3,1,2) \leq \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \leq \frac{\sqrt{6}}{\pi} |\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

because $\operatorname{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.8 (Poincaré inequality)**.** Let $D \subset \mathbb{R}^d$ be a convex domain with diameter $\operatorname{diam}(D)$. It then holds that, for $\varphi \in H^1(D)$ with $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \leq \frac{\operatorname{diam}(D)}{\pi} |\varphi|_{H^1(D)}. \tag{1.10}$$

Proof. The proof is found in [34, Theorem 3.2], also see [35]. □

Remark 1.9. The coefficient $\frac{1}{\pi}$ of (1.10) may be improved.

1.7 Abbreviated expression

FE	Finite Element
FEMs	Finite Element Methods

2 Isotropic and Anisotropic Mesh Elements

In the context of FEMs, mesh elements can be classified based on their geometric properties. *An isotropic mesh element* has equal or nearly equal edge lengths and angles, resulting in a balanced shape. In contrast, *an anisotropic mesh element* features significant variation in edge lengths and angles.

Consider the following examples: Let $s, \delta \in \mathbb{R}_+$, and $\varepsilon \geq 1$, $\varepsilon \in \mathbb{R}$.

Example 2.1. In the case of the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$, the triangle is classified as follows:

- If $\delta \approx 1$, the triangle T is considered an isotropic mesh element.
- Conversely, if δ is much less than 1, i.e., $\delta \ll 1$, the triangle T becomes an anisotropic mesh element.

Example 2.2. In this case, consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Here, the vertex p_3 introduces a parameter ε that can influence the shape of the simplex. The classification of this simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle maintains a balanced shape, making it isotropic.

- If $\varepsilon \geq 1$, the triangle becomes flat when $s \ll 1$, resulting in an anisotropic mesh element.

Example 2.3. Consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. In this configuration, the classification of the simplex as isotropic or anisotropic depends on the value of δ:

- If $\delta \approx 1$, the triangle is an isotropic mesh element.
- If $\delta \ll 1$, the triangle becomes an anisotropic mesh element.

Example 2.4. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. In this case, the classification of the simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle is isotropic because the height from p_3 is equal to the base length.
- If $\varepsilon \geq 1$, the triangle will be classified as anisotropic, as the edge lengths will differ significantly when $s \ll 1$.

3 Classical Geometric Conditions

3.1 Classical Interpolation Error Estimate

Let $\widehat{T} \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^d$ be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ and $\{T, P, \Sigma\}$ with associated normed vector spaces $V(\widehat{T})$ and $V(T)$. The transformation Φ_T takes the form

$$\Phi_T : \widehat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := B_T \hat{x} + b_T \in T,$$

where $B_T \in \mathbb{R}^{d \times d}$ is an invertible matrix and $b_T \in \mathbb{R}^d$. Let $I_T : V(T) := W^{2,p}(T) \to P := \mathbb{P}^1(T)$ with $p \in [1, \infty]$ be an interpolation on T with $I_T p = p$ for any $p \in \mathcal{P}^1(T)$. According to the classical theory (e.g., see [10, 13]), there exists a positive constant c, independent of h_T, such that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \left(\|B_T\|_2 \|B_T^{-1}\|_2 \right) \|B_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity $\|B_T\|_2 \|B_T^{-1}\|_2$ is called the *Euclidean condition number* of B_T. By standard estimates (e.g., see [13, Lemma 1.100]), we have

$$\|B_T\|_2 \|B_T^{-1}\|_2 \leq c \frac{h_T}{\rho_T}, \quad \|B_T\|_2 \leq c h_T.$$

It thus holds that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.1}$$

As a geometric condition, the *shape-regularity condition* is well known to obtain global interpolation error estimates. This condition is stated as follows.

Condition 3.1 (Shape-regularity condition)**.** There exists a constant $\gamma_1 > 0$ such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{3.2}$$

Under Condition 3.1, that is, when the quantity $\frac{h_T}{\rho_T}$ is bounded on each T, it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq c h |\varphi|_{W^{2,p}(\Omega)},$$

where $I_h \varphi$ is the standard global linear interpolation of φ on $\mathbb{T}_h$.

3.2 Regular Mesh Conditions

Geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity condition (3.2). A proof can be found in [7, Theorem 1].

Condition 3.2 (Zlámal's condition)**.** There exists a constant $\gamma_2 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$, any simplex $T \in \mathbb{T}_h$ and any dihedral angle ψ and for $d = 3$, also any solid angle θ of T, we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \tag{3.1}$$

Condition 3.3. There exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_3 h_T^d. \tag{3.2}$$

Condition 3.4. There exists a constant $\gamma_4 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_4 |B_d^T|, \tag{3.3}$$

where $B^T \supset T$ is the circumscribed ball of T.

Note 3.5. If Condition 3.1 or 3.2 or 3.3 or 3.4 holds, a family of simplicial partitions is called *regular*.

