# NUEN 647
# Uncertainty Quantification for Nuclear Engineering
# Assignment 1

Due on Tuesday, October 4, 2016

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Complete the exercises in the Chapter 2 notes. Be sure to include discussion of results where appropriate. You may use any tools that are approrpriate to solving the problem.

# Problem 1

Show that the transformation in equation 1 results in a standard normal random variable by computing the mean and variance of z.

$$z = \frac{x-\mu}{\sigma} \tag{1}$$

An important special case of the expectation value is the mean which is the expected value of $x$. It is often denoted as $\mu$,

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

where $x$ is a realization of a random sample and $f(x)$ is the probability density function (PDF) for the random variable. For a normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

For the sake of the transformation, the value of z substitutes for $x$, the realization of a random sample (not the PDF because we are transforming that distribution). Therefore, the mean for z is:

$$\mu_z = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

If $u = (x-\mu)^2$ and $\frac{du}{2} = (x-\mu)dx$ (note that the limits change from $(-\infty, \infty)$ to $(\infty, \infty)$ - but that seems fishy to me so I will change it back after integration).

$$\mu_z = \int_{\infty}^{\infty} \frac{1}{2\sigma^2\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} du = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} \right|_{\infty}^{\infty}$$

$$\mu_z = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \right|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}} (e^{-\infty} - e^{-\infty}) = \boxed{0}$$

The variance is defined as:

$$Var(X) = E[(X - \mu_X)^2]$$

Substituting Eq. 1 for $X$, (but not for the pdf)

$$Var(Z) = E[(\frac{x-\mu_X}{\sigma_X} - \mu_Z)^2] = E\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] = \frac{1}{\sigma_X^2}(E[x^2] - 2\mu_X E[x] + \mu^2 E[1])$$

Noting that above it was proven that $E[x] = \mu_X$ and given that the definition of $E[1] = 1$ and assuming that $E[x^2] = \sigma_X^2 + \mu_X^2$ (will solve on next page)

$$\frac{1}{\sigma_X^2}(\sigma_X^2 + \mu_X^2 - 2\mu_X^2 + \mu_X^2) = \boxed{1}$$

$$E[x^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

If $t = \frac{(x-\mu)}{\sqrt{2}\sigma}$ and $\sqrt{2}\sigma dt = dx$ and $x = t\sqrt{2}\sigma + \mu$ then (limits of integration don't change)

$$E[x^2] = \int_{-\infty}^{\infty} \frac{\left(t\sqrt{2}\sigma + \mu\right)^2}{\sqrt{\pi}} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(2\sigma^2 \left(t^2 e^{-t^2}\right) + 2\sqrt{2}\sigma\mu \left(te^{-t^2}\right) + \mu^2 \left(e^{-t^2}\right)\right)$$

According to wolfram alpha

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} te^{-t^2} = 0$$

$$\int_{-\infty}^{\infty} e^{-t^2} = \sqrt{\pi}$$

Which simplifies the above to $\sigma^2 + \mu^2$.

# Problem 2

Consider the random variables $X \sim U(-1, 1)$ and $Y \sim X^2$. Are these independent random variables? What is their covariance?

**Marginal and Joint PDFs**

The PDF for X is:

$$f_X(x) = \frac{1}{(1-(-1))} = 0.5 \quad x \in [-1, 1]$$

The PDF for Y is: $^{link}$

$$f_Y(y) = \frac{1}{2\sqrt{y}} \quad y \in [0, 1]$$

Without using the handy reference, this may be derived from the joint PDF, $f(x, y)$, defined as (between McClarrens Eq. 2.30 and 2.31):

$$f(x, y) = f(y|x) f_X(x)$$

From the definition of Y, $f(y|x)$ is 0 except when $y = x^2$. I think this would be.

$$f(y|x) = \delta(y - x^2)$$

Which means,

$$f(x, y) = 0.5\delta(y - x^2)$$

To calculate the PDF for Y (f(y)), we need to integrate over all other variables (in this case, X).

$$f(y) = \int_{-1}^{1} f(x, y)dx = \int_{-1}^{1} 0.5\delta(y - x^2)dx$$

Wolfram alpha tells me the answer is,

$$f_Y(y) = \frac{1}{2\sqrt{y}} \quad y \in [0, 1]$$

The same as above.

**Independance**

If two random variables, X and Y, are independent, they satisfy the following condition: $^{link}$

- $P(Y|X) = P(Y)$, for all values of $X$ and $Y$.

Because $P(Y|X) = \delta(y - xY2) \neq P(Y) = \frac{1}{2\sqrt{y}}$ (at least not for ALL values of x and y), these two variables are $\boxed{dependent}$.

Problem 2 continued on next page...

**Covariance**

The covariance for two random variables is defined as:

$$\sigma_{XY} = E[(x - \mu_X)(y - \mu_Y)]$$

This simplifies down to:

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \int_{-1}^{1} dx \int_{0}^{1} dy \quad xyf(x,y) - \mu_X\mu_Y$$

Because $\mu_X = 0$ this reduces to

$$\begin{aligned}
\sigma_{XY} = E(XY) &= \int_{-1}^{1} dx \int_{0}^{1} dy \quad xyf(x,y) \\
&= \int_{-1}^{1} dx \int_{0}^{1} dy \quad xy0.5\delta(y - x^2) \\
&= \int_{-1}^{1} dx \quad 0.5x^3 dx \\
&= \boxed{0}
\end{aligned}$$

Wolfram alpha gave the step between the second and third line. These variables are dependant, but have a zero covariance.

# Problem 3

Show that a general covariance matrix must be positive definite, i.e. $\vec{x}^T \Sigma \vec{x} > 0$ for any vector $\vec{x}$ that is not all zeros.

