Reliability Analysis Methods

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There are numerous sources of uncertainties that should be considered in engineering design. Reliability analysis methods provide a framework to account for these uncertainties in a rational manner. This chapter presents the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM). These methods are illustrated by academic examples.

1. Introduction

Engineering design aims at providing minimum levels of serviceability and safety during the structural lifetime. This is a difficult task because there are important sources of uncertainty that could lead to over- or under-design solutions. For example, there are uncertainties related to environmental exposure, loading, material properties, engineering models, etc. Reliability analysis methods offer the theoretical framework for considering uncertainties in a comprehensive decision scheme. The main goal of reliability analysis methods is to evaluate the ability of systems or components to remain safe and operational during their lifecycle.

The main objective of this chapter is to present and illustrate various reliability analysis methods that can be used for engineering or research purposes. The chapter starts with a description of fundamental concepts for reliability analysis. After, we present the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM). These methods are illustrated by academic examples.

2. Basic concepts for reliability analysis

Reliability methods have been established to take into account, in a rigorous manner, the uncertainties involved in the analysis of an engineering problem. The *failure probability* and the *reliability index* are used to quantify risks and therefore evaluate the consequences of failure. In this approach, the governing parameters of the prob-

lem are modeled as random variables. Random variables can be grouped in a random vector $\mathbf{X}$ where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF).

For reliability analysis, the space D of random variables may be divided into the *failure* and the *safety regions*. The failure region D_f is defined by $D_f = \{\mathbf{X} \mid g(\mathbf{X}) \leq 0\}$ and the *safety region*, D_s, by $D_s = \{\mathbf{X} \mid g(\mathbf{X}) > 0\}$ where $g(\mathbf{X})$ represents the performance function. Notice that $g(\mathbf{X})=0$ is the boundary between failure and safety regions and it is called the *limit state surface*.

In the simplest case, the performance function $g(\mathbf{X})$ is expressed as the difference between the resistance $R(\mathbf{X})$ and the demand or solicitation on the system $S(\mathbf{X})$ – i.e., $g(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X})$. In reliability engineering analysis, $g(\mathbf{X})$ is usually expressed in terms of displacement, strain, stress, etc. The performance functions can be related to the following structural conditions:

1. *Serviceability limit state*: under this condition, 'failure' is related to a serviceability loss that does not imply a significant decay of structural safety. For example, if the reliability analysis of a given structural component focuses on a maximum displacement v_{max}, the performance function can write:

$$g(X) = v_{max} - v(\mathbf{X}) \tag{1}$$

   where v_{max} could be fixed by standards or particular serviceability constraints and $v(\mathbf{X})$ is the displacement of the point of interest that depends on $\mathbf{X}$ random variables (material strength, geometry, load, etc.). In the case of failure, $v(\mathbf{X}) > v_{max}$, but the structural component is still considered safe.

2. *Ultimate limit state*: this condition describes the state at which structural safety is highly affected and may lead to total failure or collapse. For instance, if the reliability analysis focuses on the bending moment of a beam, the performance function is:

$$g(X) = M_r(\mathbf{X}) - M_s \tag{2}$$

   where $M_r(\mathbf{X})$ is the resistant bending moment of the beam that depends on $\mathbf{X}$ random variables (material strength, sectional geometry, etc.), and M_s is the soliciting bending moment. Notice that although M_s is assumed to be deterministic in eq. (2), this variable may also be considered as a random variable. In the case of failure, $M_s > M_r(\mathbf{X})$, leading to the collapse of the beam.

By accounting for these definitions, the failure probability, P_f, is determined by:

$$P_f = \mathrm{P}[g(\mathbf{X}) \leq 0] = \int_{g(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) dx_1 \dots dx_n \tag{3}$$

Notice that the limit state function can be a linear or a nonlinear function of the basic variables. FORM can be used to evaluate eq. (3) when the limit state function is a linear function of uncorrelated normal variables or when the nonlinear limit state function is represented by a first-order (linear) approximation with equivalent normal variables. SORM estimates the probability of failure by approximating the nonlinear limit state function (including a linear limit state function with correlated non-normal variables) by a second-order representation.

3. First Order Reliability Methods (FORM)

The First Order Reliability Method (FORM) makes use of the first and second moments of the random variables. This method includes two approaches [Hal00]. These are First-Order Second-Moment (FOSM) and Advanced First-Order Second-Moment (AFOSM) approaches. In FOSM, the information on the distribution of random variables is ignored; however, in AFOSM, the distributional information is appropriately used.

3.1. First Order Second Moment (FOSM)

The First Order Second Moment (FOSM) method makes use of only second-moment statistics (i.e. mean and standard deviation) of the random variables and it requires a linearized form of the performance function at the mean values of the random variables. A first-order Taylor series approximation is used to linearize the performance function at the mean values of the random variables.

Cornell [Cor69] proposed the original FOSM formulation. Let us consider an elementary reliability case where a weight is hung by a cable. The load-carrying capacity or the resistance of the cable is R and the load effect is S. The resistance and the load will be modeled as independent Gaussian random variables with $N(\mu_R, \sigma_R)$ and $N(\mu_S, \sigma_S)$, respectively. In this case, the failure probability P_f is related to the failure event $R < S$, and is computed as:

$$P_f = \mathrm{P}(R < S) = \mathrm{P}(R - S < 0) \tag{4}$$

A new random variable Z (called performance function) can be introduced:

$$Z = R - S \tag{5}$$

The performance function Z is characterized by a mean $\mu_Z = \mu_R - \mu_S$ and a standard deviation $\sigma_Z^2 = \sigma_R^2 + \sigma_S^2$. Since R and S are Gaussian, it can be demonstrated that Z also follows a Gaussian distribution. Figure 1 presents the PDF of Z. It is observed that the failure probability is related to the event $\mathrm{P}(Z < 0)$. Consequently, P_f could be estimated directly from:

$$P_f = \Phi[(0 - \mu_Z)/\sigma_Z] = \Phi[-\mu_Z/\sigma_Z] = \Phi[-\beta] \tag{6}$$

where $\Phi[.]$ represents the standard normal cumulative distribution function (CDF) and $\beta = \mu_Z / \sigma_Z$ is the 'reliability index' that is also used to quantify risks of failure.
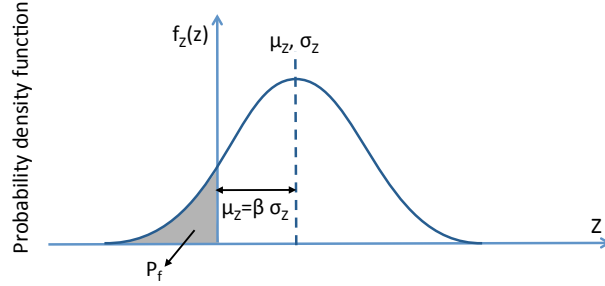

Figure 1: Probability density function of Z

As may be easily seen from Figure 1, the reliability index computed by FOSM represents the number of standard deviations that separate the mean value of the performance function from the limit state surface $Z = 0$.

