Chapter 7
Reliability Methods

It is often the case that a sensitivity/UQ analysis needs to determine how likely a quantity of interest is to cross some threshold. In these types of situations reliability methods, the subject of this chapter, can be a fast, though sometimes fragile, approach to answer such questions. These methods are popular in civil engineering where the reliability of a structure is of fundamental importance: this is where the method gets its name. Specifically, in structural analysis one wants to know when the stress on a structure exceeds the yield stress. Many of the topics in this section are presented in the book on reliability analysis by Haldar and Mahadevan [Haldar and Mahadevan(2000)].

7.1 A Simple Example: The infinite multiplication factor, k_∞

Consider that situation where an engineer is producing a solution of ^{235}U, a fissile material, in water. For reasons of safety, it must not be possible to make a critical mass with this solution regardless of the size of the container. In this situation we are interested in keeping the infinite multiplication factor, k_∞, less than one. A simple model of the infinite multiplication factor is

$$k_\infty = \frac{\nu\Sigma_\mathrm{f}}{\Sigma_\mathrm{a}}, \tag{7.1}$$

where $\nu\Sigma_\mathrm{f} f$ is the number of neutrons emitted for fission times the macroscopic cross-section for fission and Σ_a is the macroscopic absorption cross-section of the mixture.

For this problem our threshold is clearly $k_\infty = 1$. We now make the assumption that $\nu\Sigma_\mathrm{f}$ and Σ_a are lognormal random variables (i.e. their logarithms are normally distributed):

$$\nu\Sigma_\mathrm{f} \sim \mathrm{LN}(\lambda_\mathrm{f}, \zeta_\mathrm{f}), \qquad \Sigma_\mathrm{a} \sim \mathrm{LN}(\lambda_\mathrm{a}, \zeta_\mathrm{a}).$$

Recall that for a lognormal distribution for a random variable X

$$\lambda_X = \mathrm{E}(\log X) = \log \mu_X - \frac{1}{2}\zeta_X, \tag{7.2}$$

and

$$\zeta_X^2 = \mathrm{Var}(\log X) = \log\left(1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right), \tag{7.3}$$

where μ_X is the mean value of X and σ_X is the standard deviation of X.

We proceed by taking the logarithm of Eq. (7.1) to get

$$Z \equiv \log k_\infty = \log \nu\Sigma_\mathrm{f} - \log \Sigma_\mathrm{a}. \tag{7.4}$$

The random variable Z will be normally distributed because the difference of two normally distributed random variables is normally distributed with $Z \sim N(\lambda_\mathrm{f} -$

$\lambda_a, \sqrt{\zeta_f^2 + \zeta_a^2}$). Based on our safety threshold we want to know when $Z > 0$ because $\log(1) = 0$. Therefore, the probably of exceeding our threshold or probability of failure in the reliability method parlance can be expressed as in terms of the standard normal CDF, $\Phi(x)$, and the satandard normal PDF, $\phi(x)$, as

$$\begin{aligned} p_{\text{fail}} &= P(\text{failure}) = P(Z > 0) = P(\log \Sigma_a - \log \nu\Sigma_f < 0) \\ &= \int_{-\infty}^{0} \phi\left(\frac{s - (\lambda_a - \lambda_f)}{\sqrt{\zeta_a^2 + \zeta_f^2}}\right) ds \\ &= \Phi\left(\frac{0 - (\lambda_a - \lambda_f)}{\sqrt{\zeta_a^2 + \zeta_f^2}}\right) \\ &= 1 - \Phi\left(\frac{(\lambda_a - \lambda_f)}{\sqrt{\zeta_a^2 + \zeta_f^2}}\right). \end{aligned} \tag{7.5}$$

Let's put some numbers in here to get an idea of how this works. Let's have $\nu\Sigma_f$ be lognormally distributed with a mean of $\mu_X = 150\,\text{cm}^{-1}$ with a standard deviation of 10%, corresponding to $\sigma_X = 15\,\text{cm}^{-1}$. Also, for Σ_a we will use $\mu_X = 200\,\text{cm}^{-1}$ and $\sigma_X = 20\,\text{cm}^{-1}$. Using Eqs. (7.2) and (7.3), we get that $\nu\Sigma_f \sim \text{LN}(5.0057, 0.1)$ and $\Sigma_a \sim \text{LN}(5.2933, 0.1)$. A plot of these distributions is shown in Figure 7.1; notice that the overlap of the two distributions is where $\nu\Sigma_f$ is greater than Σ_a, i.e., the failure region. The area of this overlapping region is equal to the expression derived in Eq. (7.5). From that equation we get that the probability of failure is 0.0207 or about 2%. We can check this result against a Monte Carlo simulation where we draw a sample from the distributions of $\nu\Sigma_f$ and Σ_a and then evaluate Z.

