Reconsidered error analysis in the finite element methods

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Abstract

This article presents novel proof methods for estimating interpolation errors, predicated on the understanding that one has already studied foundational error analysis using the finite element method.

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Part I

Interpolation Error Analysis using a New Geometric Parameter

1 Preliminalies

1.1 General Convention

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants c are bounded if the maximum angle is bounded. These values vary across different contexts.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert \cdot \rvert_d$	d-dimensional Hausdorff measure
$v\lvert_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x \rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x \rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
A, B	Matrices
A_{ij} or $[A]_{ij}$	Entry of A in the ith and the jth column
$A^\top$	Transpose of the matrix A
$\mathrm{Tr}(A)$	Trace of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A_{ii}$
$\det(A)$	Determinant of A
$\mathrm{diag}(A)$	Diagonal of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{diag}(A)_{ij} := \delta_{ij} A_{ij}$, $1 \le i, j \le d$
Ax	Matrix-vector product: For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, $(Ax)_i := \sum_{j=1}^{d} A_{ij} x_j$ for $1 \le i \le d$
$A : B$	Double contraction:

	For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, $A : B := \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$
$\|A\|_2$	Operator norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_2 := \sup_{0\neq x\in\mathbb{R}^d} \frac{\|Ax\|_E}{\|x\|_E}$
$\|A\|_{\max}$	Max norm of A: For $A \in \mathbb{R}^{d\times d}$, $\|A\|_{\max} := \max_{1\leq i,j\leq d} \|A_{ij}\|$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this article, we use the following facts.

For $A \in \mathbb{R}^{m\times n}$, it holds that

$$\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn}\|A\|_{\max}, \tag{1.1}$$

e.g., see [21, p. 56]. For $A, B \in \mathbb{R}^{m\times m}$, it holds that

$$\|AB\|_2 \leq \|A\|_2\|B\|_2. \tag{1.2}$$

If $A^\top A$ is a positive definite matrix in $\mathbb{R}^{d\times d}$, the spectral norm of the matrix $A^\top A$ is the largest eigenvalue of $A^\top A$; i.e.,

$$\|A\|_2 = \big(\lambda_{\max}(A^\top A)\big)^{1/2} = \sigma_{\max}(A), \tag{1.3}$$

where $\lambda_{\max}(A)$ and $\sigma_{\max}(A)$ are respectively the largest eigenvalues and singular values of A.

If $A \in O(d)$, because $A^\top = A^{-1}$ and

$$|Ax|_E^2 = (Ax)^\top(Ax) = x^\top A^\top Ax = x^\top A^{-1}Ax = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0\neq x\in\mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0\neq x\in\mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

1.4 Function Spaces

This article uses standard Sobolev spaces with associated norms (e.g., see [8, 16, 17]).

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \ldots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T)$, $\mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$) is spanned by the restriction to T of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$)

1.5.2 Meshes

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \mathrm{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $p_1, \ldots, p_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex p_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^\partial$ be the set of faces on boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1,\infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h;\mathbb{R}^q) := \{v \in L^p(\Omega;\mathbb{R}^q) : \ v|_T \in W^{s,p}(T;\mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} := \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \quad \text{if } p \in [1,\infty),$$
$$\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} := \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h;\mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h;\mathbb{R}^q) := W^{s,2}(\mathbb{T}_h;\mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h;\mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla\varphi\|^2_{L^2(T)^d} \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.$$

For a boundary face $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$[\![v \cdot n]\!] := [\![v \cdot n]\!]_F := v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp,$$
$$[\![v]\!] := [\![v]\!]_F := v_\natural - v_\sharp, \quad \natural > \sharp,$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_{\omega} := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F}v_\natural + \omega_{T_\sharp,F}v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi) \cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1, \infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div}; T)$ space by

$$H(\mathrm{div}; \mathbb{T}_h) := \left\{ v \in L^2(\Omega)^d;\ v|_T \in H(\mathrm{div}; T)\ \forall T \in \mathbb{T}_h \right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div}; \mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div}; \mathbb{T}_h)$,

$$(\mathrm{div}_h\, v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.5.4 Barycentric Coodinates

For a simplex $T \subset \mathbb{R}^d$, let $\{p_i\}_{i=1}^{d+1}$ be vertices of T and $(x_1^{(i)}, \ldots, x_d^{(i)})^T$ coordinates of p_i. We set

$$\Delta := \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ of the point $p(x_1, \ldots, x_d)$ with respect to $\{p_i\}_{i=1}^{d+1}$ are then defined as

$$\lambda_i(x) := \frac{1}{\Delta} \det \begin{pmatrix} 1 & \cdots & \overset{\overset{i}{\smile}}{1} & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(p_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

Let r, s be two nonnegative real numbers and $\{x_i\}_{i\in I}$ be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r \leq s, \\ \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \leq \operatorname{card}(I)^{\frac{r-s}{rs}} \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \tag{1.4}$$

see [17, Exercise 12.1].

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.1. Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) & \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.5}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.6}$$

Proof. See, for example, [16, Corollary B.43, Theorem B.40] and [17, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.2. Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.7}$$

Proof. See, for example, [38, Theorem 2.1 (p. 61)] and the references therein. □

1.6.3 Trace Theorem

Theorem 1.3 (Trace on low-dimensional manifolds)**.** Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0, \ldots, d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.

Proof. See [17, Theorem 3.15]. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [15, 10]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [2, Lemma 2.1].

Lemma 1.4. Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped concerning balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le \ell$, $p \in [1, \infty]$. It then holds that

$$\|\partial^\gamma (\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \le C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.8}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \le \ell-1} \int_B \eta(y)(\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \tag{1.9}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [14, Theorem 1.1] and [15, 10, 43] which are variants of the Bramble–Hilbert lemma.

Lemma 1.5. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.10}$$

Proof. The proof is found in [14, Theorem 1.1]. □

Remark 1.6. In [10, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m,\gamma) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.11}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\operatorname{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d,m,\gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d,m)$ of the estimate (1.10) does not depend on the geometric parameter γ.

Remark 1.7. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d,m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf*ü*rth [43].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \le C^{BH}(d,k,m) \operatorname{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0, 1, \dots, m-1. \tag{1.12}$$

Verf*ü*rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d,k,m) \le \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3, 1, 2) \leq \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \leq \frac{\sqrt{6}}{\pi}|\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

because $\mathrm{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.8 (Poincaré inequality)**.** Let $D \subset \mathbb{R}^d$ be a convex domain with diameter $\mathrm{diam}(D)$. It then holds that, for $\varphi \in H^1(D)$ with $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \leq \frac{\mathrm{diam}(D)}{\pi}|\varphi|_{H^1(D)}. \tag{1.13}$$

Proof. The proof is found in [39, Theorem 3.2], also see [40]. □

Remark 1.9. The coefficient $\frac{1}{\pi}$ of (1.13) may be improved.

1.7 Abbreviated expression

FE	Finite Element
FEMs	Finite Element Methods

2 Isotropic and Anisotropic Mesh Elements

In the context of FEMs, mesh elements can be classified based on their geometric properties. *An isotropic mesh element* has equal or nearly equal edge lengths and angles, resulting in a balanced shape. In contrast, *an anisotropic mesh element* features significant variation in edge lengths and angles.

Consider the following examples: Let $s, \delta \in \mathbb{R}_+$, and $\varepsilon \geq 1$, $\varepsilon \in \mathbb{R}$.

Example 2.1. In the case of the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, \delta s)^\top$, the triangle is classified as follows:

- If $\delta \approx 1$, the triangle T is considered an isotropic mesh element.
- Conversely, if δ is much less than 1, i.e., $\delta \ll 1$, the triangle T becomes an anisotropic mesh element.

Example 2.2. In this case, consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0, 0)^\top$, $p_2 := (2s, 0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Here, the vertex p_3 introduces a parameter ε that can influence the shape of the simplex. The classification of this simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle maintains a balanced shape, making it isotropic.

- If $\varepsilon > 1$, the triangle becomes flat when $s \ll 1$, resulting in an anisotropic mesh element.

Example 2.3. Consider the simplex $T \subset \mathbb{R}^2$ defined by the vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. In this configuration, the classification of the simplex as isotropic or anisotropic depends on the value of δ:

- If $\delta \approx 1$, the triangle is an isotropic mesh element.
- If $\delta \ll 1$, the triangle becomes an anisotropic mesh element.

Example 2.4. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. In this case, the classification of the simplex as isotropic or anisotropic depends on the value of ε:

- If $\varepsilon = 1$, the triangle is isotropic because the height from p_3 is equal to the base length.
- If $\varepsilon > 1$, the triangle will be classified as anisotropic, as the edge lengths will differ significantly when $s \ll 1$.

Example 2.5. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$. The classification of the simplex into two types of anisotropic structures is determined by the values of δ and ε:

- If $1 < \varepsilon < \delta$, the triangle is flattened so that the point p_3 approaches the point p_1, i.e. the origin as $s \to 0$.
- If $1 < \delta < \varepsilon$, the triangle is flattened so that point p_3 approaches a point on the straight line $\overline{p_1 p_2}$ that does not include points p_1 and p_2 as $s \to 0$.

3 Classical Geometric Conditions

3.1 Classical Interpolation Error Estimate

Let $\widehat{T} \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^d$ be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ and $\{T, P, \Sigma\}$ with associated normed vector spaces $V(\widehat{T})$ and $V(T)$. The transformation Φ_T takes the form

$$\Phi_T : \widehat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := B_T \hat{x} + b_T \in T,$$

where $B_T \in \mathbb{R}^{d \times d}$ is an invertible matrix and $b_T \in \mathbb{R}^d$. Let $I_T : V(T) := W^{2,p}(T) \to P := \mathbb{P}^1(T)$ with $p \in [1, \infty]$ be an interpolation on T with $I_T p = p$ for any $p \in \mathcal{P}^1(T)$. According to the classical theory (e.g., see [13, 16]), there exists a positive constant c, independent of h_T, such that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \left(\|B_T\|_2 \|B_T^{-1}\|_2 \right) \|B_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity $\|B_T\|_2 \|B_T^{-1}\|_2$ is called the *Euclidean condition number* of B_T. By standard estimates (e.g., see [16, Lemma 1.100]), we have

$$\|B_T\|_2 \|B_T^{-1}\|_2 \leq c \frac{h_T}{\rho_T}, \quad \|B_T\|_2 \leq c h_T.$$

It thus holds that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.1}$$

As a geometric condition, the *shape-regularity condition* is well known to obtain global interpolation error estimates. This condition is stated as follows.

Condition 3.1 (Shape-regularity condition)**.** There exists a constant $\gamma_1 > 0$ such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{3.2}$$

Under Condition 3.1, that is, when the quantity $\frac{h_T}{\rho_T}$ is bounded on each T, it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)},$$

where $I_h \varphi$ is the standard global linear interpolation of φ on $\mathbb{T}_h$.

3.2 Regular Mesh Conditions

Geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity condition (3.2). A proof can be found in [9, Theorem 1].

Condition 3.2 (Zlámal's condition)**.** There exists a constant $\gamma_2 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$, any simplex $T \in \mathbb{T}_h$ and any dihedral angle ψ and for $d = 3$, also any solid angle θ of T, we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \tag{3.3}$$

Condition 3.3. There exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_3 h_T^d. \tag{3.4}$$

Condition 3.4. There exists a constant $\gamma_4 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T|_d \geq \gamma_4 |B_d^T|, \tag{3.5}$$

where $B^T \supset T$ is the circumscribed ball of T.

Note 3.5. If Condition 3.1 or 3.2 or 3.3 or 3.4 holds, a family of simplicial partitions is called *regular*.

Note 3.6. Condition 3.2 was presented by Zlámal [44] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on $\mathbb{R}^2$. Zlámal's condition can be generalised into $\mathbb{R}^n$ for any $n \in \{2, 3, \ldots\}$. Later, the shape-regularity condition (the inscribed ball condition) was introduced; see [13]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

Note 3.7. Condition 3.3 seems to be simpler than Condition 3.1, Condition 3.2 and Condition 3.4. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

3.3 What happens when anisotropic meshes are used?

Using the equivalence conditions in Section 3.2, the error estimate (3.1) is rewritten as

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T^2}{|T|_2} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.6}$$

We considered the following five anisotropic elements as in Section 2: Let $0 < s, \delta \ll 1$, $s, \delta \in \mathbb{R}$, and $\varepsilon > 1$, $\varepsilon \in \mathbb{R}$.

Example 3.8. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, \delta s)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = \delta s^2$, and

$$\frac{h_T^2}{|T|_2} = \frac{4}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \le \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}.$$

When $\delta \ll 1$, the interpolation error (3.6) may be large.

Example 3.9. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$, and $p_3 := (s, s^\varepsilon)^\top$. Then, we have that $h_T = 2s$, $|T|_2 = s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = 4 s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity is not satisfied. In this case, when $\varepsilon > 2$, the estimate (3.6) diverges as $s \to 0$.

Example 3.10. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, \delta s)^\top$. Then, we have that $h_T = s\sqrt{1+\delta^2} \approx s$, $|T|_2 = \frac{1}{2}\delta s^2$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(1+\delta^2)}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \le \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}. \tag{3.7}$$

It is implied that the interpolation error (3.7) may be large when $\delta \ll 1$.

Example 3.11. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, and $p_3 := (0, s^\varepsilon)^\top$. Subsequently, we obtain $h_T = \sqrt{s^2 + s^{2\varepsilon}} \approx s$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s^2 + s^{2\varepsilon})}{s^{1+\varepsilon}} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$.

Example 3.12. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$. If $1 < \varepsilon < \delta$, we have $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \le c s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$. If $1 < \delta < \varepsilon$, we have $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $|T|_2 = \frac{1}{2} s^{1+\varepsilon}$ and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \le c s^{1-\varepsilon} \to \infty \text{ as } s \to 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as $s \to 0$.

Remark 3.13. As will be explained later, the factor $\frac{1}{\delta}$ in Example 3.10 is violated. The interpolation error estimate converges in the cases of Example 3.11 and Example 3.12 with $1 < \varepsilon < \delta$ using new precise interpolation error estimates under more relaxed geometric conditions.

4 Classical Relaxed Geometric Conditions

4.1 Semi-regular Mesh Conditions for $d = 2$

In 1957, Synge [41, Section 3.8] proposed the following condition.

Condition 4.1 (Synge's condition)**.** There exists $\frac{\pi}{3} \leq \gamma_5 < \pi$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_5, \tag{4.1}$$

where $\theta_{T,\max}$ is the maximal angle of T.

Under Condition 4.1, Synge proved an optimal interpolation error estimate as follows.

$$\|\varphi - I_h \varphi\|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)} \quad \text{for } p = \infty.$$

The inequality (4.1) is called *Synge's condition* or the *maximum-angle condition*. In 1976, several author's [5, 7, 22, 35] independently proved the convergence of finite element for $p < \infty$. It ensures that finite elements converge effectively when the minimum angle approaches zero as the mesh size decreases. If this condition is not met, the accuracy of interpolation for linear triangular elements can suffer, similar to the absence of Zlámal's condition, see e.g. [5, p. 223]. This underscores the importance of keeping proper geometric constraints to ensure reliable outcomes in numerical methods. Synge's condition is essential in finite element analysis.

In [36], Křížek proposed the following circumscribed ball condition for $d = 2$ which is equivalent to Synge's condition.

Condition 4.2. There exists $\gamma_6 > 0$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\frac{R_2}{h_T} \leq \gamma_6, \tag{4.2}$$

where R_2 is the radius of the circumscribed ball of $T \subset \mathbb{R}^2$.

Note 4.3. If Condition 4.1 or 4.2 holds, the associated families of partitions are called *semi-regular*.

Remark 4.4. Assume that Condition 3.3 holds, that is, there exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_3 h_T^2.$$

Let $T \subset \mathbb{R}^2$ be the triangle with vertices P_1, P_2 and P_3 such that the maximum angle $\theta_{T,\max}$ of T is $\angle P_2 P_1 P_3$. We then have $h_T = |P_2 P_3|$ and

$$\frac{R_2}{h_T} = \frac{|P_2 P_3|}{2 h_T \sin \theta_{T,\max}} = \frac{|P_1 P_2||P_1 P_3|}{2|P_1 P_2||P_1 P_3| \sin \theta_{T,\max}} \leq c \frac{h_T^2}{|T|} \leq \frac{c}{\gamma_3} =: \gamma_6.$$

This implies that each regular family is semi-regular. However, the converse implication does not hold.

4.2 Semi-regular Mesh Conditions for $d = 3$

Synge's condition (4.1) is extended to the case of tetrahedra in [37].

Condition 4.5. There exists a constant $0 < \gamma_7 < \pi$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_7, \tag{4.3a}$$

$$\psi_{T,\max} \leq \gamma_7, \tag{4.3b}$$

where $\theta_{T,\max}$ is the maximum angle of all triangular faces of the tetrahedron T and $\psi_{T,\max}$ is the maximum dihedral angle of T.

Remark 4.6. The theory of anisotropic interpolation has been advanced through extensive research ([3, 2, 11]).

Question 4.7. Is there a semi-regularity condition which equivalent to Synge's condition (4.3) for $d = 3$?

Remark 4.8. This article introduces a novel geometric condition intended to serve as an alternative to Synge's condition specifically for three-dimensional cases.

5 Settings for New Interpolation Theory

5.1 Reference Elements

We first define the reference elements $\widehat{T} \subset \mathbb{R}^d$.

Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be a reference triangle with vertices $\hat{p}_1 := (0,0)^\top$, $\hat{p}_2 := (1,0)^\top$, and $\hat{p}_3 := (0,1)^\top$.

Three-dimensional case

In the three-dimensional case, we consider the following two cases: (i) and (ii); see Condition 5.2.

Let $\widehat{T}_1$ and $\widehat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\widehat{T}_1$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (0,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$;

(ii) $\widehat{T}_2$ has vertices $\hat{p}_1 := (0,0,0)^\top$, $\hat{p}_2 := (1,0,0)^\top$, $\hat{p}_3 := (1,1,0)^\top$, and $\hat{p}_4 := (0,0,1)^T$.

Therefore, we set $\widehat{T} \in \{\widehat{T}_1, \widehat{T}_2\}$. Note that the case (i) is called *the regular vertex property*, see [1].

5.2 Two-step Affine Mapping

To an affine simplex $T \subset \mathbb{R}^d$, we construct two affine mappings $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$. First, we define the affine mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ as

$$\Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto \tilde{x} := \Phi_{\widetilde{T}}(\hat{x}) := A_{\widetilde{T}} \hat{x} \in \widetilde{T}, \tag{5.1}$$

where $A_{\widetilde{T}} \in \mathbb{R}^{d\times d}$ is an invertible matrix. We then define the affine mapping $\Phi_T : \widetilde{T} \to T$ as follows:

$$\Phi_T : \widetilde{T} \ni \tilde{x} \mapsto x := \Phi_T(\tilde{x}) := A_T\tilde{x} + b_T \in T, \tag{5.2}$$

where $b_T \in \mathbb{R}^d$ is a vector and $A_T \in O(d)$ denotes the rotation and mirror-imaging matrix. We define the affine mapping $\Phi : \widehat{T} \to T$ as

$$\Phi := \Phi_T \circ \Phi_{\widetilde{T}} : \widehat{T} \ni \hat{x} \mapsto x := \Phi(\hat{x}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\hat{x}) = A\hat{x} + b_T \in T,$$

where $A := A_T A_{\widetilde{T}} \in \mathbb{R}^{d\times d}$.

Construct mapping $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$

We consider the affine mapping (5.1). We define the matrix $A_{\widetilde{T}} \in \mathbb{R}^{d\times d}$ as follows. We first define the diagonal matrix as

$$\widehat{A} := \operatorname{diag}(h_1, \dots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i, \tag{5.3}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

For $d = 2$, we define the regular matrix $\widetilde{A} \in \mathbb{R}^{2\times 2}$ as

$$\widetilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \tag{5.4}$$

with the parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For the reference element $\widehat{T}$, let $\mathfrak{T}^{(2)}$ be a family of triangles.

$$\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}) = A_{\widetilde{T}}(\widehat{T}), \quad A_{\widetilde{T}} := \widetilde{A}\widehat{A}$$

with the vertices $\tilde{p}_1 := (0, 0)^\top$, $\tilde{p}_2 := (h_1, 0)^\top$ and $\tilde{p}_3 := (h_2 s, h_2 t)^\top$. Then, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$ and $h_2 = |\tilde{p}_1 - \tilde{p}_3| > 0$.

For $d = 3$, we define the regular matrices $\widetilde{A}_1, \widetilde{A}_2 \in \mathbb{R}^{3\times 3}$ as follows:

$$\widetilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{A}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \tag{5.5}$$

with the parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ h_2 s_1 \le h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0, \ h_3 s_{21} \le h_1/2. \end{cases}$$

Therefore, we set $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$. For the reference elements $\widehat{T}_i$, $i = 1, 2$, let $\mathfrak{T}_i^{(3)}$, $i = 1, 2$, be a family of tetrahedra.

$$\widetilde{T}_i = \Phi_{\widetilde{T}_i}(\widehat{T}_i) = A_{\widetilde{T}_i}(\widehat{T}_i), \quad A_{\widetilde{T}_i} := \widetilde{A}_i\widehat{A}, \quad i = 1, 2,$$

with the vertices

$$\begin{aligned} &\tilde{p}_1 := (0, 0, 0)^\top, \ \tilde{p}_2 := (h_1, 0, 0)^\top, \ \tilde{p}_4 := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^\top, \\ &\begin{cases} \tilde{p}_3 := (h_2 s_1, h_2 t_1, 0)^\top & \text{for case (i)}, \\ \tilde{p}_3 := (h_1 - h_2 s_1, h_2 t_1, 0)^\top & \text{for case (ii)}. \end{cases} \end{aligned}$$

Subsequently, $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$, $h_3 = |\tilde{p}_1 - \tilde{p}_4| > 0$, and

$$h_2 = \begin{cases} |\tilde{p}_1 - \tilde{p}_3| > 0 & \text{for case (i)}, \\ |\tilde{p}_2 - \tilde{p}_3| > 0 & \text{for case (ii)}. \end{cases}$$

Construct mapping $\Phi_T : \widetilde{T} \to T$

We determine the affine mapping (5.2) as follows. Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \ldots, d+1)$. Let $b_T \in \mathbb{R}^d$ be the vector and $A_T \in O(d)$ be the rotation and mirror imaging matrix such that

$$p_i = \Phi_T(\tilde{p}_i) = A_T \tilde{p}_i + b_T, \quad i \in \{1, \ldots, d+1\},$$

where vertices p_i $(i = 1, \ldots, d+1)$ satisfy the following conditions:

Condition 5.1 (Case in which $d = 2$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \ldots, 3)$. We assume that $\overline{p_2 p_3}$ is the longest edge of T, that is, $h_T := |p_2 - p_3|$. We set $h_1 = |p_1 - p_2|$ and $h_2 = |p_1 - p_3|$. We then assume that $h_2 \leq h_1$. Because $\frac{1}{2} h_T < h_1 \leq h_T$, $h_1 \approx h_T$.

Condition 5.2 (Case in which $d = 3$)**.** Let $T \in \mathbb{T}_h$ have vertices p_i $(i = 1, \ldots, 4)$. Let L_i $(1 \leq i \leq 6)$ be the edges of T. We denote by $L_{\min}$ the edge of T with the minimum length; that is, $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$. We set $h_2 := |L_{\min}|$ and assume that

$$\text{the endpoints of } L_{\min} \text{ are either } \{p_1, p_3\} \text{ or } \{p_2, p_3\}.$$

Among the four edges sharing an endpoint with $L_{\min}$, we consider the longest edge $L_{\max}^{(\min)}$. Let p_1 and p_2 be the endpoints of edge $L_{\max}^{(\min)}$. Thus, we have

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting $\mathbb{R}^3$ with a plane that contains the midpoint of the edge $L_{\max}^{(\min)}$ and is perpendicular to the vector $p_1 - p_2$. Thus, there are two cases.

(Type i) p_3 and p_4 belong to the same half-space;

(Type ii) p_3 and p_4 belong to different half-spaces.

In each case, we set

(Type i) p_1 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_1 - p_3|$;

(Type ii) p_2 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_2 - p_3|$.

Finally, we set $h_3 = |p_1 - p_4|$. We implicitly assume that p_1 and p_4 belong to the same half-space. Additionally, note that $h_1 \approx h_T$.

Note 5.3. As an example, we define the matrices A_T as

$$A_T := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad A_T := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ denotes the angle.

Note 5.4. None of the lengths of the edges of a simplex or the measures of the simplex are changed by the transformation, i.e.,

$$h_i \leq h_T, \quad i = 1, \ldots, d. \tag{5.6}$$

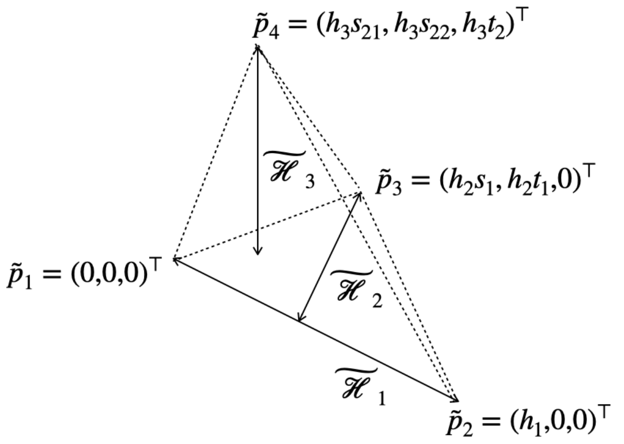


Fig. 1: New parameters $\widetilde{\mathscr{H}_i}$, $i = 1, 2, 3$

5.3 Additional Notations and Assumptions

For convenience, we introduce the following additional notation. We define a parameter $\widetilde{\mathscr{H}_i}$, $i = 1, \ldots, d$, as

$$\begin{cases} \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t \quad \text{if } d = 2, \\ \widetilde{\mathscr{H}_1} := h_1, \quad \widetilde{\mathscr{H}_2} := h_2 t_1, \quad \widetilde{\mathscr{H}_3} := h_3 t_2 \quad \text{if } d = 3, \end{cases}$$

see Fig. 1.