For lognormal random variables, an alternative formulation to eq. (6) could be derived as follows: Assume that R and S are statically independent lognormal variables, that is, $LN(\lambda_R, \xi_R)$ and $LN(\lambda_S, \xi_S)$. In this case, another random variable Y can be introduced as

$$Y = R / S \tag{7}$$

or

$$\ln Y = Z = \ln R - \ln S \tag{8}$$

The failure event can be defined as $Y < 1.0$ or $Z < 0$. Since R and S are lognormal, $\ln R$ and $\ln S$ are normal; therefore, $\ln Y$ or Z is a normal random variable with mean $\lambda_R - \lambda_S$ and standard deviation $\sqrt{\xi_R^2 + \xi_S^2}$. The probability of failure can be defined (similar to eq. (6)) by

$$P_f = \Phi\left[\frac{0-(\lambda_R-\lambda_S)}{\sqrt{\xi_R^2+\xi_S^2}}\right] = \Phi\left[-\frac{\mu_Z}{\sigma_Z}\right] = \Phi[-\beta] \tag{9}$$

In the general case where the performance function Z is a function of a vector of n random variables, i.e.,

$$Z = g(\mathbf{X}) = g(X_1, X_2, \ldots, X_n) \tag{10}$$

the Taylor series expansion about the mean value gives:

$$Z = g(\mu_{\mathbf{X}}) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_{X_i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})(X_j - \mu_{X_j}) + \dots \quad (11)$$

where the derivatives are evaluated at the mean values of the random variables (X_1, $X_2,\dots, X_n$), and μ_{X_i} is the mean value of X_i. By truncating eq. (11) at the linear terms, it is possible to obtain first-order approximations of the mean and variance of Z as follows:

$$\mu_Z \approx g(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n}) \quad (12)$$

and

$$\sigma_Z^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial X_i} \frac{\partial g}{\partial X_j} \text{Cov}(X_i, X_j) \quad (13)$$

where $\text{Cov}(X_i, X_j)$ is the covariance of X_i and X_j. For uncorrelated random variables, the variance becomes:

$$\sigma_Z^2 \approx \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)^2 \text{Var}(X_i) \quad (14)$$

Consequently, by estimating μ_Z and σ_Z from eqs. (12) and (13), respectively, the reliability index can be computed as $\beta = \mu_Z / \sigma_Z$.

The exact failure probability could be derived from the reliability index only in few cases:
1. if all the X_i's are statically independent normal variables and if Z is a linear function of the X_i, then, Z is normal and the probability of failure is given by eq. (6);
2. if all the X_i's are statically independent lognormal variables and if $g(\mathbf{X})$ is a multiplicative function of the X_i's, then $Z = \ln g(\mathbf{X})$ is normal and the probability of failure is also given by $P_f = \Phi(-\beta)$.

To conclude, in most cases it is not likely that all the variables are statically independent normal or lognormal. Nor is it likely that the performance function is a simple additive or multiplicative function of these variables. In such cases, the reliability index cannot be directly related to the probability of failure; nevertheless, the equation $P_f = \Phi(-\beta)$ does provide a rough idea of the level of risk. Notice finally that FOSM approach has the following shortcomings:
- the information about the distribution of the independent variables is not considered,
- the truncation errors may be significant if $g(.)$ is non-linear,

- the assessment of reliability index varies when the limit state function is expressed in different but mechanically equivalent formulations – e.g., $(R - S = 0)$ or $(R / S = 1)$.

Example 1 *Reliability index of a steel beam using FOSM*

Let us consider a steel beam subjected to a deterministic bending moment of M_a =130 kNm. The yield stress F_y and the plastic modulus Z_p of the beam are considered as random variables following normal distributions with the following parameters:

$$\mu_{F_y} = 250 \text{ MPa}, \sigma_{F_y} = 25 \text{ MPa}$$

$$\mu_{Z_p} = 9 \times 10^{-4} \text{ m}^3, \sigma_{Z_p} = 4.5 \times 10^{-5} \text{ m}^3$$

Question 1: Write two different performance functions by considering strength and stress formulations.

For the strength formulation, the resistance of the beam is a random variable defined as $R = F_y Z_p$ and the solicitation S is deterministic where $S = M_a$. Then, the performance function becomes

$$g(F_y, Z_p) = R - S = F_y Z_p - M_a \tag{15}$$

For the stress formulation, the yield stress becomes the resistance of the performance function, i.e. $R = F_y$ and the solicitation is computed as $S = M_a/Z$. In this case, both R and S are random variables and the performance function becomes

$$g(F_y, Z_p) = R - S = F_y - \frac{M_a}{Z_p} \tag{16}$$

Question 2: Estimate the reliability index using the strength formulation.

We assume that the random variables are independent. We will first estimate μ_R and σ_R by using eqs. (12) and (14) respectively:

$$\mu_R \approx \mu_{F_y} \mu_{Z_p} = (250)(9 \times 10^{-4}) = 225 \text{kN} \cdot \text{m}$$

$$\sigma_R \approx \left[\text{Var}(F_y) \left(\frac{\partial R}{\partial F_y} \right)^2 + \text{Var}(Z_p) \left(\frac{\partial R}{\partial Z_p} \right)^2 \right]^{\frac{1}{2}} = \left[25^2 \left(9 \times 10^{-4} \right)^2 + \left(4.5 \times 10^{-5} \right)^2 250^2 \right]^{\frac{1}{2}}$$

$$= 25.16 \text{ kN} \cdot \text{m}$$

The sollicitation is deterministic, then, $\mu_S = M_a$ and $\sigma_S = 0$. Consequently, the reliability index is given by :

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} = \frac{225 - 130}{\sqrt{25.15^2 + 0^2}} = 3.77$$

Question 3: Estimate the reliability index using the stress formulation.

