1.

– IS THE MARKET COMPLETE?

$Im(A)$ HAS MAX SPAN?

CHECK THAT ALL COLUMNS ARE LIN. INDEPENDENT, GIVING AN IMAGE OF dim EQUAL TO # OUTCOMES

(IF A SQUARE CHECK $\det(A) \neq 0$)

IF # STOCKS < # OUTCOMES, MARKET MUST BE INCOMPLETE

DIMENSION OF THE SPAN OF PAYOFFS EQUALS # Ω

A HAS MAX DIM $Ker(A) = \{0\}$

COMPLETE $\Rightarrow$ SDF EXISTS

– LOP HOLD?

COMPLETE $\Rightarrow$ LOP HOLDS

A HAS FULL RANK

A IS INVERTIBLE

NA $\Rightarrow$ LOP

DISPROVE IF YOU CAN FIND 2 (LIN. COMB.) OF THE SECURITIES HAVE SAME PAYOFF BUT $\neq$ PRICE

$\exists m, \psi, q \Downarrow$ LOP HOLDS

WITH α PARAMETRISATION OF THE PAYOFFS

– α ST. A COMPLETE $\Rightarrow$ LOP HOLDS

– ELSE $\Rightarrow$ EXPRESS ONE PAYOFF IN TERMS OF OTHERS, CHECK THAT COST HAS THE SAME COEFFICIENTS.

LOP HOLDS IF $\forall i, j$ IF $S_j(1) = k S_i(1)$ $\Downarrow$ $S_j(0) = k S_i(0)$

PAYOFF MATRIX

$$A = \left[\underline{B}(1)\ S_1(1) \dots S_N(1)\right] \in \mathbb{R}^{K \times (N+1)} \qquad S_j(1) = \begin{pmatrix} S_j(1)(\omega_1) \\ \vdots \\ S_j(1)(\omega_K) \end{pmatrix}$$

CASHFLOW MATRIX ($t=0$)

$$M = \begin{bmatrix} -B(0)\ -S_1(0) \dots -S_N(0) \\ A \end{bmatrix} \in \mathbb{R}^{(K+1)(N+1)}$$

IS x A TRADED PAYOFF?

EQUIVALENT TO SHOWING THAT $x \in span(\{S_i(1)\})$

IS THE SECURITY REDUNDANT?

IFF PAYOFF CAN BE WRITTEN AS A LINEAR COMBINATION OF OTHER SECURITIES' PAYOFF

– IS MARKET ARBITRAGE-FREE?

$\exists \psi, Q, m$ STRICT. POS. $\iff$ NA — 1^{ST} FFTAP

(ALL ARE EQUIVALENT AND RELATED)

MAY BE GOOD TO PLOT THE m LINEAR CONSTRAINTS

IF m EXISTS AS $f(a_1 \dots a_L)$, N-L SECURITIES ARE REDUNDANT

$$NA \iff \left\{\begin{bmatrix} -V_\vartheta(0) \\ V_\vartheta(1) \end{bmatrix}\right\} \cap \mathbb{R}^{K+1}_{+} = \{0\}$$

– DOES ADDING A DERIVATIVE KEEP NA? (GIVEN PAYOFF)

1. COMPUTE PAYOFF $X(1)$
2. ADD $\mathbb{E}^{\mathbb{P}}[mX(1)] = X(0)$ TO THE CONSTRAINTS ON m CALCULATIONS
3. CHECK $\exists m$ STRICT POS. ($\Rightarrow$ NA)

– FIND ARBITRAGE STRATEGY:

$$\begin{cases} V_\vartheta(0) = S(0)\vartheta \leq 0 & \text{TOTAL COST} \leq 0 \\ V_\vartheta(1)(\omega_k) \geq 0\ \forall k & \text{MAKE A PROFIT} \end{cases}$$

AT LEAST ONE HOLDS:

$$\begin{cases} V_\vartheta(0) < 0 \\ V_\vartheta(1)(\omega_k) > 0 \end{cases}$$

$\rightarrow$ WRITE THE CONSTRAINTS, SOLVE THE SYSTEM (MAYBE PLOT IT)

REMEMBER: $\vartheta = (1, a, \dots z)$ TO RELY ON ONE LESS CONSTRAINT

1. FIND ϑ ST. COST ≤ 0 / < 0 (LIN. DEP. PAYOFF)
2. SHOW PAYOFF > 0 / ≥ 0

FIND TWO ASSETS WITH SAME PAYOFF, BUY ONE AND SHORT SELL OTHER (AKA $\vartheta_1 = POS$, $\vartheta_2 = NEG$)

IF YOU KNOW THE REDUNDANT S_i:

i, j ST. $S_j(1) = \alpha S_i(1)$ WITH $S_j(0) \neq \alpha S_i(0)$

IF $\alpha > 0$, $S_j(0) < \alpha S_i(0)$ SHORT i, BUY j:

$$\vartheta^* = \begin{pmatrix} 0 \\ \vdots \\ -\alpha \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \rightarrow i^{TH} \\ \rightarrow j^{TH} \end{matrix} \quad \text{ALL OTHERS ARE ZERO}$$

$$\begin{cases} V_{\vartheta^*}(0) = -\alpha S_i(0) + S_j(0) < 0 \\ V_{\vartheta^*}(1) = -\alpha S_i(1) + S_j(1) = 0 \end{cases}$$

EXISTENCE OF ONE IS EQUIVALENT TO OTHERS

STATE PRICE VECTOR (SPV, ψ)	STOCHASTIC DISCOUNT FACTOR (SDF, m)	RISK NEUTRAL PROBABILITY (RNP, q)
$\psi \in \mathbb{R}^K$ (# OUTCOMES)	$m \in \mathbb{R}^K$ (# OUTCOMES)	$q \in \mathbb{R}^K$ (# OUTCOMES)
$\forall k$, $\psi_k \geq 0$	$\forall i = 1 \dots N$ (SECURITIES) $\mathbb{E}^{\mathbb{P}}[m S_i(1)] = S_i(0)$	$S_j(0) = \mathbb{E}^{Q}\left[\frac{S_j(1)}{1+r}\right]$
$\forall j$, $S_j(0) = \psi^T S_j(1)$ ($N+1$ ASSETS)	$\mathbb{E}(m) = \frac{1}{1+r}$	$q_k > 0$
$\boxed{(1 \mid \psi^T) M = 0^T}$	SDF IS ≥ 0	$q_k = \psi_k (1+r)$
$\boxed{(A^T)\underline{\psi} = -\begin{pmatrix} B(0) \\ \vdots \\ S_N(0) \end{pmatrix}}$	$m_k = \frac{1}{1+r}\frac{q_k}{p_k}$	$\forall k$ $\mathbb{E}^q(R_k) = r$
$\psi_k = m_k p_k$	$V_\vartheta(0) = \mathbb{E}[m V_\vartheta(1)]$	$\left(R_k = \frac{S_k(1) - S_k(0)}{S_k(0)}\right)$
$\sum_k \psi_k = \frac{1}{1+r}$	$\mathbb{E}[m(R_j - r)] = 0$	
$V_\vartheta(0) = \psi^T V_\vartheta(1)$		
$\psi^T(R_j - r) = 0$		

UNIQUENESS OF m:

$\boxed{NA + COMPLETE \iff \exists! m, \psi, q}$ 2^{nd} FTAP

LOP + COMPLETE

$\Uparrow$

NA + COMPLETE $\Rightarrow \dim M = \dim A = K$

IF YOU KNOW MISPRICED ASSET S_i:

IE $S_i(0) \neq \psi^T S_i(1)$

1. IF $S_i(0) > \psi^T S_i(1)$ $\rightarrow$ OVERPRICED SO SELL NOW AS STRATEGY ($\binom{TRUE}{PRICE} > \binom{FAIR}{PRICE}$)
2. ϑ HAS -1 AT i ENTRY
3. FIND THE ϑ_j ENTRIES: $Prices_0 \cdot \vartheta = 0$ SO $\vartheta = \begin{pmatrix} -1 \\ f(\alpha) \end{pmatrix}$
4. WRITE $A\vartheta \geq 0$
5. SOLVE FOR α ST. AT LEAST ONE INEQUALITY IS STRICT

GIVEN $f \in \mathbb{R}^K$, FIND $\gamma, \lambda \in \mathbb{R}$ ST.

$$\mathbb{E}[R] = \gamma + \beta_{R,f}\lambda \quad \Big| \quad \beta_{R,f} = \frac{cov[R,f]}{Var[f]}$$

1. $\gamma = \frac{1}{a + b\mathbb{E}[f]}$, $\lambda = -bVar[f]\gamma$
2. COMPUTE $\mathbb{E}[f] \rightarrow \gamma \rightarrow Var[f] \rightarrow \lambda$
3. COMPUTE $\mathbb{E}[R]$

USE THE HISTORICAL DISTRIBUTION HERE

GIVEN $Var(f)$, $R_2 = \frac{S_2(1)}{S_2(0)}$, TEST FOR

1. COMPUTE $\mathbb{E}[R_2 f]$, $\mathbb{E}[R_2]$
2. COMPUTE $cov[R_2, f]$ — $cov(A,B) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]$ $\rightarrow$ SEE $\mathbb{E}[R_2 f]$, $\mathbb{E}[R_2]$, $\mathbb{E}[f]$
3. $\beta_{R_2,f} = \frac{cov[R,f]}{Var[f]} = \dots$
4. $\gamma + \beta_{R_2,f}\lambda = [\dots] = \mathbb{E}[R_2]$

(MARKET WITH S_1, S_2 * GIVEN $(\mathbb{E}[xx^T])^{-1}$ $x = (S_1(1), S_2(1))$)

– FIND m_π^* ONLY SDF TRADED

1. WRITE A, $S = (S_1(0), S_2(0))$
2. $\boxed{\vartheta_{m^*} = (\mathbb{E}[xx^T])^{-1} S}$ OR $\mathbb{E}[xx^T]\vartheta_{m^*} = S$
3. $m^* = A\vartheta_{m^*}$

