SUBSPACES DEPENDENCE

CONSTRUCT WITH INTERSECTION OF HYPERPLANES IF $\dim \leq d-1$ IN $\mathbb{R}^d$ →

ROUCHÉ-CAPELLI
$\{x \in \mathbb{R}^d \mid Ax = b\}$ IF $\text{RANK}(A) = \text{RANK}(A|b)$ HAS $\dim = d - \text{RANK}(A)$, IS AFFINE

THE KER. OF A MATRIX IS A SUBSPACE, HENCE AFFINE.

CONVEX $\subsetneq$ **AFFINE** $\subsetneq$ **LINEAR**

- $co(\text{FINITE}) = \text{COMPACT}$
- $d+1$ AFF. INDEP. VECTORS IN $\mathbb{R}^d$
- $d+1$ POINTS CAN DESCRIBE A d-DIM SPACE

LINEAR: ALWAYS HAS ZERO

$\text{CONE}(S) \neq \text{CONE}(co(S))$

CARATHEODORY

$\text{CONV}(X) := \left\{ \sum_{i=1}^{d+1} \alpha_i x_i \,\middle|\, x_i \in X, \alpha_i \geq 0, \sum \alpha_i = 1 \right\}$

CONV IS THE SET OF POINTS YOU CAN OBTAIN BY COMBINING AT MOST $d+1$ POINTS

- HYPERPLANE = ALL POINTS PROJECTED LINEARLY TO A VALUE ON $\hat{c}$ → $a^T x = b$
- HALF-SPACE = ALL POINTS ON ONE SIDE OF THE $\hat{c}$ VECTOR → $a^T x \leq b$

($a^T \neq \underline{0}$)

- CONE = ALL POINTS ON $0 \to s \in S$ RAYS UP TO INFINITY → $\{\alpha x + \beta y \mid \alpha, \beta \geq 0;\ x, y \in S\}$
- PRIMITIVE CONE = CONE FROM LIN. INDEP. VECTORS; PRIMITIVE CONE ⇒ CLOSED

FINITELY GEN CONE $= \bigcup$ FINITE GEN PRIMITIVE CONES

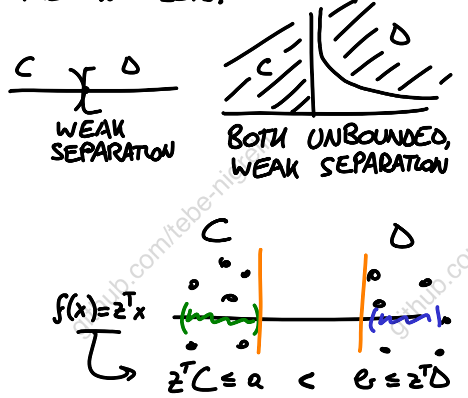
THE CONE OF ANY CROSS-SECTION OF THE CONE GIVES THE CONE ITSELF

SEPARATION THM

$C, D \subseteq \mathbb{R}^d$, C BOUNDED

SETS ARE: NONEMPTY, CLOSED, CONVEX, DISJOINT

$\exists$ HYPERPLANE STRONGLY SEPARATING THE TWO SETS:



LP
$\max c^T x$
$Ax \leq b$

$(LP^*)^* = LP$

EQUATION

- $Ax \leq b \to Ax + z = b$
- $x_i \in \mathbb{R} \to x_i^+ - x_i^- = x_i$, $x_i^+, x_i^- \geq 0$

DUAL
$\min b^T y$
$A^T y = c$
$y \geq 0$

$\dim(\emptyset) = -1$
$\dim(\{p\}) = 0$

CONVERSION:

$x_{1\ldots n}$	→ $y_{1\ldots m}$
A	→ A^T
b	→ c
$\max c^T x$	→ $\min b^T y$
$\leq$	→ $\geq$
$\geq$	→ $\leq$
$=$	→ $\in \mathbb{R}$

NONSTRICT INEQUALITIES ENSURE CLOSURE
⇓
MAX EXISTS (OTHERWISE $\exists$ SUP ONLY)

WEAK LP DUALITY
FEASIBLE SOLUTIONS
PRIMAL $c^T x$ $\leq$ DUAL $b^T y$

STRONG LP DUALITY
OPTIMA EXIST
PRIMAL $c^T x^*$ $=$ DUAL $b^T y^*$

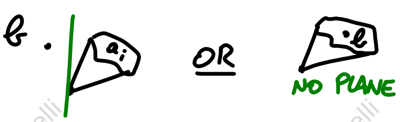
OPTIMALITY CHECK:
$y^T(b - Ax) = 0$
(SLACK VECTOR)