According to the performance function given by eq. (16), we obtain directly the mean and the standard deviation of the resistance: $\mu_R = \mu_{F_y}$ and $\sigma_R = \sigma_{F_y}$. For the sollicitation, μ_S and σ_S are computed from eqs. (12) and (14) respectively:

$$\mu_S \approx M_a / \mu_{Z_p} = 130 / (9 \times 10^{-4}) = 144.4 \text{ MPa}$$

$$\sigma_S \approx \left[\text{Var}(Z_p) \left(-\frac{M_a}{\mu_{Z_p}^2} \right)^2 \right]^{\frac{1}{2}} = \sigma_{Z_p} \frac{M_a}{\mu_{Z_p}^2} = 4.5 \times 10^{-5} \frac{130}{\left(9 \times 10^{-4}\right)^2} = 7.22 \text{ MPa}$$

And the reliability index becomes

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} = \frac{250 - 144.4}{\sqrt{25^2 + 7.22^2}} = 4.06$$

By comparing the reliability indexes, it is noted that the result depends on the formulation of the performance function. This lack of invariance motivated the development of other reliability methods such as that presented in the following sections.

3.2. Advanced First Order Second Moment (AFOSM)

AFOSM is also called 'Hasofer-Lind' method. In this method, the assessment of the reliability index is mainly based on the transformation/reduction of the problem to a standardized coordinate system. Thus, a random variable X_i is reduced as:

$$X_i' = (X_i - \mu_{X_1}) / \sigma_{X_1} \quad (i = 1, 2, ..., n) \tag{17}$$

where X_i' is a random variable with zero mean and unit standard deviation. Thus, Eq. (17) is used to transform the original limit state surface $g(\mathbf{X}) = 0$ into a reduced limit state surface $g(\mathbf{X'}) = 0$. Consequently, $\mathbf{X}$ denotes 'original coordinate system' and $\mathbf{X'}$ describes the 'transformed or reduced coordinate system'. In the standardized coordinate system, the Hasofer-Lind reliability index β_{HL} corresponds to the minimum distance from the origin of the axes (in the reduced coordinates system) to the limit state surface:

$$\beta_{HL} = \sqrt{(\mathbf{x'*})^T(\mathbf{x'*})} \tag{18}$$

The minimum distance point on the limit state surface is called the 'design point'. It is denoted by vector $\mathbf{x}^*$ in the original coordinate system and by vector $\mathbf{x'}^*$ in the reduced coordinate system. These vectors represent the values of all the random variables, *i.e.* X_1, X_2, …, X_n at the design point corresponding to the coordinate system being used.

Figure 2 illustrates the reduction of the random variables R and S for a linear performance function such as that described by eq. (5). According to eq. (17), R and S can be reduced as:

$$R' = (R - \mu_R)/\sigma_R \quad \text{and} \quad S' = (S - \mu_S)/\sigma_S \tag{19}$$

The substitution of R' and S' into the limit state surface ($Z=0$) gives the new limit state surface in the reduced coordinate system (Figure 2b):

$$Z = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S = 0 \tag{20}$$
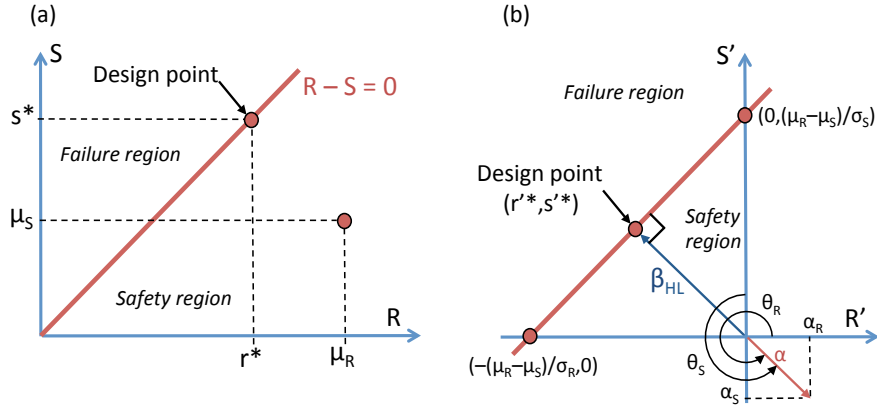

Figure 2: Reduction of coordinates: (a) original coordinates, (b) reduced coordinates

The reliability of the problem is estimated by using eq. (18). It corresponds to the minimum distance between the limit state surface and the origin (in the reduced coordinates system). By using simple trigonometry, this distance (reliability index) can be estimated as:

$$\beta_{HL} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{21}$$

It should be emphasized here that in the present case of a linear limit state surface, the Hasofer-Lind reliability index corresponds exactly to the reliability index com-

puted from FOSM if both R and S are normal variables. However, this is not the case for other limit state surfaces or random variable distributions.

From Figure (2b), it is apparent that if the limit state line is closer to the origin in the reduced coordinate system, the failure region is larger, and if it is farther away from the origin, the failure region is smaller. Thus, the position of the limit state surface relative to the origin in the reduced coordinate system is a measure of the reliability of the system. Notice that the Hasofer-Lind reliability index can be used to calculate the failure probability as $P_f = \Phi(-\beta_{HL})$. This is the integral of the standard normal density function along the ray joining the origin and $\mathbf{x'}^*$. It is obvious that the nearer $\mathbf{x'}^*$ is to the origin, the larger is also the most probable failure point. The point of minimum distance from the origin to the limit state surface, $\mathbf{x'}^*$, represents the worst combination of the stochastic variables and is appropriately named the 'design point' or the 'most probable point MPP' of failure.

Finally, it should be noted that the Hasofer-Lind reliability index is invariant, because regardless of the form in which the limit state equation is written [e.g., ($R–S=0$) or ($R/S=1$)], its geometric shape and the distance from the origin remain constant.

For the general case of a non-linear limit state surface, the assessment of the minimum distance can be written as an optimization problem:

$$
\begin{aligned}
&\text{minimize } D = \sqrt{\mathbf{x'}^T \mathbf{x'}} \\
&\text{subject to } g(\mathbf{x'}) = 0
\end{aligned}
\tag{22}
$$

By using Lagrange's multipliers, the minimum distance (for n random variables) could be estimated as:

$$
\beta_{HL} = -\frac{\displaystyle\sum_{i=1}^{n} x_i'^* \left(\frac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\displaystyle\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i'}\right)^{2*}}}
\tag{23}
$$

where $(\partial g/\partial X_i')^*$ is the ith partial derivative evaluated at the design point $(x_1'^*, x_2'^*, \ldots, x_n'^*)$. The design point in the reduced coordinates is:

$$
x_i'^* = -\alpha_i \beta_{HL} \quad (i = 1, 2, \ldots, n)
\tag{24}
$$

where α_i are the direction cosines along the coordinate axes X_i'. They are given by:

$$\alpha_i = \frac{\left(\dfrac{\partial g}{\partial X_i'}\right)^*}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_i'}\right)^{2*}}} \tag{25}$$

By using eq. (17), the design point (in the original space) is given by:

$$x_i^* = \mu_{X_i} - \alpha_i \sigma_{X_i} \beta_{HL} \tag{26}$$

An algorithm was formulated by [Rac76] to compute β_{HL} and $x_i'^*$ as follows:

Step 1: Define the appropriate performance function $g(\mathbf{X})$.

