Reconsidered error analysis for finite element methods

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Abstract

This article presents novel proof methods for estimating finite element errors, predicated on the understanding that one has already studied foundational error analysis using the finite element method. This article summarises References [7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18]. We are also correcting any typos found in each paper as we find them. The purpose is to make an easy-to-understand note of 'Special Topics in Finite Element Methods.' The latest version is available on the web.

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Part I
Preliminaries

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise, all constants c are bounded if the maximum angle is bounded. These values vary across different contexts. Furthermore, we use some abbreviations.

FE	Finite Element
FEMs	Finite Element Methods
CR	Crouzeix–Raviart
RT	Raviart–Thomas

If nothing is stated, the symbols are the same as in [14].

1 Meshes, Mesh faces, Averages and Jumps

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \mathrm{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such a mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^\partial$ be the set of faces on the boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^\partial$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

Furthermore, for a simplex $T \subset \mathbb{R}^d$, let $\mathcal{F}_T$ be the collection of the faces of T.

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1,\infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h;\mathbb{R}^q) := \{v \in L^p(\Omega;\mathbb{R}^q) : \ v|_T \in W^{s,p}(T;\mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} := \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \quad \text{if } p \in [1,\infty),$$
$$\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} := \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h;\mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h;\mathbb{R}^q) := W^{s,2}(\mathbb{T}_h;\mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h;\mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|^2_{L^2(T)^d} \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$\begin{aligned} [\![v \cdot n]\!] &:= [\![v \cdot n]\!]_F := v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp, \\ [\![v]\!] &:= [\![v]\!]_F := v_\natural - v_\sharp, \quad \natural > \sharp, \end{aligned}$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_\omega := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F} v_\natural + \omega_{T_\sharp,F} v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi) \cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1,\infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div};T)$ space by

$$H(\mathrm{div};\mathbb{T}_h) := \left\{ v \in L^2(\Omega)^d;\ v|_T \in H(\mathrm{div};T)\ \forall T \in \mathbb{T}_h \right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div};\mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div};\mathbb{T}_h)$,

$$(\mathrm{div}_h\, v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

2 Various FE Spaces

2.1 Spaces of Polynomials

Let $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$. Let $\mathbb{P}^k$ be the space of polynomials in the variables $x_1, \ldots, x_d$, with real coefficients and of global degree at most k,

$$\mathbb{P}^k := \left\{ p(x) = \sum_{0 \le i_1,\ldots,i_d \le k, i_1+\cdots+i_d \le k} \alpha_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d};\ \alpha_{i_1,\ldots,i_d} \in \mathbb{R} \right\},$$

with

$$N^{(d,k)} := \dim \mathbb{P}^k = \binom{d+k}{k}.$$

For a simplex $T \subset \mathbb{R}^d$, $\mathbb{P}^k(T)$ is spanned by the restriction to T of polynomials in $\mathbb{P}^k$. Furthermore, we set

$$\mathbb{R}^k(\partial T) := \{\varphi_h \in L^2(\partial T) : \ \varphi_h|_F \in \mathbb{P}^k(F) \ \forall F \in \mathcal{F}_T\}.$$

Then, the RT polynomial space of order $k \in \mathbb{N}_0$ is defined as

$$\mathbb{RT}^k(T) := \{q \in (\mathbb{P}^k(T)^d + x\mathbb{P}^k(T)) : q \cdot n \in \mathbb{R}^k(\partial T)\},$$

with

$$N^{(RT)} := \dim \mathbb{RT}^k(T).$$

2.2 FE Spaces

We define various FE spaces as follows.

2.2.1 Lagrange FE Spaces

For $k \in \mathbb{N}$, a broken FE space V_h^{kDC} is defined as

$$V_h^{kDC} := \{\varphi_h \in L^\infty(\Omega); \ \varphi_h|_T \in \mathbb{P}^k(T) \quad \forall T \in \mathbb{T}_h\}. \tag{2.1}$$

Then, the Lagrange FE spaces V_h^{kL} and V_{h0}^{kL} are defined as

$$V_h^{kL} := \left\{\varphi_h \in V_h^{kDC} : \ [\![\varphi_h]\!]_F = 0 \quad \forall F \in \mathcal{F}_h^i\right\} \subset H^1(\Omega), \tag{2.2}$$

$$V_{h0}^{kL} := \left\{\varphi_h \in V_h^{kL} : \ \varphi_h|_{\partial\Omega} = 0\right\} \subset H_0^1(\Omega). \tag{2.3}$$