Note 3.6. Condition 3.2 was presented by Zlámal [38] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on $\mathbb{R}^2$. Zlámal's condition can be generalised into $\mathbb{R}^n$ for any $n \in \{2, 3, \ldots\}$. Later, the shape-regularity condition (the inscribed ball condition) was introduced; see [10]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

Note 3.7. Condition 3.3 seems to be simpler than Condition 3.1, Condition 3.2 and Condition 3.4. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

3.3 What happens when anisotropic meshes are used?

Using the equivalence conditions in Section 3.2, the error estimate (3.1) is rewritten as

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T^2}{|T|_2} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.1}$$

We considered the following four anisotropic elements as in Section 2: Let $0 < s, \delta \ll 1$, $s, \delta \in \mathbb{R}$, and $\varepsilon > 1$, $\varepsilon \in \mathbb{R}$.

Example 3.8. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = \delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{4}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.1) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}.$$

When $\delta \ll 1$, the interpolation error (3.1) may be large.

Example 3.9. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = s^{1+\varepsilon}$, and

$$\frac{h_T^2}{|T|_2} = 4s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity is not satisfied. In this case, when $\varepsilon > 2$, the estimate (3.1) diverges as $s \to 0$.

Example 3.10. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. Then, we have that $h_T = s\sqrt{1+\delta^2} \approx s$, and $|T|_2 = \frac{1}{2}\delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{2(1+\delta^2)}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.1) is as follows:

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.2}$$

It is implied that the interpolation error (3.2) may be large when $\delta \ll 1$.

Example 3.11. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. Subsequently, we obtain $h_T = \sqrt{s^2 + s^{2\varepsilon}} \approx s$, and $|T|_2 = \frac{1}{2}s^{1+\varepsilon}$, and

$$\frac{h_T^2}{|T|_2} = \frac{2(s^2 + s^{2\varepsilon})}{s^{1+\varepsilon}} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.1) diverges as $s \to 0$.

Remark 3.12. As will be explained later, the factor $\frac{1}{\delta}$ in Example 3.10 is violated and the interpolation error estimate converges in the case of Example 3.11 using new precise interpolation error estimates under more relaxed geometric conditions.

4 Classical Relaxed Geometric Conditions

4.1 Semi-regular Mesh Conditions for $d = 2$

In 1957, Synge [36, Section 3.8] proposed the following condition.

Condition 4.1 (Synge's condition)**.** There exists $\frac{\pi}{3} \leq \gamma_5 < \pi$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_5, \tag{4.1}$$

where $\theta_{T,\max}$ is the maximal angle of T.

Under Condition 4.1, Synge proved an optimal interpolation error estimate as follows.

$$\|\varphi - I_h\varphi\|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)} \quad \text{for } p = \infty.$$

The inequality (4.1) is called *Synge's condition* or the *maximum-angle condition.* In 1976, several author's [4, 5, 18, 30] independently proved the convergence of finite element for $p < \infty$. It ensures that finite elements converge effectively when the minimum angle approaches zero as the mesh size decreases. If this condition is not met, the accuracy of interpolation for linear triangular elements can suffer, similar to the absence of Zlámal's condition, see e.g. [4, p. 223]. This underscores the importance of keeping proper geometric constraints to ensure reliable outcomes in numerical methods. Synge's condition is essential in finite element analysis.

In [31], Křížek proposed the following circumscribed ball condition for $d = 2$ which is equivalent to Synge's condition.

Condition 4.2. There exists $\gamma_6 > 0$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\frac{R_2}{h_T} \leq \gamma_6, \tag{4.2}$$

where R_2 is the radius of the circumscribed ball of $T \subset \mathbb{R}^2$.

Note 4.3. If Condition 4.1 or 4.2 holds, the associated families of partitions are called *semi-regular*.

Remark 4.4. Assume that Condition 3.3 holds, that is, there exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_3 h_T^2.$$

Let $T \subset \mathbb{R}^2$ be the triangle with vertices P_1, P_2 and P_3 such that the maximum angle $\theta_{T,\max}$ of T is $\angle P_2 P_1 P_3$. We then have $h_T = |P_2 P_3|$ and

$$\frac{R_2}{h_T} = \frac{|P_2 P_3|}{2 h_T \sin \theta_{T,\max}} = \frac{|P_1 P_2||P_1 P_3|}{2|P_1 P_2||P_1 P_3| \sin \theta_{T,\max}} \leq c \frac{h_T^2}{|T|} \leq \frac{c}{\gamma_3} =: \gamma_6.$$

This implies that each regular family is semi-regular. However, the converse implication does not hold.

4.2 Semi-regular Mesh Conditions for $d = 3$

Synge's condition (4.1) is extended to the case of tetrahedra in [32].

Condition 4.5. There exists a constant $0 < \gamma_7 < \pi$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_7, \tag{4.1a}$$
$$\psi_{T,\max} \leq \gamma_7, \tag{4.1b}$$

where $\theta_{T,\max}$ is the maximum angle of all triangular faces of the tetrahedron T and $\psi_{T,\max}$ is the maximum dihedral angle of T.

Remark 4.6. The theory of anisotropic interpolation has been advanced through extensive research ([3, 2, 9]).

Question 4.7. Is there a semi-regularity condition which equivalent to Synge's condition (4.1) for $d = 3$?

Remark 4.8. This article introduces a novel geometric condition intended to serve as an alternative to Synge's condition specifically for three-dimensional cases.