Great, we have a number, but what does this result really tell us. There is a 2% chance of have a $k_\infty > 1$, but is there a way to get a measure of the reliability of the system without having to actually evaluate the standard normal CDF? There is, and to do this we rewrite Eq. (7.5) as

$$\lambda_a = \lambda_f + \Phi^{-1}(1 - p_{\text{fail}})\sqrt{\zeta_a^2 + \zeta_f^2}. \tag{7.6}$$

If we define,

$$\beta = \Phi^{-1}(1 - p_{\text{fail}}) = \frac{(\lambda_a - \lambda_f)}{\sqrt{\zeta_a^2 + \zeta_f^2}}, \tag{7.7}$$

we can rewrite Eq. (7.6) as

$$\begin{aligned} \lambda_a - \lambda_f &= 0 + \beta\sigma_Z \\ &= \mu_Z, \end{aligned} \tag{7.8}$$

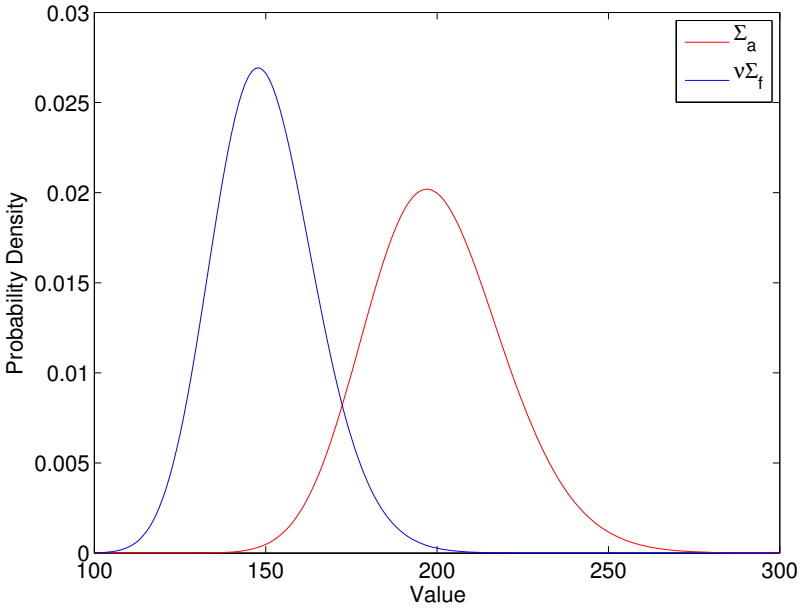

Fig. 7.1 Distributions of $\nu\Sigma_{\mathrm{f}}$ and Σ_{a} used in the example. The area of overlap of the two curves is the region of failure.

that is, β tells us how many standard deviations above zero is the mean of Z. Obviously a large value of β, corresponding to a small value of p_{fail}, means the probability of failure is small. In the example problem we did above, $\beta = 2.03958$. Therefore, the point of failure is two standard deviations above the mean of Z.

7.2 General Statement of Reliability Analysis

In the previous section we solved a simplified example problem to demonstrate the concepts of reliability analysis. We had an objective function and a value that when the function exceeded that threshold, failure was said to occur. We then were able to compute the probability that the threshold was exceeded given uncertainties to the inputs for the model. We then found a measure, β, of how close to failure the mean state of our system was.