2.2.2 RT FE spaces

For $k \in \mathbb{N}_0$, we define a broken finite element space as

$$RT^k(\mathbb{T}_h) := \left\{v_h \in L^1(\Omega)^d; \ v_h|_T \in \mathbb{RT}^k(T) \quad \forall T \in \mathbb{T}_h\right\}. \tag{2.4}$$

The RT finite element space is defined as

$$V_h^{kRT} := \{v_h \in RT^k(\mathbb{T}_h); \ [\![v_h \cdot n]\!]_F = 0, \quad \forall F \in \mathcal{F}_h^i\} \subset H(\mathrm{div};\Omega). \tag{2.5}$$

2.2.3 Discontinuous FE space

For $k \in \mathbb{N}$, we define the standard discontinuous space as

$$W_h^{kDC} := \left\{p_h \in L^1(\Omega); \ p_h|_T \in \mathbb{P}^k(T) \quad \forall T \in \mathbb{T}_h\right\}. \tag{2.6}$$

2.2.4 CR FE spaces

We define the CR finite element spaces as

$$V_h^{CR} := \left\{\varphi_h \in W_h^{1DC}; \ \int_F [\![\varphi_h]\!]_F ds = 0 \quad \forall F \in \mathcal{F}_h^i\right\}, \tag{2.7}$$

$$V_{h0}^{CR} := \left\{\varphi_h \in V_h^{CR}; \ \int_F [\![\varphi_h]\!]_F ds = 0 \quad \forall F \in \mathcal{F}_h^\partial\right\}. \tag{2.8}$$

2.2.5 Morley FE spaces

The Morley finite element spaces are as follows:

$$V_h^M := \Big\{ \varphi_h \in W_h^{2DC} : \int_F \left[\!\!\left[\frac{\partial \varphi_h}{\partial n} \right]\!\!\right] ds = 0 \ \forall F \in \mathcal{F}_h^i,$$
$$\text{the integral average of } \varphi_h \text{ over each } (d-2)\text{-dimensional}$$
$$\text{subsimplex of } T \in \mathbb{T}_h \text{ is continuous} \Big\}.$$

In particular, for $d = 2$, the space V_{h0}^M is defined as

$$V_{h0}^M := \Big\{ \varphi_h \in W_h^{2DC} : \int_F \left[\!\!\left[\frac{\partial \varphi_h}{\partial n} \right]\!\!\right] ds = 0 \ \forall F \in \mathcal{F}_h,$$
$$\varphi_h \text{ is continuous at each vertex in } \Omega, \ \varphi_h(p) = 0, \ \ p \in \partial\Omega \Big\}.$$

3 Interpolation Error Estimates

Our strategy for interpolation errors on anisotropic meshes was proposed by [7, 8, 14, 16, 18].

3.1 Edge Characterisation on a Simplex

For $T \in \mathbb{T}_h$, we characterise the edges of T.

Condition 3.1 (Case in which $d = 2$)**.** We assume that $\overline{p_2 p_3}$ is the longest edge of T, that is, $h_T := |p_2 - p_3|$. We assume that $h_2 \leq h_1$. We then have $h_1 = |p_1 - p_2|$ and $h_2 = |p_1 - p_3|$. Because $\frac{1}{2} h_T < h_1 \leq h_T$, $h_1 \approx h_T$.

Condition 3.2 (Case in which $d = 3$)**.** Let $T \in \mathbb{T}_h$ contain vertices p_i $(i = 1, \ldots, 4)$. Let L_i $(1 \leq i \leq 6)$ be the edges of T. We denote by $L_{\min}$ the edge of T with the minimum length; i.e. $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$. We set $h_2 := |L_{\min}|$ and assume that

$$\text{the endpoints of } L_{\min} \text{ are either } \{p_1, p_3\} \text{ or } \{p_2, p_3\}.$$

Among the four edges that share an endpoint with $L_{\min}$, we consider the longest edge $L_{\max}^{(\min)}$. Let p_1 and p_2 be the endpoints of edge $L_{\max}^{(\min)}$. Thus, we have

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting $\mathbb{R}^3$ with a plane that contains the midpoint of edge $L_{\max}^{(\min)}$ and is perpendicular to vector $p_1 - p_2$. Thus, there are two cases.

(Type i) p_3 and p_4 belong to the same half-space;

(Type ii) p_3 and p_4 belong to different half-spaces.