CHECK IF m^* IS SDF FOR THE MARKET

1. KNOW m SET OF LINEAR CONSTRAINTS
2. $m = A\vartheta_{m^*}$
3. SOLVE SYSTEM FOR m^*, ϑ_{m^*}

(THERE SHOULD BE ENOUGH CONSTRAINTS TO SOLVE SYSTEM, IMPLYING SET OF SDF IS $\{m^*\}$)

FIND SET OF SDF/m:

$\forall j$ SECURITIES $S_j(0) = \mathbb{E}[m S_j(1)] = \sum_{k=1}^{K} \mathbb{P}(\omega_k) m(\omega_k) S_j(1)(\omega_k)$ (NOTE $m \in \mathbb{R}^K$)

CHARACTERISE ALL m:

$m = m^* + m^\perp$

$m^\perp = \{\varepsilon : \mathbb{E}[\varepsilon^T A] = 0\}$

ε SAME SHAPE AS m^*

CHECK THAT $\hat{R}$ BELONGS TO MV FRONTIER:

$$\sigma^2[\hat{R}] = \frac{c}{d}\mathbb{E}[\hat{R}]^2 - 2\frac{a}{d}\mathbb{E}[\hat{R}] + \frac{b}{d}$$

$a = \mathbb{E}[R^*]$

$b = \mathbb{E}[R^{e*}]\mathbb{E}[(R^*)^2] + \mathbb{E}[R^*]^2$

$c = \mathbb{E}[R_f] - 1 - \mathbb{E}[R^{e*}]$

$d = \mathbb{E}[R^{e*}]$

$\hat{R} = R_f + \lambda R^e$

$R^e = R^* - R_f$

λ SCALAR

FIND $\vartheta_{R^{e*}}$:

SOLVE $R^{e*} = A\vartheta_{R^{e*}}$

* FIND R^*

1. DO $\boxed{\pi[m^*] = \mathbb{E}[(m^*)^2]}$ — WRT $\mathbb{P}(\omega_i)$ HISTORICAL PROBABILITY
2. $R^* = \frac{m^*}{\pi[m^*]}$

$$\vartheta_{R^*} = \frac{\vartheta_{m^*}}{\pi[m^*]}$$

* FIND REPLICATING STRATEGY OF R^*:

1. FIND ϑ_{R^*} IS ST. PAYOFF IS THE SAME
2. EXPRESS R^* IN TERMS OF COLUMNS OF 'A' MATRIX
3. WRITE VECTOR OF THE COEFFICIENTS

* GIVEN $\mathbb{E}[R^*]$, R^{e*}, k:

FIND R^{MV} ST $\mathbb{E}[R^{MV}] = k$

1. $\boxed{R^{MV} = R^* + \omega^{MV} R^{e*}}$ (MUST BE IN MV FRONTIER) $R^{MV} = (1-\alpha)R_f \mathbb{1} + \alpha R^*$, $\alpha = \frac{\mathbb{E}[R^{MV}] - R_f}{\mathbb{E}[R^*] - R_f}$
2. $\boxed{\omega^{MV} = \frac{k - \mathbb{E}[R^*]}{\mathbb{E}[R^{e*}]}}$ $Var[R^{MV}] = \alpha^2 Var[R^*]$
3. COMPUTE $\mathbb{E}[R^{e*}]$, ω^{MV}, R^{MV}

* GIVEN R^{e*}, FIND R^{CMR} CONST. MIMICKING PORTFOLIO

1. $$\boxed{R^{CMR} = R^* + \frac{\mathbb{E}[(R^*)^2]}{\mathbb{E}[R^*]} R^{e*}}$$
2. COMPUTE $\mathbb{E}[R^*]$, $\mathbb{E}[(R^*)^2]$

R^*, R^{e*} PROPERTIES

$R_f = 1 + r$ (RISK-FREE RATE)

$$R^* = \frac{m^*}{\mathbb{E}[m^*]} = \frac{m^*}{\mathbb{E}[(m^*)^2]}, \quad R^{e*} = Proj[1|A_0] \in A_0$$

$$\mathbb{E}[R^*] = \frac{\mathbb{E}[m^*]}{\mathbb{E}[(m^*)^2]}, \quad m^* = \frac{R^*}{\mathbb{E}[(R^*)^2]}$$

$$\mathbb{E}[R^* R^{e*}] = 0 \quad \forall R^{e*} \in A_0$$

$$\mathbb{E}[(R^{e*})^2] = \mathbb{E}[R^{e*}]$$

$$A = \langle\alpha R^*\rangle \oplus A_0$$

$$Cov(A,B) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]$$

$$R^{MIN} = R^* + \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}[R^{e*}]} R^{e*}$$

$$\mathbb{E}[R^{MIN}] = \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}[R^{e*}]} = \omega^{MIN}$$

$$A_1 = \{R \in A \mid \pi[R] = 1\}$$

$$A_0 = \{R^e \in A \mid \pi[R^e] = 0\}$$

INTERPRET γ WRT m^*

"γ IS SHADOW GROSS RISK-FREE RATE"

THIS CAN BE SHOWN:

1. TRADING RISKLESS SECURITY, $\mathbb{E}[m] = \frac{1}{R}$
2. IF THE RISKLESS ASSET IS NOT TRADED, USING m^* FROM BETA PRICING RELATIONSHIP
3. COMPUTE $\mathbb{E}[m^*] = \dots$, HENCE $R = \frac{1}{\mathbb{E}[m^*]} = \gamma$

* GIVEN $V(0)$,

FIND ϑ^{MV} IN TERMS OF S_1, S_2 DELIVERING R^{MV} OPTIMAL (GROSS RETURN)

1. $R_i(1) = \frac{S_i(1)}{S_i(0)} \rightarrow \sum_{i=0}^{N} \omega_i^{MV} \cdot R_i(1) = R^{MV}$
2. SOLVE FOR ALL ω_i^{MV} (SCALARS)
3. $\vartheta_i^{MV} = \frac{\omega_i^{MV} V(0)}{S_i(0)} \rightarrow \vartheta_i^{MV}$ IN UNITS OF $S_i(0)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

IT'S A CALCULATOR EXAM: DON'T TRY TO RESOLVE THE WHOLE ALGEBRAIC EQUATION WHEN YOU CAN DUMP IT IN THE CALCULATOR, WITH LESS RISK OF MAKING MISTAKES

SDF DECOMPOSITION

$\forall m$, $\exists! m^*$ ST. $m = m^* + \varepsilon$, $\varepsilon \in A^\perp$

DECOMPOSE $m^* = a\mathbb{1} + bf$, GIVEN $f = (:)$, m^*

SOLVE SYSTEM BY USING ZEROES TO FIND THE VARIABLES

EXAMS COVERED

- DEC 2023 ✓
- JAN 2023 ✓
- MOCK ✓
- QF SHEET ✓

2.

$t=0,1,2$

$\mathcal{f}^0_1 \to \mathcal{f}^1_1 \to \omega_1, \omega_2$; $\mathcal{f}^1_2 \to \omega_3, \omega_4$

B LOCALLY RISKLESS, KNOW $r(0)$, $r(1)(f_1^1)$, $r(1)(f_2^1)$

PRICES: S RISKY: YOU KNOW $S(0), S(1)(f_1^1), S(1)(f_2^1)$, $S(2)(\omega_1) \dots S(2)(\omega_4)$

FIND PRICE PROCESS OF $B=\{B(t)\}_{t=0,1,2}$

1. $B(0)=1$
2. $B(1)=B(0)(1+r(0))$
3. $B(2)=\begin{cases} B(1)(1+r(1)(f_1^1)) & f_1^1 \\ B(1)(1+r(1)(f_2^1)) & f_2^1 \end{cases}$ TRACE THE TRAJECTORY FOR EACH TREE NODE

- FIND ARBITRAGE STRATEGY:

1st TYPE $\begin{cases} V_\vartheta(0)=S(0)\vartheta \le 0 \\ V_\vartheta(t)(f_h^t) \ge 0 \ \forall h,t \\ V_\vartheta(\tau)(f_\ell^\tau) > 0 \ \exists f_\ell^\tau \end{cases}$ BAILOUT

2nd TYPE $\begin{cases} V_\vartheta(0)<0 \\ V_\vartheta(t)(f_h^t) \ge 0 \ \forall h,t \end{cases}$

$\nexists$ 2nd TYPE $\Downarrow$ LOP | $\nexists$ NA (1,2) $\Downarrow$ NA

LOP VIOLATION: FIND ϑ, ϑ' ST. $V_\vartheta(0) \ne V_{\vartheta'}(0)$ BUT SAME $C_\vartheta(t)(f_h^t)\ \forall t,h$

FIND SET OF RNP $\mathbb{Q}$

1. SOLVE FOR m_0

$$\begin{cases} S(0) = \frac{1}{1+r(0)}\left[S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1]\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S(1)}{1+r(1)}\right] \to \text{GIVES } \mathbb{Q}[f_i^1] \\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

2. SOLVE FOR $m_{1,1}$ AND $m_{1,2}$:

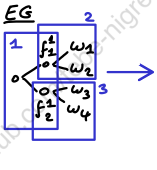
EG $S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)}\left[\sum_m S(2)(\omega_m)\mathbb{Q}(\omega_m|f_1^1)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)}{1+r(1)}\right](f_1^1)$ GIVES $\mathbb{Q}[\omega_i|f_1^1]$ NOTE $r(1)$ DEPENDS ON f_1^1

3. USE $\mathbb{Q}[\omega_i] = \mathbb{Q}[f_1^1]\mathbb{Q}[\omega_i|f_1^1]$

CHECK DYNAMIC COMPLETENESS

FOR EACH STEP PAYOFF MATRIX A IS COMPLETE $\to$ THIS IS ENOUGH TO DISPROVE, NOT PROVE: (YOU COULD CHECK THAT $\det \ne 0$) DYN. COMPL. $\Rightarrow$ COMPL. IN EACH SUBMARKET

EG



3x SUBMARKETS: IF B, S TRADED,

$A_1 = \begin{bmatrix} B(0) & S(1)(f_1^1) \\ B(0) & S(1)(f_2^1)\end{bmatrix}$

$A_2 = \begin{bmatrix} B(1)(f_1^1) & S(1)\dots(\omega_1) \\ B(1)(f_1^1) & S(1)\dots(\omega_2)\end{bmatrix}$ ...

