16/03 2025

Topological Space

Space on which limits, convergence, neighbourhoods is defined

Definition approaches:
- metric
- sequence → for algorithms
- open/closed → simplest sets

Open and closed sets are duals — you generally get one's properties from the other

$d: X\times X \to \mathbb{R}_+ = [0,\infty)$ → A metric space is (X,d)
- $d(x,y)=0$ if $x=y$
- $d(x,y) = d(y,x)$
- $d(x,z) \le d(x,y) + d(y,z)$

→ can also be discrete: $d(x,y) = \mathbb{1}(x\neq y)$

On S^1: $d(\vartheta,\eta) = |e^{i\vartheta}-e^{i\eta}| = \sqrt{2}\sqrt{1-\cos(\vartheta-\eta)}$

Cantor space: $(\{0,1\}^{\mathbb{N}}, d)$, $d(a,b) = \sum_{n=1}^{\infty} 2^{-n}|a_n-b_n|$

Hilbert space: $\left(\{a:\mathbb{N}\to\mathbb{R} \mid \sum|a_n|^2<\infty\}, \ell_2\right)$, $d(a,b) = \sqrt{\sum_{n=1}^{\infty}|a_n-b_n|^2}$

$(\mathcal{C}([a,b]), d)$ — $\mathcal{C}^0$ functions on $[a,b]$, $d(f,g) = \max_{x\in[a,b]}|f(x)-g(x)|$

Note that any subset of a metric space is a metric space — metric space requires no closure

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From a Metric, get a Neighbourhood:

→ Open ball / (closed ball: use $\le$)

$x_0\in X$, $r>0$ $B(x_0,r) := \{x\in X : d(x,x_0)<r\}$ → Take $(\mathbb{Q},d)$

→ Neighbourhood:

$U\subset X$ is ... of x_0 if it contains a $B(x_0,r)$ for $r>0$ — Any superset of open ball $B(x_0,r)$ is a neighbourhood

Equivalent Metrics

$d_1\sim d_2$ iff $\begin{cases} A \text{ open under } d_1 \\ \Updownarrow \\ A \text{ open under } d_2\end{cases}$

Equivalently, A is $\mathcal{N}(x)$ with d_1 iff with d_2

$\mathcal{B}_1 \sim \mathcal{B}_2$ bases

$\forall x\in X$

for $B_1\in\mathcal{B}_1(x)$, $\exists B_2\in\mathcal{B}_2 : B_2\subset B_1$ — and vice versa

→ you can "sandwich" the bases

Neighbourhood "Algebra"

$\mathcal{N}(x)$ collection of neighbourhoods of x.
- $U\in\mathcal{N}(x) \Rightarrow x\in U$ — always contain x
- $U\in\mathcal{N}(x), U\subset V \Rightarrow V\in\mathcal{N}(x)$ — nested outwardly
- $U,V\in\mathcal{N}(x) \Rightarrow U\cap V\in\mathcal{N}(x)$ — closed in intersection
- $\forall U\in\mathcal{N}(x), \exists V\subset U : \forall y\in V,\ U\in\mathcal{N}(y)$

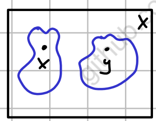
(you can find some V small enough that the original U is one of $\mathcal{N}(y)$ $\forall y\in V$) — (you can find a subset close enough to x so U is a $\mathcal{N}(y)$ for all $y\in$ subs.)

Separation Property

$\forall x,y\in X$, $x\neq y$

$\exists V\in\mathcal{N}(x), V\in\mathcal{N}(y)$:

$U\cap V\neq\emptyset$



Neighbourhood Basis

$\mathcal{B}\subset\mathcal{N}(x)$ st.

