

**NUEN 647**  
**Uncertainty Quantification for Nuclear Engineering**  
**Homework 2**

Due on Wednesday, October 19, 2016

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## Problem 1

Consider a covariance function between points in 2-D space:

$$k(x_1, y_1, x_2, y_2) = \exp[-|x_1 - x_2| - |y_1 - y_2|]$$

Generate 4 realizations of a Gaussian random process with zero mean,  $\mu(x, y) = 0$ , and this covariance function defined on the unit square,  $x, y \in [0, 1]$ . For the realizations, evaluate the process at 50 equally space points in each direction. Plot the realizations.

## Problem 2

Assume you have 100 samples of a pair of random variables  $(X_1, X_2)$  that have a positive correlation, call this set of pairs,  $\mathbf{A}_1$ . You then draw another 100 samples and call this set  $\mathbf{A}_2$ . The Pearson correlation between  $(X_1, X_2)$  in  $\mathbf{A}_1$  is positive and the Pearson correlation between  $(X_1, X_2)$  in  $\mathbf{A}_2$  is negative. What can you say about the Pearson correlation for all 200 samples?

A normalized measure of the relation between two random variables, is the Pearson correlation coefficient,  $\rho$ . Oftentimes, this is simply called the correlation coefficient or correlation.

$$\rho(X_1, X_2) = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X1}\sigma_{X2}}$$

The expectation value for a series of realizations is defined:

$$E[g(x)] \approx \frac{1}{N} \sum_{i=1}^N g(x_i)$$

For the first 100 values:

$$\begin{aligned} \rho_1 &= \frac{\frac{1}{100} \sum_{i=1}^{100} X_{1,i} X_{2,i} - \frac{1}{10000} \sum_{i=1}^{100} X_{1,i} \sum_{i=1}^{100} X_{2,i}}{\sigma_{X1,A1} \sigma_{X2,A1}} \\ 100 \sigma_{X1,A1} \sigma_{X2,A1} \rho_1 &= \sum_{i=1}^{100} X_{1,i} X_{2,i} - \frac{1}{100} \sum_{i=1}^{100} X_{1,i} \sum_{i=1}^{100} X_{2,i} \\ 100 \sigma_{X1,A1} \sigma_{X2,A1} \rho_1 + \frac{1}{100} \sum_{i=1}^{100} X_{1,i} \sum_{i=1}^{100} X_{2,i} &= \sum_{i=1}^{100} X_{1,i} X_{2,i} \end{aligned}$$

Similarly for the second 100 values:

$$\sum_{i=101}^{200} X_{1,i} X_{2,i} = 100 \sigma_{X1,A2} \sigma_{X2,A2} \rho_2 + \frac{1}{100} \sum_{i=101}^{200} X_{1,i} \sum_{i=101}^{200} X_{2,i}$$

The Pearson coefficient for all 200 values:

$$\begin{aligned} \rho_3 &= \frac{\frac{1}{200} \sum_{i=1}^{200} X_{1,i} X_{2,i} - \frac{1}{40000} \sum_{i=1}^{200} X_{1,i} \sum_{i=1}^{200} X_{2,i}}{\sigma_{X1,A3} \sigma_{X2,A3}} \\ 200 \sigma_{X1,A3} \sigma_{X2,A3} \rho_3 &= \sum_{i=1}^{200} X_{1,i} X_{2,i} - \frac{1}{200} \sum_{i=1}^{200} X_{1,i} \sum_{i=1}^{200} X_{2,i} \\ 200 \sigma_{X1,A3} \sigma_{X2,A3} \rho_3 + \frac{1}{200} \sum_{i=1}^{200} X_{1,i} \sum_{i=1}^{200} X_{2,i} &= \sum_{i=1}^{200} X_{1,i} X_{2,i} \end{aligned}$$