Given that $\vec{Y}$ is a vector of random variables and $\vec{\mu}_Y$ is a vector of the mean values for the random variables found in $\vec{Y}$.

$$\begin{aligned}\vec{x}^T \Sigma \vec{x} &= \vec{x}^T E[(\vec{Y} - \vec{\mu}_Y)(\vec{Y} - \vec{\mu}_Y)^T]\vec{x} \\ &= E[\vec{x}^T (\vec{Y} - \vec{\mu}_Y)(\vec{Y} - \vec{\mu}_Y)^T \vec{x}]\end{aligned}$$

The last step above puts a constant inside the expectation value integral. Notice

$$\vec{x}^T (\vec{Y} - \vec{\mu}_Y) = (\vec{Y} - \vec{\mu}_Y)^T \vec{x}$$

and that both are scaler functions of the random variables. Therefore,

$$\begin{aligned}\vec{x}^T \Sigma \vec{x} &= E[(\vec{x}^T (\vec{Y} - \vec{\mu}_Y))^2] \\ &= E[g(\vec{Y})^2] = \sigma_f^2\end{aligned}$$

The expectation value for a multivariate distribution is defined as

$$E[g(\vec{Y})] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \; ... \int_{-\infty}^{\infty} dy_p \; g(\vec{y}) f(\vec{y})$$

Where $f(\vec{y})$ is the multivariate PDF for the random variables of $\vec{Y}$. If a number of samples is given, rather than functions that can be integrated, the expectation value for a multivariate distribution is defined as:

$$E[g(\vec{Y})] = SC$$

To prove that the covariance matrix is positive definite the above integral must be proved to be positive with $g(x) = (\vec{x}^T (\vec{Y} - \vec{\mu}_Y))^2$. Explicitly,

$$\begin{aligned}E[g(Y)] &= \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \; ... \int_{-\infty}^{\infty} dy_p \; (\vec{x}^T (\vec{Y} - \vec{\mu}_Y))^2 f(y) \\ &= \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \; ... \int_{-\infty}^{\infty} dy_p \; ((y_1 - \mu_1)x_1 + (y_2 - \mu_2)x_2 + \; ... \; + (y_p - \mu_p)x_p)^2 f(y)\end{aligned}$$

# Problem 4

Use rejection sampling to sample from a Gamma random variable $X \sim \mathscr{G}(\alpha, \beta)$ where

$$f(x) = \frac{\theta^{\alpha-1}e^{-\theta\beta}}{\Gamma(\alpha)\beta^{-\alpha}} \quad \alpha, \beta > 0$$

Let $\alpha = 1$ and $\beta = 0.5$. From rejection sampling with a $N = 10^4$, compute a rejection rate for the sampling procedure. Now draw a triangle around the function and do rejection sampling. Compare the rejection rate from the triangle versus the rectangle. You may consider that the PDF is zero if $f(x) < 10^{-6}$.

Python script for rejection sampling.

Listing 1: Python Script for problem

```python
#!/usr/bin/env python3

#################################################################
#################### Import packages ########################
#################################################################

import time
start_time = time.time()
import scipy.special as sps
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
import random as rn
import Functions as fun
import copy

#################################################################
####################### Calculations ########################
#################################################################

#Values go to 10^-6 around 26.245
N=100;alpha=1;beta=0.5;a=0;b=26.245;h=0.5;Nsamples=10**4

theta=np.linspace(a,b,N)
f_x=fun.GammaPDF(alpha,beta,theta)

(fig,ax)=fun.Plot(theta,f_x)

OutsideSquare=0;OutsideTri=0
for i in range(0,Nsamples):
    X=rn.uniform(a,b);Xt=copy.copy(X)
    Y=rn.uniform(0,h);Yt=copy.copy(Y)

    if Y>(-h/b)*X+h: #Triangular
        Xt=b-Xt
        Yt=h-Yt

    H=fun.GammaPDF(alpha,beta,X) #Square
    Ht=fun.GammaPDF(alpha,beta,Xt) #Triangle

```

```
        if(Y<H): #Square
            ax=fun.PlotaxIn(X,Y,i,ax,1)
        else:
            OutsideSquare=OutsideSquare+1
            ax=fun.PlotaxOut(X,Y,i,ax,1)

        if(Yt<Ht): #Triangular
            ax=fun.PlotaxIn(Xt,Yt,i,ax,2)
        else:
            OutsideTri=OutsideTri+1
            ax=fun.PlotaxOut(Xt,Yt,i,ax,2)

ax=fun.Plotlegend(ax,theta,f_x)

RejectionSq=OutsideSquare/Nsamples;RejectionTri=OutsideTri/Nsamples

print("Square = "+str(RejectionSq)+" Triangle = "+str(RejectionTri))
plt.savefig('P4F1.pdf')

##################### Time To execute ################
print("--- %s seconds ---" % (time.time() - start_time))
```

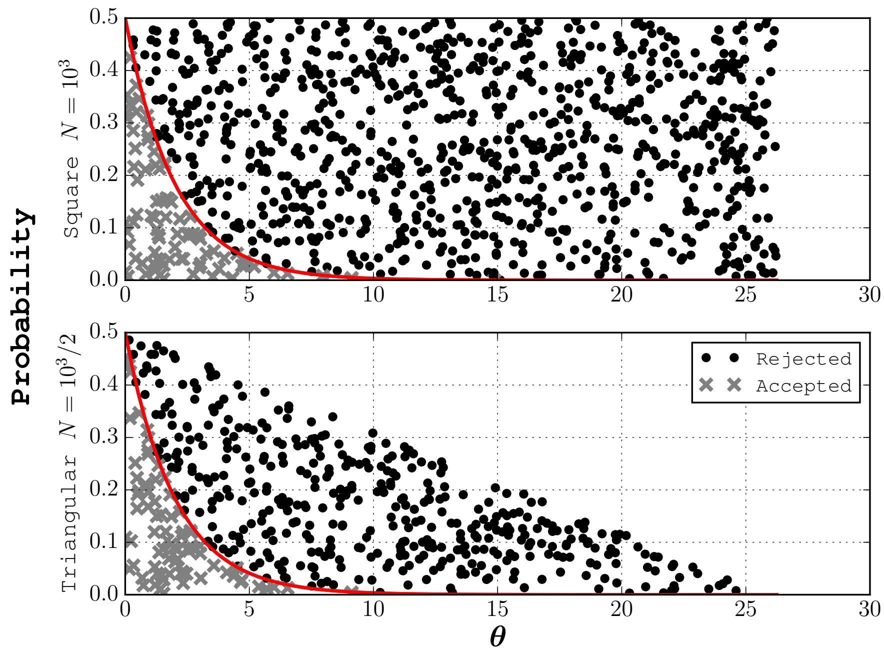


Figure 1: Square (top) and triangular (bottom) rejection sampling for the Gamma random variable.