All of these concepts can be generalized to deal with more complicated situations. Firstly, we define a performance function as

$$Z = g(X_1, X_2, \ldots, X_n),$$

where the X_i are random variables that are input to the model. We then define a failure surface as the locations where $Z = 0$ and say that $Z < 0$ represents failure and $Z > 0$ represents success. Then to compute the probability of failure (i.e., the probability that $Z < 0$), we need to compute the integral

$$p_{\text{fail}} = \int \dots \int f_X(x_1, x_2, \dots, x_n)\, dx_1 dx_2 \dots dx_n, \tag{7.9}$$

where the integration is over all regions where $Z < 0$ (i.e., the failure region) and $f_X(x1, x2, \dots, x_n)$ is the joint probability density function for the input random variables.

In general, Eq. (7.9) is difficult to compute because one does not necessarily know what the failure region is, and determining the failure region is likely to be more computationally expensive than the integral itself in the case where evaluating Z is expensive as in the case of each evaluation representing a large scale simulation. In the next section we will discuss on way to deal with such a situation.

7.3 First-Order Reliability Methods (FORM)

In this section we will discuss a particular kind of first-order reliability method called the first-order second-moment method (FOSM). In this method we approximate the performance function Z by a linear Taylor-series expansion (hence, first-order) and then use the mean and standard deviation of this approximate performance function to estimate the probability of failure.

Looking back in our example above we found that

$$\beta = \frac{\mu_Z}{\sigma_Z}, \tag{7.10}$$

where μ_Z was the mean and σ_Z was the standard deviation of the Z. We can also see from Eq. (7.7) that

$$p_{\text{fail}} = 1 - \Phi(\beta). \tag{7.11}$$

This last relation, that the probability to failure is related to the standard normal CDF evaluated at β is true only if Z is normally distributed, otherwise it is an approximation. Though it is an approximation it can be an efficient way to estimate the probability of failure.

To compute β we need μ_Z and σ_Z. We can compute these by first taking a linear Taylor series expansion of Z about the mean of the X's:

$$Z \approx g(\mu_{\mathbf{X}}) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\mu_{\mathbf{X}}} (X_i - \mu_{X_i}), \tag{7.12}$$

and then noting that the mean and variance of this expansion can be written as

$$\mu_Z \approx g(\mu_{\mathbf{X}}), \tag{7.13}$$

and

$$\sigma_Z^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_{\mu_{\mathbf{X}}} \left.\frac{\partial g}{\partial X_j}\right|_{\mu_{\mathbf{X}}} \mathrm{Cov}(X_i, X_j), \tag{7.14}$$

where $\mathrm{Cov}(\mathrm{X_i}, \mathrm{X_j})$ is the covariance of variables X_i and X_j. When the input variables are uncorrelated, then we get

$$\sigma_Z^2 = \sum_{i=1}^{n} \left(\left.\frac{\partial g}{\partial X_i}\right|_{\mu_{\mathbf{X}}} \right)^2 \sigma_{X_i}^2. \tag{7.15}$$

Therefore, to compute β and compute an estimate of the probability of failure we need to compute the mean of Z and its standard deviation. To estimate the standard deviation we also required knowledge of the derivatives of Z with respect to the inputs. Of course, as mentioned above the value of p_{fail} in Eq. (7.11) is exact only in the case where Z is normal. The only situations where this will be the case is 1) when Z is a linear function of the X_i's and the X_i's are independent normal variables or 2) when Z is a multiplicative function of the X_i's and the X_i's are independent lognormal variables. It is unlikely that for a given problem either of these conditions will be satisfied.

As an example we will now apply FOSM to a more complicated problem than the k_∞ problem we treated above. Here we will consider the moderator temperature coefficient for a thermal reactor [Lewis(2008)]. In a nuclear reactor negative feedback is an important mechanism for safety: if the temperature increases, one wants the reactivity to go down as a result (otherwise, an increase in reactivity that caused an increased temperature would cause a further increase in reactivity and the system would be unstable). One can assure such negative feedback by making the logarithmic derivative of the multiplication factor with respect to the moderator temperature negative. We call this logarithmic derivative the moderator temperature coefficient[1]. This coefficient, denoted by α_{m}, is given by

$$\alpha_{\mathrm{m}} = \frac{1}{k_\infty}\frac{\partial k_\infty}{\partial T_{\mathrm{m}}} = \beta_{\mathrm{m}}\left(\log p + (1-f)\right). \tag{7.16}$$

In this equation k_∞ is the infinite multiplication factor for the reactor, T_{m} is the moderator temperature, p is the resonance escape probability for a fission neutron, f is the thermal utilization (i.e., the likelihood that given a thermal neutron is absorbed, that it is absorbed in fuel), and β_{m} is the volumetric coefficient of thermal expansion at constant pressure:

$$\beta_{\mathrm{m}} = -\frac{1}{N_{\mathrm{m}}}\frac{\partial N_{\mathrm{m}}}{\partial T_{\mathrm{m}}}, \tag{7.17}$$

with N_{m} the number density of the moderator.