5 Settings for New Interpolation Theory

5.1 Reference Elements

We first define the reference elements $\widehat{T} \subset \mathbb{R}^d$.

Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be a reference triangle with vertices $\hat{p}_1 := (0,0)^\top$, $\hat{p}_2 := (1,0)^\top$, and $\hat{p}_3 := (0,1)^\top$.

Three-dimensional case

In the three-dimensional case, we consider the following two cases: (i) and (ii); see Condition 5.2.

Let $\widehat{T}_1$ and $\widehat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\widehat{T}_1$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (0,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$;

(ii) $\widehat{T}_2$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (1,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$.

Therefore, we set $\widehat{T} \in \{\widehat{T}_1, \widehat{T}_2\}$. Note that the case (i) is called *the regular vertex property*, see [1].

5.2 Two-step Affine Mapping

To an affine simplex $T \subset \mathbb{R}^d$, we construct two affine mappings $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$. First, we define the affine mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ as

$$\Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto \tilde{x} := \Phi_{\widetilde{T}}(\hat{x}) := A_{\widetilde{T}} \hat{x} \in \widetilde{T}, \tag{5.1}$$

where $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ is an invertible matrix. We then define the affine mapping $\Phi_T : \widetilde{T} \to T$ as follows:

$$\Phi_T : \widetilde{T} \ni \tilde{x} \mapsto x := \Phi_T(\tilde{x}) := A_T \tilde{x} + b_T \in T, \tag{5.2}$$

where $b_T \in \mathbb{R}^d$ is a vector and $A_T \in O(d)$ denotes the rotation and mirror-imaging matrix. We define the affine mapping $\Phi : \widehat{T} \to T$ as

$$\Phi := \Phi_T \circ \Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto x := \Phi(\hat{x}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\hat{x}) = A\hat{x} + b_T \in T,$$

where $A := A_T A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$.

Construct mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$

We consider the affine mapping (5.1). We define the matrix $A_{\widetilde{T}} \in \mathbb{R}^{d \times d}$ as follows. We first define the diagonal matrix as

$$\widehat{A} := \operatorname{diag}(h_1, \dots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i, \tag{5.3}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

For $d = 2$, we define the regular matrix $\widetilde{A} \in \mathbb{R}^{2 \times 2}$ as

$$\widetilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \tag{5.4}$$

with the parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For the reference element $\widehat{T}$, let $\mathfrak{T}^{(2)}$ be a family of triangles.

$$\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}) = A_{\widetilde{T}}(\widehat{T}), \quad A_{\widetilde{T}} := \widetilde{A}\widehat{A}$$

with the vertices $\tilde{p}_1 := (0,0)^\top$, $\tilde{p}_2 := (h_1, 0)^\top$ and $\tilde{p}_3 := (h_2 s, h_2 t)^\top$. Then, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$ and $h_2 = |\tilde{p}_1 - \tilde{p}_3| > 0$.

For $d = 3$, we define the regular matrices $\widetilde{A}_1, \widetilde{A}_2 \in \mathbb{R}^{3\times 3}$ as follows:

$$\widetilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{A}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \tag{5.5}$$

with the parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ h_2 s_1 \le h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0, \ h_3 s_{21} \le h_1/2. \end{cases}$$

Therefore, we set $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$. For the reference elements $\widehat{T}_i$, $i = 1, 2$, let $\mathfrak{T}_i^{(3)}$, $i = 1, 2$, be a family of tetrahedra.

$$\widetilde{T}_i = \Phi_{\widetilde{T}_i}(\widehat{T}_i) = A_{\widetilde{T}_i}(\widehat{T}_i), \quad A_{\widetilde{T}_i} := \widetilde{A}_i\widehat{A}, \quad i = 1, 2,$$

with the vertices

$$\begin{aligned} &\tilde{p}_1 := (0,0,0)^\top, \ \tilde{p}_2 := (h_1,0,0)^\top, \ \tilde{p}_4 := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^\top, \\ &\begin{cases} \tilde{p}_3 := (h_2 s_1, h_2 t_1, 0)^\top & \text{for case (i)}, \\ \tilde{p}_3 := (h_1 - h_2 s_1, h_2 t_1, 0)^\top & \text{for case (ii)}. \end{cases} \end{aligned}$$

Subsequently, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$, $h_3 = |\tilde{p}_1 - \tilde{p}_4| > 0$, and

$$h_2 = \begin{cases} |\tilde{p}_1 - \tilde{p}_3| > 0 & \text{for case (i)}, \\ |\tilde{p}_2 - \tilde{p}_3| > 0 & \text{for case (ii)}. \end{cases}$$

Construct mapping $\Phi_T : \widetilde{T} \to T$

We determine the affine mapping (5.2) as follows. Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \dots, d+1)$. Let $b_T \in \mathbb{R}^d$ be the vector and $A_T \in O(d)$ be the rotation and mirror imaging matrix such that

$$p_i = \Phi_T(\tilde{p}_i) = A_T \tilde{p}_i + b_T, \quad i \in \{1, \dots, d+1\},$$

where vertices p_i $(i = 1, \dots, d+1)$ satisfy the following conditions:

Condition 5.1 (Case in which $d = 2$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \dots, 3)$. We assume that $\overline{p_2 p_3}$ is the longest edge of T, that is, $h_T := |p_2 - p_3|$. We set $h_1 = |p_1 - p_2|$ and $h_2 = |p_1 - p_3|$. We then assume that $h_2 \le h_1$. Because $\frac{1}{2} h_T < h_1 \le h_T$, $h_1 \approx h_T$.