BINOMIAL MODEL (P)

- NA $\iff d < 1+r < u$
- COMPLETE $\iff u \ne d$
- RNP: BINOMIAL DISTRIBUTION IN P

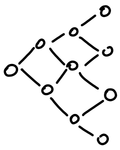
$\mathbb{P}(S(t)=S(0)u^h d^{n-h}) = \binom{n}{h} p^h (1-p)^{n-h}$

USE IT TO COMPUTE (VALUE, $\mathbb{P}$) PAIRS IN $\Rightarrow$ ANY PUT/CALL OPTION'S VALUATION

DETERMINE HISTORICAL PROBABILITY $\to$ SAME BINOMIAL BUT USING q

$\mathbb{P}(u) = \frac{(1+r)-d}{u-d} = q$

$\mathbb{P}(d) = \frac{u-(1+r)}{u-d} = 1-q$



NA PRICE OF PUT OPTION:

$P(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{P(T)}{(1+3)^T}\right]$ (USING HISTORICAL PROBABILITY)

PRICES ARE CALCULATED AT $t=0,1$ AND NOT $t=2$

AMERICAN PUT OPTION, IS THERE AN OPTIMAL EARLY EXERCISE OPPORTUNITY?

1. COMPUTE PRICES AT MATURITY ($t=1$) (SAME AS EU OPTION)
2. COMPUTE PRICE AT $t=0$
3. CHECK IF $P_{USA}(t) > P_{EU}(t)$:

EARLY EXERCISE PREMIUM $= P_{Am}(t) - P_{EU}(t)$

2nd FTAP NA, DYNAMICALLY COMPLETE $\Leftrightarrow \exists! \psi, \mathbb{Q}$

IS THERE NA? IFF $\exists \psi, \mathbb{Q}$ (1st FTAP)

* USING M RUNNING MAXIMUM OF S,

BARRIER CLAIM: $X(t) = \begin{cases} c\cdot(1+r(t-1)) & M(t) \ge b \\ 0 & \text{ELSE} \end{cases}$

FOR c, b GIVEN.

COMPUTE RUNNING MAX OF S:

$\begin{cases} M(1) = \max_{t=0,1} S(t) \\ M(2) = \max_{t=0,1,2} S(t) \end{cases}$

1. GROUP f_i^t OR ω_i
2. SELECT MAX: $M(1) = \begin{cases} \dots & \omega_1 \\ \dots & \omega_n \end{cases}$

* **COMPUTE CASHFLOW**

$\boxed{\text{VALUE OF ASSET AS } f(t)}$

1. COMPUTE DIRECTLY $t=2$ USING THE FORMULA
2. COMPOSE THE PAYOFF BY PARTITION:

$X(t) = \begin{cases} \dots & \omega_{\dots} \in f_1^t \\ \vdots \\ 0 & \omega_{\dots} \in f_M \end{cases}$

$\mathcal{P}_t$ IS THE PARTITION

$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{P}_t](f_h^t) = \sum_{\omega\in\Omega} X(\omega)\mathbb{Q}(\omega|f_h^t) = \sum_{\omega\in f_h^t} X(\omega)\frac{\mathbb{Q}(\omega)}{\mathbb{Q}(f_h^t)}$

EU PUT OPTION (K,T)

$x(2) = \max(K-S(T), 0) = (K-S(T))^+$

EU CALL OPTION (K,T)

$x(2) = \max(S(T)-K, 0) = (S(T)-K)^+$

MULTIVARIATE KNOCK-IN CALL OPTION S, (K,T)

$C_{KI}(2) = \begin{cases} (S(2)-K)^+ & \max\{r(0), r(1)\} \ge \bar{r} \\ 0 & \text{ELSE} \end{cases}$

USE THE RECURSIVE FORMULA BECAUSE YOU GO FROM $x(2)$ TO $x(0)$

\+ $\Omega = \{\omega_1 \dots \omega_4\}$ GIVEN $\mathcal{P}_t$: $\mathbb{P}(\omega_i|f_j^1)$

AT $t=1$, AN INVESTOR WITH LOG-UTILITY MAXIMISES $\mathbb{E}[\text{FINAL UTILITY}]$ FOR WEALTH $= W$ AT $t=0$

\+ **FIND OPTIMAL DISTRIBUTION OF WEALTH AT $t=1$, FOR f_i^1, ASSUMING B,S (ASSETS), $\mathbb{P}(\omega_h)$** USING THE HISTORICAL PROBABILITY

1. $x_1^* = \frac{R}{u-d}\cdot\frac{p-q}{q(1-q)}$
2. FIXING f_1^1 GIVES

$R = 1+r(1)$

$u = \frac{S(2)(\omega_1)}{S(1)(f_1^1)}$??

$d = \frac{S(2)(\omega_2)}{S(1)(f_1^1)}$

1. FIND $\mathbb{P}(\omega_i|f_j^1) \to \mathbb{P}$ AT f_i^1
2. FIND $\vartheta(1)(f_j^1)$: $v = V(1) \to$ WEALTH CONSTRAINT $(\vartheta_0(1)B(1) + \vartheta_1(1)S(1) = V(1))$
3. $\boxed{x = \frac{\vartheta_1(1)S(1)}{v}}$ $\boxed{1-x = \frac{\vartheta_0(1)B(1)}{v}}$ % WEALTH SPENT ON EACH ASSET
4. $V(2) = \vartheta_1(1)S(2) + \vartheta_0(1)B(2) = x\frac{v}{S(1)}S(2) + (1-x)\frac{v}{B(1)}B(2)$
5. $F(1,v) = \max_x(\mathbb{E}(\ln(V(2)))) \to$ max $\mathbb{E}$ log utility
6. APPLY f_i^1 VALUES IN $B(1), S(1)$
7. MAXIMISE $F(1,v)$ IN x.
8. $\vartheta_1(1)(f_i^1) = x^* \frac{v}{S(1)(f_i^1)}$

GENERAL SETTING

$V_\vartheta(2) = \vartheta_0(1)B(2) + \vartheta_1(1)S(2)$

$V_\vartheta(1) = \vartheta_0(0)B(1) + \vartheta_1(0)S(1)$

2 ASSETS ϑ WEIGHTS:

$x(t) = \frac{\vartheta_1(t)S(t)}{V_\vartheta(t)}$

$1-x(t) = \frac{\vartheta_0(t)B(t)}{V_\vartheta(t)}$

SOLVE FOR $\vartheta_i(t)$

$V(2)\big|_{V(1)=v} = v(1+r(1)+x(\frac{S(2)}{S(1)}-(1+r(1))))$

$F(1,v) = \ln(v) + \max_x \underbrace{\mathbb{E}^{\mathbb{P}}[F(2,V_\vartheta(2))|V_\vartheta(1)=v]}_{f(x)}$

$t=1$ CASE

$\max_{x\in[0,1]} f(x) \to \vartheta_1(1)(f_h^1) = \frac{v x^*}{S(1)(f_h^1)}$

$\vartheta_0(1)(f_h^1) = \frac{v(1-x^*)}{B(1)(f_h^1)}$

* **IS $X(t)$ MEASURABLE WRT $\mathcal{P}_{t-1}$ AT $t=1,2$?**

- CHECK THAT $X(t)$ IS CONSTANT IN ALL $\mathcal{P}_t$:

EG. $\to X(2) = \begin{cases} 1 & \omega_1 \\ 1 & \omega_2 \\ 0 & \omega_3 \end{cases}$

WITH $\{\omega_1,\omega_2\} = f_1^1$, $\{\omega_3\} = f_2^1$; $X(2)$ CONST IN f_i^1, SO IT IS MEASURABLE

$\to X(1)$ MUST BE CONSTANT BETWEEN f_1^1, f_2^1 TO BE MEASURABLE

* **COMPUTE NA PRICE FOR $t=0,1$** (IF YOU START FROM THE FINAL PRICES)

1. CALCULATE TERMINAL PAYOFF
2. COMPUTE $t-1$ PRICE, FOR EACH $\mathcal{P}_i$

$S_x(t-1)(f_i^{t-1}) = \mathbb{E}^{\mathbb{Q}}\left[\frac{x(t)}{1+r(t-1)}\middle|\mathcal{P}_i\right]$

SAME TERMINAL PRICE $\Downarrow$ SAME INITIAL PRICE (ASSUMING SAME INFO STRUCTURE)

EG $S_x(1) = \mathbb{E}\left[\frac{X(2)}{1+r(1)}\middle|\mathcal{P}_1\right] = \begin{cases} \dots & f_1^1 \\ \dots & f_2^1 \end{cases}$ DEPENDS ON $\mathcal{P}_1$ $\to$ USE THE $\mathbb{P}$ CONDITIONED ON THE f'

$S_x(0) = \mathbb{E}\left[\frac{S_x(1)}{1+r(0)}\right]$

* THE SUM COMES FROM THE UTILITY OF HAVING DIFFERENT UTILITY FOR WEALTH AT DIFFERENT TIMES

INTEREST-RATE SWAP

$\boxed{T=2 \text{ MATURITY}, \ x(1) = \bar{r}-r(0),\ x(2) = \bar{r}-r(1)}$

USE GS FORMULA BECAUSE YOU KNOW $x(2), x(1)$

COMPUTE NA PRICE AT $t=0$

$S_x(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{x(1)}{1+r(0)} + \frac{x(2)}{(1+r(0))(1+r(1))}\right]$ F

1. $S_x(0) = k$
2. SOLVE FOR $\bar{r}$ (USE THE DEFINITION)

FIND REPLICATING STRATEGY

1. FIND PAYOFF MATRIX FOR EACH SUBMARKET
2. FIND ϑ ST. $A_i\vartheta = \vartheta'$ WITH ϑ, ϑ' HAVING COST EQUAL TO THE OPTION

$V_\vartheta(t) = \binom{B(t)}{S(t)}(\text{OPTION})$

3. COMPOSE THE REPLICATING STRATEGIES: THE AGENT KNOWS HOW TO SWITCH ASSETS BASED ON THE INFORMATION PARTITION

DETERMINE BUY-AND-HOLD ϑ THAT REPLICATES ... ($T=2$, $K=100$):

1. WRITE PAYOFF OF ... AS A SUM OF OTHER PAYOFFS
2. COMPUTE INITIAL COST OF ϑ

\+ **FIND # RISKY S UNITS TRADED AT $t=1$ IN f_i^1, GIVEN $\omega=1$ AT $t=1$**

$\vartheta_1^*(1) = \frac{\alpha_1(1)V_\vartheta(1)}{S(1)}$

$\alpha_1(1) = x_1^*$

$\omega = V_\vartheta(1) = 1$

A REPLICATING STRATEGY TO SOME DERIVATIVE HAS COST ST.