In each case, we set

(Type i) p_1 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_1 - p_3|$;

(Type ii) p_2 and p_3 as the endpoints of $L_{\min}$, that is, $h_2 = |p_2 - p_3|$.

Finally, we set $h_3 = |p_1 - p_4|$. We implicitly assume that p_1 and p_4 belong to the same half-space. Additionally, note that $h_1 \approx h_T$.

3.2 Additional Notations and Assumptions

We define the vectors $r_n \in \mathbb{R}^d$ and $n = 1, \ldots, d$ as follows: If $d = 2$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

and if $d = 3$,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \dfrac{p_3 - p_1}{|p_3 - p_1|}, & \text{for case (i)}, \\ r_2 := \dfrac{p_3 - p_2}{|p_3 - p_2|} & \text{for case (ii)}. \end{cases}$$

For a sufficiently smooth function φ and a vector function $v := (v_1, \ldots, v_d)^T$, we define the directional derivative for $i \in \{1, \ldots, d\}$ as

$$\begin{aligned} \frac{\partial \varphi}{\partial r_i} &:= (r_i \cdot \nabla_x)\varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}}, \\ \frac{\partial v}{\partial r_i} &:= \left(\frac{\partial v_1}{\partial r_i}, \ldots, \frac{\partial v_d}{\partial r_i} \right)^T = ((r_i \cdot \nabla_x)v_1, \ldots, (r_i \cdot \nabla_x)v_d)^T. \end{aligned}$$

For a multiindex $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we use the notation

$$\partial^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|}\varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}}, \quad h^\beta := h_1^{\beta_1} \cdots h_d^{\beta_d}.$$

We note that $\partial^\beta \varphi \neq \partial_r^\beta \varphi$.

3.3 New Geometric Parameter and Condition

We proposed a new geometric parameter H_T in [16].

Definition 3.3. Parameter H_T is defined as follows:

$$H_T := \frac{\prod_{i=1}^d h_i}{|T|_d} h_T.$$

We introduce geometric conditions to obtain the optimal convergence rate of the anisotropic error estimates.

Condition 3.4. A family of meshes $\{\mathbb{T}_h\}$ is semi-regular if there exists $\gamma_0 > 0$ such that

$$\frac{H_T}{h_T} \leq \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{3.1}$$

Remark 3.5. The geometric condition in (3.1) is equivalent to the maximum angle condition ([14, Section 7]).

3.4 Lagrange Interpolation Error Estimates

Let $T \in \mathbb{T}_h$. The local Lagrange interpolation operator is defined as

$$I_T^L : \mathcal{C}(T) \ni \varphi \mapsto I_T^L \varphi := \sum_{i=1}^{N^{(d,k)}} \varphi(p_i)\theta_i \in \mathbb{P}^k,$$

where p_i ($i = 1, \dots, N^{(d,k)}$) are Langange nodes and θ_i ($i = 1, \dots, N^{(d,k)}$) are Lagrange basis functions, see [14, Section 16.2]. We also define the global interpolation I_h^L to space $V_{h,k}^L$ as

$$(I_h^L \varphi)|_T := I_T^L(\varphi|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall \varphi \in \mathcal{C}(\overline{\Omega}).$$

We have the following local Lagrange interpolation error estimate.

Theorem 3.6. Let $k \in \mathbb{N}$. Let $m \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, and $p \in \mathbb{R}$ be such that $0 \leq m \leq \ell \leq k+1$ and

$$d = 2 : \begin{cases} p \in (2, \infty] & \text{if } m = 0,\ \ell = 1, \\ p \in [1, \infty] & \text{if } m = 0,\ \ell \geq 2 \text{ or } m \geq 1,\ \ell - m \geq 1, \end{cases}$$

$$d = 3 : \begin{cases} p \in \left(\frac{3}{\ell}, \infty\right] & \text{if } m = 0,\ \ell = 1, 2, \\ p \in (2, \infty] & \text{if } m \geq 1,\ \ell - m = 1, \\ p \in [1, \infty] & \text{if } m = 0,\ \ell \geq 3 \text{ or } m \geq 1,\ \ell - m \geq 2. \end{cases}$$

Setting $q \in [1, \infty)$ be such that

$$W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T}), \tag{3.2}$$

that is $(\ell - m) - \frac{d}{p} \geq -\frac{d}{q}$. Then, for all $\varphi \in W^{\ell,p}(T)$, we have

$$|\varphi - I_T^L \varphi|_{W^{m,q}(T)} \leq c|T|_d^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon|=\ell-m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)}. \tag{3.3}$$