Step 2: Assume initial values for the design point $\mathbf{x}_i^*$. The initial design point is usually assumed to be at the mean value of the $n-1$ random variables. For the last random variable, its value is obtained from the performance function to ensure that the design point is located on the limit state surface $g(\mathbf{X}) = 0$.

Step 3: Obtain the design point in the reduced space $\mathbf{x}'^* = [x_1'^*, x_2'^*, \ldots, x_n'^*]$ as

$$x_i'^* = (x_i^* - \mu_{X_i}) / \sigma_{X_i} \quad (i = 1, 2, \ldots, n) \tag{27}$$

Step 4: Estimate the partial derivatives of the performance function $(\partial g/\partial X_i')^*$ with respect to the variables in the reduced space and evaluate them at $x_i'^*$. These derivatives can be estimated from the performance function in the original space by using the chain rule of differentiation

$$\frac{\partial g}{\partial X_i'} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial X_i'} = \frac{\partial g}{\partial X_i}\sigma_{X_i} \tag{28}$$

Define the column vector $\mathbf{A}$ such that

$$A_i = \left(\frac{\partial g}{\partial X_i'}\right)^* \tag{29}$$

Step 5: Compute the reliability index as

$$\beta_{HL} = -\frac{\mathbf{A}^T\mathbf{x}'^*}{\sqrt{\mathbf{A}^T\mathbf{A}}} \tag{30}$$

Step 6: Determine a vector of directional cosines as

$$\boldsymbol{\alpha} = \frac{\mathbf{A}}{\sqrt{\mathbf{A}^T \mathbf{A}}} \qquad (31)$$

Step 7: Obtain the new design point $x_i'^*$ for the $n-1$ random variables.

Step 8: Determine the coordinates of the new design point in the original space for the $n-1$ random variables considered in Step 7 as

$$x_i^* = \mu_{X_i} + x_i'^* \sigma_{X_i} \quad (i = 1,2,...,n) \qquad (32)$$

Step 9: Determine the value of the last random variable in the original space such that the estimated point belongs to the limit state surface $g(\mathbf{X}) = 0$.

Step 10: Repeat Steps 3 to 9 until convergence of β_{HL}.

Finally, notice that the Hasofer-Lind reliability index β_{HL} can be used to estimate a first-order approximation of the failure probability as $P_f \approx \Phi(-\beta_{HL})$.

Example 2 *Assessment of the reliability index by using FORM (Adapted from [San10])*

Let us suppose that the performance function of a problem is defined by

$$g(X_1, X_2, X_3) = 6.2\, X_1 - X_2\, X_3^2$$

where the random variables X_1, X_2, X_3 follow normal distributions with means $\mu_{X1} = 20$, $\mu_{X2} = 5$, and $\mu_{X3} = 4$; and standard deviations $\sigma_{X1} = 3.5$, $\sigma_{X2} = 0.8$, and $\sigma_{X3} = 0.4$.

Question 1: Estimate the reliability index by using FORM.

Once the performance function is defined (Step 1), we define the coordinates of the design point in the original space (Step 2):

$$\mathbf{x}^* = \left[x_1^* = \mu_{X_1}, x_2^* = \mu_{X_2}, x_3^* = \sqrt{\frac{6.2 x_1^*}{x_2^*}} \right] = [20, 5, 4.979]$$

The Step 3 consists of obtaining the design points in the reduced space:

$$x_1'^* = (x_1^* - \mu_{X_1})/\sigma_{X_1} = 0$$

$$x_2'^* = (x_2^* - \mu_{X_2})/\sigma_{X_2} = 0$$

$$x_3'^* = (4.979 - \mu_{X_3})/\sigma_{X_3} = 2.448$$

Then, $(\mathbf{x'}^*)^T = [0, 0, 2.448]$. In the Step 4, we estimate the vector containing the derivatives of the performance function evaluated at the design point (eqs. 28-29):

$$A_1 = \left(\frac{\partial g}{\partial X_1'}\right)^* = \left(\frac{\partial g}{\partial X_1}\right)^* \sigma_{X_1} = (6.2)(3.5) = 21.7$$

$$A_2 = \left(\frac{\partial g}{\partial X_2'}\right)^* = \left(\frac{\partial g}{\partial X_2}\right)^* \sigma_{X_2} = -(x_3^*)^2 \sigma_{X_2} = -19.84$$

$$A_3 = \left(\frac{\partial g}{\partial X_3'}\right)^* = \left(\frac{\partial g}{\partial X_3}\right)^* \sigma_{X_3} = -2x_2^* x_3^* \sigma_{X_3} = -19.92$$

Therefore, $\mathbf{A}^T = [21.7, -19.84, -19.92]$. The first estimate of the reliability index becomes (Step 5):

$$\beta_{HL} = -\frac{\mathbf{A}^T \mathbf{x}'^*}{\sqrt{\mathbf{A}^T \mathbf{A}}} = 1.374$$

We estimate the vector of directional cosines in Step 6:

$$\boldsymbol{\alpha} = \frac{\mathbf{A}}{\sqrt{\mathbf{A}^T \mathbf{A}}} = \begin{bmatrix} 0.611 \\ -0.559 \\ -0.561 \end{bmatrix}$$

The new design point in the reduced space is estimated for the $n - 1$ random variables (Step 7):

$$x_1'^* = -\alpha_1 \beta_{HL} = (-0.611)(1.374) = -0.840$$

$$x_2'^* = -\alpha_2 \beta_{HL} = (0.559)(1.374) = 0.768$$

Consequently, the corresponding values in the original space are (Step 8):

$$x_1^* = \mu_{X_1} + x_1'^* \sigma_{X_1} = 20 + (-0.840)3.5 = 17.06$$

$$x_2^* = \mu_{X_2} + x_2'^* \sigma_{X_2} = 5 + (0.768)0.8 = 5.614$$

We estimate afterwards the value of x_3^* such that it belongs to the limit state function (Step 9):

$$x_3^* = \sqrt{\frac{6.2 x_1^*}{x_2^*}} = 4.34$$

The algorithm is repeated until convergence of the reliability index (Step 10). Table 1 presents the results of various iterations. Convergence is reached after three iterations for this example. We found a reliability index $\beta_{HL} = 1.413$ that corresponds to a failure probability $P_f \approx 0.079$ computed based on a first-order approximation.