If we plug in the Pearson for the first 100 and second 100 for the right side of the equation,

$$\begin{aligned}
& 200\sigma_{X1,A3}\sigma_{X2,A3}\rho_3 + \frac{1}{200} \sum_{i=1}^{200} X_{1,i} \sum_{i=1}^{200} X_{2,i} \\
& = \\
& 100(\sigma_{X1A1}\sigma_{X2A1}\rho_1 + \sigma_{X1A2}\sigma_{X2A2}\rho_2) + \frac{1}{100} \left( \sum_{i=1}^{100} x_{1,i} \sum_{i=1}^{100} x_{2,i} + \sum_{i=101}^{200} x_{1,i} \sum_{i=101}^{200} x_{2,i} \right)
\end{aligned}$$

Grouping and setting:

$$\begin{aligned}
\sum_{i=1}^{100} x_{1,i} &= X_{1,1} \\
\sum_{i=1}^{100} x_{2,i} &= X_{2,1} \\
\sum_{i=101}^{200} x_{1,i} &= X_{1,2} \\
\sum_{i=101}^{200} x_{2,i} &= X_{2,2} \\
\sum_{i=1}^{200} x_{1,i} &= X_{1,3} \\
\sum_{i=1}^{200} x_{2,i} &= X_{2,3}
\end{aligned}$$

$$200\sigma_{X1,A3}\sigma_{X2,A3}\rho_3 - 100(\sigma_{X1A1}\sigma_{X2A1}\rho_1 + \sigma_{X1A2}\sigma_{X2A2}\rho_2) = \frac{1}{100} (X_{1,1}X_{2,1} + X_{1,2}X_{2,2}) - \frac{1}{200} X_{1,3}X_{2,3}$$

Setting:

$$\begin{aligned}
\sigma_{X1A1} &= \frac{1}{100} \sum_{i=1}^{100} (x_{1,i}^2 - \mu_{X_{11}}^2) = \frac{1}{100} \sigma'_{X1A1} \\
\sigma_{X2A1} &= \frac{1}{100} \sum_{i=1}^{100} (x_{2,i}^2 - \mu_{X_{21}}^2) = \frac{1}{100} \sigma'_{X2A1} \\
\sigma_{X1A2} &= \frac{1}{100} \sum_{i=101}^{200} (x_{1,i}^2 - \mu_{X_{12}}^2) = \frac{1}{100} \sigma'_{X1A2} \\
\sigma_{X2A2} &= \frac{1}{100} \sum_{i=101}^{200} (x_{2,i}^2 - \mu_{X_{22}}^2) = \frac{1}{100} \sigma'_{X2A2} \\
\sigma_{X1A3} &= \frac{1}{200} \sum_{i=1}^{200} (x_{1,i}^2 - \mu_{X_{13}}^2) = \frac{1}{200} \sigma'_{X1A3} \\
\sigma_{X2A3} &= \frac{1}{200} \sum_{i=1}^{200} (x_{2,i}^2 - \mu_{X_{23}}^2) = \frac{1}{200} \sigma'_{X2A3}
\end{aligned}$$

Where  $A3$  and  $\rho_3$  are for the series added to 200. Plugging these in, and multiplying both sides of the equation by 200.

$$\sigma'_{X1,A3}\sigma'_{X2,A3}\rho_3 - 2(\sigma'_{X1A1}\sigma'_{X2A1}\rho_1 + \sigma'_{X1A2}\sigma'_{X2A2}\rho_2) = 2(X_{1,1}X_{2,1} + X_{1,2}X_{2,2}) - X_{1,3}X_{2,3}$$

Note:  $X_{1,3} = X_{1,1} + X_{1,2}$  and  $X_{2,3} = X_{2,1} + X_{2,2}$  and that the right side of the equation simplifies to:  $(X_{1,1} - X_{1,2})(X_{2,1} - X_{2,2})$ . Then  $\rho_3$  is:

$$\rho_3 = \frac{(X_{1,1} - X_{1,2})(X_{2,1} - X_{2,2}) + 2(\sigma'_{X1A1}\sigma'_{X2A1}\rho_1 + \sigma'_{X1A2}\sigma'_{X2A2}\rho_2)}{\sigma'_{X1,A3}\sigma'_{X2,A3}}$$

