Problem Sets

Kohei Izumi

Current version: November 12, 2022

The order of contents is arbitrary.

Contents

1 Probabilities

1.1

Show that $[EX > \infty \implies \Pr(X = \infty)]$ is not correct.

1.2 *Negative binomial distribution*

Let X be the number of failures until n times success and p be the probability of success. Write down the negative binomial distribution. And derive its expected value.

hint: Factor the combination function so that you can use general binomial theorem.

1.3 *Chebychev's inequality*

$X \geq 0, a > 0$. Prove that $\Pr(X \geq a) \leq \frac{1}{a} EX$.

1.4 *Geometric distribution and negative binomial distribution*

X follows the geometric distribution. Calculate $\sum_{n=1}^{\infty} \Pr(X \geq n)$. Then compare the result from question 1.2 and check that $EX = \sum_{n=1}^{\infty} \Pr(X \geq n)$.

1.5

Let X and Y be independent random variable. Let G_X and G_Y be the generating function. Show that $G_{X+Y}(x) = G_X(x) G_Y(x)$.

1.6

Let X and Y be independent random variable, and they follow Poisson distribution with average λ and μ respectively. Show that the distribution of $X + Y$ equals Poisson distribution with average $\lambda + \mu$.

1.7

Show that conditional expected value exists if $E|X| < \infty$.

1.8 *Memorylessness*

Random variable X follows the geometric distribution, then $P(X \geq n) = q^n$. (i) Show that memorylessness $P(X \geq n+m \mid X \geq n) = P(X \geq m)$. (ii) Show that if X is memoryless then X follows the geometric distribution.

hint: Put $q_n \equiv P(X \geq n)$ then you can get $q_{n+m} = q_n q_m$ by memorylessness.

1.9 *Chapman-Kolomogorov equation*

Consider a Markov chain. Show that for $0 \leq r \leq n$, $p_{ij}^{(n)} = \sum_{k \in S} p_{ik}^{(r)} p_{kj}^{(n-r)}$ where $p_{ij}^{(n)}$ is the probability of moving to state i from state j in step n. Moreover, $p_{ij}^{(n+m)} \geq p_{ik}^{(n)} p_{kj}^{(m)}$.

1.10

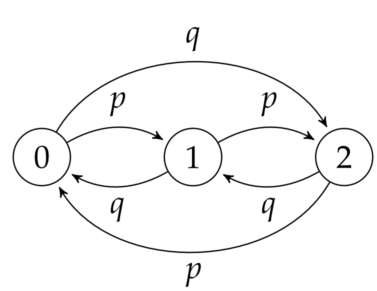
Show that $\Pr(A \cup B \mid C) = \Pr(A \mid B \cup C) \Pr(B \mid C)$.

1.11

$\|x\|_{\mathbb{R}^d} = \left(\sum_{i \in S} x_i^2\right)^{1/2}$, $\|x\| = \frac{1}{2} \sum_{i \in S} |x_i|$. Show that $\|x\|_{\mathbb{R}^d} \leq 2\|x\| \leq \sqrt{d}\|x\|_{\mathbb{R}^d}$.

hint: Cauchy-Schwarz inequality.

1.12 *Markov chain*



$0 < p < 1, p + q = 1$. (i) Check that the Markov chain is irreducible and aperiodic. (ii) Compute the stationary distribution and the average recursion time.

1.13 *State i is reccurent*

Consider a Markov chain. Suppose that $\lim_{n\to\infty} p_{ii}^{(n)} = \pi_i > 0$. Put $f_{ii}^{(k)}$ such that $p_{ii}^{(k)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}$. Show that $\sum_{k=1}^{N} f_{ii}^{(k)} = 1$ for some N.

hint: You cannot directly put "limit" into an infinite sum.

1.14

Let generating function of probability distribution $\{p_k\}$ on $\mathbb{N}_0$ be $G(x) = \sum_{n=0}^{\infty} p_n x^n$. Show that $G'(1-0)$ equals to the mean of G, including cases where both sides are infinity.

1.15

(i) Show that $u_{2n} \equiv \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{(2n)!}{n!n!} = (-1)^n\binom{-\frac{1}{2}}{n}$. (ii) Show that $\sum_{n=0}^{\infty} u_{2n}x^{2n} = \frac{1}{\sqrt{1-x^2}}$.

hint: This is the part of the theorem: one-dimentional simple random walk is reccurent and zero-reccurent. Use general binomial theorem.

1.16 *Weak Law of Large Numbers*

State the weak law of large number and show its proof.

hint: Chebychev's inequality.