The rejection rate for the square is $\boxed{92.65\%}$.
The rejection rate for the triangle is $\boxed{85.04\%}$.

The acceptance rate about doubled from the square to the triangle ($(1 - 0.9265) * 2 = 0.147 \approx 0.1496 = (1 - 0.8504)$). This is what is expected because we cut the sampling area in half. This could also be used to verify the PDF is properly normalized. $0.1496 * 0.5 * 26.245 * 1/2 = 0.98$.

# Problem 5

Consider a random variable, $X > 0$, that has it's logarithm distributed by a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. Such a distribution is called a log-normal distribution. Compute this distribution's a) mean, b) variance, c) median, d) mode, e) skew, and d) kurtosis.

The PDF for the log-normal distribution, found on wikipedia, is:

$$f(X) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}} \quad x \in [0, \infty)$$

For a standard log-normal distribution this is

$$f(X) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{ln(x)^2}{2}} \quad x \in [0, \infty)$$

Which simplifies to:

$$\begin{aligned} f(X) &= \frac{1}{x\sqrt{2\pi}} \left(e^{-ln(x)}\right)^{\frac{ln(x)}{2}} \\ &= \frac{1}{x\sqrt{2\pi}} \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} \quad x \in [0, \infty) \end{aligned}$$

**a) mean**

The mean, $\mu$, is defined as:

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

Using the log-normal distribution the mean is:

$$\begin{aligned} \mu = E[x] &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx \end{aligned}$$

Wolfram alpha says

$$\int_0^{\infty} \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx = \sqrt{2e\pi}$$

Therefore the answer is $\boxed{\mu = \sqrt{e}}$

**b) Variance**

Variance, $\sigma^2$, is defined as:

$$\begin{aligned}\sigma^2 &= E[(X-\mu)^2]\\ &= E[X^2] - 2E[\mu X] + E[\mu^2]\\ &= E[X^2] - \mu^2\\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2\end{aligned}$$

Using the log-normal distribution the variance is:

$$\begin{aligned}\sigma^2 = E[x^2] - \mu^2 &= \int_0^{\infty} \frac{x}{\sqrt{2\pi}} \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx - \mu^2\\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx - \mu^2\end{aligned}$$

Wolfram alpha says

$$\int_0^{\infty} x \left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx = e^2\sqrt{2\pi}$$

Therefore the answer is $\boxed{\sigma^2 = e(e-1)}$

**c) Median**

The median is defined as the point where the CDF is equal to one-half. The CDF, in terms of the PDF is:

$$F_X(x) = \int_{-\infty}^{x} f(x')dx'$$

For the log-normal distribution

$$\begin{aligned}F_X(x) &= \int_0^{x} \frac{1}{x'\sigma\sqrt{2\pi}} e^{-\frac{(ln(x')-\mu)^2}{2\sigma^2}} dx'\\ &= \frac{1}{2} + \frac{1}{2} erf\left[\frac{ln(x)-\mu}{\sqrt{2}\sigma}\right]\end{aligned}$$

If $\sigma = 1$ and $\mu = 0$,

$$F_X(x) = \frac{1}{2} + \frac{1}{2} erf\left[\frac{ln(x)}{\sqrt{2}}\right]$$

If $F_X(x) = \frac{1}{2}$ then

$$\begin{aligned}erf^{-1}[0]\sqrt{2} &= ln(x)\\ e^{erf^{-1}[0]\sqrt{2}} &= x\end{aligned}$$

$erf^{-1}[0] = 0$, because wolfram says so...

Therefore the answer is $\boxed{Median = 1}$

**d) Mode**

The mode is the point where the PDF takes its maximum value, the most likely value of the distribution. This could also be the point where its derivative goes to 0.

$$f(X) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{ln(x)^2}{2}}$$

Derivatives:

$$e^{-\frac{ln(x)^2}{2}} \frac{d}{dx} = \frac{-ln(x)}{x} e^{-\frac{ln(x)^2}{2}}$$

$$\frac{1}{x\sqrt{2\pi}} \frac{d}{dx} = -\frac{1}{x^2\sqrt{2\pi}}$$

Combining:

$$f'(X) = \left(\frac{-ln(x) - 1}{x^2\sqrt{2\pi}}\right) e^{-\frac{ln(x)^2}{2}}$$

Setting Equal to Zero

$$0 = \left(\frac{-ln(x) - 1}{x^2\sqrt{2\pi}}\right) e^{-\frac{ln(x)^2}{2}}$$

$$0 = (-ln(x) - 1)e^{-\frac{ln(x)^2}{2}}$$

$$0 = (-ln(x) - 1)$$

$$-1 = ln(x)$$

$$x \boxed{= e^{-1}}$$

Problem 5 continued on next page. . . 