Condition 5.2 (Case in which $d = 3$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \ldots, 4)$. Let L_i $(1 \leq i \leq 6)$ be the edges of T. We denote by $L_{\min}$ the edge of T with the minimum length; that is, $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$. We set $h_2 := |L_{\min}|$ and assume that

$$\text{the endpoints of } L_{\min} \text{ are either } \{p_1, p_3\} \text{ or } \{p_2, p_3\}.$$

Among the four edges sharing an endpoint with $L_{\min}$, we consider the longest edge $L_{\max}^{(\min)}$. Let p_1 and p_2 be the endpoints of edge $L_{\max}^{(\min)}$. Thus, we have

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting $\mathbb{R}^3$ with a plane that contains the midpoint of the edge $L_{\max}^{(\min)}$ and is perpendicular to the vector $p_1 - p_2$. Thus, there are two cases.

(Type i) p_3 and p_4 belong to the same half-space;

(Type ii) p_3 and p_4 belong to different half-spaces.

In each case, we set

(Type i) p_1 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_1 - p_3|$;

(Type ii) p_2 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_2 - p_3|$.

Finally, we set $h_3 = |p_1 - p_4|$. We implicitly assume that p_1 and p_4 belong to the same half-space. Additionally, note that $h_1 \approx h_T$.

Note 5.3. As an example, we define the matrices A_T as

$$A_T := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad A_T := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ denotes the angle.

Note 5.4. None of the lengths of the edges of a simplex or the measures of the simplex are changed by the transformation, i.e.,

$$h_i \leq h_T, \quad i = 1, \ldots, d. \tag{5.6}$$

5.3 Additional Notations and Assumptions

For convenience, we introduce the following additional notation. We define a parameter $\widetilde{\mathscr{H}_i}$, $i = 1, \ldots, d$, as

$$\begin{cases} \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t \quad \text{if } d = 2, \\ \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t_1, \quad \widetilde{\mathscr{H}_3} := h_3 t_2 \quad \text{if } d = 3, \end{cases}$$

Assumption 5.5. In an anisotropic interpolation error analysis, we impose a geometric condition for the simplex $\widetilde{T}$:

1. If $d = 2$, there are no additional conditions;
2. If $d = 3$, there exists a positive constant M independent of $h_{\widetilde{T}}$ such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$. Note that if $s_{22} \neq 0$, this condition means that the order concerning h_T of h_3 coincides with the order of h_2, and if $s_{22} = 0$, the order of h_3 may be different from that of h_2.

We define the vectors $r_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows: If $d = 2$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

see Fig. 1, and if $d = 3$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \dfrac{p_3 - p_1}{|p_3 - p_1|}, & \text{for case (i),} \\ r_2 := \dfrac{p_3 - p_2}{|p_3 - p_2|} & \text{for case (ii),} \end{cases}$$

see Fig 2 for (Type i) and Fig 3 for (Type ii). Furthermore, we define the vectors $\tilde{r}_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows. If $d = 2$,

$$\tilde{r}_1 := (1, 0)^\top, \quad \tilde{r}_2 := (s, t)^\top,$$

and if $d = 3$,

$$\tilde{r}_1 := (1, 0, 0)^\top, \quad \tilde{r}_3 := (s_{21}, s_{22}, t_2)^\top, \quad \begin{cases} \tilde{r}_2 := (s_1, t_1, 0)^\top & \text{for case (i),} \\ \tilde{r}_2 := (-s_1, t_1, 0)^\top & \text{for case (ii).} \end{cases}$$