$\forall V\in\mathcal{N}(x), \exists U\in\mathcal{B}$ with $U\subset V$

Infinite restriction around x:

→ $\{B(x,r), r>0\}$

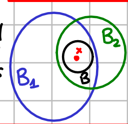
→ $\{B[x,r], r>0\}$

(Note it does not need to be efficient)

$\mathcal{B}$ basis of topology $\mathcal{C}$

- $\forall x\in X$, $\exists B\in\mathcal{B} : x\in B$
- if $B_1,B_2\in\mathcal{B}$, $x\in B_1\cap B_2$, $\exists B\in\mathcal{B}(x) : B\subset B_1\cap B_2$ (*)

In $\mathbb{R}^d$ open boxes form a basis



→ There is always this notion of infinitesimal refinement

You go from basis of each $\mathcal{N}(x)$ to basis of $\mathcal{C}$ by adding the notion of "dense interaction" (*)

Continuity

$(X,d_X), (Y,d_Y)$ then $f: X\to Y$

is continuous at $x_0\in X$ if:

$\forall\varepsilon>0, \exists\delta>0$:
- $\forall x\in X$, $d_X(x,x_0)<\delta \Rightarrow d_Y(f(x),f(x_0))<\varepsilon$
- $f(B(x_0,\delta))\subset B(f(x_0),\varepsilon)$
- $f^{-1}(B(f(x_0),\varepsilon)) \supset B(x_0,\delta)$

OR
- $\forall V\in\mathcal{N}(f(x_0)), \exists U\in\mathcal{N}(x_0) : f(U)\subset V$
- $\forall V\in\mathcal{N}(f(x_0)), f^{-1}(V)\in\mathcal{N}(x_0)$

you can basically drop ε,δ distances for $\mathcal{N}(\dots)$ notion

Continuity is useful because $f\in C^0$ are closed for $+, -, \cdot, \frac{1}{\cdots}$, $\circ$

$\begin{cases} f: X\to Y \\ g: Y\to Z\end{cases}$ both C^0 $\Rightarrow g\circ f: X\to Z$ is continuous

Continuity preserves open/closed through the inverse

$f^{-1}(o/c) = o/c$

All closed:
- $\{x\in X : f(x)\ge a\} = f^{-1}([a,\infty))$
- $\{x\in X : f(x)=0\} = f^{-1}(\{a\})$
- $\{x\in X : f_i(x)\ge a_i, i\in[m]\} = \bigcap_{i\in[m]} f_i^{-1}([a_i,\infty))$
- $\{x\in X : |f(x)|\le r\} = f^{-1}(B[0,r])$

Uniform continuity

$\forall\varepsilon>0, \exists\delta: x_1,x_2\in X$, $d_X(x_1,x_2)<\delta \Rightarrow d_Y(f(x_1),f(x_2))<\varepsilon$

L-Lipschitz ($\forall x_1,x_2$)

$d_Y(f(x_1),f(x_2)) \le L\cdot d_X(x_1,x_2)$

Lipschitz ⇒ Uniformly continuous ⇒ continuous at every $x_0\in X$

- $(X,d_X) \to A\subset X$ then $(A,d|_A)$ is metric space
- $(X_1,d_1), (X_2,d_2)$ metric spaces then $X = X_1\times X_2$ is a MP.

with $\begin{cases} d_p(x,y) = (d_1(x_1,y_1)^p + d_2(x_2,y_2)^p)^{\frac{1}{p}} \\ 1\le p<\infty\end{cases}$

$\sim d_\infty(x,y) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$

- $\bar{d}(x,y) = \min(d(x,y),1)$ is an equivalent bounded metric
- $h: [0,\infty)\to[0,\infty)$

C^0, increasing, concave, $\not\equiv 0$, $0\mapsto 0$

$d_h(x,y) := h(d(x,y))$ is metric (equivalent)

Open Set

$G\subset X$ open iff

$\forall x_0\in G, \exists r>0$ st. $B(x_0,r)\subset G$

OR it is a neighbourhood of all its points

- $\emptyset, X$ open
- $\bigcap^{m}$ open = open
- $\bigcup^{\infty}$ open = open

- $\emptyset, X$ closed
- $\bigcup^{m}$ closed = closed
- $\bigcap^{\infty}$ closed = closed

$\mathring{A}$ / int(A)
- largest open set in A
- union of all open subsets
- $\{x_0 \mid \exists r>0 : B(x_0,r)\subset A\}$
- $\{x_0 \mid A\in\mathcal{N}(x)\}$