Assumption 5.5. In an anisotropic interpolation error analysis, we impose a geometric condition for the simplex $\widetilde{T}$:

1. If $d = 2$, there are no additional conditions;
2. If $d = 3$, there exists a positive constant M independent of $h_{\widetilde{T}}$ such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$. Note that if $s_{22} \neq 0$, this condition means that the order concerning h_T of h_3 coincides with the order of h_2, and if $s_{22} = 0$, the order of h_3 may be different from that of h_2.

We define the vectors $r_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows: If $d = 2$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

see Fig. 2, and if $d = 3$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \dfrac{p_3 - p_1}{|p_3 - p_1|}, \quad \text{for case (i)}, \\ r_2 := \dfrac{p_3 - p_2}{|p_3 - p_2|} \quad \text{for case (ii)}, \end{cases}$$

see Fig 3 for (Type i) and Fig 4 for (Type ii). Furthermore, we define the vectors $\tilde{r}_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows. If $d = 2$,

$$\tilde{r}_1 := (1, 0)^\top, \quad \tilde{r}_2 := (s, t)^\top,$$

and if $d = 3$,

$$\tilde{r}_1 := (1, 0, 0)^\top, \quad \tilde{r}_3 := (s_{21}, s_{22}, t_2)^\top, \quad \begin{cases} \tilde{r}_2 := (s_1, t_1, 0)^\top \quad \text{for case (i)}, \\ \tilde{r}_2 := (-s_1, t_1, 0)^\top \quad \text{for case (ii)}. \end{cases}$$

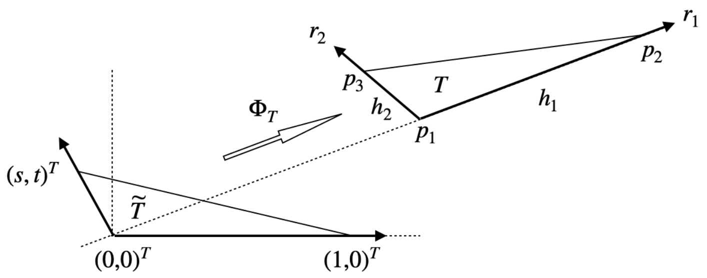
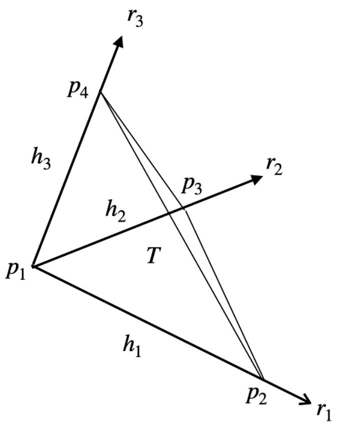


Fig. 2: Affine mapping Φ_T and vectors r_i, $i = 1, 2$



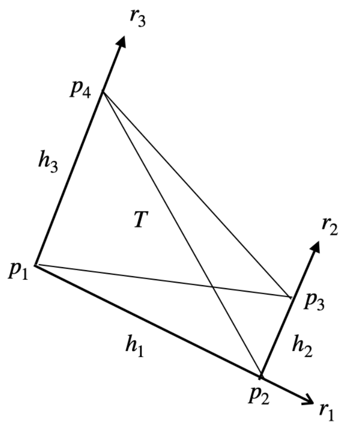


Fig. 3: (Type i) Vectors r_i, $i = 1, 2, 3$ Fig. 4: (Type ii) Vectors r_i, $i = 1, 2, 3$

Remark 5.6. The vectors $\tilde{r}_i$, $i \in \{1, \dots, d\}$ are unit vectors. Indeed, if $d = 2$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s^2 + t^2} = 1,$$

if $d = 3$,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s_1^2 + t_1^2} = 1, \quad |\tilde{r}_3|_E = \sqrt{s_{21}^2 + s_{22}^2 + t_2^2} = 1.$$

For a sufficiently smooth function φ and vector function $v := (v_1, \dots, v_d)^\top$, we define the directional derivative of $i \in \{1, \dots, d\}$ as:

$$\frac{\partial \varphi}{\partial r_i} := (r_i \cdot \nabla_x)\varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}},$$
$$\frac{\partial v}{\partial r_i} := \left(\frac{\partial v_1}{\partial r_i}, \dots, \frac{\partial v_d}{\partial r_i} \right)^\top = ((r_i \cdot \nabla_x) v_1, \dots, (r_i \cdot \nabla_x) v_d)^\top.$$

For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we use the following notation.

$$\partial^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial r_1^{\beta_1} \dots \partial r_d^{\beta_d}}.$$

Note that $\partial^\beta \varphi \neq \partial_r^\beta \varphi$.

6 New Semi-regularity Condition

6.1 New Geometric Parameter and Condition

We proposed a new geometric parameter H_T in [32].

Definition 6.1. Parameter H_T is defined as follows:

$$H_T := \frac{\prod_{i=1}^{d} h_i}{|T|_d} h_T.$$

We introduce geometric conditions to obtain the optimal convergence rate of the anisotropic error estimates.

Condition 6.2. A family of meshes $\{\mathbb{T}_h\}$ is semi-regular if there exists $\gamma_0 > 0$ such that

$$\frac{H_T}{h_T} \leq \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{6.1}$$

Remark 6.3. The geometric condition in (6.1) is equivalent to the maximum angle condition (Section 7).

Remark 6.4. We consider the good elements on the meshes in Section 8. On anisotropic meshes, good elements may satisfy the following conditions:

$(d = 2)$ $h_2 \approx h_2 t$;

$(d = 3)$ $h_2 \approx h_2 t_1$ and $h_3 \approx h_3 t_2$.

6.2 Properties of the New Geometric Parameter

We first show the relation between h_T and H_T.

Lemma 6.5. It holds that

$$h_T \leq \frac{1}{2} H_T \quad \text{if } d = 2, \tag{6.2}$$

$$h_T < \frac{1}{6} H_T \quad \text{if } d = 3. \tag{6.3}$$

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. By constructing the standard element in the two-dimensional case, the angle $\theta_{\max} := \angle p_2 p_1 p_3$ is the maximum angle of T. We then have $\frac{\pi}{3} < \theta_{\max} < \pi$, that is, $0 < \sin\theta_{\max} \leq 1$. Therefore, it holds that

$$H_T = \frac{h_1 h_2}{|T|_2} h_T = \frac{2}{\sin\theta_{\max}} h_T \geq 2 h_T.$$

We here used the fact that $|T|_2 = \frac{1}{2} h_1 h_2 \sin\theta_{\max}$.

Three-dimensional case. We denote by ϕ_T the angle between the base $\triangle p_1 p_2 p_3$ of T and the segment $\overline{p_1 p_4}$. Recall that there are two types of standard elements, (Type i) or (Type ii). We denote by θ_T

(Type i) the angle between the segments $\overline{p_1 p_2}$ and $\overline{p_1 p_3}$, that is, $\theta_T := \angle p_2 p_1 p_3$, or

(Type ii) the angle between the segments $\overline{p_2 p_1}$ and $\overline{p_2 p_3}$, that is, $\theta_T := \angle p_1 p_2 p_3$.

We set $t_1 := \sin\theta_T$ and $t_2 := \sin\phi_T$. By constructing the standard element in the three-dimensional case, the angle $\angle p_1 p_3 p_2$ is the maximum angle of the base $\triangle p_1 p_2 p_3$ of T. Therefore, we have $0 < \theta_T < \frac{\pi}{2}$. Because $0 < \phi_T < \pi$, it holds that

$$H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T = \frac{6}{\sin\theta_T \sin\phi_T} h_T > 6 h_T.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin\theta_T \sin\phi_T$. □

We introduce another geometric parameter regarding Definition 6.1.

Definition 6.6 (Another parameter H_T^*)**.** For $T \in \mathbb{T}_h$, we denote by L_i edges of the simplex T. We define the new parameter H_T^* as

$$H_T^* := \frac{h_T^2}{|T|_2} \min_{1 \le i \le 3} |L_i| \quad \text{if } d = 2, \quad H_T^* := \frac{h_T^2}{|T|_3} \min_{1 \le i,j \le 6, i \neq j} |L_i||L_j| \quad \text{if } d = 3. \tag{6.4}$$

The parameters H_T^* and H_T are equivalent.

Lemma 6.7. It holds that

$$\frac{1}{2} H_T^* < H_T < 2 H_T^*. \tag{6.5}$$

Furthermore, H_T^* is equivalent to the circumradius R_2 of T in the two-dimensional case.

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Let L_i $(i = 1, 2, 3)$ denote edges of the triangle T with $|L_1| \le |L_2| \le |L_3|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_3| = h_T$. Because $h_2 \le h_1 < 2h_T$ and $h_T < h_1 + h_2 \le 2h_1$ for the triangle $\triangle p_1 p_2 p_3$, it holds that

$$\frac{1}{2} h_T < h_1 = |L_2| < 2h_T = 2h_T.$$

We thus have

$$\frac{1}{2} H_T^* = \frac{1}{2} \frac{|L_1|}{|T|_2} h_T^2 < H_T = \frac{h_1 h_2}{|T|_2} h_T < 2 \frac{|L_1|}{|T|_2} h_T^2 = 2 H_T^*.$$

Furthermore, it holds that

$$2R_2 = 2 \frac{|L_1||L_2||L_3|}{4|T|_2} < H_T^* = \frac{|L_1|}{|T|_2} h_T^2 < 8 \frac{|L_1||L_2||L_3|}{4|T|_2} = 8R_2.$$

Three-dimensional case. Let L_i $(i = 1, \dots, 6)$ denote edges of the triangle T with $|L_1| \le |L_2| \le \cdots \le |L_6|$. It obviously holds that $h_2 = |L_1|$ and $h_T = |L_6|$. Recall that there are two types of standard elements, (Type i) or (Type ii).

(Type i) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$ and $h_6 := |p_2 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \le \min\{h_3, h_4, h_6\} \le \max\{h_3, h_4, h_6\} \le h_1. \tag{6.6}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1 p_2 p_4$, it holds that

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T & \text{or} \\ h_3 \le h_1 \le h_5 = h_T. \end{cases}$$

We thus have

$$\begin{cases} h_3 \le h_5 \le h_1 = h_T \quad \text{or} \\ h_3 \le h_1 \le h_T < 2h_1, \quad \frac{1}{2}h_T < h_1 \le h_T. \end{cases}$$

Because $h_3 \le h_5$, the length of the edge L_2 is equal to the one of h_3, h_4 or h_6.

Assume that $|L_2| = h_3$. We then have

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_T \le \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T^*(< 2H_T^*).$$

Assume that $|L_2| = h_4$. We consider the triangle $\triangle p_1p_3p_4$. From the assumption, we have $h_2 \le h_4 \le h_3$ and $\frac{1}{2}h_3 < h_4 \le h_3$. We then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_{T^s} < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

Assume that $|L_2| = h_6$. We consider the triangle $\triangle p_1p_2p_3$. Because p_1 and p_3 belong to the same half-space for the triangle $\triangle p_1p_2p_3$, it holds that $h_2 \le h_6 \le h_1$ and $\frac{1}{2}h_1 < h_6 \le h_1$. From (6.6), we have

$$\frac{1}{2}h_3 \le \frac{1}{2}h_1 < h_6 \le h_1.$$

Because $h_6 \le h_3$, we then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2}\frac{|L_1||L_2|}{|T|_3}h_T^2 < H_T = \frac{h_1h_2h_3}{|T|_3}h_T < 2\frac{|L_1||L_2|}{|T|_3}h_T^2 = 2H_T^*.$$

(Type ii) We set $h_4 := |p_3 - p_4|$, $h_5 := |p_2 - p_4|$, and $h_6 := |p_1 - p_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$ is the longest edge among the four edges that share an endpoint with L_1, it holds that

$$h_2 \le \min\{h_4, h_5, h_6\} \le \max\{h_4, h_5, h_6\} \le h_1. \tag{6.7}$$

Because p_1 and p_4 belong to the same half-space for the triangle $\triangle p_1p_2p_4$ and (6.7), it holds that

$$h_3 \le h_5 \le h_1.$$

This implies that $h_1 = h_T$. Therefore, the length of the edge L_2 is equal to the one of h_3, h_4, or h_6.

Assume that $|L_2| = h_3$. We then have

$$\begin{aligned}\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T &= \frac{h_1h_2h_3}{|T|_3}h_T \\ &= \frac{|L_1||L_2|}{|T|_3}h_T^2 = H_T^*(< 2H_T).\end{aligned}$$

Assume that $|L_2| = h_4$. For the triangle $\triangle p_2p_3p_4$, we have

$$h_2 \le h_4 \le h_5 < 2h_4.$$

Because $h_3 \leq h_5$ and $h_4 \leq h_3$, it holds that

$$\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2\frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.$$

Assume that $|L_2| = h_6$. We have $h_1 < h_2 + h_6 < 2h_6$ for the triangle $\triangle p_1 p_2 p_3$. Therefore, since $h_6 \leq h_3 \leq h_1$, we obtain

$$\left(\frac{1}{2}H_T^* <\right) H_T^* = \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2\frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.$$

□

6.3 Euclidean Condition Number

Examining the Euclidean condition number is useful for deriving appropriate interpolation error estimates.

Lemma 6.8. It holds that

$$\|\widehat{A}\|_2 \leq h_T, \quad \|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}, \tag{6.8a}$$

$$\|\widetilde{A}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3, \end{cases} \qquad \|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 \leq \begin{cases} \frac{h_1 h_2}{|T|_2} = \frac{H_T}{h_T} & \text{if } d = 2, \\ \frac{2}{3}\frac{h_1 h_2 h_3}{|T|_3} = \frac{2}{3}\frac{H_T}{h_T} & \text{if } d = 3, \end{cases} \tag{6.8b}$$

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1. \tag{6.8c}$$

where a parameter H_T is defined in Definition 6.1. Furthermore, we have

$$|\det(A_{\widetilde{T}})| = |\det(\widetilde{A})||\det(\widehat{A})| = \frac{|T|_d}{|\widetilde{T}|_d}\frac{|\widetilde{T}|_d}{|\widehat{T}|_d} = d!|T|_d, \quad |\det(A_T)| = 1. \tag{6.9}$$

Proof. We first show the equality (6.9). Because

$$\int_T dx = \int_{\widetilde{T}} |\det(A_T)| d\tilde{x}, \quad \int_{\widetilde{T}} d\tilde{x} = \int_{\widehat{T}} |\det(A_{\widetilde{T}})| d\hat{x},$$

and $|T|_d = |\widetilde{T}|_d$, we conclude (6.9).

We show the equality (6.8a). From

$$(\widehat{A})^\top \widehat{A} = \mathrm{diag}(h_1^2, \ldots, h_d^2), \quad \widehat{A}^{-1}\widehat{A}^{-\top} = \mathrm{diag}(h_1^{-2}, \ldots, h_d^{-2}),$$

we have

$$\|\widehat{A}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} = \max\{h_1, \cdots, h_d\} \leq h_T,$$

and

$$\|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} \lambda_{\max}(\widehat{A}^{-1}\widehat{A}^{-\top})^{\frac{1}{2}} = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}},$$

which leads to (6.8a).

We next show the equality (6.8b). We consider for each dimension, $d = 2, 3$.

Two-dimensional case. Because

$$\widetilde{A}^\top \widetilde{A} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad \widetilde{A}^{-1}\widetilde{A}^{-\top} = \frac{1}{t^2}\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}, \quad |s| \le 1,$$

we have

$$\|\widetilde{A}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \le (1 + |s|)^{\frac{1}{2}} \le \sqrt{2},$$

and

$$\|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}^{-1}\widetilde{A}^{-\top})^{\frac{1}{2}} \le \frac{2}{t} = \frac{h_1 h_2}{|T|_d},$$

which leads to (6.8b) for $d = 2$. Here, we used the fact that $|\widetilde{T}|_d = \frac{1}{2} h_1 h_2 t$ and $|T|_d = |\widetilde{T}|_d$.

Three-dimensional case. The matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ introduced in (5.5) can be decomposed as $\widetilde{A}_1 = \widetilde{M}_0 \widetilde{M}_1$ and $\widetilde{A}_2 = \widetilde{M}_0 \widetilde{M}_2$ with

$$\widetilde{M}_0 := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{M}_1 := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{M}_2 := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\widetilde{M}_2^\top \widetilde{M}_2$ coincide with those of $\widetilde{M}_1^\top \widetilde{M}_1$, and only Case (i) is shown.

We have the inequalities

$$\begin{aligned} \|\widetilde{A}_1\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \le \lambda_{\max}(\widetilde{M}_0^\top \widetilde{M}_0)^{\frac{1}{2}} \lambda_{\max}(\widetilde{M}_1^\top \widetilde{M}_1)^{\frac{1}{2}} \\ &\le \left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)^{\frac{1}{2}} (1 + |s_1|)^{\frac{1}{2}} \le 2, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{A}_1\|_2 \|\widetilde{A}_1^{-1}\|_2 &= \lambda_{\max}(\widetilde{A}_1^\top \widetilde{A}_1)^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}_1^{-1} \widetilde{A}_1^{-\top})^{\frac{1}{2}} \\ &\le \frac{\left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)(1 + |s_1|)}{t_1 t_2} \le \frac{4}{t_1 t_2} = \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_d}, \end{aligned}$$

where we used the fact that $|\widetilde{T}|_d = \frac{1}{6} h_1 h_2 h_3 t_1 t_2$ and $|T|_d = |\widetilde{T}|_d$.

Because the length of all edges of a simplex and measure of the simplex is not changed by a rotation and mirror imaging matrix and $A_T, A_T^{-1} \in O(d)$,

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1,$$

which is (6.8c). $\square$

7 New Geometric Mesh Condition and the Maximum-angle Condition

7.1 Statements

We state the following theorems concerning the new condition.

Theorem 7.1. Condition 6.2 holds if and only if Condition 4.1 holds when $d = 2$.

Proof. In the case of $d = 2$, we use the previous result presented in [36]; i.e., there exists a constant $\gamma_6 > 0$ such that

$$\frac{R_2}{h_T} \leq \gamma_6 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h,$$

if and only if Condition 4.1 is satisfied. Combining this result with H_T being equivalent to the circumradius R_2 of T (Lemma 6.7), we have the desired conclusion. □

Theorem 7.2. Condition 6.2 holds if and only if Condition 4.5 holds when $d = 3$.

The proof can be found in [31]. Preparation is needed to prove the three-dimensional case. The following subsection shows the symbols used only in this section.

7.2 Notation

Let $T \in \mathbb{T}_h$ be the standard element in $\mathbb{R}^3$ with vertices, P_1, P_2, P_3 and P_4. Let F_i be the face of a simplex T opposite to the vertex P_i. We denote by $\psi^{i,j}$ (Table 5) the angle between the face F_i and the face F_j, see Figure 5. Note that $\psi^{i,j} = \psi^{j,i}$. Furthermore, we denote by θ^i_j (Table 6) the internal angle at the vertex P_j on the face F_i and by ϕ^i_j (Table 7) the angle between the face F_i and the segment $\overline{P_j P_i}$.

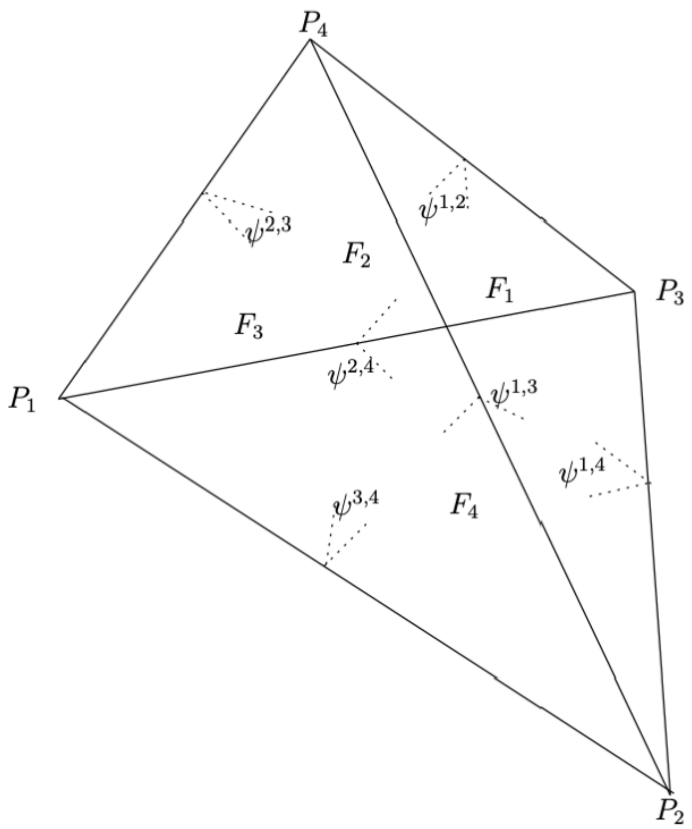


Fig. 5: Tetrahedra

7.3 Preliminaries: Part 1

We introduce three lemmata.

Table 5: $\psi^{i,j}$

	F_1	F_2	F_3	F_4
F_1	-	$\psi^{1,2}$	$\psi^{1,3}$	$\psi^{1,4}$
F_2	$\psi^{2,1}$	-	$\psi^{2,3}$	$\psi^{2,4}$
F_3	$\psi^{3,1}$	$\psi^{3,2}$	-	$\psi^{3,4}$
F_4	$\psi^{4,1}$	$\psi^{4,2}$	$\psi^{4,3}$	-

Table 6: θ^i_j

	F_1	F_2	F_3	F_4
P_1	-	θ^2_1	θ^3_1	θ^4_1
P_2	θ^1_2	-	θ^3_2	θ^4_2
P_3	θ^1_3	θ^2_3	-	θ^4_3
P_4	θ^1_4	θ^2_4	θ^3_4	-

Table 7: ϕ^i_j

	F_1	F_2	F_3	F_4
P_1	-	ϕ^2_1	ϕ^3_1	ϕ^4_1
P_2	ϕ^1_2	-	ϕ^3_2	ϕ^4_2
P_3	ϕ^1_3	ϕ^2_3	-	ϕ^4_3
P_4	ϕ^1_4	ϕ^2_4	ϕ^3_4	-

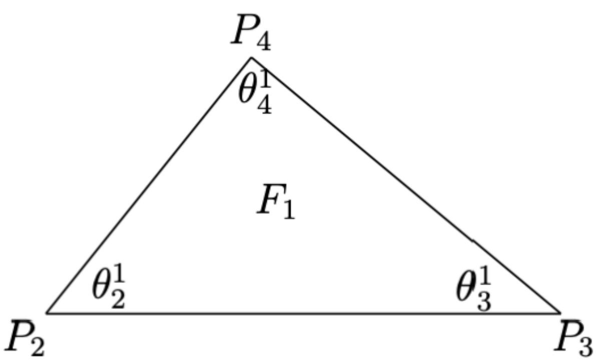


Fig. 6: Face 1

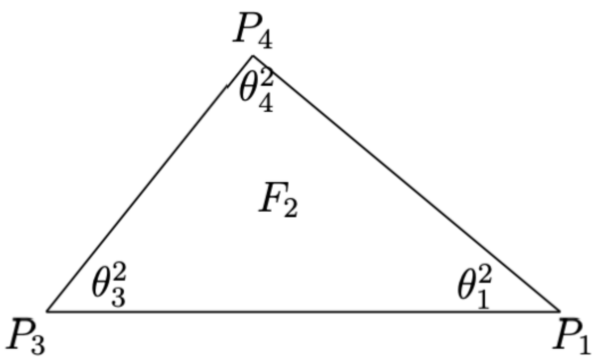


Fig. 7: Face 2

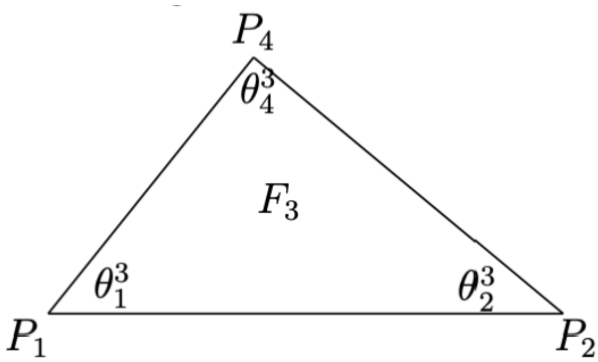


Fig. 8: Face 3

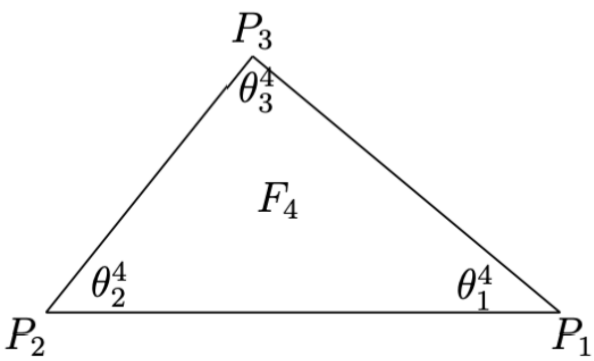


Fig. 9: Face 4

Lemma 7.3. Let $K \subset \mathbb{R}^2$ be a simplex and let θ_1, θ_2 and θ_3 be internal angles of K with $\theta_1 \le \theta_2 \le \theta_3$. If there exists $0 < \theta_0 < \pi$, $\theta_0 \in \mathbb{R}$, such that $\theta_3 \le \theta_0$, we then have

$$\sin\theta_2,\ \sin\theta_3 \ge \min\left\{\sin\frac{\pi-\theta_0}{2}, \sin\theta_0\right\}.$$

Proof. Because $\theta_1+\theta_2+\theta_3 = \pi$ and $\theta_1 \le \theta_2 \le \theta_3$, we have

$$\theta_0 \ge \theta_3 \ge \theta_2 \ge \frac{\theta_1+\theta_2}{2} \ge \frac{\pi-\theta_3}{2} \ge \frac{\pi-\theta_0}{2},$$

which leads to the target inequality. □

Lemma 7.4. Let $K \subset \mathbb{R}^2$ be a simplex with internal angles θ_1, θ_2 and θ_3. For any fixed $\gamma \in \mathbb{R}$ with $0 < \gamma < \pi$, we assume that $\pi - \gamma \le \theta_i$, $i \in \{1, 2, 3\}$. We then have $\theta_{i+1},\ \theta_{i+2} \le \gamma$, where the indices i, $i+1$ and $i+2$ have to be understood "mod 3".

