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INVERSE FUNCTION THEOREM → $f: U \to \mathbb{R}^n$, C^1, U OPEN, $x_0 \in U$, $Df(x_0)$ INVERTIBLE AT x_0 POINT ⇒ $\exists U'$ OPEN $\subseteq U$, $x_0 \in U'$, $f(U')$ OPEN, $f|_{U'}$ DIFFEOM.

LOCAL INVERTIBILITY OF f FROM INVERTIBILITY OF TAYLOR 1st ORDER EXPANSION

FUNCTION BECOMES INVERTIBLE AS C^1 AT A NEIGHBOURHOOD NEAR x_0

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IMPLICIT FUNCTION THEOREM: $P \subseteq \mathbb{R}^k$, $S, V \subseteq \mathbb{R}^n$ → OPEN, $f: P \times S \to V$ C^1 } $D_s f(p_0, s_0)$ INVERTIBLE, $v_0 = f(p_0, s_0)$

⇓

v_0, p_0, s_0 HAVE OPEN NEIGHBOURHOODS ST. $(p, v) \mapsto s$ ST. $f(p,s) = v$ IS A DIFFEOMORPHISM

GENERALISES * TO FIND SOLUTION TO FUNCTION $f(p, s) = v$

MANIFOLD DEFINITIONS

- REMEMBER ALL DEF. NEED
- $f \in C^1$
- ALWAYS A DIFFERENT KIND OF INVERTIBILITY
- $M \cap U$ IN TERMS OF f EXPRESS. ↑ EXACT EQUALITY

$M \subseteq \mathbb{R}^n$ IS k-DIM. MAN. $\iff$ $\forall p \in M$, $\exists U \subseteq \mathbb{R}^n$ OPEN: $p \in U$, $\exists \phi: U \to \mathbb{R}^{k+1}$, $\phi \in C^1$ DIFFEOMORPHISM WITH ITS IMAGE:

$\phi(M \cap U) = [\mathbb{R}^k \times \{0\}] \cap \phi(U)$

DEFORMATION; M CONTAINED IN U; PLANE INCLUDED IN MAPPING OF U

→ JUST FIND $\phi: U \to \mathbb{R}^n$, $p \in U$ ST. $D\phi(p)$ INVERTIBLE, $\forall p \in M$, $M \cap U = \{x \in U \mid \phi(x) \in \mathbb{R}^k \times \{0\}\}$ (SOME KIND OF INVERSE GRAPH TRANSFORMATION)

THE NEIGHBOURHOOD OF EVERY POINT CAN BE SMOOTHLY MORPHED TO AND FROM THE GRAPH OF A k-DIM. HYPERPLANE

$M \subseteq \mathbb{R}^n$ IS k-DIM. MAN. $\iff$ $\forall p \in M$, $\exists U \subseteq \mathbb{R}^n$ OPEN: $p \in U$, $\exists h: U \to \mathbb{R}^{n-k}$, $h \in C^1$ ST

- $Dh(p)$ FULL RANK MATRIX
- $M \cap U = \{x \in U : h(x) = \underline{0}\}$

k DEG. OF FREEDOM

EVERY NEIGHBOURHOOD OF M IS ZERO SET OF A FUNCTION WITH k DEG. FREEDOM AND INVERTIBLE JACOBIAN

THE INTERSECTION OF OPEN MANIFOLDS IS AN OPEN MANIFOLD

FULL RANK ⇕ INVERTIBLE WITH IMAGE ⇕ SURJECTIVE

$M \subseteq \mathbb{R}^n$ IS k-DIM. MAN. $\iff$ $\forall p \in M$, $\exists U \subseteq \mathbb{R}^n$ OPEN: $p \in U$, $\exists V \subseteq \mathbb{R}^k$, $\psi: V \to U$, $\psi \in C^1$

- $D\psi(x)$ ALWAY INJECTIVE $\forall x$
- ψ HOMEOMORPHISM WITH ITS IMAGE ← C^0 WITH C^0 INVERSE
- $M \cap U = \phi(V)$

