Computational Methods for Nuclear Reactor Physics

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September 1, 2015

Chapter 1

The Transport of Neutrons

1.1 The Angular Flux

There are seven independent variables needed to specify the state of the free neutron field in a nuclear system: three spatial unknowns, $\mathbf{x} = (x, y, z)^{\mathtt{t}}$, time, t, two directional variables, $\hat{\Omega}$, and a neutron speed that is a function of the neutron energy, $v(E)$. The directional, or angular, variable, $\hat{\Omega}$, is a point on the unit sphere . This point is generally represented using a polar angle, θ, and an azimuthal angle φ. How this point on the unit sphere is represented depends on the geometry of interest. In general three dimensional coordinates, with these angles defined such that the polar axis points in the z direction, the point on the unit sphere in (x, y, z) coordinates is given by

Figure 1.1: The coordinate system for $\hat{\Omega}$ that we will use has θ as the polar angle relative to the z-axis, and φ as the azimuthal angle relative to the x-axis.

(1.1)

It will be convenient to define the quantity $\mu = \cos\theta$ to write

(1.2)

The speed, v acts as an energy variable because there is a one-to-one correspondence between speed and energy.

The most basic quantity that describes a radiation field is the *kinetic density*, $f(\mathbf{x}, \hat{\Omega}, v, t)$. This kinetic density times a differential volume of phase space gives

the number of neutrons in that piece of phase space[1]. Specifically,

is the number of neutrons with position in the volume element $d\mathbf{x}$ about $\mathbf{x}$, traveling in directions within $d\hat{\Omega}$ about $\hat{\Omega}$, with frequency in dv about v at time time t. The differential element $d\hat{\Omega}$ in our polar coordinate system is given by $d\hat{\Omega} = \sin\theta d\theta d\varphi$ or $d\hat{\Omega} = d\mu d\varphi$. Using the kinetic density, we define a *spectral neutron density*, $N_v(\mathbf{x}, v, t)$ as

$$\begin{aligned} N_v(\mathbf{x}, v, t) &\equiv \int_{4\pi} d\hat{\Omega}\, f(\mathbf{x}, \hat{\Omega}, v, t) \\ &= \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu\, f(\mathbf{x}, \hat{\Omega}, v, t), \end{aligned}$$

with $N_v(\mathbf{x}, v, t)d\mathbf{x}dv$ as the number of neutrons with position in the volume element $d\mathbf{x}$ about $\mathbf{x}$, with frequency in dv about v at time time t. The notation $\int_{4\pi} d\hat{\Omega}$ indicates that the integration is carried out over the entire unit sphere. Similarly, the *neutron density*, $N(\mathbf{x}, t)$ is defined as

where $N(\mathbf{x}, t)d\mathbf{x}$ is the number of neutrons with position in the volume element $d\mathbf{x}$ about $\mathbf{x}$ at time t.

These densities are important quantities in neutron transport, however it is more common to work with angular fluxes rather than densities. The *angular flux*, $\psi(\mathbf{x}, \hat{\Omega}, v, t)$, is defined as

The angular flux allows us to quantify the flow of neutrons across a surface:

is the number of neutrons traveling with directions in $d\hat{\Omega}$ about $\hat{\Omega}$ with speed in dv about v that flows across a differential area element dA with outward normal

[1] Technically, this is the expected or mean number of neutrons in this piece of phase space. There can be fluctuations from the mean that are significant at small neutron numbers. We will ignore these fluctuations our study because in almost all cases there are many neutrons in the important pieces of phase space. We could not do this if, for example, we were interested in a small number of neutrons entering a detector or the probability of a chain reaction starting from a single neutron in supercritical system.

$\hat{n}$ in time dt. This relation is also the reason that the angular flux has the unwieldy name that it does: the angular flux can tell us the number of neutrons that cross a surface (that is the flux across that surface) in a particular angle.