Proof. Because $\theta_1+\theta_2+\theta_3 = \pi$, we have

$$\theta_{i+1} = \pi - \theta_i - \theta_{i+2} < \pi - \theta_i \le \pi - (\pi-\gamma) = \gamma.$$

□

Lemma 7.5. Let $\gamma \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma < \pi$. It then holds that

$$0 < \frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} \le 1.$$

Proof. Because $\cos\gamma = 1 - 2\sin^2\frac{\gamma}{2}$, we have

$$\frac{\cos\gamma+1}{\sin\frac{\gamma}{2}+1} = \frac{2-2\sin^2\frac{\gamma}{2}}{\sin\frac{\gamma}{2}+1} = 2\left(1-\sin\frac{\gamma}{2}\right).$$

Therefore, for $\frac{\pi}{3} \le \gamma < \pi$, the target inequality holds. □

7.4 Preliminaries: Part 2

Lemma 7.6 (Cosine rules for the sides and for the angles)**.** It holds that

$$\begin{aligned}
\cos\theta_j^{j+3} &= \cos\theta_j^{j+1}\cos\theta_j^{j+2} + \sin\theta_j^{j+1}\sin\theta_j^{j+2}\cos\psi^{j+1,j+2}, && (7.1a)\\
\cos\theta_j^{j+1} &= \cos\theta_j^{j+2}\cos\theta_j^{j+3} + \sin\theta_j^{j+2}\sin\theta_j^{j+3}\cos\psi^{j+2,j+3}, && (7.1b)\\
\cos\theta_j^{j+2} &= \cos\theta_j^{j+3}\cos\theta_j^{j+1} + \sin\theta_j^{j+3}\sin\theta_j^{j+1}\cos\psi^{j+3,j+1}, && (7.1c)\\
\cos\psi^{j+1,j+2} &= \sin\psi^{j+2,j+3}\sin\psi^{j+3,j+1}\cos\theta_j^{j+3} - \cos\psi^{j+2,j+3}\cos\psi^{j+3,j+1}, && (7.1d)\\
\cos\psi^{j+2,j+3} &= \sin\psi^{j+3,j+1}\sin\psi^{j+1,j+2}\cos\theta_j^{j+1} - \cos\psi^{j+3,j+1}\cos\psi^{j+1,j+2}, && (7.1e)\\
\cos\psi^{j+3,j+1} &= \sin\psi^{j+1,j+2}\sin\psi^{j+2,j+3}\cos\theta_j^{j+2} - \cos\psi^{j+1,j+2}\cos\psi^{j+2,j+3}, && (7.1f)
\end{aligned}$$

where the indices j, $j+1$, $j+2$ and $j+3$ have to be understood "mod 4".

Proof. A proof can be found in [20, 42]. □

Lemma 7.7. Let $\gamma_{\max} \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma_{\max} < \pi$ satisfy Condition 4.5 for the maximum solid $\theta_{T,\max}$ and the maximum dihedral $\psi_{T,\max}$ of T. Assume that for each $j = 1, 2$, θ_j^4 is not the minimum angle of $\triangle P_1 P_2 P_3$ and $\theta_j^4 < \frac{\pi}{2}$. Then, setting $\delta := \delta(\gamma_{\max})$, $0 < \delta \le \frac{\pi}{2}$ such that

$$\sin\delta = \left(\frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{j+1,4} \ge \delta, \quad \text{or} \quad \psi^{3,4} \ge \delta, \tag{7.2}$$

where the indices j and $j+1$ have to be understood "mod 2".

Proof. From Lemma 7.5, we have

$$0 < \frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1} \le 1,$$

because $\frac{\pi}{3} \le \gamma_{\max} < \pi$. Therefore, δ is well-defined.

We use proof by contradiction. Suppose that

$$0 < \psi^{j+1,4} < \delta, \quad 0 < \psi^{3,4} < \delta,$$

that is,

$$0 < \sin\psi^{j+1,4} \sin\psi^{3,4} < \sin^2\delta, \text{ and } 1 > \cos\psi^{j+1,4} \cos\psi^{3,4} > \cos^2\delta \ge 0.$$

From Lemma 7.3 and assumption, we have

$$\frac{\pi - \gamma_{\max}}{2} \le \theta_j^4 < \frac{\pi}{2},$$

which implies

$$0 < \cos\theta_j^4 \le \cos\left(\frac{\pi - \gamma_{\max}}{2} \right) = \sin\frac{\gamma_{\max}}{2}.$$

We thus obtain

$$\sin\psi^{j+1,4} \sin\psi^{3,4} \cos\theta_j^4 < \sin^2\delta \sin\frac{\gamma_{\max}}{2}.$$

Using the cosine rule (7.1d) with $j = 1$ and the above inequalities yield

$$\begin{aligned}
\cos\psi_{2,3} &= \sin\psi^{3,4} \sin\psi^{4,2} \cos\theta_1^4 - \cos\psi^{3,4} \cos\psi^{4,2} \\
&< \sin^2\delta \sin\frac{\gamma_{\max}}{2} - (1 - \sin^2\delta) \\
&= \frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1} \left(\sin\frac{\gamma_{\max}}{2} + 1 \right) - 1 = \cos\gamma_{\max}.
\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{2,3} \le \gamma_{\max} < \pi$, that is, $\cos\psi^{2,3} \ge \cos\gamma_{\max}$.

Analogously, using the cosine rule (7.1f) with $j = 2$ and the above inequalities yield

$$\begin{aligned}
\cos\psi^{1,3} &= \sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 - \cos\psi^{3,4} \cos\psi^{4,1} \\
&< \cos\gamma_{\max}.
\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{1,3} \le \gamma_{\max} < \pi$, that is, $\cos\psi^{1,3} \ge \cos\gamma_{\max}$. $\square$

Corollary 7.8. For each $j = 1, 2$, under assumptions in Lemma 7.7, it holds that setting $C_0 := \min\{\delta, \gamma_{\max}\}$,

$$\sin \psi^{j+1,4} \geq C_0, \quad \text{or} \quad \sin \psi^{3,4} \geq C_0$$

where the indices j and $j+1$ have to be understood "mod 2".

Lemma 7.9. For any $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ and $k \in \{1, 2, 3, 4\}$, $k \neq i, j$, it holds that

$$\sin \phi_j^i = \sin \theta_j^k \sin \psi^{k,i}.$$

Proof. We only show the case $i = 4$, $j = 1$ and $k = 2$. We then have

$$\sin \phi_1^4 = |\overline{P_1P_4}| \sin \theta_1^2 \times \frac{1}{|\overline{P_1P_4}|} \sin \psi^{2,4} = \sin \theta_1^2 \sin \psi^{2,4}.$$

□

Lemma 7.10. Assume that there exists a positive constant M_j $(j = 1, 2)$ with $0 < M_j < 1$ such that

$$\sin \theta_j^4 \sin \phi_1^4 > M_j, \quad j = 1, 2.$$

Setting $\gamma(M_j) := \pi - \sin^{-1} M_j$ $(j = 1, 2)$, we have $\frac{\pi}{2} < \gamma(M_j) < \pi$ and it holds that for each $j = 1, 2$,

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(M_j), \\
&\theta_2^3,\ \theta_4^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \theta_1^3,\ \psi^{2,4},\ \psi^{3,4} < \gamma(M_j).
\end{aligned}$$

Proof. From assumption, we have, for each $j = 1, 2$,

$$\begin{aligned}
&\sin \theta_j^4 \geq \sin \theta_j^4 \sin \phi_1^4 > M_j, \\
&\sin \phi_1^4 > M_j.
\end{aligned}$$

The definition of $\gamma(M_j)$ and Lemma 7.4 yield, for each $j = 1, 2$,

$$\begin{aligned}
&\pi - \gamma < \theta_j^4 < \gamma(M_j), \quad \theta_{j+1}^4 < \gamma(M_j), \quad \theta_{j+2}^4 < \gamma(M_j), \\
&\pi - \gamma < \phi_1^4 < \gamma(M_j),
\end{aligned}$$

where the indices j, $j+1$ and $j+2$ have to be understood "mod 3".

We obtain, from Lemma 7.9,

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} = \sin \theta_1^3 \sin \psi^{3,4} > M_j, \quad j = 1, 2.$$

We then have, for each $j = 1, 2$,

$$\sin \theta_1^2,\ \sin \psi^{2,4},\ \sin \theta_1^3,\ \sin \psi^{3,4} > M_j,$$

that is,

$$\pi - \gamma(M_j) < \theta_1^2,\ \theta_1^3,\ \psi^{2,4},\ \psi^{3,4} < \gamma(M_j).$$

On $\triangle P_1P_2P_4$ and $\triangle P_1P_3P_4$, using Lemma 7.4 yields

$$\theta_2^3,\ \theta_4^3,\ \theta_3^2,\ \theta_4^2 < \gamma(M_j), \quad j = 1, 2.$$

□

By analogous argument with Lemma 7.10, we get the following two lemmata.

Lemma 7.11. Assume that there exists M_3 with $0 < M_3 < 1$ such that

$$\sin\theta_3^1 \sin\phi_3^1 > M_3.$$

Setting $\gamma(M_3) := \pi - \sin^{-1} M_3$, we have $\frac{\pi}{2} < \gamma(M_3) < \pi$ and it holds that

$$\theta_3^2,\ \theta_3^4,\ \theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Proof. From assumption, we have

$$\sin\theta_3^1 \geq \sin\theta_3^1 \sin\phi_3^1 > M_3, \quad \sin\phi_3^1 > M_3.$$

Using the definition of $\gamma(M_3)$ yields

$$\pi - \gamma < \theta_1^3 < \gamma(M_3), \quad \pi - \gamma < \phi_1^3 < \gamma(M_3).$$

We obtain, from Lemma 7.9,

$$\sin\phi_3^1 = \sin\theta_3^2 \sin\psi^{2,1} = \sin\theta_3^4 \sin\psi^{4,1} > M_3.$$

We then have

$$\sin\theta_3^2,\ \sin\psi^{2,1},\ \sin\theta_3^4,\ \sin\psi^{4,1} > M_3,$$

that is,

$$\pi - \gamma(M_3) < \theta_3^2,\ \theta_3^4,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Meanwhile, on $\triangle P_2P_3P_4$, using Lemma 7.4, we have

$$\theta_2^1,\ \theta_4^1 < \gamma(M_3).$$

□

Lemma 7.12. Assume that there exists M_4 with $0 < M_4 < 1$ such that

$$\sin\theta_2^1 \sin\phi_4^1 > M_4.$$

Setting $\gamma(M_4) := \pi - \sin^{-1} M_4$, we have $\frac{\pi}{2} < \gamma(M_4) < \pi$ and it holds that

$$\theta_4^2,\ \theta_4^3,\ \theta_2^1,\ \theta_3^1,\ \theta_4^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(M_4).$$

Proof. The proof is obtained by using an analogous argument with Lemma 7.11. □

7.5 Proof of Theorem 7.2 in (Type i)

7.5.1 Condition 4.5 ⇒ Condition 6.2

We set $t_1 := \sin\theta_1^4$ and $t_2 := \sin\phi_1^4$. We then have

$$\frac{H_T}{h_T} = \frac{h_1h_2h_3}{|T|_3} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4}.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin\theta_1^4 \sin\phi_1^4$. By construct of the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1 P_2 P_3$ of T. We hence have $\theta_1^4 < \frac{\pi}{2}$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^4 \leq \gamma_{11}, \quad \sin\theta_1^4 \geq \min\left\{ \sin\frac{\pi - \gamma_{11}}{2}, \sin\gamma_{11} \right\} =: C_1.$$

Due to Lemma 7.7, setting $\delta := \delta(\gamma_{11})$, $0 < \delta \leq \frac{\pi}{2}$ such that

$$\sin\delta = \left(\frac{\cos\gamma_{11} + 1}{\sin\frac{\gamma_{11}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that $\psi^{2,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_1^4 = \sin\theta_1^2 \sin\psi^{2,4} \geq C_0 \sin\theta_1^2.$$

By construct of the standard element (Type i), the angle θ_1^2 is not the minimum angle of $\triangle P_1 P_3 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^2 \leq \gamma_{11}, \quad \sin\theta_1^2 \geq C_1.$$

We thus obtain

$$\sin\phi_1^4 \geq C_0 C_1.$$

Suppose that $\psi^{3,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin\phi_1^4 = \sin\theta_1^3 \sin\psi^{3,4} \geq C_0 \sin\theta_1^3.$$

By construct of the standard element (Type i), the angle θ_1^3 is not the minimum angle of $\triangle P_1 P_2 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin\theta_1^3 \geq C_1.$$

We thus obtain

$$\sin\phi_1^4 \geq C_0 C_1.$$

In both cases

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yield

$$\frac{H_T}{h_T} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \leq \frac{6}{C_0 C_1^2} =: D_1 > 0,$$

that is, Condition 6.2 holds. $\square$

7.5.2 Condition 6.2 $\Rightarrow$ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \leq \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_1^4 < \frac{\pi}{2}$ and $\sin\theta_1^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_1^4 \sin\phi_1^4 \geq \frac{6}{\gamma_9} =: C_2.$$

From Lemma 7.10 with $j = 1$, setting $\gamma(C_2) := \pi - \sin^{-1} C_2$, we have $\frac{\pi}{2} < \gamma(C_2) < \pi$ and it holds that

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2), \\
&\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
|T|_3 &= \frac{1}{3} \times \frac{1}{2} |\overline{P_2 P_3}| |\overline{P_3 P_4}| \sin\theta_3^1 \times h_2 \sin\phi_3^1 = \frac{1}{6} h_2 |\overline{P_2 P_3}| |\overline{P_3 P_4}| \sin\theta_3^1 \sin\phi_3^1 \\
&< \frac{1}{3} h_1 h_2 h_3 \sin\theta_3^1 \sin\phi_3^1,
\end{aligned}$$

where we used the fact that $|\overline{P_3 P_4}| < |\overline{P_1 P_4}| + |\overline{P_1 P_3}| \leq 2h_3$ on $\triangle P_1 P_3 P_4$ and $|\overline{P_2 P_3}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_T}{h_T} > \frac{3}{\sin\theta_3^1 \sin\phi_3^1},$$

that is,

$$\sin\theta_3^1 \sin\phi_3^1 > \frac{3}{\gamma_9} =: C_3.$$

From Lemma 7.11, setting $\gamma(C_3) := \pi - \sin^{-1} C_3$, we have $\frac{\pi}{2} < \gamma(C_3) < \pi$ and it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(C_3).$$

Due to the cosine rule (7.1f) with $j = 2$, we get

$$\cos\psi^{1,3} = \sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 - \cos\psi^{3,4} \cos\psi^{4,1}.$$

By constructing the standard element (Type i), the angle θ_2^4 is the minimum angle of $\triangle P_1 P_2 P_3$. Therefore, we have

$$\begin{aligned}
&\cos\theta_2^4 \geq \frac{1}{2} \quad \text{because } \theta_2^4 \leq \frac{\pi}{3}, \\
&\sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 > 0, \quad \text{because } \sin\psi^{3,4} \sin\psi^{4,1} > 0,
\end{aligned}$$

and thus

$$\cos\psi^{1,3} > -\cos\psi^{3,4} \cos\psi^{4,1}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,1} > C_3$ yields

$$\begin{aligned}
\cos \psi^{1,3} &> - \cos \psi^{3,4} \cos \psi^{4,1} \\
&\geq -|\cos \psi^{3,4}||\cos \psi^{4,1}| = -\sqrt{1-\sin^2 \psi^{3,4}}\sqrt{1-\sin^2 \psi^{4,1}} \\
&> -\sqrt{1-C_2^2}\sqrt{1-C_3^2} =: C_4 > -1.
\end{aligned}$$

Setting $\gamma(C_4) := \cos^{-1} C_4$, it holds that

$$\psi^{1,3} < \gamma(C_4) < \pi.$$

Due to the cosine rule (7.1d) with $j = 1$, we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1P_2P_3$ of T^s. We hence have $\theta_1^4 < \frac{\pi}{2}$. Therefore, we have

$$\begin{aligned}
&\cos \theta_1^4 > 0 \quad \text{because } \theta_1^4 \leq \frac{\pi}{2}, \\
&\sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 > 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0,
\end{aligned}$$

and thus

$$\cos \psi^{2,3} > - \cos \psi^{3,4} \cos \psi^{4,2}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,2} > C_2$ yield

$$\begin{aligned}
\cos \psi^{2,3} &> - \cos \psi^{3,4} \cos \psi^{4,2} \\
&\geq -|\cos \psi^{3,4}||\cos \psi^{4,2}| = -\sqrt{1-\sin^2 \psi^{3,4}}\sqrt{1-\sin^2 \psi^{4,2}} \\
&> -(1-C_2^2) =: C_5 > -1.
\end{aligned}$$

Setting $\gamma(C_5) := \cos^{-1} C_5$, it holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_3), \gamma(C_4), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, Condition 4.5 holds. □

7.6 Proof of Theorem 7.2 in (Type ii)

7.6.1 Condition 4.5 ⇒ Condition 6.2

We set $t_1 := \sin \theta_2^4$ and $t_2 := \sin \phi_1^4$. We then have

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4}.$$

We here used the fact that $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_2^4 \sin \phi_1^4$. By construct of the standard element (Type ii), the angle θ_3^4 and θ_1^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1P_2P_3$ of T^s. We hence have $\theta_2^4 < \frac{\pi}{2}$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^4 \leq \gamma_{11}, \quad \sin \theta_2^4 \geq C_1.$$

Due to Lemma 7.7, it holds that

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that $\psi^{1,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_2^4 = \sin \theta_2^1 \sin \psi^{1,4} \geq C_0 \sin \theta_2^1.$$

Furthermore, it holds that

$$\sin \phi_1^4 = \frac{|\overline{P_2 P_4}| \sin \phi_2^4}{h_3}.$$

By construct of the standard element (Type ii), the angle θ_2^1 is not the minimum angle of $\triangle P_2 P_3 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^1 \leq \gamma_{11}, \quad \sin \theta_2^1 \geq C_1.$$

Because $h_3 = |\overline{P_1 P_4}| < |\overline{P_2 P_4}|$ on $\triangle P_1 P_2 P_4$, we thus obtain

$$\sin \phi_1^4 = \frac{|\overline{P_2 P_4}|}{h_3} \sin \phi_2^4 > C_0 C_1.$$

Suppose that $\psi^{3,4} \geq \delta$. By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_1^4 = \sin \theta_1^3 \sin \psi^{3,4} \geq C_0 \sin \theta_1^3.$$

By constructing the standard element (Type ii), the angle θ_1^3 is not the minimum angle of $\triangle P_1 P_2 P_4$. From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin \theta_1^3 \geq C_1.$$

We thus obtain

$$\sin \phi_1^4 > C_0 C_1.$$

In both cases

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yields

$$\frac{H_T}{h_T} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq \frac{6}{C_0 C_1^2} = D_1 > 0,$$

that is, Condition 6.2 holds. □

7.6.2 Condition 6.2 ⇒ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_2^4 < \frac{\pi}{2}$ and $\sin\theta_2^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_2^4 \sin\phi_1^4 \geq \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.10 with $j = 2$, it holds that

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2),\\
&\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
|T|_3 &= \frac{1}{3} \times \frac{1}{2} |\overline{P_2P_4}||\overline{P_2P_3}| \sin\theta_2^1 \times h_3 \sin\phi_4^1\\
&< \frac{1}{6} h_1 h_2 h_3 \sin\theta_2^1 \sin\phi_4^1,
\end{aligned}$$

where we used the fact that $|\overline{P_3P_2}| = h_2$ and $|\overline{P_2P_4}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_{T^s}}{h_{T^s}} > \frac{6}{\sin\theta_2^1 \sin\phi_4^1},$$

that is,

$$\sin\theta_2^1 \sin\phi_4^1 > \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.12, it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(C_2).$$

Due to the cosine rule (7.1e) with $j = 2$, we get

$$\cos\psi^{4,1} = \sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 - \cos\psi^{1,3} \cos\psi^{3,4}.$$

By constructing the standard element (Type ii), the angle θ_2^3 is the minimum angle of $\triangle P_1P_2P_4$. Therefore, we have

$$\begin{aligned}
&\cos\theta_2^3 \geq \frac{1}{2} \quad \text{because } \theta_2^3 \leq \frac{\pi}{3},\\
&\sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 > 0, \quad \text{because } \sin\psi^{1,3} \sin\psi^{3,4} > 0,
\end{aligned}$$

and thus

$$\cos\psi^{4,1} > -\cos\psi^{1,3} \cos\psi^{3,4}.$$

Using $\sin\psi^{1,3} > C_2$ and $\sin\psi^{3,4} > C_2$ yield

$$\begin{aligned}
\cos\psi^{4,1} &> -\cos\psi^{1,3} \cos\psi^{3,4}\\
&\geq -\sqrt{1 - \sin^2\psi^{1,3}}\sqrt{1 - \sin^2\psi^{3,4}}\\
&> -(1 - C_2^2) = C_5 > -1.
\end{aligned}$$

It then holds that

$$\psi^{4,1} < \gamma(C_5) < \pi.$$

Due to the cosine rule (7.1d) with $j = 1$, we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type ii), the angle θ_1^4 is the minimum angle of $\triangle P_1 P_2 P_3$. We hence have $\theta_1^4 < \frac{\pi}{3}$. Therefore, we have

$$\begin{aligned} &\cos \theta_1^4 \geq \frac{1}{2} \quad \text{because } \theta_1^4 \leq \frac{\pi}{3}, \\ &\sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 > 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0, \end{aligned}$$

and thus

$$\cos \psi^{2,3} > - \cos \psi^{3,4} \cos \psi^{4,2}.$$

Using $\sin \psi^{3,4} > C_2$ and $\sin \psi^{4,2} > C_2$ yield

$$\begin{aligned} \cos \psi^{2,3} &> - \cos \psi^{3,4} \cos \psi^{4,2} \\ &\geq - \sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,2}} \\ &> -(1 - C_2^2) = C_5 > -1. \end{aligned}$$

It then holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_2), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, Condition 4.5 holds. $\square$

8 Good Elements or not for $d = 2, 3$?

In this subsection, we consider good elements on meshes. In this paper, we define 'good elements' on meshes as the existence of a positive constant $\gamma_0 > 0$ satisfying (6.1). We treat a "Right-angled triangle", "Blade" and "Dagger" for $d = 2$, and "Spire", "Spear", "Spindle", "Spike", "Splinter" and "Sliver" for $d = 3$, which are introduced in [12]. We give the quantities $h_{\max}/h_{\min}$ and H_T/h_T for those elements. The parameters $h_{\max}$ and $h_{\min}$ are defined as

$$h_{\max} := \max\{h_1, \ldots, h_d\}, \quad h_{\min} := \min\{h_1, \ldots, h_d\}. \tag{8.1}$$

8.1 Isotropic Mesh Elements

Recall that an isotropic mesh element has equal or nearly equal edge lengths and angles, resulting in a balanced shape. Then, the geometric condition (3.4) is satisfied. Therefore, it holds that

$$\frac{H_T}{h_T} \leq \frac{h_T^d}{|T|_d} \leq \frac{1}{\gamma_3}, \quad \frac{h_{\max}}{h_{\min}} \leq c \frac{h_T^d}{|T|_d} \leq \frac{c}{\gamma_3}.$$

In this case, elements satisfying the geometric condition (3.4) are "good."

8.2 Anisotropic mesh: two-dimensional case

Let $S \subset \mathbb{R}^2$ be a triangle. Let $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon, \delta, \gamma \in \mathbb{R}$.

Example 8.1 (Right-angled triangle)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = s$ and $h_2 = s^\varepsilon$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \le s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element S is "good."

Example 8.2 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ and $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \le c s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \le c. \end{aligned}$$

In this case, the element S is "good."

Remark 8.3. In the above examples, $h_2 \approx \widetilde{\mathscr{H}_2}$ holds. That is, the good element $S \subset \mathbb{R}^2$ may satisfy conditions such as $h_2 \approx \widetilde{\mathscr{H}_2}$.

Example 8.4 (Blade)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = h_2 = \sqrt{s^2 + s^{2\varepsilon}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \to \infty \quad \text{as } s \to 0.$$

In this case, the element S is "not good."

Example 8.5 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ and $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \le c s^{1-\delta} \to \infty \quad \text{as } s \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \le c s^{\delta-\varepsilon} \to \infty \quad \text{as } s \to 0. \end{aligned}$$

In this case, the element S is "not good."

Anisotropic elements in the next two examples are also "good." However, these examples differ slightly from Examples 8.1 and 8.4.

Example 8.6 (Right-angled triangle)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = s$ and $h_2 = \delta s$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{1}{\delta}, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element S is "good." However, the factor $\frac{1}{\delta}$ is very large.

Example 8.7 (Blade)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = h_2 = s\sqrt{1 + \delta^2}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2(1+\delta^2)}{\delta s^2} \le \frac{c}{\delta},$$

In this case, the element S is "good." However, the factor $\frac{1}{\delta}$ is very large.

8.3 Anisotropic mesh: three-dimensional case

Example 8.8. Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let S be the base of T; i.e., $S = \triangle p_1 p_2 p_3$. Recall that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{h_1 h_2}{\frac{1}{2} h_1 h_2 t_1} \frac{h_3}{\frac{1}{3} h_3 t_2} \leq \frac{H_S}{h_S} \frac{h_3}{\frac{1}{3} \widetilde{\mathscr{H}_3}}. \tag{8.2}$$

If the triangle S is "not good" such as in Examples 8.4 and 8.5, the quantity (8.2) may diverge. In the following, we consider the case that the triangle S is "good".