FARKAS LEMMA

EXACTLY ONE IS TRUE:

A) $b \in \text{CONE}(\{a_i\})$

B) A HYPERPLANE PASSING THROUGH ZERO WEAKLY SEPARATES: → $\{x \mid y^T x = 0\}$
$b \to y^T b < 0$
$\text{CONE}(\{a_i\}) \to y^T a_i \geq 0 \quad \forall i = 1 \ldots n$



CASES (SYMMETRY)

- BOTH INFEASIBLE
- ONE UNBOUNDED → OTHER INFEASIBLE
- BOTH FEASIBLE

⇕

$(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m)$

ONE AND ONLY ONE:

A) $\exists x \in \mathbb{R}^n$ ST. $Ax = b$ AND $x \geq 0$

B) $\exists y \in \mathbb{R}^m$ ST. $y^T A \geq \underline{0}^T$ AND $y^T b < 0$

POLYTOPES ⊆ POLYHEDRA

- POLYTOPES: BOUNDED ⇒ COMPACT; $\max c^T x$ EXISTS
- POLYHEDRA: CLOSED; INTERSECTION OF A FINITE NUMBER OF HALFSPACES ⇓ CONVEX

SUPPORTING HYPERPLANE $\cap$

HYPERPLANE TOUCHING P } $h \cap P \neq \emptyset$
AND WHOSE HALFSPACE CONTAIN ALL P } $\forall x \in P \; c^T x \leq \delta$

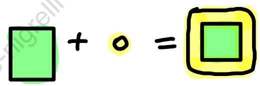
MAY NOT HAVE VERTICES ↙ **POLYHEDRON** = **FACE OF P** → IMPROPER = $\emptyset$, P ; → PROPER

ONE VERTEX, ↖ MULTIPLE SUPPORTING HYPERPLANES

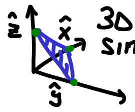
- ALSO A POLYHEDRON
- $\dim < \dim P$ →

"FACET" IFF $\dim = \dim(P) - 1$
HIGHEST dim FACE

MINKOWSKI SUM



- CUBE $[-1,1]^n \rightarrow \forall i = 1..n \begin{cases} x_i \leq 1 \\ x_i \geq -1 \end{cases}$
- CROSS POLYTOPE $\{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$ 
  $\sum |x_i| \leq 1 \longrightarrow \forall \sigma \in \{\pm 1\}^n : \sigma^n \leq 1$
- REGULAR n-SIMPLEX $\forall i = 1 \dots n+1 \;\; x_i \geq 0, \;\; \sum_i^{n+1} x_i = 1$
  SPACE OF DISCRETE P. DISTRIBUTIONS → 3D SIMPLEX
  HIGH-DIMENSIONAL "SAIL" COVERING A NOOK



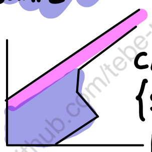
MINKOWSKI-WEYL *

$P \subseteq \mathbb{R}^n$ POLYTOPE $\iff$ FINITE SET $\exists V = \{\text{VERTICES}\}$, $P = \text{conv}(V)$

$P \subseteq \mathbb{R}^n$ POLYHEDRA $\iff$ FINITE SETS $\exists V = \{\text{VERTICES}\}$, $Y = \{\text{WINDOW}\}$

$P = \text{conv}(V) + \text{cone}(Y)$

PROVIDES THE BOUNDED REGION'S SHAPE ↑ ; ↑ PROVIDES THE UNBOUNDED REGION'S SHAPE



CHAR CONE(Y) IS $\{y \mid Ay \leq 0\}$ IF $P \neq \emptyset$
FOR $\{y_i\}$ FINITE

YOU CAN EXPRESS POLYTOPES AS:
$P = \text{conv}(V)$ (1)
$\; = \{x \mid Ax \leq b\}$ (2)

(1) UP TO $2^{|V|}$ INEQUALITIES $\times O(n)$

(2) TRY ALL VERTICES $2^{m/2}$

(2) BETTER: USE LP

A FACE MAY BE REPRESENTED AS A SUBSYSTEM: $A'x \leq b'$ FROM $Ax \leq b$

DESCRIPTION OF P IS MINIMAL:
- ROWS OF A ARE LIN. INDEP. AND REMOVING ANY INEQUALITY CHANGES P
- CONVERTING $\geq \to =$ CHANGES P