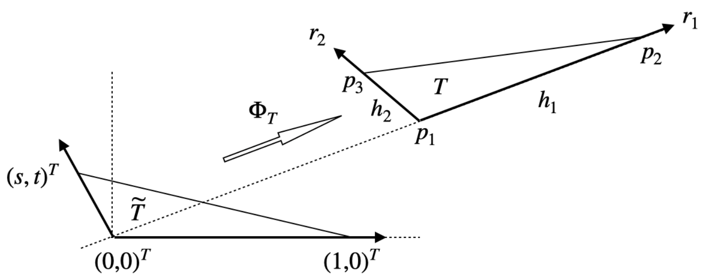

Fig. 1: Affine mapping Φ_T and vectors r_i, $i = 1, 2$

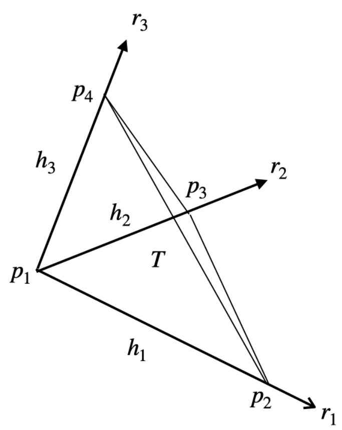

Fig. 2: (Type i) Vectors r_i, $i = 1, 2, 3$

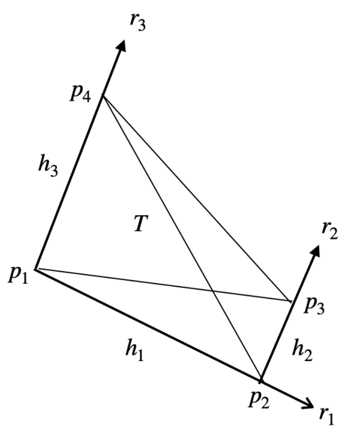

Fig. 3: (Type ii) Vectors r_i, $i = 1, 2, 3$

Remark 5.6. The vectors $\tilde{r}_i$, $i \in \{1, \dots, d\}$ are unit vectors. Indeed, if $d = 2$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s^2 + t^2} = 1,$$

if $d = 3$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s_1^2 + t_1^2} = 1, \quad |\tilde{r}_3|_E = \sqrt{s_{21}^2 + s_{22}^2 + t_2^2} = 1.$$

For a sufficiently smooth function φ and vector function $v := (v_1, \dots, v_d)^\top$, we define the directional derivative of $i \in \{1, \dots, d\}$ as:

$$\begin{aligned} \frac{\partial \varphi}{\partial r_i} &:= (r_i \cdot \nabla_x)\varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}}, \\ \frac{\partial v}{\partial r_i} &:= \left(\frac{\partial v_1}{\partial r_i}, \dots, \frac{\partial v_d}{\partial r_i} \right)^\top = ((r_i \cdot \nabla_x)v_1, \dots, (r_i \cdot \nabla_x)v_d)^\top. \end{aligned}$$

For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we use the following notation.

$$\partial^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial r_1^{\beta_1} \dots \partial r_d^{\beta_d}}.$$

Note that $\partial^\beta \varphi \neq \partial_r^\beta \varphi$.

6 New Semi-regularity Condition

6.1 New Geometric Parameter and Condition

We proposed a new geometric parameter H_T in [27].

Definition 6.1. Parameter H_T is defined as follows:

$$H_T := \frac{\prod_{i=1}^d h_i}{|T|_d} h_T.$$

We introduce geometric conditions to obtain the optimal convergence rate of the anisotropic error estimates.

Assumption 6.2. A family of meshes $\{\mathbb{T}_h\}$ is semi-regular if there exists $\gamma_0 > 0$ such that

$$\frac{H_T}{h_T} \leq \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{6.1}$$

Remark 6.3. We consider the good elements on the meshes in [29]. On anisotropic meshes, good elements may satisfy the following conditions:

$(d = 2)$ $h_2 \approx h_2 t$;

$(d = 3)$ $h_2 \approx h_2 t_1$ and $h_3 \approx h_3 t_2$.

Remark 6.4. The geometric condition in (6.1) is equivalent to the maximum angle condition ([29, Theorem 1]).

6.2 Properties of the New Geometric Parameter

We first show the relation between h_T and H_T.

Lemma 6.5. It holds that

$$\begin{cases} h_T \leq \frac{1}{2} H_T & \text{if } d = 2, \\ h_T < \frac{1}{6} H_T & \text{if } d = 3. \end{cases} \tag{6.1}$$

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. By constructing the standard element in the two-dimensional case, the angle $\theta_{\max} := \angle p_2 p_1 p_3$ is the maximum angle of T. We then have $\frac{\pi}{3} < \theta_{\max} < \pi$, that is, $0 < \sin \theta_{\max} \leq 1$. Therefore, it holds that

$$H_T = \frac{h_1 h_2}{|T|_2} h_T = \frac{2}{\sin \theta_{\max}} h_T \geq 2 h_T.$$

We here used the fact that $|T|_2 = \frac{1}{2} h_1 h_2 \sin \theta_{\max}$.

Three-dimensional case. We denote by ϕ_T the angle between the base $\triangle p_1 p_2 p_3$ of T and the segment $\overline{p_1 p_4}$. Recall that there are two types of standard elements, (Type i) or (Type ii). We denote by θ_T

(Type i) the angle between the segments $\overline{p_1 p_2}$ and $\overline{p_1 p_3}$, that is, $\theta_T := \angle p_2 p_1 p_3$, or

(Type ii) the angle between the segments $\overline{p_2 p_1}$ and $\overline{p_2 p_3}$, that is, $\theta_T := \angle p_1 p_2 p_3$.

We set $t_1 := \sin \theta_T$ and $t_2 := \sin \phi_T$. By constructing the standard element in the three-dimensional case, the angle $\angle p_1 p_3 p_2$ is the maximum angle of the base $\triangle p_1 p_2 p_3$ of T. Therefore, we have $0 < \theta_T < \frac{\pi}{2}$. Because $0 < \phi_T < \pi$, it holds that

$$H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T = \frac{6}{\sin \theta_T \sin \phi_T} h_T > 6 h_T.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_T \sin \phi_T$. $\square$

We introduce another geometric parameter regarding Definition 6.1.