Assume that there exists a positive constant M such that $\frac{H_S}{h_S} \leq M$. For simplicity, we set $p_1 := (0,0,0)^\top$, $p_2 := (2s,0,0)^\top$, and $p_3 := (2s - \sqrt{4s^2 - s^{2\gamma}}, s^\gamma, 0)^\top$ with $1 < \gamma$. Then,

$$h_1 = 2s, \quad h_2 = \sqrt{\frac{4s^{2\gamma}}{2 + \sqrt{4 - s^{2\gamma-2}}}},$$

and because $h_{\max} \approx cs$,

$$\frac{h_{\max}}{h_{\min}} \leq \frac{cs}{h_2} \leq cs^{1-\gamma} \to \infty \quad \text{as } s \to 0.$$

If we set $p_4 := (s, 0, s^\varepsilon)^\top$ with $1 < \varepsilon$, the triangle $\triangle p_1 p_2 p_4$ is the blade (Example 8.4). Then,

$$h_3 = \sqrt{s^2 + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{2+\gamma}}{s^{1+\gamma+\varepsilon}} \leq cs^{1-\varepsilon} \to \infty \quad \text{as } s \to 0.$$

In this case, the element T is "not good."

If we set $p_4 := (s^\delta, 0, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon < \gamma$, the triangle $\triangle p_1 p_2 p_4$ is the dagger (Example 8.5, Fig. 10). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\delta}}{s^{1+\gamma+\varepsilon}} \leq cs^{\delta-\varepsilon} \to \infty \quad \text{as } s \to 0.$$

In this case, the element T is "not good."

If we set $p_4 := (s^\delta, 0, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta < \gamma$, the triangle $\triangle p_1 p_2 p_4$ is the dagger (Example 8.2). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\varepsilon}}{s^{1+\gamma+\varepsilon}} \leq c.$$

In this case, the element T is "good" and $h_3 \approx h_3 t_2 = \widetilde{\mathscr{H}_3}$ holds.

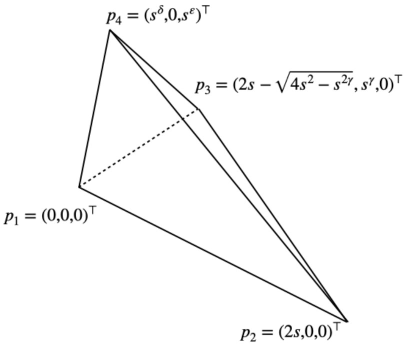


Fig. 10: Example 8.8

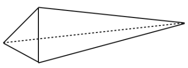


Fig. 11: Spire

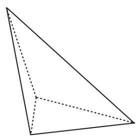


Fig. 12: Spear

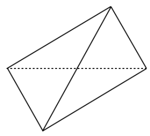


Fig. 13: Spindle

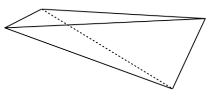


Fig. 14: Spike

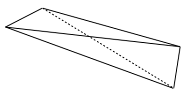


Fig. 15: Splinter

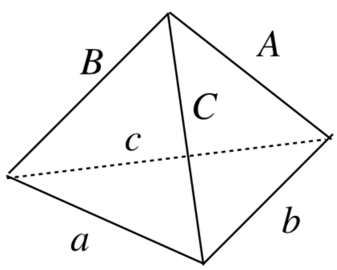


Fig. 16: R_3

Example 8.9. In [12], the spire has a cycle of three daggers among its four triangles; see Figure 11. The splinter has four daggers; see Figure 15. The spear and spike have two daggers and two blades as triangles; see Figures 12, 14. The spindle has four blades as triangles; see Figure 13.

Remark 8.10. The above examples reveal that the good element $T \subset \mathbb{R}^3$ may satisfy conditions such as $h_2 \approx \tilde{\mathscr{H}_2}$ and $h_3 \approx \tilde{\mathscr{H}_3}$.

Example 8.11. Using an element T called *Sliver*, we compare the three quantities $\frac{h_T^3}{|T|_3}$, $\frac{H_T}{h_T}$, and $\frac{R_3}{h_T}$, where the formulation of the circumradius R_3 of a tetrahedron T is as follows, e.g., see [23]. Let a, b and c be the lengths of the three edges of T and A, B, C the length of the opposite edges of a, b, c, respectively. Then,

$$R_3 = \frac{\sqrt{(aA + bB + cC)(aA + bB - cC)(aA - bB + cC)(-aA + bB + cC)}}{24|T|_3},$$

see Fig. 16.

Let $T \subset \mathbb{R}^3$ be the simplex with vertices $p_1 := (s^{\varepsilon_2}, 0, 0)^\top$, $p_2 := (-s^{\varepsilon_2}, 0, 0)^\top$, $p_3 := (0, -s, s^{\varepsilon_1})^\top$, and $p_4 := (0, s, s^{\varepsilon_1})^\top$ $(\varepsilon_1, \varepsilon_2 > 1)$, where $s := \frac{1}{N}$, $N \in \mathbb{N}$, see Fig. 17. Let L_i $(1 \le i \le 6)$ be the edges of T with $h_{\min} = L_1 \le L_2 \le \cdots \le L_6 = h_T$. Recall that $h_{\max} \approx h_T$ and

$$\frac{h_{\max}}{h_{\min}} \le c\frac{L_6}{L_1}, \quad \frac{H_T}{h_T} = \frac{L_1 L_2}{|T|_3} h_T.$$

Table 8: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.5$, $\varepsilon_2 = 1.0$)

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	1.4033	6.7882e+01	3.4471e+01	5.0195e-01
64	1.5625e-02	1.4087	9.6000e+01	4.8375e+01	5.0098e-01
128	7.8125e-03	1.4115	1.3576e+02	6.8147e+01	5.0049e-01

Table 9: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.0$, $\varepsilon_2 = 1.5$)

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	6.7882e+01	8.5513	5.0006e-01
64	1.5625e-02	8.0000	9.6000e+01	8.5184	5.0002e-01
128	7.8125e-03	1.1314e+01	1.3576e+02	8.5018	5.0000e-01

Table 10: $h_T^3/|T|_3$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.5$, $\varepsilon_2 = 1.5$)

N	s	L_6/L_1	$h_T^3/\|T\|_3$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	3.8400e+02	3.4986e+01	1.4170
64	1.5625e-02	8.0000	7.6800e+02	4.8744e+01	2.0010
128	7.8125e-03	1.1314e+01	1.5360e+03	6.8411e+01	2.8288

In Table 8, the angle between $\triangle p_1p_2p_3$ and $\triangle p_1p_2p_4$ tends to π as $s \to 0$, and the simplex T is "not good." In Table 9, the angle between $\triangle p_1p_3p_4$ and $\triangle p_2p_3p_4$ tends to 0 as $s \to 0$, the simplex T is "good." In Table 10, from the numerical result, the simplex T is "not good."

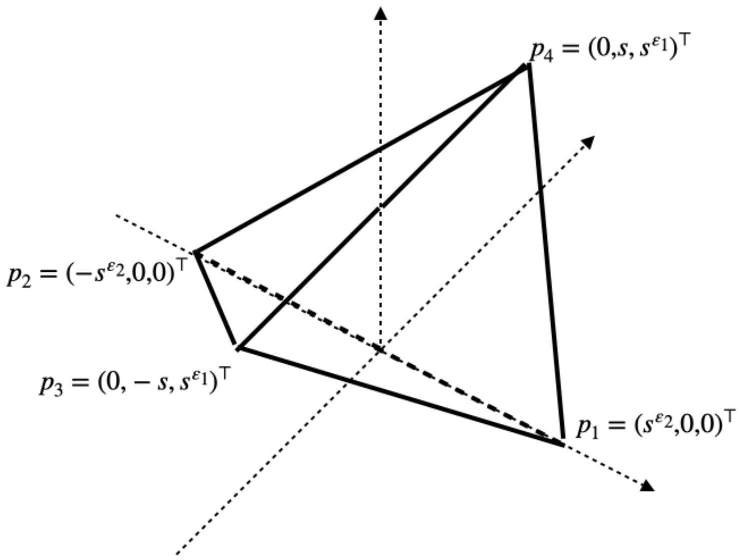


Fig. 17: Sliver

9 FE Generation

We follow the procedure described in [17, Chapter 9] and [16, Section 1.4.1 and 1.2.1]; also see [32, Section 3.5]. The definition of a *finite element* can be found in [13, p. 78] and [17, Definition 5.2].

For the reference element $\widehat{T}$ defined in Sections 5.1, let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a fixed reference finite element, where $\widehat{P}$ is a vector space of functions $\hat{q} : \widehat{T} \to \mathbb{R}^n$ for some positive integer n (typically $n = 1$ or $n = d$) and $\widehat{\Sigma}$ is a set of n_0 linear forms $\{\hat{\chi}_1, \ldots, \hat{\chi}_{n_0}\}$ such that

$$\widehat{P} \ni \hat{q} \mapsto (\hat{\chi}_1(\hat{q}), \ldots, \hat{\chi}_{n_0}(\hat{q}))^\top \in \mathbb{R}^{n_0}$$

is bijective; i.e., $\widehat{\Sigma}$ is a basis for $\mathcal{L}(\widehat{P};\mathbb{R})$. Further, we denote by $\{\hat{\theta}_1, \ldots, \hat{\theta}_{n_0}\}$ in $\widehat{P}$ the local ($\mathbb{R}^n$-valued) shape functions such that

$$\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}, \quad 1 \le i, j \le n_0.$$

Let $V(\widehat{T})$ be a normed vector space of functions $\hat{\varphi} : \widehat{T} \to \mathbb{R}^n$ such that $\widehat{P} \subset V(\widehat{T})$ and the linear forms $\{\hat{\chi}_1, \ldots, \hat{\chi}_{n_0}\}$ can be extended to $V(\widehat{T})'$, i.e., there exist $\{\bar{\chi}_1, \ldots, \bar{\chi}_{n_0}\}$ and c_χ such that $\hat{\chi}_i(\hat{q}) = \bar{\chi}_i(\hat{q})$ for any $\hat{q} \in \widehat{P}$, and $|\bar{\chi}_i(\hat{v})| \le c_\chi \|\hat{v}\|_{V(\widehat{T})}$ and for $i \in \{1, \ldots, n_0\}$. We use the same symbol $\hat{\chi}_i$ instead of $\bar{\chi}_i$. The local interpolation operator $I_{\widehat{T}}$ is then defined by

$$I_{\widehat{T}} : V(\widehat{T}) \ni \hat{\varphi} \mapsto \sum_{i=1}^{n_0} \hat{\chi}_i(\hat{\varphi})\hat{\theta}_i \in \widehat{P}. \tag{9.1}$$

It obviously holds that, for any $\hat{\varphi} \in V(\widehat{T})$,

$$\hat{\chi}_i(I_{\widehat{T}}\hat{\varphi}) = \hat{\chi}_i(\hat{\varphi}) \quad i = 1, \ldots, n_0. \tag{9.2}$$

Proposition 9.1. $\widehat{P}$ is invariant under $I_{\widehat{T}}$, that is,

$$I_{\widehat{T}}\hat{q} = \hat{q} \quad \forall \hat{q} \in \widehat{P}. \tag{9.3}$$

Proof. Let $\hat{q} = \sum_{j=1}^{n_0} \alpha_j \hat{\theta}_j$ for $\alpha_j \in \mathbb{R}$, $1 \le j \le n_0$. Then,

$$I_{\widehat{T}} \hat{q} = \sum_{i,j=1}^{n_0} \alpha_j \hat{\chi}_i(\hat{\theta}_j) \hat{\theta}_i = \hat{q}.$$

$\square$

Let $\Phi_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ and $\Phi_T : \widetilde{T} \to T$ be the two affine mappings defined in Section 5.2. For any $T \in \mathbb{T}_h$ with $T = \Phi(\widehat{T}) = (\Phi_T \circ \Phi_{\widetilde{T}})(\widehat{T})$, we fdefine a Banach space $V(T)$ of $\mathbb{R}^n$-valued functions that is the counterpart of $V(\widehat{T})$ and define a linear bijection mapping by

$$\psi := \psi_{\widehat{T}} \circ \psi_{\widetilde{T}} : V(T) \ni \varphi \mapsto \hat{\varphi} := \psi(\varphi) := \varphi \circ \Phi \in V(\widehat{T}),$$

with two linear bijection mappings:

$$\begin{aligned} \psi_{\widetilde{T}} &: V(T) \ni \varphi \mapsto \tilde{\varphi} := \psi_{\widetilde{T}}(\varphi) := \varphi \circ \Phi_T \in V(\widetilde{T}), \\ \psi_{\widehat{T}} &: V(\widetilde{T}) \ni \tilde{\varphi} \mapsto \hat{\varphi} := \psi_{\widehat{T}}(\tilde{\varphi}) := \tilde{\varphi} \circ \Phi_{\widetilde{T}} \in V(\widehat{T}). \end{aligned}$$

Triples $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T, P, \Sigma\}$ are defined as follows:

$$\begin{cases} \widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T}); \\ \widetilde{P} = \{\psi_{\widehat{T}}^{-1}(\hat{q});\ \hat{q} \in \widehat{P}\}; \\ \widetilde{\Sigma} = \{\{\tilde{\chi}_i\}_{1 \le i \le n_0};\ \tilde{\chi}_i = \hat{\chi}_i(\psi_{\widehat{T}}(\tilde{q})), \forall \tilde{q} \in \widetilde{P}, \hat{\chi}_i \in \widehat{\Sigma}\}, \end{cases}$$

and

$$\begin{cases} T = \Phi_T(\widetilde{T}); \\ P = \{\psi_{\widetilde{T}}^{-1}(\tilde{q});\ \tilde{q} \in \widetilde{P}\}; \\ \Sigma = \{\{\chi_i\}_{1 \le i \le n_0};\ \chi_i = \tilde{\chi}_i(\psi_{\widetilde{T}}(q)), \forall q \in P, \tilde{\chi}_i \in \widetilde{\Sigma}\}. \end{cases}$$

Proposition 9.2. The triples $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T, P, \Sigma\}$ are finite elements.

Proof. A proof can be obtained similarly for [17, Proposition 9.2]. $\square$

The local shape functions are $\tilde{\theta}_i = \psi_{\widehat{T}}^{-1}(\hat{\theta}_i)$ and $\theta_i = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$, $1 \le i \le n_0$, and the associated local interpolation operators are respectively defined by

$$I_{\widetilde{T}} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}} \tilde{\varphi} := \sum_{i=1}^{n_0} \tilde{\chi}_i(\tilde{\varphi}) \tilde{\theta}_i \in \widetilde{P}, \tag{9.4}$$

$$I_T : V(T) \ni \varphi \mapsto I_T \varphi := \sum_{i=1}^{n_0} \chi_i(\varphi) \theta_i \in P. \tag{9.5}$$

The following diagrams play an important role in analysing the interpolation error.

Proposition 9.3 (Commuting diagrams)**.** The diagrams

$$\begin{tikzcd} V(T) \arrow[r, "\psi_{\widetilde{T}}"] \arrow[d, "I_T"'] & V(\widetilde{T}) \arrow[r, "\psi_{\widehat{T}}"] \arrow[d, "I_{\widetilde{T}}"'] & V(\widehat{T}) \arrow[d, "I_{\widehat{T}}"] \\ P \arrow[r, "\psi_{\widetilde{T}}"'] & \widetilde{P} \arrow[r, "\psi_{\widehat{T}}"'] & \widehat{P} \end{tikzcd}$$

commute. Furthermore, $\widetilde{P}$ and P are respectively invariant under $I_{\widetilde{T}}$ and I_T.

Proof. A proof can be obtained similarly for [17, Proposition 9.3].

Let $\tilde{\varphi} \in V(\widetilde{T})$. The definition of $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ implies that

$$I_{\widehat{T}}(\psi_{\widehat{T}}(\tilde{\varphi})) = \sum_{i=1}^{n_0} \hat{\chi}_i(\psi_{\widehat{T}}(\tilde{\varphi}))\hat{\theta}_i = \sum_{i=1}^{n_0} \tilde{\chi}_i(\tilde{\varphi})\psi_{\widehat{T}}(\tilde{\theta}_i) = \psi_{\widehat{T}}(I_{\widetilde{T}}\tilde{\varphi}).$$

Here, we used the linearity of $\psi_{\widehat{T}}$. Therefore, the right diagram commutes.

Let $\tilde{q} \in \widetilde{P}$. Because $\psi_{\widehat{T}}(\tilde{q}) \in \widehat{P}$ and $\widehat{P}$ is invariant under $I_{\widehat{T}}$,

$$I_{\widetilde{T}}(\tilde{q}) = \psi_{\widehat{T}}^{-1}(I_{\widehat{T}}(\psi_{\widehat{T}}(\tilde{q}))) = \psi_{\widehat{T}}^{-1}(\psi_{\widehat{T}}(\tilde{q})) = \tilde{q}.$$

Another diagram can be proved in the same way. □

Example 9.4. Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element.

1. For the Lagrange finite element of degree k, we set $V(\widehat{T}) := \mathcal{C}^0(\widehat{T})$.
2. For the Hermite finite element, we set $V(\widehat{T}) := \mathcal{C}^1(\widehat{T})$.
3. For the Crouzeix–Raviart finite element with $k = 1$, we set $V(\widehat{T}) := W^{1,1}(\widehat{T})$.

10 New Scaling Argument: Part 1

This section gives estimates related to a scaling argument corresponding to [16, Lemma 1.101].

10.1 Preliminalies

10.1.1 Additional New Condition

The following condition is used for obtaining optimal interpolation error estimates.

Condition 10.1. In anisotropic interpolation error analysis, we impose the following geometric condition for the simplex T:

1. If $d = 2$, there are no additional conditions;
2. If $d = 3$, there must exist a positive constant M independent of h_T such that $|s_{22}| \leq M\frac{h_2 t_1}{h_3}$. Note that if $s_{22} \neq 0$, this condition means that the order of h_3 with respect to h_T coincides with the order of h_2, and if $s_{22} = 0$, the order of h_3 may be different from that of h_2.

Recall that

$$\begin{aligned} &|s| \leq 1, \quad h_2 \leq h_1 \quad \text{if } d = 2, \\ &|s_1| \leq 1, \ |s_{21}| \leq 1, \quad h_2 \leq h_3 \leq h_1 \quad \text{if } d = 3. \end{aligned}$$

When $d = 3$, if Condition 10.1 is imposed, there exists a positive constant M independent of h_T such that $|s_{22}| \leq M\frac{h_2 t_1}{h_3}$. We thus have, if $d = 2$,

$$h_1|[\widetilde{A}]_{j1}| \leq \widetilde{\mathscr{H}_j}, \quad h_2|[\widetilde{A}]_{j2}| \leq \widetilde{\mathscr{H}_j}, \quad j = 1, 2,$$

and, if $d = 3$, for $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$ and $j = 1, 2, 3$,

$$h_1|[\widetilde{A}]_{j1}| \leq \widetilde{\mathscr{H}_j}, \quad h_2|[\widetilde{A}]_{j2}| \leq \widetilde{\mathscr{H}_j}, \quad h_3|[\widetilde{A}]_{j3}| \leq \max\{1, M\}\widetilde{\mathscr{H}_j}, \quad j = 1, 2, 3.$$

10.1.2 Calculations 1

We use the following calculations in (10.2). Recall that $\tilde{x} = A_{\widetilde{T}}\hat{x}$ with $A_{\widetilde{T}} = \widetilde{A}\widehat{A}$ and $x = A_T\tilde{x} + b_T$. For any multi-indices β and γ, we have

$$
\begin{aligned}
\partial_{\hat{x}}^{\beta+\gamma} &= \frac{\partial^{|\beta|+|\gamma|}}{\partial\hat{x}_1^{\beta_1}\cdots\partial\hat{x}_d^{\beta_d}\partial\hat{x}_1^{\gamma_1}\cdots\partial\hat{x}_d^{\gamma_d}} \\
&= \underbrace{\sum_{i_1^{(1)},i_1^{(0,1)}=1}^{d} h_1[\widetilde{A}]_{i_1^{(1)}1}[A_T]_{i_1^{(0,1)}i_1^{(1)}}\cdots\sum_{i_{\beta_1}^{(1)},i_{\beta_1}^{(0,1)}=1}^{d} h_1[\widetilde{A}]_{i_{\beta_1}^{(1)}1}[A_T]_{i_{\beta_1}^{(0,1)}i_{\beta_1}^{(1)}}\cdots}_{\beta_1\text{times}} \\
&\quad \underbrace{\sum_{i_1^{(d)},i_1^{(0,d)}=1}^{d} h_d[\widetilde{A}]_{i_1^{(d)}d}[A_T]_{i_1^{(0,d)}i_1^{(d)}}\cdots\sum_{i_{\beta_d}^{(d)},i_{\beta_d}^{(0,d)}=1}^{d} h_d[\widetilde{A}]_{i_{\beta_d}^{(d)}d}[A_T]_{i_{\beta_d}^{(0,d)}i_{\beta_d}^{(d)}}}_{\beta_d\text{times}} \\
&\quad \underbrace{\sum_{j_1^{(1)},j_1^{(0,1)}=1}^{d} h_1[\widetilde{A}]_{j_1^{(1)}1}[A_T]_{j_1^{(0,1)}j_1^{(1)}}\cdots\sum_{j_{\gamma_1}^{(1)},j_{\gamma_1}^{(0,1)}=1}^{d} h_1[\widetilde{A}]_{j_{\gamma_1}^{(1)}1}[A_T]_{j_{\gamma_1}^{(0,1)}j_{\gamma_1}^{(1)}}\cdots}_{\gamma_1\text{times}} \\
&\quad \underbrace{\sum_{j_1^{(d)},j_1^{(0,d)}=1}^{d} h_d[\widetilde{A}]_{j_1^{(d)}d}[A_T]_{j_1^{(0,d)}j_1^{(d)}}\cdots\sum_{j_{\gamma_d}^{(d)},j_{\gamma_d}^{(0,d)}=1}^{d} h_d[\widetilde{A}]_{j_{\gamma_d}^{(d)}d}[A_T]_{j_{\gamma_d}^{(0,d)}j_{\gamma_d}^{(d)}}}_{\gamma_d\text{times}} \\
&\quad \underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}}\cdots\partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1\text{times}}\cdots\underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}}\cdots\partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d\text{times}}\underbrace{\frac{\partial^{\gamma_1}}{\partial x_{j_1^{(0,1)}}\cdots\partial x_{j_{\gamma_1}^{(0,1)}}}}_{\gamma_1\text{times}}\cdots\underbrace{\frac{\partial^{\gamma_d}}{\partial x_{j_1^{(0,d)}}\cdots\partial x_{j_{\gamma_d}^{(0,d)}}}}_{\gamma_d\text{times}}.
\end{aligned}
$$

Let $\hat{\varphi} \in \mathcal{C}^2(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi}\circ\Phi_{\widetilde{T}}^{-1}$ and $\varphi = \tilde{\varphi}\circ\Phi_T^{-1}$. Then, for $1 \le i \le d$,

$$
\begin{aligned}
\left|\frac{\partial\hat{\varphi}}{\partial\hat{x}_i}\right| &= \left|\sum_{i_1^{(1)}=1}^{d}\sum_{i_1^{(0,1)}=1}^{d} h_i[\widetilde{A}]_{i_1^{(1)}i}[A_T]_{i_1^{(0,1)}i_1^{(1)}}\frac{\partial\varphi}{\partial x_{i_1^{(0,1)}}}\right| \\
&= h_i\left|\sum_{i_1^{(1)}=1}^{d}\sum_{i_1^{(0,1)}=1}^{d}[A_T]_{i_1^{(0,1)}i_1^{(1)}}(r_i)_{i_1^{(1)}}\frac{\partial\varphi}{\partial x_{i_1^{(0,1)}}}\right| = h_i\left|\frac{\partial\varphi}{\partial r_i}\right| \\
&\le h_i\|\widetilde{A}\|_{\max}\|A_T\|_{\max}\sum_{i_1^{(1)}=1}^{d}\sum_{i_1^{(0,1)}=1}^{d}\left|\frac{\partial\varphi}{\partial x_{i_1^{(0,1)}}}\right|,
\end{aligned}
$$

and for $1 \le i,j \le d$,

$$
\begin{aligned}
\left|\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_i\partial\hat{x}_j}\right| = \Bigg|&\sum_{i_1^{(1)},j_1^{(1)}=1}^{d}\sum_{i_1^{(0,1)},j_1^{(0,1)}=1}^{d} h_ih_j[\widetilde{A}]_{i_1^{(1)}i}[\widetilde{A}]_{j_1^{(1)}j} \\
&[A_T]_{i_1^{(0,1)}i_1^{(1)}}[A_T]_{j_1^{(0,1)}j_1^{(1)}}\frac{\partial^2\varphi}{\partial x_{i_1^{(0,1)}}\partial x_{j_1^{(0,1)}}}\Bigg| = h_ih_j\left|\frac{\partial^2\varphi}{\partial r_i\partial r_j}\right|
\end{aligned}
$$

$$\leq h_i h_j \sum_{j_1^{(1)}=1}^{d} |[\widetilde{A}]_{j_1^{(1)} j}| \left| \sum_{j_1^{(0,1)}=1}^{d} [A_T]_{j_1^{(0,1)} j_1^{(1)}} \frac{\partial^2 \varphi}{\partial r_i \partial x_{j_1^{(0,1)}}} \right|$$

$$\leq h_i h_j \|\widetilde{A}\|_{\max} \|A_T\|_{\max} \sum_{j_1^{(0,1)}=1}^{d} \left| \frac{\partial^2 \varphi}{\partial r_i \partial x_{j_1^{(0,1)}}} \right|$$

$$\leq h_i h_j \|\widetilde{A}\|_{\max}^2 \|A_T\|_{\max}^2 \sum_{i_1^{(0,1)}, j_1^{(0,1)}=1}^{d} \left| \frac{\partial^2 \varphi}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} \right|.$$