} AROUND ANY $p \in M$, M CAN BE PARAMETRISED WITH SOME ψ DIFFEOMORPHISM

$M \subseteq \mathbb{R}^n$ IS k-DIM. MAN. $\iff$ $\forall p \in M$ (UP TO PERMUTATION OF COORDINATES) $\exists U \subseteq \mathbb{R}^n$ OPEN: $p \in U$, $U = V \times W$, $\begin{cases} V \subseteq \mathbb{R}^k \\ W \subseteq \mathbb{R}^{n-k} \end{cases}$ OPEN, $\exists \tilde{\varphi}: V \to W$, $\tilde{\varphi} \in C^1$, $M \cap U = \{(x, \tilde{\varphi}(x)) \mid x \in V\}$

} THE NEIGHBOURHOOD OF EACH POINT IS THE GRAPH OF A k-INPUT C^1 FUNCTION WHOSE OUTPUTS HAVE ENOUGH COORDINATES SO THE SUM GOES TO n

TANGENT SPACE (VECTOR SPACE OF TANGENT VECTORS AT $p \in M$)

$T_pM := \{\gamma'(0) \mid \gamma: [0, \varepsilon) \to M, C^1, \gamma(0) = p\}$

$\varepsilon: [\gamma|_0, \gamma|_\varepsilon] \in M$

$\dim(T_pM) = \dim(M)$

↳ NECESSARY BUT NOT SUFF TO PROVE THAT M IS A MANIFOLD

$Df(p): T_pM \to T_{f(p)}N$

$D(f \circ \gamma)(0) = Df(p)[\gamma'(0)]$

EXISTS, IS UNIQUE UP TO CHANGE OF BASIS → ALL C^1 EXTENSIONS AT POINT ARE THE SAME

IT IS A V.S, NOT AN AFFINE SPACE

EXISTS BECAUSE THE OBJECT M IS SMOOTH AT THE SURFACE

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$f: U \to N$, $N \subseteq \mathbb{R}^n$ MANIFOLD

$f \in C^1$: f DIFFERENTIABLE, $Df \in C^0$

$f: M \to \mathbb{R}^n$, $f \in C^1$

$\Updownarrow$ ALWAYS EXTENSIBLE IN A C^1 WAY

$\forall p \in M$, $\exists U$ OPEN, $p \in U$, $\exists \tilde{f}: U \to \mathbb{R}^n$, $\tilde{f} \in C^1$, $\tilde{f}|_{M \cap U} = f|_{M \cap U}$

f IS C^1 OVER M PARAMETR.

$M = \bigcup_i \varphi^{(i)}(V^{(i)})$

$f: M \to \mathbb{R}^n$ IS C^1 IF $\forall i \;\; f \circ \varphi^{(i)} \in C^1$

COMPOSITION IS C^1 (USEFUL TO DISPROVE)

$f: M \to \mathbb{R}^n$, $M \subseteq \mathbb{R}^m$ k-DIM MANIFOLD,

$f \in C^1$: $\forall \varphi: V \to M$, $\varphi \in C^1$, $V \subseteq \mathbb{R}^k$

$f \circ \varphi: V \to \mathbb{R}^n$ IS C^1

$f: M \to \mathbb{R}^n$, $f \in C^1$

$\Downarrow$

$f: M \to N \subseteq \mathbb{R}^n$, $f \in C^1$

$f: M \to N$ DIFFEOMORPHISM:

- BIJECTIVE
- $f, f^{-1} \in C^1$

TANGENT CONE AT p

$T_p S := \{\lambda v \mid \lambda > 0,\ v = \lim_{i \to \infty} \frac{p^i - p}{\|p^i - p\|}\}$

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DIFFERENTIAL OF $f|_p$

$f: M \to N$, $\forall p \in M$

$Df(p): T_p M \to T_{f(p)} N$

WELL DEFINED, UNIQUE LOCALLY $\to$ $\forall p \in M$, $\tilde{f} \in C^1$ EXTENSION OF f NEAR p, $\forall v \in T_p M$

$Df(p)[v] = D\tilde{f}(p)[v]$

HOW TO COMPUTE

M OR N 0-DIM $\Rightarrow$ $T_p M$ OR $T_{f(p)} N = \{0\}$ $\Rightarrow$ $Df(p)$ IS ZERO MAP

$f: M \to N$ INJECTIVE, C^1, M k-DIM MANIF., $k \geq 1$, $S \subseteq M$ BOREL SET

$\underbrace{\mathcal{H}^k(f(S))}_{\text{PARAM.}} = \int_S J_f(x)\, d\underbrace{\mathcal{H}^k(x)}_{\text{PARAM.}}$

$= \sum_i \int_{S^{(i)}} J_{\varphi^{(i)}}(x)\, dx$ USING MULTIPLE PARAM.