**e) Skew**

The skewness, $\gamma_1$, is related to the third moment of $f(x)$, that is the expected value of $X^3$.

$$\gamma_1 = \frac{E[(X-\mu)^3]}{Var(X)^{3/2}}$$

Earlier it was proven that $Var(X) = e^2 - e$. Expanding the numerator...

$$\begin{aligned}(X-\mu)^3 &= (X-\mu)(X-\mu)(X-\mu)\\ &= X^3 - 3X^2\mu + 3X\mu^2 - \mu^3\\ E[(X-\mu)^3] &= E[X^3] - 3E[X^2]\mu + 2\mu^3\end{aligned}$$

The last two terms have $\mu^3$ in them because $E[X] = \mu$. Recall, that earlier, we discovered $E[X^2] = e^2$ and that $\mu = \sqrt{e}$. Plugging in what we know,

$$\gamma_1 = \frac{E[X^3] - 3e^2\sqrt{e} + 2e^{3/2}}{(e^2 - e)^{3/2}}$$

To determine $E[X^3]$,

$$\begin{aligned}E[X^3] &= \int_0^\infty \frac{x^2}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty x^2\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx\end{aligned}$$

Wolfram alpha says

$$\int_0^\infty x^2\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx = e^{9/2}\sqrt{2\pi}$$

Which Means:

$$E[X^3] = e^{9/2}$$

Plugging this in,

$$\gamma_1 = \boxed{\frac{e^{9/2} - 3e^{5/2} + 2e^{3/2}}{(e^2 - e)^{3/2}} = (e+2)\sqrt{e-1} = 6.1849}$$

The standard log-normal distribution has a positive skew, which means that the distribution goes to zero more slowly to the right of the mean (look up log-normal distribution on wikipedia, and you'll be able to see that is the case).

**f) Kurtosis**

The excess kurtosis is a measure of "tail fatness" for a distribution, related to the fourth moment of a random variable's PDF:

$$Kurt(x) = \frac{E[(X-\mu)^4]}{Var(X)^2} - 3 = \frac{E[(X-\mu)^4]}{\sigma^4} - 3$$

It should be noted that these $\mu$'s and $\sigma$'s are the $\mu$'s and $\sigma$'s from the log-normal distribution, not the $\mu$'s and $\sigma$'s from the normal distribution with $\mu = 0$ and $\sigma = 1$ by which the logarithm was distributed. Earlier it was proven that $\sigma^2 = e^2 - e$, $E[X] = \mu = \sqrt{e}$, $E[X^2] = e^2$, and that $E[X^3] = e^{9/2}$. Expanding the numerator...

$$\begin{aligned}(X-\mu)^4 &= (X-\mu)(X-\mu)(X-\mu)(X-\mu)\\ &= X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4\end{aligned}$$

To determine $E[X^4]$,

$$\begin{aligned}E[X^4] &= \int_0^\infty \frac{x^3}{\sqrt{2\pi}}\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty x^3\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx\end{aligned}$$

Wolfram alpha says

$$\int_0^\infty x^3\left(\frac{1}{x}\right)^{\frac{ln(x)}{2}} dx = e^8\sqrt{2\pi}$$

Which Means:

$$E[X^4] = e^8$$

Plugging everything in:

$$\begin{aligned}Kurt(x) &= \frac{E[(X-\mu)^4]}{\sigma^4} - 3\\ &= \frac{E[X^4] - 4E[X^3]\mu + 6E[X^2]\mu^2 - 4E[X]\mu^3 + \mu^4]}{\sigma^4} - 3\\ &= \boxed{\frac{e^8 - 4e^{9/2}\sqrt{e} + 6e^2e - 4\sqrt{e}e^{3/2} + e^2]}{(e^2-e)^2} - 3}\\ &= \boxed{e^4 + 2e^3 + 3e^2 - 6 = 110.94}\end{aligned}$$

Because the kurtosis is positive, we can deduce that the standard log-normal distribution has heavier tails than a normal distribution. Meaning that the PDF doesn't approach zero as quickly as for a normal distribution.

# Problem 6

(Monte Hall Problem) You are on a game show and are presented with three doors from which to choose. One of the doors contains a prize and the other two have nothing. You pick a door (say door 1), and then the host opens another door (say door 3), and asks you if you want to switch to door number 2. What should you do?

a Using Bayes' theorem give the probability of winning if you switch.

b Write a simulation code to show that by randomly assigning a prize to a door, then opening either door 2 or 3 depending on which has the prize, and then either switching or not. Compute the likelihood of winning if you stick, versus the likelihood of winning if you switch.

**Bayes' theorem**

$$f(x|y) = \frac{f(y|x)f_X(x)}{f_Y(y)}$$

- Define $f(x|y)$ as $f(P2|D3)$, the probability of the prize being behind door 2, given that door 3 was opened to show nothing.
- Define $f(y|x)$ as $f(D3|P2)$, the probability of opening door 3 to reveal nothing, given that the prize is behind door 2. This probability is 1, because the host will either open door 2 or door 3 (we picked door 1). Since good ol host will not open the door with the prize, he will open door 3.
- Define $f_X(x)$ as $f(P2)$, the probability of finding the prize behind door 2. This probability is 1/3 because, all things being equal, the prize is equally likely to be found behind any of the three doors
- Define $f_Y(y)$ as $f(D3)$, the probabilty the host openings door 3 to show nothing. There are three options for this. First, if the prize is behind door 1, the host will open door 3 half the time. Second, if the prize is behind door 2, the host will open door 3 every time. Third, if the prize is behind door 3, the host will never open door 3. This equates to:

$$f(D3) = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{1}{2}$$

$$f(P2|D3) = \frac{f(D3|P2)f(P2)}{f(D3)} = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$$

Python script computing the likelihood of winning.

Listing 2: Python Script for problem

```python
#!/usr/bin/env python3

#################################################################
#################### Import packages #########################
#################################################################

import time
start_time = time.time()
import random as rn
import copy

#################################################################
```

```
######################## Functions ###########################
##############################################################

def Remove(Not,List):
    List2=copy.copy(List)
    List2.remove(Not)
    return(List2)

def SelectOneRandomly(list):
    sum=len(list)
    rand=rn.uniform(0,1)
    for i in range(0,len(list)):
        if(rand> i/sum and rand <= (i+1)/sum):
            Selection=list[i]
    return(Selection)

##############################################################
####################### Calculations #########################
##############################################################

N=100000;Stay=0;Switch=0
for i in range(0,N):
    MyPick=SelectOneRandomly([1,2,3])
    DoorPrize=SelectOneRandomly([1,2,3])
    AvaDoors=[1,2,3]

    # Don't need this, but to remove one door randomly
    if(DoorPrize==MyPick):
        CanRemove=Remove(MyPick,AvaDoors)
        ToRemove=SelectOneRandomly(CanRemove)
        AvaDoors.remove(ToRemove)
    else:
        CanRemove=Remove(DoorPrize,AvaDoors)
        CanRemove=Remove(MyPick,CanRemove)
        AvaDoors.remove(CanRemove[0])

    # All we really need, I think
    if(DoorPrize==MyPick):
        Stay=Stay+1
    else:
        Switch=Switch+1

print("You win "+str(round(100*(Stay/N)))+"% of the time if you stay")
print("You win "+str(round(100*(Switch/N)))+"% of the time if you switch")

##################### Time To execute ################
print("--- %s seconds ---" % (time.time() - start_time))
```

Output of code:

You win 33% of the time if you stay.
You win 67% of the time if you switch.