[1] There is an analagous coefficient for the fuel.

For safety purposes, as noted above, $\alpha_{\rm m}$ should be negative. In reality, the value of $\beta_{\rm m}$ is positive, and both p and f are between 0 and 1 (they are probabilities). For the purposes of using a reliability method we define Z as

$$Z = -\alpha_{\rm m} = -\beta_{\rm m}\left(\log p + (1-f)\right), \tag{7.18}$$

with failure denoted by $Z < 0$. In this example we will use values typical for a pressurized, light water reactor [Lewis(2008)]: $\beta_{\rm m} = 0.004\ {\rm K}^{-1}$, $p = 0.63$, and $f = 0.94$. For each of these parameters we assume independence, and a standard deviation of 5% for each: $\sigma_{\beta_{\rm m}} = 0.0004\ {\rm K}^{-1}$, $\sigma_p = 0.063$, and $\sigma_f = .094$. Notice that for this method we do not need to specify what kind of distribution each of these parameters takes—this can be both a strength of the method, one does not need to assume a distribution, but also is a weakness as no information about the distribution of the parameters can be used.

The mean value of Z is given by

$$\mu_Z = -(0.004)\left[\log 0.63 + (1-0.94)\right] = 0.0020881.$$

To compute the variance of Z we use Eq. (7.15) because the variables are independent:

$$\begin{aligned}\sigma_Z^2 &= \left((.94-1) - \log 0.63\right)^2 (.0004)^2 \\ &\quad + \left(-\frac{0.004}{0.94}\right)^2 (0.063)^2 + (0.004)^2 (0.094)^2 = 3.27237 \times 10^{-7}.\end{aligned} \tag{7.19}$$

Therefore, we can compute β from Eq. (7.10) as

$$\beta = \frac{0.0020881}{\sqrt{3.27237 \times 10^{-7}.}} = 3.65023.$$

This corresponds to a probability of failure of

$$p_{\rm fail} = 1 - \Phi\left(3.65023\right) = 0.000131003,$$

that is, given these parameters and their underlying standard deviations we get a probability of failure of about 1 in 10,000. Note that though this probability of failure might seem small, in terms of reactor safety a 1 in 10,000 chance of a severe accident is not small.

Given this probability of failure from the FOSM method, we can test the prediction of the probability of failure using Monte Carlo sampling. To do this we need to give a distribution to each parameter. We will do two cases: one where each distribution is uniform over some interval consistent with the mean and standard deviations quoted above, and one where the distribution for $\beta_{\rm m}$ is normal and the two probability parameters, f and p, are given by a beta distribution. The parameters for the uniform case are

$$\begin{aligned}\beta_{\mathrm{m}} &\sim \mathrm{U}(0.00330718, 0.00469282),\\ p &\sim \mathrm{U}(0.520881, 0.739119),\\ f &\sim \mathrm{U}(0.777187, 1),\end{aligned}$$

the distribution for f had to be truncated at 1 because it represents a probability. In this case the probability of failure is 0, i.e., there is no way that Z will be less than zero (this can be checked without Monte Carlo by evaluating Z at the extreme points of the parameters).