FOR NONSQUARE J_f USE $\sqrt{\det(J_f^T J_f)}$

INTEGRATE h OVER M: $\int_S h(f(s)) J_f(s)\, ds$

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ALL f CONVEX, C^1, h LINEAR, g CONVEX $\to$ $\min_{x \in \mathbb{R}^n} f(x)$ ST. $\forall i\ h_i(x) = 0$, $\forall j\ g_j(x) \leq 0$

METHOD:

1. CHECK WEIERSTRASS (UNDO f: INCREASING $(x)^3 \approx x$)
2. FIND CRIT. POINTS / SOLUTIONS TO EQUATION
3. CHECK $p \in S$ (FEASIBLE)

x_0 IS CONSTRAINED MINIMUM

$\Updownarrow$

x_0 SATISFIES:

- $\exists \underbrace{\lambda_{1 \dots k}}_{g_i} \geq 0$, $\exists \underbrace{\mu_{1 \dots \ell}}_{h_j}$ ST.
- $0 = \nabla f(x_0) + \underline{\lambda} \cdot \nabla \underline{g}(x_0) + \underline{\mu} \cdot \nabla \underline{h}(x_0)$
- $\forall i \;\; \lambda_i g_i(x_0) = 0$ (FOR DUALITY)

WEIERSTRASS

SINCE T COMPACT $\Downarrow$ $f|_T$ ADMITS max/min

USE IT TO CHECK $\exists$ SOL.

x_0 CONSTRAINED MINIMUM FOR $f: M \to \mathbb{R}$ UNDER $\forall i\ h_{(i)}(x) = 0$ $\Rightarrow$ LAGRANGE MULT $\exists \mu_1 \dots \mu_\ell \in \mathbb{R}$

$\nabla f(x_0) + \underline{\mu} \cdot \nabla \underline{h}(x_0) = 0$

$\left(\{\nabla h_{(i)}(x_0)\}\ \text{LIN. INDEP}\right) \Rightarrow \ell \leq n$

DEFINES A MANIFOLD

CRITICAL POINTS : $\{x \mid Df(x) = 0\}$

SPECTRAL THEOREM

$A^T = A \Rightarrow A$ IS DIAGONALISABLE

- f LINEAR
- M UNBOUNDED

$\longrightarrow$ USUALLY f UNBOUNDED

DO ONE STEP, THEN ONE CHECK

REMEMBER TO CHECK

- SLATER
- $f, g, h \in C^1$
- $\lambda_i g_i(x_0) = 0$

PROJECTS $Df(p)$ ONTO $T_p M$ SO $u(t) = -\nabla f(u)$ IS LIMITED TO TRAVERSING M AND BECOMES $u' = \nabla^M f(p) u$

$\nabla^M f(p) = \nabla \tilde{f}(p) - (\nabla \tilde{f}(p) \cdot \nabla h(p)) \cdot \boxed{\frac{\nabla h(p)}{\|\nabla h(p)\|^2}}$ $\hat{n}$

EFFICIENCY BOUND

$\frac{d}{dt} f(u(t)) = -|\nabla^M f(u(t))|^2$

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$\nabla^M f(p)$ ORTHOGONAL PROJECTION

GRADIENT FLOW

$u' = -\nabla f(u)$ MINIMISE $f: M \to \mathbb{R}$

M COMPACT $\Downarrow$ $\forall t$ $u(t)$ IS WELL DEFINED

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$f: M \to N$, $f \in C^\infty$

$p \in M$ IS REGULAR: $Df(p): T_p M \to T_{f(p)} N$ INVERTIBLE

FINDING $Df(p)$: MUST BE ORTHONORMAL

IF $T_{f(p)} N$ USES $\{e_i\}$ BASIS, DO $Df(p)[v_i]$ FOR $T_p M$ BASIS

OTHERWISE SOLVE FOR A ST $\forall i$ $A[v^i] = w^i$ } USE AS J_f TO EVALUATE AREA