PROFIT = COST OF DERIV. + COST OF STRATEGY $= X(1) + V_\vartheta(1)$

USES

GORDON-SHAPIRO FORMULA F $\to$ DOING VALUATION WHEN YOU ARE TOLD THE PRICES IN DIFFERENT MOMENTS IN TIME

$S_x(t) = \mathbb{E}^{\mathbb{Q}}_t\left[\sum_{\tau=t+1}^{T} \frac{x(\tau)}{B(\tau)}B(t)\right]$

$S_x(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{x(1)}{B(1)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{x(2)}{B(2)}\right]$

USE RNP, ENSURES NO ARBITRAGE

$t=0$ CASE

$F(0,v_0) = \max_{\vartheta_0,\vartheta_1} \mathbb{E}^{\mathbb{P}}[F(1,V_\vartheta(1))|V_\vartheta(0)=v_0]$

$= \ln(v_0) + \max_x \underbrace{\mathbb{E}^{\mathbb{P}}[\ln(1+r(0)+x(\frac{S(1)}{S(0)}-(1+r(0))))]}_{f(x)}$

$\max_x f(x) \to x^*$

BINOMIAL POWER MODEL, UTILITY

$F(2,v) = \frac{v^{1-\gamma}-1}{1-\gamma}$

$F(1,v) = \max_x v^{1-\gamma}\mathbb{E}^{\mathbb{P}}\left[\frac{((1+r)+x(u-(1+r)))^{1-\gamma}}{1-\gamma}p + \frac{((1+r)+x(d-(1+r)))^{1-\gamma}}{1-\gamma}(1-p)\right] - \frac{1}{1-\gamma}$

$x = \frac{\left(\left(\frac{((1+r)-d(1-p))}{(u-(1+r))p}\right)^{1/\gamma}-1\right)(1+r)}{(d-(1+r))-(u-(1+r))\left(\frac{((1+r)-d(1-p))}{(u-(1+r))p}\right)^{1/\gamma}}$

VALUE PROCESS OF STRATEGY ϑ

$V_\vartheta(t) = \begin{cases} \vartheta_0(t)B(t) + \sum_{j=1}^N \vartheta_j(t)S_j(t) & t=0,\dots,T-1 \\ \vartheta_0(T-1)B(T) + \sum_{j=1}^N \vartheta_j(T-1)S_j(T) & t=T \end{cases}$

CASHFLOW PROCESS

$C_\vartheta(t) = \sum_{j=0}^{N}\left[(\vartheta_0(t-1)-\vartheta_0(t))S_j(t)\right]$

SELF FINANCING ϑ:

$C_\vartheta(t) = 0 \quad t=1,\dots,T-1$

$V_\vartheta(t) = \vartheta(t-1)B(t) + \sum_j \vartheta_j(t-1)S_j(t)$

The return of the portfolio is $w^T R$. Its expected value is $w^T \mu$ and its variance is $w^T \Sigma w$.

The mean-variance problem as stated by Markowitz (1952) reads

$$\min_{w \in R^N} w^T \Sigma w \tag{4.1}$$
$$\text{s.t.} \quad w^T \mathbf{1} = 1 \tag{4.2}$$
$$w^T \mu = c \tag{4.3}$$

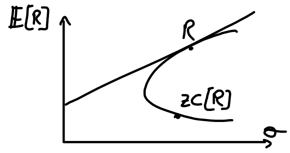
Therefore, we state the mean-variance allocation problem when a risk-free security is traded as

$$\min_{w \in \mathbb{R}^N} w^T \Sigma w \tag{4.4}$$
$$\text{s.t.} \quad w^T \mu + (1 - w^T \mathbf{1}) R_f = c \tag{4.5}$$

Zero-beta portfolio return on the mean-variance frontier

Given any return $R \neq R^{MIN}$ on the mean-variance frontier, there exists a unique uncorrelated portfolio return $zc[R]$ on the frontier.

Given R on the frontier with $R = R^* + wR^{e^*}$, we determine $zc[R]$ as $zc[R] = R^* + w^{zc[R]} R^{e^*}$ imposing



$$\mathbb{C}ov\,[R, zc[R]] = 0.$$

i.e.

$$\begin{aligned} & \mathbb{C}ov\left[R^* + wR^{e^*}, R^* + w^{zc[R]} R^{e^*}\right] \\ = \; & \mathbb{V}ar\,[R^*] + w^{zc[R]} w \mathbb{V}ar\left[R^{e^*}\right] - w\mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right] - w^{zc[R]}\mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right] = 0 \end{aligned}$$

so that

$$w^{zc[R]} = \frac{w\mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right] - \mathbb{V}ar\,[R^*]}{w\mathbb{V}ar\left[R^{e^*}\right] - \mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right]}.$$

Constant mimicking portfolio return

The third proxy for the riskless return is the traded return of the payoff that gets closer to 1,

$$proj\,[1|A]\,.$$

Its return is

$$R^{CMR} = \frac{proj\,[1|A]}{\pi\,[proj\,[1|A]]},$$

where $\pi\,[proj\,[1|A]] \neq 0$ as we will check later on. Recalling from Theorem 54 that $A = \langle \alpha R^* \rangle \oplus A_0$, we also have that $\mathcal{L}^2 = A^{\perp} \oplus A = A^{\perp} \oplus \langle \alpha R^* \rangle \oplus A_0$.

$$R^{CMR} = \frac{proj\,[1|A]}{\pi\,[proj\,[1|A]]} = R^* + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\,[R^*]} R^{e^*}$$

which implies $w^{CMR} = \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]}$.

$$\begin{aligned} \mathbb{E}\left[R^{CMR}\right] &= \mathbb{E}\left[R^* + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\,[R^*]} R^{e^*}\right] \\ &= \mathbb{E}\,[R^*] + \frac{\mathbb{E}\left[(R^*)^2\right]\mathbb{E}\left[R^{e^*}\right]}{\mathbb{E}\,[R^*]} \\ &= \frac{\mathbb{E}\,[R^*]^2 + \mathbb{E}\left[(R^*)^2\right]\mathbb{E}\left[R^{e^*}\right]}{\mathbb{E}\,[R^*]} \end{aligned}$$

Remark 65 *The constant mimicking portfolio return satisfies* $\mathbb{E}\left[zc\left[R^{CMR}\right]\right] = 0$. *Indeed,* $w^{zc\left[R^{CMR}\right]} = \frac{w^{CMR}\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right] - \mathbb{V}ar[R^*]}{w^{CMR}\mathbb{V}ar\left[R^{e^*}\right] - \mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right]}$ *with* $w^{CMR} = \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]}$ *so that*

$$\begin{aligned} \mathbb{E}\left[zc\left[R^{CMR}\right]\right] &= \mathbb{E}\,[R^*] + \frac{w^{CMR}\mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right] - \mathbb{V}ar\,[R^*]}{w^{CMR}\mathbb{V}ar\left[R^{e^*}\right] - \mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right]}\mathbb{E}\left[R^{e^*}\right] \\ &= \mathbb{E}\,[R^*] + \frac{\frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]}\mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right] - \mathbb{V}ar\,[R^*]}{\frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]}\mathbb{V}ar\left[R^{e^*}\right] - \mathbb{E}\,[R^*]\,\mathbb{E}\left[R^{e^*}\right]}\mathbb{E}\left[R^{e^*}\right] \\ &= \frac{\mathbb{E}\left[(R^*)^2\right]\left(1 - \mathbb{E}\left[R^{e^*}\right]\right) + \mathbb{E}\left[(R^*)^2\right]\left(-1 + \mathbb{E}\left[R^{e^*}\right]\right)}{\frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]}\left(1 - \mathbb{E}\left[R^{e^*}\right]\right) - \mathbb{E}\,[R^*]} = 0 \end{aligned}$$

FRONTIER WITH RISK-FREE ASSET

We label by $\widetilde{\pi}$ the extension of the pricing functional π to $\widetilde{A}$ such that

$$\widetilde{\pi}\,[1] = \frac{1}{R_f}$$

and by $\widetilde{m}^*$ the (unique) traded stochastic discount factor for this extended market, such that

$$R_f = \frac{1}{\mathbb{E}\,[\widetilde{m}^*]}.$$

Accordingly, we define

$$\begin{aligned} \widetilde{A}_1 &= \left\{R \in \widetilde{A} \,\middle|\, \widetilde{\pi}\,[R] = 1\right\} \\ \widetilde{A}_0 &= \left\{R^e \in \widetilde{A} \,\middle|\, \pi\,[R^e] = 0\right\} \\ \widetilde{R}^* &= \frac{\widetilde{m}^*}{\pi\,[\widetilde{m}^*]} \\ \widetilde{R}^{e^*} &= proj\left[1|\widetilde{A}_0\right] \in \widetilde{A}_0. \end{aligned}$$

Notice that, in general, $R^* \neq \widetilde{R}^*$ and $R^{e^*} \neq \widetilde{R}^{e^*}$.