Assuming that:

$$\frac{(X_{1,1} - X_{1,2})(X_{2,1} - X_{2,2})}{\sigma'_{X1,A3}\sigma'_{X2,A3}} \approx 0$$

and

$$\begin{aligned}\sigma'_{X1,A3}\sigma'_{X2,A3} &\approx 4\sigma'_{X1A1}\sigma'_{X2A1} \\ \text{or} &\approx 4\sigma'_{X1A2}\sigma'_{X2A2}\end{aligned}$$

The above would simplify to:

$$\rho_3 \approx \frac{\rho_1 + \rho_2}{2}$$

Meaning,  $\rho_3$  will usually be inside the interval  $\rho_2 < \rho_3 < \rho_1$ , I was curious, and wrote a script, to check to see if it would ever be outside. There could be an error with my script, but I found with the below script that a small percentage (less than 1% of the time around 0.2-0.4%), it would be outside the above interval. I also made a histogram plot...because I like wasting time.

Listing 1: Script for Problem

```
#!/usr/bin/env python3

#####
##### Import packages #####
5 #####

import numpy as np
import time
start_time = time.time()
10 import Functions as Fun

#####
##### Calculations #####
#####

15 Error=[];Ntimes=1000;Nsamples=100;CountOut=0

for i in range(0,Ntimes):

20     Positive=True
    Negative=True
    while(Positive or Negative):

        X1=np.random.uniform(-1,1,Nsamples)
        X2=np.random.uniform(-1,1,Nsamples)
25         rho=Fun.CalculateRho(X1,X2)

        if rho>0:
            rho1=rho;X11=X1;X21=X2;
            Positive=False
30         if rho<0:
            rho2=rho;X12=X1;X22=X2;
```

```

        Negative=False

35
        rho_Guess=(rho1+rho2)/2

        X13=np.append(X11,X12)
        X23=np.append(X21,X22)
40
        rho=Fun.CalculateRho(X13,X23)

        if(rho>rho1 or rho<rho2):
            CountOut=CountOut+1
            Error.append((abs(rho_Guess-rho)/rho)*100)
45
        Fun.PlotHistSave(Error,Ntimes)

        print("Percent outside rho1 and rho2: "+str(100*CountOut/Ntimes)+"%")

50
        ##### Time To execute #####

        print("--- %s seconds ---" % (time.time() - start_time))

```

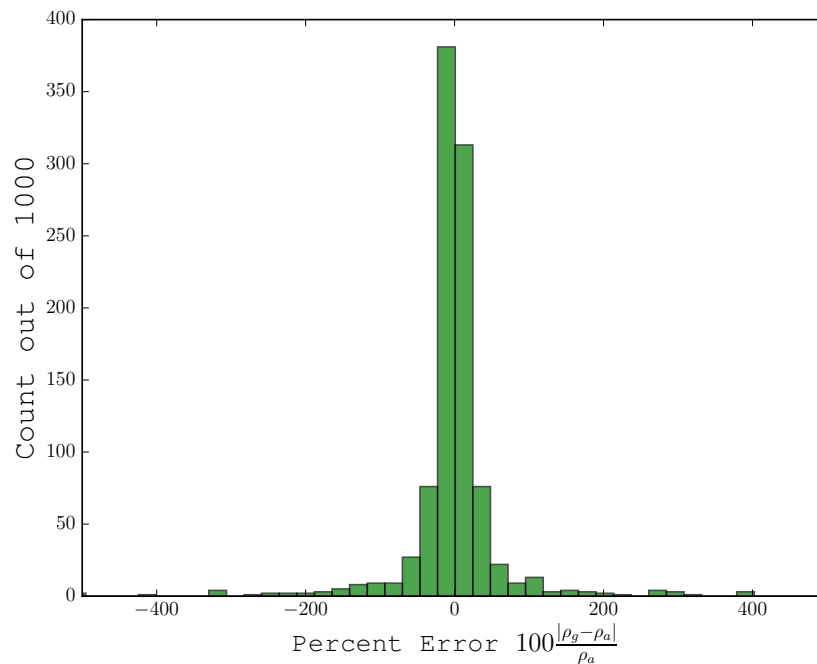


Figure 1: Histogram plot showing error  $\rho_g$  is the approximated guess at  $\rho_3$  and  $\rho_a$  is the actual calculated  $\rho_3$ .