# Problem 7

Consider a variable $Y$ distributed by a normal distribution with mean given by $\theta$:

$$f(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)}$$

Now consider $\theta$ to be a random variable as well, and $\sigma$ to be a known constant. Then say $\theta$ is normally distributed, with mean $\mu$ and variance $\tau^2$ to give

$$\pi(\theta) = \frac{1}{\tau\sqrt{2\pi}} e^{\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right)}$$

The parameters $\mu$ and $\tau$ are called hyperparameters. Using Bayes' theorem find $p(\theta|y)$, and show that it is a normal distribution.

$$p(\theta|y) = \frac{f(y|\theta)\cdot\pi(\theta)}{f(y)}$$

$f(y)$ will be a function of y, but is a constant for a given y, which is the case for $p(\theta|y)$.

$$\begin{aligned}
p(\theta|y) &= cf(y|\theta)\cdot\pi(\theta) \\
&= ce^{\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)} e^{\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right)} \\
&= ce^{\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)+\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right)} \\
&= ce^{c}e^{\frac{(\theta'-\mu')^2}{2\sigma'^2}} \\
&= ce^{\frac{(\theta'-\mu')^2}{2\sigma'^2}}
\end{aligned}$$

This looks like a normal. In order to combine the two fractions in the exponent completion of squares needs to be done.

# Problem 8

Suppose that X is the number of people arriving at a particular tavern during a given hour. This type of arrival process is naturally described by a Poisson process:

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x \in \{0, 1, 2, ...\}, \quad \theta > 0.$$

We then say that our prior distribution of $\theta$ is a Gamma distribution.

$$\pi(\theta) = \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}, \qquad \theta, \alpha, \beta > 0.$$

Therefore, we say that $\theta \sim G(\alpha, \beta)$.

- Show using Bayes' theorem that the posterior distribution for $\theta$ given $x$ is proportional to a Gamma distribution.
- Suppose you observe 42 people arriving in one hour, and the prior distribution has $\alpha = 5$ and $\beta = 6$. Generate samples from the posterior distribution and show graphically how the prior has changed given the observation.

**Bayes' Theorem for posterior**

$$f(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{f(x)}$$

Where $f(x)$:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} d\theta f(x,\theta) \\ &= \int_{-\infty}^{\infty} d\theta f(x|\theta)\pi(\theta) \\ &= \int_0^{\infty} d\theta \frac{e^{-\theta}\theta^x}{x!} \cdot \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} \\ &= \frac{\int_0^\infty d\theta e^{\theta(-1-1/\beta)}\theta^{\alpha+x-1}}{x!\Gamma(\alpha)\beta^\alpha} \end{aligned}$$

The deonminator will cancel when the Posterior is solved for. And the numerator, without the integral, will be the numerator for the posterior derivation.

Problem 8 continued on next page... 

If:

$$y =\theta(1+1/\beta) \rightarrow \theta = \frac{y}{1+1/\beta}$$
$$dy =(1+1/\beta)d\theta$$
$$\frac{dy}{1+1/\beta} =d\theta$$

then:

$$\int_0^\infty d\theta e^{\theta(-1-1/\beta)}\theta^{\alpha+x-1}$$
$$=\int_0^\infty \frac{dy}{1+1/\beta}e^{-y}\frac{y^{\alpha+x-1}(1+1/\beta)}{(1+1/\beta)^{\alpha+x}}$$
$$=\frac{1}{(1+1/\beta)^{\alpha+x}}\int_0^\infty e^{-y}y^{\alpha+x-1}dy$$
$$=\frac{1}{(1+1/\beta)^{\alpha+x}}\Gamma(\alpha+x)$$

Therefore the posterior is:

$$f(\theta|x) =\frac{f(x|\theta)\pi(\theta)}{f(x)}$$
$$=\frac{\frac{e^{-\theta}\theta^x}{x!}\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}}{\frac{\frac{1}{(1+1/\beta)^{\alpha+x}}\Gamma(\alpha+x)}{x!\Gamma(\alpha)\beta^\alpha}}$$
$$=\frac{e^{-\theta(1+1/\beta)}\theta^{\alpha+x-1}}{\frac{1}{(1+1/\beta)^{\alpha+x}}\Gamma(\alpha+x)}$$
$$=(1+1/\beta)^{\alpha+x}\frac{e^{-\theta(1+1/\beta)}\theta^{\alpha+x-1}}{\Gamma(\alpha+x)}$$

If we write a gamma distribution as:

$$GammaDist(z) = \frac{1}{\Gamma(a)b^a}z^{a-1}e^{-z/b}$$

Then if: $a=\alpha+x$ and $b=(1+1/\beta)^{-1}$:

$$GammaDist(z) = (1+1/\beta)^{\alpha+x}\frac{e^{-z(1+1/\beta)}z^{\alpha+x-1}}{\Gamma(\alpha+x)}$$

The same as the posterior where $\theta = z$.

Python script for plotting.