*V ARE CHOSEN AS EXTREMES: $V_{EXT} = \{x \in P \mid x \notin co(P \setminus \{x\})\}$

F FACE = INTERSECTION OF FACES CONTAINING F
$\quad\; = co(\text{VERTICES})$

→ "MINIMAL" FACE: NONEMPTY, DOES NOT CONTAIN PROPER FACE ←

THERE IS A CHAIN $\emptyset \subset F_1 \dots \subset P$ CONTAINING ONE FACE PER dim FROM t = min FACE TO dim(P)

ALL MINIMAL FACES HAVE THE SAME DIM, DEPENDING ON THE SHAPE

(MINIMAL REPRESENTATION → SPLIT $Ax \leq b$ IN $A'x = b'$, $A''x \leq b''$)

- $P \neq \emptyset \rightarrow \exists z \in P$ ST $Az < b$ (P HAS NONEMPTY INTERIOR) ⇒ IN GENERAL: $\dim(P) = n - \text{RANK}(A')$; $z \notin$ PROPER FACE OF P

F, G FACES ⇒ $M = F \cap G$ FACE EXISTS, J CONTAINING BOTH, IS SUBSET OF ALL FACES CONTAINING BOTH

- ONE-TO-ONE CORRESPONDENCE BETWEEN FACETS AND INEQUALITIES IN $Ax \leq b$

$\dim(P) = n \Rightarrow$ MIN DESCRIPTION IS UNIQUE UP TO CONSTANT MULTIPLICATION OF INEQUALITIES

SEPARATION THEOREM
- CONVEX, CLOSED SET
- POINT NOT IN SET

$\exists$ HYPERPLANE STRONGLY SEPARATING THE SETS

⇓

A PROPER FACE IS THE INTERSECTION OF SOME OF ITS FACETS (SUBSYSTEM) ⇒ F FACE OF P $\iff$ $F \neq \emptyset$, $F \subseteq P$, $F = \{A'x = b', \tilde{A}x = \tilde{b}\}$

$\{\text{FACET}\} \longleftrightarrow \{\text{INEQUALITY}\}$
⇓
$\bigcap$ FACET = PROPER FACE
↑ HIGHEST DIMENSION FACE

(FORM AFFINE SPACES BY INTERSECTING HYPERPLANES)

MINIMAL FACES OF P HAVE $\dim = n - \text{RANK}\begin{pmatrix} A' \\ A'' \end{pmatrix}$

IF LP HAS OPTIMAL SOLUTION:
- $\{\text{OPTIMAL SOLUTION}\}$ = FACE
- $\exists$ MINIMAL FACE E, $\forall x \in E \;\; x \in E$ IS OPTIMAL
- IF P HAS VERTEX, THERE IS OPTIMAL SOLUTION (VERTEX)

SIMPLEX

BASIC FEASIBLE SOLUTION: $\begin{pmatrix} \max c^T x \\ \text{ST. } Ax = b \\ x \geq 0 \end{pmatrix}$ ← EQUALITY AFTER ADDING $s_1 \dots s_n$

x: $\exists B \subseteq \{1 \dots n\}$ ST. A_B INVERTIBLE, $x_j = 0 \ \forall j \notin B$ } B GIVES A SUBSYSTEM WITH A BASIS TO DESCRIBE A FEASIBLE SOLUTION

SIMPLEX TABLEAU

SYSTEM OF $m+1$ EQUATIONS GIVEN BY F. BASIS B.

USES VARIABLES $x_{1 \dots n}$, z

$$\begin{cases} Ax = b \\ z = c^T x \end{cases} \text{(OBJECTIVE)} \longrightarrow \begin{cases} x_B = p + Q x_N \\ z = z_0 + r^T x_N \end{cases}$$

T(B) IS UNIQUE UP TO CHOICE OF B

IF $r \leq 0$ THE SOLUTION IS OPTIMAL

≥ 0 "BASIC" FEASIBLE SOLUTION (SATISFIES CONSTR.)

