NUEN 647
Uncertainty Quantification for Nuclear Engineering
Assignment 1

Due on Tuesday, October 4, 2016

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Contents

Complete the exercises in the Chapter 2 notes. Be sure to include discussion of results where appropriate. You may use any tools that are approrpriate to solving the problem.

Problem 1

Show that the transformation in equation 1 results in a standard normal random variable by computing the mean and variance of z.

$$z = \frac{x-\mu}{\sigma} \tag{1}$$

An important special case of the expectation value is the mean which is the expected value of x. It is often denoted as μ,

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

where x is a realization of a random sample and $f(x)$ is the probability density function (PDF) for the random variable. For a normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

For the sake of the transformation, the value of z substitutes for x, the realization of a random sample (not the PDF because we are transforming that distribution). Therefore, the mean for z is:

$$\mu_z = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

If $u = (x-\mu)^2$ and $\frac{du}{2} = (x-\mu)dx$ (note that the limits change from $(-\infty, \infty)$ to (∞, ∞) - but that seems fishy to me so I will change it back after integration).

$$\mu_z = \int_{\infty}^{\infty} \frac{1}{2\sigma^2\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} du = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} \right|_{\infty}^{\infty}$$

$$\mu_z = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \right|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(e^{-\infty} - e^{-\infty}) = \boxed{0}$$

The variance is defined as:

$$Var(X) = E[(X-\mu)^2]$$

Substituting Eq. 1 for X, (but not for the pdf - I could be wrong about that)

$$Var(X) = E[(\frac{x-\mu}{\sigma} - \mu)^2] = E\left[\left(\frac{x-\mu-\mu\sigma}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}(E[x^2] - 2\mu E[x] - 2\mu\sigma E[x] + \mu^2 E[1] + \mu^2\sigma^2 + 2\mu^2\sigma)$$

Noting that above it was proven that $E[x] = \mu$ and given that the definition of $E[1] = 1$ and assuming that $E[x^2] = \sigma^2 + \mu^2$ (will solve on next page)

$$\frac{1}{\sigma^2}(\sigma^2 + \mu^2 - 2\mu^2 - 2\mu^2\sigma + \mu^2 + \mu^2\sigma^2 + 2\mu^2\sigma) = \frac{1}{\sigma^2}(\sigma^2 + \mu^2\sigma^2) = \boxed{1 + \mu^2 = 1}$$

This is assuming that $\mu = 0$. Which was shown above.

$$E[x^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

If $t = \frac{(x-\mu)}{\sqrt{2}\sigma}$ and $\sqrt{2}\sigma dt = dx$ and $x = t\sqrt{2}\sigma + \mu$ then (limits of integration don't change)

$$E[x^2] = \int_{-\infty}^{\infty} \frac{\left(t\sqrt{2}\sigma + \mu\right)^2}{\sqrt{\pi}} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(2\sigma^2 \left(t^2 e^{-t^2}\right) + 2\sqrt{2}\sigma\mu \left(te^{-t^2}\right) + \mu^2 \left(e^{-t^2}\right)\right)$$

According to wolfram alpha

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} te^{-t^2} = 0$$

$$\int_{-\infty}^{\infty} e^{-t^2} = \sqrt{\pi}$$

Which simplifies the above to $\sigma^2 + \mu^2$.

Problem 2

Consider the random variables $X \sim U(-1,1)$ and $Y \sim X^2$. Are these independent random variables? What is their covariance?

If two random variables, X and Y, are independent, they satisfy the following condition: ^{link}

- $P(X|Y) = P(X)$, for all values of X and Y.

The PDF for X is:

$$f_X(x) = \frac{1}{(1-(-1))} = 0.5 \quad x \in [-1,1]$$

The PDF for Y is:

$$f_Y(x) = f_X(x)^2 = 0.25 \quad x \in [-1,1]$$

I am confused, because at this point the PDF for Y does not integrate to one, but to 0.5...I probably did something wrong (anyway continue)

There is a special case for a collection of random variables where the joint PDF can be factored into the product of individual PDFs as:

$$f(\vec{x}) = f(x,y) = \prod_{i=1}^{p} f(x_i) = f_X(x) * f_Y(x) = 0.5 * 0.25 = 0.125 \quad x,y \in [-1,1]$$

We can define the probability distribution of Y provided X=x as

$$f(y|X=x) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dy} = \frac{0.125}{\int_{-1}^{1} 0.125dy} = 0.5$$

Are X and Y independant?

$$P(Y|X) \stackrel{?}{=} P(Y)$$
$$0.5 \neq 0.25$$

If what I did was correct...then X and Y shown to be dependant.

The measure for how random variables change together is called the covariance and the covariance between X and Y is written as σ_{XY}.

$$\begin{aligned}
\sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \int_{-1}^{1} dx \int_{-1}^{1} dy(x-\mu_x)(y-\mu_y)f(x,y) \\
&= \int_{-1}^{1} dx \int_{-1}^{1} dy(xy - 0.125x - 0.5y + 0.0625)0.125 \\
&= 0.125 \int_{-1}^{1} dx \left| \left(\frac{xy^2}{2} - 0.125xy - 0.25y^2 + 0.0625y\right) \right|_{-1}^{1}
\end{aligned}$$

This is an example citation [**?**].