Listing 3: Python Script for problem

```
#!/usr/bin/env python3

#################################################################
#################### Import packages ##########################
#################################################################

import time
start_time = time.time()
```

```python
import numpy as np
import Functions as fun

##################################################################
####################### Calculations #########################
##################################################################
N=100

#Prior
alpha=5
beta=6
theta=np.linspace(0,80,N)
pi=fun.GammaPDF(alpha,beta,theta)

#Posterior
a=alpha+42
b=(1+1/beta)**-1
f_theta=fun.GammaPDF(a,b,theta)

fun.Plot(theta,pi,f_theta)

###################### Time To execute #################
print("--- %s seconds ---" % (time.time() - start_time))
```

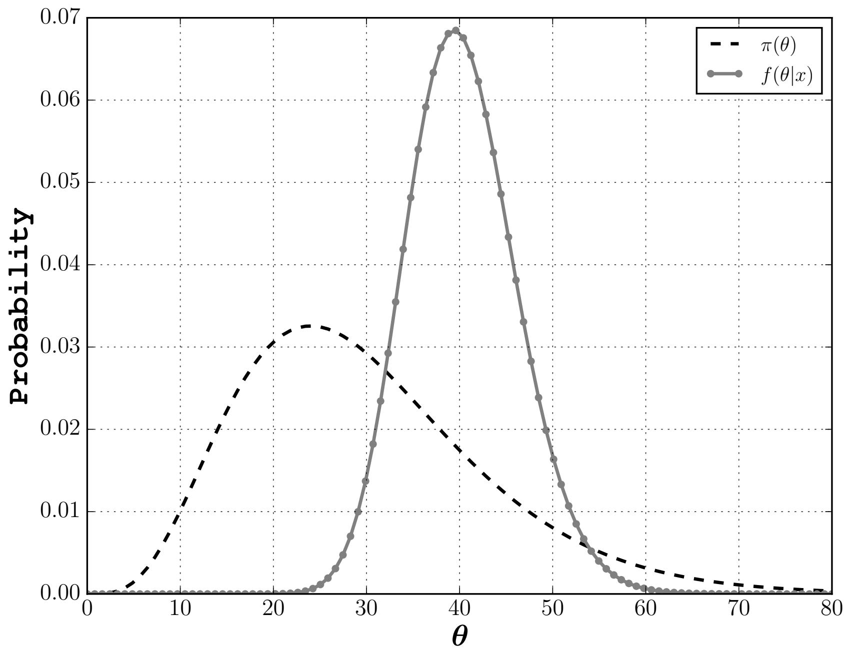


Figure 2: Plots of prior and posterior.

The plot shifts over closer to 42.

# Problem 9

Generate $N$ samples from a standard normal random variable and estimate the mean, variance, skewness, and kurtosis from the samples. Use $N = 10, 10^2, \cdots, 10^4$, and discuss how the errors in the approximations behave as a function of N.

Python script for sampling, calculaiton and plotting.

Listing 4: Python Script for problem

```python
#!/usr/bin/env python3

#################################################################
#################### Import packages ##########################
#################################################################

import time
start_time = time.time()
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
import Functions as fun

#################################################################
####################### Calculations ##########################
#################################################################

N=np.array([10,100,1000,10000]);N=np.logspace(1,4,100)
fig,ax=fun.PlotSetup()

for i in range(0,len(N)):
    s=np.random.normal(0,1,N[i])
    mean=sum(s)/N[i]
    variance=(1/N[i])*sum((s-mean)**2)
    skew=(1/N[i])*sum((s-mean)**3)/((variance)**(3/2))
    kurtosis=(1/N[i])*sum((s-mean)**4)/(variance**2)-3
    #Print if you want to
    if N[i]==10 or N[i]==100 or N[i]==1000 or N[i]==10000:
        print("N = "+str(N[i])+" mean = "+str(mean)+\
              " variance = "+str(variance)+\
              " skew = "+str(skew)+\
              " kurtosis = "+str(kurtosis))
    Emean=abs(mean-0)
    Evariance=abs(variance-1)
    Eskew=abs(skew-0)
    Ekurtosis=abs(kurtosis-0)
    ax=fun.Plot(N[i],Emean,Evariance,Eskew,Ekurtosis,ax)

plt.savefig('P9F1.pdf')

##################### Time To execute ################
print("--- %s seconds ---" % (time.time() - start_time))
```

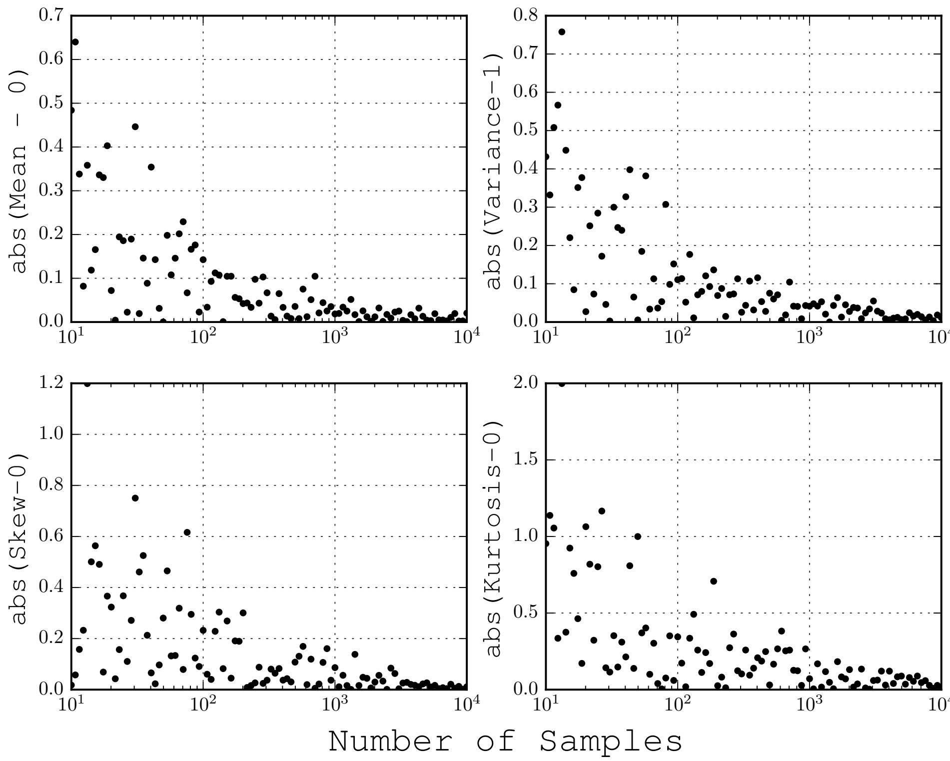


Figure 3: Sample parameter differences from expected values.

