Reconsidered error analysis in the finite element methods

Hiroki ISHIZAKA

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Abstract

We introduce new proof methods for interpolation error estimates.

Contents

1 Preliminalies

1.1 General Convention

Throughout this article, we denote by c a constant independent of h (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants c are bounded if the maximum angle is bounded. These values vary across different contexts.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert \cdot \rvert_d$	d-dimensional Hausdorff measure
$v\|_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x \rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x \rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
A, B	Matrices
A_{ij} or $[A]_{ij}$	Entry of A in the ith and the jth column
$A^\top$	Transpose of the matrix A
$\mathrm{Tr}(A)$	Trace of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A_{ii}$
$\det(A)$	Determinant of A
$\mathrm{diag}(A)$	Diagonal of A: For $A \in \mathbb{R}^{m \times n}$, $\mathrm{diag}(A)_{ij} := \delta_{ij} A_{ij}$, $1 \leq i, j \leq d$
Ax	Matrix-vector product: For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, $(Ax)_i := \sum_{j=1}^{d} A_{ij} x_j$ for $1 \leq i \leq d$
$A : B$	Double contraction: For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$, $A : B := \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$
$\lVert A \rVert_2$	Operator norm of A: For $A \in \mathbb{R}^{d \times d}$, $\lVert A \rVert_2 := \sup_{0 \neq x \in \mathbb{R}^d} \frac{\lvert Ax \rvert_E}{\lvert x \rvert_E}$
$\lVert A \rVert_{\max}$	Max norm of A: For $A \in \mathbb{R}^{d \times d}$, $\lVert A \rVert_{\max} := \max_{1 \leq i,j \leq d} \lvert A_{ij} \rvert$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this article, we use the following facts.

For $A \in \mathbb{R}^{m \times n}$, it holds that

$$\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn}\|A\|_{\max}, \tag{1.1}$$

e.g., see [10, p. 56]. For $A, B \in \mathbb{R}^{m \times m}$, it holds that

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2. \tag{1.2}$$

If $A^\top A$ is a positive definite matrix in $\mathbb{R}^{d \times d}$, the spectral norm of the matrix $A^\top A$ is the largest eigenvalue of $A^\top A$; i.e.,

$$\|A\|_2 = \left(\lambda_{\max}(A^\top A)\right)^{1/2} = \sigma_{\max}(A), \tag{1.3}$$

where $\lambda_{\max}(A)$ and $\sigma_{\max}(A)$ are respectively the largest eigenvalues and singular values of A.

If $A \in O(d)$, because $A^\top = A^{-1}$ and

$$|Ax|_E^2 = (Ax)^\top (Ax) = x^\top A^\top A x = x^\top A^{-1} A x = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0 \neq x \in \mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0 \neq x \in \mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

1.4 Function Spaces

In this article, we use standard Sobolev spaces with associated norms (e.g., see [2, 6, 7]).

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \ldots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T)$, $\mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$) is spanned by the restriction to T of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$)

1.5.2 Meshes

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \operatorname{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $p_1, \dots, p_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex p_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let $\mathcal{F}_h^i$ be the set of interior faces, and $\mathcal{F}_h^{\partial}$ be the set of faces on boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^{\partial}$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$, let n_F be the unit normal vector from $T_\natural$ to $T_\sharp$. (ii) If $F \in \mathcal{F}_h^{\partial}$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1, \infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h; \mathbb{R}^q) := \{v \in L^p(\Omega; \mathbb{R}^q) : \ v|_T \in W^{s,p}(T; \mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\begin{aligned} \|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} &:= \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \qquad \text{if } p \in [1, \infty), \\ \|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} &:= \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}. \end{aligned}$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h; \mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h; \mathbb{R}^q) := W^{s,2}(\mathbb{T}_h; \mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h; \mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|^2_{L^2(T)^d} \right)^{1/2} \qquad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in H^1(\mathbb{T}_h)$. Suppose that $F \in \mathcal{F}_h^i$ with $F = T_\natural \cap T_\sharp$, $T_\natural, T_\sharp \in \mathbb{T}_h$, $\natural > \sharp$. We set $\varphi_\natural := \varphi|_{T_\natural}$ and $\varphi_\sharp := \varphi|_{T_\sharp}$. The jump in φ across F is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_\natural - \varphi_\sharp, \quad \natural > \sharp.$$