The following decompositions from the previous sessions hold

$$\begin{aligned} \widetilde{A} &= \left\langle \alpha \widetilde{R}^* \right\rangle \oplus \widetilde{A}_0 \\ \widetilde{A}_0 &= \left\langle \beta \widetilde{R}^e \right\rangle \oplus \left\{\widetilde{n} \in \widetilde{A}_0 : \quad \mathbb{E}\left[\widetilde{R}^e \widetilde{n}\right] = 0\right\}. \end{aligned}$$

Claim 66 $R \in \widetilde{A}_1 \iff$ *there exist* $w \in \mathbb{R}$, $n \in \widetilde{A}_0$ *such that* $\mathbb{E}\,[n] = \mathbb{E}\left[\widetilde{R}^* n\right] = \mathbb{E}\left[\widetilde{R}^e n\right] = 0$ *and*

$$R = \widetilde{R}^* + w\widetilde{R}^e + n.$$

Claim 67 $R^{MV} \in A_1$ *is on the mean variance frontier if and only if*

$$R^{MV} = \widetilde{R}^* + w\widetilde{R}^e$$

for some $w \in \mathbb{R}$.

Proposition 68 R_f *is on the mean-variance frontier and can be written as*

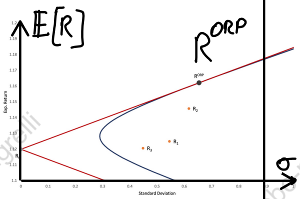
$$R_f = \widetilde{R}^* + R_f \widetilde{R}^e.$$

We now draw the frontier on the standard deviation-expected value plane when the risk-free security is traded. Since $\widetilde{R}^*$ and R_f are both on the frontier, according to the Two-fund Separation Theorem, R^{MV} is on the frontier if and only if

$$R^{MV} = (1 - \alpha)\, R_f + \alpha \widetilde{R}^*.$$

Then, we can compute the expected value, the variance and the standard deviation of any return on the frontier as

$$\begin{aligned} \mathbb{E}\left[R^{MV}\right] &= (1 - \alpha)\, R_f + \alpha \mathbb{E}\left[\widetilde{R}^*\right] \\ &= R_f + \alpha\left[\mathbb{E}\left[\widetilde{R}^*\right] - R_f\right], \\ \mathbb{V}ar\left[R^{MV}\right] &= \alpha^2 \mathbb{V}ar\left[\widetilde{R}^*\right] \\ \sigma\left[R^{MV}\right] &= |\alpha|\, \sigma\left[\widetilde{R}^*\right] \end{aligned}$$



Definition 70 *Assume* $R_f < \mathbb{E}\left[R^{MIN}\right]$, *and consider in the* $\sigma\,[R]$-$\mathbb{E}\,[R]$

plane the tangent from $(0, R_f)$ *to the efficient frontier with risky assets only. We call optimal risky portfolio return* R^{ORP}, *the return in* A_1 *such that* $\left(\sigma\left[R^{ORP}\right], \mathbb{E}\left[R^{ORP}\right]\right)$ *are the coordinates of the tangency point.*

Remark 72 *As a consequence of the previous result and of the Two-fund Separation Theorem,* $R \in \widetilde{A}_1$ *is a mean-variance return if and only if*

$$R = (1 - \alpha)\, R_f + \alpha R^{ORP}$$

As we will see in Chapter 6, this fact will be crucial to establish the classical CAPM equation.

With a single factor, the beta-pricing equation boils down to

$$\mathbb{E}[R] = \gamma + \beta_{R,f}\lambda, \qquad \forall R \in A_1$$

Most importantly, in this case the factor loading takes a very simple and intuitive form, that is

$$\beta_{R,f} = \frac{\mathbb{C}ov[R,f]}{\mathbb{V}ar[f]}$$

Theorem 73 *Assume $\mathbb{V}ar[f] \neq 0$. There exist $\gamma, \lambda \in \mathbb{R}$, $\gamma \neq 0$ such that*

$$\mathbb{E}[R] = \gamma + \beta_{R,f}\lambda, \qquad \forall R \in A_1$$

with

$$\beta_{R,f} = \frac{\mathbb{C}ov[R,f]}{\mathbb{V}ar[f]}$$

if and only if there exist $a, b \in \mathbb{R}$ with $a + b\mathbb{E}[f] \neq 0$ s.t.

$$m = a + bf \quad \textit{is a SDF}$$

We assume now that the factor is a traded return, $\widehat{R} \in A_1$, in a market that does not trade the risk-free rate R_f. Recall from Subsection 4.2.4 that in this case the global minimum variance return takes the form

$$R^{MIN} = R^* + \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}[R^{e^*}]} R^{e^*},$$

while the constant-mimicking portfolio return on the MV frontier takes the form

$$R^{CMR} = R^* + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]} R^{e^*}.$$

Theorem 74 *Given any return $\widehat{R} = R^* + \widehat{w}R^{e^*} + \widehat{n} \in A_1$, $\exists a, b \in \mathbb{R}$ such that*

$$m = a + b\hat{R}$$

is a SDF if and only if

$$\widehat{n} = 0 \quad \textit{and} \quad \widehat{w} \neq \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}[R^*]},$$

that is $\hat{R}$ is a MV return different from the constant-mimicking return R^{CMR}.

Theorem 75 *Given any return $\widehat{R} = R^* + \widehat{w}R^{e^*} + \widehat{n} \in A_1$,*

$$\mathbb{E}[R] = \mathbb{E}\left[zc\left[\widehat{R}\right]\right] + \beta_{R,\hat{R}}\left(\mathbb{E}\left[\hat{R}\right] - \mathbb{E}\left[zc\left[\widehat{R}\right]\right]\right), \quad \forall R \in A_1$$

with

$$\beta_{R,\hat{R}} = \frac{\mathbb{C}ov\left[R,\hat{R}\right]}{\mathbb{V}ar\left[\hat{R}\right]}$$

if and only if

$$\widehat{n} = 0 \quad \textit{and} \quad \widehat{w} \neq \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}[R^{e^*}]},$$

that is $\hat{R}$ is a MV return different from the global minimum variance return R^{MIN}.

Theorem 76 *Given any return $\widehat{R} \in A_1$ the following statements are equivalent:*

1. $\exists a, b \in \mathbb{R}$ such that
$$a + bE(\widehat{R}) \neq 0 \quad \text{and} \quad m = a + b\hat{R} \quad \textit{is a SDF;}$$
2. $\hat{R}$ is a MV return with $\widehat{R} \neq R^{CMR}$ and $\widehat{R} \neq R^{MIN}$;
3. The frontier return uncorrelated with $\widehat{R}$ has non-vanishing mean, i.e. $\mathbb{E}\left[zc\left[\widehat{R}\right]\right] \neq 0$, and
$$\mathbb{E}[R] = \mathbb{E}\left[zc\left[\widehat{R}\right]\right] + \beta_{R,\hat{R}}\left(\mathbb{E}\left[\widehat{R}\right] - \mathbb{E}\left[zc\left[\widehat{R}\right]\right]\right) \quad \forall R \in A_1$$
where as usual
$$\beta_{R,\hat{R}} = \frac{\mathbb{C}ov\left[R,\hat{R}\right]}{\mathbb{V}ar\left[\hat{R}\right]}$$

established in the previous Section for the case without a risk-free rate can be restated as follows when a risk-free return is available.

Theorem 77 *Given any return $\hat{R} \in \widetilde{A}_1$ the following statements are equivalent:*

1. $\exists a, b \in \mathbb{R}$ such that
$$m = a + b\hat{R} \quad \textit{is a SDF}$$
2. $\hat{R}$ is a MV return and $\hat{R} \neq R_f$;
3. $$\mathbb{E}[R] = R_f + \beta_{R,\hat{R}}\left[\mathbb{E}\left[\hat{R}\right] - R_f\right], \qquad \forall R \in \widetilde{A_1}$$
where as usual
$$\beta_{R,\hat{R}} = \frac{\mathbb{C}ov\left[R,\hat{R}\right]}{\mathbb{V}ar\left[\hat{R}\right]}$$

By Theorem 73, therefore, we conclude that, in the presence of a risk-free return, the beta-pricing equation

$$\mathbb{E}[R] = R_f + \beta_{R,\hat{R}}\left[\mathbb{E}\left[\hat{R}\right] - R_f\right], \qquad \forall R \in \widetilde{A}_1$$

maps in a one-to-one way into the SDFs

$$m = \frac{1}{R_f} + \mathbb{E}\left[\widehat{R}\right]\frac{\mathbb{E}\left[\widehat{R}\right] - R_f}{R_f\mathbb{V}ar\left[\widehat{R}\right]} - \frac{\mathbb{E}\left[\widehat{R}\right] - R_f}{R_f\mathbb{V}ar\left[\widehat{R}\right]}\hat{R}.$$

The Multi-period Binomial Model

6.1 Description of the Model

The multi-period binomial model involves two securities.

The first one is the *risk-free* asset B yielding a constant one-period interest rate, i.e. $r(t) = r > 0$ for $t = 0, ..., T-1$, employing the notation introduced in the previous chapters. The risk-free asset B at a generic time t will then have price $B(t) = (1+r)^t$.