Table 1: Summary of the iterative process to estimate β_{HL}

Variable	Iteration Number		
	1	2	3
x_1^*	20	17.06	16.732
x_2^*	5	5.614	5.519
x_3^*	4.98	4.34	4.335
β_{HL}	1.374	1.413	1.413
x_1^*	17.06	16.732	16.709
x_2^*	5.614	5.519	5.521
x_3^*	4.34	4.335	4.332

3.2.1. Extension of AFOSM to the case of non-normal random variables

To extend the Hasofer-Lind method to the case of non-normal random variables, Rackwitz and Fiessler [Rac78] proposed to transform each non-normal random variable into an equivalent normal random variable with a mean $\mu_{X_i}^e$ and standard deviation $\sigma_{X_i}^e$. This transformation allows estimating a solution in the reduced space by using the procedure explained in the previous paragraphs. The equivalent parameters evaluated at the design point x_i^* are given by:

$$\mu_{X_i}^e = x_i^* - \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right]\sigma_{X_i}^e \tag{33}$$

$$\sigma_{X_i}^e = \frac{\phi\left[\Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right]\right]}{f_{X_i}\left(x_i^*\right)} \tag{34}$$

where $\Phi[.]$ and $\phi[.]$ are the CDF and the PDF of the standard variate, respectively, and $F_{X_i}(.)$ and $f_{X_i}(.)$ are the CDF and PDF of the original non-normal random vari-

able (some useful Matlab commandes may be found in Appendix). Notice that eqs. (33-34) are derived by equating the cumulative distribution functions and the probability density functions of the actual variables and the equivalent normal variables at the design point on the limit state surface.

Since the equivalent parameters are evaluated at the design point, each iteration should include the assessment of the equivalent parameters. The before presented algorithm is thus modified as follows:

- Steps 1 and 2 remain similar.

- Step 3 should include the assessment of the equivalent parameters $\mu_{X_i}^e$ and $\sigma_{X_i}^e$ at the design point for each non-normal random variable. These equivalent parameters will be used to determine the design point in the reduced space as follows

$$x_i'^* = (x_i^* - \mu_{X_i}^e)/\sigma_{X_i}^e \qquad (35)$$

- Steps 4 to 7 remain exactly similar.

- In Step 8, the assessment of the coordinates in the original space becomes

$$x_i^* = \mu_{X_i}^e + x_i'^* \sigma_{X_i}^e \qquad (36)$$

- Finally, Steps 9 and 10 remain similar.

3.3. Ellipsoid Approach

The Hasofer-Lind reliability index can be formulated in a matrix form:

$$\beta_{HL} = \min_{g(\mathbf{X})=0} \sqrt{(\mathbf{X}-\boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{X}-\boldsymbol{\mu})} \qquad (37)$$

where $\mathbf{X}$ is a vector containing the n random variables, $\boldsymbol{\mu}$ is a vector containning their mean values, and $\mathbf{C}$ is the covariance matrix.

An intuitive interpretation of the reliability index was suggested in Low and Tang (cf. [Low97] and [Low04]), where the concept of an expanding ellipse (Figure 3) led to a simple method for computing the Hasofer-Lind reliability index in the original space of the random variables using an optimization tool available in most spreadsheet software packages.

When there are only two uncorrelated non-normal random variables X_1 and X_2, these variables span a two-dimensional random space, with an equivalent one-sigma dispersion ellipse (corresponding to $\beta_{HL}=1$ in Eq. (37) without the min.), centered at the equivalent mean values μ_1^N and μ_2^N and whose axes are parallel to the coordinate axes of the original space. For correlated variables, a tilted ellipse is obtained.

Low and Tang (cf. [Low97] and [Low04]) reported that the Hasofer-Lind reliability index β_{HL} may be regarded as the codirectional axis ratio of the smallest ellipse (which is either an expansion or a contraction of the $1-\sigma$ ellipse) that just touches the limit state surface to the $1-\sigma$ dispersion ellipse. They also stated that finding the smallest ellipse that is tangent to the limit state surface is equivalent to finding the most probable failure point.
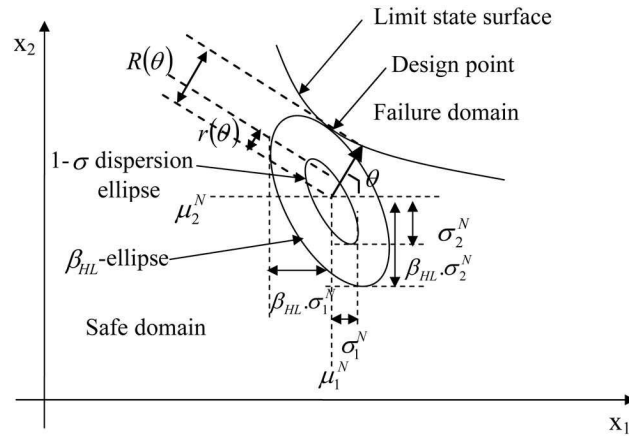

Figure 3: Ellipsoid approach for the computation of the Hasofer-Lind reliability index

3.3.1. Ellipsoid Approach via Spreadsheet

Low and Tang (cf. [Low97] and [Low04]) showed that the minimization of the Hasofer-Lind reliability index can be efficiently carried out in the Excel spreadsheet environment. The spreadsheet approach is simple and easy to understand because it works in the original space of random variables and does not require the additional step of transforming **X** to **X**' where **X**' is a transformed vector of the random variables in the uncorrelated Gaussian space. Notice however that the optimization in original space is not preferred from a computational perspective. This is because the optimization in a standardized space is mathematically more desirable in nonlinear optimization. For example, when minimizing the quadratic form of Eq. (37) in the original space; in some cases, the correct solution is obtained only when the solver option "use automatic scaling" is activated. As an alternative, Cholesky factorization of the covariance matrix can be used.

