

# Reconsidered error analysis in the finite element methods

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## Abstract

This article presents novel proof methods for estimating interpolation errors, predicated on the understanding that one has already studied foundational error analysis using the finite element method.

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## Part I

# Interpolation Error Analysis using a New Geometric Parameter

## 1 Preliminaries

### 1.1 General Convention

Throughout this article, we denote by  $c$  a constant independent of  $h$  (defined later) and the angles and aspect ratios of simplices, unless specified otherwise all constants  $c$  are bounded if the maximum angle is bounded. These values vary across different contexts.

### 1.2 Basic Notation

$d$	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	$d$ -dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$ \cdot _d$	$d$ -dimensional Hausdorff measure
$v _D$	Restriction of the function $v$ to the set $D$
$\dim(V)$	Dimension of the vector space $V$
$\delta_{ij}$	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \dots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

### 1.3 Vectors and Matrices

$(v_1, \dots, v_d)^T$	Cartesian components of the vector $v$ in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$ : $x \cdot y := \sum_{i=1}^d x_i y_i$
$ x _E$	Euclidean norm in $\mathbb{R}^d$ : $ x _E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
$A, B$	Matrices
$A_{ij}$ or $[A]_{ij}$	Entry of $A$ in the $i$ th and the $j$ th column
$A^\top$	Transpose of the matrix $A$
$\text{Tr}(A)$	Trace of $A$ : For $A \in \mathbb{R}^{m \times n}$ , $\text{Tr}(A) := \sum_{i=1}^d A_{ii}$
$\det(A)$	Determinant of $A$
$\text{diag}(A)$	Diagonal of $A$ : For $A \in \mathbb{R}^{m \times n}$ , $\text{diag}(A)_{ij} := \delta_{ij} A_{ij}$ , $1 \leq i, j \leq d$
$Ax$	Matrix-vector product: For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ , $(Ax)_i := \sum_{j=1}^d A_{ij} x_j$ for $1 \leq i \leq d$
$A : B$	Double contraction:

	For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ , $A : B := \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$
$\ A\ _2$	Operator norm of $A$ : For $A \in \mathbb{R}^{d \times d}$ , $\ A\ _2 := \sup_{0 \neq x \in \mathbb{R}^d} \frac{ Ax _E}{ x _E}$
$\ A\ _{\max}$	Max norm of $A$ : For $A \in \mathbb{R}^{d \times d}$ , $\ A\ _{\max} := \max_{1 \leq i, j \leq d}  A_{ij} $
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant $\pm 1$

In this article, we use the following facts.

For  $A \in \mathbb{R}^{m \times n}$ , it holds that

$$\|A\|_{\max} \leq \|A\|_2 \leq \sqrt{mn} \|A\|_{\max}, \quad (1.1)$$

e.g., see [19, p. 56]. For  $A, B \in \mathbb{R}^{m \times m}$ , it holds that

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2. \quad (1.2)$$

If  $A^\top A$  is a positive definite matrix in  $\mathbb{R}^{d \times d}$ , the spectral norm of the matrix  $A^\top A$  is the largest eigenvalue of  $A^\top A$ ; i.e.,

$$\|A\|_2 = (\lambda_{\max}(A^\top A))^{1/2} = \sigma_{\max}(A), \quad (1.3)$$

where  $\lambda_{\max}(A)$  and  $\sigma_{\max}(A)$  are respectively the largest eigenvalues and singular values of  $A$ .

If  $A \in O(d)$ , because  $A^\top = A^{-1}$  and

$$|Ax|_E^2 = (Ax)^\top (Ax) = x^\top A^\top Ax = x^\top A^{-1} Ax = |x|_E^2,$$

it holds that

$$\|A\|_2 = \sup_{0 \neq x \in \mathbb{R}^d} \frac{|Ax|_E}{|x|_E} = \sup_{0 \neq x \in \mathbb{R}^d} \frac{|x|_E}{|x|_E} = 1.$$

## 1.4 Function Spaces

This article uses standard Sobolev spaces with associated norms (e.g., see [6, 14, 15]).

## 1.5 Finite-Element-Methods-Related Symbols

### 1.5.1 Symbols

$\mathbb{P}^k$	Vector space of polynomials in the variables $x_1, \dots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathbb{P}^k) = \binom{d+k}{k}$
$\mathbb{RT}^k$	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $\mathbb{RT}^k := (\mathbb{P}^k)^d + x\mathbb{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim \mathbb{RT}^k$
$T, \tilde{T}, \hat{T}, K$	Closed simplices in $\mathbb{R}^d$
$\mathbb{P}^k(T), \mathbb{RT}^k(T)$	$\mathbb{P}^k(T)$ (or $\mathbb{RT}^k(T)$ ) is spanned by the restriction to $T$ of polynomials in $\mathbb{P}^k$ (or $\mathbb{RT}^k$ )

### 1.5.2 Meshes

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be a bounded polyhedral domain. Furthermore, we assume that  $\Omega$  is convex if necessary. Let  $\mathbb{T}_h = \{T\}$  be a simplicial mesh of  $\overline{\Omega}$  made up of closed  $d$ -simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with  $h := \max_{T \in \mathbb{T}_h} h_T$ , where  $h_T := \text{diam}(T)$ . We also use a symbol  $\rho_T$  which means the radius of the largest ball inscribed in  $T$ . We assume that each face of any  $d$ -simplex  $T_1$  in  $\mathbb{T}_h$  is either a subset of the boundary  $\partial\Omega$  or a face of another  $d$ -simplex  $T_2$  in  $\mathbb{T}_h$ . That is,  $\mathbb{T}_h$  is a simplicial mesh of  $\overline{\Omega}$  without hanging nodes. Such mesh  $\mathbb{T}_h$  is said to be conformal. Let  $\{\mathbb{T}_h\}$  be a family of conformal meshes.

Let  $T$  be a simplex of  $\mathbb{T}_h$  which is a convex hull of  $d + 1$  vertices,  $p_1, \dots, p_{d+1}$ , that do not belong to the same hyperplane. Let  $S_i$  be the face of a simplex  $T$  opposite to the vertex  $p_i$ . For  $d = 3$ , angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

### 1.5.3 Broken Sobolev Spaces, Mesh faces, Averages and Jumps

Let  $\mathcal{F}_h^i$  be the set of interior faces, and  $\mathcal{F}_h^\partial$  be the set of faces on boundary  $\partial\Omega$ . We set  $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$ . For any  $F \in \mathcal{F}_h$ , we define the unit normal  $n_F$  to  $F$  as follows: (i) If  $F \in \mathcal{F}_h^i$  with  $F = T_{\natural} \cap T_{\sharp}$ ,  $T_{\natural}, T_{\sharp} \in \mathbb{T}_h$ ,  $\natural > \sharp$ , let  $n_F$  be the unit normal vector from  $T_{\natural}$  to  $T_{\sharp}$ . (ii) If  $F \in \mathcal{F}_h^\partial$ ,  $n_F$  is the unit outward normal  $n$  to  $\partial\Omega$ . We also use the following set. For any  $F \in \mathcal{F}_h$ ,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : F \subset T\}.$$

We consider  $\mathbb{R}^q$ -valued functions for some  $q \in \mathbb{N}$ . Let  $p \in [1, \infty]$  and  $s > 0$  be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h; \mathbb{R}^q) := \{v \in L^p(\Omega; \mathbb{R}^q) : v|_T \in W^{s,p}(T; \mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\begin{aligned} \|v\|_{W^{s,p}(\mathbb{T}_h; \mathbb{R}^q)} &:= \left( \sum_{T \in \mathbb{T}_h} \|v\|_{W^{s,p}(T; \mathbb{R}^q)}^p \right)^{1/p} \quad \text{if } p \in [1, \infty), \\ \|v\|_{W^{s,\infty}(\mathbb{T}_h; \mathbb{R}^q)} &:= \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T; \mathbb{R}^q)}. \end{aligned}$$

When  $q = 1$ , we denote  $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h; \mathbb{R})$ . When  $p = 2$ , we write  $H^s(\mathbb{T}_h)^q := H^s(\mathbb{T}_h; \mathbb{R}^q) := W^{s,2}(\mathbb{T}_h; \mathbb{R}^q)$  and  $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h; \mathbb{R})$ . We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left( \sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|_{L^2(T)^d}^2 \right)^{1/2} \quad \varphi \in H^1(\mathbb{T}_h).$$

Let  $\varphi \in H^1(\mathbb{T}_h)$ . Suppose that  $F \in \mathcal{F}_h^i$  with  $F = T_{\natural} \cap T_{\sharp}$ ,  $T_{\natural}, T_{\sharp} \in \mathbb{T}_h$ ,  $\natural > \sharp$ . We set  $\varphi_{\natural} := \varphi|_{T_{\natural}}$  and  $\varphi_{\sharp} := \varphi|_{T_{\sharp}}$ . The jump in  $\varphi$  across  $F$  is defined as

$$[\![\varphi]\!] := [\![\varphi]\!]_F := \varphi_{\natural} - \varphi_{\sharp}, \quad \natural > \sharp.$$

For a boundary face  $F \in \mathcal{F}_h^\partial$  with  $F = \partial T \cap \partial\Omega$ ,  $\llbracket \varphi \rrbracket_F := \varphi|_T$ . For any  $v \in H^1(\mathbb{T}_h)^d$ , the notations

$$\begin{aligned}\llbracket v \cdot n \rrbracket &:= \llbracket v \cdot n \rrbracket_F := v_{\natural} \cdot n_F - v_{\sharp} \cdot n_F, \quad \natural > \sharp, \\ \llbracket v \rrbracket &:= \llbracket v \rrbracket_F := v_{\natural} - v_{\sharp}, \quad \natural > \sharp,\end{aligned}$$

denote the jump in the normal component of  $v$  and the jump of  $v$ . Set two nonnegative real numbers  $\omega_{T_{\natural},F}$  and  $\omega_{T_{\sharp},F}$  such that

$$\omega_{T_{\natural},F} + \omega_{T_{\sharp},F} = 1.$$

The skew-weighted average of  $\varphi$  across  $F$  is then defined as

$$\{\{\varphi\}\}_{\bar{\omega}} := \{\{\varphi\}\}_{\bar{\omega},F} := \omega_{T_{\natural},F}\varphi_{\natural} + \omega_{T_{\sharp},F}\varphi_{\sharp}.$$

For a boundary face  $F \in \mathcal{F}_h^\partial$  with  $F = \partial T \cap \partial\Omega$ ,  $\{\{\varphi\}\}_{\bar{\omega}} := \varphi|_T$ . Furthermore,

$$\{\{v\}\}_{\omega} := \{\{v\}\}_{\omega,F} := \omega_{T_{\natural},F}v_{\natural} + \omega_{T_{\sharp},F}v_{\sharp},$$

for the weighted average of  $v$ . For any  $v \in H^1(\mathbb{T}_h)^d$  and  $\varphi \in H^1(\mathbb{T}_h)$ ,

$$\llbracket (v\varphi) \cdot n \rrbracket_F = \{\{v\}\}_{\omega,F} \cdot n_F \llbracket \varphi \rrbracket_F + \llbracket v \cdot n \rrbracket_F \{\{\varphi\}\}_{\bar{\omega},F}.$$

We define a broken gradient operator as follows. Let  $p \in [1, \infty]$ . For  $\varphi \in W^{1,p}(\mathbb{T}_h)$ , the broken gradient  $\nabla_h : W^{1,p}(\mathbb{T}_h) \rightarrow L^p(\Omega)^d$  is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken  $H(\text{div}; T)$  space by

$$H(\text{div}; \mathbb{T}_h) := \{v \in L^2(\Omega)^d; v|_T \in H(\text{div}; T) \quad \forall T \in \mathbb{T}_h\},$$

and the broken divergence operator  $\text{div}_h : H(\text{div}; \mathbb{T}_h) \rightarrow L^2(\Omega)$  such that, for all  $v \in H(\text{div}; \mathbb{T}_h)$ ,

$$(\text{div}_h v)|_T := \text{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

#### 1.5.4 Barycentric Coordinates

For a simplex  $T \subset \mathbb{R}^d$ , let  $\{p_i\}_{i=1}^{d+1}$  be vertices of  $T$  and  $(x_1^{(i)}, \dots, x_d^{(i)})^T$  coordinates of  $p_i$ . We set

$$\Delta := \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates  $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \rightarrow \mathbb{R}$  of the point  $p(x_1, \dots, x_d)$  with respect to  $\{p_i\}_{i=1}^{d+1}$  are then defined as

$$\lambda_i(x) := \frac{1}{\Delta} \det \begin{pmatrix} 1 & \cdots & \overset{i}{\underbrace{1}} & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(p_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1.$$

## 1.6 Useful Tools for Analysis

### 1.6.1 Jensen-type Inequality

Let  $r, s$  be two nonnegative real numbers and  $\{x_i\}_{i \in I}$  be a finite sequence of nonnegative numbers. It then holds that

$$\begin{cases} (\sum_{i \in I} x_i^s)^{\frac{1}{s}} \leq (\sum_{i \in I} x_i^r)^{\frac{1}{r}} & \text{if } r \leq s, \\ (\sum_{i \in I} x_i^s)^{\frac{1}{s}} \leq \text{card}(I)^{\frac{r-s}{rs}} (\sum_{i \in I} x_i^r)^{\frac{1}{r}} & \text{if } r > s, \end{cases} \quad (1.4)$$

see [15, Exercise 12.1].

### 1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

**Theorem 1.1.** Let  $d \geq 2$ ,  $s > 0$ , and  $p \in [1, \infty]$ . Let  $D \subset \mathbb{R}^d$  be a bounded open subset of  $\mathbb{R}^d$ . If  $D$  is a Lipschitz set, we then have

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap C^{0,\xi}(\overline{D}) & \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \quad (1.5)$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap C^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \quad (1.6)$$

**Proof.** See, for example, [14, Corollary B.43, Theorem B.40] and [15, Theorem 2.31] and the references therein.  $\square$

The following is the embedding theorem related to operator from  $W^{s,p}(D)$  into  $L^q(S_r)$ , where  $S_r$  is some plane  $r$ -dimensional piece belonging to  $D$  with dimensions  $r < d$ .

**Theorem 1.2.** Let  $p, q \in [1, +\infty]$  and  $s \geq 1$  be an integer. Let  $D \subset \mathbb{R}^d$  be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s}, r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \quad (1.7)$$

**Proof.** See, for example, [36, Theorem 2.1 (p. 61)] and the references therein.  $\square$

### 1.6.3 Trace Theorem

**Theorem 1.3** (Trace on low-dimensional manifolds). Let  $p \in [1, \infty)$  and let  $D$  be a Lipschitz domain in  $\mathbb{R}^d$ . Let  $M$  be a smooth, or polyhedral, manifold of dimension  $r$  in  $\overline{D}$ ,  $r \in \{0, \dots, d\}$ . Then, there exists a bounded trace operator from  $W^{s,p}(D)$  to  $L^p(M)$ , provided  $sp > d - r$ , or  $s \geq d - r$  if  $p = 1$ .