- ENTERING RULE → NONBASIC ENTERS IFF ITS z COEFF IS POS (AKA YOU CAN ++ OBJ.)
- LEAVING RULE → x_h ST. INCREASES z, } GIVES COLUMN; PICK S_i ST. MOST STRICT INCREASE UNIT } GIVES ROW

RAISE (NEXT LOWEST UPPER BOUND) $x_{\ell\beta}$: $0 \to \frac{p_\alpha}{-q_{\alpha\beta}}$

$x_{k\alpha}$ ST. $q_{\alpha\beta} < 0 \ \wedge \ \frac{p_\alpha}{-q_{\alpha\beta}} = \min$

IF $\forall i$, $\frac{p_i}{-q_{i\beta}} \geq 0$, LP UNBOUNDED

} YOU OBTAIN B' FEASIBLE BASIS

- STOP THE ALGORITHM → IN THE z ROW ALL NONBASIC $(x_1 \dots x_n)$ VARIABLES HAVE NEGATIVE COEFF. / IF ALL COEFF ARE < 0 (STRICT), UNIQUE OPTIMAL
- UNBOUNDED CASE → YOU CAN INCREASE VARIABLES FOREVER

- BASIC VARIABLE = SLACK VARIABLES
- NONSING. MATRIX = INVERTIBLE MATRIX
- FEASIBLE SOLUTION = SATISFIES CONSTRAINTS
- BASIC SOLUTION = ≥ 0
- FEASIBLE BASIS = ALL VECTORS SATISFY THE CONSTRAINTS

$\exists$ OPTIMAL $\Downarrow$ $\exists$ BASIC OPTIMAL

EVERY VERTEX OF P IS A BASIC FEASIBLE SOLUTION WHEN WRITTEN IN TERMS OF SOME B

AT EACH STEP SIMPLEX MOVES BETWEEN VERTICES OVER EDGES OR IT STAYS ON THE VERTEX.

↓

STEPS: x_1, x_2

EITHER $x_1 = x_2$ OR $\text{conv}(x_1, x_2)$ IS AN EDGE OF P

IF THE ORIGIN IS NOT A VECTOR YOU CAN'T START FROM IT.

↓

SOLVE AUXILIARY LP FOR A STARTING P.:

$$\max c^T x \quad \text{ST. } Ax = b, \ x \geq 0 \ \longrightarrow \ \max -\sum_{i=n+1}^{n+m} w_i \ \text{(MINIMISE SLACK VARIABLES)} \quad \text{ST. } \bar{A}\bar{x} = b \ \text{(START FROM 0)}, \ \bar{x} \geq 0$$

(WLOG ASSUME $b \geq 0$)

USING $\bar{A} = [A \mid I_m]$

$\bar{x} = (x_1 \dots x_n, x_{n+1} \dots x_{n+m})$

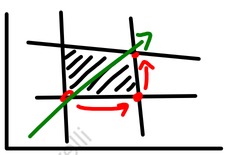
AUXILIARY HAS SOLUTION WITH OBJ=0 $\Leftrightarrow$ ORIGINAL LP HAS A FEASIBLE SOLUTION

FOR $B \subseteq \{1 \dots m\}$, WITH A_B INVERTIBLE $\exists$ AT MOST ONE FEASIBLE SOLUTION x ST $x_j = 0 \ \forall j \in B$ AND IT IS A VERTEX

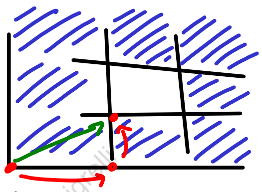
WORST-CASE EFFICIENCY $O(2^n)$

↓

STUDY OF FINDING SHARPER BOUNDS FOR MIN # STEPS TO GET TO OPTIMAL



LP WITHOUT 0



AUX. SIMPLEX

PIVOT RULES

- DANZIG: LARGEST COEFF. IN z ROW
- LARGEST INCREASE: FOR EACH VAR WITH POS COEFF, PICK THE BIGGEST INCREASE
- STEEPEST EDGE: PICK VAR ST. $(x_2 - x_1)$ SAME DIR AS c^T (OBJ.) } "DEVEX" FAST APPROXIMATION
- RANDOM EDGE
- BLAND'S (PREVENTS CYCLING): ENTERING HAS SMALLEST INDEX. LEAVING WITH SMALLER INDEX, IF THERE ARE ++ CHOICES

DUAL SIMPLEX

- B ST. $r \leq 0$
- LEAVING $x_{k\alpha}$ ST. $p_\alpha < 0$

(IMPOSSIBLE ⇒ P FEASIBLE D OPTIMAL)

- ENTERING: $x_{\ell\beta}$ ST. $x_{\ell\beta}$ INCREASE TURNS $x_{k\alpha} = 0$, KEEPING LAST ROW ≤ 0