10.1.3 Calculations 2

We use the following calculations in (10.3). Recall that $\tilde{x} = A_{\widetilde{T}} \hat{x}$ with $A_{\widetilde{T}} = \widetilde{A}\widehat{A}$. For any multi-indices β and γ, we have

$$\partial_{\hat{x}}^{\beta+\gamma} = \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}}$$

$$= \underbrace{\sum_{i_1^{(1)}=1}^{d} h_1 [\widetilde{A}]_{i_1^{(1)} 1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_1 [\widetilde{A}]_{i_{\beta_1}^{(1)} 1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_d [\widetilde{A}]_{i_1^{(d)} d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_d [\widetilde{A}]_{i_{\beta_d}^{(d)} d}}_{\beta_d \text{times}}$$

$$\underbrace{\sum_{j_1^{(1)}=1}^{d} h_1 [\widetilde{A}]_{j_1^{(1)} 1} \cdots \sum_{j_{\gamma_1}^{(1)}=1}^{d} h_1 [\widetilde{A}]_{j_{\gamma_1}^{(1)} 1}}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^{d} h_d [\widetilde{A}]_{j_1^{(d)} d} \cdots \sum_{j_{\gamma_d}^{(d)}=1}^{d} h_d [\widetilde{A}]_{j_{\gamma_d}^{(d)} d}}_{\gamma_d \text{times}}$$

$$\frac{\partial^{\beta_1}}{\underbrace{\partial \tilde{x}_{i_1^{1)}} \cdots \partial \tilde{x}_{i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}}} \cdots \frac{\partial^{\beta_d}}{\underbrace{\partial \tilde{x}_{i_1^{(d)}} \cdots \partial \tilde{x}_{i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}} \frac{\partial^{\gamma_1}}{\underbrace{\partial \tilde{x}_{j_1^{(1)}} \cdots \partial \tilde{x}_{j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}}} \cdots \frac{\partial^{\gamma_d}}{\underbrace{\partial \tilde{x}_{j_1^{(d)}} \cdots \partial \tilde{x}_{j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}}.$$

Let $\hat{\varphi} \in \mathcal{C}^2(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$. Then, for $1 \leq i \leq d$,

$$\left| \frac{\partial \hat{\varphi}}{\partial \hat{x}_i} \right| \leq \sum_{i_1^{(1)}=1}^{d} h_i \left| [\widetilde{A}]_{i_1^{(1)} i} \right| \left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}}} \right| \leq \begin{cases} h_i \|\widetilde{A}\|_{\max} \sum_{i_1^{(1)}=1}^{d} \left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}}} \right| & \text{or,} \\ c \sum_{i_1^{(1)}=1}^{d} \widetilde{\mathscr{H}}_{i_1^{(1)}} \left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}}} \right|, \end{cases}$$

and for $1 \leq i, j \leq d$,

$$\left| \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_i \partial \hat{x}_j} \right| = \left| \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} h_i h_j [\widetilde{A}]_{i_1^{(1)} i} [\widetilde{A}]_{j_1^{(1)} j} \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}} \partial \tilde{x}_{j_1^{(1)}}} \right|$$

$$\leq \begin{cases} h_i h_j \|\widetilde{A}\|_{\max}^2 \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} \left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}} \partial \tilde{x}_{j_1^{(1)}}} \right| & \text{or,} \\ h_j \sum_{j_1^{(1)}=1}^{d} |[\widetilde{A}]_{j_1^{(1)} j}| \left| \sum_{i_1^{(1)}=1}^{d} h_i [\widetilde{A}]_{i_1^{(1)} i} \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}} \partial \tilde{x}_{j_1^{(1)}}} \right| & \\ \quad \leq c h_j \|\widetilde{A}\|_{\max} \sum_{j_1^{(1)}=1}^{d} \sum_{i_1^{(1)}=1}^{d} \widetilde{\mathscr{H}}_{i_1^{(1)}} \left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}} \partial \tilde{x}_{j_1^{(1)}}} \right| & \text{or,} \\ c \sum_{i_1^{(1)}=1}^{d} \sum_{j_1^{(1)}=1}^{d} \widetilde{\mathscr{H}}_{i_1^{(1)}} \widetilde{\mathscr{H}}_{j_1^{(1)}} \left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_{i_1^{(1)}} \partial \tilde{x}_{j_1^{(1)}}} \right|. \end{cases}$$

10.1.4 Calculations 3

We use the following calculations in (10.1). Recall that $\hat{x} = A_{\widetilde{T}}^{-1}\tilde{x}$ with $A_{\widetilde{T}} = \widetilde{A}\widehat{A}$. For any multi-indices β, we have

$$\partial_{\tilde{x}}^{\beta} = \frac{\partial^{|\beta|}}{\partial \tilde{x}_1^{\beta_1} \cdots \partial \tilde{x}_d^{\beta_d}}$$
$$= \underbrace{\sum_{i_1^{(1)}=1}^{d} h_{i_1^{(1)}}^{-1} [\widetilde{A}^{-1}]_{i_1^{(1)}1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_{i_{\beta_1}^{(1)}}^{-1} [\widetilde{A}^{-1}]_{i_{\beta_1}^{(1)}1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_{i_1^{(d)}}^{-1} [\widetilde{A}^{-1}]_{i_1^{(d)}d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_{i_{\beta_d}^{(d)}}^{-1} [\widetilde{A}^{-1}]_{i_{\beta_d}^{(d)}d}}_{\beta_d \text{times}}$$
$$\underbrace{\frac{\partial^{\beta_1}}{\partial \hat{x}_{i_1^{1)}} \cdots \partial \hat{x}_{i_{\beta_1}^{(1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial \hat{x}_{i_1^{(d)}} \cdots \partial \hat{x}_{i_{\beta_d}^{(d)}}}}_{\beta_d \text{times}}.$$

Let $\tilde{\varphi} \in \mathcal{C}^2(\widetilde{T})$ with $\hat{\varphi} = \tilde{\varphi} \circ \Phi_{\widetilde{T}}$. Then, for $1 \leq i \leq d$,

$$\left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_i} \right| \leq \sum_{i_1^{(1)}=1}^{d} h_{i_1^{(1)}}^{-1} \left| [\widetilde{A}^{-1}]_{i_1^{(1)}i} \right| \left| \frac{\partial \hat{\varphi}}{\partial \hat{x}_{i_1^{(1)}}} \right| \leq \|\widetilde{A}^{-1}\|_{\max} \sum_{i_1^{(1)}=1}^{d} h_{i_1^{(1)}}^{-1} \left| \frac{\partial \hat{\varphi}}{\partial \hat{x}_{i_1^{(1)}}} \right|,$$

and for $1 \leq i, j \leq d$,

$$\begin{aligned} \left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_i \partial \tilde{x}_j} \right| &= \left| \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} h_{i_1^{(1)}}^{-1} h_{j_1^{(1)}}^{-1} [\widetilde{A}^{-1}]_{i_1^{(1)}i} [\widetilde{A}^{-1}]_{j_1^{(1)}j} \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_{i_1^{(1)}} \partial \hat{x}_{j_1^{(1)}}} \right| \\ &\leq \|\widetilde{A}^{-1}\|_{\max}^2 \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} h_{i_1^{(1)}}^{-1} h_{j_1^{(1)}}^{-1} \left| \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_{i_1^{(1)}} \partial \hat{x}_{j_1^{(1)}}} \right|. \end{aligned}$$

10.2 Main Results

Lemma 10.2. Let $m, \ell \in \mathbb{N}_0$ with $\ell \geq m$. Let $\beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ and $\gamma := (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$ be multi-indices with $|\beta| = m$ and $|\gamma| = \ell - m$. Then, for any $\hat{\varphi} \in W^{m,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$, it holds that

$$|\tilde{\varphi}|_{W^{m,p}(\widetilde{T})} \leq c |\det(A_{\widetilde{T}})|^{\frac{1}{p}} \|\widetilde{A}^{-1}\|_2^m \left(\sum_{|\beta|=m} (h^{-\beta})^p \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|_{L^p(\widehat{T})}^p \right)^{\frac{1}{p}} \quad \text{if } p \in [1, \infty), \tag{10.1a}$$

$$|\tilde{\varphi}|_{W^{m,\infty}(\widetilde{T})} \leq c \|\widetilde{A}^{-1}\|_2^m \max_{|\beta|=m} \left(h^{-\beta} \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|_{L^\infty(\widehat{T})} \right) \quad \text{if } p = \infty. \tag{10.1b}$$

Let $p \in [0, \infty]$. Furthermore, for any $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$ and $\varphi = \tilde{\varphi} \circ \Phi_T^{-1}$, it holds that

$$\|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m h^{\beta} \sum_{|\epsilon|=|\gamma|} h^{\varepsilon} |\partial_r^{\epsilon} \varphi|_{W^{m,p}(T)}. \tag{10.2}$$

In particular, if Condition 10.1 is imposed, then for any $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$, it holds that

$$\|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m h^{\beta} \sum_{|\epsilon|=|\gamma|} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\varepsilon} \tilde{\varphi}|_{W^{m,p}(\widetilde{T})}. \tag{10.3}$$

Here, for $p = \infty$ and any positive real x, $x^{-\frac{1}{p}} = 1$.

Proof. We divide the proof into three parts.

Proof of (10.1)**.** Let $p \in [1,\infty)$. Because the space $\mathcal{C}^m(\widehat{T})$ is dense in the space $W^{m,p}(\widehat{T})$, we show (10.1) for $\hat{\varphi} \in \mathcal{C}^m(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$. Through the calculation (Section 10.1.4) and (1.1), we have for any multi-index γ with $|\gamma| = m$,

$$|\partial_{\tilde{x}}^{\gamma} \tilde{\varphi}| \leq c \|\widetilde{A}^{-1}\|_2^m \sum_{|\beta|=m} h^{-\beta} |\partial_{\hat{x}}^{\beta} \hat{\varphi}|.$$

Through a change in a variable, we obtain

$$|\tilde{\varphi}|^p_{W^{m,p}(\widetilde{T})} = \sum_{|\gamma|=m} \|\partial_{\tilde{x}}^{\gamma} \tilde{\varphi}\|^p_{L^p(\widetilde{T})} \leq c |\det(A_{\widetilde{T}})| \|\widetilde{A}^{-1}\|_2^{mp} \sum_{|\beta|=m} (h^{-\beta})^p \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|^p_{L^p(\widehat{T})},$$

which leads to the inequality (10.1a). We consider the case that $p = \infty$. A function $\hat{\varphi} \in W^{m,\infty}(\widehat{T})$ belongs to the space $W^{m,p}(\widehat{T})$ for any $p \in [1,\infty)$. It therefore holds that $\tilde{\varphi} \in W^{m,p}(\widetilde{T})$ for any $p \in [1,\infty)$ and, from (1.4),

$$\begin{aligned}
\|\partial_{\tilde{x}}^{\gamma} \tilde{\varphi}\|_{L^p(\widetilde{T})} &\leq |\tilde{\varphi}|_{W^{|\gamma|,p}(\widetilde{T})} \\
&\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{p}} \|\widetilde{A}^{-1}\|_2^m \left(\sum_{|\beta|=|\gamma|} (h^{-\beta})^p \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|^p_{L^p(\widehat{T})} \right)^{\frac{1}{p}} \\
&\leq c \left(\sup_{1 \leq p} |\det(A_{\widetilde{T}})|^{\frac{1}{p}} \right) \|\widetilde{A}^{-1}\|_2^m \sum_{|\beta|=|\gamma|} h^{-\beta} \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\leq c \left(\sup_{1 \leq p} |\det(A_{\widetilde{T}})|^{\frac{1}{p}} \right) \|\widetilde{A}^{-1}\|_2^m \sum_{|\beta|=|\gamma|} h^{-\beta} \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|_{L^\infty(\widehat{T})} < +\infty, \qquad (10.4)
\end{aligned}$$

for multi-index $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq m$. This implies that the function $\partial_{\tilde{x}}^{\gamma} \tilde{\varphi}$ is in the space $L^\infty(\widetilde{T})$ for each $|\gamma| \leq m$. We therefore have $\tilde{\varphi} \in W^{m,\infty}(\widetilde{T})$. By passing to the limit $p \to \infty$ in (10.4) and because $\lim_{p\to\infty} \|\cdot\|_{L^p(\widetilde{T})} = \|\cdot\|_{L^\infty(\widetilde{T})}$, we have

$$|\tilde{\varphi}|_{W^{m,\infty}(\widetilde{T})} \leq c \|\widetilde{A}^{-1}\|_2^m \max_{|\beta|=m} \left(h^{-\beta} \|\partial_{\hat{x}}^{\beta} \hat{\varphi}\|_{L^\infty(\widehat{T})} \right),$$

which is (10.1b).

Proof of (10.3)**.** Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}_0^d$ and $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}_0^d$ be multi-indies with $|\varepsilon| = |\gamma|$ and $|\delta| = |\beta|$. Let $p \in [1,\infty)$. Because the space $\mathcal{C}^\ell(\widehat{T})$ is dense in the space $W^{\ell,p}(\widehat{T})$, we show (10.3) for $\hat{\varphi} \in \mathcal{C}^\ell(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$. Through a simple calculation, we have

$$\begin{aligned}
&|\partial_{\hat{x}}^{\beta+\gamma} \hat{\varphi}| = \left| \frac{\partial^\ell \hat{\varphi}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \right| \\
&\leq c h^\beta \|\widetilde{A}\|_{\max}^{|\beta|} \underbrace{\sum_{i_1^{(1)}=1}^{d} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d}}_{\beta_d \text{times}} \underbrace{\sum_{j_1^{(1)}=1}^{d} \cdots \sum_{j_{\gamma_1}^{(1)}=1}^{d}}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^{d} \cdots \sum_{j_{\gamma_d}^{(d)}=1}^{d}}_{\gamma_d \text{times}} \\
&\underbrace{\widetilde{\mathscr{H}}_{j_1^{(1)}} \cdots \widetilde{\mathscr{H}}_{j_{\varepsilon_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots \underbrace{\widetilde{\mathscr{H}}_{j_1^{(d)}} \cdots \widetilde{\mathscr{H}}_{j_{\varepsilon_d}^{(d)}}}_{\gamma_d \text{times}} \\
&\left| \underbrace{\frac{\partial^{\beta_1}}{\partial \tilde{x}_{i_1^{(1)}} \cdots \partial \tilde{x}_{i_{\beta_1}^{(1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial \tilde{x}_{i_1^{(d)}} \cdots \partial \tilde{x}_{i_{\beta_d}^{(d)}}}}_{\beta_d \text{times}} \underbrace{\frac{\partial^{\gamma_1}}{\partial \tilde{x}_{j_1^{(1)}} \cdots \partial \tilde{x}_{j_{\gamma_1}^{(1)}}}}_{\gamma_1 \text{times}} \cdots \underbrace{\frac{\partial^{\gamma_d}}{\partial \tilde{x}_{j_1^{(d)}} \cdots \partial \tilde{x}_{j_{\gamma_d}^{(d)}}}}_{\gamma_d \text{times}} \tilde{\varphi} \right|
\end{aligned}$$

$$\leq ch^{\beta}\|\widetilde{A}\|_{\max}^{|\beta|}\sum_{|\delta|=|\beta|}\sum_{|\varepsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon}}|\partial_{\tilde{x}}^{\delta}\partial_{\tilde{x}}^{\varepsilon}\tilde{\varphi}|.$$

We then have, using (1.1),

$$\begin{aligned}\int_{\widehat{T}}|\partial_{\hat{x}}^{\beta}\partial_{\hat{x}}^{\gamma}\hat{\varphi}|^{p}d\hat{x} &\leq c\|\widetilde{A}\|_{2}^{mp}h^{\beta p}\sum_{|\delta|=|\beta|}\sum_{|\epsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon p}}\int_{\widehat{T}}|\partial_{\tilde{x}}^{\delta}\partial_{\tilde{x}}^{\varepsilon}\tilde{\varphi}|^{p}d\hat{x}\\ &= c|\det(A_{\widetilde{T}})|^{-1}\|\widetilde{A}\|_{2}^{mp}h^{\beta p}\sum_{|\delta|=|\beta|}\sum_{|\epsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon p}}\int_{\widetilde{T}}|\partial_{\tilde{x}}^{\delta}\partial_{\tilde{x}}^{\varepsilon}\tilde{\varphi}|^{p}d\tilde{x}.\end{aligned}$$

Therefore, using (1.4), we have

$$\|\partial_{\hat{x}}^{\beta}\partial_{\hat{x}}^{\gamma}\hat{\varphi}\|_{L^{p}(\widehat{T})}\leq c|\det(A_{\widetilde{T}}))|^{-\frac{1}{p}}\|\widetilde{A}\|_{2}^{m}h^{\beta}\sum_{|\epsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon}}|\partial_{\tilde{x}}^{\epsilon}\tilde{\varphi}|_{W^{m,p}(\widetilde{T})},$$

which concludes (10.3). We consider the case that $p=\infty$. A function $\varphi\in W^{\ell,\infty}(T)$ belongs to the space $W^{\ell,p}(T)$ for any $p\in[1,\infty)$. It therefore holds that $\hat{\varphi}\in W^{\ell,p}(\widehat{T})$ for any $p\in[1,\infty)$ and thus

$$\begin{aligned}\|\partial_{\hat{x}}^{\beta}\partial_{\hat{x}}^{\gamma}\hat{\varphi}\|_{L^{p}(\widehat{T})} &\leq c|\det(A_{\widetilde{T}})|^{-\frac{1}{p}}\|\widetilde{A}\|_{2}^{m}h^{\beta}\sum_{|\epsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon}}|\partial_{\tilde{x}}^{\epsilon}\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}\\ &\leq c\|\widetilde{A}\|_{2}^{m}h^{\beta}\sum_{|\epsilon|=|\gamma|}\widetilde{\mathscr{H}^{\varepsilon}}|\partial_{\tilde{x}}^{\epsilon}\tilde{\varphi}|_{W^{m,\infty}(\widetilde{T})}<\infty.\end{aligned}\tag{10.5}$$

This implies that the function $\partial_{\hat{x}}^{\beta}\partial_{\hat{x}}^{\gamma}\hat{\varphi}$ is in the space $L^{\infty}(\widehat{T})$. Inequality (10.3) for $p=\infty$ is obtained by passing to the limit $p\to\infty$ in (10.5) on the basis that $\lim_{p\to\infty}\|\cdot\|_{L^{p}(\widehat{T})}=\|\cdot\|_{L^{\infty}(\widehat{T})}$.

Proof of (10.2)**.** We follow the proof of (10.3). Let $p\in[1,\infty)$. Because the space $\mathcal{C}^{\ell}(\widehat{T})$ is dense in the space $W^{\ell,p}(\widehat{T})$, we show (10.2) for $\hat{\varphi}\in\mathcal{C}^{\ell}(\widehat{T})$ with $\tilde{\varphi}=\hat{\varphi}\circ\Phi_{\widetilde{T}}^{-1}$ and $\varphi=\tilde{\varphi}\circ\Phi_{T}^{-1}$, it holds that, for $1\leq i,k\leq d$,

$$\left|\partial_{\hat{x}}^{\beta+\gamma}\hat{\varphi}\right|\leq ch^{\beta}\|\widetilde{A}\|_{\max}^{|\beta|}\|A_{T}\|_{\max}^{|\beta|}\sum_{|\delta|=|\beta|}\sum_{|\varepsilon|=|\gamma|}h^{\varepsilon}\left|\partial_{x}^{\delta}\partial_{r}^{\varepsilon}\varphi\right|.$$

Using (6.8c) and (1.1), we obtain (10.2) for $p\in[1,\infty]$ by an argument analogous to the proof of (10.3). $\square$

Remark 10.3. In inequality (10.3), it is possible to obtain the estimates in T by specifically determining the matrix $\mathcal{A}_{T}$.

Let $\ell=2$, $m=1$ and $p=q=2$. Recall that

$$\Phi_{T}:\widetilde{T}\ni\tilde{x}\mapsto x=\Phi_{T}(\tilde{x})=A_{T}\tilde{x}+b_{T}\in T.$$

For $\tilde{\varphi}\in\mathcal{C}^{2}(\widetilde{T})$ with $\varphi=\tilde{\varphi}\circ\Phi_{T}^{-1}$ and $1\leq i,j\leq d$, we have

$$\left|\frac{\partial^{2}\tilde{\varphi}}{\partial\tilde{x}_{i}\partial\tilde{x}_{j}}(x)\right|=\left|\sum_{i_{1}^{(1)},j_{1}^{(1)}=1}^{2}[A_{T}]_{i_{1}^{(1)}i}[A_{T}]_{j_{1}^{(1)}j}\frac{\partial^{2}\varphi}{\partial x_{i_{1}^{(1)}}\partial x_{j_{1}^{(1)}}}(x)\right|.$$

Let $d=2$. We define the matrix A_{T} as

$$A_{T}:=\begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}.$$

Because $\|A_T\|_{\max} = 1$, we have

$$\left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_i \partial \tilde{x}_j}(x) \right| \leq \left| \frac{\partial^2 \varphi}{\partial x_{i+1} \partial x_{j+1}}(x) \right|,$$

where the indices i, $i+1$ and j, $j+1$ have to be understood mod 2. Because $|\det(A_T)| = 1$, it holds that

$$\left\| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_i \partial \tilde{x}_j} \right\|_{L^2(\widetilde{T})} \leq \left\| \frac{\partial^2 \varphi}{\partial x_{i_{i+1}} \partial x_{j+1}} \right\|_{L^2(T)}.$$

We then have

$$\sum_{j=1}^{2} \widetilde{\mathscr{H}_j} \left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_j} \right|_{H^1(\widetilde{T})} \leq \sum_{j=1}^{2} \widetilde{\mathscr{H}_j} \left| \frac{\partial \varphi}{\partial x_{j+1}} \right|_{H^1(T)},$$

where the indices j, $j+1$ have to be understood mod 2.

We define the matrix A_T as

$$A_T := \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}.$$

We then have

$$\left| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_i \partial \tilde{x}_j}(x) \right| \leq \frac{1}{\sqrt{2}} \sum_{i_1^{(1)}, j_1^{(1)}=1}^{2} \left| \frac{\partial^2 \varphi}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}}(x) \right|,$$

which leads to

$$\left\| \frac{\partial^2 \tilde{\varphi}}{\partial \tilde{x}_i \partial \tilde{x}_j} \right\|^2_{L^2(\widetilde{T})} \leq c \sum_{i_1^{(1)}, j_1^{(1)}=1}^{2} \left\| \frac{\partial^2 \varphi}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \right\|^2_{L^2(T)} \leq c |\varphi|^2_{H^2(T)}.$$

We then have, using (1.4),

$$\sum_{j=1}^{2} \widetilde{\mathscr{H}_j} \left| \frac{\partial \varphi^s}{\partial x_j^s} \right|_{H^1(T^s)} \leq \sum_{j=1}^{2} \widetilde{\mathscr{H}_j} |\varphi|_{H^2(T)} \leq c h_T |\varphi|_{H^2(T)}.$$

In this case, anisotropic interpolation error estimates cannot be obtained.

Remark 10.4. We consider a general case. Let $p = q = 2$. The space $\mathcal{C}^1(\widetilde{T})$ is dense in the space $H^1(\widetilde{T})$. For $\tilde{\varphi} \in \mathcal{C}^1(\widetilde{T})$ with $\varphi = \tilde{\varphi} \circ \Phi_T^{-1}$ and $1 \leq i \leq d$, we have

$$\left| \frac{\partial \tilde{\varphi}}{\partial \tilde{x}_i}(\tilde{x}) \right| = \left| \sum_{i_1^{(1)}=1}^{d} [A_T]_{i_1^{(1)} i} \frac{\partial \varphi}{\partial x_{i_1^{(1)}}}(x) \right|.$$

Let $d = 2$. We define a rotation matrix A_T as

$$A_T := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where θ denotes the angle. We then have

$$\begin{aligned}
\left|\frac{\partial \tilde{\varphi}}{\partial \tilde{x}_1}(\tilde{x})\right| &= \left|\cos\theta \frac{\partial \varphi}{\partial x_1}(x) + \sin\theta \frac{\partial \varphi}{\partial x_2}(x)\right|, \\
\left|\frac{\partial \tilde{\varphi}}{\partial \tilde{x}_2}(\tilde{x})\right| &= \left|-\sin\theta \frac{\partial \varphi}{\partial x_1}(x) + \cos\theta \frac{\partial \varphi}{\partial x_2}(x)\right|.
\end{aligned}$$

If $|\sin\theta| \leq c\frac{\widetilde{\mathscr{H}_2}}{\widetilde{\mathscr{H}_1}}$ and $\widetilde{\mathscr{H}_2} \leq c\widetilde{\mathscr{H}_1}$, we can deduce

$$\begin{aligned}
\left|\frac{\partial \tilde{\varphi}}{\partial \tilde{x}_1}(\tilde{x})\right| &\leq \left|\frac{\partial \varphi}{\partial x_1}(x)\right| + c\frac{\widetilde{\mathscr{H}_2}}{\widetilde{\mathscr{H}_1}}\left|\frac{\partial \varphi}{\partial x_2}(x)\right|, \\
\left|\frac{\partial \tilde{\varphi}}{\partial \tilde{x}_2}(\tilde{x})\right| &\leq c\left|\frac{\partial \varphi}{\partial x_1}(x)\right| + \left|\frac{\partial \varphi}{\partial x_2}(x)\right|.
\end{aligned}$$

As $|\det(A_T)| = 1$, it holds that for $i = 1, 2$,

$$\widetilde{\mathscr{H}_i}\left\|\frac{\partial \tilde{\varphi}}{\partial \tilde{x}_i}\right\|_{L^2(\widetilde{T})} \leq c\sum_{j=1}^{2} \widetilde{\mathscr{H}_j}\left\|\frac{\partial \varphi}{\partial x_j}\right\|_{L^2(T)}.$$

Lemma 10.5. Let Φ_T be the affine mapping defined in (5.2). Let $s \geq 0$ and $1 \leq p \leq \infty$. There exists positive constants c_1 and c_2 such that, for all $T \in \mathbb{T}_h$ and $\varphi \in W^{s,p}(T)$,

$$c_1|\varphi|_{W^{s,p}(T)} \leq |\tilde{\varphi}|_{W^{s,p}(\widetilde{T})} \leq c_2|\varphi|_{W^{s,p}(T)}, \tag{10.6}$$

with $\tilde{\varphi} = \varphi \circ \Phi_T$.