In the case of non-normal distributions we get a different result. For this case the distributions of the parameters are

$$\begin{aligned}\beta_{\mathrm{m}} &\sim \mathrm{N}(.004, .0004),\\ p &\sim \mathrm{B}(36.37, 21.3602),\\ f &\sim \mathrm{B}(5.06, .322979).\end{aligned}$$

A plot of the distributions for f and p is shown in Figure 7.2. Using these distri-

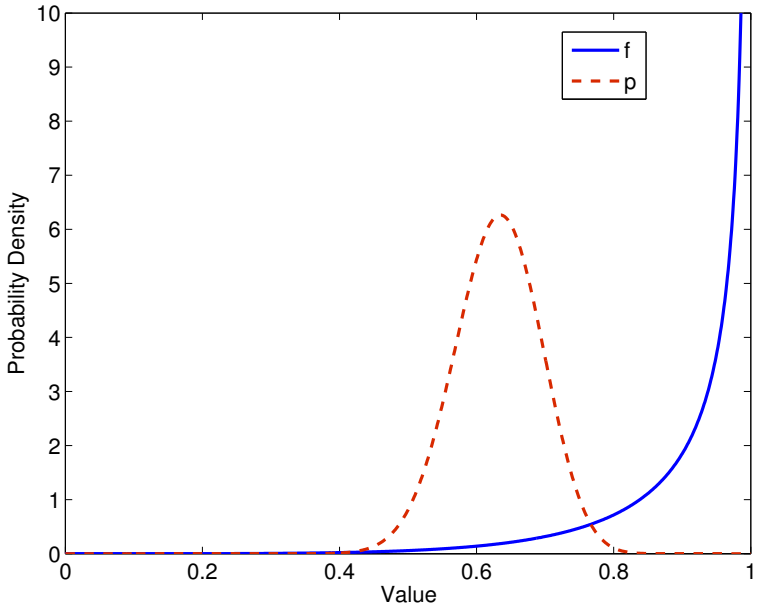

Fig. 7.2 Beta distributions of f and p used in the example.

butions distributions in a 10^7 point Monte Carlo simulation we get a probability of failure of 0.0111. The predicted probability of failure from FOSM does not change

when the underlying distributions for the inputs change: it remains 0.000131003, some two orders of magnitude too small.

In contrasting these two examples we see the fragility of the FOSM method. Depending on the distributions of the underlying parameters the results for probability of failure can be quite different in reality, but the FOSM result cannot deal with this. In one instance, the uniform distributions, the probability of failure was zero and the FOSM calculated probability was an over estimate. In the other case of beta distributions for the probability parameters, the FOSM probability significantly underestimates the probability of failure. It is also worth noting that in this example the probability of failure is independent of β_{m}, assuming that β_m is positive. Nevertheless, the value of this parameter does affect the result from FOSM as evidenced in Eq. (7.19). It is these drawbacks that led to the development of *advanced* FOSM (AFOSM) methods. In the next section we will talk about one of these.

7.4 Advanced First-Order, Second-Moment Methods

The Hasofer-Lind (H-L) method attempts to correct deficiencies in the basic FOSM method by correcting for the fact that the performance function (Z) might not be linear. What it does is find the most-probable failure point (i.e., what values of the X_i's that causes $Z < 0$ is the most probable) and then evaluate the failure probability there.

7.4.1 Hasofer-Lind for Normal Variables

In its original formulation the H-L method is only applicable to normal random variables and we will discuss that first. Consider a performance function $Z = g(\mathbf{X})$, we first normalize the random varibles as

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad (i = 1, 2, \ldots, n). \tag{7.20}$$

We then define β, the reliability index, for the H-L methods as

$$\beta_{\mathrm{HL}} = \mathrm{Min}(|\mathbf{X}'|) \text{ for } \mathbf{X}' \text{ such that } g(\mathbf{X}') = 0, \tag{7.21}$$

that is, β is the minimum distance from the origin to a point of failure. In the basic FOSM method this point was assumed to be the mean value of the random variables, in the H-L method we need to find that point using some type of optimization procedure.