### Problem 3

For the following data, compute by hand or via code you write the Pearson and Spearman correlations and Kendall's tau.

$X_1$	$X_2$
55.01	82.94
54.87	55.02
57.17	85.18
36.01	-84.27
35.88	-106.30
36.33	-119.65
43.49	-112.03
41.44	-71.69
54.43	-3.50
36.47	140.57

#### Pearson Correlation

$$\rho(X, Y) = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

Where:

$$E[g(x)] = \int_{-\infty}^{\infty} dx \, g(x) f(x) \approx \frac{1}{N} \sum_{i=1}^N g(x_i)$$

and

$$\sigma_X = \text{Var}(x) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \approx \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2 \approx \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2 \equiv s^2$$

and

$$\mu_X \approx \frac{1}{N} \sum_{i=1}^N x_i \equiv \bar{x}$$

#### Spearman Rank Correlation

$$\rho_S(X, Y) = \frac{\sum_{i=1}^N (\text{rank}(x_i) - \bar{r}_X)(\text{rank}(y_i) - \bar{r}_Y)}{\sqrt{\sum_{i=1}^N (\text{rank}(x_i) - \bar{r}_X)^2} \sqrt{\sum_{i=1}^N (\text{rank}(y_i) - \bar{r}_Y)^2}}$$

Where:

$$\text{rank}(x_i) = \text{Rank of } x_i \text{ in sample population}$$

and

$$\bar{r}_X = \frac{1}{N} \sum_{i=1}^N \text{rank}(x_i)$$

#### Kendall's Tau

**TAU!!!!!!** (Powering up)

$$\tau = \frac{(\# \text{ of concordant pairs}) - (\# \text{ of discordant pairs})}{\frac{1}{2}N(N-1)}$$



Where concordance is

$$x_i > x_j \text{ and } y_i > y_j \text{ or if } x_i < x_j \text{ and } y_i < y_j$$

for all pairs of samples ( $\frac{1}{2}N(N-1)$  of them) and discordance is

$$x_i > x_j \text{ and } y_i < y_j \text{ or if } x_i < x_j \text{ and } y_i > y_j$$

Listing 2: Script for Problem

```
#!/usr/bin/env python3

#####
##### Import packages #####
5 #####
import numpy as np
import time
start_time = time.time()
import Functions as Fun
10 #####
##### Calculations #####
#####

X1=np.array([55.01,54.87,57.17,36.01,35.88,36.33,
15           43.49,41.44,54.43,36.47])
X2=np.array([82.94,55.02,85.18,-84.27,-106.30,-119.65,
           -112.03,-71.69,-3.50,140.57])

rho,rhoNM1=Fun.CalculatePearson(X1,X2)

20
#Getting rank of each element, starting with 1
X1R=Fun.Rank(X1)
X2R=Fun.Rank(X2)

25 rhoS=Fun.CalculateSpearman(X1,X2,X1R,X2R)
tau=Fun.CalculateTau(X1,X2)

print("Pearson Var Div by N: "+str(round(rho,4)))
print("Pearson Var Div by N-1: "+str(round(rhoNM1,4)))
30 print("Spearman: "+str(round(rhoS,4)))
print("Kendall: "+str(round(tau,4)))
##### Time To execute #####
print("--- %s seconds ---" % (time.time() - start_time))
```

Code output:

Pearson Var Div by N: 0.5429  
 Pearson Var Div by N-1: 0.4886  
 Spearman: 0.5879  
 Kendall: 0.5111

## Problem 4

Demonstrate the tail dependence of a bivariate normal random variable is 0.