Proof. The following inequalities are found in [16, Lemma 1.101]. There exists a positive constant c such that, for all $T \in \mathbb{T}_h$ and $\varphi \in W^{s,p}(T)$,

$$|\tilde{\varphi}|_{W^{s,p}(\widetilde{T})} \leq c\|A_T\|_2^s|\det(A_T)|^{-\frac{1}{p}}|\varphi|_{W^{s,p}(T)}, \tag{10.7}$$

$$|\varphi|_{W^{s,p}(T)} \leq c\|A_T^{-1}\|_2^s|\det(A_T)|^{\frac{1}{p}}|\tilde{\varphi}|_{W^{s,p}(\widetilde{T})}. \tag{10.8}$$

Because the length of all edges of a simplex and measure of the simplex is not changed by a rotation and mirror imaging matrix and $A_T, A_T^{-1} \in O(d)$,

$$|\det(A_T)| = \frac{|T|_d}{|\widetilde{T}|_d} = 1, \quad \|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1. \tag{10.9}$$

From (10.7), (10.8), and (10.9), we obtain the desired inequality (10.6). $\square$

11 Classical Interpolation Error Estimates

11.1 Local Interpolation Error Estimates

The following theorem is another representation of the standard interpolation error estimates, e.g., see [16, Theorem 1.103].

Theorem 11.1. Let $1 \leq p \leq \infty$ and assume that there exists a nonnegative integer k such that

$$\mathbb{P}^k \subset \widehat{P} \subset W^{k+1,p}(\widehat{T}) \subset V(\widehat{T}).$$

Let ℓ $(0 \leq \ell \leq k)$ be such that $W^{\ell+1,p}(\widehat{T}) \subset V(\widehat{T})$ with continuous embedding. Furthermore, assume that $\ell, m \in \mathbb{N} \cup \{0\}$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell + 1$ and

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}). \tag{11.1}$$

It holds that, for any $m \in \{0, \dots, \ell+1\}$ and any $\varphi \in W^{\ell+1,p}(T)$,

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq C_*^I |T|_d^{\frac{1}{q}-\frac{1}{p}} \left(\frac{h_{\max}}{h_{\min}} \right)^m \left(\frac{H_T}{h_T} \right)^m h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)}, \tag{11.2}$$

where C_*^I is a positive constant independent of h_T and H_T, and the parameters $h_{\max}$ and $h_{\min}$ are defined by (8.1), that is,

$$h_{\max} = \max\{h_1, \dots, h_d\}, \quad h_{\min} = \min\{h_1, \dots, h_d\}.$$

Proof. Let $\hat{\varphi} \in W^{\ell+1,p}(\widehat{T})$. Because $0 \leq \ell \leq k$, $\mathbb{P}^\ell \subset \mathbb{P}^k \subset \widehat{P}$. Therefore, for any $\hat{\eta} \in \mathbb{P}^\ell$, we have $I_{\widehat{T}} \hat{\eta} = \hat{\eta}$. Using (9.3) and (11.1), we obtain

$$\begin{aligned} |\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} &\leq |\tilde{\varphi} - \hat{\eta}|_{W^{m,q}(\widehat{T})} + |I_{\widehat{T}}(\hat{\eta} - \hat{\varphi})|_{W^{m,q}(\widehat{T})} \\ &\leq c \|\tilde{\varphi} - \hat{\eta}\|_{W^{\ell+1,p}(\widehat{T})}, \end{aligned}$$

where we used the stability of the interpolation operator $I_{\widehat{T}}$, that is,

$$|I_{\widehat{T}}(\hat{\eta} - \hat{\varphi})|_{W^{m,q}(\widehat{T})} \leq \sum_{i=1}^{n_0} |\hat{\chi}_i(\hat{\eta} - \hat{\varphi})| |\hat{\theta}_i|_{W^{m,q}(\widehat{T})} \leq c \|\hat{\eta} - \tilde{\varphi}\|_{W^{\ell+1,p}(\widehat{T})}.$$

Using the classic Bramble–Hilbert–type lemma (e.g., [10, Lemma 4.3.8]), we obtain

$$|\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} \leq c \inf_{\hat{\eta} \in \mathbb{P}^\ell} \|\hat{\eta} - \tilde{\varphi}\|_{W^{\ell+1,p}(\widehat{T})} \leq c |\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})}. \tag{11.3}$$

The inequalities (10.6), (10.1), (1.4), and (11.3) yield

$$\begin{aligned} |\varphi - I_T \varphi|_{W^{m,q}(T)} &\leq c |\tilde{\varphi} - I_{\widetilde{T}} \tilde{\varphi}|_{W^{m,q}(\widetilde{T})} \\ &\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m \left(\sum_{|\beta|=m} (h^{-\beta})^q \|\partial^\beta (\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})}^q \right)^{\frac{1}{q}} \\ &\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m \max\{h_1^{-1}, \dots, h_d^{-1}\}^{|\beta|} |\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} \\ &\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m h_{\min}^{-|\beta|} |\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})}. \end{aligned} \tag{11.4}$$

Using the inequalities (1.4), (10.6) and (10.2), we have

$$\begin{aligned} |\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})} &\leq \sum_{|\gamma|=\ell+1-m} \sum_{|\beta|=m} \|\partial^\beta \partial^\gamma \hat{\varphi}\|_{L^p(\widehat{T})} \\ &\leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \sum_{|\gamma|=\ell+1-m} \sum_{|\beta|=m} h^\beta \sum_{|\epsilon|=|\gamma|} h^\varepsilon |\partial_r^\epsilon \varphi|_{W^{m,p}(T)} \\ &\leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \max\{h_1, \dots, h_d\}^{|\beta|} h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)} \\ &\leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m h_{\max}^{|\beta|} h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)}. \end{aligned} \tag{11.5}$$

From (11.4) and (11.5) together with (6.8) and (6.9), we have the desired estimate (11.2). □

Remark 11.2. We introduced the estimate (1.10), a variant of the Bramble–Hilbert lemma. However, because we prove estimate (11.3) with $p = q$ using the reference element, it is sufficient to use the standard estimate (e.g., [15, 10]) to achieve our goal.

Example 11.3. As the examples in [16, Example 1.106], we get local interpolation error estimates for a Lagrange finite element of degree k, a more general finite element, and the Crouzeix–Raviart finite element with $k = 1$.

1. For a Lagrange finite element of degree k, we set $V(\widehat{T}) := \mathcal{C}^0(\widehat{T})$. The condition on ℓ in Theorem 11.1 is $\frac{d}{p} - 1 < \ell \leq k$ because $W^{\ell+1,p}(\widehat{T}) \subset \mathcal{C}^0(\widehat{T})$ if $\ell + 1 > \frac{d}{p}$ according to the Sobolev imbedding theorem.

2. For a general finite element with $V(\widehat{T}) := \mathcal{C}^t(\widehat{T})$ and $t \in \mathbb{N}$. The condition on ℓ in Theorem 11.1 is $\frac{d}{p} - 1 + t < \ell \leq k$. When $t = 1$, there is a Hermite finite element.

3. For the Crouzeix–Raviart finite element with $k = 1$, we set $V(\widehat{T}) := W^{1,1}(\widehat{T})$. The condition on ℓ in Theorem 11.1 is $0 \leq \ell \leq 1$.

11.2 Examples of Anisotropic Elements

When $m = \ell = 1$ and $q = p$ in (11.2) of Theorem 11.1, the estimate is written as

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq C_*^I \frac{h_{\max}}{h_{\min}} \frac{H_T}{h_T} h_T |\varphi|_{W^{2,p}(T)}. \tag{11.6}$$

Let $T \subset \mathbb{R}^2$ be a triangle. As described in Section 8.1, an isotropic mesh element has equal or nearly equal edge lengths and angles, resulting in a balanced shape. Then, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq c h_T |\varphi|_{W^{2,p}(T)}. \tag{11.7}$$

We introduce typical examples of the quantities $\frac{h_{\max}}{h_{\min}}$ and $\frac{H_T}{h_T}$ in anisotropic elements. We considered the following five anisotropic elements as in Section 8.2: Let $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon, \delta, \gamma \in \mathbb{R}$.

Example 11.4 (Right-angled triangle)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = s$, $h_2 = s^\varepsilon$ and $h_T = \sqrt{s^2 + s^{2\varepsilon}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \leq s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \qquad \frac{H_T}{h_T} = 2.$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq 2C_*^I s^{2-\varepsilon} |\varphi|_{W^{2,p}(T)}.$$

When $\varepsilon > 2$, this implies that the estimate diverges as $s \to 0$. However, new interpolation error estimates will be shown to converge, see Example 14.3.

Example 11.5 (Dagger)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ and $h_T = s$; i.e.,

$$\begin{aligned}
\frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \leq c s^{1-\varepsilon} \to \infty \quad \text{as } s \to 0, \\
\frac{H_T}{h_T} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \leq c.
\end{aligned}$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq cs^{2-\varepsilon}|\varphi|_{W^{2,p}(T)}.$$

When $\varepsilon > 2$, this implies that the estimate diverges as $s \to 0$. However, new interpolation error estimates will be shown to converge, see Example 14.4.

Example 11.6 (Blade)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = h_2 = \sqrt{s^2 + s^{2\varepsilon}}$ and $h_T = 2s$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_T}{h_T} = \frac{s^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \to \infty \quad \text{as } s \to 0.$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq cs^{2-\varepsilon}|\varphi|_{W^{2,p}(T)}.$$

When $\varepsilon > 2$, this implies that the estimate diverges as $s \to 0$. In this case, the interpolation error estimate can not be improved, , see Example 14.5.

Example 11.7 (Dagger)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ and $h_T = s$; i.e.,

$$\begin{aligned}
\frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \leq cs^{1-\delta} \to \infty \quad \text{as } s \to 0, \\
\frac{H_T}{h_T} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2}s^{1+\varepsilon}} \leq cs^{\delta-\varepsilon} \to \infty \quad \text{as } s \to 0.
\end{aligned}$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq cs^{2-\varepsilon}|\varphi|_{W^{2,p}(T)}.$$

When $\varepsilon > 2$, this implies that the estimate diverges as $s \to 0$. In this case, the interpolation error estimate can not be improved, see Example 14.6.

Example 11.8 (Right-angled triangle)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = s$, $h_2 = \delta s$ and $h_T = s\sqrt{1+\delta^2}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{1}{\delta}, \quad \frac{H_T}{h_T} = 2.$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta}s|\varphi|_{W^{2,p}(T)}.$$

This implies that the estimate converges as $s \to 0$ and the error may be large. However, new interpolation error estimates remove the factor $\frac{1}{\delta}$, see Example 14.7.

Example 11.9 (Blade)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = h_2 = s\sqrt{1+\delta^2}$ and $h_T = 2s$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_T}{h_T} = \frac{s^2(1+\delta^2)}{\delta s^2} \leq \frac{c}{\delta},$$

In this case, the estimate (11.6) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta}s|\varphi|_{W^{2,p}(T)}.$$

This implies that the estimate converges as $s \to 0$ and the error may be large. Unfortunately, new interpolation error estimates do not remove the factor $\frac{1}{\delta}$, see Example 14.8.

Example 11.10 ($\mathbb{P}^1$ + bubble finite element in $\mathbb{R}^2$)**.** We give a numerical example which is not optimal in the usual sense. Let $T \subset \mathbb{R}^2$ be the triangle with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, $p_3 := (0, s^\varepsilon)^\top$ (Example 11.4), where $s := \frac{1}{N}$, $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, $1 < \varepsilon \leq 2$. Let p_4 be the barycentre of T.

Using the barycentric coordinates $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, 3$, we define the local basis functions as

$$\theta_4(x) := 27\lambda_1(x)\lambda_2(x)\lambda_3(x), \quad \theta_i(x) := \lambda_i(x) - \frac{1}{3}\theta_4(x), \quad i = 1, 2, 3.$$

The interpolation operator I_T^b defined by

$$I_T^b : H^2(T) \ni \varphi \mapsto I_T^b\varphi := \sum_{i=1}^{4} \varphi(x_i)\theta_i \in \operatorname{span}\{\theta_1, \theta_2, \theta_3, \theta_4\}.$$

From Theorem 11.1, we have

$$|\varphi - I_T^b\varphi|_{H^1(T)} \leq c h_T^{2-\varepsilon} |\varphi|_{H^2(T^s)} \quad \forall \varphi \in H^2(T).$$

Let φ be a function such that

$$\varphi(x, y) := 2x^2 - xy + 3y^2.$$

We compute the convergence order concerning the H^1 norm defined by

$$Err_s^b(H^1) := \frac{|\varphi - I_T^b\varphi|_{H^1(T)}}{|\varphi|_{H^2(T)}},$$

for the cases: $\varepsilon = 1.5$ (Table 13) and $\varepsilon = 2.0$ (Table 14). The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log\left(\frac{Err_t^b(H^1)}{Err_{t/2}^b(H^1)}\right).$$

Table 11: Error of the local interpolation operator ($\varepsilon = 1.5$)

N	s	$Err_s^b(H^1)$	r
128	7.8125e-03	2.9951e-02	
256	3.9062e-03	2.1101e-02	5.0529e-01
512	1.9531e-03	1.4874e-02	5.0452e-01
1024	9.7656e-04	1.0491e-02	5.0364e-01

Table 12: Error of the local interpolation operator ($\varepsilon = 2.0$)

N	s	$Err_s^b(H^1)$	r
128	7.8125e-03	3.3397e-01	
256	3.9062e-03	3.3366e-01	1.3398e-03
512	1.9531e-03	3.3350e-01	6.9198e-04
1024	9.7656e-04	3.3341e-01	3.8939e-04

Remark 11.11. If we are concerned with anisotropic elements, it would be desirable to remove the quantity $h_{\max}/h_{\min}$ from estimate (11.2).

12 Anisotropic Interpolation on the Reference Element

We introduce estimates on the reference element due to [3, 2] to obtain anisotropic interpolation error estimates.

For the reference element $\widehat{T}$ defined in Sections 5.1 and 5.1, let the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the reference finite element with associated normed vector space $V(\widehat{T})$.

Theorem 12.1. Let $I_{\widehat{T}} : \mathcal{C}(\widehat{T}) \to \mathbb{P}^k(\widehat{T})$ be a linear operator. Fix $m, \ell \in \mathbb{N}$ and $p, q \in [1, \infty]$ such that $0 \le m \le \ell \le k+1$ and

$$W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T}). \tag{12.1}$$

Let β be a multi-index with $|\beta| = m$. We set $j := \dim(\partial_{\hat{x}}^{\beta} \mathcal{P}^k)$. Assume that there exist linear functionals $\mathscr{F}_i$, $i = 1, \ldots, j$, such that

$$\mathscr{F}_i \in W^{\ell-m,p}(\widehat{T})', \quad \forall i = 1, \ldots, j, \tag{12.2a}$$

$$\mathscr{F}_i(\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})) = 0 \quad \forall i = 1, \ldots, j, \quad \forall \hat{\varphi} \in \mathcal{C}(\widehat{T}) : \ \partial_{\hat{x}}^{\beta}\hat{\varphi} \in W^{\ell-m,p}(\widehat{T}), \tag{12.2b}$$

$$\hat{\eta} \in \mathbb{P}^k, \quad \mathscr{F}_i(\partial_{\hat{x}}^{\beta}\hat{\eta}) = 0 \quad \forall i = 1, \ldots, j \quad \Rightarrow \quad \partial^{\beta}\hat{\eta} = 0. \tag{12.2c}$$

It holds that for all $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ with $\partial^{\beta}\hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$,

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \le C^F |\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{12.3}$$

Proof. We follow [2, Lemma 2.2].

For all $\hat{\eta} \in \mathbb{P}^{\ell-1}$, we have

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \le \|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\partial_{\hat{x}}^{\beta}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})}. \tag{12.4}$$

Note that $\hat{\eta} - I_{\widehat{T}}\hat{\varphi} \in \mathbb{P}^k$, because $\ell \le k+1$. That is, $\partial_{\hat{x}}^{\beta}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}) \in \partial_{\hat{x}}^{\beta}\mathbb{P}^k$. Because the polynomial spaces are finite-dimensional all norms are equivalent, that is, by the fact $\sum_{i=1}^{j} |\mathscr{F}_i(\hat{\eta})|$ is a norm on $\partial_{\hat{x}}^{\beta}\mathbb{P}^k$, together with (12.2a), (12.2b) and (12.2c), we have for any $\hat{\eta} \in \mathbb{P}^{\ell-1}$,

$$\begin{aligned}
\|\partial_{\hat{x}}^{\beta}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\le c \sum_{i=1}^{j} |\mathscr{F}_i(\partial_{\hat{x}}^{\beta}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}))| = c \sum_{i=1}^{j} |\mathscr{F}_i(\partial_{\hat{x}}^{\beta}(\hat{\eta} - \hat{\varphi}))| \\
&\le c \|\partial_{\hat{x}}^{\beta}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})}.
\end{aligned}$$

Using (12.4) and (15.14), it holds that for any $\hat{\eta} \in \mathbb{P}^{\ell-1}$,

$$\begin{aligned}
\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\le \|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\partial_{\hat{x}}^{\beta}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \\
&\le c \|\partial_{\hat{x}}^{\beta}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})}.
\end{aligned}$$

By Lemma 1.8, we have

$$\begin{aligned}
\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\le c \inf_{\hat{\eta} \in \mathbb{P}^{\ell-1}} \|\partial_{\hat{x}}^{\beta}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})} \\
&\le c |\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}.
\end{aligned}$$

□

Remark 12.2. Note that it is not required $I_{\widehat{T}}\hat{\eta} = \hat{\eta}$ for any $\hat{\eta} \in \mathbb{P}^{\ell-1}$.

13 Remarks on Anisotropic Interpolation Analysis

Let $\widehat{T} \subset \mathbb{R}^2$ be the reference element defined in Section 5.1. We set $k = m = 1$, $\ell = 2$, and $p = 2$. For $\hat{\varphi} \in H^2(\widehat{T})$, we set $\tilde{\varphi} = \hat{\varphi} \circ \Phi_{\widetilde{T}}^{-1}$ and $\varphi = \tilde{\varphi} \circ \Phi_T^{-1}$. Inequalities (10.1) and (10.6) yield

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c |\det(A_{\widetilde{T}})|^{\frac{1}{2}} \|\widetilde{A}^{-1}\|_2 \left(\sum_{i=1}^{2} h_i^{-2} \|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}^2 \right)^{\frac{1}{2}}. \tag{13.1}$$

The coefficient h_i^{-2} appears on the right-hand side of Eq. (13.1). A further assumption is required for this. Using Eq. (1.9) and the triangle inequality, we have

$$\|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}^2 \leq 2\|\partial_{\hat{x}_i}(\hat{\varphi} - Q^{(2)}\hat{\varphi})\|_{L^2(\widehat{T})}^2 + 2\|\partial_{\hat{x}_i}(Q^{(2)}\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}^2.$$

We use inequality (1.8) to remove the coefficient h_i^{-2}. To this end, we have to show that

$$\|\partial_{\hat{x}_i}(Q^{(2)}\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \leq c\|\partial_{\hat{x}_i}(\hat{\varphi} - Q^{(2)}\hat{\varphi})\|_{H^1(\widehat{T})}. \tag{13.2}$$

However, this is unlikely to hold because Eqs. (9.1) and (9.3) yield

$$\begin{aligned}\|\partial_{\hat{x}_i}(Q^{(2)}\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} &= \|\partial_{\hat{x}_i}(I_{\widehat{T}}(Q^{(2)}\hat{\varphi}) - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \\ &\leq c\|Q^{(2)}\hat{\varphi} - \hat{\varphi}\|_{H^2(\widehat{T})} \leq c|\hat{\varphi}|_{H^2(\widehat{T})}.\end{aligned}$$

Using the classical scaling argument (see [16, Lemma 1.101]), we have

$$|\hat{\varphi}|_{H^2(\widehat{T})} \leq c|\det(A)|^{-\frac{1}{2}}\|A\|_2|\varphi|_{H^2(T)},$$

which does not include the quantity h_i. Therefore, the quantity h_i^{-1} in Eq. (13.1) remains.

To overcome this problem, we use Theorem 12.1. That is, we assume that there exists a linear functional $\mathscr{F}_1$ such that

$$\begin{aligned}&\mathscr{F}_1 \in H^1(\widehat{T})', \\ &\mathscr{F}_1(\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})) = 0 \quad i = 1,2, \quad \forall \hat{\varphi} \in \mathcal{C}(\widehat{T}) : \ \partial_{\hat{x}_i}\hat{\varphi} \in H^1(\widehat{T}), \\ &\hat{\eta} \in \mathbb{P}^1, \quad \mathscr{F}_1(\partial_{\hat{x}_i}\hat{\eta}) = 0 \quad i = 1,2, \quad \Rightarrow \quad \partial_{\hat{x}_i}\hat{\eta} = 0.\end{aligned}$$

Because the polynomial spaces are finite-dimensional, all norms are equivalent; i.e., because $|\mathscr{F}_1(\partial_{\hat{x}_i}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}))|$ $(i = 1,2)$ is a norm on $\mathbb{P}^0$, we have that, for $i = 1,2$,

$$\begin{aligned}\|\partial_{\hat{x}_i}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} &\leq c|\mathscr{F}_1(\partial_{\hat{x}_i}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}))| = c|\mathscr{F}_1(\partial_{\hat{x}_i}(\hat{\eta} - \hat{\varphi}))| \\ &\leq c\|\partial_{\hat{x}_i}(\hat{\eta} - \hat{\varphi})\|_{H^1(\widehat{T})}.\end{aligned}$$

Setting $\hat{\eta} := Q^{(2)}\hat{\varphi}$, we obtain Eq. (13.2). Using inequality (1.8) yields

$$\|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}^2 \leq c|\partial_{\hat{x}_i}\hat{\varphi}|_{H^1(\widehat{T})}^2,$$

and so inequality (13.1) together with Eq. (1.4) can be written as

$$|\varphi - I_T\varphi|_{H^1(T)} \leq c|\det(A_{\widetilde{T}})|^{\frac{1}{2}}\|\widetilde{A}^{-1}\|_2 \sum_{i,j=1}^{2} h_i^{-1}\|\partial_{\hat{x}_i}\partial_{\hat{x}_j}\hat{\varphi}\|_{L^2(\widehat{T})}. \tag{13.3}$$

Inequality (10.2) yields

$$\|\partial_{\hat{x}_i}\partial_{\hat{x}_j}\hat{\varphi}\|_{L^2(\widehat{T})} \leq c|\det(A_{\widetilde{T}})|^{-\frac{1}{2}}\|\widetilde{A}\|_2 h_i \sum_{n=1}^{2} h_n \left|\frac{\partial\varphi}{\partial r_n}\right|_{H^1(T)}. \tag{13.4}$$

Therefore, the quantity h_i^{-1} in Eq. (13.3) and the quantity h_i in Eq. (13.4) cancel out.

14 New Interpolation Error Estimates

14.1 Local Interpolation Error Estimates

The new scaling arguments in Section 10 are the heart of the following local interpolation error estimates.

Theorem 14.1 (Local interpolation)**.** Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with the normed vector space $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ and $\widehat{P} := \mathcal{P}^k(\widehat{T})$ with $k \geq 1$. Let $I_{\widehat{T}} : V(\widehat{T}) \to \widehat{P}$ be a linear operator. Fix $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell \leq k+1$, $\ell - m \geq 1$, and the embeddings (1.5) and (1.6) with $s := \ell - m$ hold. Let β be a multi-index with $|\beta| = m$. We set $j := \dim(\partial^\beta \mathcal{P}^k)$. Assume that there exist linear functionals $\mathscr{F}_i$, $i = 1, \ldots, j$, satisfying the conditions (12.2). It then holds that, for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T}) \cap \mathcal{C}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq C_1^I |T|_d^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon| = \ell - m} h^\varepsilon |\partial_r^\varepsilon \varphi|_{W^{m,p}(T)}, \tag{14.1}$$

where C_1^I is a positive constant independent of h_T and H_T. In particular, if Condition 10.1 is imposed, it holds that, for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T}) \cap \mathcal{C}(\widehat{T})$ with $\varphi = \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq C_2^I |T|_d^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon| = \ell - m} \widetilde{\mathscr{H}^\varepsilon} |\partial_{\tilde{x}}^\varepsilon (\varphi \circ \Phi_T)|_{W^{m,p}(\Phi_T^{-1}(T))}, \tag{14.2}$$

where C_2^I is a positive constant independent of h_{T^s} and H_{T^s}.

Proof. The introduction of the functionals $\mathscr{F}_i$ follows from [3, 2], also see Theorem 12.1. Actually, under the same assumptions as in Theorem 14.1, we have

$$\|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})} \leq C^B |\partial_{\hat{x}}^\beta \hat{\varphi}|_{W^{\ell - m, p}(\widehat{T})}, \tag{14.3}$$

where $|\beta| = m$, $\hat{\varphi} \in \mathcal{C}(\widehat{T})$, and $\partial_{\hat{x}}^\beta \hat{\varphi} \in W^{\ell - m, p}(\widehat{T})$.

The inequalities in (10.6), (1.4), (10.1), and (14.3) yield

$$\begin{aligned}
|\varphi - I_T \varphi|_{W^{m,q}(T)} &\leq c |\varphi - I_T \varphi|_{W^{m,q}(T)} \\
&\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m \left(\sum_{|\beta| = m} (h^{-\beta})^q \|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})}^q \right)^{1/q} \\
&\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m \sum_{|\beta| = m} (h^{-\beta}) \|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})} \\
&\leq c |\det(A_{\widetilde{T}})|^{\frac{1}{q}} \|\widetilde{A}^{-1}\|_2^m \sum_{|\beta| = m} (h^{-\beta}) |\partial_{\hat{x}}^\beta \hat{\varphi}|_{W^{\ell - m, p}(\widehat{T})}.
\end{aligned} \tag{14.4}$$

Inequalities (1.4) and (10.2) yield

$$\begin{aligned}
&\sum_{|\beta| = m} (h^{-\beta}) |\partial_{\hat{x}}^\beta \hat{\varphi}|_{W^{\ell - m, p}(\widehat{T})} \\
&\quad \leq \sum_{|\gamma| = \ell - m} \sum_{|\beta| = m} (h^{-\beta}) \|\partial_{\hat{x}}^\beta \partial_{\hat{x}}^\gamma \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\quad \leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \sum_{|\gamma| = \ell - m} \sum_{|\beta| = m} (h^{-\beta}) h^\beta \sum_{|\epsilon| = |\gamma|} h^\varepsilon |\partial_r^\epsilon \varphi|_{W^{m,p}(T)} \\
&\quad \leq c |\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \sum_{|\epsilon| = \ell - m} h^\varepsilon |\partial_r^\varepsilon \varphi|_{W^{m,p}(T)}.
\end{aligned} \tag{14.5}$$

From (6.8), (6.9), (14.4), and (14.5), we have

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c|T|_d^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon|=\ell-m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)},$$

which is the inequality (14.1).