There are several quantities that are defined as integrals of the angular flux. The *scalar flux*, $\phi(\mathbf{x}, t)$, is the integral of the angular flux over all angles and frequencies:

Notice that ϕ is the zeroth angular moment of the angular flux. In addition to the scalar flux, we define the frequency dependent scalar flux as

Later on as we talk about numerical values of the angular flux and energy density it will be worthwhile to recall that since the angular flux is a constant multiple of the kinetic density of neutrons, it only makes physical sense to talk about non-negative fluxes (negative number densities of neutrons are physically unreasonable). This holds true for scalar fluxes as well: the number of neutrons per unit volume cannot be negative. As is the case in almost all computational physics endeavors, however, the fact that certain quantities only make sense in certain ranges (in this case non-negative values for the angular and scalar fluxes) does not guarantee that numerical methods will respect this physical truth unless forced to by design or some other intervention.

The first angular moment of the angular flux is the *net current density* (often misleadingly called just the neutron current), $\mathbf{J}(\mathbf{x}, t)$, and is defined as

Each component of $\mathbf{J}$ is given by

$$J_x = \int_0^\infty dv \int_{4\pi} d\hat{\Omega}\, \Omega_x \psi(\mathbf{x}, \hat{\Omega}, v, t), \qquad J_y = \int_0^\infty dv \int_{4\pi} d\hat{\Omega}\, \Omega_y \psi(\mathbf{x}, \hat{\Omega}, v, t),$$
$$J_z = \int_0^\infty dv \int_{4\pi} d\hat{\Omega}\, \Omega_z \psi(\mathbf{x}, \hat{\Omega}, v, t).$$

The net current density gives the net flow of neutrons across particular surfaces. For example, J_x gives the flow rate of neutrons per unit area across a plane

perpendicular to the x axis; J_y and J_z represent similar energy flows across planes perpendicular to the y and z axes. We also define an energy-dependent net current density as

The components of $\mathbf{J}$ can be negative. For example, a negative value for J_x indicates that the net flow of radiation with respect to the x-coordinate is in the negative-x direction.

The second angular moment is related the momentum flow across planes perpendicular to the coordinate axes. Following the definition of pressure in gas dynamics, we define the *neutron pressure*, $\mathbf{P}(x,t)$, as

where the outer product is given by

The P_{xy} component of the pressure tensor gives the rate of flow of current density in the x-direction across a plane perpendicular to the y axis, and the other components give similar rates of flow. Notice that the pressure tensor is symmetric, $P_{ij} = P_{ji}$; this reduces the number of independent components of the tensor from 9 down to 6. Also, the trace of the pressure tensor is the scalar flux. This property of the pressure can be found from the fact that $\Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1$ then writing

As before, we define the frequency dependent neutron pressure as

The neutron pressure is a non-negative quantity.

1.2 Neutron Interactions

In the context of nuclear reactor systems we are primarily concerned with the following types of interaction between a neutron and a nucleus: absorption, scattering, and fission. The absorption comes in several forms: radiative capture, $(n, 2n)$ interactions, and other capture reactions. The dominant scattering processes are elastic scattering and inelastic scattering. The kinematics of the scattering process depends heavily on the type of nucleus and the speed of the neutron. Calculating the rates for these interactions involves detailed atomic physics computations. For the purposes of most neutron transport calculations, these cross-sections can be thought of as available from a table of pre-calculated values. Nevertheless, it will be useful to understand the magnitudes and scalings of these interactions; a topic we shall return to later.

The principal quantity that gives the strength of interaction between radiation and matter is the total macroscopic cross-section, $\Sigma_{\mathrm{t}}(\mathbf{x}, v, t)$. This quantity has units of inverse length and is defined such that

is the probability that a neutron traveling a distance ds with speed v has an interaction with the background nuclei. Note that we do not include a dependence of $\hat{\Omega}$ in the cross-section. A cross-section independent of the direction of flight of the neutron assumes there is no special direction in the material. In certain materials such as crystals, there could be a $\hat{\Omega}$ dependence, but we do not consider this here. In problems where the background material is moving, this material flow will introduce $\hat{\Omega}$ dependence in the interaction. Also, the cross-section is a function of the material temperature, and this temperature dependence is couched in the dependent variables $\mathbf{x}$ and t because the material temperature is also a function of space and time.