The optimization procedure starts at the mean of inputs and moves toward the failure surface to find the most-probable point of failure. The algorithm is

1. Set $x_i = \mu_{X_i}$ for $i = 1, 2, \ldots, n$.

2. Normalize the x_i as
$$x_i' = \frac{x_i - \mu_{X_i}}{\sigma_{X_i}}.$$
3. Compute the n partial derivatives $\frac{\partial g}{\partial X_i}$ evaluated at $\mathbf{x}_i$.
4. Compute the partial derivatives with respect to the normalized variables:
$$\frac{\partial g}{\partial X_i'} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial X_i'} = \frac{\partial g}{\partial X_i}\sigma_{X_i}.$$
5. Update $\mathbf{x}'$ using the formula
$$\mathbf{x}'^{*} = \frac{1}{|\nabla g(\mathbf{x}')|^2}\left[\nabla g(\mathbf{x}')\cdot\mathbf{x}' - g(\mathbf{x}')\right]\nabla g(\mathbf{x}').$$
6. Compute $\beta_{\mathrm{HL}} = |\mathbf{x}'^{*}|$.
7. Update x_i as
$$x_i = \mu_{X_i} + \sigma_{X_i} x_i'^{*}.$$
8. For given tolerances δ and ε check whether the change in β_{HL} is less than δ and that $|g(\mathbf{x})| < \varepsilon$.
9. If the convergence criteria are met, then stop. Otherwise, go back to step 2.

7.4.2 Hasofer-Lind for Non-Normal Variables

To use the H-L method with non-normal random variables we need to come up with equivalent normal variables. There are several ways of doing this and we will only mention one here. If each random variable is independent, to define an equivalent normal distribution at a given point x_i, we can equate the random variable's CDF with the standard normal CDF as

$$\Phi\left(\frac{x_i - \mu_{X_i}'}{\sigma_{X_i}'}\right) = F_{X_i}(x_i), \tag{7.22}$$

where F_{X_i} is the CDF of X_i. This equation has two unknowns to solve for: μ_{X_i}' and σ_{X_i}'. Therefore, we need an additional equation. The additional equation that we will use sets μ_{X_i}' equal to the median of X_i:

$$\mu_{X_i}' = F_{X_i}^{-1}(0.5), \tag{7.23}$$

and then solve Eq. (7.22) for σ_{X_i} to get

$$\sigma_{X_i}' = \frac{x_i - \mu_{X_i}'}{\Phi^{-1}\left[F_{X_i}(x_i)\right]}. \tag{7.24}$$

We set the

We can now define an implementation of the H-L method using Newton method to find closest failure point to the origin. This algorithm is

1. Set $x_i = \mu_{X_i}$ for $i = 1, 2, \ldots, n$.
2. Compute μ'_{X_i} and σ'_{X_i} using Eqs. (7.23) and (7.24).
3. Compute x'_i as
$$x'_i = \frac{x_i - \mu'_{X_i}}{\sigma'_{X_i}}.$$
4. Compute the n partial derivatives $\frac{\partial g}{\partial X_i}$ evaluated at x_i.
5. Compute the partial derivatives in the equivalent standard normal space. Using the chain-rule we get these as (see Eq. (7.20))
$$\frac{\partial g}{\partial X'_i} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial X'_i} = \frac{\partial g}{\partial X_i}\sigma'_{X_i}.$$
6. Compute new values for the design point using a version of the formula from Newton's method
$$\mathbf{x}'^{*} = \frac{1}{|\nabla g(\mathbf{x}')|^2}\left[\nabla g(\mathbf{x}')\cdot\mathbf{x}' - g(\mathbf{x}')\right]\nabla g(\mathbf{x}').$$
7. Compute $\beta_{\mathrm{HL}} = |\mathbf{x}'^{*}|$.
8. Compute a new value for x_i as
$$x_i^{*} = \mu'_{X_i} + \sigma'_{X_i} x_i'^{*}.$$
9. For given tolerances δ and ε check whether the change in β_{HL} is less than δ and that $|g(\mathbf{x}^{*})| < \varepsilon$.
10. If the convergence criteria are met, then stop. Otherwise, set $\mathbf{x}_i = \mathbf{x}_i^{*}$ and go back to step 2.

As an example calculation using this algorithm, we will use the the moderator temperature coefficient discussed in the previous section. In this example we had

$$Z = g(\beta_{\mathrm{m}}, p, f) = -\beta_{\mathrm{m}}\left(\log p + (1-f)\right).$$

For distributions of the parameters in this model we will use

$$\begin{aligned}\beta_{\mathrm{m}} &\sim \mathrm{N}(.004, .0004),\\ p &\sim \mathrm{B}(36.37, 21.3602),\\ f &\sim \mathrm{B}(5.06, .322979).\end{aligned}$$

The results of each step in the above algorithm are detailed in Table 7.1. This table only shows four iterations, after 10 iterations the value of β_{HL} is 2.2971, corresponding to a probability of failure of 0.0108. Recall that the Monte Carlo simulation of this example in the previous section showed a probability of failure of 0.0111. The

Table 7.1 The first four iterations of Hasofer-Lind on the example problem.