Proof. A proof can be found in [14, Theorem 16.7] and [18, Corollary 1]. □

3.5 RT Interpolation Error Estimates

For $T \in \mathbb{T}_h$, let the points $\{p_1, \dots, p_{d+1}\}$ be the vertices of the simplex T. Let F_i be the face of T opposite p_i for $i \in \{1, \dots, d+1\}$. The lowest-order RT interpolation $I_T^{0RT} : H^1(T)^d \to \mathbb{RT}^0(T)$ is defined as

$$I_T^{0RT} : H^1(T)^d \ni v \mapsto I_T^{0RT} v := \sum_{i=1}^{d+1} \left(\int_{F_i} v \cdot n_{F_i} ds \right) \theta_i^{0RT} \in \mathbb{RT}^0(T),$$

where n_F denotes the outer unit normal vector of T on the face F and the local shape functions are defined as

$$\theta_i^{0RT}(x) := \frac{\iota_{F_i,T}}{d|T|_d}(x - p_i) \quad \forall i \in \{1, \dots, d+1\},$$

where $\iota_{F_i,T} := 1$ if n_{F_i} points outwards, and -1 otherwise [4, Chapter 14]. We define the global RT interpolation $I_h^{0RT} : H^1(\Omega)^d \to V_h^{0RT}$ as

$$(I_h^{0RT} v)|_T = I_T^{0RT}(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in H^1(\Omega)^d.$$

The following two theorems are divided into elements of (Type i) and (Type ii) in Section 3.1 when $d = 3$.

Theorem 3.7. Let T with $T = \Phi_T(\widetilde{T})$ and $\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T})$ be an element with Conditions 3.1 or 3.2 satisfying (Type i) in Section 3.1 when $d = 3$. Then, for any $v \in H^1(T)^d$,

$$\|I_T^{0RT} v - v\|_{L^2(T)^d} \le c \left(\frac{H_T}{h_T} \sum_{i=1}^{d} h_i \left\| \frac{\partial v}{\partial r_i} \right\|_{L^2(T)^d} + h_T \| \operatorname{div} v \|_{L^2(T)} \right). \tag{3.4}$$

Proof. A proof can be found in [14, Theorem 20.14] and [8, Theorem 2]. □

Theorem 3.8. Let $d = 3$. Let T with $T = \Phi_T(\widetilde{T})$ and $\widetilde{T} = \Phi_{\widetilde{T}}(\widehat{T})$ be an element with Condition 3.2 that satisfies (Type ii) in Section 3.1. For $v = (v_1, v_2, v_3)^T \in H^1(T)^3$,

$$\|I_T^{0RT} v - v\|_{L^2(T)^3} \le c \frac{H_T}{h_T} \left(h_T |v|_{H^1(T)^3} \right). \tag{3.5}$$

Proof. This proof is provided in [14, Theorem 20.15] and [8, Theorem 3]. □

Remark 3.9. Below, we use the interpolation error estimate (3.4) in the r_i-coordinate system for analysis.

3.6 Error Estimte of the L^2-orthogonal projection

For $T \in \mathbb{T}_h$, let $\Pi_T^0 : L^2(T) \to \mathbb{P}^0(T)$ be the L^2-orthogonal projection defined as

$$\Pi_T^0 \varphi := \frac{1}{|T|_d} \int_T \varphi dx \quad \forall \varphi \in L^2(T).$$

We also define the global interpolation Π_h^0 to space W_h^{0DC} as

$$(\Pi_h^0 \varphi)|_T := \Pi_T^0 (\varphi|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall \varphi \in L^2(\Omega).$$

The following theorem provides an anisotropic error estimate for the projection Π_T^0.

Theorem 3.10. For any $\varphi \in H^1(T)$,

$$\|\Pi_T^0 \varphi - \varphi\|_{L^2(T)} \le c \sum_{i=1}^{d} h_i \left\| \frac{\partial \varphi}{\partial r_i} \right\|_{L^2(T)}. \tag{3.6}$$

Proof. This proof can be found in [14, Theorem 17.2], [9, Theorem 2] and [10, Theorem 2]. □