When the random variables are non-normal, the Rackwitz-Fiessler equivalent normal transformation can be used to compute the equivalent normal mean μ^e and the equivalent normal standard deviation σ^e. The iterative computations of μ^e and σ^e for each trial design point are automatic during the constrained optimization search. For non-normal random variables, Eq. (37) may be rewritten as

$$\beta_{HL} = \min_{g(\mathbf{X})=0} \sqrt{\left(\frac{\mathbf{X}-\mathbf{\mu}^e}{\mathbf{\sigma}^e}\right)^T \mathbf{R}^{-1} \left(\frac{\mathbf{X}-\mathbf{\mu}^e}{\mathbf{\sigma}^e}\right)} \tag{38}$$

where $\mathbf{\mu}^e$ and $\mathbf{\sigma}^e$ are vectors containining the equivalent mean and standard deviations values, respectively, and $\mathbf{R}^{-1}$ is the inverse of the correlation matrix. This equation will be used instead of Eq. (37) since the correlation matrix $\mathbf{R}$ displays the correlation structure more explicitly than the covariance matrix $\mathbf{C}$.

Additional information on Solver's options and algorithms can be found in the Microsoft Excel Solver's help file and at www.solver.com. The implementation procedure of the ellipsoid approach in the spreadsheet is described in [Low97] and [Low04] among others. Some Excel files are available at http://alum.mit.edu/www/bklow.

3.4. Response Surface Method (RSM)

In case of analytically-unknown system response (such as the responses computed using a finite element/finite difference method), several approaches based on the Response Surface Method (RSM) can be found in the literature with the aim of calculating the reliability index and the corresponding design point. We present herein the approach by Tandjiria et al. [Tan00]. The basic idea of this method is to approximate the system response $Y(x)$ by an explicit function of random variables, and to improve the approximation *via* iterations. The system response can be approximated (in the original space of random variables) by:

$$Y(x) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i^2 \tag{39}$$

where x_i are the random variables (μ_i and σ_i being their mean and standard deviation values, respectively); n is the number of random variables; and (a_i, b_i) are coefficients to be determined. The RSM algorithm is summarized as follows:

Step 1: Evaluate the system response $Y(x)$ at the mean value point μ and at the $2n$ points each at $\mu \pm k\sigma$ where k can be arbitrarily chosen ($k = 1$).

Step 2: The above $2n+1$ values of $Y(x)$ are used to solve eq. (39) and find the coefficients (a_i, b_i). Then, the performance function g(x) can be constructed to give a tentative response surface function.

Step 3: Solve eq. (37) to obtain a tentative design point and a tentative β_{HL} subjected to the constraint that the tentative performance function of step 2 be equal to zero.

Step 4: Repeat Steps 1 to 3 until convergence of β_{HL}. Each time, Step 1 is repeated, the $2n + 1$ sampled points are centered at the new tentative design point of Step 3.

Concerning the numerical implementation of the RSM algorithm described above, the determination of β_{HL} requires (i) the resolution of eq. (39) for the $2n+1$ sampled points, and (ii) the minimization of β_{HL} given by Eq. (37). These two operations constitute a single iteration and can be done using the optimization toolbox available in Excel. Several iterations should be performed until convergence of β_{HL}. Convergence is considered to be reached when the absolute difference between two successive values of the reliability index becomes smaller than 10^{-2}.

4. Second Order Reliability Method (SORM)

Reliability assessment is relatively simple if the limit state function is linear. However, most of the limit state functions are nonlinear. The nonlinearity is due to nonlinear relationship between random variables, to the consideration of non-normal random variables, and/or to the transformation from correlated to uncorrelated random variables. Indeed, a linear limit state in the original space becomes non-linear when transformed to the standard normal space if any of the variables is non-normal. Also, the transformation from correlated to uncorrelated variables might induce nonlinearity.

Figure 4 presents examples of linear and nonlinear limit state functions in the reduced space. Both limit state functions have the same minimum distance point β, but the failure regions are different in both cases. The failure probability of the nonlinear limit state should be less than that of the linear limit state. The FORM approach approximates the limit state function with a linear function and will therefore provide the same assessment of the probability of failure for both cases. This approximation introduces errors in the assessment of the probability of failure. Consequently, it is preferable to use a higher order approximation for the failure probability computation.

The SORM method improves the assessment given by FORM by including information about the curvature (which is related to the second-order derivatives of the limit state function with respect to the basic variables). The Taylor series expansion of a general nonlinear function $g(X_1, X_2,\ldots, X_n)$ at the design point $(x^*_1, x^*_2,\ldots, x^*_n)$ is given by:

$$g(X_1, X_2, ..., X_n) = g(x_1^*, x_2^*, ..., x_n^*) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(x_i - x_i^*)$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(x_i - x_i^*)(x_j - x_j^*) + ...$$
(40)

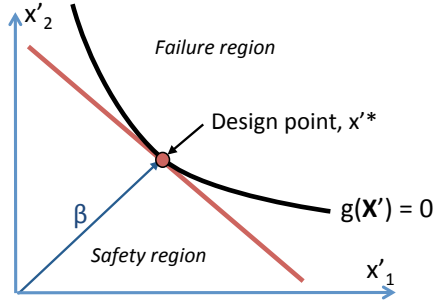where the derivatives are evaluated at the design point.

Figure 4: Linear and nonlinear limit state functions

For reliability analysis, the space of standard normal variables is more convenient for a second order approximation of $g()$. In the following, X_i and Y_i will refer to random variables in the original and equivalent uncorrelated standard normal spaces, respectively. If all the variables are uncorrelated, $Y_i = (X_i - \mu_{X_i}^e)/\sigma_{X_i}^e$ where $\mu_{X_i}^e$ and $\sigma_{X_i}^e$ are the equivalent normal mean and standard deviation of X_i at the design point x_i^*.

In the Taylor series approximation given by eq. (40), FORM ignores the terms beyond the first order term, and SORM ignores the terms beyond the second-order term (involving second-order derivatives).

Breitung [Bre84] proposed a simple closed-form solution for the probability computation using the theory of asymptotic approximation as:

$$P_f \approx \Phi(-\beta_{\text{FORM}}) \prod_{i=1}^{n-1} (1 + \beta_{\text{FORM}} \kappa_i)^{-1/2}$$
(41)

where κ_i represents the principal curvatures of the limit state function at the minimum distance point, and β_{FORM} is the reliability index computed by the FORM method.

The assessment of P_f requires the computation of κ_i. Towards this aim, the random variables Y_i (in the $\mathbf{Y}$ reduced space) are rotated to another set of variables Y_i', such that the last Y_i' variable coincides with the vector $\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is the unit gradient vector of the limit state at the minimum distance point.