The total macroscopic cross-section consists of two components, the macroscopic absorption cross-section, $\Sigma_{\mathrm{a}}(\mathbf{x}, v, t)$, and the macroscopic scattering cross-section, $\Sigma_{\mathrm{s}}(\mathbf{x}, v, t)$ such that $\Sigma_{\mathrm{t}}(\mathbf{x}, v, t) = \Sigma_{\mathrm{a}}(\mathbf{x}, v, t) + \Sigma_{\mathrm{s}}(\mathbf{x}, v, t)$. Each of these cross-sections can be written as a number density times an area. For example,

(1.3)

where N_a is the number density of nuclei σ_a is the microscopic cross-section of absorption given in units of area. Typically, the microscopic cross-section is given in barns (b) where 1 b = 10^{-24} cm^2. It is also common to drop the modifier "macro" or "micro" unless it is needed to avoid confusion. In our study we will primarily use macroscopic cross-sections. The primary absorption reactions of interest to us are radiative capture, where a neutron absorbs a neutron and releases a gamma ray (photon), and neutron-induced fission.

The cross-section multiplied by a distance, ds, gives the probability of a neutron being absorbed by a nucleus. The microscopic cross-sections are strongly dependent on the speed of the neutron. This is due to quantum physics considerations which give rise to resonances. These resonances are particular neutron energies where a reaction is favorable, and cause the cross-section to increase by several orders of magnitude. Also, when a neutron moves more slowly, a reaction tends to be more likely because a slower neutron spends more time near a nucleus.

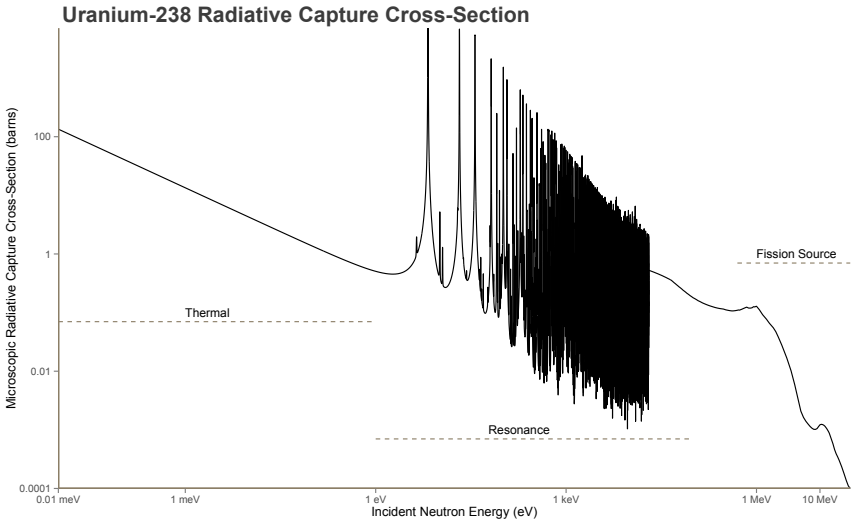

Figure 1.2: The radiative capture microscopic cross-section of ^{238}U. Radiative capture is the primary absorption process of neutrons. The names for different energy ranges are denoted in the figure.

Neutron-induced fission deserves special attention as it is the reason that nuclear reactors exist. The fission reaction is an absorption reaction where the neutron is absorbed by the nucleus and then splits into two smaller nuclei–called daughter products–and emits several neutrons in the process. The fission cross-section is written as $\Sigma_f(\mathbf{x}, v)$ and the mean number of neutrons emitted in the

fission reaction is denoted by $\bar{\nu}(\mathbf{x}, v)$. This definition allows us to define

as the expected number of neutrons emitted from fission reactions per unit distance travelled per neutron with speed v. We have made the relationship between v and E explicit in this definition. There is also a spectrum that fission neutrons have. That is, when fission neutrons are emitted their energy is governed by a distribution that we denote as $\chi(\mathbf{x}, v)$. With this definition we write

as the expected density of fission neutrons emitted into dE about E per unit distance travelled by neutrons with speed $v(E')$. Note that χ is a probability density function in energy so that

The nature of neutron transport inside a reactor leads to several definitions of energy ranges. Fission neutrons are typically born from fission with energy in the MeV range. In thermal reactors these neutrons are slowed through the resonance range where resonance interactions are possible, before reaching the thermal range where neutrons are in near equilibrium with the vibration energy of the reactor's nuclei. Typical values for these ranges are given in Figure 1.2.