The second security is the risky stock S. Given $S(t)$, the time-t price of the security S can take only two values at the following date $t+1$:

$$
\begin{array}{ccll}
 & \nearrow & S(t)\,u & \text{with probability } p \\
S(t) & & & \\
 & \searrow & S(t)\,d & \text{with probability } 1-p \\
t & & t+1 &
\end{array}
$$

for $t = 0, ..., T-1$.

$$\mathcal{A}(t)(f_h^t) = \begin{bmatrix} (1+r)^{t+1} & S(t)(f_h^t)\cdot u \\ (1+r)^{t+1} & S(t)(f_h^t)\cdot d \end{bmatrix}$$

$$
\begin{array}{ccccccc}
 & & & & & & Su^3 \\
 & & & & Su^2 & & \\
 & & Su & & & & Su^2d \\
S & & & & Sud & & \\
 & & Sd & & & & Sud^2 \\
 & & & & Sd^2 & & \\
 & & & & & & Sd^3 \\
t=0 & & t=1 & & t=2 & & t=3
\end{array}
$$

We finally write the *relative increment* of S between t and $t+1$:

$$\frac{\Delta S(t)}{S(t)} = \frac{S(t+1)-S(t)}{S(t)} = \begin{cases} u-1 & \text{with probability } p \\ d-1 & \text{with probability } 1-p \end{cases}$$

$$\mathbb{P}\left[S(t) = Su^k d^{t-k}\right] = \binom{t}{k} p^k (1-p)^{t-k}$$

We know that no-arbitrage holds in the one-period binomial model if:

$$d < 1+r < u,$$

probability in every one-period submarket. In particular, the risk-neutral probability of an up movement of $S(t)$ between t and $t+1$ is:

$$\mathbb{Q}\left[S(t+1) = S(t)\cdot u\right] = q = \frac{1+r-d}{u-d}$$

while that of a down movement is

$$\mathbb{Q}\left[S(t+1) = S(t)\cdot d\right] = 1-q = \frac{u-(1+r)}{u-d}$$

We now want to price the same European call option on S with strike price K and maturity T, by employing point 3. in Proposition 43. That result guarantees that the unique no-arbitrage price process of the option is given by the conditional expected value, under the risk-neutral measure, of the discounted future cashflows of the option. In particular, from equation (5.4) we get

$$
\begin{aligned}
c(t) &= S_X(t) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{\tau=t+1}^{T}(1+r)^{-(\tau-t)}X(\tau)\middle|\mathcal{P}_t\right] = \\
&= \mathbb{E}^{\mathbb{Q}}\left[(1+r)^{-(T-t)}\max(S(T)-K;0)\middle|\mathcal{P}_t\right]
\end{aligned}
$$

7.1 Features of European Options

Consider a financial market that is complete and free of arbitrage opportunities. We denote by $\mathbb{Q}$ the unique risk neutral measure used to price derivatives and by $\mathcal{P}$ the information structure of investors. Due to market completeness, a European derivative X can be replicated by a self-financing strategy ϑ. In this section we emphasize some important features of European derivatives that will be useful when dealing with American options. The no-arbitrage price $V^X(t)$ of X and the value of the replicating strategy coincide and are given by

$$\frac{V^X(t)}{B(t)} = \frac{V_\vartheta(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{B(T)}\middle|\mathcal{P}_t\right] \tag{7.1}$$

for any $t = 0, ..., T$.

In order to compute $V^X(0)$ we can either compute the expectation of $\frac{X(T)}{B(T)}$ under $\mathbb{Q}$, or exploit the following *backward recursion in time*:

$$\frac{V^X(T)}{B(T)} = \frac{X(T)}{B(T)}$$

7.2 Valuation of American Options

Let X be an **American** option with maturity T. The option can be exercised *at every intermediate date*

$$t = 1, ..., T-1, T$$

If the holder exercises at the date τ, the time-t No-Arbitrage value of the discounted payoff he gets is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X(\tau)}{B(\tau)}\middle|\mathcal{P}_t\right]$$

If the option was European with maturity τ, this would be the value of the option at time t, that is the value of the holder position. By properly choosing the exercise time τ, the holder of the option can increase the time t value of his position. Hence, if the holder is smart enough, he will maximize his expected payoff by choosing the suitable τ. Hence the value of the American option at time t is

$$\widetilde{V}(t) = \sup_{t\le\tau\le T}\mathbb{E}^{\mathbb{Q}}\left[\frac{X(\tau)}{B(\tau)}\middle|\mathcal{P}_t\right] \tag{7.4}$$

Proposition 48 ***(Backward recursive formula for the American option)*** *The discounted value of the American option $\widetilde{V}$ defined in (7.4) is given by*

$$\widetilde{V}(T) = \widetilde{X}(T)$$

and

$$\widetilde{V}(t) = \max\left\{\widetilde{X}(t); \mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}(t+1)\middle|\mathcal{P}_t\right]\right\}$$

for $t = T-1, ..., 0$

- $B(t)\cdot\mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}(t+1)\middle|\mathcal{P}_t\right]$ *is called continuation value at time t : it is the fair value in t of what the holder gains if he decides to wait till $t+1$.*
- $X(t)$ *is the immediate payoff at time t : what the holder gains if he exercises immediately.*

8.1.2 Maximizing the log-utility of terminal wealth in the binomial market with $T = 2$ via DP.

We first look for the maximizing weights. Since there are only two assets, we just have to determine the weights of the risky asset at $t = 0$ and $t = 1$. We denote these unknowns with $\alpha_1^*(0) = x_0$, $\alpha_1^*(1) = x_1$. All intermediate utilities are null. The value function at $T = 2$ is

$$F(2, v) = \ln v$$

and for $t = 1$, recalling that $V_\alpha(2) = V_\alpha(1)(\alpha_0(1)(1+r) + \alpha_1(1)\xi(2))$, we obtain

$$F(1, v) = \max_{\alpha(1)}\left(\mathbb{E}\left[F(2, V(2))\mid V(1) = v\right]\right)$$

$$x_1^* = R\frac{(u-R)p + (d-R)(1-p)}{(R-d)(u-R)}$$

$$
\begin{aligned}
\alpha_1^*(0) &= \alpha_1^*(1) = x_1^* = R\frac{(u-R)p + (d-R)(1-p)}{(R-d)(u-R)} \\
\alpha_0^*(0) &= \alpha_0^*(1) = 1 - x_1^*
\end{aligned}
$$

$$\vartheta_1(0) = \frac{\alpha_1^*(0)}{S(0)} = \frac{x_1^*}{S(0)} \text{ and } \vartheta_0(0) = 1 - \alpha_1^*(0) = 1 - x_1^*$$

; to

$$
\begin{aligned}
V_\vartheta(1) &= \vartheta_0(0)R + \vartheta_1(0)S(0)\xi \\
&= (1-x_1^*)R + x_1^*\xi
\end{aligned}
$$

rategy at $t = 1$ is

$$
\begin{aligned}
\vartheta_1(1) &= \frac{\alpha_1(1)V_\vartheta(1)}{S(1)} = \frac{x_1^*((1-x_1^*)R + x_1^*\xi)}{S(0)\xi} \\
\vartheta_0(1) &= \frac{\alpha_0(1)V_\vartheta(1)}{B(1)} = \frac{(1-x_1^*)((1-x_1^*)R + x_1^*\xi)}{R}
\end{aligned}
$$

In this section we maximize the utility from terminal wealth with a generic power utility in a two-period binomial model. Our main goal is to show how the investor's risk aversion γ modifies the optimal portfolio strategy. The terminal utility is

$$u(w) = \frac{w^{1-\gamma}-1}{1-\gamma} \quad \text{with } \gamma > 0$$

$$F(1, v) = \max_{\alpha(1)}\left(\mathbb{E}\left[F(2, V(2))\mid V(1) = v\right]\right)$$

$$A = \left(\frac{(R-d)}{(u-R)}\frac{(1-p)}{p}\right)^{\frac{1}{\gamma}} = \left(\frac{\frac{(R-d)}{u-d}}{\frac{(u-R)}{u-d}}\frac{(1-p)}{p}\right)^{\frac{1}{\gamma}} = \left(\frac{q}{(1-q)}\frac{(1-p)}{p}\right)^{\frac{1}{\gamma}}$$

The function we have to maximize over x_0 is

$$\mathbb{E}\left[((1-x_0)R + x_0\xi)^{1-\gamma}\right] = h(x_0) \text{ that is max for } x_0 = x_1^* = \frac{R(1-A)}{A(u-R)+R-d}.$$

We therefore get

$$F(0, v) = v^{1-\gamma}(h(x_1^*))^2 - \frac{1}{1-\gamma}$$

But the most important result is the optimal weight

$$\alpha_1(0) = \alpha_1(1) = x_1^* = \frac{R(1-A)}{A(u-R)+R-d}$$

a) The payoff matrix of this financial market is

$$\mathcal{A}_\alpha = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 2\alpha \\ 1.25 & \alpha & \alpha \end{bmatrix}$$

and its determinant is equal to

$$\begin{aligned} \det \mathcal{A}_\alpha &= 1.25 \det \begin{bmatrix} 1 & 15 & 5 \\ 1 & 0 & 2\alpha \\ 1 & \alpha & \alpha \end{bmatrix} \\ &= 1.25(-2\alpha^2 - 20\alpha). \end{aligned}$$

It holds

$$\det \mathcal{A}_\alpha = 0 \Longleftrightarrow \alpha(\alpha - 10) = 0$$

Therefore, if $\alpha \notin \{0, 10\}$, $rk(\mathcal{A}_\alpha) = 3 = |\Omega|$ and the market is complete. If $\alpha \in \{0, 10\}$, as the minor given by the first two rows and the first two columns is different from zero and $\det \mathcal{A}_\alpha = 0$, we have $rk(\mathcal{A}_\alpha) = 2 < 3 = |\Omega|$ and the market is not complete.

c) When $\alpha = 10$ and $\beta = 15$, the LOP does not hold and the market cannot be arbitrage free. In particular, we know that the only initial price of S_2 which is compatible with the LOP is $S_2(0) = 11.5$. Therefore, if S_2 trades at the initial price of 15, it is actually too expensive and an arbitrage strategy consists on (short) selling it while buying its replicating strategy computed above. Formally, the strategy underlying this arbitrage opportunity is given by

$$\underbrace{\vartheta_0 = 16,\ \ \vartheta_1 = -1}_{\text{replication of } S_2} \ \text{ and } \ \underbrace{\vartheta_2 = -1}_{\text{short position on } S_2}.$$

b) For $\alpha \notin \{0, 10\}$, there are no redundant securities and, therefore, the LOP holds.

If $\alpha = 0$ the payoff matrix becomes

$$\mathcal{A}_0 = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 0 \\ 1.25 & 0 & 0 \end{bmatrix}.$$

In this case, $S_2(1) = 3S_1(1)$ and the LOP holds if and only if $S_2(0) = 3S_1(0)$. This constraint delivers

$$4.5 = 3\beta$$

or $\beta = 1.5$. Therefore, if $\alpha = 0$ the LOP holds if and only if $\beta = 1.5$.