**Proof.** See [15, Theorem 3.15].  $\square$

### 1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [13, 8]) plays a major role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [2, Lemma 2.1].

**Lemma 1.4.** Let  $D \subset \mathbb{R}^d$  be a connected open set that is star-shaped concerning balls  $B$ . Let  $\gamma$  be a multi-index with  $m := |\gamma|$  and  $\varphi \in L^1(D)$  be a function with  $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$ , where  $\ell \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$ ,  $0 \leq m \leq \ell$ ,  $p \in [1, \infty]$ . It then holds that

$$\|\partial^\gamma(\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \leq C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \quad (1.8)$$

where  $C^{BH}$  depends only on  $d$ ,  $\ell$ ,  $\text{diam } D$ , and  $\text{diam } B$ , and  $Q^{(\ell)}\varphi$  is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \leq \ell-1} \int_B \eta(y) (\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathbb{P}^{\ell-1}, \quad (1.9)$$

where  $\eta \in \mathcal{C}_0^\infty(B)$  is a given function with  $\int_B \eta dx = 1$ .

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [12, Theorem 1.1] and [13, 8, 41] which are variants of the Bramble–Hilbert lemma.

**Lemma 1.5.** Let  $D \subset \mathbb{R}^d$  be a bounded convex domain. Let  $\varphi \in W^{m,p}(D)$  with  $m \in \mathbb{N}$  and  $1 \leq p \leq \infty$ . There exists a polynomial  $\eta \in \mathbb{P}^{m-1}$  such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d, m) \text{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \quad (1.10)$$

**Proof.** The proof is found in [12, Theorem 1.1].  $\square$

**Remark 1.6.** In [8, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let  $B$  be a ball in  $D \subset \mathbb{R}^d$  such that  $D$  is star-shaped with respect to  $B$  and its radius  $r > \frac{1}{2}r_{\max}$ , where  $r_{\max} := \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$ . Let  $\varphi \in W^{m,p}(D)$  with  $m \in \mathbb{N}$  and  $1 \leq p \leq \infty$ . There exists a polynomial  $\eta \in \mathbb{P}^{m-1}$  such that

$$|\varphi - \eta|_{W^{k,p}(D)} \leq C^{BH}(d, m, \gamma) \text{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \quad (1.11)$$

Here,  $\gamma$  is called the chunkiness parameter of  $D$ , which is defined by

$$\gamma := \frac{\text{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant  $C^{BH}(d, m, \gamma)$  depends on the chunkiness parameter. Meanwhile, the constant  $C^{BH}(d, m)$  of the estimate (1.10) does not depend on the geometric parameter  $\gamma$ .

**Remark 1.7.** For general Sobolev spaces  $W^{m,p}(\Omega)$ , the upper bounds on the constant  $C^{BH}(d, m)$  are not given, as far as we know. However, when  $p = 2$ , the following result has been obtained by Verfürth [41].

Let  $D \subset \mathbb{R}^d$  be a bounded convex domain. Let  $\varphi \in H^m(D)$  with  $m \in \mathbb{N}$ . There exists a polynomial  $\eta \in \mathbb{P}^{m-1}$  such that

$$|\varphi - \eta|_{H^k(D)} \leq C^{BH}(d, k, m) \text{diam}(D)^{m-k} |\varphi|_{H^m(D)}, \quad k = 0, 1, \dots, m-1. \quad (1.12)$$

Verfürth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d, k, m) \leq \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{[\frac{m-k}{d}]\}^{d/2}},$$



where  $[x]$  denotes the largest integer less than or equal to  $x$ .

As an example, let us consider the case  $d = 3$ ,  $k = 1$ , and  $m = 2$ . We then have

$$C^{BH}(3, 1, 2) \leq \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element  $\hat{T}$ , we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\hat{T})} \leq \frac{\sqrt{6}}{\pi} |\hat{\varphi}|_{H^2(\hat{T})} \quad \forall \hat{\varphi} \in H^2(\hat{T}),$$

because  $\text{diam}(\hat{T}) = \sqrt{2}$ .

### 1.6.5 Poincaré inequality

**Theorem 1.8** (Poincaré inequality). Let  $D \subset \mathbb{R}^d$  be a convex domain with diameter  $\text{diam}(D)$ . It then holds that, for  $\varphi \in H^1(D)$  with  $\int_D \varphi dx = 0$ ,

$$\|\varphi\|_{L^2(D)} \leq \frac{\text{diam}(D)}{\pi} |\varphi|_{H^1(D)}. \quad (1.13)$$

**Proof.** The proof is found in [37, Theorem 3.2], also see [38]. □

**Remark 1.9.** The coefficient  $\frac{1}{\pi}$  of (1.13) may be improved.

### 1.7 Abbreviated expression

FE	Finite Element
FEMs	Finite Element Methods

## 2 Isotropic and Anisotropic Mesh Elements

In the context of FEMs, mesh elements can be classified based on their geometric properties. An *isotropic mesh element* has equal or nearly equal edge lengths and angles, resulting in a balanced shape. In contrast, an *anisotropic mesh element* features significant variation in edge lengths and angles.

Consider the following examples: Let  $s, \delta \in \mathbb{R}_+$ , and  $\varepsilon \geq 1$ ,  $\varepsilon \in \mathbb{R}$ .

**Example 2.1.** In the case of the simplex  $T \subset \mathbb{R}^2$  defined by the vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (2s, 0)^\top$ , and  $p_3 := (s, \delta s)^\top$ , the triangle is classified as follows:

- If  $\delta \approx 1$ , the triangle  $T$  is considered an isotropic mesh element.
- Conversely, if  $\delta$  is much less than 1, i.e.,  $\delta \ll 1$ , the triangle  $T$  becomes an anisotropic mesh element.

**Example 2.2.** In this case, consider the simplex  $T \subset \mathbb{R}^2$  defined by the vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (2s, 0)^\top$ , and  $p_3 := (s, s^\varepsilon)^\top$ . Here, the vertex  $p_3$  introduces a parameter  $\varepsilon$  that can influence the shape of the simplex. The classification of this simplex as isotropic or anisotropic depends on the value of  $\varepsilon$ :

- If  $\varepsilon = 1$ , the triangle maintains a balanced shape, making it isotropic.

- If  $\varepsilon > 1$ , the triangle becomes flat when  $s \ll 1$ , resulting in an anisotropic mesh element.

**Example 2.3.** Consider the simplex  $T \subset \mathbb{R}^2$  defined by the vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$ , and  $p_3 := (0, \delta s)^\top$ . In this configuration, the classification of the simplex as isotropic or anisotropic depends on the value of  $\delta$ :

- If  $\delta \approx 1$ , the triangle is an isotropic mesh element.
- If  $\delta \ll 1$ , the triangle becomes an anisotropic mesh element.

**Example 2.4.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$ , and  $p_3 := (0, s^\varepsilon)^\top$ . In this case, the classification of the simplex as isotropic or anisotropic depends on the value of  $\varepsilon$ :

- If  $\varepsilon = 1$ , the triangle is isotropic because the height from  $p_3$  is equal to the base length.
- If  $\varepsilon > 1$ , the triangle will be classified as anisotropic, as the edge lengths will differ significantly when  $s \ll 1$ .

**Example 2.5.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$  and  $p_3 := (s^\delta, s^\varepsilon)^\top$ . The classification of the simplex into two types of anisotropic structures is determined by the values of  $\delta$  and  $\varepsilon$ :

- If  $1 < \varepsilon < \delta$ , the triangle is flattened so that the point  $p_3$  approaches the point  $p_1$ , i.e. the origin as  $s \rightarrow 0$ .
- If  $1 < \delta < \varepsilon$ , the triangle is flattened so that point  $p_3$  approaches a point on the straight line  $\overline{p_1 p_2}$  that does not include points  $p_1$  and  $p_2$  as  $s \rightarrow 0$ .

## 3 Classical Geometric Conditions

### 3.1 Classical Interpolation Error Estimate

Let  $\hat{T} \subset \mathbb{R}^d$  and  $T \subset \mathbb{R}^d$  be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements  $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$  and  $\{T, P, \Sigma\}$  with associated normed vector spaces  $V(\hat{T})$  and  $V(T)$ . The transformation  $\Phi_T$  takes the form

$$\Phi_T : \hat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := B_T \hat{x} + b_T \in T,$$

where  $B_T \in \mathbb{R}^{d \times d}$  is an invertible matrix and  $b_T \in \mathbb{R}^d$ . Let  $I_T : V(T) := W^{2,p}(T) \rightarrow P := \mathbb{P}^1(T)$  with  $p \in [1, \infty]$  be an interpolation on  $T$  with  $I_T p = p$  for any  $p \in \mathcal{P}^1(T)$ . According to the classical theory (e.g., see [11, 14]), there exists a positive constant  $c$ , independent of  $h_T$ , such that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c (\|B_T\|_2 \|B_T^{-1}\|_2) \|B_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity  $\|B_T\|_2 \|B_T^{-1}\|_2$  is called the *Euclidean condition number* of  $B_T$ . By standard estimates (e.g., see [14, Lemma 1.100]), we have

$$\|B_T\|_2 \|B_T^{-1}\|_2 \leq c \frac{h_T}{\rho_T}, \quad \|B_T\|_2 \leq c h_T.$$

It thus holds that

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}. \quad (3.1)$$

As a geometric condition, the *shape-regularity condition* is well known to obtain global interpolation error estimates. This condition is stated as follows.

**Condition 3.1** (Shape-regularity condition). There exists a constant  $\gamma_1 > 0$  such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \quad (3.2)$$

Under Condition 3.1, that is, when the quantity  $\frac{h_T}{\rho_T}$  is bounded on each  $T$ , it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq ch |\varphi|_{W^{2,p}(\Omega)},$$

where  $I_h \varphi$  is the standard global linear interpolation of  $\varphi$  on  $\mathbb{T}_h$ .

### 3.2 Regular Mesh Conditions

Geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity condition (3.2). A proof can be found in [7, Theorem 1].

**Condition 3.2** (Zlámal's condition). There exists a constant  $\gamma_2 > 0$  such that for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$ , any simplex  $T \in \mathbb{T}_h$  and any dihedral angle  $\psi$  and for  $d = 3$ , also any solid angle  $\theta$  of  $T$ , we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \quad (3.3)$$

**Condition 3.3.** There exists a constant  $\gamma_3 > 0$  such that for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ , we have

$$|T|_d \geq \gamma_3 h_T^d. \quad (3.4)$$

**Condition 3.4.** There exists a constant  $\gamma_4 > 0$  such that for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ , we have

$$|T|_d \geq \gamma_4 |B_d^T|, \quad (3.5)$$

where  $B^T \supset T$  is the circumscribed ball of  $T$ .

**Note 3.5.** If Condition 3.1 or 3.2 or 3.3 or 3.4 holds, a family of simplicial partitions is called *regular*.

**Note 3.6.** Condition 3.2 was presented by Zlámal [42] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on  $\mathbb{R}^2$ . Zlámal's condition can be generalised into  $\mathbb{R}^n$  for any  $n \in \{2, 3, \dots\}$ . Later, the shape-regularity condition (the inscribed ball condition) was introduced; see [11]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

**Note 3.7.** Condition 3.3 seems to be simpler than Condition 3.1, Condition 3.2 and Condition 3.4. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

### 3.3 What happens when anisotropic meshes are used?

Using the equivalence conditions in Section 3.2, the error estimate (3.1) is rewritten as

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq c \frac{h_T^2}{|T|_2} h_T |\varphi|_{W^{2,p}(T)}. \quad (3.6)$$

We considered the following five anisotropic elements as in Section 2: Let  $0 < s, \delta \ll 1$ ,  $s, \delta \in \mathbb{R}$ , and  $\varepsilon > 1$ ,  $\varepsilon \in \mathbb{R}$ .

**Example 3.8.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0,0)^\top$ ,  $p_2 := (2s,0)^\top$ , and  $p_3 := (s, \delta s)^\top$ . Then, we have that  $h_T = 2s$ ,  $|T|_2 = \delta s^2$ , and

$$\frac{h_T^2}{|T|_2} = \frac{4}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}.$$

When  $\delta \ll 1$ , the interpolation error (3.6) may be large.

**Example 3.9.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0,0)^\top$ ,  $p_2 := (2s,0)^\top$ , and  $p_3 := (s, s^\varepsilon)^\top$ . Then, we have that  $h_T = 2s$ ,  $|T|_2 = s^{1+\varepsilon}$  and

$$\frac{h_T^2}{|T|_2} = 4s^{1-\varepsilon} \rightarrow \infty \text{ as } s \rightarrow 0.$$

Therefore, the shape-regularity is not satisfied. In this case, when  $\varepsilon > 2$ , the estimate (3.6) diverges as  $s \rightarrow 0$ .

**Example 3.10.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0,0)^\top$ ,  $p_2 := (s,0)^\top$ , and  $p_3 := (0, \delta s)^\top$ . Then, we have that  $h_T = s\sqrt{1+\delta^2} \approx s$ ,  $|T|_2 = \frac{1}{2}\delta s^2$  and

$$\frac{h_T^2}{|T|_2} = \frac{2(1+\delta^2)}{\delta} < +\infty.$$

Therefore, the shape regularity is satisfied. The estimate (3.6) is as follows:

$$|\varphi - I_T \varphi|_{W^{1,p}(T)} \leq \frac{c}{\delta} h_T |\varphi|_{W^{2,p}(T)}. \quad (3.7)$$

It is implied that the interpolation error (3.7) may be large when  $\delta \ll 1$ .

**Example 3.11.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0,0)^\top$ ,  $p_2 := (s,0)^\top$ , and  $p_3 := (0, s^\varepsilon)^\top$ . Subsequently, we obtain  $h_T = \sqrt{s^2 + s^{2\varepsilon}} \approx s$ ,  $|T|_2 = \frac{1}{2}s^{1+\varepsilon}$  and

$$\frac{h_T^2}{|T|_2} = \frac{2(s^2 + s^{2\varepsilon})}{s^{1+\varepsilon}} \rightarrow \infty \text{ as } s \rightarrow 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as  $s \rightarrow 0$ .