The bivariate Gaussian copula is defined as:

$$C_N(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

Where:

$$\Phi^{-1}(q) = \mu + \sigma\sqrt{2}\text{erf}^{-1}(2q - 1)$$

Evaluated at  $q = 0$  and  $q = 1$ :

$$\Phi^{-1}(0) = -\infty \quad \Phi^{-1}(1) = \infty$$

Also where:

$$\Phi_\rho(x, y) = \int_{-\infty}^x dx' \int_{-\infty}^y dy' \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right]$$

with

$$z = \frac{(x' - \mu_x)^2}{\sigma_x^2} - \frac{2\rho(x' - \mu_x)(y' - \mu_y)}{\sigma_x\sigma_y} + \frac{(y' - \mu_y)^2}{\sigma_y^2}$$

and

$$\rho = \frac{E[XY] - E[X]E[Y]}{\sigma_x\sigma_y}$$

**Note:**

$$\Phi_\rho(-\infty, -\infty) = \int_{-\infty}^{-\infty} dx' \int_{-\infty}^{-\infty} dy' \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right] = 0$$

Because integrating over zero domain is 0.

$$\Phi_\rho(\infty, \infty) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right] = 1$$

Because integrating over the entire domain of a PDF is 1.

**Lower Tail Dependence:**

$$\begin{aligned} \lambda_l &= \lim_{q \rightarrow 0} \frac{C(q, q)}{q} \\ &= \lim_{q \rightarrow 0} \frac{\Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q))}{q} \end{aligned}$$

Applying L'Hôpital (Pronounced Hospital - like the place you go when you get sick - not really).

$$\begin{aligned} \lambda_l &= \lim_{q \rightarrow 0} \frac{\frac{d}{dq} \Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q))}{1} \\ &= \lim_{q \rightarrow 0} \frac{d}{dq} \Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q)) \end{aligned}$$

Evaluating at  $q = 0$

$$\begin{aligned}\lambda_l &= \frac{d}{dq} \Phi_\rho(\Phi^{-1}(0), \Phi^{-1}(0)) \\ &= \frac{d}{dq} \Phi_\rho(-\infty, -\infty) \\ &= \frac{d}{dq} 0 \\ &= 0\end{aligned}$$

The problem with this is, I do not think I can just plug in  $q = 0$  inside the whole mess, but rather like this:

$$\lambda_l = \left| \frac{d}{dq} \Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q)) \right|_{q=0}$$

Here, I would want to say that the integral cancels with  $\frac{d}{dq}$  (not sure if thats correct - because there are two integrals), to get

$$\begin{aligned}\lambda_l &= \left| \phi(\Phi^{-1}(q), \Phi^{-1}(q)) \right|_{q=0} \\ &= \phi(\Phi^{-1}(0), \Phi^{-1}(0)) \\ &= \phi(-\infty, -\infty) \\ &= 0\end{aligned}$$

Where  $\phi$  is the PDF of a bivariate normal ( $\Phi$  without the integrals). Noting  $\phi(-\infty, -\infty)$  ends up with a  $e^{-\infty}$  term.

but if we apply the limit before L'Hôspital, then we get:

$$\begin{aligned}\lambda_l &= \lim_{q \rightarrow 0} \frac{C(q, q)}{q} \\ &= \lim_{q \rightarrow 0} \frac{\Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q))}{q} \\ &= \frac{\Phi_\rho(\Phi^{-1}(0), \Phi^{-1}(0))}{0} \\ &= \frac{\Phi_\rho(-\infty, -\infty)}{0} \\ &= \frac{0}{0}\end{aligned}$$

I am not sure if thats healthy.

**Upper Tail Dependence:**

$$\begin{aligned}\lambda_u &= \lim_{q \rightarrow 1} \frac{1 - 2q + C(q, q)}{1 - q} \\ &= \lim_{q \rightarrow 1} \frac{1 - 2q + \Phi_\rho(\Phi^{-1}(q), \Phi^{-1}(q))}{1 - q}\end{aligned}$$

Applying the limit:

$$\begin{aligned}\lambda_u &= \frac{1 - 2 + \Phi_\rho(\Phi^{-1}(1), \Phi^{-1}(1))}{0} \\ &= \frac{1 - 2 + \Phi_\rho(\infty, \infty)}{0} \\ &= \frac{1 - 2 + 1}{0} \\ &= \frac{0}{0}\end{aligned}$$

Again, I do not think this is healthy, but I don't know what else to do. Could look at similar things as above, but I think they wouldn't even give 0...I could try and evaluate the intervals and then differentiate but that sounds like a mess.