If $\alpha = 10$ the payoff matrix becomes

$$\mathcal{A}_{10} = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 20 \\ 1.25 & 10 & 10 \end{bmatrix}.$$

As we know that one of the three securities is redundant, say S_2, we could solve the linear system

$$[B(1) \quad S_1(1)] \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \end{bmatrix} = S_2(1)$$

which is equivalent to

$$\begin{cases} 1.25\vartheta_0 + 15\vartheta_1 = 5 \\ 1.25\vartheta_0 + 0\vartheta_1 = 20 \\ 1.25\vartheta_0 + 10\vartheta_1 = 10 \end{cases}$$
$$\vartheta_0 = 16 \ \text{ and } \ \vartheta_1 = -1.$$

Now, as $S_2(1) = 16B(1) - S_1(1)$, the LOP holds if and only if this relationship is verified also at $t = 0$, namely if

$$\begin{aligned} S_2(0) &= 16B(0) - S_1(0) \\ \beta &= 16 - 4.5 = 11.5. \end{aligned}$$

Therefore, if $\alpha = 10$, the LOP holds if and only if $\beta = 11.5$.

d) When $\alpha = 10$ and $\beta = 11.5$, the LOP holds. To check whether NA holds in the market as well, we look for state price vectors, solving the system

$$(\mathcal{A}_{10})^T \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} B(0) \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

which is equivalent to

$$\begin{cases} 1.25\psi_1 + 1.25\psi_2 + 1.25\psi_3 = 1 \\ 15\psi_1 + 0\psi_2 + 10\psi_3 = 4.5 \\ 5\psi_1 + 20\psi_2 + 10\psi_3 = 11.5 \end{cases}$$

which is solved by

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} - \frac{2}{3}\psi_3 \\ \frac{1}{2} - \frac{1}{3}\psi_3 \\ \psi_3 \end{bmatrix}.$$

we know one security is redundant

These state price vectors have strictly positive component as long as

$$\psi_1 = \frac{3}{10} - \frac{2}{3}\psi_3 > 0 \Longleftrightarrow \psi_3 < \frac{9}{20} = 0.45$$

If $0 < \psi_3 < \frac{9}{20}$, the market admits infinitely many strictly positive state price vectors and, according to the FFTAP, the market is arbitrage free (and incomplete).

new contingent claim, find NA price

e) As

$$\min\{S_1(1), S_2(1)\} = \begin{cases} \min\{15, 5\} = 5 & \text{on } \omega_1 \\ \min\{0, 20\} = 0 & \text{on } \omega_2 \\ \min\{10, 10\} = 10 & \text{on } \omega_3 \end{cases}$$

we have

$$X(1) = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}.$$

The no arbitrage price at $t = 0$ is equal to

$$\begin{aligned} X(0) &= [\ \psi_1 \ \ \psi_2 \ \ \psi_3\ ]\, X(1) \\ &= 10\psi_1 + 10\psi_3 \\ &= 10\left(\frac{3}{10} - \frac{2}{3}\psi_3\right) + 10\psi_3 \\ &= \frac{10}{3}\psi_3 + 3. \end{aligned}$$

As NA is preserved as long as $0 < \psi_3 < \frac{9}{20}$, the set of admissible no arbitrage prices at $t = 0$ is

$$X(0) \in (3, 4.5)$$

as $\frac{10}{3} \cdot 0 + 3 = 3$ and $\frac{10}{3}\frac{9}{20} + 3 = 4.5$.

f) If the contingent claim trades at $4 \in (3, 4.5)$, the extended market is arbitrage free. Moreover,

$$\frac{10}{3}\psi_3 + 3 = 4$$

delivers $\psi_3 = \frac{3}{10}$ and

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} - \frac{2}{3}\frac{3}{10} = \frac{1}{10} = 0.1 \\ \frac{1}{2} - \frac{1}{3}\frac{3}{10} = \frac{4}{10} = 0.4 \\ \frac{3}{10} = 0.3 \end{bmatrix}.$$

As there exists one and only one strictly positive state price vector, the market is arbitrage free and complete (according to the SFTAP). From the relationships

$$\psi_k = \frac{q_k}{1 + r} \ \text{ and } \ m_k = \frac{q_k}{p_k(1 + r)}$$

we get

$$m_k = \frac{q_k}{p_k(1 + r)} = \frac{\psi_k(1 + r)}{p_k(1 + r)} = \frac{\psi_k}{p_k}$$

Unique traded SDF m^*

a) We know from the Lecture Notes that it holds

$$\begin{aligned} \vartheta_{m^*} &= \left(\mathbb{E}\left[xx^T\right]\right)^{-1} S \\ m^* &= \mathcal{A}\left(\mathbb{E}\left[xx^T\right]\right)^{-1} S \end{aligned}$$

where

$$\mathcal{A} = \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \ \text{ and } \ S = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore, we get

$$\begin{aligned} \vartheta_{m^*} &= \begin{bmatrix} 1.4 & -3.4 \\ -3.4 & 8.4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} m^* &= \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \\ &= \begin{bmatrix} 1.0 \\ 0.8 \\ 0.4 \end{bmatrix}. \end{aligned}$$

Suppose the option of point 4 is now of American type. Is there any optimal early exercise opportunity? If your answer is positive, find the early exercise premium of the option.

The American option coincides with the European one at maturity.

$$p_{Am}(1) = \max\left((K - S(1))^+; p(1)\right) = \begin{cases} \max(0; 0) = 0 & \text{if } f_1^1 \\ \max(1.9; 1.7157) = 1.9 > 1.7157 & \text{if } f_2^1 \end{cases}.$$

and we see that there is an optimal early exercise opportunity at $t = 1$ on f_2^1. The value at $t = 0$ of the American option is

$$p_{Am}(0) = \max\left((K - S(0))^+; \mathbb{E}^{\mathbb{Q}}\left[\frac{p_{Am}(1)}{1 + r(0)}\right]\right) = \max(0; 1.1068) = 1.1068$$

since

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{p_{Am}(1)}{1 + r(0)}\right] = \frac{0 \cdot 0.4 + 1.9 \cdot 0.6}{1.03} = 1.1068$$

Hence the early exercise premium is

$$p_{Am}(0) - p(0) = 1.1068 - 0.99944 = 0.10736$$

$\mathbb{P}(\omega_1 | f_1^1) = \mathbb{P}(\omega_2 | f_1^1) = 50\%$, $\mathbb{P}(\omega_3 | f_2^1) = 70\%$ and $\mathbb{P}_1(\omega_4 | f_2^1) = 30\%$

r setting we have to look for $\vartheta(1)(f_2^1)$ such that

$$\vartheta_0(1) B(1) + \vartheta_1(1) S(1) = V(1) = v$$
$$x = \frac{\vartheta_1(1) S(1)}{v} \ \text{ and } \ 1 - x = \frac{\vartheta_0(1) B(1)}{v}$$
$$\vartheta_1(1) = x\frac{v}{S(1)} \ \text{ and } \ \vartheta_0(1) = (1 - x)\frac{v}{B(1)}$$
$$V(2) = \vartheta_1(1) S(2) + \vartheta_0(1) B(2)$$

and

$$\begin{aligned} F(1, v) &= \max_x \left(\mathbb{E}_1\left[F(2, V(2))\right]\right) \\ &= \max_x \left(\mathbb{E}_1\left[\ln\left(v\left(x\frac{S(2)}{S(1)} + (1 - x)(1 + r(1))\right)\right)\right]\right) \\ &= \ln v + \max_x \left(\mathbb{E}_1\left[\ln\left(x\frac{S(2)}{S(1)} + (1 - x)(1 + r(1))\right)\right]\right) \end{aligned}$$

On f_2^1 the value function $F(1, v)$ becomes

$$\begin{aligned} F(1, v) &= \ln v + \max_x \left(\mathbb{E}_1\left[\ln\left(x\frac{S(2)}{S(1)} + (1 - x)(1 + r(1))\right)\right]\right) \\ &= \max_x \left(\mathbb{E}_1\left[\ln\left(v\left(x\frac{S(2)}{S(1)} + (1 - x)(1 + r(1))\right)\right)\right]\right) \\ &= \ln v + \max_x \left(\mathbb{E}_1\left[F(1, v)\right]\right) \\ &= \ln v + \max_x \left(0.7\ln(x1.1 + (1 - x)1.02) + 0.3\ln(x0.9 + (1 - x)1.02)\right) \end{aligned}$$

The function we have to maximize is

$$f(x) = 0.7\ln(0.08x + 1.02) + 0.3\ln(1.02 - 0.12x)$$

b) Determine R^* and its replicating strategy.

As m^* can be replicated by 0.2 units of S_1 and -0.2 units of S_2 we have

$$R^* = \frac{m^*}{\pi[m^*]}$$

$$\vartheta_{R^*} = \begin{bmatrix} \frac{0.2}{0.6} \\ \frac{-0.2}{0.6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

and

$$= \frac{m^*}{\mathbb{E}[(m^*)^2]}$$

$$R^* = \underbrace{\frac{1}{0.6}}_{\frac{1}{\pi[m^*]}} \underbrace{\begin{bmatrix} 1.0 \\ 0.8 \\ 0.4 \end{bmatrix}}_{m^*} = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}.$$

or

$$\begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can focus on the first two equations

$$\begin{cases} -\frac{2}{9} = 9x + 4y \\ \frac{2}{9} = 6x + 2y \end{cases}$$

solved by

$$x = \frac{2}{9} \text{ and } y = -\frac{5}{9}$$

that (of course) satisfy also the last equation $\frac{1}{9} = 3 \cdot \frac{2}{9} + 1 \cdot (-\frac{5}{9}) = \frac{1}{9}$. Therefore,

$$\vartheta_{R^{e^*}} = \begin{bmatrix} \frac{2}{9} \\ -\frac{5}{9} \end{bmatrix}$$

and

$$\vartheta_{R^{CMR}} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} + \frac{15}{11}\begin{bmatrix} \frac{2}{9} \\ -\frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ -\frac{12}{11} \end{bmatrix}.$$

EU PUT replication (choosing 2/5)

6. The terminal payoff of a *European put option on* S with maturity $T = 2$ and strike 100 is

$$put\,(2) = (100 - S\,(2))^+ = \begin{cases} 100 - S\,(2) & \text{if } S\,(2) < 100 \\ 0 & \text{otherwise} \end{cases}$$

This payoff is equal to the constant amount 100 plus the final payoff of a short position on X, because

$$-X\,(2) = \begin{cases} -S\,(2) & \text{if } S\,(2) < 100 \\ -100 & \text{otherwise} \end{cases}$$

and

$$100 - X\,(2) = \begin{cases} 100 - S\,(2) & \text{if } S\,(2) < 100 \\ 100 - 100 = 0 & \text{otherwise} \end{cases}$$