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MANIFOLD

(DISPROVE) | (PROVE) ← YOU CAN USE ANY OTHER MANIFOLD DEFINITION ACTUALLY

IMPLIES $Dh(p)$ NOT INVERTIBLE, CAN'T APPLY IMPLICIT FUNCTION THEOREM — IF $M=\{x\mid h(x)=0\}$ AND $Dh'(p)=0$, AT POINT $p\in M$, NOT MANIFOLD

$M=(x,\varphi(x))$ FOR $\varphi\in C^1$

$M=\{x\mid h(x)=0\}$. $Dh'(p)\neq 0$ $\forall p\in M$ ← WORKS TRIVIALLY IF h_p IS THE SAME FOR ALL p

FOR SET-DEFINED MANIFOLD — THERE ARE MORE LIN. INDEP VECTORS THAN $\dim(M)$

IF YOU HAVE THE TANGENTS (PARAMETRISATION) → IF TANGENT CONE IS NOT A V.S, YOU CAN FIND A DIRECTION ST. $-a\notin$ CONE

FIND $p\in M$, AND SEQUENCE ST. $Dh(p^i)\mapsto 0$. HENCE $\nexists T_pM$ DESPITE M BEING k-DIM → YOU KNOW $\dim(T_pM)=k$ BUT $\exists$ $k+1$ ⊥ VECT. SO DIM IS INCONSISTENT

IF A PROPERTY CLAIMS $g(x,y)=0$ $\forall x,y$ THEN DO $g(y,x)$ ALSO TO SHOW SOME EQUALITY CONDITION → I.E. f CONVEX, x LOCAL MAX $\Downarrow$ $f\equiv c$

IN MULTIPLE CHOICE, CHECK HOW TO DISPROVE EACH, DOING SIMPLER FIRST, BY EXCLUSION. DISPROVING IS GENERALLY QUICKER

$h: M\to\mathbb{R}$
$f: V\to M$
$\int h(f(v))\,J_{f(v)}\,dv$ INVARIANT

1. FIND ORTHONORMAL BASES OF T_pM, $T_{f(p)}N$ ST. $Df(v_1)=w_1$
3. COMPUTE $J_{f(v)}=\|w_1\|\cdot\|w_2\|$

(IF J_f GIVEN, USE $\sqrt{J^TJ}$ IF J NONSQUARE) - YOU MAY DERIVE J AS MATRIX ST $v_1\mapsto w_1$, $v_2\mapsto w_2$

MANIFOLD GIVEN AS PARAMETRISATION $\Rightarrow$ (WRITE AS $x\mapsto y$) FIND POINTS WITH INDEP. TANGENT VECTORS } $\dim(T_pM)=k$

TANGENT CONE: CONICAL SPAN OF TANGENT VECTORS } USE PARAM. → USE GRAPHS TO CHECK PROPERTIES

CHECK PARAMETRIC CURVE IS INJECTIVE: → f INJECTIVE, DERIVATIVE NEVER VANISHES $\Rightarrow$ DIFFEOMORPHISM

$\varphi(t)$ INJ $\Longleftrightarrow$ THE FUNCTIONS ARE INJECTIVE TOGETHER
↓
$t\mapsto(x(t),y(t))$ DO $t=x^{-1}(x)$, $y=y(x^{-1}(x))$

USE TO SHOW THAT $x=s$: $\begin{cases}x(s)=x(t)\\ y(s)=y(t)\end{cases} \overset{?}{\Rightarrow}$ (SHOW IT BY ALGEBRA)

COMMON TRANSFORM → $\begin{cases}x(s)+y(s)=x(t)+y(t)\\ x(s)-y(s)=x(t)-y(t)\end{cases}$

$y_1=y_2$ $\Downarrow$ $t_1=t_2$

REMEMBER TO CHECK $\{D_ih\}$ INDEPENDENCE THROUGH ALL THE PARTIALS

CHECK M IS MANIFOLD: $(M=\{h=0\})$
→ KNOWING $\{x_i\}$ ST $\begin{cases}h(x_i)=0,\\ Dh(x_i)=0\end{cases}$
→ $\exists x_i$ IS NECESSARY BUT NOT SUFFICIENT
→ FIND TAYLOR APPROX AT x_i TO UNDERSTAND LOCAL STRUCTURE
→ IF THE NEIGHBOURHOOD CAN'T BE SMOOTHLY PARAMETRISED, IT IS NOT A MANIFOLD