**Example 3.12.** Let  $T \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0,0)^\top$ ,  $p_2 := (s,0)^\top$  and  $p_3 := (s^\delta, s^\varepsilon)^\top$ . If  $1 < \varepsilon < \delta$ , we have  $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ ,  $|T|_2 = \frac{1}{2}s^{1+\varepsilon}$  and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \leq cs^{1-\varepsilon} \rightarrow \infty \text{ as } s \rightarrow 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as  $s \rightarrow 0$ . If  $1 < \delta < \varepsilon$ , we have  $h_T = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$ ,  $|T|_2 = \frac{1}{2}s^{1+\varepsilon}$  and

$$\frac{h_T^2}{|T|_2} = \frac{2(s - s^\delta)^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \leq cs^{1-\varepsilon} \rightarrow \infty \text{ as } s \rightarrow 0.$$

Therefore, the shape-regularity condition is not satisfied. In this case, it is implied that the estimate (3.6) diverges as  $s \rightarrow 0$ .

**Remark 3.13.** As will be explained later, the factor  $\frac{1}{\delta}$  in Example 3.10 is violated. The interpolation error estimate converges in the cases of Example 3.11 and Example 3.12 with  $1 < \varepsilon < \delta$  using new precise interpolation error estimates under more relaxed geometric conditions.

## 4 Classical Relaxed Geometric Conditions

### 4.1 Semi-regular Mesh Conditions for $d = 2$

In 1957, Syngé [39, Section 3.8] proposed the following condition.

**Condition 4.1** (Syngé's condition). There exists  $\frac{\pi}{3} \leq \gamma_5 < \pi$  such that, for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ ,

$$\theta_{T,\max} \leq \gamma_5, \quad (4.1)$$

where  $\theta_{T,\max}$  is the maximal angle of  $T$ .

Under Condition 4.1, Syngé proved an optimal interpolation error estimate as follows.

$$\|\varphi - I_h \varphi\|_{W^{1,p}(\Omega)} \leq ch |\varphi|_{W^{2,p}(\Omega)} \quad \text{for } p = \infty.$$

The inequality (4.1) is called *Syngé's condition* or the *maximum-angle condition*. In 1976, several author's [4, 5, 20, 33] independently proved the convergence of finite element for  $p < \infty$ . It ensures that finite elements converge effectively when the minimum angle approaches zero as the mesh size decreases. If this condition is not met, the accuracy of interpolation for linear triangular elements can suffer, similar to the absence of Zlámal's condition, see e.g. [4, p. 223]. This underscores the importance of keeping proper geometric constraints to ensure reliable outcomes in numerical methods. Syngé's condition is essential in finite element analysis.

In [34], Křížek proposed the following circumscribed ball condition for  $d = 2$  which is equivalent to Syngé's condition.

**Condition 4.2.** There exists  $\gamma_6 > 0$  such that, for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ ,

$$\frac{R_2}{h_T} \leq \gamma_6, \quad (4.2)$$

where  $R_2$  is the radius of the circumscribed ball of  $T \subset \mathbb{R}^2$ .

**Note 4.3.** If Condition 4.1 or 4.2 holds, the associated families of partitions are called *semi-regular*.

**Remark 4.4.** Assume that Condition 3.3 holds, that is, there exists a constant  $\gamma_3 > 0$  such that for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ , we have

$$|T| \geq \gamma_3 h_T^2.$$

Let  $T \subset \mathbb{R}^2$  be the triangle with vertices  $P_1, P_2$  and  $P_3$  such that the maximum angle  $\theta_{T,\max}$  of  $T$  is  $\angle P_2 P_1 P_3$ . We then have  $h_T = |P_2 P_3|$  and

$$\frac{R_2}{h_T} = \frac{|P_2 P_3|}{2h_T \sin \theta_{T,\max}} = \frac{|P_1 P_2| |P_1 P_3|}{2|P_1 P_2| |P_1 P_3| \sin \theta_{T,\max}} \leq c \frac{h_T^2}{|T|} \leq \frac{c}{\gamma_3} =: \gamma_6.$$

This implies that each regular family is semi-regular. However, the converse implication does not hold.

## 4.2 Semi-regular Mesh Conditions for $d = 3$

Synge's condition (4.1) is extended to the case of tetrahedra in [35].

**Condition 4.5.** There exists a constant  $0 < \gamma_7 < \pi$  such that for any  $\mathbb{T}_h \in \{\mathbb{T}_h\}$  and any simplex  $T \in \mathbb{T}_h$ ,

$$\theta_{T,\max} \leq \gamma_7, \quad (4.3a)$$

$$\psi_{T,\max} \leq \gamma_7, \quad (4.3b)$$

where  $\theta_{T,\max}$  is the maximum angle of all triangular faces of the tetrahedron  $T$  and  $\psi_{T,\max}$  is the maximum dihedral angle of  $T$ .

**Remark 4.6.** The theory of anisotropic interpolation has been advanced through extensive research ([3, 2, 9]).

**Question 4.7.** Is there a semi-regularity condition which equivalent to Synge's condition (4.3) for  $d = 3$ ?

**Remark 4.8.** This article introduces a novel geometric condition intended to serve as an alternative to Synge's condition specifically for three-dimensional cases.

## 5 Settings for New Interpolation Theory

### 5.1 Reference Elements

We first define the reference elements  $\widehat{T} \subset \mathbb{R}^d$ .

#### Two-dimensional case

Let  $\widehat{T} \subset \mathbb{R}^2$  be a reference triangle with vertices  $\hat{p}_1 := (0, 0)^\top$ ,  $\hat{p}_2 := (1, 0)^\top$ , and  $\hat{p}_3 := (0, 1)^\top$ .

#### Three-dimensional case

In the three-dimensional case, we consider the following two cases: (i) and (ii); see Condition 5.2.

Let  $\widehat{T}_1$  and  $\widehat{T}_2$  be reference tetrahedra with the following vertices:

- (i)  $\widehat{T}_1$  has vertices  $\hat{p}_1 := (0, 0, 0)^\top$ ,  $\hat{p}_2 := (1, 0, 0)^\top$ ,  $\hat{p}_3 := (0, 1, 0)^\top$ , and  $\hat{p}_4 := (0, 0, 1)^\top$ ;
- (ii)  $\widehat{T}_2$  has vertices  $\hat{p}_1 := (0, 0, 0)^\top$ ,  $\hat{p}_2 := (1, 0, 0)^\top$ ,  $\hat{p}_3 := (1, 1, 0)^\top$ , and  $\hat{p}_4 := (0, 0, 1)^\top$ .

Therefore, we set  $\widehat{T} \in \{\widehat{T}_1, \widehat{T}_2\}$ . Note that the case (i) is called *the regular vertex property*, see [1].

### 5.2 Two-step Affine Mapping

To an affine simplex  $T \subset \mathbb{R}^d$ , we construct two affine mappings  $\Phi_{\widehat{T}} : \widehat{T} \rightarrow \widetilde{T}$  and  $\Phi_T : \widetilde{T} \rightarrow T$ . First, we define the affine mapping  $\Phi_{\widehat{T}} : \widehat{T} \rightarrow \widetilde{T}$  as

$$\Phi_{\widehat{T}} : \widehat{T} \ni \hat{x} \mapsto \tilde{x} := \Phi_{\widehat{T}}(\hat{x}) := A_{\widehat{T}} \hat{x} \in \widetilde{T}, \quad (5.1)$$

where  $A_{\tilde{T}} \in \mathbb{R}^{d \times d}$  is an invertible matrix. We then define the affine mapping  $\Phi_T : \tilde{T} \rightarrow T$  as follows:

$$\Phi_T : \tilde{T} \ni \tilde{x} \mapsto x := \Phi_T(\tilde{x}) := A_T \tilde{x} + b_T \in T, \quad (5.2)$$

where  $b_T \in \mathbb{R}^d$  is a vector and  $A_T \in O(d)$  denotes the rotation and mirror-imaging matrix. We define the affine mapping  $\Phi : \hat{T} \rightarrow T$  as

$$\Phi := \Phi_T \circ \Phi_{\tilde{T}} : \hat{T} \ni \hat{x} \mapsto x := \Phi(\hat{x}) = (\Phi_T \circ \Phi_{\tilde{T}})(\hat{x}) = A\hat{x} + b_T \in T,$$

where  $A := A_T A_{\tilde{T}} \in \mathbb{R}^{d \times d}$ .

### Construct mapping $\Phi_{\tilde{T}} : \hat{T} \rightarrow \tilde{T}$

We consider the affine mapping (5.1). We define the matrix  $A_{\tilde{T}} \in \mathbb{R}^{d \times d}$  as follows. We first define the diagonal matrix as

$$\hat{A} := \text{diag}(h_1, \dots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i, \quad (5.3)$$

where  $\mathbb{R}_+$  denotes the set of positive real numbers.

For  $d = 2$ , we define the regular matrix  $\tilde{A} \in \mathbb{R}^{2 \times 2}$  as

$$\tilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \quad (5.4)$$

with the parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For the reference element  $\hat{T}$ , let  $\mathfrak{T}^{(2)}$  be a family of triangles.

$$\tilde{T} = \Phi_{\tilde{T}}(\hat{T}) = A_{\tilde{T}}(\hat{T}), \quad A_{\tilde{T}} := \tilde{A}\hat{A}$$

with the vertices  $\tilde{p}_1 := (0, 0)^\top$ ,  $\tilde{p}_2 := (h_1, 0)^\top$  and  $\tilde{p}_3 := (h_2 s, h_2 t)^\top$ . Then,  $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$  and  $h_2 = |\tilde{p}_1 - \tilde{p}_3| > 0$ .

For  $d = 3$ , we define the regular matrices  $\tilde{A}_1, \tilde{A}_2 \in \mathbb{R}^{3 \times 3}$  as follows:

$$\tilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \tilde{A}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \quad (5.5)$$

with the parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, & s_1 > 0, & t_1 > 0, & h_2 s_1 \leq h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, & t_2 > 0, & h_3 s_{21} \leq h_1/2. \end{cases}$$

Therefore, we set  $\tilde{A} \in \{\tilde{A}_1, \tilde{A}_2\}$ . For the reference elements  $\hat{T}_i$ ,  $i = 1, 2$ , let  $\mathfrak{T}_i^{(3)}$ ,  $i = 1, 2$ , be a family of tetrahedra.

$$\tilde{T}_i = \Phi_{\tilde{T}_i}(\hat{T}_i) = A_{\tilde{T}_i}(\hat{T}_i), \quad A_{\tilde{T}_i} := \tilde{A}_i \hat{A}, \quad i = 1, 2,$$

with the vertices

$$\begin{aligned} \tilde{p}_1 &:= (0, 0, 0)^\top, \quad \tilde{p}_2 := (h_1, 0, 0)^\top, \quad \tilde{p}_4 := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^\top, \\ \begin{cases} \tilde{p}_3 &:= (h_2 s_1, h_2 t_1, 0)^\top & \text{for case (i),} \\ \tilde{p}_3 &:= (h_1 - h_2 s_1, h_2 t_1, 0)^\top & \text{for case (ii).} \end{cases} \end{aligned}$$

Subsequently,  $h_1 = |\tilde{p}_1 - \tilde{p}_2| > 0$ ,  $h_3 = |\tilde{p}_1 - \tilde{p}_4| > 0$ , and

$$h_2 = \begin{cases} |\tilde{p}_1 - \tilde{p}_3| > 0 & \text{for case (i),} \\ |\tilde{p}_2 - \tilde{p}_3| > 0 & \text{for case (ii).} \end{cases}$$

### Construct mapping $\Phi_T : \tilde{T} \rightarrow T$

We determine the affine mapping (5.2) as follows. Let  $T \in \mathbb{T}_h$  have vertices  $p_i$  ( $i = 1, \dots, d+1$ ). Let  $b_T \in \mathbb{R}^d$  be the vector and  $A_T \in O(d)$  be the rotation and mirror imaging matrix such that

$$p_i = \Phi_T(\tilde{p}_i) = A_T \tilde{p}_i + b_T, \quad i \in \{1, \dots, d+1\},$$

where vertices  $p_i$  ( $i = 1, \dots, d+1$ ) satisfy the following conditions:

**Condition 5.1** (Case in which  $d = 2$ ). Let  $T \in \mathbb{T}_h$  have vertices  $p_i$  ( $i = 1, \dots, 3$ ). We assume that  $\overline{p_2 p_3}$  is the longest edge of  $T$ , that is,  $h_T := |p_2 - p_3|$ . We set  $h_1 = |p_1 - p_2|$  and  $h_2 = |p_1 - p_3|$ . We then assume that  $h_2 \leq h_1$ . Because  $\frac{1}{2}h_T < h_1 \leq h_T$ ,  $h_1 \approx h_T$ .

**Condition 5.2** (Case in which  $d = 3$ ). Let  $T \in \mathbb{T}_h$  have vertices  $p_i$  ( $i = 1, \dots, 4$ ). Let  $L_i$  ( $1 \leq i \leq 6$ ) be the edges of  $T$ . We denote by  $L_{\min}$  the edge of  $T$  with the minimum length; that is,  $|L_{\min}| = \min_{1 \leq i \leq 6} |L_i|$ . We set  $h_2 := |L_{\min}|$  and assume that

$$\text{the endpoints of } L_{\min} \text{ are either } \{p_1, p_3\} \text{ or } \{p_2, p_3\}.$$

Among the four edges sharing an endpoint with  $L_{\min}$ , we consider the longest edge  $L_{\max}^{(\min)}$ . Let  $p_1$  and  $p_2$  be the endpoints of edge  $L_{\max}^{(\min)}$ . Thus, we have

$$h_1 = |L_{\max}^{(\min)}| = |p_1 - p_2|.$$

We consider cutting  $\mathbb{R}^3$  with a plane that contains the midpoint of the edge  $L_{\max}^{(\min)}$  and is perpendicular to the vector  $p_1 - p_2$ . Thus, there are two cases.

**(Type i)**  $p_3$  and  $p_4$  belong to the same half-space;

**(Type ii)**  $p_3$  and  $p_4$  belong to different half-spaces.

In each case, we set

**(Type i)**  $p_1$  and  $p_3$  as the endpoints of  $L_{\min}$ , that is,  $h_2 = |p_1 - p_3|$ ;

**(Type ii)**  $p_2$  and  $p_3$  as the endpoints of  $L_{\min}$ , that is,  $h_2 = |p_2 - p_3|$ .

Finally, we set  $h_3 = |p_1 - p_4|$ . We implicitly assume that  $p_1$  and  $p_4$  belong to the same half-space. Additionally, note that  $h_1 \approx h_T$ .

**Note 5.3.** As an example, we define the matrices  $A_T$  as

$$A_T := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad A_T := \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $\theta$  denotes the angle.