Figure 5 describes the problem for two random variables indicating that the problem consists of a simply rotation of coordinates. This rotation can be carried out by an orthogonal transformation:

$$\mathbf{Y}' = \mathbf{RY} \tag{42}$$

where $\mathbf{R}$ is the rotation matrix. For instance, in the case of two random variables:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{43}$$

where θ is the counterclockwise angle of rotation of the axes (Figure 5). For n random variables, the reader may refer to [Hal00] for the determination of the matrix $\mathbf{R}$.
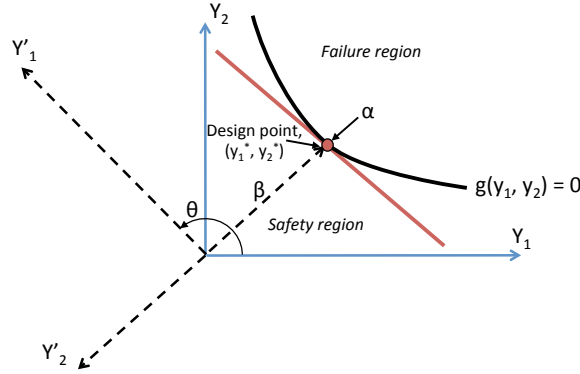

Figure 5: Rotation of axis in the standardized space

The matrix $\mathbf{R}$ is after used to estimate a matrix $\mathbf{A}$, whose elements are denoted a_{ij}, as follows:

$$a_{ij} = \frac{\left(\mathbf{RDR}^T\right)_{ij}}{\left|\nabla G(\mathbf{y}^*)\right|}, \quad i,j = 1,2,...,n-1 \tag{44}$$

where $\mathbf{D}$ is the $n \times n$ second-derivative matrix of the limit state function in the standard normal space evaluated at the design point and $|\nabla G(\mathbf{y}^*)|$ is the length of the gradient vector in the standard normal space.

The last variable Y_n coincides with the β vector computed with the FORM approach in the rotated normal space. In the next step, the last row and last column in the **A** matrix and the last row in the **Y'** vector are dropped to take this factor into account. The limit state can be rewritten in terms of a second-order approximation in this rotated standard normal space **Y'** as:

$$y'_n = \beta + \frac{1}{2} \mathbf{y'}^T \mathbf{A} \mathbf{y'} \tag{45}$$

where the matrix **A** has now the size $(n-1) \times (n-1)$. Afterwards, the curvatures κ_i of eq. (41) are computed as the eigenvalues of the matrix **A**, to estimate the probability of failure.

Breitung's SORM method uses a parabolic approximation (it does not consider a general second order approximation) by ignoring the mixed terms and their derivatives in the Taylor series approximation in eq. (40). This approach uses the theory of asymptotic approximation to derive the probability estimate. This approximation is accurate for large values of β (which is the case for engineering purposes). However, the assessment is less accurate for smaller β.

Example 3   *Assessment of the reliability index by using SORM (Adapted from [San10])*

Suppose that the performance function of a problem is defined by

$$g(X_1, X_2) = X_1 X_2 - 80$$

where X_1 follows a normal distribution with mean $\mu_{X_1} = 20$ and standard deviation $\sigma_{X_1} = 2$. X_2 follows a lognormal distribution with mean $\mu_{X_2} = 7$ and standard deviation $\sigma_{X_2} = 1.4$.

The results of the FORM approach provide a reliability index $\beta_{FORM} = 2.402$ and the two following vectors x^* and α:

$$\mathbf{x}^* = (17.612, 4.542)$$
$$\boldsymbol{\alpha} = (0.497, 0.868)$$

where x^* is the final design point in the original space and α is the vector of direction cosines.

Question 1: Estimate the probability of failure by using SORM and compare the results with the probability of failure estimated from FORM.

We already have the results of the FORM approach. We will transform X_1 and X_2 from the original to the standard normal space. Since X_2 is lognormal, we use the Rackwitz and Fiessler procedure to estimate the equivalent parameters:

$$\sigma_{X_2}^e = \frac{\phi\left[\Phi^{-1}\left[F_{X_2}\left(x_2^*\right)\right]\right]}{f_{X_2}\left(x_2^*\right)} = \frac{\phi\left[\Phi^{-1}\left[F_{X_2}\left(4.542\right)\right]\right]}{f_{X_2}\left(4.542\right)} = 0.9$$

$$\mu_{X_2}^e = x_2^* - \Phi^{-1}\left[F_{X_2}\left(x_2^*\right)\right]\sigma_{X_2}^e = 4.542 - \Phi^{-1}\left[F_{X_2}\left(4.542\right)\right]0.9 = 6.418$$

Transforming X_1 and X_2 from the original to the reduced space gives:

$$Y_1 = (X_1 - \mu_{X_1})/\sigma_{X_1}$$

$$Y_2 = (X_2 - \mu_{X_2}^e)/\sigma_{X_2}^e$$

Then the values of the design point for each random variable in the reduced space are:

$$y_1^* = (17.612 - 20)/2 = -1.194$$

$$y_2^* = (4.542 - 6.418)/0.9 = -2.085$$

The first step in SORM is to determine the matrix $\mathbf{R}$ from eq. (43). In the rotated coordinates, the second coordinate need to coincide with the unit gradient vector $\boldsymbol{\alpha}$. Figure 6 presents the example in the reduced space including the performance function and the design point. It also includes the geometrical description of the assessment of θ. Thus, it can be noted that $\theta = 270° + \tan^{-1}(-2.085/-1.194) = 330.194°$, and $\mathbf{R}$ becomes

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.868 & -0.497 \\ 0.497 & 0.868 \end{bmatrix}$$

Notice that the elements of $\mathbf{R}$ are also easily available from the direction cosines, i.e. the components of the unit gradient vector $\boldsymbol{\alpha}$.
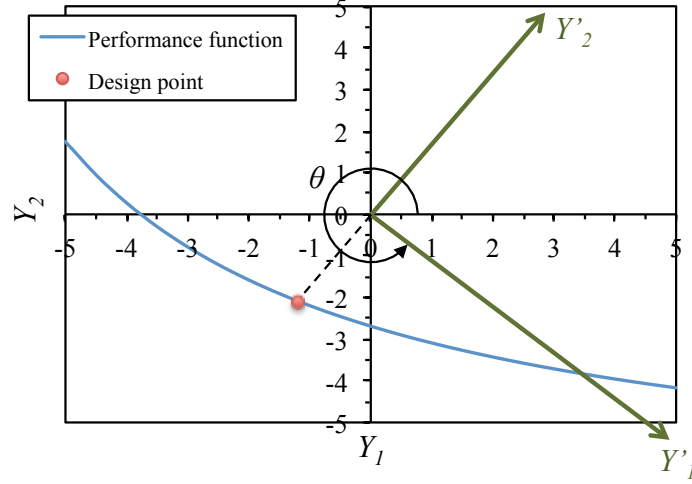
Figure 6: Rotation of the axis in the reduced space