$\ker(Dh(p))$ → FINDING T_pM: $T_pM=\{x\mid Dh(p)\cdot x=0\}$ ← NORMAL TO ∂M
FIND BASIS OF THIS SPACE

DETERMINE MAX/MIN FOR A FUNCTION DEFINED ON A SET
$\nabla f(x)+\mu\nabla g(x)+\lambda h(x)=0$
SOLVE FOR μ,λ,x. ↳ EQUATE BY ROW
↓
CHECK IF MIN OR MAX, IF $x\in S$

∇g IS DIRECTION OF g INCREASE. $g=0$ SO ∇g DIRECTION IS FORBIDDEN. IF $\nabla f\parallel\nabla g$, THE DIRECTION YOU NEED TO GO TO IS THE ONE YOU CANNOT GO TO

$M,M'\subseteq\mathbb{R}^n$, k-DIM $\Rightarrow$ $M\cap M'$, $M\cup M'$ MAY NOT BE MANIFOLDS
⊕ COMPACT (NOT ALWAYS DIFFEOMORPHIC)

CHECK $F(N)=M$ → ALGEBRA SUBSTITUTION
SUBSTITUTE $f_1,f_2\dots$ INTO M DEFINITION AND CHECK THAT M IS SATISFIED FOR $x_1\dots x_n\in M$

FIND VALUES c ST $\{x\mid h(x)=c\}$ IS A MANIFOLD ($h\in C^1$) → FIND x_i ST. $Dh(x)=0$ ⊕ X SET IS MANIFOLD EXCEPT FOR $h(x_i)$
↳ FIND c SO $x_i\notin\{x\mid h(x)=c\}$

FIND MATRIX $Df(p)$ FOR $M=\{h=0\}$, $N=\{g=0\}$
1. FIND BASIS OF $T_pM=\ker(Dh(p))$
2. FIND BASIS OF $T_pN=\ker(Dg(p))$ ← YOU MAY BE GIVEN THE BASIS, BUT THE PROCESS IS THE SAME
3. FIND MATRIX $A=Df(p)$ ST. $T_pM\mapsto T_pN$

WHETHER P IS LOCAL MIN./MAX WRT. TANGENT

SIGN OF A FUNCTION AT A REGULAR POINT] → $\nabla f(p)\neq 0$
- $f(p)\oplus\nabla f(p)(x-p)$ ← $\forall x\in B_\varepsilon(p)$
→ $f(p)>0$, $|f(p)|\geq|\nabla f(p)\cdot(x-p)|$: POSITIVE
→ $f(p)<0$, $|f(p)|\geq|\nabla f(p)\cdot(x-p)|$: NEGATIVE
→ $|f(p)|<|\nabla f(p)\cdot(x-p)|$: CHANGES SIGN

IF $f|_S$ HAS $\nabla f|_S\equiv 0$, $f|_S\equiv c$
CRITICAL POINTS DON'T HAVE TAYLOR EXPANSIONS

DEGREE OF A MAP BETWEEN MANIFOLDS (NUMBER OF TIMES THE DOMAIN TRAVERSES THE MANIFOLD) → $f: M\to N$, $f\in C^0$, BETWEEN M,N ORIENTED, CONNECTED, CLOSED

DEFINE HOMOTOPY: $H\in C^0$ ST.
$H(0,x)=f(x)$] → DEG ??
$H(1,x)=x$] → DEG. 1

OR: (EQUIVALENTLY)
1. FIND y REG. VALUE
2. $f^{-1}(y)=\{x_i\}$ ⊕ $x_i\in M$
3. $\deg(f)=\sum \mathrm{sign}(\det(Df(x_i)))$

→ ALL REGULAR POINTS WILL HAVE THE SAME DEG

SPECTRAL THEOREM: $A=RDR'$ FOR: D DIAGONAL, R,R' ST. $B^T=B^{-1}$ → SOME MATRICES ARE NEVER DIAGONALISABLE (EVEN IN $\mathbb{C}$)

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