**Note 5.4.** None of the lengths of the edges of a simplex or the measures of the simplex are changed by the transformation, i.e.,

$$h_i \leq h_T, \quad i = 1, \dots, d. \tag{5.6}$$



### 5.3 Additional Notations and Assumptions

For convenience, we introduce the following additional notation. We define a parameter  $\widetilde{\mathcal{H}}_i$ ,  $i = 1, \dots, d$ , as

$$\begin{cases} \widetilde{\mathcal{H}}_1 := h_1, & \widetilde{\mathcal{H}}_2 := h_2 t & \text{if } d = 2, \\ \widetilde{\mathcal{H}}_1 := h_1, & \widetilde{\mathcal{H}}_2 := h_2 t_1, & \widetilde{\mathcal{H}}_3 := h_3 t_2 & \text{if } d = 3, \end{cases}$$

**Assumption 5.5.** In an anisotropic interpolation error analysis, we impose a geometric condition for the simplex  $\widetilde{T}$ :

1. If  $d = 2$ , there are no additional conditions;
2. If  $d = 3$ , there exists a positive constant  $M$  independent of  $h_{\widetilde{T}}$  such that  $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$ . Note that if  $s_{22} \neq 0$ , this condition means that the order concerning  $h_T$  of  $h_3$  coincides with the order of  $h_2$ , and if  $s_{22} = 0$ , the order of  $h_3$  may be different from that of  $h_2$ .

We define the vectors  $r_n \in \mathbb{R}^d$  and  $n = 1, \dots, d$  as follows: If  $d = 2$ ,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_2 := \frac{p_3 - p_1}{|p_3 - p_1|},$$

see Fig. 1, and if  $d = 3$ ,

$$r_1 := \frac{p_2 - p_1}{|p_2 - p_1|}, \quad r_3 := \frac{p_4 - p_1}{|p_4 - p_1|}, \quad \begin{cases} r_2 := \frac{p_3 - p_1}{|p_3 - p_1|}, & \text{for case (i),} \\ r_2 := \frac{p_3 - p_2}{|p_3 - p_2|} & \text{for case (ii),} \end{cases}$$

see Fig 2 for (Type i) and Fig 3 for (Type ii). Furthermore, we define the vectors  $\tilde{r}_n \in \mathbb{R}^d$  and  $n = 1, \dots, d$  as follows. If  $d = 2$ ,

$$\tilde{r}_1 := (1, 0)^\top, \quad \tilde{r}_2 := (s, t)^\top,$$

and if  $d = 3$ ,

$$\tilde{r}_1 := (1, 0, 0)^\top, \quad \tilde{r}_3 := (s_{21}, s_{22}, t_2)^\top, \quad \begin{cases} \tilde{r}_2 := (s_1, t_1, 0)^\top & \text{for case (i),} \\ \tilde{r}_2 := (-s_1, t_1, 0)^\top & \text{for case (ii).} \end{cases}$$

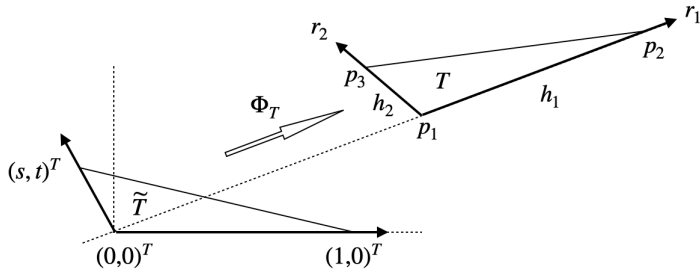


Fig. 1: Affine mapping  $\Phi_T$  and vectors  $r_i$ ,  $i = 1, 2$

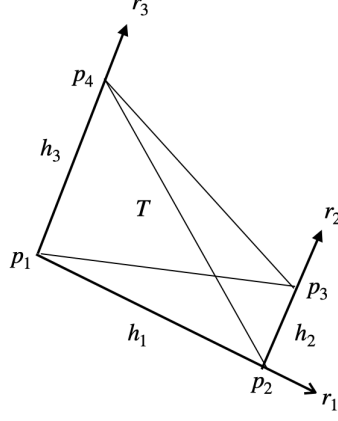
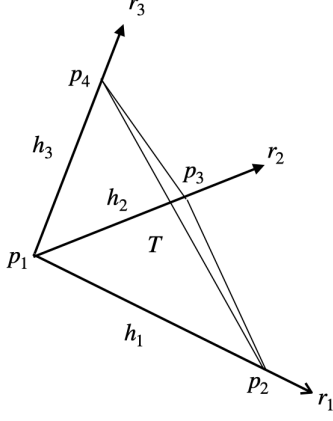


Fig. 2: (Type i) Vectors  $r_i$ ,  $i = 1, 2, 3$  Fig. 3: (Type ii) Vectors  $r_i$ ,  $i = 1, 2, 3$

**Remark 5.6.** The vectors  $\tilde{r}_i$ ,  $i \in \{1, \dots, d\}$  are unit vectors. Indeed, if  $d = 2$ ,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s^2 + t^2} = 1,$$

if  $d = 3$ ,

$$|\tilde{r}_1|_E = 1, \quad |\tilde{r}_2|_E = \sqrt{s_1^2 + t_1^2} = 1, \quad |\tilde{r}_3|_E = \sqrt{s_{21}^2 + s_{22}^2 + t_2^2} = 1.$$

For a sufficiently smooth function  $\varphi$  and vector function  $v := (v_1, \dots, v_d)^\top$ , we define the directional derivative of  $i \in \{1, \dots, d\}$  as:

$$\begin{aligned} \frac{\partial \varphi}{\partial r_i} &:= (r_i \cdot \nabla_x) \varphi = \sum_{i_0=1}^d (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}}, \\ \frac{\partial v}{\partial r_i} &:= \left( \frac{\partial v_1}{\partial r_i}, \dots, \frac{\partial v_d}{\partial r_i} \right)^\top = ((r_i \cdot \nabla_x) v_1, \dots, (r_i \cdot \nabla_x) v_d)^\top. \end{aligned}$$

For a multi-index  $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ , we use the following notation.

$$\partial^\beta \varphi := \frac{\partial^{|\beta|} \varphi}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}, \quad \partial_r^\beta \varphi := \frac{\partial^{|\beta|} \varphi}{\partial r_1^{\beta_1} \dots \partial r_d^{\beta_d}}.$$

Note that  $\partial^\beta \varphi \neq \partial_r^\beta \varphi$ .

## 6 New Semi-regularity Condition

### 6.1 New Geometric Parameter and Condition

We proposed a new geometric parameter  $H_T$  in [30].

**Definition 6.1.** Parameter  $H_T$  is defined as follows:

$$H_T := \frac{\prod_{i=1}^d h_i}{|T|_d} h_T.$$

We introduce geometric conditions to obtain the optimal convergence rate of the anisotropic error estimates.

**Condition 6.2.** A family of meshes  $\{\mathbb{T}_h\}$  is semi-regular if there exists  $\gamma_0 > 0$  such that

$$\frac{H_T}{h_T} \leq \gamma_0 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \quad (6.1)$$

**Remark 6.3.** The geometric condition in (6.1) is equivalent to the maximum angle condition (Section 7).

**Remark 6.4.** We consider the good elements on the meshes in Section 8. On anisotropic meshes, good elements may satisfy the following conditions:

( $d = 2$ )  $h_2 \approx h_2 t$ ;

( $d = 3$ )  $h_2 \approx h_2 t_1$  and  $h_3 \approx h_3 t_2$ .

## 6.2 Properties of the New Geometric Parameter

We first show the relation between  $h_T$  and  $H_T$ .

**Lemma 6.5.** It holds that

$$h_T \leq \frac{1}{2} H_T \quad \text{if } d = 2, \quad (6.2)$$

$$h_T < \frac{1}{6} H_T \quad \text{if } d = 3. \quad (6.3)$$

**Proof.** We consider for each dimension,  $d = 2, 3$ .

**Two-dimensional case.** By constructing the standard element in the two-dimensional case, the angle  $\theta_{\max} := \angle p_2 p_1 p_3$  is the maximum angle of  $T$ . We then have  $\frac{\pi}{3} < \theta_{\max} < \pi$ , that is,  $0 < \sin \theta_{\max} \leq 1$ . Therefore, it holds that

$$H_T = \frac{h_1 h_2}{|T|_2} h_T = \frac{2}{\sin \theta_{\max}} h_T \geq 2 h_T.$$

We here used the fact that  $|T|_2 = \frac{1}{2} h_1 h_2 \sin \theta_{\max}$ .

**Three-dimensional case.** We denote by  $\phi_T$  the angle between the base  $\triangle p_1 p_2 p_3$  of  $T$  and the segment  $\overline{p_1 p_4}$ . Recall that there are two types of standard elements, (Type i) or (Type ii). We denote by  $\theta_T$

(**Type i**) the angle between the segments  $\overline{p_1 p_2}$  and  $\overline{p_1 p_3}$ , that is,  $\theta_T := \angle p_2 p_1 p_3$ , or

(**Type ii**) the angle between the segments  $\overline{p_2 p_1}$  and  $\overline{p_2 p_3}$ , that is,  $\theta_T := \angle p_1 p_2 p_3$ .

We set  $t_1 := \sin \theta_T$  and  $t_2 := \sin \phi_T$ . By constructing the standard element in the three-dimensional case, the angle  $\angle p_1 p_3 p_2$  is the maximum angle of the base  $\triangle p_1 p_2 p_3$  of  $T$ . Therefore, we have  $0 < \theta_T < \frac{\pi}{2}$ . Because  $0 < \phi_T < \pi$ , it holds that

$$H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T = \frac{6}{\sin \theta_T \sin \phi_T} h_T > 6 h_T.$$

We here used the fact that  $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_T \sin \phi_T$ . □

We introduce another geometric parameter regarding Definition 6.1.

**Definition 6.6** (Another parameter  $H_T^*$ ). For  $T \in \mathbb{T}_h$ , we denote by  $L_i$  edges of the simplex  $T$ . We define the new parameter  $H_T^*$  as

$$H_T^* := \frac{h_T^2}{|T|_2} \min_{1 \leq i \leq 3} |L_i| \quad \text{if } d = 2, \quad H_T^* := \frac{h_T^2}{|T|_3} \min_{1 \leq i, j \leq 6, i \neq j} |L_i| |L_j| \quad \text{if } d = 3. \quad (6.4)$$

The parameters  $H_T^*$  and  $H_T$  are equivalent.

**Lemma 6.7.** It holds that

$$\frac{1}{2}H_T^* < H_T < 2H_T^*. \quad (6.5)$$

Furthermore,  $H_T^*$  is equivalent to the circumradius  $R_2$  of  $T$  in the two-dimensional case.

**Proof.** We consider for each dimension,  $d = 2, 3$ .

**Two-dimensional case.** Let  $L_i$  ( $i = 1, 2, 3$ ) denote edges of the triangle  $T$  with  $|L_1| \leq |L_2| \leq |L_3|$ . It obviously holds that  $h_2 = |L_1|$  and  $h_T = |L_3| = h_T$ . Because  $h_2 \leq h_1 < 2h_T$  and  $h_T < h_1 + h_2 \leq 2h_1$  for the triangle  $\triangle p_1 p_2 p_3$ , it holds that

$$\frac{1}{2}h_T < h_1 = |L_2| < 2h_T = 2h_T.$$

We thus have

$$\frac{1}{2}H_T^* = \frac{1}{2} \frac{|L_1|}{|T|_2} h_T^2 < H_T = \frac{h_1 h_2}{|T|_2} h_T < 2 \frac{|L_1|}{|T|_2} h_T^2 = 2H_T^*.$$

Furthermore, it holds that

$$2R_2 = 2 \frac{|L_1||L_2||L_3|}{4|T|_2} < H_T^* = \frac{|L_1|}{|T|_2} h_T^2 < 8 \frac{|L_1||L_2||L_3|}{4|T|_2} = 8R_2.$$

**Three-dimensional case.** Let  $L_i$  ( $i = 1, \dots, 6$ ) denote edges of the triangle  $T$  with  $|L_1| \leq |L_2| \leq \dots \leq |L_6|$ . It obviously holds that  $h_2 = |L_1|$  and  $h_T = |L_6|$ . Recall that there are two types of standard elements, (Type i) or (Type ii).

**(Type i)** We set  $h_4 := |p_3 - p_4|$ ,  $h_5 := |p_2 - p_4|$  and  $h_6 := |p_2 - p_3|$ . Because  $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$  is the longest edge among the four edges that share an endpoint with  $L_1$ , it holds that

$$h_2 \leq \min\{h_3, h_4, h_6\} \leq \max\{h_3, h_4, h_6\} \leq h_1. \quad (6.6)$$

Because  $p_1$  and  $p_4$  belong to the same half-space for the triangle  $\triangle p_1 p_2 p_4$ , it holds that

$$\begin{cases} h_3 \leq h_5 \leq h_1 = h_T & \text{or} \\ h_3 \leq h_1 \leq h_5 = h_T. \end{cases}$$

We thus have

$$\begin{cases} h_3 \leq h_5 \leq h_1 = h_T & \text{or} \\ h_3 \leq h_1 \leq h_T < 2h_1, & \frac{1}{2}h_T < h_1 \leq h_T. \end{cases}$$

Because  $h_3 \leq h_5$ , the length of the edge  $L_2$  is equal to the one of  $h_3$ ,  $h_4$  or  $h_6$ .

Assume that  $|L_2| = h_3$ . We then have

$$\frac{1}{2}H_T^* = \frac{1}{2} \frac{|L_1||L_2|}{|T|_3} h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T \leq \frac{|L_1||L_2|}{|T|_3} h_T^2 = H_T^* (< 2H_T^*).$$

Assume that  $|L_2| = h_4$ . We consider the triangle  $\triangle p_1 p_3 p_4$ . From the assumption, we have  $h_2 \leq h_4 \leq h_3$  and  $\frac{1}{2}h_3 < h_4 \leq h_3$ . We then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2} \frac{|L_1||L_2|}{|T|_3} h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.$$

Assume that  $|L_2| = h_6$ . We consider the triangle  $\triangle p_1 p_2 p_3$ . Because  $p_1$  and  $p_3$  belong to the same half-space for the triangle  $\triangle p_1 p_2 p_3$ , it holds that  $h_2 \leq h_6 \leq h_1$  and  $\frac{1}{2}h_1 < h_6 \leq h_1$ . From (6.6), we have

$$\frac{1}{2}h_3 \leq \frac{1}{2}h_1 < h_6 \leq h_1.$$

Because  $h_6 \leq h_3$ , we then obtain

$$\frac{1}{2}H_T^* = \frac{1}{2} \frac{|L_1||L_2|}{|T|_3} h_T^2 < H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T < 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*.$$

**(Type ii)** We set  $h_4 := |p_3 - p_4|$ ,  $h_5 := |p_2 - p_4|$ , and  $h_6 := |p_1 - p_3|$ . Because  $h_1 = |E_{\max}^{(\min)}| = |p_1 - p_2|$  is the longest edge among the four edges that share an endpoint with  $L_1$ , it holds that

$$h_2 \leq \min\{h_4, h_5, h_6\} \leq \max\{h_4, h_5, h_6\} \leq h_1. \quad (6.7)$$

Because  $p_1$  and  $p_4$  belong to the same half-space for the triangle  $\triangle p_1 p_2 p_4$  and (6.7), it holds that

$$h_3 \leq h_5 \leq h_1.$$

This implies that  $h_1 = h_T$ . Therefore, the length of the edge  $L_2$  is equal to the one of  $h_3$ ,  $h_4$ , or  $h_6$ .