The next step is to construct the matrix $\mathbf{D}$ that contains the second derivatives of the performance function with respect to each variable in the standard normal space. We use the chain rule of differentiation for determining the derivatives from the limit state function in the original space

$$\frac{\partial^2 g}{\partial Y_1^2} = \frac{\partial}{\partial X_1}\left[\frac{\partial g}{\partial X_1}\frac{\partial X_1}{\partial Y_1}\right]\frac{\partial X_1}{\partial Y_1} = \frac{\partial}{\partial X_1}\left[X_2\sigma_{X_1}\right]\sigma_{X_1} = 0$$

$$\frac{\partial^2 g}{\partial Y_2^2} = \frac{\partial}{\partial X_2}\left[\frac{\partial g}{\partial X_2}\frac{\partial X_2}{\partial Y_2}\right]\frac{\partial X_2}{\partial Y_2} = \frac{\partial}{\partial X_2}\left[X_1\sigma_{X_2}^e\right]\sigma_{X_2} = 0$$

$$\frac{\partial^2 g}{\partial Y_1 Y_2} = \frac{\partial}{\partial X_1}\left[\frac{\partial g}{\partial X_2}\frac{\partial X_2}{\partial Y_2}\right]\frac{\partial X_1}{\partial Y_1} = \frac{\partial}{\partial X_1}\left[X_1\sigma_{X_2}^e\right]\sigma_{X_1} = \sigma_{X_2}^e\sigma_{X_1}$$

Consequently

$$\mathbf{D} = \begin{bmatrix} \dfrac{\partial^2 g}{\partial Y_1^2} & \dfrac{\partial^2 g}{\partial Y_1 Y_2} \\ \dfrac{\partial^2 g}{\partial Y_1 Y_2} & \dfrac{\partial^2 g}{\partial Y_2^2} \end{bmatrix} = \begin{bmatrix} 0 & \sigma_{X_2}^e\sigma_{X_1} \\ \sigma_{X_2}^e\sigma_{X_1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1.799 \\ 1.799 & 0 \end{bmatrix}$$

We estimate the following partial derivatives for the assessment of $\mathbf{A}$

$$\frac{\partial g}{\partial Y_1} = \frac{\partial g}{\partial X_1}\frac{\partial X_1}{\partial Y_1} = X_2\sigma_{X_1}$$

$$\frac{\partial g}{\partial Y_2} = \frac{\partial g}{\partial X_2}\frac{\partial X_2}{\partial Y_2} = X_1\sigma_{X_2}^e$$

These derivatives evaluated at the design point are

$$\nabla G(\mathbf{y}^*) = \begin{bmatrix} x_2^*\sigma_{X_1} \\ x_1^*\sigma_{X_2}^e \end{bmatrix} = \begin{bmatrix} (17.612)(2) \\ (4.542)(0.9) \end{bmatrix} = \begin{bmatrix} 35.223 \\ 4.086 \end{bmatrix}$$

and the length of the vector is

$$\left|\nabla G(\mathbf{y}^*)\right| = \sqrt{35.223^2 + 4.086^2} = 35.46$$

Therefore by applying eq. (44), the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} -0.044 & 0.026 \\ 0.026 & 0.044 \end{bmatrix}$$

Afterwards, we modify the matrix $\mathbf{A}$ by deleting the last row and column. In this case, we have only one element $a_{11} = -0.044$. Then, the eigenvalue of this one element matrix is simply the same element: $\kappa_1 = a_{11} = -0.044$.

The approximation of the failure probability is estimated from eq. (41)

$$P_f \approx \Phi(-\beta_{\text{FORM}})\prod_{i=1}^{n-1}(1 + \beta_{\text{FORM}}\kappa_i)^{-1/2}$$

$$\approx \Phi(-2.402)\frac{1}{\sqrt{1 + (2.402)(-0.044)}} \approx 8.609 \cdot 10^{-3}$$

The probability of failure estimated from FORM is $P_f = \Phi(-2.402) = 8.144 \cdot 10^{-3}$. The probability of failure estimate from the SORM procedure is about 5% larger. This result is expected because of the curvature of the performance function (Figure 6). ∎

5. Conclusions

This chapter presented the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM). The First Order Reliability Method (FORM) makes use of the first and second moments of the random variables. This method includes two approaches. These are First-Order Second-Moment (FOSM) and Ad-

vanced First-Order Second-Moment (AFOSM) approaches. In FOSM, the information on the distribution of random variables is ignored; however, in AFOSM (called also Hasofer-Lind approach), the distributional information is appropriately used. It was shown that contrary to FOSM, the Hasofer-Lind method led to an invariant reliability index regardless of the form in which the limit state equation is written.

While AFOSM requires the transformation of the limit state surface to a standard space of random variables, the recent ellipsoid approach led to a simple method for computing the Hasofer-Lind reliability index in the original space of random variables using an optimization tool available in most spreadsheet software packages.

In case of analytically-unknown system response (i.e. when the system response is computed using a finite element/finite difference method), the Response Surface Method (RSM) can be used to calculate the Hasofer-Lind reliability index and the corresponding design point. The basic idea of this method is to approximate the system response by an explicit function of random variables, and to improve the approximation via iterations.

For the computation of the failure probability, it was shown that the Hasofer-Lind reliability index can be used to evaluate the failure probability when the limit state function is a linear function of uncorrelated normal variables or when the nonlinear limit state function is represented by a first-order (linear) approximation with equivalent normal variables. SORM estimates the probability of failure by approximating the nonlinear limit state function (including a linear limit state function with correlated non-normal variables) by a second-order representation.

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Appendix: Summary of Matlab® functions for the normal distribution

Function	Description
normpdf(X,mu,sigma)	computes the PDF at each of the values in X using the normal distribution with mean mu and standard deviation sigma. X, mu, and sigma can be vectors, matrices, or multidimensional arrays that all have the same size. A scalar input is expanded to a constant array with the same dimensions as the other inputs. The parameters in sigma must be positive.
normcdf(X)	returns the standard normal CDF at each value in X. The standard normal distribution has parameters mu = 0 and sigma = 1.
norminv(P,mu,sigma)	computes the inverse of the normal CDF using the corresponding mean mu and standard deviation sigma at the corresponding probabilities in P. P, mu, and sigma can be vectors, matrices, or multidimensional arrays that all have the same size. A scalar input is expanded to a constant array with the same dimensions as the other inputs. The parameters in sigma must be positive, and the values in P must lie in the interval [0 1].