1.17

$\lambda > 0$. Show that

$$\lim_{n\to\infty} e^{-n\lambda} \sum_{0\le k\le nT} \frac{(n\lambda)^k}{k!} \to \begin{cases} 1 \text{ if } T > \lambda, \\ 0 \text{ if } T < \lambda. \end{cases}$$

hint: Let $\{X_n\}$ is the independent random variable of mean λ Poisson.

2 Mechanism Design

2.1

Suppose that G is a mean-preserving spread of F. Show that why G is more informative than F. Note that we say A is more informed than B if A's information about some state of the wold is more accurate than B's in the sense of Blackwell (1953). Check Blackwell (1951) and Brooks, Frankel, and Kamenica (forthcoming, Ecta) for more discussion.

2.2 *Hopenhayn and Saeedi (2022)*

$p(z) = P(Q) + z,\ P(Q) = p(0) = \Psi^{-1}(1 - \frac{Q}{M}),\ Q_i = \int S(p(z))dG_i(z),\ U(z, \theta, p) = z + \theta - p(z)$ where $z \sim G$ is quality and $\theta \sim \Psi$ is type. G_2 is a MPS of G_1 so that G_2 is more informative than G_1. (i) Show that if supply function S is concave, $Q_1 \geq Q_2$. (ii) Show that consumer surplus is increasing in Q. Third, what implication do you get from these results?

hint: For the second question, integration by substitution might be helpful.

2.3 *Kamenica and Gentzkow (2011)*

Suppose that there exists τ such that $\sum_{\mu \in \{\mu_s\}_s} \mu\tau(\mu) = \mu_0$ and $E_\tau \hat{v}(\mu) = v^*$. Construct a signal with value v^* and its realization space is at most $|\Omega| + 1$ cardinality.

hint: Caratheodory's theorem.

3 Linear Algebra

3.1

Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. Compute A^n.

4 Random Questions

4.1

X is rv. Show that $\mathrm{E}\left[\frac{f(x)^2}{g(X)}\right] \geq \frac{\mathrm{E}[f(X)]^2}{\mathrm{E}[g(X)]}$.

4.2

$p > \frac{1}{2}$. Show that $p(1-p) < \frac{1}{4}$.

4.3

(i) Why is $\log(1+x) \leq x$ obvious? (ii) Show that $x - x^2 \leq \log(1+x)$ with $|x| < 1$.

4.4

Show that $\forall x \in [0, \frac{\pi}{2}]$, $\frac{2}{\pi}x \leq \sin x \leq x$.

4.5

Show that $\int_{x^2 \leq 1} \frac{dx}{x^2} = \infty$.

5 Answers and Hints

1.1

ongoing.

1.2

ongoing.

1.3

ongoing.

1.4

ongoing.

1.5

ongoing.

1.6

ongoing.

1.7

ongoing.

1.8

ongoing.

1.9

ongoing.

1.10

ongoing.

1.11

ongoing.

1.12

ongoing.

1.13

ongoing.

1.14

ongoing.

1.15

ongoing.

1.16

If independent random variables $\{X_n\}$ satisfy $EX_n = m$ and $VX_n \leq L < \infty \ (n \in \mathbb{N})$, then $\forall \varepsilon > 0$, $\mathrm{P}\left(|\frac{S_n}{n} - m| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2} \to 0 \ (n \to \infty)$ where $S_n = X_1 + \cdots + X_n$.

Proof. $ES_n = nm$ and $VS_n \leq nL$. From Chebychev's inequality,

$$\mathrm{LHS} = \mathrm{P}\left(|S_n - nm| \geq n\varepsilon\right) \leq \frac{1}{\varepsilon^2 n^2} E(S_n - nm)^2 \leq \frac{L}{\varepsilon^2 n} \to 0 \ (n \to \infty).$$

$\square$

1.17

ongoing.

2.1

ongoing.

2.2

ongoing.

2.3

ongoing.

3.1

ongoing.

4.1

By Jensen's inequality,

$$\mathrm{E}\left[\frac{f(x)^2}{g(X)}\right] \geq \mathrm{E}\left[\mathrm{E}\left[\frac{f(X)^2}{g(X)}\right]\right] = \mathrm{E}\left[f(X)\right]^2 \mathrm{E}\left[\frac{1}{g(X)}\right] \geq \frac{\mathrm{E}\left[f(X)\right]^2}{\mathrm{E}\left[g(X)\right]}.$$

4.2

Immediate from the inequality of arithmetic and geometric means.

4.3

(i) Just graph it. (ii) Taylor expansion.

4.4

Just graph it.

4.5

Divide the interval to $[-1, -\varepsilon]$ and $[\varepsilon, 1]$, then this is immediate.