Assume that  $|L_2| = h_3$ . We then have

$$\begin{aligned} \left(\frac{1}{2}H_T^* < \right) H_T^* &= \frac{|L_1||L_2|}{|T|_3} h_T^2 = H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &= \frac{|L_1||L_2|}{|T|_3} h_T^2 = H_T^* (< 2H_T). \end{aligned}$$

Assume that  $|L_2| = h_4$ . For the triangle  $\triangle p_2 p_3 p_4$ , we have

$$h_2 \leq h_4 \leq h_5 < 2h_4.$$

Because  $h_3 \leq h_5$  and  $h_4 \leq h_3$ , it holds that

$$\begin{aligned} \left(\frac{1}{2}H_T^* < \right) H_T^* &= \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &< 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*. \end{aligned}$$

Assume that  $|L_2| = h_6$ . We have  $h_1 < h_2 + h_6 < 2h_6$  for the triangle  $\triangle p_1 p_2 p_3$ . Therefore, since  $h_6 \leq h_3 \leq h_1$ , we obtain

$$\begin{aligned} \left(\frac{1}{2}H_T^* < \right) H_T^* &= \frac{|L_1||L_2|}{|T|_3} h_T^2 \leq H_T = \frac{h_1 h_2 h_3}{|T|_3} h_T \\ &< 2 \frac{|L_1||L_2|}{|T|_3} h_T^2 = 2H_T^*. \end{aligned}$$

□

### 6.3 Euclidean Condition Number

Examining the Euclidean condition number is useful for deriving appropriate interpolation error estimates.

**Lemma 6.8.** It holds that

$$\|\widehat{A}\|_2 \leq h_T, \quad \|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \frac{\max\{h_1, \dots, h_d\}}{\min\{h_1, \dots, h_d\}}, \quad (6.8a)$$

$$\|\widetilde{A}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3, \end{cases} \quad \|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 \leq \begin{cases} \frac{h_1 h_2}{|T|_2} = \frac{H_T}{h_T} & \text{if } d = 2, \\ \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_3} = \frac{2}{3} \frac{H_T}{h_T} & \text{if } d = 3, \end{cases} \quad (6.8b)$$

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1. \quad (6.8c)$$

where a parameter  $H_T$  is defined in Definition 6.1. Furthermore, we have

$$|\det(A_{\widetilde{T}})| = |\det(\widetilde{A})| |\det(\widehat{A})| = \frac{|T|_d |\widetilde{T}|_d}{|\widehat{T}|_d} = d! |T|_d, \quad |\det(A_T)| = 1. \quad (6.9)$$

**Proof.** We first show the equality (6.9). Because

$$\int_T dx = \int_{\widetilde{T}} |\det(A_T)| d\widetilde{x}, \quad \int_{\widetilde{T}} d\widetilde{x} = \int_{\widehat{T}} |\det(A_{\widetilde{T}})| d\widehat{x},$$

and  $|T|_d = |\widetilde{T}|_d$ , we conclude (6.9).

We show the equality (6.8a). From

$$(\widehat{A})^\top \widehat{A} = \text{diag}(h_1^2, \dots, h_d^2), \quad \widehat{A}^{-1} \widehat{A}^{-\top} = \text{diag}(h_1^{-2}, \dots, h_d^{-2}),$$

we have

$$\|\widehat{A}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} = \max\{h_1, \dots, h_d\} \leq h_T,$$

and

$$\|\widehat{A}\|_2 \|\widehat{A}^{-1}\|_2 = \lambda_{\max}(\widehat{A}^\top \widehat{A})^{\frac{1}{2}} \lambda_{\max}(\widehat{A}^{-1} \widehat{A}^{-\top})^{\frac{1}{2}} = \frac{\max\{h_1, \dots, h_d\}}{\min\{h_1, \dots, h_d\}},$$

which leads to (6.8a).

We next show the equality (6.8b). We consider for each dimension,  $d = 2, 3$ .

**Two-dimensional case.** Because

$$\widetilde{A}^\top \widetilde{A} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad \widetilde{A}^{-1} \widetilde{A}^{-\top} = \frac{1}{t^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}, \quad |s| \leq 1,$$

we have

$$\|\widetilde{A}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \leq (1 + |s|)^{\frac{1}{2}} \leq \sqrt{2},$$

and

$$\|\widetilde{A}\|_2 \|\widetilde{A}^{-1}\|_2 = \lambda_{\max}(\widetilde{A}^\top \widetilde{A})^{\frac{1}{2}} \lambda_{\max}(\widetilde{A}^{-1} \widetilde{A}^{-\top})^{\frac{1}{2}} \leq \frac{2}{t} = \frac{h_1 h_2}{|T|_d},$$

which leads to (6.8b) for  $d = 2$ . Here, we used the fact that  $|\widetilde{T}|_d = \frac{1}{2} h_1 h_2 t$  and  $|T|_d = |\widetilde{T}|_d$ .

**Three-dimensional case.** The matrices  $\tilde{A}_1$  and  $\tilde{A}_2$  introduced in (5.5) can be decomposed as  $\tilde{A}_1 = \tilde{M}_0 \tilde{M}_1$  and  $\tilde{A}_2 = \tilde{M}_0 \tilde{M}_2$  with

$$\tilde{M}_0 := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \tilde{M}_1 := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{M}_2 := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of  $\tilde{M}_2^\top \tilde{M}_2$  coincide with those of  $\tilde{M}_1^\top \tilde{M}_1$ , and only Case (i) is shown.

We have the inequalities

$$\begin{aligned} \|\tilde{A}_1\|_2 &= \lambda_{\max}(\tilde{A}_1^\top \tilde{A}_1)^{\frac{1}{2}} \leq \lambda_{\max}(\tilde{M}_0^\top \tilde{M}_0)^{\frac{1}{2}} \lambda_{\max}(\tilde{M}_1^\top \tilde{M}_1)^{\frac{1}{2}} \\ &\leq \left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right)^{\frac{1}{2}} (1 + |s_1|)^{\frac{1}{2}} \leq 2, \end{aligned}$$

and

$$\begin{aligned} \|\tilde{A}_1\|_2 \|\tilde{A}_1^{-1}\|_2 &= \lambda_{\max}(\tilde{A}_1^\top \tilde{A}_1)^{\frac{1}{2}} \lambda_{\max}(\tilde{A}_1^{-1} \tilde{A}_1^{-\top})^{\frac{1}{2}} \\ &\leq \frac{\left(1 + \sqrt{s_{21}^2 + s_{22}^2}\right) (1 + |s_1|)}{t_1 t_2} \leq \frac{4}{t_1 t_2} = \frac{2}{3} \frac{h_1 h_2 h_3}{|T|_d}, \end{aligned}$$

where we used the fact that  $|\tilde{T}|_d = \frac{1}{6} h_1 h_2 h_3 t_1 t_2$  and  $|T|_d = |\tilde{T}|_d$ .

Because the length of all edges of a simplex and measure of the simplex is not changed by a rotation and mirror imaging matrix and  $A_T, A_T^{-1} \in O(d)$ ,

$$\|A_T\|_2 = 1, \quad \|A_T^{-1}\|_2 = 1,$$

which is (6.8c). □

## 7 New Geometric Mesh Condition equivalent to the Maximum-angle Condition

### 7.1 Statements

We state the following theorems concerning the new condition.

**Theorem 7.1.** Condition 6.2 holds if and only if Condition 4.1 holds when  $d = 2$ .

**Proof.** In the case of  $d = 2$ , we use the previous result presented in [34]; i.e., there exists a constant  $\gamma_6 > 0$  such that

$$\frac{R_2}{h_T} \leq \gamma_6 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h,$$

if and only if Condition 4.1 is satisfied. Combining this result with  $H_T$  being equivalent to the circumradius  $R_2$  of  $T$  (Lemma 6.7), we have the desired conclusion. □

**Theorem 7.2.** Condition 6.2 holds if and only if Condition 4.5 holds when  $d = 3$ .

The proof can be found in [29]. Preparation is needed to prove the three-dimensional case. The following subsection shows the symbols used only in this section.

## 7.2 Notation

Let  $T \in \mathbb{T}_h$  be the standard element in  $\mathbb{R}^3$  with vertices,  $P_1, P_2, P_3$  and  $P_4$ . Let  $F_i$  be the face of a simplex  $T$  opposite to the vertex  $P_i$ . We denote by  $\psi^{i,j}$  (Table 5) the angle between the face  $F_i$  and the face  $F_j$ , see Figure 4. Note that  $\psi^{i,j} = \psi^{j,i}$ . Furthermore, we denote by  $\theta_j^i$  (Table 6) the internal angle at the vertex  $P_j$  on the face  $F_i$  and by  $\phi_j^i$  (Table 7) the angle between the face  $F_i$  and the segment  $\overline{P_j P_i}$ .

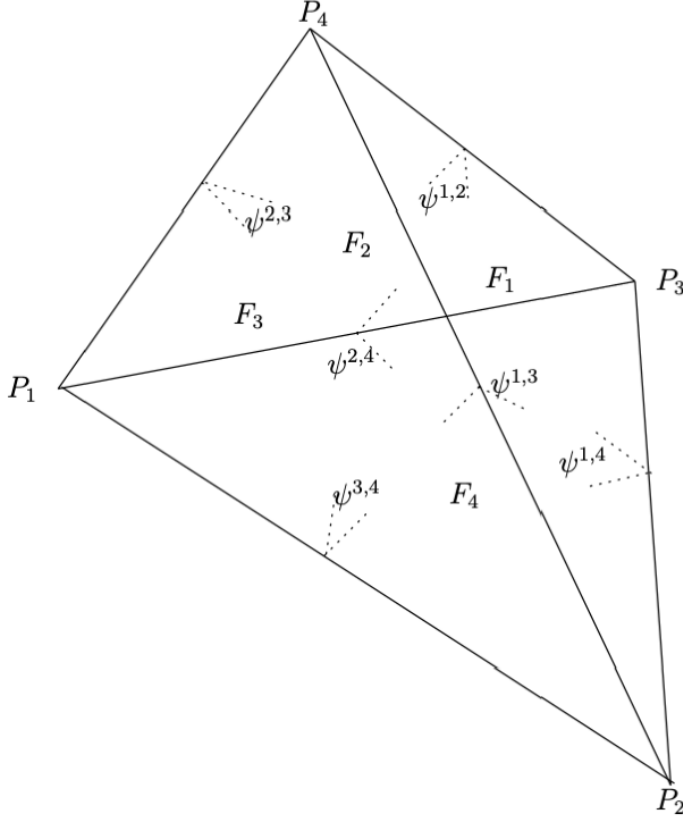


Fig. 4: Tetrahedra

Table 5:  $\psi^{i,j}$

	$F_1$	$F_2$	$F_3$	$F_4$
$F_1$	-	$\psi^{1,2}$	$\psi^{1,3}$	$\psi^{1,4}$
$F_2$	$\psi^{2,1}$	-	$\psi^{2,3}$	$\psi^{2,4}$
$F_3$	$\psi^{3,1}$	$\psi^{3,2}$	-	$\psi^{3,4}$
$F_4$	$\psi^{4,1}$	$\psi^{4,2}$	$\psi^{4,3}$	-

Table 6:  $\theta_j^i$

	$F_1$	$F_2$	$F_3$	$F_4$
$P_1$	-	$\theta_1^2$	$\theta_1^3$	$\theta_1^4$
$P_2$	$\theta_2^1$	-	$\theta_2^3$	$\theta_2^4$
$P_3$	$\theta_3^1$	$\theta_3^2$	-	$\theta_3^4$
$P_4$	$\theta_4^1$	$\theta_4^2$	$\theta_4^3$	-

## 7.3 Preliminaries: Part 1

We introduce three lemmata.

**Lemma 7.3.** Let  $K \subset \mathbb{R}^2$  be a simplex and let  $\theta_1, \theta_2$  and  $\theta_3$  be internal angles of  $K$  with  $\theta_1 \leq \theta_2 \leq \theta_3$ . If there exists  $0 < \theta_0 < \pi$ ,  $\theta_0 \in \mathbb{R}$ , such that  $\theta_3 \leq \theta_0$ , we then have

$$\sin \theta_2, \sin \theta_3 \geq \min \left\{ \sin \frac{\pi - \theta_0}{2}, \sin \theta_0 \right\}.$$



Table 7:  $\phi_j^i$

	$F_1$	$F_2$	$F_3$	$F_4$
$P_1$	-	$\phi_1^2$	$\phi_1^3$	$\phi_1^4$
$P_2$	$\phi_2^1$	-	$\phi_2^3$	$\phi_2^4$
$P_3$	$\phi_3^1$	$\phi_3^2$	-	$\phi_3^4$
$P_4$	$\phi_4^1$	$\phi_4^2$	$\phi_4^3$	-