For a boundary face $F \in \mathcal{F}_h^{\partial}$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$. For any $v \in H^1(\mathbb{T}_h)^d$, the notations

$$\begin{aligned} [\![v \cdot n]\!] &:= [\![v \cdot n]\!]_F := v_\natural \cdot n_F - v_\sharp \cdot n_F, \quad \natural > \sharp, \\ [\![v]\!] &:= [\![v]\!]_F := v_\natural - v_\sharp, \quad \natural > \sharp, \end{aligned}$$

denote the jump in the normal component of v and the jump of v. Set two nonnegative real numbers $\omega_{T_\natural,F}$ and $\omega_{T_\sharp,F}$ such that

$$\omega_{T_\natural,F} + \omega_{T_\sharp,F} = 1.$$

The skew-weighted average of φ across F is then defined as

$$\{\{\varphi\}\}_{\overline{\omega}} := \{\{\varphi\}\}_{\overline{\omega},F} := \omega_{T_\sharp,F}\varphi_\natural + \omega_{T_\natural,F}\varphi_\sharp.$$

For a boundary face $F \in \mathcal{F}_h^\partial$ with $F = \partial T \cap \partial\Omega$, $\{\{\varphi\}\}_{\overline{\omega}} := \varphi|_T$. Furthermore,

$$\{\{v\}\}_\omega := \{\{v\}\}_{\omega,F} := \omega_{T_\natural,F}v_\natural + \omega_{T_\sharp,F}v_\sharp,$$

for the weighted average of v. For any $v \in H^1(\mathbb{T}_h)^d$ and $\varphi \in H^1(\mathbb{T}_h)$,

$$[\![(v\varphi)\cdot n]\!]_F = \{\{v\}\}_{\omega,F} \cdot n_F[\![\varphi]\!]_F + [\![v\cdot n]\!]_F\{\{\varphi\}\}_{\overline{\omega},F}.$$

We define a broken gradient operator as follows. Let $p \in [1,\infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h\varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div};T)$ space by

$$H(\mathrm{div};\mathbb{T}_h) := \left\{v \in L^2(\Omega)^d;\ v|_T \in H(\mathrm{div};T)\ \forall T \in \mathbb{T}_h\right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div};\mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div};\mathbb{T}_h)$,

$$(\mathrm{div}_h v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

Let r, s be two nonnegative real numbers and $\{x_i\}_{i\in I}$ be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \le \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r \le s, \\ \left(\sum_{i\in I} x_i^s\right)^{\frac{1}{s}} \le \mathrm{card}(I)^{\frac{r-s}{rs}} \left(\sum_{i\in I} x_i^r\right)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \tag{1.4}$$

see [7, Exercise 12.1].

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.1. Let $d \ge 2$, $s > 0$, and $p \in [1,\infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) \quad \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) \quad \forall q \in [p,\infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) \quad \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.5}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.6}$$

Proof. See, for example, [6, Corollary B.43, Theorem B.40] and [7, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.2. Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.7}$$

Proof. See, for example, [22, Theorem 2.1 (p. 61)] and the references therein. □

1.6.3 Trace Theorem

Theorem 1.3 (Trace on low-dimensional manifolds)**.** Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0, \dots, d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.

Proof. See [7, Theorem 3.15]. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [5, 3]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [1, Lemma 2.1].

Lemma 1.4. Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped with respect to balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leq m \leq \ell$, $p \in [1, \infty]$. It then holds that

$$\|\partial^\gamma(\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \leq C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.8}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \leq \ell - 1} \int_B \eta(y)(\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \tag{1.9}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [4, Theorem 1.1] and [5, 3, 25] which are variants of the Bramble–Hilbert lemma.

Lemma 1.5. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d, m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.10}$$

Proof. The proof is found in [4, Theorem 1.1]. □

Remark 1.6. In [3, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d, m, \gamma) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.11}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\operatorname{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d, m, \gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d, m)$ of the estimate (1.10) does not depend on the geometric parameter γ.

Remark 1.7. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d, m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf*ü*rth [25].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathbb{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \leq C^{BH}(d, k, m) \operatorname{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0, 1, \dots, m-1. \tag{1.12}$$

Verf*ü*rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d, k, m) \leq \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3, 1, 2) \leq \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \leq \frac{\sqrt{6}}{\pi} |\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

becase $\operatorname{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.8 (Poincaré inequality)**.** Let $D \subset \mathbb{R}^d$ be a convex domain with diameter $\operatorname{diam}(D)$. It then holds that, for $\varphi \in H^1(D)$ with $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \leq \frac{\operatorname{diam}(D)}{\pi} |\varphi|_{H^1(D)}. \tag{1.13}$$

Proof. The proof is found in [23, Theorem 3.2], also see [24]. □

Remark 1.9. The coefficient $\frac{1}{\pi}$ of (1.13) may be improved.

2 Geometric Conditions

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