A feature of the fission reaction is that most of the neutrons are emitted very soon after the nucleus splits (on the order of 10^{-14} s). These are the so-called prompt neutrons. There is a fraction of neutrons, denoted by β that emerge later on and are called delayed neutrons. These neutrons are emitted by certain daughter products called delayed-neutron precursors. The delayed neutron precursor density c is the density of nuclei that will emit neutrons. These precursors are generally classified by the time-scale at which they emit neutrons, and each has a fraction of delayed neutrons associated with it:

There is also a spectrum for each c_i given by χ_i. The c_i's are commonly called delayed neutron precursor groups, but to avoid confusion with another important division, we will call them delayed-neutron precursor "flavors". This is also a more accurate term as there are 5 (or 6 depending on culture) basic tastes, and usually we use $I = 6$ in practice.

The treatment of the scattering cross-section is somewhat more complicated than absorption reactions. This is due to the fact that a neutron entering a scattering interaction exits with a different direction of flight and frequency. To accurately describe the transport of neutrons we must take this change into account. The scattering cross-section is defined in the same fashion as the other cross-sections: $\Sigma_s ds$ is the probability that a neutron traveling a distance ds undergoes a scattering interaction. However, we must also define a differential scattering cross-section that takes into account the change in $\hat{\Omega}$ and v. The double-differential scattering cross-section, $\Sigma_s(\mathbf{x}, \hat{\Omega}' \cdot \hat{\Omega}, v' \to v)$, is defined such that

gives the probability that a neutron with direction of flight $\hat{\Omega}'$ and frequency v' traveling a distance ds undergoes a scattering collision and exits with $\hat{\Omega}$ in $d\hat{\Omega}$ and v in dv. The double-differential scattering cross-section is a function of $\hat{\Omega}' \cdot \hat{\Omega}$ rather than a function of the two angles independently. This is equivalent to saying that the differential scattering cross-section depends only on the scattered angle. We can make this implication because, as previously assumed, the cross-sections do not depend on the direction of flight of the incident neutron.

The double-differential scattering cross-section is related to the scattering cross-section by the relation

$$\begin{aligned}\Sigma_s(\mathbf{x}, v', t) &= \int_0^\infty dv \int_{4\pi} d\hat{\Omega} \Sigma_s(\mathbf{x}, \hat{\Omega}' \cdot \hat{\Omega}, v' \to v) \\ &= 2\pi \int_0^\infty dv \int_{-1}^1 d(\cos\Theta) \Sigma_s(\mathbf{x}, \cos\Theta, v' \to v),\end{aligned}$$

where $\cos\Theta = \hat{\Omega}' \cdot \hat{\Omega}$. The double-differential scattering cross-section can also be written using a scattering kernel as

(1.4)

where

An important concept when dealing with the material interaction is the mean-free path. The mean-free path is the average distance that a neutron travels before having a given interaction. The total mean-free path, $\lambda(\mathbf{x}, v, t)$, is the reciprocal of the total cross-section:

The absorption and scattering mean-free paths are similarly defined,

and give the average distance a neutron travels before an absorption or scattering interaction respectively.

Besides absorbing or scattering neutrons, the material can emit neutrons. This can be from a number of possible reactions including, but not limited to, spontaneous fission radionuclides, sources which combine alpha emitters with beryllium, and even large particle accelerators with the right targets. We will write a source of neutrons as $q(\mathbf{x}, \hat{\Omega}, v, t)d\mathbf{x}d\hat{\Omega}dv$, which emits q neutrons at $d\mathbf{x}$ about $\mathbf{x}$ into $d\hat{\Omega}$ about $\hat{\Omega}$ with frequency in dv about v at time t. Though we have allowed the source to emit particles anisotropically in angle, in most situations we will not require this anisotropy.