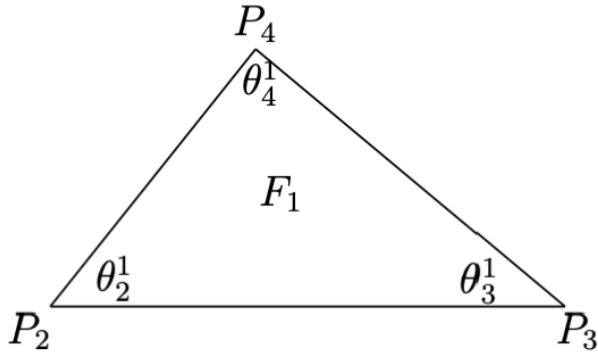


Fig. 5: Face 1

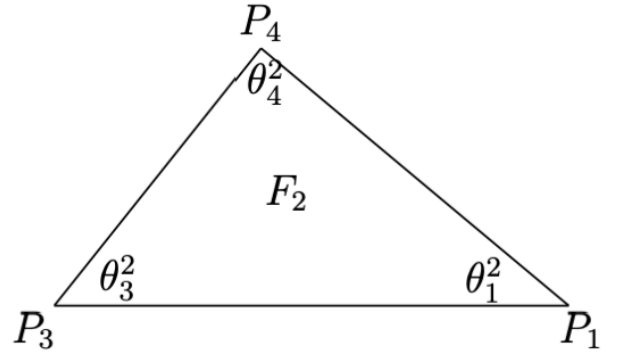


Fig. 6: Face 2

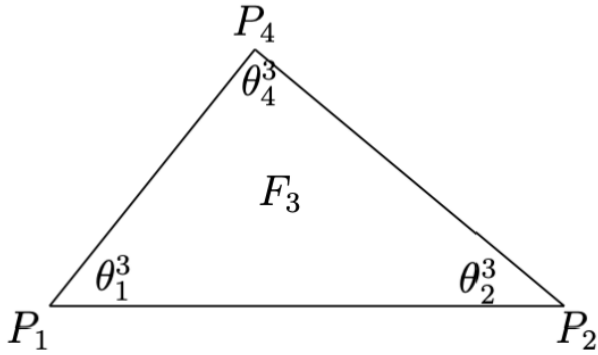


Fig. 7: Face 3

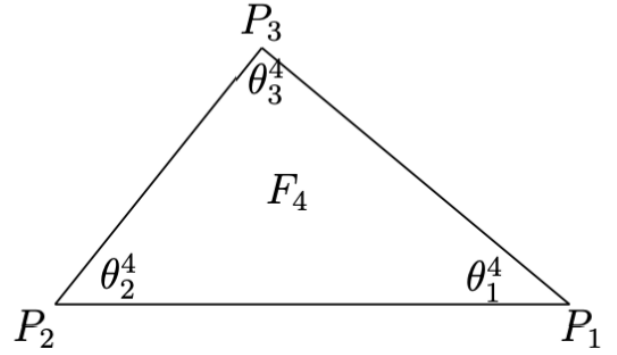


Fig. 8: Face 4

**Proof.** Because  $\theta_1 + \theta_2 + \theta_3 = \pi$  and  $\theta_1 \leq \theta_2 \leq \theta_3$ , we have

$$\theta_0 \geq \theta_3 \geq \theta_2 \geq \frac{\theta_1 + \theta_2}{2} \geq \frac{\pi - \theta_3}{2} \geq \frac{\pi - \theta_0}{2},$$

which leads to the target inequality.  $\square$

**Lemma 7.4.** Let  $K \subset \mathbb{R}^2$  be a simplex with internal angles  $\theta_1, \theta_2$  and  $\theta_3$ . For any fixed  $\gamma \in \mathbb{R}$  with  $0 < \gamma < \pi$ , we assume that  $\pi - \gamma \leq \theta_i, i \in \{1, 2, 3\}$ . We then have  $\theta_{i+1}, \theta_{i+2} \leq \gamma$ , where the indices  $i, i+1$  and  $i+2$  have to be understood "mod 3".

**Proof.** Because  $\theta_1 + \theta_2 + \theta_3 = \pi$ , we have

$$\theta_{i+1} = \pi - \theta_i - \theta_{i+2} < \pi - \theta_i \leq \pi - (\pi - \gamma) = \gamma.$$

$\square$

**Lemma 7.5.** Let  $\gamma \in \mathbb{R}$  with  $\frac{\pi}{3} \leq \gamma < \pi$ . It then holds that

$$0 < \frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} \leq 1.$$

**Proof.** Because  $\cos \gamma = 1 - 2 \sin^2 \frac{\gamma}{2}$ , we have

$$\frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} = \frac{2 - 2 \sin^2 \frac{\gamma}{2}}{\sin \frac{\gamma}{2} + 1} = 2 \left( 1 - \sin \frac{\gamma}{2} \right).$$

Therefore, for  $\frac{\pi}{3} \leq \gamma < \pi$ , the target inequality holds.  $\square$

## 7.4 Preliminaries: Part 2

**Lemma 7.6** (Cosine rules for the sides and for the angles). It holds that

$$\cos \theta_j^{j+3} = \cos \theta_j^{j+1} \cos \theta_j^{j+2} + \sin \theta_j^{j+1} \sin \theta_j^{j+2} \cos \psi^{j+1,j+2}, \quad (7.1a)$$

$$\cos \theta_j^{j+1} = \cos \theta_j^{j+2} \cos \theta_j^{j+3} + \sin \theta_j^{j+2} \sin \theta_j^{j+3} \cos \psi^{j+2,j+3}, \quad (7.1b)$$

$$\cos \theta_j^{j+2} = \cos \theta_j^{j+3} \cos \theta_j^{j+1} + \sin \theta_j^{j+3} \sin \theta_j^{j+1} \cos \psi^{j+3,j+1}, \quad (7.1c)$$

$$\cos \psi^{j+1,j+2} = \sin \psi^{j+2,j+3} \sin \psi^{j+3,j+1} \cos \theta_j^{j+3} - \cos \psi^{j+2,j+3} \cos \psi^{j+3,j+1}, \quad (7.1d)$$

$$\cos \psi^{j+2,j+3} = \sin \psi^{j+3,j+1} \sin \psi^{j+1,j+2} \cos \theta_j^{j+1} - \cos \psi^{j+3,j+1} \cos \psi^{j+1,j+2}, \quad (7.1e)$$

$$\cos \psi^{j+3,j+1} = \sin \psi^{j+1,j+2} \sin \psi^{j+2,j+3} \cos \theta_j^{j+2} - \cos \psi^{j+1,j+2} \cos \psi^{j+2,j+3}, \quad (7.1f)$$

where the indices  $j, j+1, j+2$  and  $j+3$  have to be understood "mod 4".

**Proof.** A proof can be found in [18, 40].  $\square$

**Lemma 7.7.** Let  $\gamma_{\max} \in \mathbb{R}$  with  $\frac{\pi}{3} \leq \gamma_{\max} < \pi$  satisfy Condition 4.5 for the maximum solid  $\theta_{T,\max}$  and the maximum dihedral  $\psi_{T,\max}$  of  $T$ . Assume that for each  $j = 1, 2$ ,  $\theta_j^4$  is not the minimum angle of  $\triangle P_1 P_2 P_3$  and  $\theta_j^4 < \frac{\pi}{2}$ . Then, setting  $\delta := \delta(\gamma_{\max})$ ,  $0 < \delta \leq \frac{\pi}{2}$  such that

$$\sin \delta = \left( \frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{j+1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta, \quad (7.2)$$

where the indices  $j$  and  $j+1$  have to be understood "mod 2".

**Proof.** From Lemma 7.5, we have

$$0 < \frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \leq 1,$$

because  $\frac{\pi}{3} \leq \gamma_{\max} < \pi$ . Therefore,  $\delta$  is well-defined.

We use proof by contradiction. Suppose that

$$0 < \psi^{j+1,4} < \delta, \quad 0 < \psi^{3,4} < \delta,$$

that is,

$$0 < \sin \psi^{j+1,4} \sin \psi^{3,4} < \sin^2 \delta, \quad \text{and} \quad 1 > \cos \psi^{j+1,4} \cos \psi^{3,4} > \cos^2 \delta \geq 0.$$

From Lemma 7.3 and assumption, we have

$$\frac{\pi - \gamma_{\max}}{2} \leq \theta_j^4 < \frac{\pi}{2},$$

which implies

$$0 < \cos \theta_j^4 \leq \cos \left( \frac{\pi - \gamma_{\max}}{2} \right) = \sin \frac{\gamma_{\max}}{2}.$$

We thus obtain

$$\sin \psi^{j+1,4} \sin \psi^{3,4} \cos \theta_j^4 < \sin^2 \delta \sin \frac{\gamma_{\max}}{2}.$$

Using the cosine rule (7.1d) with  $j = 1$  and the above inequalities yield

$$\begin{aligned} \cos \psi_{2,3} &= \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2} \\ &< \sin^2 \delta \sin \frac{\gamma_{\max}}{2} - (1 - \sin^2 \delta) \\ &= \frac{\cos \gamma_{\max} + 1}{\sin \frac{\gamma_{\max}}{2} + 1} \left( \sin \frac{\gamma_{\max}}{2} + 1 \right) - 1 = \cos \gamma_{\max}. \end{aligned}$$

This is contradiction for the maximum-angle condition  $0 < \psi^{2,3} \leq \gamma_{\max} < \pi$ , that is,  $\cos \psi^{2,3} \geq \cos \gamma_{\max}$ .

Analogously, using the cosine rule (7.1f) with  $j = 2$  and the above inequalities yield

$$\begin{aligned} \cos \psi^{1,3} &= \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 - \cos \psi^{3,4} \cos \psi^{4,1} \\ &< \cos \gamma_{\max}. \end{aligned}$$

This is contradiction for the maximum-angle condition  $0 < \psi^{1,3} \leq \gamma_{\max} < \pi$ , that is,  $\cos \psi^{1,3} \geq \cos \gamma_{\max}$ .  $\square$

**Corollary 7.8.** For each  $j = 1, 2$ , under assumptions in Lemma 7.7, it holds that setting  $C_0 := \min\{\delta, \gamma_{\max}\}$ ,

$$\sin \psi^{j+1,4} \geq C_0, \quad \text{or} \quad \sin \psi^{3,4} \geq C_0$$

where the indices  $j$  and  $j + 1$  have to be understood "mod 2".

**Lemma 7.9.** For any  $i, j \in \{1, 2, 3, 4\}$ ,  $i \neq j$  and  $k \in \{1, 2, 3, 4\}$ ,  $k \neq i, j$ , it holds that

$$\sin \phi_j^i = \sin \theta_j^k \sin \psi^{k,i}.$$

**Proof.** We only show the case  $i = 4$ ,  $j = 1$  and  $k = 2$ . We then have

$$\sin \phi_1^4 = |\overline{P_1 P_4}| \sin \theta_1^2 \times \frac{1}{|\overline{P_1 P_4}|} \sin \psi^{2,4} = \sin \theta_1^2 \sin \psi^{2,4}.$$

□

**Lemma 7.10.** Assume that there exists a positive constant  $M_j$  ( $j = 1, 2$ ) with  $0 < M_j < 1$  such that

$$\sin \theta_j^4 \sin \phi_1^4 > M_j, \quad j = 1, 2.$$

Setting  $\gamma(M_j) := \pi - \sin^{-1} M_j$  ( $j = 1, 2$ ), we have  $\frac{\pi}{2} < \gamma(M_j) < \pi$  and it holds that for each  $j = 1, 2$ ,

$$\begin{aligned} \theta_1^4, \theta_2^4, \theta_3^4 &< \gamma(M_j), \\ \theta_2^3, \theta_4^3, \theta_3^2, \theta_4^2, \theta_1^2, \theta_1^3, \psi^{2,4}, \psi^{3,4} &< \gamma(M_j). \end{aligned}$$

**Proof.** From assumption, we have, for each  $j = 1, 2$ ,

$$\begin{aligned} \sin \theta_j^4 &\geq \sin \theta_j^4 \sin \phi_1^4 > M_j, \\ \sin \phi_1^4 &> M_j. \end{aligned}$$

The definition of  $\gamma(M_j)$  and Lemma 7.4 yield, for each  $j = 1, 2$ ,

$$\begin{aligned} \pi - \gamma < \theta_j^4 < \gamma(M_j), \quad \theta_{j+1}^4 < \gamma(M_j), \quad \theta_{j+2}^4 < \gamma(M_j), \\ \pi - \gamma < \phi_1^4 < \gamma(M_j), \end{aligned}$$

where the indices  $j$ ,  $j + 1$  and  $j + 2$  have to be understood "mod 3".

We obtain, from Lemma 7.9,

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} = \sin \theta_1^3 \sin \psi^{3,4} > M_j, \quad j = 1, 2.$$

We then have, for each  $j = 1, 2$ ,

$$\sin \theta_1^2, \sin \psi^{2,4}, \sin \theta_1^3, \sin \psi^{3,4} > M_j,$$

that is,

$$\pi - \gamma(M_j) < \theta_1^2, \theta_1^3, \psi^{2,4}, \psi^{3,4} < \gamma(M_j).$$

On  $\triangle P_1 P_2 P_4$  and  $\triangle P_1 P_3 P_4$ , using Lemma 7.4 yields

$$\theta_2^3, \theta_4^3, \theta_3^2, \theta_4^2 < \gamma(M_j), \quad j = 1, 2.$$

□

By analogous argument with Lemma 7.10, we get the following two lemmata.

**Lemma 7.11.** Assume that there exists  $M_3$  with  $0 < M_3 < 1$  such that

$$\sin \theta_3^1 \sin \phi_3^1 > M_3.$$

Setting  $\gamma(M_3) := \pi - \sin^{-1} M_3$ , we have  $\frac{\pi}{2} < \gamma(M_3) < \pi$  and it holds that

$$\theta_3^2, \theta_3^4, \theta_2^1, \theta_4^1, \theta_3^1, \psi^{2,1}, \psi^{4,1} < \gamma(M_3).$$

**Proof.** From assumption, we have

$$\sin \theta_3^1 \geq \sin \theta_3^1 \sin \phi_3^1 > M_3, \quad \sin \phi_3^1 > M_3.$$

Using the definition of  $\gamma(M_3)$  yields

$$\pi - \gamma < \theta_1^3 < \gamma(M_3), \quad \pi - \gamma < \phi_1^3 < \gamma(M_3).$$

We obtain, from Lemma 7.9,

$$\sin \phi_3^1 = \sin \theta_3^2 \sin \psi^{2,1} = \sin \theta_3^4 \sin \psi^{4,1} > M_3.$$

We then have

$$\sin \theta_3^2, \sin \psi^{2,1}, \sin \theta_3^4, \sin \psi^{4,1} > M_3,$$

that is,

$$\pi - \gamma(M_3) < \theta_3^2, \theta_3^4, \psi^{2,1}, \psi^{4,1} < \gamma(M_3).$$

Meanwhile, on  $\triangle P_2 P_3 P_4$ , using Lemma 7.4, we have

$$\theta_2^1, \theta_4^1 < \gamma(M_3).$$

□

**Lemma 7.12.** Assume that there exists  $M_4$  with  $0 < M_4 < 1$  such that

$$\sin \theta_2^1 \sin \phi_4^1 > M_4.$$

Setting  $\gamma(M_4) := \pi - \sin^{-1} M_4$ , we have  $\frac{\pi}{2} < \gamma(M_4) < \pi$  and it holds that

$$\theta_4^2, \theta_4^3, \theta_2^1, \theta_3^1, \theta_4^1, \psi^{1,2}, \psi^{1,3} < \gamma(M_4).$$

**Proof.** The proof is obtained by using an analogous argument with Lemma 7.11. □

## 7.5 Proof of Theorem 7.2 in (Type i)

### 7.5.1 Condition 4.5 $\Rightarrow$ Condition 6.2

We set  $t_1 := \sin \theta_1^4$  and  $t_2 := \sin \phi_1^4$ . We then have

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4}.$$

We here used the fact that  $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_1^4 \sin \phi_1^4$ . By construct of the standard element (Type i), the angle  $\theta_3^4$  and  $\theta_2^4$  are respectively the maximum angle and the minimum angle of the base  $\triangle P_1 P_2 P_3$  of  $T$ . We hence have  $\theta_1^4 < \frac{\pi}{2}$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^4 \leq \gamma_{11}, \quad \sin \theta_1^4 \geq \min \left\{ \sin \frac{\pi - \gamma_{11}}{2}, \sin \gamma_{11} \right\} =: C_1.$$