Boundary

$\partial A := \bar{A}\setminus\mathring{A}$

- points whose $\mathcal{N}(x)$ have both A, $X\setminus A$

Note $\begin{cases}\mathring{\mathbb{Q}} = \emptyset \\ \bar{\mathbb{Q}} = \mathbb{R} \\ \partial\mathbb{Q} = \mathbb{R}\end{cases}$

$\bar{A}$, Closure
- smallest closed superset of A
- intersection of all closed supersets of A
- $\{x \mid \forall r>0\ B(x,r)\cap A\neq\emptyset\}$ — adds boundary

A closed $\iff \bar{A} = A$

A open $\iff \mathring{A} = A$

TOPOLOGY (X SET)

$\mathcal{C}$ ON X IS A COLLECTION OF SUBSETS OF $X \rightarrow (\mathcal{C} \subset 2^X)$ ST.

i) $\emptyset, X \in \mathcal{C}$

ii) $G_1 \dots G_N \in \mathcal{C} \Rightarrow \bigcap_{n=1}^{N} G_n \in \mathcal{C}$

iii) $G_\alpha \in \mathcal{C}, \alpha \in I \Rightarrow \bigcup_{\alpha \in I} G_\alpha \in \mathcal{C}$

- $\rightarrow G \in \mathcal{C}$ ARE CALLED OPEN
- $\rightarrow (X, \mathcal{C})$ IS A TOPOLOGICAL SPACE
  - $\hookrightarrow$ AKA: SPACE + TOPOLOGY
- $\rightarrow$ (+ SEPARABILITY $\Rightarrow$ HAUSDORFF T. SPACE)
- $\rightarrow F \subset X$ CLOSED IFF $X \setminus F$ OPEN

$U \subset X$ IS A NEIGHBOURHOOD OF x IF $\exists G \in \mathcal{C}: U \supset G \ni x$

DISCRETE TOPOLOGY ($\mathcal{C} = 2^X$)

- BASIS: $\mathcal{B}(x) = \{x\}$ $\leftarrow$ REMEMBER SINGLETON IS OPEN
- $f: X \rightarrow Y$ ARE ALWAYS CONTINUOUS
- $f: Y \rightarrow X$ IS CONTINUOUS IFF $f^{-1}(\{x\})$ OPEN $\forall x$
  - ALWAYS OPEN

THERE IS THIS NOTION OF MINIMUM DISTANCE BETWEEN POINTS $\Rightarrow$

TAKE $\mathbb{N}$ WITH $d(m,n) = |m-n|$

$\forall n \in \mathbb{N}$, $\{n\}$ OPEN SINCE $B(n, \frac{1}{2}) = \{n\}$

$\mathbb{R} \cup \{\pm\infty\} \overset{hom}{\sim} [-1, 1]$

$\mathbb{R} \cup \{\infty\} \overset{hom}{\sim} S^1$

TOPOLOGY ON $\mathbb{R} \cup \{\infty\}$ USING (a,b), $(a, \infty]$

$f: X \rightarrow \mathbb{R}$ CONT. IF
- $f|_{\mathbb{R}}$ CONT.
- $\lim_{x \to \infty} f(x) = f(\infty)$

SETS ACT AS "STITCHES" BETWEEN POINTS

$\mathbb{N} \cup \{+\infty\}$ WITH $(k, \infty]$

$a: \mathbb{N} \cup \{+\infty\} \rightarrow \mathbb{R}$ IS CONT. IFF $\lim_{n \to \infty} a_n = a(\infty)$

INDISCRETE TOPOLOGY

$\mathcal{C} = \{\emptyset, X\}$

$f: Y \rightarrow X$ CONTINUOUS IF $f: X \rightarrow Y$ IS CONT.

