NUEN 647
Uncertainty Quantification for Nuclear Engineering
Assignment 1

Due on Tuesday, October 4, 2016

Dr. McClarren

Paul Mendoza

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Complete the exercises in the Chapter 2 notes. Be sure to include discussion of results where appropriate. You may use any tools that are approrpriate to solving the problem.

Problem 1

Show that the transformation in equation 1 results in a standard normal random variable by computing the mean and variance of z.

$$z = \frac{x-\mu}{\sigma} \tag{1}$$

An important special case of the expectation value is the mean which is the expected value of x. It is often denoted as μ,

$$\mu = E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

where x is a realization of a random sample and $f(x)$ is the probability density function (PDF) for the random variable. For a normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

For the sake of the transformation, the value of z substitutes for x, the realization of a random sample (not the PDF because we are transforming that distribution). Therefore, the mean for z is:

$$\mu_z = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

If $u = (x-\mu)^2$ and $\frac{du}{2} = (x-\mu)dx$ (note that the limits change from $(-\infty, \infty)$ to (∞, ∞) - but that seems fishy to me so I will change it back after integration).

$$\mu_z = \int_{\infty}^{\infty} \frac{1}{2\sigma^2\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} du = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-u}{2\sigma^2}} \right|_{\infty}^{\infty}$$

$$\mu_z = \left| \frac{-1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \right|_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(e^{-\infty} - e^{-\infty}) = \boxed{0}$$

The variance is defined as:

$$Var(X) = E[(X-\mu)^2]$$

Substituting Eq. 1 for X, (but not for the pdf - I could be wrong about that)

$$Var(X) = E[(\frac{x-\mu}{\sigma} - \mu)^2] = E\left[\left(\frac{x-\mu-\mu\sigma}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}(E[x^2] - 2\mu E[x] - 2\mu\sigma E[x] + \mu^2 E[1] + \mu^2\sigma^2 + 2\mu^2\sigma)$$

Noting that above it was proven that $E[x] = \mu$ and given that the definition of $E[1] = 1$ and assuming that $E[x^2] = \sigma^2 + \mu^2$ (will solve on next page)

$$\frac{1}{\sigma^2}(\sigma^2 + \mu^2 - 2\mu^2 - 2\mu^2\sigma + \mu^2 + \mu^2\sigma^2 + 2\mu^2\sigma) = \frac{1}{\sigma^2}(\sigma^2 + \mu^2\sigma^2) = \boxed{1 + \mu^2 = 1}$$

This is assuming that $\mu = 0$. Which was shown above.

$$E[x^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

If $t = \frac{(x-\mu)}{\sqrt{2}\sigma}$ and $\sqrt{2}\sigma dt = dx$ and $x = t\sqrt{2}\sigma + \mu$ then (limits of integration don't change)

$$E[x^2] = \int_{-\infty}^{\infty} \frac{\left(t\sqrt{2}\sigma + \mu\right)^2}{\sqrt{\pi}} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(2\sigma^2 \left(t^2 e^{-t^2}\right) + 2\sqrt{2}\sigma\mu \left(te^{-t^2}\right) + \mu^2 \left(e^{-t^2}\right)\right)$$

According to wolfram alpha

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} te^{-t^2} = 0$$

$$\int_{-\infty}^{\infty} e^{-t^2} = \sqrt{\pi}$$

Which simplifies the above to $\sigma^2 + \mu^2$.

Problem 2

Consider the random variables $X \sim U(-1, 1)$ and $Y \sim X^2$. Are these independent random variables? What is their covariance?

If two random variables, X and Y, are independent, they satisfy the following condition: ^{link}

- $P(X|Y) = P(X)$, for all values of X and Y.

The PDF for X is:

$$f_X(x) = \frac{1}{(1 - (-1))} = 0.5 \quad x \in [-1, 1]$$

The PDF for Y is: ^{link}

$$f_Y(y) = \frac{1}{2\sqrt{y}} \quad y \in [0, 1]$$

If the covariance is non zero, then these two variables are independant. The covariance of two random variables can be given by: ^{link}

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \int_{-1}^{1} dx \int_{0}^{1} dy \;\; xyf(x, y) - \mu_X\mu_Y$$

Because $\mu_X = 0$ this reduces to

$$\sigma_{XY} = E(XY) = \int_{-1}^{1} dx \int_{0}^{1} dy \;\; xyf(x, y)$$

Where $f(x, y)$ is:

$$f(x, y) = f(y|x)f_X(x)$$

From the definition of Y, $f(y|x)$ is 0 except when $y = x^2$. I think this would be.

$$f(y|x) = \delta(y - x^2)$$

Which means,

$$f(x, y) = 0.5\delta(y - x^2)$$

I am not 100% sure to do here, but after using wolfram, I think its 0.25.

Problem 3

Show that a general covariance matrix must be positive definite, i.e. $\vec{x}^T \Sigma \vec{x} > 0$ for any vector $\vec{x}$ that is not all zeros.

Given that $\vec{Y}$ is a vector of random variables and $\vec{\mu}_Y$ is a vector of the mean values for the random variables found in $\vec{Y}$.

$$\begin{aligned}\vec{x}^T \Sigma \vec{x} &= \vec{x}^T E[(\vec{Y} - \vec{\mu}_Y)(\vec{Y} - \vec{\mu}_Y)^T]\vec{x} \\ &= E[\vec{x}^T(\vec{Y} - \vec{\mu}_Y)(\vec{Y} - \vec{\mu}_Y)^T \vec{x}]\end{aligned}$$

The last step above puts a constant inside the expectation value integral. Notice

$$\vec{x}^T(\vec{Y} - \vec{\mu}_Y) = (\vec{Y} - \vec{\mu}_Y)^T \vec{x}$$

and that both are scaler functions of the random variables. Therefore,

$$\begin{aligned}\vec{x}^T \Sigma \vec{x} &= E[(\vec{x}^T(\vec{Y} - \vec{\mu}_Y))^2] \\ &= E[g(Y)^2] = \sigma_f^2\end{aligned}$$

The expectation value for a multivariate distribution is defined as

$$E[g(Y)] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \; ... \int_{-\infty}^{\infty} dy_p \; g(y) f(y)$$

Where $f(y)$ is the multivariate PDF for the random variables of $\vec{Y}$. To prove that the integral

This is an example citation [1].

References

[1] E. T. Tatro, S. Hefler, S. Shumaker-Armstrong, B. Soontornniyomkij, M. Yang, A. Yermanos, N. Wren, D. J. Moore, and C. L. Achim. Modulation of bk channel by microrna-9 in neurons after exposure to hiv and methamphetamine. *J Neuroimmune Pharmacol*, 2013. Tatro, Erick T Hefler, Shannon Shumaker-Armstrong, Stephanie Soontornniyomkij, Benchawanna Yang, Michael Yermanos, Alex Wren, Nina Moore, David J Achim, Cristian L R03 DA031591/DA/NIDA NIH HHS/United States U19 AI096113/AI/NIAID NIH HHS/United States Journal article Journal of neuroimmune pharmacology : the official journal of the Society on NeuroImmune Pharmacology J Neuroimmune Pharmacol. 2013 Mar 19.