Due to Lemma 7.7, setting  $\delta := \delta(\gamma_{11})$ ,  $0 < \delta \leq \frac{\pi}{2}$  such that

$$\sin \delta = \left( \frac{\cos \gamma_{11} + 1}{\sin \frac{\gamma_{11}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that  $\psi^{2,4} \geq \delta$ . By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} \geq C_0 \sin \theta_1^2.$$

By construct of the standard element (Type i), the angle  $\theta_1^2$  is not the minimum angle of  $\triangle P_1 P_3 P_4$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^2 \leq \gamma_{11}, \quad \sin \theta_1^2 \geq C_1.$$

We thus obtain

$$\sin \phi_1^4 \geq C_0 C_1.$$

Suppose that  $\psi^{3,4} \geq \delta$ . By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_1^4 = \sin \theta_1^3 \sin \psi^{3,4} \geq C_0 \sin \theta_1^3.$$

By construct of the standard element (Type i), the angle  $\theta_1^3$  is not the minimum angle of  $\triangle P_1 P_2 P_4$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin \theta_1^3 \geq C_1.$$

We thus obtain

$$\sin \phi_1^4 \geq C_0 C_1.$$

In both cases

$$\psi^{2,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yield

$$\frac{H_T}{h_T} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq \frac{6}{C_0 C_1^2} =: D_1 > 0,$$

that is, Condition 6.2 holds. □

### 7.5.2 Condition 6.2 $\Rightarrow$ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq \gamma_9.$$

Remark that  $\frac{6}{\gamma_9} < 1$  because  $\theta_1^4 < \frac{\pi}{2}$  and  $\sin \theta_1^4 \sin \phi_1^4 < 1$ . Therefore, we have

$$\sin \theta_1^4 \sin \phi_1^4 \geq \frac{6}{\gamma_9} =: C_2.$$

From Lemma 7.10 with  $j = 1$ , setting  $\gamma(C_2) := \pi - \sin^{-1} C_2$ , we have  $\frac{\pi}{2} < \gamma(C_2) < \pi$  and it holds that

$$\begin{aligned} \theta_1^4, \theta_2^4, \theta_3^4 &< \gamma(C_2), \\ \theta_2^3, \theta_4^3, \theta_1^3, \theta_3^2, \theta_4^2, \theta_1^2, \psi^{2,4}, \psi^{3,4} &< \gamma(C_2). \end{aligned}$$

Furthermore, we write

$$\begin{aligned} |T|_3 &= \frac{1}{3} \times \frac{1}{2} |\overline{P_2 P_3}| |\overline{P_3 P_4}| \sin \theta_3^1 \times h_2 \sin \phi_3^1 = \frac{1}{6} h_2 |\overline{P_2 P_3}| |\overline{P_3 P_4}| \sin \theta_3^1 \sin \phi_3^1 \\ &< \frac{1}{3} h_1 h_2 h_3 \sin \theta_3^1 \sin \phi_3^1, \end{aligned}$$

where we used the fact that  $|\overline{P_3 P_4}| < |\overline{P_1 P_4}| + |\overline{P_1 P_3}| \leq 2h_3$  on  $\triangle P_1 P_3 P_4$  and  $|\overline{P_2 P_3}| \leq h_1$ . We thus have

$$\gamma_9 \geq \frac{H_T}{h_T} > \frac{3}{\sin \theta_3^1 \sin \phi_3^1},$$

that is,

$$\sin \theta_3^1 \sin \phi_3^1 > \frac{3}{\gamma_9} =: C_3.$$

From Lemma 7.11, setting  $\gamma(C_3) := \pi - \sin^{-1} C_3$ , we have  $\frac{\pi}{2} < \gamma(C_3) < \pi$  and it holds that

$$\theta_2^1, \theta_4^1, \theta_3^1, \psi^{2,1}, \psi^{4,1} < \gamma(C_3).$$

Due to the cosine rule (7.1f) with  $j = 2$ , we get

$$\cos \psi^{1,3} = \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 - \cos \psi^{3,4} \cos \psi^{4,1}.$$

By constructing the standard element (Type i), the angle  $\theta_2^4$  is the minimum angle of  $\triangle P_1 P_2 P_3$ . Therefore, we have

$$\begin{aligned} \cos \theta_2^4 &\geq \frac{1}{2} \quad \text{because } \theta_2^4 \leq \frac{\pi}{3}, \\ \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 &> 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,1} > 0, \end{aligned}$$

and thus

$$\cos \psi^{1,3} > -\cos \psi^{3,4} \cos \psi^{4,1}.$$

Using  $\sin \psi^{3,4} > C_2$  and  $\sin \psi^{4,1} > C_3$  yields

$$\begin{aligned} \cos \psi^{1,3} &> -\cos \psi^{3,4} \cos \psi^{4,1} \\ &\geq -|\cos \psi^{3,4}| |\cos \psi^{4,1}| = -\sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,1}} \\ &> -\sqrt{1 - C_2^2} \sqrt{1 - C_3^2} =: C_4 > -1. \end{aligned}$$

Setting  $\gamma(C_4) := \cos^{-1} C_4$ , it holds that

$$\psi^{1,3} < \gamma(C_4) < \pi.$$

Due to the cosine rule (7.1d) with  $j = 1$ , we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type i), the angle  $\theta_3^4$  and  $\theta_2^4$  are respectively the maximum angle and the minimum angle of the base  $\triangle P_1 P_2 P_3$  of  $T^s$ . We hence have  $\theta_1^4 < \frac{\pi}{2}$ . Therefore, we have

$$\begin{aligned} \cos \theta_1^4 &> 0 \quad \text{because } \theta_1^4 \leq \frac{\pi}{2}, \\ \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 &> 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0, \end{aligned}$$

and thus

$$\cos \psi^{2,3} > -\cos \psi^{3,4} \cos \psi^{4,2}.$$

Using  $\sin \psi^{3,4} > C_2$  and  $\sin \psi^{4,2} > C_2$  yield

$$\begin{aligned} \cos \psi^{2,3} &> -\cos \psi^{3,4} \cos \psi^{4,2} \\ &\geq -|\cos \psi^{3,4}| |\cos \psi^{4,2}| = -\sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,2}} \\ &> -(1 - C_2^2) =: C_5 > -1. \end{aligned}$$

Setting  $\gamma(C_5) := \cos^{-1} C_5$ , it holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set  $\gamma_{\max} := \max\{\gamma(C_3), \gamma(C_4), \gamma(C_5)\}$ . We then have  $0 < \gamma_{\max} < \pi$ , that is, Condition 4.5 holds.  $\square$

## 7.6 Proof of Theorem 7.2 in (Type ii)

### 7.6.1 Condition 4.5 $\Rightarrow$ Condition 6.2

We set  $t_1 := \sin \theta_2^4$  and  $t_2 := \sin \phi_1^4$ . We then have

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4}.$$

We here used the fact that  $|T|_3 = \frac{1}{6} h_1 h_2 h_3 \sin \theta_2^4 \sin \phi_1^4$ . By construct of the standard element (Type ii), the angle  $\theta_3^4$  and  $\theta_1^4$  are respectively the maximum angle and the minimum angle of the base  $\triangle P_1 P_2 P_3$  of  $T^s$ . We hence have  $\theta_2^4 < \frac{\pi}{2}$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^4 \leq \gamma_{11}, \quad \sin \theta_2^4 \geq C_1.$$

Due to Lemma 7.7, it holds that

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that  $\psi^{1,4} \geq \delta$ . By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_2^4 = \sin \theta_2^1 \sin \psi^{1,4} \geq C_0 \sin \theta_2^1.$$

Furthermore, it holds that

$$\sin \phi_1^4 = \frac{|P_2 P_4| \sin \phi_2^4}{h_3}.$$



By construct of the standard element (Type ii), the angle  $\theta_2^1$  is not the minimum angle of  $\triangle P_2 P_3 P_4$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^1 \leq \gamma_{11}, \quad \sin \theta_2^1 \geq C_1.$$

Because  $h_3 = |\overline{P_1 P_4}| < |\overline{P_2 P_4}|$  on  $\triangle P_1 P_2 P_4$ , we thus obtain

$$\sin \phi_1^4 = \frac{|\overline{P_2 P_4}|}{h_3} \sin \phi_2^4 > C_0 C_1.$$

Suppose that  $\psi^{3,4} \geq \delta$ . By Corollary 7.8 and Lemma 7.9, we have

$$\sin \phi_1^4 = \sin \theta_1^3 \sin \psi^{3,4} \geq C_0 \sin \theta_1^3.$$

By constructing the standard element (Type ii), the angle  $\theta_1^3$  is not the minimum angle of  $\triangle P_1 P_2 P_4$ . From Lemma 7.3, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin \theta_1^3 \geq C_1.$$

We thus obtain

$$\sin \phi_1^4 > C_0 C_1.$$

In both cases

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yields

$$\frac{H_T}{h_T} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq \frac{6}{C_0 C_1^2} = D_1 > 0,$$

that is, Condition 6.2 holds. □

### 7.6.2 Condition 6.2 $\Rightarrow$ Condition 4.5

From assumption, it holds that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \leq \gamma_9.$$

Remark that  $\frac{6}{\gamma_9} < 1$  because  $\theta_2^4 < \frac{\pi}{2}$  and  $\sin \theta_2^4 \sin \phi_1^4 < 1$ . Therefore, we have

$$\sin \theta_2^4 \sin \phi_1^4 \geq \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.10 with  $j = 2$ , it holds that

$$\begin{aligned} \theta_1^4, \theta_2^4, \theta_3^4 &< \gamma(C_2), \\ \theta_2^3, \theta_4^3, \theta_1^3, \theta_3^2, \theta_4^2, \theta_1^2, \psi^{2,4}, \psi^{3,4} &< \gamma(C_2). \end{aligned}$$

Furthermore, we write

$$\begin{aligned} |T|_3 &= \frac{1}{3} \times \frac{1}{2} |\overline{P_2 P_4}| |\overline{P_2 P_3}| \sin \theta_2^1 \times h_3 \sin \phi_4^1 \\ &< \frac{1}{6} h_1 h_2 h_3 \sin \theta_2^1 \sin \phi_4^1, \end{aligned}$$

where we used the fact that  $|\overline{P_3P_2}| = h_2$  and  $|\overline{P_2P_4}| \leq h_1$ . We thus have

$$\gamma_9 \geq \frac{H_{T^s}}{h_{T^s}} > \frac{6}{\sin \theta_2^1 \sin \phi_4^1},$$

that is,

$$\sin \theta_2^1 \sin \phi_4^1 > \frac{6}{\gamma_9} = C_2.$$

From Lemma 7.12, it holds that

$$\theta_2^1, \theta_4^1, \theta_3^1, \psi^{1,2}, \psi^{1,3} < \gamma(C_2).$$

Due to the cosine rule (7.1e) with  $j = 2$ , we get

$$\cos \psi^{4,1} = \sin \psi^{1,3} \sin \psi^{3,4} \cos \theta_2^3 - \cos \psi^{1,3} \cos \psi^{3,4}.$$

By constructing the standard element (Type ii), the angle  $\theta_2^3$  is the minimum angle of  $\triangle P_1P_2P_4$ . Therefore, we have

$$\begin{aligned} \cos \theta_2^3 &\geq \frac{1}{2} \quad \text{because } \theta_2^3 \leq \frac{\pi}{3}, \\ \sin \psi^{1,3} \sin \psi^{3,4} \cos \theta_2^3 &> 0, \quad \text{because } \sin \psi^{1,3} \sin \psi^{3,4} > 0, \end{aligned}$$

and thus

$$\cos \psi^{4,1} > -\cos \psi^{1,3} \cos \psi^{3,4}.$$

Using  $\sin \psi^{1,3} > C_2$  and  $\sin \psi^{3,4} > C_2$  yield

$$\begin{aligned} \cos \psi^{4,1} &> -\cos \psi^{1,3} \cos \psi^{3,4} \\ &\geq -\sqrt{1 - \sin^2 \psi^{1,3}} \sqrt{1 - \sin^2 \psi^{3,4}} \\ &> -(1 - C_2^2) = C_5 > -1. \end{aligned}$$

It then holds that

$$\psi^{4,1} < \gamma(C_5) < \pi.$$

Due to the cosine rule (7.1d) with  $j = 1$ , we get

$$\cos \psi^{2,3} = \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 - \cos \psi^{3,4} \cos \psi^{4,2}.$$

By constructing the standard element (Type ii), the angle  $\theta_1^4$  is the minimum angle of  $\triangle P_1P_2P_3$ . We hence have  $\theta_1^4 < \frac{\pi}{3}$ . Therefore, we have

$$\begin{aligned} \cos \theta_1^4 &\geq \frac{1}{2} \quad \text{because } \theta_1^4 \leq \frac{\pi}{3}, \\ \sin \psi^{3,4} \sin \psi^{4,2} \cos \theta_1^4 &> 0, \quad \text{because } \sin \psi^{3,4} \sin \psi^{4,2} > 0, \end{aligned}$$

and thus

$$\cos \psi^{2,3} > -\cos \psi^{3,4} \cos \psi^{4,2}.$$

Using  $\sin \psi^{3,4} > C_2$  and  $\sin \psi^{4,2} > C_2$  yield

$$\begin{aligned} \cos \psi^{2,3} &> -\cos \psi^{3,4} \cos \psi^{4,2} \\ &\geq -\sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,2}} \\ &> -(1 - C_2^2) = C_5 > -1. \end{aligned}$$

It then holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set  $\gamma_{\max} := \max\{\gamma(C_2), \gamma(C_5)\}$ . We then have  $0 < \gamma_{\max} < \pi$ , that is, Condition 4.5 holds.  $\square$

## 8 Good Elements or not for $d = 2, 3$ ?

In this subsection, we consider good elements on meshes. In this paper, we define 'good elements' on meshes as the existence of a positive constant  $\gamma_0 > 0$  satisfying (6.1). We treat a "Right-angled triangle", "Blade" and "Dagger" for  $d = 2$ , and "Spire", "Spear", "Spindle", "Spike", "Splinter" and "Sliver" for  $d = 3$ , which are introduced in [10]. We give the quantities  $h_{\max}/h_{\min}$  and  $H_T/h_T$  for those elements. The parameters  $h_{\max}$  and  $h_{\min}$  are defined as

$$h_{\max} := \max\{h_1, \dots, h_d\}, \quad h_{\min} := \min\{h_1, \dots, h_d\}. \quad (8.1)$$

### 8.1 Isotropic Mesh Elements

Recall that an isotropic mesh element has equal or nearly equal edge lengths and angles, resulting in a balanced shape. Then, the geometric condition (3.4) is satisfied. Therefore, it holds that

$$\frac{H_T}{h_T} \leq \frac{h_T^d}{|T|_d} \leq \frac{1}{\gamma_3}, \quad \frac{h_{\max}}{h_{\min}} \leq c \frac{h_T^d}{|T|_d} \leq \frac{c}{\gamma_3}.$$

In this case, elements satisfying the geometric condition (3.4) are "good."