		Iteration 1	Iteration 2	Iteration 3	Iteration 4
Step 2	$(\mu'_{\beta_m}, \sigma'_{\beta_m})$	$(0.004, 0.0004)$	$(0.004, 0.0004)$	$(0.004, 0.0004)$	$(0.004, 0.0004)$
	(μ'_p, σ'_p)	$(0.6315, 0.06334)$	$(0.6315, 0.05778)$	$(0.6315, 0.06103)$	$(0.6315, 0.061814)$
	(μ'_f, σ'_f)	$(0.9815, 0.07807)$	$(0.9815, 0.1230)$	$(0.9815, 0.1629)$	$(0.9815, 0.167876)$
Step 3	β'_m	0	-0.9952	-0.1549	0.0027
	p'	-0.02385	2.7285	1.4349	1.0753
	f'	-0.53148	-1.2263	-1.9234	-2.0288
Step 4	$\left(\frac{\partial g}{\partial \beta_m}\right)$	0.402035	0.0674	-0.0021	0.0005
	$\left(\frac{\partial g}{\partial p}\right)$	-0.00634921	-0.0046	-0.0055	-0.0057
	$\left(\frac{\partial g}{\partial f}\right)$	0.004	0.0036	0.0039	0.0040
Step 5	$\left(\frac{\partial g}{\partial \beta'_m}\right)$	1.6082×10^{-4}	2.7×10^{-5}	-8×10^{-7}	2×10^{-7}
	$\left(\frac{\partial g}{\partial p'}\right)$	-4.0217×10^{-4}	-2.637×10^{-4}	3.342×10^{-4}	-3.543×10^{-4}
	$\left(\frac{\partial g}{\partial f'}\right)$	3.1227×10^{-4}	4.432×10^{-4}	6.4616×10^{-4}	6.717×10^{-4}
Step 6	$\mathbf{x}'^* = (\beta'_m, p', f')$	$(-0.9952, 2.4889, -1.9326)$	$(-0.1549, 1.5156, -2.5469)$	$(0.0027, 1.0890, -2.0905)$	$(-0.0006, 1.0725, -2.0331)$
Step 7	β_{HL}	3.3046	2.9678	2.3571	2.2986
Step 8	$\mathbf{x}^* = (\beta'_m, p', f')$	$(0.0036, 0.7892, 0.8306)$	$(0.0039, 0.7191, 0.6681)$	$(0.0040, 0.6980, 0.6409)$	$(0.0040, 0.6978, 0.6402)$
Step 9	$\Delta\beta_{HL}$	-	0.3367	0.6107	0.0585
	$\lvert g(\mathbf{x}^*)\rvert$	2.4276×10^{-4}	-8.2553×10^{-6}	1.8925×10^{-6}	7.4211×10^{-10}

Hasofer-Lind method therefore agrees with the "true" probability of failure to two significant digits. This is a great improvement over the FOSM method, which said the probability of failure was 0.00131003.

Though the Hasofer-Lind method can greatly improve on the accuracy of the FOSM method, it still has limitations in that it assumes a linear expansion of Z about the mean value, implcitly assuming that Z can be approximated by a normal distribution. Other reliability methods have been developed to address this problem, such as *Second*-order reliability methods where Z is expanded through the second-order term. Though in principle this can be more accurate it begins to defeat the purpose of reliability methods in that more evaluations of Z are required to compute the expansion of Z. Also, other methods that take into account nonlinear effects in the quantity of interest by performing an orthogonal moment expansion. These methods, broadly called polynomial chaos expansions, are attractive because they do not require the problem to be posed as a failure problem (though they can treat this type of problem) and they are based on orthogonal expansions of the quantity of interest and therefore will converge better than a Taylor series if the quantity of interest is smooth.