NUEN 647 Homework 3

Due: December 10, 2016

Short Problems

1. Fit the data in Table 1 to a linear model using

   (a) least squares

   (b) ridge regression

   (c) lasso regression

   Be sure to do cross-validation for each fit, and for each method present your best estimate of the model.

2. Derive the adjoint operator for the equation

$$-\nabla^2\phi(x,y,z) + \frac{1}{L^2}\phi(x,y,z) = \frac{Q}{D},$$

$$\phi(0,y,z) = \phi(x,0,z) = \phi(x,y,0) = \phi(X,y,z) = \phi(x,Y,z) = \phi(x,y,Z).$$

   Compute the sensitivity to the QoI:

$$\text{QoI} = \int_0^X dx \int_0^Y dy \int_0^Z dz\, \frac{D}{L^2}\phi(x,y,z),$$

   for X, Y, Z, L, D, and Q.

3. For the random variable $X \sim N(0,1)$ draw fifty samples and generate histograms using the following sampling techniques

   (a) Simple random sampling,

   (b) Stratified sampling,

	x_1	x_2	y
1	0.99	0.98	6.42
2	-0.75	-0.76	0.20
3	-0.50	-0.48	0.80
4	-1.08	-1.08	-0.57
5	0.09	0.09	4.75
6	-1.28	-1.27	-1.42
7	-0.79	-0.79	1.07
8	-1.17	-1.17	0.20
9	-0.57	-0.57	1.08
10	-1.62	-1.62	-0.15
11	0.34	0.35	2.90
12	0.51	0.51	3.37
13	-0.91	-0.92	0.05
14	1.85	1.86	5.50
15	-1.12	-1.12	0.17
16	-0.70	-0.70	1.72
17	1.19	1.18	3.97
18	1.24	1.23	6.38
19	-0.52	-0.52	3.29
20	-1.41	-1.44	-1.49

Table 1: Data to fit to linear model $y = a + bx_1 + cx_2$

(c) A van der Corput sequence of base 2,

(d) A van der Corput sequence of base 3.

4. Consider the Rosenbrock function: $f(x, y) = (1-x)^2 + 100(y-x^2)^2$. Assume that $x = 2t-1$, where $T \sim B(3, 2)$ and $y = 2s-1$, where $S \sim B(1.1, 2)$. Estimate the probability that $f(x, y)$ is less than 10 using

   (a) a first-order second-moment reliability method

   (b) Latin hypercube sampling using 50 points.

   (c) A Halton sequence using 50 points.

   Compare this with the probability you calculate using 10^5 random samples. (*Hint: Matlab has a built-in function for sampling beta R.V.'s "betarnd"*).

5. Consider the exponential integral function, $\mathrm{E}_n(x)$,

$$\mathrm{E}_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n}\, dt.$$

   This function is involved in the solution to many pure-absorbing transport problems. Use this function to solve the transport problem,

$$\mu \frac{\partial \psi}{\partial x} + \sigma \psi = 0,$$

$$\psi(0, \mu > 0) = 1, \psi(10, \mu < 0) = 0,$$

   for the scalar flux $\phi(x) = \int_{-1}^{1} \psi(x, \mu)\, d\mu$. Assume that $\sigma \sim \mathrm{GAM}(10, .1)$. Use a PCE expansion to estimate the distribution, mean, and variance of $\phi(x)$ at $x = 1, 1.5, 3, 5$. Also, plot the mean value of ϕ as a function of x.

6. You perform a measurement of a beam of radiation satifying the boundary condition in problem 5 hitting a slab, and somehow are able to measure the scalar flux at $x = 1, 1.5, 3, 5$:

$$\phi(1) = 0.201131, \quad \phi(1.5) = 0.110135, \quad \phi(3) = 0.0228748, \quad \phi(5) = 0.00328249.$$

   Using the prior distribution for σ from problem 1, and the experimental data just given, derive a posterior distribution for σ (i.e., calibrate σ). You may assume that the measurement has an error distributed by $N(0, \sigma = 0.001)$.

Long Problems

Select three from the following list.