The plot shows that the error is decreasing as the sample size increases. The mean and variance decrease as a function of 1/N.

# Problem 10

Consider the joint PDF

$$f(x,y) = e^{-x/y}, \quad x \in [0,\infty) \;\; y \in [0,\sqrt{2}].$$

Compute and plot the marginal PDFs for X and Y. Additionally, compute the conditional probability distributions, and make plots for $f(y|X = \mu_x)$ and $f(x|y = \mu_y)$.

**Marginal PDFs**

For y:

$$\begin{aligned} f(y) &= \int_0^\infty e^{-x/y} dx \\ &= -y \left| e^{-x/y} \right|_0^\infty \\ &= -y(e^{-\infty} - e^{-0}) \\ &= y, \quad y \in [0,\sqrt{2}] \end{aligned}$$

For x (use transform $1/y = t$ and $\frac{-dt}{t^2} = dy$):

$$\begin{aligned} f(x) &= \int_0^{\sqrt{2}} e^{-x/y} dy \\ &= \int_\infty^{1/\sqrt{2}} \frac{-e^{-xt}}{t^2} dt \\ &= \int_{1/\sqrt{2}}^\infty \frac{e^{-xt}}{t^2} dt \end{aligned}$$

Use $u = e^{-xt}$, $du = -xe^{-xt}$, $v = -1/t$, $dv = \frac{1}{t^2}dt$,

$$\begin{aligned} f(x) &= \left| \frac{-e^{-xt}}{t} \right|_{\frac{1}{\sqrt{2}}}^\infty - x \int_{\frac{1}{\sqrt{2}}}^\infty \frac{e^{-xt}}{t} dt \\ &= \left( \frac{-1}{e^\infty \infty} + \sqrt{2} e^{\frac{-x}{\sqrt{2}}} \right) - x \int_{\frac{1}{\sqrt{2}}}^\infty \frac{e^{-xt}}{t} dt \\ &= \sqrt{2} e^{\frac{-x}{\sqrt{2}}} - x \int_{\frac{1}{\sqrt{2}}}^\infty \frac{e^{-xt}}{t} dt \end{aligned}$$

Wolfram alpha says that:

$$\int_{\frac{1}{\sqrt{2}}}^\infty \frac{e^{-xt}}{t} dt = \Gamma\left(0, \frac{x}{\sqrt{2}}\right)$$

Where $\Gamma(a,x)$ is the incomplete gamma function...Python has a function for that, lets hope it works and we will leave it at that.

Therefore $f(x)$ is:

$$f(x) = \sqrt{2} e^{\frac{-x}{\sqrt{2}}} - x\Gamma\left(0, \frac{x}{\sqrt{2}}\right), \quad x \in [0,\infty)$$

Problem 10 continued on next page. . . 

**Conditional Probability Distributions**

The conditional probability distribution is defined as:

$$f(y|X=x) = \frac{f(x,y)}{f_X(x)}$$

Plugging in the above for f(y—X=x):

$$f(y|X=x) = \frac{e^{-x/y}}{\sqrt{2}e^{\frac{-x}{\sqrt{2}}} - x\Gamma\left(0, \frac{x}{\sqrt{2}}\right)}, \quad x \in [0,\infty) \;\; y \in [0,\sqrt{2}]$$

Plugging in for f(x—Y=y):

$$f(x|Y=y) = \frac{e^{-x/y}}{y}, \quad x \in [0,\infty) \;\; y \in [0,\sqrt{2}]$$

**Mean Values**

The mean, $\mu$, is defined as:

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

For $f(y)$

$$\begin{aligned}
\mu_Y = E[y] &= \int_0^{\sqrt{2}} y f(y) dy \\
&= \int_0^{\sqrt{2}} yy dy \\
&= \left|\frac{y^3}{3}\right|_0^{\sqrt{2}} \\
&= \frac{2^{3/2}}{3} \\
&= \boxed{0.9428}
\end{aligned}$$

For $f(x)$

$$\begin{aligned}
\mu_X = E[x] &= \int_0^{\infty} x f(x) dx \\
&= \int_0^{\infty} x \left(\sqrt{2}e^{\frac{-x}{\sqrt{2}}} - x\Gamma\left(0, \frac{x}{\sqrt{2}}\right)\right) dx \\
&= \int_0^{\infty} x\sqrt{2}e^{\frac{-x}{\sqrt{2}}} - x^2\Gamma\left(0, \frac{x}{\sqrt{2}}\right) dx
\end{aligned}$$

Wolfram alpha says:

$$\int_0^{\infty} x e^{-\frac{x}{\sqrt{2}}} dx = 2$$

This means:

$$\mu_X = 2\sqrt{2} - \int_0^{\infty} x^2\Gamma\left(0, \frac{x}{\sqrt{2}}\right) dx$$

Problem 10 continued on next page. . . 

