Chapter 7

The Method of Characteristics

7.1 The Foundations of the Method of Characteristics

The starting point for the method of characteristics (MOC) is the characteristic transformation. We will be able to demonstrate how MOC works by beginning with the single-group, steady-state transport equation without scattering (this is equivalent to the equation we solve via sweeps in source iteration),

$$\left(\hat{\Omega}\cdot\nabla+\Sigma_{\mathrm{t}}\right)\psi(\mathbf{x},\hat{\Omega})=q(\mathbf{x},\hat{\Omega}). \tag{7.1}$$

We then consider the change of variables to a coordinate system where space is parameterized relative to a reference point along all directions. The parameter is typically denoted as s and is used as

$$\mathbf{x}(s)=\mathbf{x}_0+s\hat{\Omega}=\left(x_0+s\Omega_x,y_0+s\Omega_y,z_0+s\Omega_z\right). \tag{7.2}$$

To use this parameterization in the transport equation, we need to express spatial derivatives in terms of derivatives of s. For the x-derivative we have, using the chain-rule

$$\frac{\partial}{\partial x}\psi(s,\hat{\Omega})=\frac{\partial s}{\partial x}\frac{\partial}{\partial s}\psi(s,\hat{\Omega})=\hat{\Omega}_x^{-1}\frac{\partial}{\partial s}\psi(s,\hat{\Omega}). \tag{7.3}$$

This relation makes the $ho\cdot\nabla\psi$ term in Eq. (7.1) simplify so that the transport equation becomes

$$\left(\frac{\partial}{\partial s}+\Sigma_{\mathrm{t}}\right)\psi(s,\hat{\Omega})=q(s,\hat{\Omega}). \tag{7.4}$$

The characteristic transform writes the transport equation in terms of a simple parameterization that represents the particles streaming along a particular direction. What we will do is solve the transport equation exactly along a particular direction. This can be seen by considering a region with constant cross-sections and sources where in the direction $\hat{\Omega}$ the distance across the region is S. In such a case we call $s=0$ the point at which the characteristic enters the region and assume that we know the value of $\psi(s_0,\hat{\Omega})$. We then multiply Eq. (7.4) by integrating factor, $\exp(\Sigma_{\mathrm{t}}s)$ and divide by Σ_{t}, to get a simple ODE

for the angular flux:

$$\frac{1}{\Sigma_t}\frac{\partial}{\partial s}e^{\Sigma_t s}\psi(s,\hat{\Omega}) = \frac{e^{\Sigma_t s}}{\Sigma_t}q(s,\hat{\Omega}). \tag{7.5}$$

Taking the integral of Eq. (7.5), we get

$$\frac{e^{\Sigma_t s}}{\Sigma_t}\psi(s,\hat{\Omega}) - \frac{1}{\Sigma_t}\psi(0,\hat{\Omega}) = \int_0^s ds' \frac{e^{\Sigma_t s'}}{\Sigma_t}q(s',\hat{\Omega}), \qquad s \in [0,S]. \tag{7.6}$$

Solving this for $\psi(s,\hat{\Omega})$ gives (for a constant source)

$$\psi(s,\hat{\Omega}) = \psi(0,\hat{\Omega})e^{-\Sigma_t s} + \frac{q}{\Sigma_t}\left(1 - e^{-\Sigma_t s}\right), \qquad s \in [0,S]. \tag{7.7}$$

Therefore, the angular flux exiting the region of constant cross-sections and source is

$$\psi(S,\hat{\Omega}) = \psi(0,\hat{\Omega})e^{-\Sigma_t S} + \frac{q}{\Sigma_t}\left(1 - e^{-\Sigma_t S}\right). \tag{7.8}$$

This leads to the definition of the change in the angular flux along the path as

$$\Delta\psi(\hat{\Omega}) = \psi(S,\hat{\Omega}) - \psi(0,\hat{\Omega}) = \left(\psi(0,\hat{\Omega}) - \frac{q}{\Sigma_t}\right)\left(1 - e^{-\Sigma_t S}\right). \tag{7.9}$$

We can also compute the average value of the angular flux as

$$\bar{\psi}(\hat{\Omega}) = \frac{1}{S}\int_0^S ds\, \psi(s,\hat{\Omega}) = \frac{\psi(0,\hat{\Omega})}{\Sigma_t}\left(1 - e^{-\Sigma_t S}\right) + \frac{q}{\Sigma_t^2}\left(e^{-\Sigma_t S} - 1 + S\Sigma_t\right). \tag{7.10}$$

Therefore, we can write the solution at any point s inside the region given the incoming angular flux. If we think of a transport problem as a series of regions of constant cross-sections and sources, then we can solve the transport equation along a given characteristic by starting at the boundary, then computing the exiting angular flux to the first region, using this as the starting value for the next region, and continuing on until we reach the other problem boundary.

For almost any problem we want to solve, the source will be a function of the scalar flux through scattering and fission. Therefore, we need to compute the scalar flux inside each region. We first write the scalar flux at s using a simple integral:

$$\phi(s) = \int_{4\pi} d\hat{\Omega}\, \psi(s, \hat{\Omega}) = \int_{4\pi} d\hat{\Omega} \left[\psi(0, \hat{\Omega}) e^{-\Sigma_{\mathrm{t}} s} + \frac{q}{\Sigma_{\mathrm{t}}} \left(1 - e^{-\Sigma_{\mathrm{t}} s}\right) \right]. \tag{7.11}$$

We will approximate this integral using a tensor product of a quadrature rule in the polar and azimuthal angles:

$$\phi(s) = \sum_{n_p=1}^{N_p} \sum_{n_a=1}^{N_a} w_{n_p} w_{n_a} \left[\psi(0, \hat{\Omega}_{pa}) e^{-\Sigma_{\mathrm{t}} s} + \frac{q}{\Sigma_{\mathrm{t}}} \left(1 - e^{-\Sigma_{\mathrm{t}} s}\right) \right], \tag{7.12}$$

where N_p and N_a are the number of polar and azimuthal angles, w_{n_p} and w_{n_a} are the polar and azimuthal weights, and we denote the angle as $\hat{\Omega}_{pa} = (\theta_{n_p}, \varphi_{n_a})$.