NON-HAUSDORFF TOPOLOGY:

EG $\mathbb{R}$, $\mathcal{B} = \{(a, \infty) | a \in \mathbb{R}\}$

THEN $f: X \rightarrow \mathbb{R}$ CONT. IFF $\{x : f(x) > a\}$ OPEN, AKA. $\{x : f(x) \leq a\}$ CLOSED

CONTINUITY IN X IS "LOWER-SEMICONTINUITY"

COMPARING $\{\mathcal{C}_i\}_k$ ON X

$\mathcal{C}_1$ COARSER $\mathcal{C}_2$ $\Big]\rightarrow$ $\mathcal{C}_2$ FINER $\mathcal{C}_1$

$(\mathcal{C}_1 \subset \mathcal{C}_2)$ — $\mathcal{C}_2$ IS FINER IF IT HAS SOME MORE SETS

AFFECTS EASE OF CONTINUITY:

$f: X \rightarrow Y$ CONTINUOUS MORE EASILY IF
- X FINER
- Y COARSER

$\Big\}$ ALWAYS REASON BY $f^{-1}(\dots) \subset \dots$

$\rightarrow$ IF X IS COARSER, THERE ARE FEWER CONSTRAINTS FOR THE CONTINUITY OF f TO HOLD $f^{-1}(V) \subset U \;\Big|\; \begin{matrix} V \in Y \\ U \in X \end{matrix}$

THE IDENTITY IS CONTINUOUS

$i: (X, \mathcal{C}_2) \rightarrow (X, \mathcal{C}_1)$

$\Updownarrow$

$\mathcal{C}_2$ FINER THAN $\mathcal{C}_1$

SUBSPACE

$(X, \mathcal{C})$, $A \subset X$ | RESTRICTION OF THE SUPPORT

THE NATURAL INDUCED $(\mathcal{C}_A, A)$ IS THE COARSEST TOPOLOGY ST. $j: A \rightarrow X$ IS CONTINUOUS

$\leftarrow$ JUST REQUIRED TO GUARANTEE THAT THE RESTRICTIONS REMAIN CONTINUOUS

$j^{-1}(a) = a$ $\Big] \rightarrow j^{-1}(G) = G \cap A$
- $\in A$, $\in X$

RESTRICTION PRESERVES OPEN/CLOSED $\neq$ EXPANSION CAN CHANGE $\sim$ A CONTINUOUS FUNCTION MAY NOT HAVE A CONTINUOUS EXTENSION TO A SUPERSPACE

$\mathcal{B}$ BASE OF OPEN $\mathcal{N}$.

- THEN $\forall x \in A$ $\mathcal{B}' = \mathcal{B} \cap A$, $\mathcal{B} \in \mathcal{B}(x)$ FORMS A BASE IN A
- $\mathcal{C}_A = \{G \cap A : G \in \mathcal{C}\}$
  (OBTAIN RELATIVELY OPEN SETS BY THE INTERSECTION WITH A)
- F CLOSED IN A IFF $F = F' \cap A$ FOR F' CLOSED IN A — CLOSURE IS CONSERVED BY RESTRICTION
- $f: T \rightarrow A$ CONTINUOUS
  $\Updownarrow$
  $(j \circ f): T \rightarrow X$ CONTINUOUS
  AKA. $\begin{cases} f': T \rightarrow X \text{ CONTINUOUS}, \\ f'(T) \subset A, \; f: T \rightarrow A \text{ CONTINUOUS} \end{cases}$

HOMEOMORPHISM

$f: X \rightarrow Y$

- CONTINUOUS
- BIJECTIVE
- $f^{-1}: Y \rightarrow X$ CONTINUOUS

| CONTINUOUS, BIJECTIVE, MAPS OPEN TO OPEN AND CLOSED TO CLOSED

$\rightarrow$

EMBEDDING

(HOMEOMORPHISM BETWEEN X, $f(X) \subset Y$)

$f: X \rightarrow Y$

- CONTINUOUS
- INJECTIVE
- $f^{-1}: f(X) \rightarrow X$ CONTINUOUS

PRODUCT

$\pi_i: X \rightarrow Y_i$, $i \in I$, $(Y_i, \mathcal{C}_i)$

THE TOPOLOGY INDUCED BY $\{\pi_i\}$ ON X IS COARSEST ST. ALL π_i ARE CONTINUOUS

$\uparrow$

$\mathcal{C}$ WILL CONTAIN ALL SETS $\pi_i^{-1}(G) \;\Big|\; G$ OPEN IN Y_i AND THEIR INTERSECTIONS