Definition 6.6 (Another parameter H_T^*)**.** For $T \in \mathbb{T}_h$, we denote by L_i edges of the simplex T. We define the new parameter H_T^* as

$$H_T^* := \frac{h_T^2}{|T|_2} \min_{1 \leq i \leq 3} |L_i| \quad \text{if } d = 2, \quad H_T^* := \frac{h_T^2}{|T|_3} \min_{1 \leq i, j \leq 6, i \neq j} |L_i||L_j| \quad \text{if } d = 3. \tag{6.2}$$

The parameters H_T^* and H_T are equivalent.

Lemma 6.7. It holds that

$$\frac{1}{2} H_T^* < H_T < 2 H_T^*. \tag{6.3}$$

Furthermore, H_T^* is equivalent to the circumradius R_2 of T in the two-dimensional case.

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Let L_i $(i = 1, 2, 3)$ denote edges of the triangle T with $|L_1| \leq |L_2| \leq |L_3|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_3| = h_T$. Because $h_2 \leq h_1 < 2 h_T$ and $h_T < h_1 + h_2 \leq 2 h_1$ for the triangle $\triangle p_1 p_2 p_3$, it holds that

$$\frac{1}{2} h_T < h_1 = |L_2| < 2 h_T = 2 h_T.$$

We thus have

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1|}{|T|_2}h_T^2 < H_T = \frac{h_1 h_2}{|T|_2}h_T < 2\frac{|L_1|}{|T|_2}h_T^2 = 2H_T^*.$$

Furthermore, it holds that

$$2R_2 = 2\frac{|L_1||L_2||L_3|}{4|T|_2} < H_T^* = \frac{|L_1|}{|T|_2}h_T^2 < 8\frac{|L_1||L_2||L_3|}{4|T|_2} = 8R_2.$$

Three-dimensional case. Let L_i $(i = 1, \dots, 6)$ denote edges of the triangle T with $|L_1| \le |L_2| \le \cdots \le |L_6|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_6|$. Recall that there are two types of standard elements, (Type i) or (Type ii).

(Type i) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$ and $h_6 := |p_2 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \le \min\{h_3, h_4, h_6\} \le \max\{h_3, h_4, h_6\} \le h_1. \tag{6.4}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1 p_2 p_4$, it holds that

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T & \text{or} \\ h_3 \le h_1 \le h_5 = h_T. \end{cases}$$

We thus have

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T & \text{or} \\ h_3 \le h_1 \le h_T < 2h_1, & \frac{1}{2}h_T < h_1 \le h_T. \end{cases}$$

Because $h_3 \le h_5$, the length of the edge L_2 is equal to the one of h_3, h_4 or h_6.

Assume that $|L_2| = h_3$. We then have

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3}h_T \le \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T^*(< 2H_T^*).$$

Assume that $|L_2| = h_4$. We consider the triangle $\triangle p_1 p_3 p_4$. From the assumption, we have $h_2 \le h_4 \le h_3$ and $\frac{1}{2}h_3 < h_4 \le h_3$. We then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3}h_{T^s} < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

Assume that $|L_2| = h_6$. We consider the triangle $\triangle p_1 p_2 p_3$. Because p_1 and p_3 belong to the same half-space for the triangle $\triangle p_1 p_2 p_3$, it holds that $h_2 \le h_6 \le h_1$ and $\frac{1}{2}h_1 < h_6 \le h_1$. From (6.4), we have

$$\frac{1}{2}h_3 \le \frac{1}{2}h_1 < h_6 \le h_1.$$

Because $h_6 \le h_3$, we then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3}h_T < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

(Type ii) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$, and $h_6 := |p_1 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \leq \min\{h_4, h_5, h_6\} \leq \max\{h_4, h_5, h_6\} \leq h_1. \tag{6.5}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1 p_2 p_4$ and (6.5), it holds that

$$h_3 \leq h_5 \leq h_1.$$

This implies that $h_1 = h_T$. Therefore, the length of the edge L_2 is equal to the one of h_3, h_4, or h_6.

Assume that $|L_2| = h_3$. We then have

$$\begin{aligned}\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 = H_T &= \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &= \frac{|L_1||L_2|}{|T|_3} h_T^2 = H_T^* (< 2H_T).\end{aligned}$$

Assume that $|L_2| = h_4$. For the triangle $\triangle p_2 p_3 p_4$, we have

$$h_2 \leq h_4 \leq h_5 < 2h_4.$$

Because $h_3 \leq h_5$ and $h_4 \leq h_3$, it holds that

$$\begin{aligned}\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T &= \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &< 2\frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.\end{aligned}$$

Assume that $|L_2| = h_6$. We have $h_1 < h_2 + h_6 < 2h_6$ for the triangle $\triangle p_1 p_2 p_3$. Therefore, since $h_6 \leq h_3 \leq h_1$, we obtain

$$\begin{aligned}\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T &= \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &< 2\frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.\end{aligned}$$

$\square$

6.3 Euclidean Condition Number

Examining the Euclidean condition number is useful for deriving appropriate interpolation error estimates.