Let us work on that incomplete Gamma:

$$-\int_0^\infty x^2\Gamma\left(0,\frac{x}{\sqrt{2}}\right)dx$$

Before it was an incomplete Gamma, it was this:

$$-\int_0^\infty x^2 \quad \int_{\frac{1}{\sqrt{2}}}^\infty \frac{e^{-xt}}{t}dtdx$$

This looks more doable (McClarren is a mean old man).

$$\begin{aligned}
&-\int_0^\infty dx\int_{\frac{1}{\sqrt{2}}}^\infty dt \quad \frac{x^2e^{-xt}}{t}\\
=&-\int_{\frac{1}{\sqrt{2}}}^\infty dt\int_0^\infty dx \quad \frac{x^2e^{-xt}}{t}\\
=&-\int_{\frac{1}{\sqrt{2}}}^\infty dt\int_0^\infty \frac{x^2e^{-xt}}{t}dx
\end{aligned}$$

Wolfram Alpha says:

$$\int_0^\infty \frac{x^2e^{-xt}}{t}dx=\frac{2}{t^4}$$

Plugging this in:

$$\begin{aligned}
&-\int_{\frac{1}{\sqrt{2}}}^\infty \frac{2}{t^4}dt\\
=&\frac{2}{3}\left|\frac{1}{t^3}\right|_{\frac{1}{\sqrt{2}}}^\infty\\
=&\frac{-4\sqrt{2}}{3}=-1.8856
\end{aligned}$$

Therefore:

$$\boxed{\mu_X=0.942809}$$

**To Plot**

$$f(y|X=\mu_X)=\frac{e^{-\mu_X/y}}{\sqrt{2}e^{\frac{-\mu_X}{\sqrt{2}}}-\mu_X\Gamma\left(0,\frac{\mu_X}{\sqrt{2}}\right)},\qquad y\in[0,\sqrt{2}]$$

$$f(x|Y=\mu_Y)=\frac{e^{-x/\mu_Y}}{\mu_Y},\qquad x\in[0,\infty)$$

Where $\mu_X=0.942809$ and $\mu_Y=0.9428$

Python script for plotting

Listing 5: Python Script for problem

```
#!/usr/bin/env python3

#################################################################
#################### Import packages ##########################
#################################################################
```

```
import time
start_time = time.time()
import scipy.special as sps
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
import Functions as fun

################################################################
######################## Calculations #########################
################################################################

N=100
#Can't use 0 in some calculations below, because division by zero
notzero=0.000001

x=np.linspace(0,8,N)
y=np.linspace(notzero,2**0.5,N)
ux=0.942809
uy=0.942809

f_X=np.exp(-x/uy)/uy

C=(2**0.5)*np.exp(-ux/(2**0.5))
#Note gammaincc has two c values
C2=ux*sps.gammaincc(notzero,ux/(2**0.5))*sps.gamma(notzero)
f_Y=np.exp(-ux/y)/(C-C2)

fig,ax=fun.Plot()

ax=fun.Plotax(x,f_X,ax,2)

ax=fun.Plotax(y,f_Y,ax,1)

plt.savefig('P10F1.pdf')

###################### Time To execute #################
print("--- %s seconds ---" % (time.time() - start_time))
```

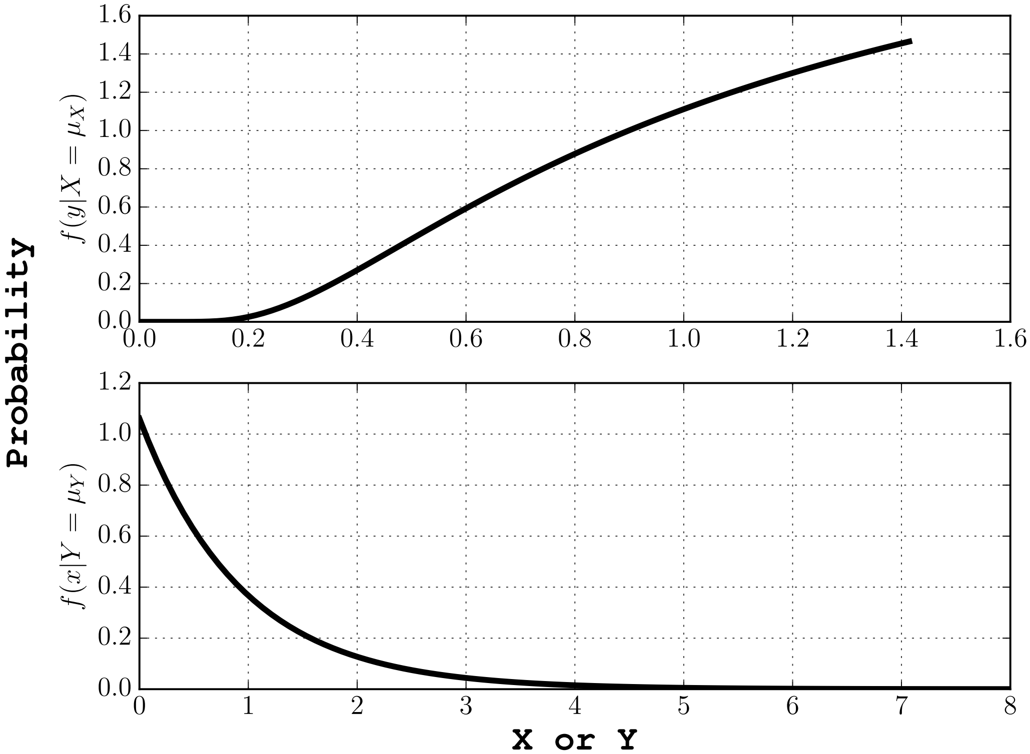


Figure 4: Conditional probability plots.

This is an example citation [1].

## References

[1] E. T. Tatro, S. Hefler, S. Shumaker-Armstrong, B. Soontornniyomkij, M. Yang, A. Yermanos, N. Wren, D. J. Moore, and C. L. Achim. Modulation of bk channel by microrna-9 in neurons after exposure to hiv and methamphetamine. *J Neuroimmune Pharmacol*, 2013. Tatro, Erick T Hefler, Shannon Shumaker-Armstrong, Stephanie Soontornniyomkij, Benchawanna Yang, Michael Yermanos, Alex Wren, Nina Moore, David J Achim, Cristian L R03 DA031591/DA/NIDA NIH HHS/United States U19 AI096113/AI/NIAID NIH HHS/United States Journal article Journal of neuroimmune pharmacology : the official journal of the Society on NeuroImmune Pharmacology J Neuroimmune Pharmacol. 2013 Mar 19.