1.3 The Transport Equation

We shall now derive an equation for the transport of neutrons through a stationary background material. The derivation will be require only an accounting of how neutrons move through phase space and are created / destroyed. To begin the derivation we examine the time rate of change in the kinetic density which is

One way that neutrons leave phase space is through material interactions, i.e. collisions. The rate at which collisions with nuclei occur is

(1.5)

That Eq. (1.5) gives the rate of interactions with the material can be seen by noting that vf is the distance neutrons in a particular region of phase space travel in a unit of time. Therefore, $vf\sigma$ is the rate of interactions.

Another way that particles can leave a region of phase space is by streaming. When a particle streams, both v and $\hat{\Omega}$ remain constant, only the particle's position changes. The total amount of streaming is then

There are three ways that neutrons enter a region of phase space: scattering, fission, and sources. For scattering, neutrons in another region of phase space can scatter into the region of phase space of interest. The rate of scattered particles entering phase space is

(1.6)

where $v' = v(E')$. To treat the creation of neutrons from fission we need to count the number of neutrons that enter our phase space using our definition above to write the number of fission neutrons enter phase space as

The factor of 4π is required because the fission neutrons are assumed to be emitted isotropically in angle.

As for particles entering phase space from sources, their rate is given by

(1.8)

Combining all these contributions to the change in neutrons in a piece of phase space we get

Then multiplying Eq. (1.9) by $\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \vec{\Omega}} \frac{\partial}{\partial E}$ we get the conventional form of the transport equation:

This equation is probably the most concise statement of the governing equation for neutrons moving through a stationary material. We give it an equally weighty title: the neutron transport equation. Nevertheless, there is a significant amount of detail couched in the interaction terms in this equation. For instance we have said nothing about the form the scattering kernel can take. We can also modify this equation to explicitly show the contribution from delayed

neutrons:

1.3.1 Boundary and Initial Conditions

The boundary conditions for the neutron transport equation, Eq. (1.10), are different than those for many other areas of computational physics. If we imagine an experiment where we have an object and want to observe the change in the number of neutrons in that object, what can we prescribe in that experiment? Obviously we can say what the material, initial number of neutrons and any sources of radiation are. We can also specify what the *incoming* neutrons that hit the object are. In that respect we specify the neutrons that are entering the body. The same is true for the boundary conditions for Eq. (1.10): we can specify the neutrons that enter the system. We cannot specify the neutrons that exit the system because the neutrons that exit the system are a function of the incoming neutrons as well as the other features of the system—specifying the outgoing neutrons is over specifying the problem and is ill-posed in general. As a result boundary conditions for Eq. (1.10) take the form

(1.12)

where $\hat{n}$ is the outward normal to the boundary. The fact that proper boundary conditions only specify incoming neutrons can lead to complications in numerical methods where only integrals over all $\hat{\Omega}$ are computed. This is especially true in diffusion calculations, as we will see. Initial conditions are more straightforward than boundary conditions. At the beginning of a problem we can specify

the entire angular flux, $\psi(\mathbf{x}, \hat{\Omega}, v, 0)$. Typically we will only specify the initial radiation as being a isotropic, but this is not a requirement.

The conditions on the angular flux at a material interface are that the angular flux is continuous as the interface is crossed. To see this, consider a planar interface that is crossed in the z-direction at $z = 0$. We will integrate the transport equation over a range $z \in [-\epsilon/2.\epsilon]$ and take the limit as $z \to 0$. This operation leads to

This limit is given by ψ being continuous across the interface when $\mu \neq 0$. When $\mu = 0$, the limit equation is satisfied no matter the value of ψ on each side of the interface. In other words, there can be a jump in the angular flux because this is a direction parallel to the interface and no information crosses the interface in this direction.

1.3.2 Hyperbolicity of the Transport Equation

The transport equation is hyperbolic in the sense that information travels at a finite speed. What this means physically is that when a neutron is released at point $\mathbf{x}$ at time $t = 0$ cannot travel beyond a point beyond $\hat{\Omega} \cdot \mathbf{x} > vt$, where c is the speed of light. In other words the transport equation does not allow information to travel faster than than v.

