

NUEN 647
Uncertainty Quantification for Nuclear Engineering
Assignment 1

Due on Tuesday, October 4, 2016

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Complete the exercises in the Chapter 2 notes. Be sure to include discussion of results where appropriate. You may use any tools that are appropriate to solving the problem.

Problem 1

Show that the transformation in equation 1 results in a standard normal random variable by computing the mean and variance of z .

$$z = \frac{x - \mu}{\sigma} \quad (1)$$

An important special case of the expectation value is the mean which is the expected value of x . It is often denoted as μ ,

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

where x is a realization of a random sample and $f(x)$ is the probability density function (PDF) for the random variable. For a normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

For the sake of the transformation, the value of z substitutes for x , the realization of a random sample (not the PDF because we are transforming that distribution). Therefore, the mean for z is:

$$\mu_z = \int_{-\infty}^{\infty} \frac{x - \mu}{\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

If $u = (x - \mu)^2$ and $\frac{du}{2} = (x - \mu)dx$ (note that the limits change from $(-\infty, \infty)$ to (∞, ∞) - but that seems fishy to me so I will change it back after integration).

$$\begin{aligned} \mu_z &= \int_{-\infty}^{\infty} \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{u}{2\sigma^2}} du = \left| \frac{-1}{\sqrt{2\pi}} e^{-\frac{u}{2\sigma^2}} \right|_{-\infty}^{\infty} \\ \mu_z &= \left| \frac{-1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}} (e^{-\infty} - e^{-\infty}) = \boxed{0} \end{aligned}$$

The variance is defined as:

$$\text{Var}(X) = E[(X - \mu)^2]$$

Substituting Eq. 1 for X , (but not for the pdf - I could be wrong about that)

$$\text{Var}(X) = E\left[\left(\frac{x - \mu}{\sigma} - \mu\right)^2\right] = E\left[\left(\frac{x - \mu - \mu\sigma}{\sigma}\right)^2\right] = \frac{1}{\sigma^2} (E[x^2] - 2\mu E[x] - 2\mu\sigma E[x] + \mu^2 E[1] + \mu^2\sigma^2 + 2\mu^2\sigma)$$

Noting that above it was proven that $E[x] = \mu$ and given that the definition of $E[1] = 1$ and assuming that $E[x^2] = \sigma^2 + \mu^2$ (will solve on next page)

$$\frac{1}{\sigma^2} (\sigma^2 + \mu^2 - 2\mu^2 - 2\mu^2\sigma + \mu^2 + \mu^2\sigma^2 + 2\mu^2\sigma) = \frac{1}{\sigma^2} (\sigma^2 + \mu^2\sigma^2) = \boxed{1 + \mu^2 = 1}$$

This is assuming that $\mu = 0$. Which was shown above.

$$E[x^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

If $t = \frac{(x-\mu)}{\sqrt{2}\sigma}$ and $\sqrt{2}\sigma dt = dx$ and $x = t\sqrt{2}\sigma + \mu$ then (limits of integration don't change)

$$E[x^2] = \int_{-\infty}^{\infty} \frac{(t\sqrt{2}\sigma + \mu)^2}{\sqrt{\pi}} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(2\sigma^2 (t^2 e^{-t^2}) + 2\sqrt{2}\sigma\mu (te^{-t^2}) + \mu^2 (e^{-t^2}) \right) dt$$

According to wolfram alpha

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} te^{-t^2} dt = 0$$

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

Which simplifies the above to $\sigma^2 + \mu^2$.

Problem 2

Consider the random variables $X \sim U(-1, 1)$ and $Y \sim X^2$. Are these independent random variables? What is their covariance?

If two random variables, X and Y, are independent, they satisfy the following condition: *link*

- $P(X|Y) = P(X)$, for all values of X and Y .

The PDF for X is:

$$f_X(x) = \frac{1}{(1 - (-1))} = 0.5 \quad x \in [-1, 1]$$

The PDF for Y is:

$$f_Y(x) = f_X(x)^2 = 0.25 \quad x \in [-1, 1]$$

I am confused, because at this point the PDF for Y does not integrate to one, but to 0.5...I probably did something wrong.

This is an example citation [1].

References

- [1] E. T. Tatro, S. Heffler, S. Shumaker-Armstrong, B. Soontornniyomkij, M. Yang, A. Yermanos, N. Wren, D. J. Moore, and C. L. Achim. Modulation of bk channel by microrna-9 in neurons after exposure to hiv and methamphetamine. *J Neuroimmune Pharmacol*, 2013. Tatro, Erick T Heffler, Shannon Shumaker-Armstrong, Stephanie Soontornniyomkij, Benchawanna Yang, Michael Yermanos, Alex Wren, Nina Moore, David J Achim, Cristian L R03 DA031591/DA/NIDA NIH HHS/United States U19 AI096113/AI/NIAID NIH HHS/United States Journal article Journal of neuroimmune pharmacology : the official journal of the Society on NeuroImmune Pharmacology J Neuroimmune Pharmacol. 2013 Mar 19.