In the extended market this final payoff $100 - X\,(2)$ is obtained by buying at the initial date 100 units of the zero coupon bond of point 5. and by selling 1 unit of the derivative X. More formally consider the *buy-and-hold strategy* in the extended market

$$\begin{aligned} \vartheta_0\,(t) &= \vartheta_0 = 0 \quad \text{units of } B \\ \vartheta_1\,(t) &= \vartheta_1 = 0 \quad \text{units of } S \\ \vartheta_X\,(t) &= \vartheta_X = -1 \quad \text{units of } X \\ \vartheta_Y\,(t) &= \vartheta_Y = 0 \quad \text{units of } Y \\ \vartheta_{ZCB}\,(t) &= \vartheta_{ZCB} = 100 \quad \text{units of } ZCB \end{aligned}$$

for $t = 0, 1$. Then

$$\begin{aligned} C_\vartheta\,(2) &= V_\vartheta\,(2) = -1 \cdot X\,(2) + 100 \cdot ZCB\,(2) = -1 \cdot X\,(2) + 100 \cdot 1 \\ &= \begin{cases} 100 - S\,(2) & \text{if } S\,(2) < 100 \\ 100 - 100 = 0 & \text{otherwise} \end{cases} = (100 - S\,(2))^+ = put\,(2)\,. \end{aligned}$$

At $t = 1$ the cashflow of the strategy $C_\vartheta\,(1) = 0$, because ϑ is buy-and-hold. Therefore, the cashflow process of ϑ coincides with the cashflow process of the European put option: hence ϑ replicates the put option.

c) Knowing that

$$R^{e^*} = \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

find the constant mimicking portfolio return, R^{CMR}, and its replicating strategy.

$$R^{CMR} = R^* + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} R^{e^*}.$$

Then,

$$\begin{aligned} \mathbb{E}\left[R^*\right] &= \frac{1}{3}\left(\frac{1}{3}(5+4+2)\right) = \frac{11}{9} \\ \mathbb{E}\left[(R^*)^2\right] &= \frac{1}{3}\left(\frac{1}{3^2}\left(5^2+4^2+2^2\right)\right) = \frac{5}{3} \\ \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} &= \frac{\frac{5}{3}}{\frac{11}{9}} = \frac{15}{11} \end{aligned}$$

Therefore, we have

$$R^{CMR} = \underbrace{\frac{1}{3}\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}}_{R^*} + \underbrace{\frac{15}{11}}_{\frac{\mathbb{E}[(R^*)^2]}{\mathbb{E}[R^*]}} \underbrace{\frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}_{R^{e^*}} = \begin{bmatrix} \frac{15}{11} \\ \frac{18}{11} \\ \frac{9}{11} \end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} + \frac{15}{11}\frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

As for the replicating strategy of R^{CMR} we have

$$\vartheta_{R^{CMR}} = \vartheta_{R^*} + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} \vartheta_{R^{e^*}}.$$

To compute this, we must find $\vartheta_{R^{e^*}}$ such that

$$R^{e^*} = A\vartheta_{R^{e^*}}$$

Verifying process M a MARTINGALE (over Q)

4. The European digital option can be replicated by a dynamic investment strategy ϑ because the market is complete. The cost of replication V_ϑ equals the price of the option S_X by no-arbitrage. Hence

$$\begin{aligned} V_\vartheta(1)(f_1^1) &= S_X(1)(f_1^1) = 0 \\ V_\vartheta(1)(f_2^1) &= S_X(1)(f_2^1) = 0.5 \end{aligned}$$

and at $t = 0$

$$V_\vartheta(0) = S_X(0) = 0.196\,08$$

5. The process M is a martingale with respect to $\mathbb{Q}^S$ if

$$M\,(0) = \mathbb{E}^{\mathbb{Q}^S}\left[M\,(1)\right]$$

and

$$\begin{aligned} M\,(1)\,(f_1^1) &= \mathbb{E}^{\mathbb{Q}^S}\left[M\,(2)|\,\mathcal{P}_1\right](f_1^1) \qquad (*) \\ M\,(1)\,(f_2^1) &= \mathbb{E}^{\mathbb{Q}^S}\left[M\,(2)|\,\mathcal{P}_1\right](f_2^1). \qquad (**) \end{aligned}$$

From the first equation we obtain

$$\begin{aligned} \frac{1}{S\,(0)} &= M\,(0) = \mathbb{E}^{\mathbb{Q}^S}\left[M\,(1)\right] = \frac{1+r\,(0)}{S(1)(f_1^1)}\mathbb{Q}^S[f_1^1] + \frac{1+r\,(0)}{S(1)(f_2^1)}\mathbb{Q}^S[f_2^1] \\ \frac{1}{10} &= \frac{1.02}{11}\mathbb{Q}^S[f_1^1] + \frac{1.02}{9}\left(1 - \mathbb{Q}^S[f_1^1]\right) \end{aligned}$$

that delivers $\mathbb{Q}^S[f_1^1] = 0.64706$ and $\mathbb{Q}^S[f_2^1] = 1 - \mathbb{Q}^S[f_1^1] = 1 - 0.64706 = 0.352\,94$.

Conditions $(*)$ and $(**)$ deliver two different equations to determine (resp.) $\mathbb{Q}^S[\omega_1|f_1^1]$ and $\mathbb{Q}^S[\omega_3|f_2^1]$. In fact, from equation $(*)$ we get

$$\frac{1.02}{11} = \left\{\frac{1.0404}{12.1} \cdot \mathbb{Q}^S[\omega_1|f_1^1] + \frac{1.0404}{9.9} \cdot \left(1 - \mathbb{Q}^S[\omega_1|f_1^1]\right)\right\}$$

that is solved by

$$\begin{aligned} \mathbb{Q}^S[\omega_1|f_1^1] &= 0.647\,06 \\ \mathbb{Q}^S[\omega_2|f_1^1] &= 1 - 0.647\,06 = 0.35294. \end{aligned}$$

From equation $(**)$ we get

$$\frac{1.02}{9} = \left\{\frac{1.02}{9.9} \cdot \mathbb{Q}^S[\omega_3|f_2^1] + \frac{1.02}{8.1} \cdot \left(1 - \mathbb{Q}^S[\omega_3|f_2^1]\right)\right\}$$

leading to

$$\begin{aligned} \mathbb{Q}^S[\omega_3|f_2^1] &= 0.55 \\ \mathbb{Q}^S[\omega_4|f_2^1] &= 1 - 0.55 = 0.45. \end{aligned}$$

Therefore

$$\mathbb{Q}^S[\omega_1] = 0.64706 \cdot 0.64706 = 0.41869$$

R^{MV} for given k + optimal allocation to achieve it

Question.

7. It holds $\mathbb{E}[R^*] = 0.8206$ and
$$R^{e^*} = \begin{bmatrix} 0.276 \\ -0.129 \\ 0.074 \\ -0.110 \end{bmatrix}.$$
Let $k = 1.1$. Find R^{MV}, the return of the mean variance portfolio such that $\mathbb{E}\left[R^{MV}\right] = k$.

8. Assuming an initial endowment equal to $V(0) = 10$, find the optimal allocation $\vartheta^{MV} = (\vartheta_1^{MV}, \vartheta_2^{MV})$, in terms of the number of units of S_1 and S_2 to trade, that delivers the optimal gross return R^{MV} found in the previous point.

Answer

7. From the Lecture Notes we know that
$$R^{MV} = R^* + w^{MV} R^{e^*}$$
with
$$w^{MV} = \frac{k - \mathbb{E}[R^*]}{\mathbb{E}[R^{e^*}]}.$$
We need to compute
$$\mathbb{E}\left[R^{e^*}\right] = \frac{1}{4}(0.276 - 0.129 + 0.074 - 0.110) = 0.0278.$$
Therefore
$$w^{MV} = \frac{1.1 - 0.8206}{0.0278} = 10.05$$
and
$$R^{MV} = \frac{1}{1.0432}\begin{bmatrix} 0.736 \\ 0.352 \\ 0.544 \\ 1.792 \end{bmatrix} + 10.05 \begin{bmatrix} 0.276 \\ -0.129 \\ 0.074 \\ -0.110 \end{bmatrix} = \begin{bmatrix} 3.4793 \\ -0.9590 \\ 1.2652 \\ 0.6123 \end{bmatrix}.$$

8. It holds
$$R_1(1) = \frac{S_1(1)}{S_1(0)} = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 2 \end{bmatrix}$$
$$R_2(1) = \frac{S_2(1)}{S_2(0)} = \frac{1}{2.8}\begin{bmatrix} 4 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{4}{2.8} \\ 0 \\ \frac{2}{2.8} \\ \frac{4}{2.8} \end{bmatrix}.$$
Since it must be
$$\begin{aligned} w_1^{MV} R_1(1) + w_2^{MV} R_2(1) &= R^{MV} \\ w_1^{MV}\begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 2 \end{bmatrix} + w_2^{MV}\begin{bmatrix} \frac{4}{2.8} \\ 0 \\ \frac{2}{2.8} \\ \frac{4}{2.8} \end{bmatrix} &= \begin{bmatrix} 3.4793 \\ -0.9590 \\ 1.2652 \\ 0.6123 \end{bmatrix} \end{aligned}$$
we can focus on the first and second entry and immediately get
$$\begin{aligned} w_2^{MV} &= \frac{3.4793}{\frac{4}{2.8}} = 2.4355 \\ w_1^{MV} &= \frac{-0.9590}{\frac{2}{3}} = -1.4385. \end{aligned}$$
Finally, given $V(0) = 10$, we get
$$\begin{aligned} \vartheta_1^{MV} &= \frac{w_1^{MV} V(0)}{S_1(0)} = \frac{-1.4385 \cdot 10}{3} = -4.795 \\ \vartheta_2^{MV} &= \frac{w_2^{MV} V(0)}{S_2(0)} = \frac{2.4355 \cdot 10}{2.8} = 8.6982. \end{aligned}$$