## Problem 5

Another Archimedean copula is the Joe copula with generator

$$\varphi_J(t) = -\log(1 - (1 - t)^\theta),$$

and

$$\varphi_J^{-1} = 1 - (1 - \exp(-t))^{1/\theta}.$$

- (a) Compute the bivariate copula for this generator

The Archimedean copula for  $\varphi$  is

$$C_\varphi(u, v) = \hat{\varphi}^{-1}(\varphi(u) + \varphi(v))$$

- (b) Derive the upper and lower tail dependence for this copula

$$\lambda_l = \lim_{q \rightarrow 0} \frac{C(q, q)}{q}$$

$$\lambda_u = \lim_{q \rightarrow 1} \frac{1 - 2q + C(q, q)}{1 - q}$$

- (c) Compute the value of Kendall's tau for this copula

$$\tau(U, V) = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt$$

- (d) Generate 1000 samples from the copula with standard normal margins and a value of Kendall's tau of 0.6.

**(a)**

$$\begin{aligned} C_\varphi(u, v) &= \hat{\varphi}^{-1}(\varphi(u) + \varphi(v)) \\ &= \hat{\varphi}^{-1}(-\log(1 - (1 - u)^\theta) - \log(1 - (1 - v)^\theta)) \\ &= \hat{\varphi}^{-1}(-\log([1 - (1 - u)^\theta][1 - (1 - v)^\theta])) \\ &= 1 - (1 - \exp(-(-\log([1 - (1 - u)^\theta][1 - (1 - v)^\theta])))^{1/\theta} \\ &= 1 - (1 - [1 - (1 - u)^\theta][1 - (1 - v)^\theta])^{1/\theta} \\ &= 1 - [(1 - v)^\theta + (1 - u)^\theta - (1 - u)^\theta(1 - v)^\theta]^{1/\theta} \end{aligned}$$

**(b) Lower**

$$\begin{aligned} \lambda_l &= \lim_{q \rightarrow 0} \frac{C(q, q)}{q} \\ &= \lim_{q \rightarrow 0} \frac{1 - [(1 - q)^\theta + (1 - q)^\theta - (1 - q)^\theta(1 - q)^\theta]^{1/\theta}}{q} \\ &= \lim_{q \rightarrow 0} \frac{1 - [2(1 - q)^\theta - (1 - q)^{2\theta}]^{1/\theta}}{q} \\ &= \lim_{q \rightarrow 0} \frac{1 - [(1 - q)^\theta(2 - (1 - q)^\theta)]^{1/\theta}}{q} \\ &= \lim_{q \rightarrow 0} \frac{1 - (1 - q) [(2 - (1 - q)^\theta)]^{1/\theta}}{q} \end{aligned}$$

Applying L'Hôpital (did it in my head - not really)

$$\begin{aligned}\lambda_l &= \lim_{q \rightarrow 0} -2(2 - (1 - q))^{\frac{1}{\theta} - 1}((1 - q)^\theta - 1) \\ &= -2(2 - (1 - 0))^{\frac{1}{\theta} - 1}((1 - 0)^\theta - 1) \\ &= 0\end{aligned}$$

**Upper**

$$\begin{aligned}\lambda_u &= \lim_{q \rightarrow 1} \frac{1 - 2q + C(q, q)}{1 - q} \\ &= \lim_{q \rightarrow 1} \frac{1 - 2q + 1 - (1 - q) [(2 - (1 - q)^\theta)]^{1/\theta}}{1 - q} \\ &= \lim_{q \rightarrow 1} \frac{(1 - q) (2 - [(2 - (1 - q)^\theta)]^{1/\theta})}{1 - q} \\ &= \lim_{q \rightarrow 1} 2 - [(2 - (1 - q)^\theta)]^{1/\theta} \\ &= 2 - 2^{1/\theta}\end{aligned}$$

**(c)**

$$\begin{aligned}\varphi'(t) &= -\frac{\theta(1-t)^{\theta-1}}{1-(1-t)^\theta} \\ \tau(U, V) &= 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt \\ &= 1 + 4 \int_0^1 \frac{-\log(1 - (1-t)^\theta)}{-\frac{\theta(1-t)^{\theta-1}}{1-(1-t)^\theta}} dt \\ &= 1 + 4 \int_0^1 \frac{-\log(1 - (1-t)^\theta)}{-\frac{\theta(1-t)^{\theta-1}}{1-(1-t)^\theta}} dt\end{aligned}$$