3.7 CR Interpolation Error Estimates

For $T \in \mathbb{T}_h$, let the points $\{p_1, \dots, p_{d+1}\}$ be the vertices of the simplex T. Let F_i be the face of T opposite p_i for $i \in \{1, \dots, d+1\}$. The CR interpolation operator $I_T^{CR} : H^1(T) \to \mathbb{P}^1(T)$ is defined as, for any $\varphi \in H^1(T)$,

$$I_T^{CR} : H^1(T) \ni \varphi \mapsto I_T^{CR} \varphi := \sum_{i=1}^{d+1} \left(\frac{1}{|F_i|_{d-1}} \int_{F_i} \varphi ds \right) \theta_i^{CR} \in \mathbb{P}^1(T).$$

where θ_i^{CR} is the basis of the CR finite element such that

$$\theta_i^{CR}(x) := d \left(\frac{1}{d} - \lambda_i(x) \right) \quad \forall i \in \{1, \dots, d+1\}.$$

Here, $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ denote the barycentric coordinates. We define the global interpolation operator $I_h^{CR} : H^1(\Omega) \to V_h^{CR}$ as

$$(I_h^{CR} \varphi)|_T = I_T^{CR} (\varphi|_T), \quad \forall T \in \mathbb{T}_h, \quad \forall \varphi \in H^1(\Omega).$$

We then present the estimates of the anisotropic CR interpolation errors.

Theorem 3.11. For any $\varphi \in H^2(T)$,

$$|I_T^{CR}\varphi - \varphi|_{H^1(T)} \leq c \sum_{i=1}^{d} h_i \left| \frac{\partial \varphi}{\partial r_i} \right|_{H^1(T)}, \tag{3.7}$$

$$\|I_T^{CR}\varphi - \varphi\|_{L^2(T)} \leq c \sum_{|\varepsilon|=2} h^{\varepsilon} \|\partial_r^{\varepsilon}\varphi\|_{L^2(T)}. \tag{3.8}$$

Proof. The proof of (3.7) can be found in [9, Theorem 3] and [10, Theorem 3]. The proof for (3.8) can be found in [11, Theorem 2]. See also [14, Theorem 18.3]. □

3.8 Morley Interpolation Error Estimates

Let $T \in \mathbb{T}_h$. Let F_i, $1 \leq i \leq d+1$ be the $(d-1)$-dimensional subsimplex of T without vertices p_i and $S_{i,j}$, $1 \leq i < j \leq d+1$ be the $(d-2)$-dimensional subsimplex of T without vertices p_i and p_j. The Morley interpolation operator I_T^M is defined as

$$I_T^M : H^2(T) \ni \varphi \mapsto I_T^M \varphi \in \mathbb{P}^2,$$

with

$$I_T^M \varphi := \sum_{1 \leq i < j \leq d+1} \left(\frac{1}{|S_{i,j}|_{d-2}} \int_{S_{i,j}} \varphi ds \right) \theta_{i,j}^{(1)} + \sum_{1 \leq i \leq d+1} \left(\frac{1}{|F_i|_{d-1}} \int_{F_i} \frac{\partial \varphi}{\partial n_i} ds \right) \theta_i^{(2)},$$

where $\frac{\partial}{\partial n_i} = n_{T,i} \cdot \nabla$, and $n_{T,i}$ is the unit outer normal to $F_i \subset \partial T$ and the nodal basis functions are defined as follows:

$$\begin{aligned}
\theta_{i,j}^{(1)} &:= 1 - (d-1)(\lambda_i + \lambda_j) + d(d-1)\lambda_i\lambda_j \\
&\quad - (d-1)(\nabla\lambda_i)^{\top}\nabla\lambda_j \sum_{k=i,j} \frac{\lambda_k(d\lambda_k - 2)}{2|\nabla\lambda_k|_E^2}, \quad 1 \leq i < j \leq d+1, \\
\theta_i^{(2)} &:= \frac{\lambda_i(d\lambda_i - 2)}{2|\nabla\lambda_i|_E}, \quad 1 \leq i \leq d+1.
\end{aligned}$$

We also define the global interpolation $I_h^M : H^2(\Omega) \to V_h^M$ as follows.

$$(I_h^M \varphi)|_T := I_T^M(\varphi|_T) \quad \forall T \in \mathbb{T}_h, \ \forall \varphi \in H^2(\Omega).$$

The anisotropic Morley interpolation error estimate is expressed as

Theorem 3.12. For any $\varphi \in H^3(T)$, we have

$$|I_T^M \varphi - \varphi|_{H^2(T)} \leq c \sum_{i=1}^{d} h_i \left| \frac{\partial \varphi}{\partial r_i} \right|_{H^2(T)}. \tag{3.10}$$

Proof. A proof can be found in [14, Theorem 18.9] and [10, Theorem 4]. □

Part II

Continuous Problems

4 Poisson Equation

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