$\downarrow$

BASIS OF TOPOLOGY

CARTESIAN PRODUCT

$X = X_1 \times X_2$

$\begin{cases} \pi_1: X \rightarrow X_1 \text{ AS } (x_1, x_2) \mapsto x_1 \\ \pi: X \rightarrow X_2 \text{ AS } (x_1, x_2) \mapsto x_2 \end{cases}$

BASE $\mathcal{B} = \{G_1 \times G_2 : G_i \in \mathcal{C}_i\}$

FINAL TOPOLOGY

$(X, \mathcal{C})$ HAS $\pi: X \rightarrow Y$

$\mathcal{C}'$ IS FINEST ST. π IS CONTINUOUS

CONNECTED SPACES

(YOU CAN REACH $X \rightarrow Y$ CONTINUOUSLY)

PATH-CONNECTED

$\forall x, y \in X$

$\exists \alpha: [0,1] \rightarrow X$ CONTINUOUS ST. $\alpha(0) = x$, $\alpha(1) = y$ $\Big]$ "PATH", INJECTIVITY NOT REQUIRED

DISCONNECTED

$\exists f: X \rightarrow \mathbb{R}$ C^0, SURJECTIVE $\Updownarrow$ ST. $f(X) = \{a, b\}$

CONTRAPOSITIVE

EVERY FUNCTION $f: X \rightarrow \{a, b\}$ IS NOT SURJECTIVE $\Rightarrow$ X IS CONNECTED

CONTINUITY PRESERVES CONNECTEDNESS

DISCONNECTED

$\exists G_1, G_2$ NONEMPTY, OPEN ST.

$X = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$

"SEPARATION" DEFINITION IS EQUIVALENT WITH CLOSED

IF NO SEPARATION EXISTS, X IS CONNECTED

IT IS ALSO EQUAL TO FIND $A \subset X$, $A \notin \{\emptyset, X\}$, BOTH OPEN AND CLOSED

THEN G_1, G_2 ARE ALSO CLOSED

(IN $\mathbb{R}$), $C \subset \mathbb{R}$ NONEMPTY

CONNECTED $\Longleftrightarrow$ CONVEX (AKA. INTERVAL)

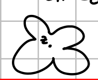
$C_\lambda \subset X$ CONNECTED, $\lambda \in \Lambda$ — IF $\bigcap C_\lambda \neq \emptyset$, $\Longrightarrow \bigcup C_\lambda$ IS CONNECTED

$C = \bigcup C_n$, $C_n \cap C_{n+1} \neq \emptyset$, C_n CONNECTED $\Rightarrow$ C IS CONNECTED

(IN $\mathbb{R}^d$)

EVERY STAR-SHAPED SET IS CONNECTED:

$X = \bigcup_{x \in X} [z, x]$ — CENTER



$X \subset \mathbb{R}^d$, OPEN, THEN X IS CONNECTED

IN $\mathbb{R}$, AN OPEN SET CAN BE DECOMPOSED UNIQUELY AS A UNION OF A COUNTABLE FAMILY OF DISJOINT INTERVALS

CONNECTED COMPONENTS

FOR $x \in X$, THE SET OF $y \in X$ CONNECTED TO x

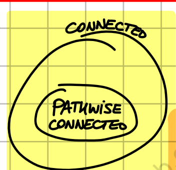
- $C[x]$ CLOSED
- $\{C[x]\}$ PARTITION
- CONTINUITY PRESERVES CONNECTEDNESS

VERY USEFUL TO DISPROVE HOMEOM.

$\rightarrow$

TOTALLY DISCONNECTED:

$C[x] = \{x\}$ $\leftarrow$ $\mathbb{Q}$ IN $\mathbb{R}$



$\{a_n\} \subset X$

$\mathbb{N} \cup \{\infty\}$

CONTINUITY:

$\begin{cases} \lim_{n \to \infty} a_n = a_\infty \\ a_n \in X, a_\infty \in \bar{X} \end{cases}$

SEQUENTIAL COMPACTNESS

EACH SEQUENCE HAS A CONVERGENT SUBSEQUENCE

$\rightarrow$ x IS A LIMIT POINT, THEN $\forall U \in \mathcal{N}(x)$, $|\{n : x_n \in U\}| = \infty$

COVER OF X

$\mathcal{C} = \{S\} \subset X$ ST.