I do not want to integrate, and will assume the answer is correct on this link.

$$\tau(U, V) = 1 - 4 \sum_{k=1}^{\infty} \frac{1}{(K(\theta K + 2)(\theta(K - 1) + 1))}$$

(d) For a Kendall's tau of 0.6,  $\theta = 3.826659$ . According to some random notes I have. In order to sample from a bivariate copula:

1. Produce  $\xi_1, \xi_2$  where  $\xi_i \sim U(0, 1)$
2. Set  $W \equiv C^{-1}(\xi_2|\xi_1)$

Where:

$$\begin{aligned} C(v|u) &= \frac{d}{du}(C(u, v)) \\ &= \frac{d}{du}(1 - [(1-v)^\theta + (1-u)^\theta - (1-u)^\theta(1-v)^\theta]^{1/\theta}) \end{aligned}$$

Wolfram gives an answer, but I want to say first, this is annoying. Okay, if  $U = (1-u)^\theta$  and  $V = (1-v)^\theta$  and  $U^* = (1-u)$ , then a reasonable looking answer is:

$$C(v|u) = \frac{U}{U^*}[V - 1][U + V - VU]^{\frac{1}{\theta}-1}$$

The next step requires us setting  $C(v|u) = \xi$  and solving for  $v$  and calling that  $C_J^{-1}(\xi|u)$ . I cannot seem to solve for  $V$ , so I'll come up with a janky way to determine  $C_J^{-1}(\xi|u)$ . Solving for a  $V$ , and remembering that  $V = (1-v)^\theta = (1 - C_J^{-1}(\xi|u))^\theta$ .

$$\begin{aligned} V &= \frac{U}{U^*}\xi[U + V - VU]^{1-\frac{1}{\theta}} + 1 \\ C_J^{-1}(\xi|u) &= 1 - \left( \frac{U}{U^*}\xi[U + V - VU]^{1-\frac{1}{\theta}} + 1 \right)^{\frac{1}{\theta}} \end{aligned}$$

Noting that we will have to iterate for a solution... this better converge, or I will destroy something.

3.  $x = F_X^{-1}(\xi_1)$   $y = F_Y^{-1}(w)$  Where:

$$F_X^{-1}(\xi) = \sqrt{2}erf^{-1}(2\xi - 1)$$

For a standard normal.

Listing 3: Script for Problem

```
#!/usr/bin/env python3

#####
##### Import packages #####
5 #####

import numpy as np
import time
import copy
import scipy.special as sps
10 import matplotlib.pyplot as plt
plt.rcParams["font.family"] = "monospace"
import matplotlib
matplotlib.rc('text',usetex=True)
matplotlib.rcParams['text.latex.preamble']=[r"\usepackage{amsmath}"]
15 start_time = time.time()
#####
```

```

##### Calculations #####
#####

20 fig=plt.figure()
   ax=fig.add_subplot(111)
   ax.set_xlabel(r'$\boldsymbol{x}$', fontsize=18)
   ax.set_ylabel(r'\textbf{y}', fontsize=18)
   ax.grid(alpha=0.8, color='black', linestyle='dotted')
25
   B=3.826659

   for i in range(0,1000):

30       l1=np.random.uniform(0,1)
       l2=np.random.uniform(0,1)

       U=(1-l2)**B
       Us=(1-l2)

35       error=100

       Cg=0.5
       count=0
40       while (error>0.001):
           V=(1-Cg)**B
           W=1-((U/Us)*l1*((U+V-V*U)**(1-1/B))+1)**(1/B)
           error=abs(Cg-W)/W
           Cg=copy.copy(W)

45           if count>1000:
               print("Did not converge after 1000")
               quit()
               count=count+1

50       W=W-1
       x=(2**0.5)*sps.erfinv(2*l1-1)
       y=(2**0.5)*sps.erfinv(2*W-1)
       ax.plot(x,y,'ko', markersize=5)

55 plt.savefig('P5.pdf')

```