1. Using a discretization of your choice, solve the equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} - \omega u,$$

for $u(x,t)$ on the spatial domain $x \in [0,10]$ with periodic boundary conditions $u(0^-) = u(10^+)$, and initial conditions

$$u(x,0) = \begin{cases} 1 & x \in [0,2.5] \\ 0 & \text{otherwise} \end{cases}.$$

Use the solution to compute the total reactions

$$\int_5^6 dx \int_0^5 dt\, \omega u(x,t).$$

Compute scaled sensitivity coefficients and sensitivity indices for normal random variables:

(a) $\mu_v = 0.5$, $\sigma_v = 0.1$,
(b) $\mu_D = 0.125$, $\sigma_D = 0.03$,
(c) $\mu_\omega = 0.1$, $\sigma_\omega = 0.05$,

How do these results change with changes in Δx and Δt?

2. Using a discretization of your choice, solve the equation

$$v\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} - \omega u + 1,$$

for $u(x)$ on the spatial domain $x \in [0,10]$ with periodic boundary conditions $u(0^-) = u(10^+)$. Use the solution to compute the total reactions

$$\int_5^6 dx\, \omega u(x).$$

Derive the adjoint equation for this equation and use it's numerical solution to compute the sensitivities to the following parameters.

(a) $\mu_v = 0.5$,
(b) $\mu_D = 0.125$,
(c) $\mu_\omega = 0.1$,

3. Using a discretization of your choice, solve the equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} - \omega u,$$

for $u(x,t)$ on the spatial domain $x \in [0,10]$ with periodic boundary conditions $u(0^-) = u(10^+)$, and initial conditions

$$u(x,0) = \begin{cases} 1 & x \in [0,2.5] \\ 0 & \text{otherwise} \end{cases}.$$

Use the solution to compute the total reactions

$$\int_5^6 dx \int_0^5 dt\, \omega u(x,t).$$

Sample values of parameters using a uniform distribution centered at the mean with upper and lower bounds $\pm 10\%$ for the following variables:

(a) $\mu_v = 0.5$,
(b) $\mu_D = 0.125$,
(c) $\mu_\omega = 0.1$,

and sample values of the following parameters in their given ranges:

(a) $\Delta x \sim [0.001, 0.5]$,
(b) $\Delta t \sim [0.001, 0.5]$.

Using regression estimate the sensitivities to each parameter.

4. Using a discretization of your choice, solve the equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} - \omega u,$$

for $u(x,t)$ on the spatial domain $x \in [0,10]$ with periodic boundary conditions $u(0^-) = u(10^+)$, and initial conditions

$$u(x,0) = \begin{cases} 1 & x \in [0, 2.5] \\ 0 & \text{otherwise} \end{cases}.$$

Use the solution to compute the total reactions

$$\int_5^6 dx \int_0^5 dt\, \omega u(x,t).$$

Compute the probability that this quantity of interest is greater than 0.035 using LHS sampling of 50 points, 50 points of a Halton sequence, and a first-order second moment method using the following distributions:

(a) $\mu_v = 0.5$, $\sigma_v = 0.1$,

(b) $\mu_D = 0.125$, $\sigma_D = 0.03$,

(c) $\mu_\omega = 0.1$, $\sigma_\omega = 0.05$,

How do these results change with changes in Δx and Δt?

5. Using a discretization of your choice, solve the equation

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} - \omega u,$$

for $u(x,t)$ on the spatial domain $x \in [0,10]$ with periodic boundary conditions $u(0^-) = u(10^+)$, and initial conditions

$$u(x,0) = \begin{cases} 1 & x \in [0, 2.5] \\ 0 & \text{otherwise} \end{cases}.$$

Using a polynomial chaos expansion estimate the mean and variance in the total number of reactions

$$\int_5^6 dx \int_0^5 dt\, \omega u(x,t).$$

Use $v = 0.5$, $D = 0.125$, and assume that ω is an uncertain parameter distributed via $\omega \sim \text{GAM}(2, .1)$. Also, report the distribution of the total number of reactions.