The results in Eqs. (7.10) and (7.13) are the two main results we will need to formulate the method of characteristics as it is used in practice to solve reactor problems.

7.2 The Method of Characteristics in 2-D Geometry

For practical application of MOC, problems are solved in 2-D Cartesian geometry where the problem is broken up into regions of constant cross-sections and sources. The constant or "flat"-source approximation means that we need to know the average scalar flux over each region. To get the average scalar flux in a region we would compute the integral of overall characteristics that enter the

region. For region i, the scalar flux would be

$$\bar{\phi}_i = \frac{1}{A_i} \int\limits_{k \in A_i} dA \int\limits_0^{S(k)} ds \sum_{n_p=1}^{N_p} \sum_{n_a=1}^{N_a} w_{n_p} w_{n_a} \sin\theta_{n_p} \left[\psi(0, \hat{\Omega}_{pa}) e^{-\Sigma_t s} + \frac{q}{\Sigma_t} \left(1 - e^{-\Sigma_t s}\right) \right], \tag{7.13}$$

where k is a track that intersects the region. The quantity $\sin\theta_{n_p} = \sqrt{1-\mu_{n_p}^2}$ projects the track onto the 2-D plane and accounts for the fact that $S(k)$ is proportionally longer the smaller the value of μ_{n_p} is.

In reality we cannot compute every track that intersects a region. Rather we estimate the integral using a discrete number of tracks and the contribution of each track is the effective width of a track times the length of the track in area A_i, $S(k)$. The contribution from each track is then the average value of the angular flux. We then can write Eq. (7.13) in fully discrete form as

$$\bar{\phi}_i = \frac{1}{A_i} \sum_{k \in A_i} \sum_{n_p=1}^{N_p} \sum_{n_a=1}^{N_a} w_k w_{n_p} w_{n_a} S(k) \sin\theta_{n_p} \bar{\psi}(\hat{\Omega}_{pa}). \tag{7.14}$$

We can rearrange this equation to give us the average scalar flux in terms of $\Delta\psi(\hat{\Omega})$:

$$\bar{\phi}_i = \frac{q}{\Sigma_t} + \frac{1}{A_i \Sigma_t} \sum_{k \in A_i} \sum_{n_p=1}^{N_p} \sum_{n_a=1}^{N_a} w_k w_{n_p} w_{n_a} S(k) \sin\theta_{n_p} \Delta\psi(\hat{\Omega}_{pa}). \tag{7.15}$$

7.2.1 Choosing the tracks

In an MOC code, the user typically enters the desired number of azimuthal angles and the desired track spacing (that is how large the distance between tracks is). There is a slight wrinkle to this, however, in that computationally we desire to have the azimuthal angles set up in such a way that for each angle

φ there is also an angle $\varphi - \pi$ in the angle set (these are called complementary angles). We then also lay the tracks out so that complementary angles intersect at problem boundaries. This allows us to easily handle reflecting boundary conditions because the end of a track directly feeds into another track at the boundary.

To make complementary tracks , we first compute the set of desired azimuthal angles based on the user specified N_a as

$$\varphi_{n_a} = \frac{2\pi}{N_a}\left(n_a - \frac{1}{2}\right). \tag{7.16}$$

Then for a domain of width W and height H where the desired track width is t_s, the number of tracks originating at the left and bottom boundaries are

$$T_x = \left\lfloor \frac{W}{t_s}|\sin\varphi_{n_a}| \right\rfloor + 1, \tag{7.17}$$

$$T_y = \left\lfloor \frac{H}{t_s}|\cos\varphi_{n_a}| \right\rfloor + 1, \tag{7.18}$$

and the delimiters $\lfloor\rfloor$ denote the floor function. To assure that there are an equal number of tracks originating at each face of the problem we tweak the azimuthal angle to be

$$\varphi_{n_a,\text{eff}} = \varphi_{n_a} + \tan^{-1}\left(\frac{HT_x}{WT_y}\right). \tag{7.19}$$

We then adjust the track spacing to assure cyclic tracks:

$$t_{s,\text{eff}} = \frac{W}{T_x}|\sin\varphi_{n_a}|. \tag{7.20}$$

Note the effective track spacing is smaller than the requested track spacing.

7.2.2 Quadrature Weights

For the polar quadrature we can use Gauss-Legendre quadrature points and weights. For the azimuthal quadrature we showed how the angles are selected, and use a midpoint rule to choose the weights. Dropping the "eff" subscript,

we write the weights as

$$w_{n_a} = \frac{1}{2\pi} \begin{cases} \frac{1}{2}(\varphi_{n_a+1} - \varphi_{n_a}) + \varphi_{n_a} & n_a = 1 \\ \frac{1}{2}(\varphi_{n_a+1} - \varphi_{n_a-1}) & 1 < n_a < N_a \\ 2\pi - \varphi_{n_a} + \frac{1}{2}(\varphi_{n_a} - \varphi_{n_a-1}) & n_a = N_a \end{cases} . \tag{7.21}$$

For the track weights, we use the effective track widths:

$$w_k = t_{s,\text{eff}}. \tag{7.22}$$