1.4 Legendre Moment Expansion of the Scattering Term

For most computational methods it will be convenient to express the scattering term in the transport equation using an orthogonal expansion of the angular part. To do this we first write the double-differential scattering kernel using a Legendre polynomial expansion

(1.13)

where $\mu_0 = \hat{\Omega}' \cdot \hat{\Omega}$ and P_l is a Legendre polynomial,

The expansion coefficients K_l are determined by the integral relation:

Then we use a similar expansion for the angular flux, except this time we need to use a spherical harmonics expansion because the angular variable in the angular flux is a point on a unit sphere and not the cosine of an angle. To that end we write

where $Y_{lm}(\hat{\Omega})$ is an orthonormal spherical harmonics function and the expansion coefficients are given by

and the asterisk denotes the complex conjugate. The spherical harmonics functions can be written as

where the associate Legendre polynomials, P_l^m are given by

With these angular expansions in hand we can write the the scattering term in the transport equation as

$$\int_0^\infty dE' \int_{4\pi} d\hat{\Omega}' \, K\left(\hat{\Omega}' \cdot \hat{\Omega}, v' \to v\right) \psi(\mathbf{x}, \hat{\Omega}', v', t) =$$
$$\int_0^\infty dE' \int_{4\pi} d\hat{\Omega}' \sum_{l'=0}^{\infty} \frac{2l+1}{4\pi} P_{l'}(\mu_0) K_l\left(v' \to v\right) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\hat{\Omega}') \phi_{lm}(\mathbf{x}, v', t)$$

The addition theorem for spherical harmonics states

The end result is that we can write the scattering terms as a sum of contributions from individual moments of the angular flux and the scattering kernel as

(1.14)

This makes the neutron transport equation

This form is attractive because many times we can approximate the scattering kernel with a few Legendre moments and then sum only has a few terms. It will also tell us how we can include some information about the scattering kernel in the diffusion approximation.

1.5 The Types of Neutron Transport Problems

The general neutron transport equation for time-dependent problems with fixed sources and delayed neutrons is actually too general for many types of analyses. For many design problems, we are interested in properties of the system such as whether the system can carry on a chain reaction or how the system will respond to changes in the neutron population. To discuss these problems we will have to introduce the concept of criticality.

A nuclear system is said to be *critical* if the system can sustain a stable chain reaction in the absence of sources. Put another way, a critical system has a non-zero, finite steady-state neutron population. For such a steady-state to exist, the creation of neutrons from fission must balance the loss of neutrons in the system due to leakage out of the boundary and absorption.

A system of said to be supercritical if a chain reaction in the system will diverge, i.e., grow without bound. Such a system will not have a finite steady state. On the other hand, a system where any chain reaction will eventually die out is called subcritical. These systems will not have a finite steady state without an external source.

1.5.1 α-Eigenvalue Problems

The α eigenvalue indicates the criticality of a system. We derive the α eigenvalue by writing the solution to the neutron transport equation as

(1.16)

We then plug this into the source-free neutron transport equation with no incident neutrons:

(1.18)

The equation will likely have solutions for multiple values of α: these are the α eigenvalues and ψ_α are the corresponding eigenfunctions. These values can

be complex; this can be seen by noting that the form of the time dependence is similar to that used in a Laplace transform. A more theoretical discussion of this can be found in the reactor theory monograph of Bell and Glasstone [1]. Given the form of the time dependence in Eq. (1.17), the α value that is the farthest to the right in the complex plane will dominate the late time behavior. It is this eigenvalue we refer to when we talk about the "α eigenvalue."

The sign of α will tell us the criticality of the system. If $\alpha = 0$, then there exists a steady-state solution to Eq. (1.17) and the system is critical. Furthermore, $\alpha > 0$ indicates that as $t \to \infty$ the solution will grow without bound, i.e., that the system is supercritical. Finally, if $\alpha < 0$, the angular flux will eventually go to 0 everywhere and the system is subcritical.