### 8.2 Anisotropic mesh: two-dimensional case

Let  $S \subset \mathbb{R}^2$  be a triangle. Let  $0 < s \ll 1$ ,  $s \in \mathbb{R}$  and  $\varepsilon, \delta, \gamma \in \mathbb{R}$

**Example 8.1** (Right-angled triangle). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$  and  $p_3 := (0, s^\varepsilon)^\top$  with  $1 < \varepsilon$ . We then have  $h_1 = s$  and  $h_2 = s^\varepsilon$ ; i.e.,

$$\frac{h_{\max}}{h_{\min}} \leq s^{1-\varepsilon} \rightarrow \infty \quad \text{as } s \rightarrow 0, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element  $S$  is "good."

**Example 8.2** (Dagger). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$  and  $p_3 := (s^\delta, s^\varepsilon)^\top$  with  $1 < \varepsilon < \delta$ . We then have  $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$  and  $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ ; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \leq cs^{1-\varepsilon} \rightarrow \infty \quad \text{as } s \rightarrow 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}} \sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2}s^{1+\varepsilon}} \leq c. \end{aligned}$$

In this case, the element  $S$  is "good."

**Remark 8.3.** In the above examples,  $h_2 \approx \widetilde{\mathcal{H}}_2$  holds. That is, the good element  $S \subset \mathbb{R}^2$  may satisfy conditions such as  $h_2 \approx \widetilde{\mathcal{H}}_2$ .

**Example 8.4** (Blade). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (2s, 0)^\top$  and  $p_3 := (s, s^\varepsilon)^\top$  with  $1 < \varepsilon$ . We then have  $h_1 = h_2 = \sqrt{s^2 + s^{2\varepsilon}}$ ; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2 + s^{2\varepsilon}}{s^{1+\varepsilon}} \rightarrow \infty \quad \text{as } s \rightarrow 0.$$

In this case, the element  $S$  is "not good."

**Example 8.5** (Dagger). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$  and  $p_3 := (s^\delta, s^\varepsilon)^\top$  with  $1 < \delta < \varepsilon$ . We then have  $h_1 = \sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}$  and  $h_2 = \sqrt{s^{2\delta} + s^{2\varepsilon}}$ ; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}}}{\sqrt{s^{2\delta} + s^{2\varepsilon}}} \leq cs^{1-\delta} \rightarrow \infty \quad \text{as } s \rightarrow 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(s - s^\delta)^2 + s^{2\varepsilon}} \sqrt{s^{2\delta} + s^{2\varepsilon}}}{\frac{1}{2}s^{1+\varepsilon}} \leq cs^{\delta-\varepsilon} \rightarrow \infty \quad \text{as } s \rightarrow 0. \end{aligned}$$

In this case, the element  $S$  is "not good."

Anisotropic elements in the next two examples are also "good." However, these examples differ slightly from Examples 8.1 and 8.4.

**Example 8.6** (Right-angled triangle). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (s, 0)^\top$  and  $p_3 := (0, \delta s)^\top$  with  $\delta \ll 1$ . We then have  $h_1 = s$  and  $h_2 = \delta s$ ; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{1}{\delta}, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element  $S$  is "good." However, the factor  $\frac{1}{\delta}$  is very large.

**Example 8.7** (Blade). Let  $S \subset \mathbb{R}^2$  be the simplex with vertices  $p_1 := (0, 0)^\top$ ,  $p_2 := (2s, 0)^\top$  and  $p_3 := (s, \delta s)^\top$  with  $\delta \ll 1$ . We then have  $h_1 = h_2 = s\sqrt{1 + \delta^2}$ ; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{s^2(1 + \delta^2)}{\delta s^2} \leq \frac{c}{\delta},$$

In this case, the element  $S$  is "good." However, the factor  $\frac{1}{\delta}$  is very large.

### 8.3 Anisotropic mesh: three-dimensional case

**Example 8.8.** Let  $T \subset \mathbb{R}^3$  be a tetrahedron. Let  $S$  be the base of  $T$ ; i.e.,  $S = \triangle p_1 p_2 p_3$ . Recall that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|_3} = \frac{h_1 h_2}{\frac{1}{2}h_1 h_2 t_1} \frac{h_3}{\frac{1}{3}h_3 t_2} \leq \frac{H_S}{h_S} \frac{h_3}{\frac{1}{3}\widetilde{\mathcal{H}}_3}. \quad (8.2)$$

If the triangle  $S$  is "not good" such as in Examples 8.4 and 8.5, the quantity (8.2) may diverge. In the following, we consider the case that the triangle  $S$  is "good".

Assume that there exists a positive constant  $M$  such that  $\frac{H_S}{h_S} \leq M$ . For simplicity, we set  $p_1 := (0, 0, 0)^\top$ ,  $p_2 := (2s, 0, 0)^\top$ , and  $p_3 := (2s - \sqrt{4s^2 - s^{2\gamma}}, s^\gamma, 0)^\top$  with  $1 < \gamma$ . Then,

$$h_1 = 2s, \quad h_2 = \sqrt{\frac{4s^{2\gamma}}{2 + \sqrt{4 - s^{2\gamma-2}}}},$$

and because  $h_{\max} \approx cs$ ,

$$\frac{h_{\max}}{h_{\min}} \leq \frac{cs}{h_2} \leq cs^{1-\gamma} \rightarrow \infty \quad \text{as } s \rightarrow 0.$$

If we set  $p_4 := (s, 0, s^\varepsilon)^\top$  with  $1 < \varepsilon$ , the triangle  $\triangle p_1 p_2 p_4$  is the blade (Example 8.4). Then,

$$h_3 = \sqrt{s^2 + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{2+\gamma}}{s^{1+\gamma+\varepsilon}} \leq cs^{1-\varepsilon} \rightarrow \infty \quad \text{as } s \rightarrow 0.$$

In this case, the element  $T$  is "not good."

If we set  $p_4 := (s^\delta, 0, s^\varepsilon)^\top$  with  $1 < \delta < \varepsilon < \gamma$ , the triangle  $\triangle p_1 p_2 p_4$  is the dagger (Example 8.5, Fig. 9). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\delta}}{s^{1+\gamma+\varepsilon}} \leq cs^{\delta-\varepsilon} \rightarrow \infty \quad \text{as } s \rightarrow 0.$$

In this case, the element  $T$  is "not good."

If we set  $p_4 := (s^\delta, 0, s^\varepsilon)^\top$  with  $1 < \varepsilon < \delta < \gamma$ , the triangle  $\triangle p_1 p_2 p_4$  is the dagger (Example 8.2). Then,

$$h_3 = \sqrt{s^{2\delta} + s^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{s^{1+\gamma+\varepsilon}}{s^{1+\gamma+\varepsilon}} \leq c.$$

In this case, the element  $T$  is "good" and  $h_3 \approx h_3 t_2 = \tilde{\mathcal{H}}_3$  holds.

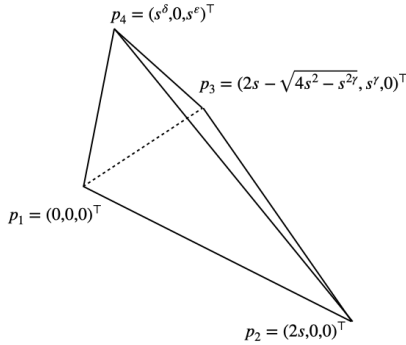


Fig. 9: Example 8.8

**Example 8.9.** In [10], the spire has a cycle of three daggers among its four triangles; see Figure 10. The splinter has four daggers; see Figure 14. The spear and spike have two daggers and two blades as triangles; see Figures 11, 13. The spindle has four blades as triangles; see Figure 12.

**Remark 8.10.** The above examples reveal that the good element  $T \subset \mathbb{R}^3$  may satisfy conditions such as  $h_2 \approx \tilde{\mathcal{H}}_2$  and  $h_3 \approx \tilde{\mathcal{H}}_3$ .

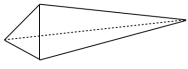


Fig. 10: Spire

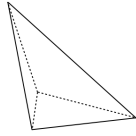


Fig. 11: Spear

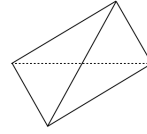


Fig. 12: Spindle

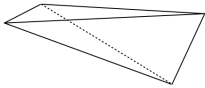


Fig. 13: Spike

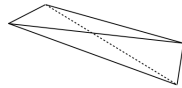


Fig. 14: Splinter

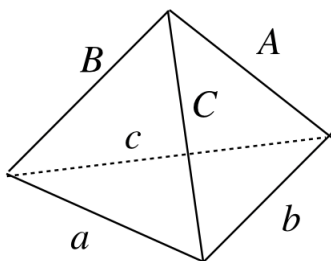


Fig. 15:  $R_3$

**Example 8.11.** Using an element  $T$  called *Sliver*, we compare the three quantities  $\frac{h_T^3}{|T|_3}$ ,  $\frac{H_T}{h_T}$ , and  $\frac{R_3}{h_T}$ , where the formulation of the circumradius  $R_3$  of a tetrahedron  $T$  is as follows, e.g., see [21]. Let  $a, b$  and  $c$  be the lengths of the three edges of  $T$  and  $A, B, C$  the length of the opposite edges of  $a, b, c$ , respectively. Then,

$$R_3 = \frac{\sqrt{(aA + bB + cC)(aA + bB - cC)(aA - bB + cC)(-aA + bB + cC)}}{24|T|_3},$$

see Fig. 15.

Let  $T \subset \mathbb{R}^3$  be the simplex with vertices  $p_1 := (s^{\varepsilon_2}, 0, 0)^\top$ ,  $p_2 := (-s^{\varepsilon_2}, 0, 0)^\top$ ,  $p_3 := (0, -s, s^{\varepsilon_1})^\top$ , and  $p_4 := (0, s, s^{\varepsilon_1})^\top$  ( $\varepsilon_1, \varepsilon_2 > 1$ ), where  $s := \frac{1}{N}$ ,  $N \in \mathbb{N}$ , see Fig. 16. Let  $L_i$  ( $1 \leq i \leq 6$ ) be the edges of  $T$  with  $h_{\min} = L_1 \leq L_2 \leq \dots \leq L_6 = h_T$ . Recall that  $h_{\max} \approx h_T$  and

$$\frac{h_{\max}}{h_{\min}} \leq c \frac{L_6}{L_1}, \quad \frac{H_T}{h_T} = \frac{L_1 L_2}{|T|_3} h_T.$$

Table 8:  $h_T^3/|T|_3$ ,  $H_T/h_T$  and  $R_3/h_T$  ( $\varepsilon_1 = 1.5$ ,  $\varepsilon_2 = 1.0$ )

$N$	$s$	$L_6/L_1$	$h_T^3/ T _3$	$H_T/h_T$	$R_3/h_T$
32	3.1250e-02	1.4033	6.7882e+01	3.4471e+01	5.0195e-01
64	1.5625e-02	1.4087	9.6000e+01	4.8375e+01	5.0098e-01
128	7.8125e-03	1.4115	1.3576e+02	6.8147e+01	5.0049e-01

Table 9:  $h_T^3/|T|_3$ ,  $H_T/h_T$  and  $R_3/h_T$  ( $\varepsilon_1 = 1.0$ ,  $\varepsilon_2 = 1.5$ )

$N$	$s$	$L_6/L_1$	$h_T^3/ T _3$	$H_T/h_T$	$R_3/h_T$
32	3.1250e-02	5.6569	6.7882e+01	8.5513	5.0006e-01
64	1.5625e-02	8.0000	9.6000e+01	8.5184	5.0002e-01
128	7.8125e-03	1.1314e+01	1.3576e+02	8.5018	5.0000e-01

Table 10:  $h_T^3/|T|_3$ ,  $H_T/h_T$  and  $R_3/h_T$  ( $\varepsilon_1 = 1.5$ ,  $\varepsilon_2 = 1.5$ )

$N$	$s$	$L_6/L_1$	$h_T^3/ T _3$	$H_T/h_T$	$R_3/h_T$
32	3.1250e-02	5.6569	3.8400e+02	3.4986e+01	1.4170
64	1.5625e-02	8.0000	7.6800e+02	4.8744e+01	2.0010
128	7.8125e-03	1.1314e+01	1.5360e+03	6.8411e+01	2.8288

In Table 8, the angle between  $\triangle p_1 p_2 p_3$  and  $\triangle p_1 p_2 p_4$  tends to  $\pi$  as  $s \rightarrow 0$ , and the simplex  $T$  is "not good." In Table 9, the angle between  $\triangle p_1 p_3 p_4$  and  $\triangle p_2 p_3 p_4$  tends to 0 as  $s \rightarrow 0$ , the simplex  $T$  is "good." In Table 10, from the numerical result, the simplex  $T$  is "not good."

## 9 Interpolation of Smooth Function

### 9.1 Finite Element Generation

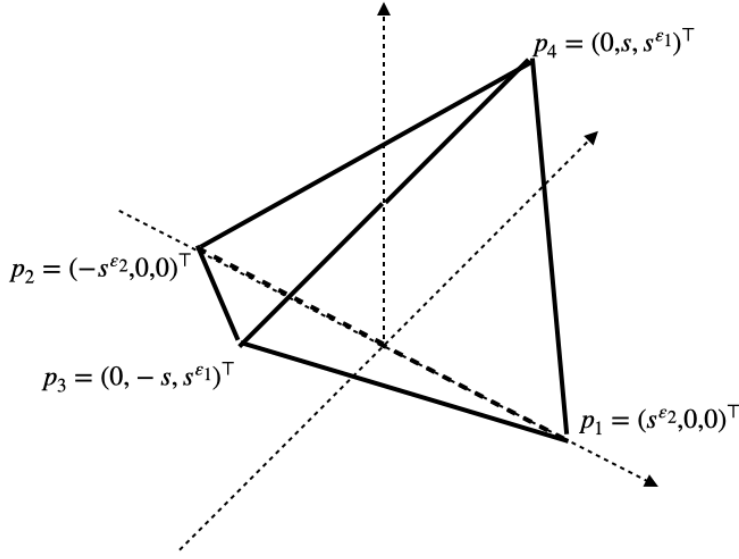


Fig. 16: Sliver

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