$X \subset \bigcup_{S \in \mathcal{C}} S$

(X, d) TOTALLY BOUNDED

$\forall \varepsilon > 0$, $\exists X_\varepsilon = \{x_1, \dots, x_N\}$ ST. $X \subset \bigcup_{n=1}^{N} B(x_n, \varepsilon)$

YOU CANNOT BE MORE THAN ε-DISTANT FROM $x \in X$ AND SOME $x_{(i)}$

COMPLETE ⊕ TOTALLY BOUNDED

$\Updownarrow$

ANY OPEN COVER HAS A FINITE SUBCOVER $\Big]$ USEFUL IN PROOFS

CAUCHY SEQUENCE

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$: $\forall m, n > N$, $d(x_m, x_n) < \varepsilon$

$\rightarrow$ (X, d) IS COMPLETE IF ALL CAUCHY SEQUENCES CONVERGE IN THE SET

LEBESGUE COVERING LEMMA

X COMPACT WITH AN OPEN COVERING: $\exists \delta$ ST. $\forall A \subset X$ CONTAINED IN SOME BALL OF RADIUS δ, THE δ BALL IS FULLY CONTAINED IN SOME U_i IN COVER

INDUCED TOPOLOGY

$\pi : X \to Y$ INDUCES ON Y THE FINEST TOPOLOGY ST. $\pi \in C^0$ ON:

$$\mathcal{C}_\pi := \{ G \subset Y \mid \pi^{-1}(G) \text{ open} \}$$

π DEFINES AN EQUIVALENCE CLASS X/f, WHICH CAN SIMPLE BE IDENTIFIED USING $(Z, \mathcal{C}_Z)$ IN A MORE TRACTABLE WAY

$$\begin{tikzcd}
(X, \mathcal{C}) \arrow[d, "\pi"', "\text{EQUIV. CLASS}"] \arrow[dr, bend left, "g", "\text{CONTINUOUS MAP}"'] & \\
(Y, \mathcal{C}_\pi) \arrow[r, "\text{HOMEO MORPH.}", "j"'] & (Z, \mathcal{C}_Z)
\end{tikzcd}$$

j IS A HOMEOMORPHISM BECAUSE IT "PRESERVES" THE TOPOLOGY OF X/f

$f : X \to Y$ CONTINUOUS, SURJECTIVE IS AN IDENTITY IF IT INDUCES ON Y THE SAME TOPOLOGY AS X

FOR A FUNCTION $\pi : X \to Y$, THEN $A \subset X$ IS SATURATED IF $(\pi^{-1} \circ \pi)(A) = A$

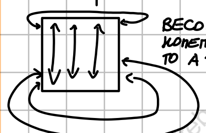
IMPORTANT BECAUSE THE SET IS THE UNION OF THE PREIMAGES (/"FIBERS") OF THE IMAGE ITSELF

FORMS A KIND OF EQUIVALENCE/ SUFFICIENT KNOWLEDGE FOR f AND TOPOLOGY

$S^2 \setminus \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \subseteq \mathbb{R}^n$

IS HOMEOMORPHIC TO $\mathbb{R}^2$

$D^2 \subseteq \mathbb{R}^2$ WITH BOUNDARY CIRCLE IDENTIFIED WITHIN THE SAME EQUIVALENCE CLASS IS HOMEOMORPHIC TO $S^2 \subseteq \mathbb{R}^3$



BECOMES HOMEOMORPHIC TO A TORUS

HOMOTOPY (F)

CONTINUOUS $F : X \times [0,1] \to Y$

ST. $\begin{cases} F(x,0) = f_0(x) \\ F(x,1) = f_1(x) \end{cases}$

FOR $f_0, f_1 : X \to Y$

$\to$ EQUIVALENCE CLASS BETWEEN C^0 FUNCTIONS $X \to Y$

$\to f_0 \overset{\text{homotopy}}{\sim} f_1$

TOPOLOGICAL SPACES ARE HOMOTOPICALLY EQUIVALENT IF THE IDENTITIES ARE IN A HOMOTOPY

HOMOTOPY IS TRANSITIVE