Assume that Condition 10.1 is imposed. Inequality (10.3) yields

$$\begin{aligned}
&\sum_{|\beta|=m} (h^{-\beta}) |\partial_{\hat{x}}^{\beta} \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})} \\
&\quad \leq \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) \|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\quad \leq c|\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) h^{\beta} \sum_{|\epsilon|=|\gamma|} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\epsilon} \tilde{\varphi}|_{W^{m,p}(\widetilde{T})} \\
&\quad \leq c|\det(A_{\widetilde{T}})|^{-\frac{1}{p}} \|\widetilde{A}\|_2^m \sum_{|\epsilon|=\ell-m} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\epsilon} \tilde{\varphi}|_{W^{m,p}(\widetilde{T})}.
\end{aligned} \tag{14.6}$$

From (6.8), (6.9), (14.4), and (14.6), we have

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c|T|_d^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon|=\ell-m} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\epsilon} \tilde{\varphi}|_{W^{m,p}(\widetilde{T})},$$

which is the inequality (14.2) using $\widetilde{T} = \Phi_T^{-1}(T)$ and $\tilde{\varphi} = \varphi \circ \Phi_T$. □

14.2 Global Interpolation Error Estimates

A global interpolation operator I_h is constructed as follows (e.g., see [16, Section 1.4.2]). Its domain is defined by

$$D(I_h) := \{\varphi \in L^1(\Omega);\ \varphi|_T \in V(T),\ \forall T \in \mathbb{T}_h\}.$$

For $T \in \mathbb{T}_h$ and $\varphi \in D(I_h)$, the quantities $\chi_i(\varphi|_T)$ are meaningful on all the mesh elements and $1 \leq i \leq n_0$. The global interpolation $I_h \varphi$ can be specified elementwise using the local interpolation operators, that is,

$$(I_h \varphi)|_T := I_T(\varphi|_T) = \sum_{i=1}^{n_0} \chi_i(\varphi|_T) \theta_i \quad \forall T \in \mathbb{T}_h, \quad \forall \varphi \in D(I_h).$$

The global interpolation operator $I_h : D(I_h) \to V_h$ is defined as

$$I_h : D(I_h) \ni \varphi \mapsto I_h \varphi := \sum_{T \in \mathbb{T}_h} \sum_{i=1}^{n_0} \chi_i(\varphi|_T) \theta_i \in V_h,$$

where V_h is defined as

$$V_h := \{\varphi_h \in L^1(\Omega)^n;\ \varphi_h|_T \in P,\ \forall T \in \mathbb{T}_h\}.$$

Corollary 14.2. Suppose that the assumptions of Theorem 14.1 are satisfied. We impose Condition 6.2. Let I_h be the corresponding global interpolation operator. It then holds that, for any $\varphi \in W^{\ell,p}(\Omega)$;

(I) if Condition 10.1 is not imposed,

$$|\varphi - I_h\varphi|_{W^{m,q}(\Omega)} \leq c \sum_{T\in\mathbb{T}_h} |T|_d^{\frac{1}{q}-\frac{1}{p}} \sum_{|\varepsilon|=\ell-m} h^{\varepsilon} |\partial_r^{\varepsilon}\varphi|_{W^{m,p}(T)}. \tag{14.7}$$

(II) if Condition 10.1 is imposed,

$$|\varphi - I_h\varphi|_{W^{m,q}(\Omega)} \leq c \sum_{T\in\mathbb{T}_h} |T|_d^{\frac{1}{q}-\frac{1}{p}} \sum_{|\varepsilon|=\ell-m} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\varepsilon}(\varphi\circ\Phi_T)|_{W^{m,p}(\Phi_T^{-1}(T))}. \tag{14.8}$$

14.3 Examples of Anisotropic Elements

When $k = 1$, $ell = 2$, $m = 1$ and $q = p$ in (14.1) of Theorem 14.1, the estimate is written as

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq C_1^I \frac{H_T}{h_T} \sum_{i=1}^{d} h_i \left|\frac{\partial\varphi}{\partial r_i}\right|_{W^{1,p}(T)}. \tag{14.9}$$

Let $T \subset \mathbb{R}^2$ be a triangle. As described in Section 8.1, an isotropic mesh element has equal or nearly equal edge lengths and angles, resulting in a balanced shape. Then, the estimate (14.9) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq c h_T |\varphi|_{W^{2,p}(T)}. \tag{14.10}$$

We considered the following five anisotropic elements as in Section 8.2: Let $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon, \delta, \gamma \in \mathbb{R}$.

Example 14.3 (Right-angled triangle)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0,s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = s$, $h_2 = s^\varepsilon$ and $h_T = \sqrt{s^2 + s^{2\varepsilon}}$; i.e.,

$$\frac{H_T}{h_T} = 2.$$

In this case, the estimate (14.9) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq 2C_1^I \sum_{i=1}^{2} h_i \left|\frac{\partial\varphi}{\partial r_i}\right|_{W^{1,p}(T)},$$

which is the anisotropic interpolation error estimate.

Example 14.4 (Dagger)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \varepsilon < \delta$. We then have $h_1 = \sqrt{(s-s^\delta)^2 + s^{2\varepsilon}}$, $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ and $h_T = s$; i.e.,

$$\frac{H_T}{h_T} = \frac{\sqrt{(s-s^\delta)^2 + s^{2\varepsilon}}\sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2}s^{1+\varepsilon}} \leq c.$$

In this case, the estimate (14.9) becomes

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq c \sum_{i=1}^{2} h_i \left|\frac{\partial\varphi}{\partial r_i}\right|_{W^{1,p}(T)},$$

which is the anisotropic interpolation error estimate.

Example 14.5 (Blade)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, s^\varepsilon)^\top$ with $1 < \varepsilon$. We then have $h_1 = h_2 = \sqrt{s^2 + s^{2\varepsilon}}$ and $h_T = 2s$; i.e.,

$$\frac{H_T}{h_T} = \frac{s^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \to \infty \quad \text{as } s \to 0.$$

In this case, the estimate (14.9) becomes

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c s^{2-\varepsilon} |\varphi|_{W^{2,p}(T)}.$$

When $\varepsilon > 2$, this implies that the estimate diverges as $s \to 0$.

Example 14.6 (Dagger)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (s^\delta, s^\varepsilon)^\top$ with $1 < \delta < \varepsilon$. We then have $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$, $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ and $h_T = s$; i.e.,

$$\frac{H_T}{h_T} = \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}} \sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2} s^{1+\varepsilon}} \leq c s^{\delta - \varepsilon} \to \infty \quad \text{as } s \to 0.$$

In this case, the estimate (14.9) becomes

$$\begin{aligned} |\varphi - I_T \varphi|_{W^{1,p}(T)} &\leq c s^{\delta-\varepsilon} \left(s \left| \frac{\partial \varphi}{\partial r_1} \right|_{W^{1,p}(T)} + s^\delta \left| \frac{\partial \varphi}{\partial r_2} \right|_{W^{1,p}(T)} \right) \\ &\leq c s^{1+\delta-\varepsilon} |\varphi|_{W^{2,p}(T)}. \end{aligned}$$

When $\varepsilon - \delta > 1$, this implies that the estimate diverges as $s \to 0$.

Example 14.7 (Right-angled triangle)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$ and $p_3 := (0, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = s$, $h_2 = \delta s$ and $h_T = s\sqrt{1 + \delta^2}$; i.e.,

$$\frac{H_T}{h_T} = 2.$$

In this case, the estimate (14.9) becomes

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \sum_{i=1}^{2} h_i \left| \frac{\partial \varphi}{\partial r_i} \right|_{W^{1,p}(T)},$$

which is the anisotropic interpolation error estimate.

Example 14.8 (Blade)**.** Let $T \subset \mathbb{R}^2$ be the simplex with vertices $p_1 := (0,0)^\top$, $p_2 := (2s,0)^\top$ and $p_3 := (s, \delta s)^\top$ with $\delta \ll 1$. We then have $h_1 = h_2 = s\sqrt{1 + \delta^2}$ and $h_T = 2s$; i.e.,

$$\frac{H_T}{h_T} = \frac{s^2(1 + \delta^2)}{\delta s^2} \leq \frac{c}{\delta},$$

In this case, the estimate (14.9) becomes

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} s |\varphi|_{W^{2,p}(T)}.$$

This implies that the estimate converges as $s \to 0$ and the error may be large. Thus, even if anisotropic mesh partitioning is used, it is unlikely to improve calculation efficiency.

14.4 Examples that do not satisfy conditions (12.2) in Theorem 12.1

The following lemma ([3, Lemma 4], [2, Lemma 2.3]) gives a criterion for the existence of linear functionals satisfying conditions (12.2b) and (12.2c).

Lemma 14.9. Let $\mathbb{P}$ be an arbitrary polynomial space and β be a multi-index. We set $j := \dim(\partial^\beta \mathbb{P})$. Assume that $I : \mathcal{C}^\mu(\widehat{T}) \to \mathbb{P}$, $\mu \in \mathbb{N}$, is a linear operator with $I\hat{\eta} = \hat{\eta}$ $\forall \hat{\eta} \in \mathbb{P}$. Then, there exist linear functionals $\mathscr{F}_i : \mathcal{C}^\infty(\widehat{T}) \to \mathbb{R}$, $i = 1, \dots, j$, such that

$$\mathscr{F}_i(\partial^\beta(\hat{\varphi} - I\hat{\varphi})) = 0 \quad \forall i = 1, \dots, j, \quad \forall \hat{\varphi} \in \mathcal{C}^\infty(\widehat{T}), \tag{14.11}$$

$$\hat{\eta} \in \mathbb{P}, \quad \mathscr{F}_i(\partial^\beta \hat{\eta}) = 0 \quad \forall i = 1, \dots, j \quad \Rightarrow \quad \partial^\beta \hat{\eta} = 0 \tag{14.12}$$

if and only if the condition

$$\hat{\varphi} \in \mathcal{C}^\infty(\widehat{T}), \quad \partial^\beta \hat{\varphi} = 0 \quad \Rightarrow \quad \partial^\beta I\hat{\varphi} = 0 \tag{14.13}$$

holds.

Proof. A proof can be found in [3, Lemma 4]. □

If Condition (14.13) is violated, estimate (12.3) does not hold. This means that one cannot obtain the estimate (14.1), which is sharper than (11.2).

The following are examples that do not satisfy (14.13). Let $\widehat{T} \subset \mathbb{R}^2$ be the reference element with vertices $\widehat{p}_1 := (0,0)^\top$, $\widehat{p}_2 := (1,0)^\top$, $\widehat{p}_3 := (0,1)^\top$. We set $\widehat{p}_4 := (1/3, 1/3)^\top$. We define the barycentric coordinates $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \dots, 3$, on the reference element as

$$\lambda_1 := 1 - \hat{x}_1 - \hat{x}_2, \quad \lambda_2 := \hat{x}_1, \quad \lambda_3 := \hat{x}_2, \quad (\hat{x}_1, \hat{x}_2)^\top \in \widehat{T}.$$

Example 14.10 ($\mathbb{P}^1$ + bubble Finite Element)**.** As mentioned in Example 11.10, we define the local basis functions as

$$\begin{aligned} \theta_4(x) &:= 27\lambda_1(x)\lambda_2(x)\lambda_3(x), \\ \theta_i(x) &:= \lambda_i(x) - \frac{1}{3}\theta_4(x), \quad i = 1, 2, 3. \end{aligned}$$

The interpolation operator I_T^b defined by

$$I^b : \mathcal{C}(\widehat{T}) \ni \hat{\varphi} \mapsto I^b \hat{\varphi} := \sum_{i=1}^{4} \hat{\varphi}(\widehat{P}_i)\theta_i \in \operatorname{span}\{\theta_1, \theta_2, \theta_3, \theta_4\}.$$

Let $\beta = (1, 0)$. Setting $\hat{\varphi}(\hat{x}_1, \hat{x}_2) := \hat{x}_2^2$, we have $\frac{\partial \hat{\varphi}}{\partial \hat{x}_1} = 0$. By simple calculation, we obtain

$$\begin{aligned} \frac{\partial}{\partial \hat{x}_1} I^b \hat{\varphi} &= \hat{\varphi}(\widehat{p}_1)\frac{\partial \theta_1}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{p}_2)\frac{\partial \theta_1}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{p}_3)\frac{\partial \theta_3}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{p}_4)\frac{\partial \theta_4}{\partial \hat{x}_1} \\ &= \frac{\partial \theta_3}{\partial \hat{x}_1} + \frac{1}{3^2}\frac{\partial \theta_4}{\partial \hat{x}_1} = -\frac{1}{3}\frac{\partial \theta_4}{\partial \hat{x}_1} + \frac{1}{3^2}\frac{\partial \theta_4}{\partial \hat{x}_1} \not\equiv 0. \end{aligned}$$

Therefore, the condition (14.13) is not satisfied. This implies that the error estimate (12.3) on the reference element does not hold for the $\mathcal{P}^1$ + bubble finite element.

Example 14.11 ($\mathbb{P}^3$ Hermite Finite Element)**.** Following [13, Theorem 2.2.8], we define the Hermite interpolation operator $I^H : H^3(T) \to \mathbb{P}^3$ as

$$
\begin{aligned}
I^H \hat{\varphi} := \sum_{i=1}^{3} &\left(-2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{1 \le j < k \le 3, j \neq i, k \neq i} \lambda_j \lambda_k \right) \hat{\varphi}(\widehat{p}_i) + 27\lambda_1\lambda_2\lambda_3 \hat{\varphi}(\widehat{p}_4) \\
&+ \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_i \lambda_j (2\lambda_i + \lambda_j - 1)(\widehat{p}_j^{(1)} - \widehat{p}_i^{(1)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_1}(\widehat{p}_i) \\
&+ \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_i \lambda_j (2\lambda_i + \lambda_j - 1)(\widehat{p}_j^{(2)} - \widehat{p}_i^{(2)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_i),
\end{aligned}
$$

where $\widehat{p}_i^{(k)}$, $1 \le k \le 2$, are the components of a point $\widehat{p}_i \in \mathbb{R}^2$. Let $\beta = (1, 0)$. Setting $\hat{\varphi}(\hat{x}_1, \hat{x}_2) := \hat{x}_2^4$, we have $\frac{\partial \hat{\varphi}}{\partial \hat{x}_1} = 0$. Furthermore, by a simple calculation, i.e.,

$$
\begin{aligned}
\frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3) &= -\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2, \\
\frac{\partial}{\partial \hat{x}_1} \{\lambda_3\lambda_1(2\lambda_3 + \lambda_1 - 1)\} &= -\hat{x}_2 + 2\hat{x}_1\hat{x}_2, \\
\frac{\partial}{\partial \hat{x}_1} \{\lambda_3\lambda_2(2\lambda_3 + \lambda_2 - 1)\} &= -\hat{x}_2 + 2\hat{x}_1\hat{x}_2 + 2\hat{x}_2^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{\partial}{\partial \hat{x}_1} I^H \hat{\varphi} = &\frac{\partial}{\partial \hat{x}_1} \left(-2\lambda_3^3 + 3\lambda_3^2 - 7\lambda_3 \sum_{1 \le j < k \le 3, j \neq 3, k \neq 3} \lambda_j \lambda_k \right) \hat{\varphi}(\widehat{p}_3) \\
&+ 27 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3)\hat{\varphi}(\widehat{p}_4) \\
&+ \frac{\partial}{\partial \hat{x}_1} \left(\sum_{j=1}^{3} \lambda_3 \lambda_j (2\lambda_3 + \lambda_j - 1)(\widehat{p}_j^{(2)} - \widehat{p}_3^{(2)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_3) \\
= &-7 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3)\hat{\varphi}(\widehat{p}_3) + 27 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3)\hat{\varphi}(\widehat{p}_4) \\
&+ \frac{\partial}{\partial \hat{x}_1} \left\{ \lambda_3\lambda_1(2\lambda_3 + \lambda_1 - 1)(\widehat{p}_1^{(2)} - \widehat{p}_3^{(2)}) \right\} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_3) \\
&+ \frac{\partial}{\partial \hat{x}_1} \left\{ \lambda_3\lambda_2(2\lambda_3 + \lambda_2 - 1)(\widehat{p}_2^{(2)} - \widehat{p}_3^{(2)}) \right\} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_3) \\
= &-7(-\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2) + \frac{1}{3}(-\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2) \\
&+ 8(\hat{x}_2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2^2) \not\equiv 0.
\end{aligned}
$$

Here, we used

$$
\begin{aligned}
&\hat{\varphi}(\widehat{p}_i) = 0, \quad \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_i) = 0, \quad i = 1, 2, \\
&\hat{\varphi}(\widehat{p}_3) = 1, \quad \hat{\varphi}(\widehat{p}_4) = \frac{1}{3^4}, \quad \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{p}_3) = 4, \\
&\widehat{p}_1^{(2)} - \widehat{p}_3^{(2)} = -1, \quad \widehat{p}_2^{(2)} - \widehat{p}_3^{(2)} = -1.
\end{aligned}
$$

Therefore, Condition (14.13) is not satisfied. This implies that error estimate (12.3) on the reference element does not hold for Hermitian finite elements.

14.5 Effect of the quantity $|T|_d^{\frac{1}{q}-\frac{1}{p}}$ in the interpolation error estimates for $d = 2, 3$

We consider the effect of the factor $|T|_d^{\frac{1}{q}-\frac{1}{p}}$.

14.5.1 Case that $q > p$

When $q > p$, the factor may affect the convergence order. In particular, the interpolation error estimate may diverge on anisotropic mesh partitions.

Let $T \subset \mathbb{R}^2$ be the triangle with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, $p_3 := (0,s^\varepsilon)^\top$ for $0 < s \ll 1$, $\varepsilon \geq 1$, $s \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. Then,

$$\frac{h_{\max}}{h_{\min}} = s^{1-\varepsilon}, \quad \frac{H_T}{h_T} = 2, \quad |T|_2 = \frac{1}{2} s^{1+\varepsilon}.$$

Let $k = 1$, $\ell = 2$, $m = 1$, $q = 2$, and $p \in (1,2)$. Then, $W^{1,p}(T) \hookrightarrow L^2(T)$ and Theorem 14.1 lead to

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c s^{-(1+\varepsilon)\frac{2-p}{2p}} \left(s \left| \frac{\partial \varphi}{\partial r_1} \right|_{W^{1,p}(T)} + s^\varepsilon \left| \frac{\partial \varphi}{\partial r_2} \right|_{W^{1,p}(T)} \right).$$

When $\varepsilon = 1$ (the case of the isotropic element), we get

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c h_T^{\frac{2(p-1)}{p}} |\varphi|_{W^{2,p}(T)}, \quad \frac{2(p-1)}{p} > 0.$$

However, when $\varepsilon > 1$ (the case of the anisotropic element), the estimate may diverge as $s \to 0$. Therefore, if $q > p$, the convergence order of the interpolation operator may deteriorate.

We next set $m = 0$, $\ell = 2$, $q = \infty$, and $p = 2$. Let

$$\varphi(x,y) := x^2 + y^2.$$

Let $I_T^L : \mathcal{C}^0(T) \to \mathbb{P}^1$ be the local Lagrange interpolation operator. For any nodes p_i of T, because $I_T^L \varphi(p_i) = \varphi(p_i)$, we have

$$I_T^L \varphi(x,y) = sx + s^\varepsilon y.$$

It thus holds that

$$(\varphi - I_T^L \varphi)(x,y) = \left(x - \frac{s}{2} \right)^2 + \left(y - \frac{s^\varepsilon}{2} \right)^2 - \frac{1}{4}(s^2 + s^{2\varepsilon}).$$

We therefore have, because $H^2(T) \hookrightarrow L^\infty(T)$,

$$\|\varphi - I_T^L \varphi\|_{L^\infty(T)} = \frac{1}{4}(s^2 + s^{2\varepsilon}), \quad \sum_{|\gamma|=2} \widetilde{\mathscr{H}}^\gamma \|\partial_x^\gamma \varphi\|_{L^2(T)} = 2|T|_2^{\frac{1}{2}} (s^2 + s^{2\varepsilon}),$$

and thus,

$$\frac{\|\varphi - I_T^L \varphi\|_{L^\infty(T)}}{|T|_2^{-\frac{1}{2}} \sum_{|\gamma|=2} \widetilde{\mathscr{H}}^\gamma \|\partial_x^\gamma \varphi\|_{L^2(T)}} = \frac{1}{8}.$$

This example implies that the convergence order is not optimal, but the estimate converges on anisotropic meshes.

14.5.2 Case that $q < p$

We consider Theorem 14.1. Let $I_T^L : \mathcal{C}(T) \to \mathbb{P}^k$ ($k \in \mathbb{N}$) be the local Lagrange interpolation operator. Let $\varphi \in W^{\ell,\infty}(T)$ be such that $\ell \in \mathbb{N}$, $2 \le \ell \le k+1$. It then holds that, for any $m \in \{0, \dots, \ell-1\}$ and $q \in [1, \infty]$,

$$|\varphi - I_T^L \varphi|_{W^{m,q}(T)} \le c|T|_d^{\frac{1}{q}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\gamma|=\ell-m} h^\gamma |\partial_r^\gamma \varphi|_{W^{m,\infty}(T)}. \tag{14.14}$$

The convergence order is therefore improved by $|T|_d^{\frac{1}{q}}$. We do numerical tests to confirm this. Let $k = 1$ and

$$\varphi(x, y, z) := x^2 + \frac{1}{4} y^2 + z^2.$$

Let $s := \frac{1}{N}$, $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, $1 < \varepsilon$. We compute the convergence order with respect to the H^1 norm defined by

$$Err_s^\varepsilon(H^1) := |\varphi - I_T^L \varphi|_{H^1(T)}.$$

The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log \left(\frac{Err_s^\varepsilon(H^1)}{Err_{s/2}^\varepsilon(H^1)} \right).$$

(I) Let $T \subset \mathbb{R}^3$ be the simplex with vertices $p_1 := (0,0,0)^\top$, $p_2 := (s,0,0)^\top$, $p_3 := (0, s^\varepsilon, 0)^\top$, and $p_4 := (0,0,s^\delta)^\top$ $(1 < \delta \le \varepsilon)$, and $0 < s \ll 1$, $s \in \mathbb{R}$. We then have $h_1 = \sqrt{s^2 + s^{2\varepsilon}}$, $h_2 = s^\varepsilon$ and $h_3 := \sqrt{s^{2\varepsilon} + s^{2\delta}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \le c s^{1-\varepsilon}, \quad \frac{H_T}{h_T} \le c.$$

From (14.14) with $m = 1$, $\ell = 2$, and $q = 2$, because $|T|_3 \approx s^{1+\varepsilon+\delta}$, we have the estimate

$$|\varphi - I_T^L \varphi|_{H^1(T)} \le c h_T^{\frac{3+\varepsilon+\delta}{2}}.$$

Computational results are for the case that $\varepsilon = 3.0$ and $\delta = 2.0$ (Table 13).

Table 13: Error of the local interpolation operator ($\varepsilon = 3.0, \delta = 2.0$)

N	s	$Err_s^{3.0}(H^1)$	r
64	1.5625e-02	2.4336e-08	
128	7.8125e-03	1.5209e-09	4.00
256	3.9062e-03	9.5053e-11	4.00

(II) Let $T \subset \mathbb{R}^3$ be the simplex with vertices $p_1 := (0,0,0)^\top$, $p_2 := (s,0,0)^\top$, $p_3 := (\frac{s}{2}, s^\varepsilon, 0)^\top$, and $p_4 := (0,0,s)^\top$ $(1 < \varepsilon \le 6)$ and $0 < s \ll 1$, $s \in \mathbb{R}$. We then have $h_1 = s$, $h_2 = \sqrt{s^2/4 + s^{2\varepsilon}}$ and $h_3 := s$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{t}{\sqrt{s^2/4 + t^{2\varepsilon}}} \le c, \quad \frac{H_T}{h_T} \le c s^{1-\varepsilon}.$$

From (14.14) with $m = 1$, $\ell = 2$, and $q = 2$, because $|T|_3 \approx s^{2+\varepsilon}$, we have the estimate

$$|\varphi - I_T^L \varphi|_{H^1(T)} \le c h_T^{3 - \frac{\varepsilon}{2}}.$$

Computational results are for the cases that $\varepsilon = 3.0, 6.0$ (Table 14).

Table 14: Error of the local interpolation operator ($\varepsilon = 3.0, 6.0$)

N	s	$Err_s^{3.0}(H^1)$	r	$Err_s^{6.0}(H^1)$	r
64	1.5625e-02	1.9934e-04		1.0206e-01	
128	7.8125e-03	7.0477e-05	1.50	1.0206e-01	0
256	3.9062e-03	2.4917e-05	1.50	1.0206e-01	0

14.6 What happens if violating the maximum-angle condition?

This subsection introduces two negative points by violating the maximum-angle condition. One is that it is practically disadvantageous. As an example, let $T \subset \mathbb{R}^2$ be the triangle with vertices $p_1 := (0,0)^\top$, $p_2 := (s,0)^\top$, $p_3 := (s/2, s^\varepsilon)^\top$ for $0 < s \ll 1$, $\varepsilon \geq 1$, $s \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. From Theorem 14.1 with $k = 1$, $\ell = 2$, $m = 1$, $p = q = 2$, we have

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c s^{2-\varepsilon} \left| \frac{\partial \varphi}{\partial r_1} \right|_{H^1(T)} + s \left| \frac{\partial \varphi}{\partial r_2} \right|_{H^1(T)}.$$

Even if one wants to reduce the step size in a specific direction (y-axis direction), the interpolation error may diverge as $s \to 0$ when $\varepsilon > 2$. This loses the benefits of using anisotropic meshes.