Finally, the α eigenvalue tells us something about the long-time behavior of systems with a source. If we think of the general solution to a source-driven system as a the sum of a homogeneous solution and a particular solution, we can write the long time behavior of the angular flux as

(1.19)

If we consider a system with a time-independent source and cross-sections, and plug in Eq. (1.19)

If the system is subcritical (i.e., the α with the largest real part is less than

zero), then as $t \to \infty$, this equation becomes

A particular solution that is constant in time can solve this equation because none of the terms depend on time, if ψ_{part} does not.

When the system is critical, that is $\alpha = 0$, as $t \to \infty$ Eq. (1.20) becomes

In this case the particular solution cannot be time independent because we know that the only solution to the equation

as $t \to \infty$ is ψ_α. Therefore, the long time behavior of the solution will still be time dependent. Physical intuition tells us that the solution should tend toward

infinity as $t \to \infty$ because each neutron that is added by the source can start a self-sustaining chain reaction. We will see this for particular transport models later on.

Finally, for a supercritical system, the solution increases exponentially over time, and the presence of a source or initial number of neutrons, will not alter that behavior.

1.5.2 k-Eigenvalue Problems

For many problems it is more useful to solve a different eigenvalue problem to determine the criticality of the system. As we will see later, it is easier to find a means to modify the number of fission neutrons emitted to get a non-trivial, steady-state solution. The k-eigenvalue problem is given by

$$(1.25)$$

In this equation k is the number we modify the fission neutron production terms to get a steady-state solution. In this case we are interested in the largest value of k that gives us a non-trivial solution. We call this value the effective multiplication factor and will denote it as k_{eff}, also known as the multiplication factor. We are interested in the largest value because this is the smallest value we would need to change the fission production by to get a solution. Furthermore, k_{eff} tells us the criticality of the system. If $k_{\text{eff}} = 0$, then there is no change needed to the fission neutron production to get a steady-state solution. When $k_{\text{eff}} > 0$, this means that we need to *decrease* the number of neutrons produced by fission to get a steady-state solution. Therefore, this system is supercritical because there is too much neutron production. Conversely, $k_{\text{eff}} < 0$ implies that the system is subcritical because there is not enough production of fission neutrons to carry on a chain reaction and give us a steady-state solution.

The eigenfunction corresponding to k_{eff} is called the fundamental mode of the reactor. This eigenfunction is important because in a critical system without sources, the angular flux will be the fundamental mode after any transients have

decayed away. Moreover, for a critical reactor, the fundamental mode is also the eigenfunction corresponding to the dominant α eigenvalue.

We can derive the interpretation of the k_{eff} as the multiplication factor. To do this consider a reactor system without sources at time $t = 0$. We define the neutrons in the system at this time as the first generation of neutrons, and we will call this first generation of neutrons with a corresponding angular flux $\psi_1(\mathbf{x}, \hat{\Omega}, v, t)$. We will follow these neutrons until they are all absorbed or leak out of the system. Therefore, the transport equation for these first-generation neutrons is

(1.27)

We then integrate this transport equation over all time to get

where $\psi_1 = vN_1$, and

There is a steady state solution to this equation because it is a fission-free system with a source.

The fission neutrons produced by first-generation neutrons are the source for second-generation neutrons. Therefore, we can write the transport equation for second-generation neutrons integrated over time as

with zero-incident neutron boundary conditions. Once again, this is a source-driven, steady-state problem for N_2 without fission. Therefore, it has a steady-state solution. We can define a general transport equation for the nth generation of neutrons as

We can write this equation in simpler form by defining the linear operators

and

to get

Notice that if we operate on both sides by $\mathcal{L}^{-1}$ we get

(1.31)

This is the same iteration as inverse power iteration to find the largest eigenvalue k to the generalized eigenvalue problem

(1.32)

Therefore, if the operator $\mathcal{L}^{-1}\mathcal{F}$ is bounded, we expect that

This relation tells us the k_{eff} is the multiplicative change in the magnitude of the number density of neutrons in the reactor. This is why we refer to it as the multiplication factor. Eq. (1.31) also indicates why k eigenvalues are the preferred method to find the criticality for many systems: simple inverse power iterations can be used to find k_{eff}. Moreover, each of those inverse power iterations involves solving a steady-state, fission-free, source-driven, neutron transport problem.