Lemma 6.8. It holds that

$$\|\widehat{A}\|_2 \leq h_T, \quad \|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}, \tag{6.1a}$$

$$\|\widetilde{A}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3, \end{cases} \qquad \|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 \leq \begin{cases} \frac{h_1 h_2}{|T|_2} = \frac{H_T}{h_T} & \text{if } d = 2, \\ \frac{2}{3}\frac{h_1 h_2 h_3}{|T|_3} = \frac{2}{3}\frac{H_T}{h_T} & \text{if } d = 3, \end{cases} \tag{6.1b}$$

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1. \tag{6.1c}$$

where a parameter H_T is defined in Definition 6.1. Furthermore, we have

$$|\det(A_{\widetilde{T}})| = |\det(\widetilde{A})||\det(\widehat{A})| = \frac{|T|_d}{|\widetilde{T}|_d}\frac{|\widetilde{T}|_d}{|\widehat{T}|_d} = d!|T|_d, \quad |\det(A_T)| = 1. \tag{6.2}$$

Proof. We first show the equality (6.2). Because

$$\int_T dx = \int_{\widetilde{T}} |\det(A_T)| d\tilde{x}, \quad \int_{\widetilde{T}} d\tilde{x} = \int_{\widehat{T}} |\det(A_{\widetilde{T}})| d\hat{x},$$

and $|T|_d = |\widetilde{T}|_d$, we conclude (6.2).

We show the equality (6.1a). From

$$(\widehat{A})^\top \widehat{A} = \mathrm{diag}(h_1^2, \ldots, h_d^2), \quad \widehat{A}^{-1}\widehat{\mathcal{A}}^{-\top} = \mathrm{diag}(h_1^{-2}, \ldots, h_d^{-2}),$$

we have

$$\|\widehat{A}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} = \max\{h_1, \cdots, h_d\} \leq h_T,$$

and

$$\|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} \lambda_{\max}(\widehat{A}^{-1}\widehat{A}^{-\top})^{\frac{1}{2}} = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}},$$

which leads to (6.1a).

We next show the equality (6.1b). We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Because

$$\widetilde{A}^\top \widetilde{A} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad \widetilde{A}^{-1}\widetilde{A}^{-\top} = \frac{1}{t^2}\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}, \quad |s| \leq 1,$$

we have

$$\|\widetilde{A}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \leq (1 + |s|)^{\frac{1}{2}} \leq \sqrt{2},$$

and

$$\|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}^{-1}\widetilde{A}^{-\top})^{\frac{1}{2}} \leq \frac{2}{t} = \frac{h_1 h_2}{|T|_d},$$

which leads to (6.1b) for $d = 2$. Here, we used the fact that $|\widetilde{T}|_d = \frac{1}{2}h_1 h_2 t$ and $|T|_d = |\widetilde{T}|_d$.

Three-dimensional case. The matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ introduced in (5.5) can be decomposed as $\widetilde{A}_1 = \widetilde{M}_0 \widetilde{M}_1$ and $\widetilde{A}_2 = \widetilde{M}_0 \widetilde{M}_2$ with

$$\widetilde{M}_0 := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{M}_1 := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{M}_2 := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\widetilde{M}_2^\top \widetilde{M}_2$ coincide with those of $\widetilde{M}_1^\top \widetilde{M}_1$, and only Case (i) is shown.

We have the inequalities

$$\begin{aligned} \|\widetilde{A}_1\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \leq \lambda_{\max}(\widetilde{M}_0^\top \widetilde{M}_0)^{\frac{1}{2}} \lambda_{\max}(\widetilde{M}_1^\top \widetilde{M}_1)^{\frac{1}{2}} \\ &\leq \left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)^{\frac{1}{2}} (1 + |s_1|)^{\frac{1}{2}} \leq 2, \end{aligned}$$

and

$$\begin{aligned}
\|\widetilde{A}_1\|_2 \|\widetilde{A}_1^{-1}\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}_1^{-1} \widetilde{A}_1^{-\top})^{\frac{1}{2}} \\
&\leq \frac{\left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)(1 + |s_1|)}{t_1 t_2} \leq \frac{4}{t_1 t_2} = \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_d},
\end{aligned}$$

where we used the fact that $|\widetilde{T}|_d = \frac{1}{6} h_1 h_2 h_3 t_1 t_2$ and $|T|_d = |\widetilde{T}|_d$.

Because the length of all edges of a simplex and measure of the simplex is not changed by a rotation and mirror imaging matrix and $A_T, A_T^{-1} \in O(d)$,

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1,$$

which is (6.1c). □

7 New Geometric Mesh Condition equivalent to the Maximum-angle Condition

8 Good Elements or not for $d = 2, 3$?

References

[1] G. Acosta, R.G. Durán: The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations, SIAM J. Numer. Anal. 37 (1999) 18-36. Zbl 0948.65115.