($q_{\alpha j} \leq 0 \ \forall j$ ⇓ D UNBOUNDED, P INFEASIBLE)

INPUT SIZE OF LP

$\langle i \rangle := \lceil \log_2(|i|+1) \rceil + 1$ (BITS) (SIGN)

- $\langle \frac{p}{q} \rangle = \langle p \rangle + \langle q \rangle$
- $\mathbb{R}^{m \times n} = \sum\sum \langle a_{ij} \rangle$
- $LP \to \langle A \rangle + \langle b \rangle + \langle c \rangle$

POLYNOMIAL ALGORITHM $\Rightarrow$ # STEPS < POLY($\langle L \rangle$)

$P \neq \emptyset \iff P \cap B(0,R) \neq \emptyset$

$R = \sqrt{n} \cdot 2^{\langle A \rangle + \langle B \rangle}$

ELLIPSOID METHOD (SHRINK AN ELLIPSOID AROUND THE SOLUTION)

ASSUMPTIONS

- $P \subseteq B_R(0)$ (STOP AT VOL < ε)
- $B_\varepsilon(p) \subseteq P$

ITERATIONS DEPENDS ON DIMENSION

$O(\text{ELLIPS}) = O(\#(n) \cdot O(\text{ORACLE}))$

$\frac{Vol(E_{k+1})}{Vol(E_k)} \leq e^{-\frac{1}{2n+2}}$

ELLIPSOID (BALL + AFFINE)

$E := \{Mx + s \mid x \in B_r(0)\}$

M INVERTIBLE MATRIX

"ORACLE" = ALGORITHM GIVING THE SUB-ELLIPSOID

SEMIDEFINITE PROGRAMMING (ELLIPSOID ALSO WORKS)

MATRIX SCALAR PRODUCT $\to A \cdot B = \sum_{ij} a_{ij} \cdot B_{ij}$

$\max\ C \cdot X$

ST $A_1 \cdot X = b_1$

$\circ\circ\circ$

$A_m \cdot X = b_m$

$X \succeq 0$ (POS. SEMID.)

INFINITE NUMBER OF CONSTR.

- POLY-TIME ORACLE EXISTS

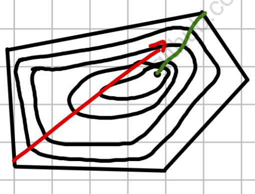
LP ALGORITHM

- **SIMPLEX**: MOVES FROM VERTEX TO VERTEX OVER THE P BOUNDARY } 2^n WORST CASE COMPLEXITY
- **ELLIPSOID**: SEQUENCE OF SMALLER ELLIPSOID CONVERGING TO OPTIMAL } POLYNOMIAL
- **INTERIOR POINT**: START FROM $p \in \mathring{P}$, FOLLOW PATH TO OPTIMUM. } POLYNOMIAL

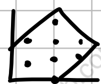
$\min c^T x$, $Ax \geq b$ $\to$ $\min c^T x - \mu \underbrace{\sum_{i=1}^{m} \ln(a_i x - b_i)}_{\text{LOG-BARRIER FOR BOUNDARY}}$

FOR $\mu >> 1$ THE MINIMISER IS THE ANALYTIC CENTER OF P

SLOWLY DECREASE μ TO GO CLOSER TO BOUNDARY } "CENTRAL PATH" $\{x^*(\mu) \mid \mu > 0\}$ MADE OF MINIMISERS $\Rightarrow$ ALGORITHM APPROXIMATES WITH JUMPS (COMPARE NEWTON ALGORITHM)



INTEGER LINEAR PROGRAMMING (ILP) / LP IN $x \in \mathbb{Z}^n$



- NP HARD $\to$ CAN BE REDUCED TO SAT
- WEAK DUALITY HOLD

IF $x \in \mathbb{Q}^n$, P IS STILL A POLYHEDRON

KNAPSACK

$\max c^T x$ (PRICE)

ST. $a^T x \leq b$ (COST)

$x \in \{0,1\}^n$

$\to$ NP-RELAXATION

$\max c^T x$

ST. $a^T x \leq b$

$x \leq 1$

$x \geq 0$

$\Downarrow$

x^* HAS AT MOST ONE NON-INTEGER COORDINATE

(NP-HARD TO GO FROM $\mathbb{R}$ TO $\mathbb{N}$)

INTEGER HULL OF P (MAY BE << P)

$P_I = Co(\text{INTEGER VECTORS}) \subseteq P$