Another is that violating the condition makes it challenging to show mathematical validity in the finite element method. One of the answers can be found in [4]. That is, the maximum-angle condition is sufficient to do numerical calculations safely.

15 Lagrange Interpolation Error Estimates

15.1 One-dimensional Lagrange Interpolation

Let $\Omega := (0,1) \subset \mathbb{R}$. For $N \in \mathbb{N}$, let $\mathbb{T}_h = \{0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1\}$ be a mesh of $\overline{\Omega}$ such as

$$\overline{\Omega} := \bigcup_{i=1}^{N} I_i, \quad \operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset \quad \text{for } i \neq j,$$

where $I_i := [x_i, x_{i+1}]$ for $0 \leq i \leq N$. We denote $h_i := x_{i+1} - x_i$ for $0 \leq i \leq N$. For $\widehat{T} := [0,1] \subset \mathbb{R}$ and $\widehat{P} := \mathbb{P}^k$ with $k \in \mathbb{N}$, let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the reference Lagrange finite element, e.g., see [16]. The corresponding interpolation operator is defined as

$$I_{\widehat{T}}^k : \mathcal{C}(\widehat{T}) \ni \hat{v} \mapsto I_{\widehat{T}}^k(\hat{v}) := \sum_{m=0}^{k} \hat{v}(\hat{\xi}_m) \widehat{\mathcal{L}}_m^k,$$

where $\hat{\xi}_m := \frac{m}{k}$ and $\{\widehat{\mathcal{L}}_0^k, \ldots, \widehat{\mathcal{L}}_k^k\}$ is the Lagrange polynomials associated with the nodes $\{\hat{\xi}_0, \ldots, \hat{\xi}_k\}$. For $i \in \{0, \ldots, N\}$, we consider the affine transformations

$$\Phi_i : \widehat{T} \ni t \mapsto x = x_i + t h_i \in I_i.$$

For $\hat{v} \in \mathcal{C}(\widehat{T})$, we set $\hat{v} = v \circ \Phi_i$.

Theorem 15.1. Let $1 \leq p \leq \infty$ and assume that there exists a nonnegative integer k such that

$$\mathcal{P}^k = \widehat{P} \subset W^{k+1,p}(\widehat{T}) \subset \mathcal{C}(\widehat{T}).$$

Let ℓ $(0 \leq \ell \leq k)$ be such that $W^{\ell+1,p}(\widehat{T}) \subset \mathcal{C}(\widehat{T})$ with continuous embedding. Furthermore, assume that $\ell, m \in \mathbb{N} \cup \{0\}$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell + 1$ and

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}).$$

It then holds that, for any $v \in W^{\ell+1,p}(I_i)$ with $\hat{v} = v \circ \Phi_i$,

$$|v - I_{I_i}^k v|_{W^{m,q}(I_i)} \leq c h_i^{\frac{1}{q} - \frac{1}{p} + \ell + 1 - m} |v|_{W^{\ell+1,p}(I_i)}. \tag{15.1}$$

Proof. We only show the outline of the proof. Scaling argument yields

$$|v - I_{I_i}^k v|_{W^{m,q}(I_i)} = h_i^{-m+\frac{1}{q}} |\hat{v} - I_{\hat{T}} \hat{v}|_{W^{m,q}(\widehat{T})},$$
$$|\hat{v}|_{W^{\ell+1,p}(\widehat{T})} = h_i^{\ell+1-\frac{1}{p}} |v|_{W^{\ell+1,p}(I_i)}.$$

Using the Sobolev embedding theorem and the Bramble–Hilbert–type lemma, we have

$$|\hat{v} - I_{\hat{T}} \hat{v}|_{W^{m,q}(\widehat{T})} \leq c |\hat{v}|_{W^{\ell+1,p}(\widehat{T})}.$$

Therefore, we obtain the estimate (15.1). □

Remark 15.2. The assumptions of Theorem 15.1 are standard; that is, there is no need to show the existence of functionals such as Theorem 14.1. Furthermore, the quantity $h_{\max}/h_{\min}$ that deteriorates the convergent order does not appear in (15.1).

Remark 15.3. If we set $x_j := \frac{j}{N+1}$, $j = 0, 1, \ldots, N, N+1$, the mesh $\mathbb{T}_h$ is said to be the uniform mesh. If we set $x_j := g\left(\frac{j}{N+1}\right)$, $j = 1, \ldots, N, N+1$ with a grading function g, the mesh $\mathbb{T}_h$ is said to be the graded mesh with respect to $x = 0$, see [6]. In particular, when one sets $g(y) := y^{\varepsilon}$ $(\varepsilon > 0)$, the mesh is called the radical mesh.

Remark 15.4 (Optimal order)**.** If $p = q$, it is possible to have the optimal error estimates even if the scale is different for each element. In the one-dimensional case, when $q > p$, the convergence order of the interpolation operator may deteriorate, see Section 14.5.1.

15.2 Lagrange Finite Element

Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 5.1 and 5.1. Let α be a multi-index. For $k \in \mathbb{N}$, we define the set of Lagrange nodes as

$$\mathscr{P} := \{\widehat{p}_i\}_{i=1}^{N^{(2,k)}} := \left\{ \left(\frac{i_1}{k}, \frac{i_2}{k} \right)^{\top} \in \mathbb{R}^2 \right\}_{0 \leq i_1 + i_2 \leq k} = \left\{ \frac{1}{k} \alpha \in \mathbb{R}^2 \right\}_{|\alpha| \leq k}, \quad \text{if } d = 2,$$
$$\mathscr{P} := \{\widehat{p}_i\}_{i=1}^{N^{(3,k)}} := \widehat{T} \cap \left\{ \left(\frac{i_1}{k}, \frac{i_2}{k}, \frac{i_3}{k} \right)^{\top} \in \mathbb{R}^3 \right\}_{0 \leq i_1, i_2, i_3 \leq k}, \quad \text{if } d = 3.$$

The Lagrange finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

1. $\widehat{P} := \mathbb{P}^k(\widehat{T})$;

2. $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \leq i \leq N^{(d,k)}}$ of $N^{(d,k)}$ linear forms $\{\hat{\chi}_i\}_{1 \leq i \leq N^{(d,k)}}$ with its components such that, for any $\hat{q} \in \widehat{P}$,

$$\hat{\chi}_i(\hat{q}) := \hat{q}(\widehat{p}_i) \quad \forall i \in \{1, \ldots, N^{(d,k)}\}. \tag{15.2}$$

The nodal basis functions associated with the degrees of freedom by (15.2) are defined as

$$\hat{\theta}_i(\widehat{p}_j) = \delta_{ij} \quad \forall i,j \in \{1,\ldots,N^{(d,k)}\}. \tag{15.3}$$

It then holds that $\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}$ for any $i,j \in \{1,\ldots,d+1\}$. Setting $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ or $V(\widehat{T}) := W^{s,p}(\widehat{T})$ with $p \in [1,\infty]$ and $ps > d$ ($s \geq d$ if $p = 1$), the local operator $I_{\widehat{T}}^L$ is defined as

$$I_{\widehat{T}}^L : V(\widehat{T}) \ni \hat{\varphi} \mapsto I_{\widehat{T}}^L \hat{\varphi} := \sum_{i=1}^{N^{(d,k)}} \hat{\varphi}(\widehat{p}_i)\hat{\theta}_i \in \widehat{P}. \tag{15.4}$$

By analogous argument in Section 9, we assume that the Lagrange finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T, P, \Sigma\}$ are constructed. The local shape functions are $\tilde{\theta}_i = \psi_{\widetilde{T}}^{-1}(\hat{\theta}_i)$ and $\theta_i = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$ for any $i \in \{1,\ldots,N^{(d,k)}\}$, and the associated local interpolation operators are respectively defined as

$$I_{\widetilde{T}}^L : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}^L \tilde{\varphi} := \sum_{i=1}^{N^{(d,k)}} \tilde{\varphi}(\widetilde{p}_i)\tilde{\theta}_i \in \widetilde{P}, \tag{15.5}$$

$$I_T^L : V(T) \ni \varphi \mapsto I_T^L \varphi := \sum_{i=1}^{N^{(d,k)}} \varphi(p_i)\theta_i \in P, \tag{15.6}$$

where $\widetilde{p}_i = \Phi_{\widetilde{T}}(\widehat{p}_i)$, $p_i = \Phi_T(\widetilde{p}_i)$ for $i \in \{1,\ldots,N^{(d,k)}\}$.

15.3 Local Interpolation Error Estimates

We first introduce the following lemmata.

Lemma 15.5 ($d = 2$)**.** Let β be a multi-index with $m := |\beta|$ and $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ a function such that $\partial_{\hat{x}}^{\beta}\hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$, where $\ell, m \in \mathbb{N}_0$, $p \in [1,\infty]$ are such that $0 \leq m \leq \ell \leq k+1$ and

$$p = \infty \quad \text{if } m = 0 \text{ and } \ell = 0, \tag{15.7a}$$

$$p > 2 \quad \text{if } m = 0 \text{ and } \ell = 1, \tag{15.7b}$$

$$m < \ell \quad \text{if } \beta_1 = 0 \text{ or } \beta_2 = 0, \text{ and } m > 0. \tag{15.7c}$$

Fix $q \in [1,\infty]$ such that $W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. It then holds that

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}^L \hat{\varphi})\|_{L^q(\widehat{T})} \leq c|\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{15.8}$$

Proof. We follow [2, Lemma 2.4]. We first give proofs in some particular cases: $k = 1, 2$.

Let $k = 1$. Let $m = 0$, that is, $\beta = (0,0)$. We then have $j = \dim \mathbb{P}^1 = 3$. From the Sobolev embedding theorem (Theorem 1.1), we have $W^{\ell,p}(\widehat{T}) \subset \mathcal{C}^0(\widehat{T})$ with $1 \leq p \leq \infty$ and $2 < \ell p$. Under this condition, we use

$$\mathscr{F}_i(\hat{\varphi}) := \hat{\varphi}(\widehat{p}_i), \quad \hat{\varphi} \in W^{\ell,p}(\widehat{T}), \quad i = 1,\ldots,3.$$

It then holds that

$$|\mathscr{F}_i(\hat{\varphi})| \leq \|\hat{\varphi}\|_{\mathcal{C}^0(\widehat{T})} \leq c\|\hat{\varphi}\|_{W^{\ell,p}(\widehat{T})},$$

which means $\mathscr{F}_i \in W^{\ell,p}(\widehat{T})'$ for $i = 1,\ldots,3$, that is, (12.2a) is satisfied. Furthermore, we have

$$\mathscr{F}_i(I_{\widehat{T}}^L \hat{\varphi}) = (I_{\widehat{T}}^L \hat{\varphi})(\widehat{p}_i) = \hat{\varphi}(\widehat{p}_i) = \mathscr{F}_i(\hat{\varphi}), \quad i = 1,\ldots,3,$$

which satisfies (12.2b). For all $\hat{\eta} \in \mathbb{P}^1$, if $\mathscr{F}_i(\hat{\eta}) = 0$ for $i = 1, \dots, 3$, it obviously holds $\hat{\eta} = 0$. This means that (12.2c) is satisfied.

Let $m = 1$. We set $\beta = (1,0)$. We then have $j = \dim(\partial^\beta \mathbb{P}^1) = 1$. We consider a functional

$$\mathscr{F}_1(\hat{\varphi}) := \int_0^1 \hat{\varphi}(\hat{x}_1, 0) d\hat{x}_1, \quad \hat{\varphi} \in W^{2,p}(\widehat{T}), \quad 1 < p.$$

We set $\widehat{I} := \{\hat{x} \in \widehat{T};\ \hat{x}_2 = 0\}$. The continuity is then shown by the trace theorem (e.g., see Theorem 1.2): if $1 = m < \ell$,

$$|\mathscr{F}_1(\hat{\varphi})| \le \|\hat{\varphi}\|_{L^1(\widehat{I})} \le c \|\hat{\varphi}\|_{W^{\ell-1,p}(\widehat{T})},$$

which means $\mathscr{F}_1 \in W^{\ell-1,p}(\widehat{T})'$, that is, (12.2a) is satisfied. Furthermore, it holds that

$$\begin{aligned} \mathscr{F}_1(\partial^{(1,0)}(\hat{\varphi} - I_{\widehat{T}}^L \hat{\varphi})) &= \int_0^1 \frac{\partial}{\partial \hat{x}_1} (\hat{\varphi} - I_{\widehat{T}}^L \hat{\varphi})(\hat{x}_1, 0) d\hat{x}_1 \\ &= \left[\hat{\varphi} - I_{\widehat{T}}^L \hat{\varphi} \right]_{(0,0)}^{(1,0)} = 0, \end{aligned}$$

which satisfy (12.2b). Let $\hat{\eta} := a\hat{x}_1 + b\hat{x}_2 + c$. We then have

$$\mathscr{F}_1(\partial^{(1,0)} \hat{\eta}) = a.$$

If $\mathscr{F}_1(\partial^{(1,0)} \hat{\eta}) = 0$, $a = 0$. This implies that $\partial^{(1,0)} \hat{\eta} = 0$. This means that (12.2c) is satisfied.

By analogous argument, the case $\beta = (0,1)$ holds.

Let $k = 2$. Let $m = 0$, that is, $\beta = (0,0)$. We then have $j = \dim \mathbb{P}^1 = 6$. Because $\dim \mathbb{P}^2 = 6$, we can show as in the case $k = 1$ and $\beta = (0,0)$.

Let $\beta := (1,0)$. We define three functionals as

$$\begin{aligned} \mathscr{F}_1(\hat{\varphi}) &:= \int_0^{\frac{1}{2}} \hat{\varphi}(\hat{x}_1, 0) d\hat{x}_1, \\ \mathscr{F}_2(\hat{\varphi}) &:= \int_{\frac{1}{2}}^1 \hat{\varphi}(\hat{x}_1, 0) d\hat{x}_1, \\ \mathscr{F}_3(\hat{\varphi}) &:= \int_0^{\frac{1}{2}} \hat{\varphi}(\hat{x}_1, 1/2) d\hat{x}_1. \end{aligned}$$

We then show (12.2a) and (12.2b) as above. Let $\hat{\eta} \in \mathbb{P}^2$ be such that

$$\mathscr{F}_i(\partial_{\hat{x}}^\beta \hat{\eta}) = 0, \quad i = 1, 2, 3. \tag{15.9}$$

We set the polynomial:

$$\begin{aligned} \hat{q} := \hat{\eta} - \hat{\eta}(1,0) \cdot 2 \left(\hat{x}_2 - \frac{1}{2} \right) (\hat{x}_2 - 1) - \hat{\eta} \left(\frac{1}{2}, \frac{1}{2} \right) \cdot [-4\hat{x}_2(\hat{x}_2 - 1)] \\ - \hat{\eta}(0,1) \cdot 2\hat{x}_2 \left(\hat{x}_2 - \frac{1}{2} \right) \in \mathcal{P}^2. \end{aligned} \tag{15.10}$$

This has the following properties:

$$\partial_{\hat{x}} \hat{\eta} = \partial_{\hat{x}} \hat{q}, \quad \hat{q}(1,0) = \hat{q}\left(\frac{1}{2}, \frac{1}{2} \right) = \hat{q}(0,1) = 0. \tag{15.11}$$

We thus have

$$0 = \mathscr{F}_3(\partial_{\hat{x}}^{\beta} \hat{\eta}) = \mathscr{F}_3(\partial_{\hat{x}}^{\beta} \hat{q}) = \hat{q}\left(\frac{1}{2}, \frac{1}{2}\right) - \hat{q}\left(0, \frac{1}{2}\right),$$

hence, $\hat{q}\left(0, \frac{1}{2}\right) = 0$. By similar way,

$$0 = \mathscr{F}_2(\partial_{\hat{x}}^{\beta} \hat{\eta}) = \mathscr{F}_2(\partial_{\hat{x}}^{\beta} \hat{q}) = \hat{q}(1,0) - \hat{q}\left(\frac{1}{2}, 0\right),$$

$$0 = \mathscr{F}_1(\partial_{\hat{x}}^{\beta} \hat{\eta}) = \mathscr{F}_1(\partial_{\hat{x}}^{\beta} \hat{q}) = \hat{q}\left(\frac{1}{2}, 0\right) - \hat{q}(0,0),$$

thus, $\hat{q}\left(\frac{1}{2}, 0\right) = 0$ and $\hat{q}(0,0) = 0$. Therefore, $\hat{q} \equiv 0$. Together with (15.13), we have $\hat{q} = \hat{q}(\hat{x}_2)$, $\partial_{\hat{x}}^{\beta} \hat{\eta} = 0$. □

Lemma 15.6 ($d = 3$)**.** Let β be a multi-index with $m := |\beta|$ and $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ a function such that $\partial_{\hat{x}}^{\beta} \hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$, where $\ell, m \in \mathbb{N}_0$, $p \in [1,\infty]$ are such that $0 \le m \le \ell \le k+1$ and

$$p = \infty \quad \text{if } m = 0 \text{ and } \ell = 0, \tag{15.12a}$$

$$p > \frac{3}{\ell} \quad \text{if } m = 0 \text{ and } \ell = 1, 2, \tag{15.12b}$$

$$m < \ell \quad \text{if } \beta_1 = 0,\ \beta_2 = 0, \text{ or } \beta_3 = 0, \tag{15.12c}$$

$$p > 2 \quad \text{if } \beta \in \{(\ell-1,0,0); (0,\ell-1,0); (0,0,\ell-1)\}. \tag{15.12d}$$

Fix $q \in [1,\infty]$ such that $W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. It then holds that

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})} \le c |\partial_{\hat{x}}^{\beta} \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{15.13}$$

Proof. A proof can be found in [2, Lemma 2.6]. □

We have the following new Lagrange interpolation error estimates.

Corollary 15.7. Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the Lagrange finite element with $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ and $\widehat{P} := \mathbb{P}^k(\widehat{T})$ with $k \ge 1$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. Let $m \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, and $p \in \mathbb{R}$ be such that $0 \le m \le \ell \le k+1$ and

$$d = 2: \begin{cases} p \in (2,\infty] & \text{if } m = 0,\ \ell = 1, \\ p \in [1,\infty] & \text{if } m = 0,\ \ell \ge 2 \text{ or } m \ge 1,\ \ell - m \ge 1, \end{cases}$$

$$d = 3: \begin{cases} p \in \left(\frac{3}{\ell},\infty\right] & \text{if } m = 0,\ \ell = 1,2, \\ p \in (2,\infty] & \text{if } m \ge 1,\ \ell - m = 1, \\ p \in [1,\infty] & \text{if } m = 0,\ \ell \ge 3 \text{ or } m \ge 1,\ \ell - m \ge 2. \end{cases}$$

Setting $q \in [1,\infty)$ be such that

$$W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T}), \tag{15.14}$$

that is $(\ell - m) - \frac{d}{p} \ge -\frac{d}{q}$. Then, for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$, we have

$$|\varphi - I_T^L \varphi|_{W^{m,q}(T)} \le c |T|_d^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon| = \ell - m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)}. \tag{15.15}$$

In particular, if Condition 10.1 is imposed, it holds that,for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T^L \varphi|_{W^{m,q}(T)} \le C_2^I |T|_d^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon| = \ell - m} \widetilde{\mathscr{H}}^{\varepsilon} |\partial_{\tilde{x}}^{\varepsilon} (\varphi \circ \Phi_{\widetilde{T}})|_{W^{m,p}(\Phi_{\widetilde{T}}^{-1}(T))}. \tag{15.16}$$

Furthermore, for any $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$, it holds that

$$\|\varphi_0 - I_T \varphi\|_{L^\infty(T)} \le c \|\varphi\|_{L^\infty(T)}.$$

References

[1] G. Acosta, R.G. Durán: The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations, SIAM J. Numer. Anal. 37 (1999) 18-36. Zbl 0948.65115.

[2] Apel, Th.: Anisotropic finite elements: Local estimates and applications. Advances in Numerical Mathematics. Teubner, Stuttgart, (1999)

[3] Apel, Th, Dobrowolski, M.: Anisotropic interpolation with applications to the finite element method. Computing **47**, 277-293 (1992)

[4] Apel, T., Eckardt, L., Hauhner, C., Kempf, V.: The maximum angle condition on finite elements: useful or not? , PAMM (2021)

[5] Babuška, I., Aziz, A.K.: On the angle condition in the finite element method. SIAM J. Numer. Anal. **13**, 214-226 (1976)

[6] Babuška, I., Suri, M.: The p and h-p versions of the finite elements method, basic principles and properties. SIAM Review. **36**, 578-632 (1994)

[7] Barnhill, R.E., Gregory, J.A.: Sard kernel theorems on triangular domains with applications to finite element error bounds. Numer. Math. **25**, 215-229 (1975/1976)

[8] Bernardi, C., Girault, V., Hecht, F., Raviart, P.-A., Rivière, B.: Mathematics and Finite Element Discretizations of Incompressible Navier–Stokes Flows. SIAM, (2024)

[9] Brandts, J., Korotov, S., Křížek, M.: On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions. Comput. Math, Appl. **55**, 2227-2233 (2008)

[10] Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, Third Edition. Springer Verlag, New York (2008)

[11] Chen, S., Shi, D., Zhao, Y.: Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes. IMA Journal of Numerical Analysis, **24**, 77-95 (2004)

[12] Cheng, S. -W., Dey, T. K., Edelsbrunner, H., Facello, M. A., Teng, S. -H.: Sliver Exudation. J. ACM, **47**, 883-904 (2000)

[13] Ciarlet, P. G.: The Finite Element Method for Elliptic problems. SIAM, New York (2002)

[14] Dekel, S., Leviatan, D.: The Bramble–Hilbert Lemma for Convex Domains, SIAM Journal on Mathematical Analysis **35** No. 5, 1203-1212 (2004)

[15] Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. Math. Comp. **34**, 441-463 (1980)

[16] Ern, A., Guermond, J.L.: Theory and Practice of Finite Elements. Springer Verlag, New York (2004)

[17] Ern, A., Guermond, J. L.: Finite Elements I: Approximation and Interpolation. Springer-Verlag, New York (2021)

[18] Ern, A., Guermond, J. L.: Finite elements II: Galerkin Approximation, Elliptic and Mixed PDEs. Springer-Verlag, New York (2021)

[19] Ern, A. and Guermond, J. L.: Finite elements III: First-Order and Time-Dependent PDEs. Springer-Verlag, New York (2021)

[20] Gellert, W., Gottwald, S., Hellwich, M., Kästner, H., Küstner, H.: The VNR Concise Encyclopedia of Mathematics, Springer, (1975)

[21] Golub, G. H., Loan, C. F. V.: Matrix Computations 3rd edition. The Johas Hopkins University Press, (1996)

[22] Gregory, J.A.: Error bounds for linear interpolation on triangles: in: J.R. Whiteman (Ed.), Proc. MAFELAP II, Academic Press, London, (1976)

[23] Hitotsumatsu, S., Kuroyanagi, K.: Geometry by barycentric coordinates, Gendaishugakusha (2014) (in Japanese)

[24] Ishizaka H. Anisotropic interpolation error analysis using a new geometric parameter and its applications. Ehime University, Ph. D. thesis (2022)

[25] Ishizaka, H.: Anisotropic Raviart–Thomas interpolation error estimates using a new geometric parameter. Calcolo 59 (4), (2022)

[26] Ishizaka, H.: Anisotropic weakly over-penalised symmetric interior penalty method for the Stokes equation. Journal of Scientific Computing 100, (2024)

[27] Ishizaka, H.: Morley finite element analysis for fourth-order elliptic equations under a semi-regular mesh condition. Applications of Mathematics **69** (6), 769–805 (2024)

[28] Ishizaka, H.: Hybrid weakly over-penalised symmetric interior penalty method on anisotropic meshes. Calcolo **61**, (2024)

[29] Ishizaka, H.: Anisotropic modified Crouzeix-Raviart finite element method for the stationary Navier-Stokes equation. Numeriche Mathmatik, (2025)

[30] Ishizaka, H.: Nitsche method under a semi-regular mesh condition. preprint (2025) `https://arxiv.org/abs/2501.06824`

[31] Ishizaka, H., Kobayashi, K., Suzuki, R., Tsuchiya, T.: A new geometric condition equivalent to the maximum angle condition for tetrahedrons. Comput. Math. Appl. 99, 323–328 (2021)

[32] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: General theory of interpolation error estimates on anisotropic meshes. Jpn. J. Ind. Appl. Math. 38 (1), 163–191 (2021)

[33] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: Crouzeix–Raviart and Raviart–Thomas finite element error analysis on anisotropic meshes violating the maximum-angle condition. Jpn. J. Ind. Appl. Math. 38 (2), 645-675 (2021)

[34] Ishizaka, H., Kobayashi, K., Tsuchiya, T.: Anisotropic interpolation error estimates using a new geometric parameter. Jpn. J. Ind. Appl. Math. 40 (1), 475–512 (2023)

[35] Jamet, P.: Estimation de I'erreur pour des éléments finis droits presque dégénérés. RAIRO Anal. Numér **10**, 43-60 (1976)

[36] Křížek, M.: On semiregular families of triangulations and linear interpolation. Appl. Math. Praha **36**, 223-232 (1991)

[37] Křížek, M.: On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. **29**, 513-520 (1992)

[38] Ladyženskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs **23**, AMS, Providence, (1968)

[39] Mario, B.: A note on the Poincaré inequality for convex domains. Z. Anal. ihre. Anwend., **22**, 751-756 (2003)

[40] Payne, L.E., Weinberger, H.F.: An optimal Poincaré-inequality for convex domains, Arch. Rational Mech. Anal. **5**, 286-292 (1960)

[41] Synge, J.L.: The Hypercircle in Mathematical Physics. Cambridge Univ. Press, Cambridge (1957)

[42] Todhunter, I.: Spherical Trigonometry: For the Use of Colleges and Schools (5th ed.), MacMillan, (1886)

[43] Verfürth, R.: A note on polynomial approximation in Sobolev spaces, Math. Modelling and Numer. Anal. **33**, 715-719 (1999)

[44] Zlámak, M.: On the finite element method, Numer. Math. **12**, 394-409 (1968)