[2] Apel, Th.: Anisotropic finite elements: Local estimates and applications. Advances in Numerical Mathematics. Teubner, Stuttgart, (1999)

[3] Apel, Th, Dobrowolski, M.: Anisotropic interpolation with applications to the finite element method. Computing **47**, 277-293 (1992)

[4] Babuška, I., Aziz, A.K.: On the angle condition in the finite element method. SIAM J. Numer. Anal. **13**, 214-226 (1976)

[5] Barnhill, R.E., Gregory, J.A.: Sard kernel theorems on triangular domains with applications to finite element error bounds. Numer. Math. **25**, 215-229 (1975/1976)

[6] Bernardi, C., Girault, V., Hecht, F., Raviart, P.-A., Rivière, B.: Mathematics and Finite Element Discretizations of Incompressible Navier–Stokes Flows. SIAM, (2024)

[7] Brandts, J., Korotov, S., Křížek, M.: On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions. Comput. Math, Appl. **55**, 2227-2233 (2008)

[8] Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, Third Edition. Springer Verlag, New York (2008)

[9] Chen, S., Shi, D., Zhao, Y.: Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes. IMA Journal of Numerical Analysis, **24**, 77-95 (2004)

[10] Ciarlet, P. G.: The Finite Element Method for Elliptic problems. SIAM, New York (2002)

[11] Dekel, S., Leviatan, D.: The Bramble–Hilbert Lemma for Convex Domains, SIAM Journal on Mathematical Analysis **35** No. 5, 1203-1212 (2004)

[12] Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. Math. Comp. **34**, 441-463 (1980)

[13] Ern, A., Guermond, J.L.: Theory and Practice of Finite Elements. Springer Verlag, New York (2004)

[14] Ern, A., Guermond, J. L.: Finite Elements I: Approximation and Interpolation. Springer-Verlag, New York (2021)

[15] Ern, A., Guermond, J. L.: Finite elements II: Galerkin Approximation, Elliptic and Mixed PDEs. Springer-Verlag, New York (2021)

[16] Ern, A. and Guermond, J. L.: Finite elements III: First-Order and Time-Dependent PDEs. Springer-Verlag, New York (2021).

[17] Golub, G. H., Loan, C. F. V.: Matrix Computations 3rd edition. The Johas Hopkins University Press, (1996)

[18] Gregory, J.A.: Error bounds for linear interpolation on triangles: in: J.R. Whiteman (Ed.), Proc. MAFELAP II, Academic Press, London, (1976)

[19] Ishizaka H. Anisotropic interpolation error analysis using a new geometric parameter and its applications. Ehime University, Ph. D. thesis (2022)

[20] Ishizaka, H.: Anisotropic Raviart–Thomas interpolation error estimates using a new geometric parameter. Calcolo 59 (4), (2022)

[21] Ishizaka, H.: Anisotropic weakly over-penalised symmetric interior penalty method for the Stokes equation. Journal of Scientific Computing 100, (2024)

[22] Ishizaka, H.: Morley finite element analysis for fourth-order elliptic equations under a semi-regular mesh condition. Applications of Mathematics **69** (6), 769–805 (2024)

[23] Ishizaka, H.: Hybrid weakly over-penalised symmetric interior penalty method on anisotropic meshes. Calcolo **61**, (2024)

[24] Ishizaka, H.: Anisotropic modified Crouzeix-Raviart finite element method for the stationary Navier-Stokes equation. Numeriche Mathmatik, (2025)

[25] Ishizaka, H.: Nitsche method under a semi-regular mesh condition. preprint (2025) `https://arxiv.org/abs/2501.06824`

[26] Ishizaka, H., Kobayashi, K., Suzuki, R., Tsuchiya, T.: A new geometric condition equivalent to the maximum angle condition for tetrahedrons. Comput. Math. Appl. 99, 323–328 (2021)

[27] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: General theory of interpolation error estimates on anisotropic meshes. Jpn. J. Ind. Appl. Math. 38 (1), 163–191 (2021)

[28] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: Crouzeix–Raviart and Raviart–Thomas finite element error analysis on anisotropic meshes violating the maximum-angle condition. Jpn. J. Ind. Appl. Math. 38 (2), 645-675 (2021)

[29] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: Anisotropic interpolation error estimates using a new geometric parameter. Jpn. J. Ind. Appl. Math. 40 (1), 475–512 (2023)

[30] Jamet, P.: Estimation de I'erreur pour des éléments finis droits presque dégénérés. RAIRO Anal. Numér **10**, 43-60 (1976)

[31] Křížek, M.: On semiregular families of triangulations and linear interpolation. Appl. Math. Praha **36**, 223-232 (1991)

[32] Křížek, M.: On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. **29**, 513-520 (1992)

[33] Ladyženskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs **23**, AMS, Providence, (1968)

[34] Mario, B.: A note on the Poincaré inequality for convex domains. Z. Anal. ihre. Anwend., **22**, 751-756 (2003)

[35] Payne, L.E., Weinberger, H.F.: An optimal Poincaré-inequality for convex domains, Arch. Rational Mech. Anal. **5**, 286-292 (1960)

[36] Synge, J.L.: The Hypercircle in Mathematical Physics. Cambridge Univ. Press, Cambridge (1957)

[37] Verfürth, R.: A note on polynomial approximation in Sobolev spaces, Math. Modelling and Numer. Anal. **33**, 715-719 (1999)

[38] Zlámak, M.: On the finite element method, Numer. Math. **12**, 394-409 (1968)