2.1

$co(S) =$ SET OF CONVEX COMBINATIONS OF S POINTS

PROOF:

$\tilde{C} =$ SET OF CONVEX COMBINATIONS

- $S \subseteq \tilde{C}$: $\forall p \in S,\ p = 1 \cdot p$
  SO $S \subseteq \tilde{C}$

- $\tilde{C}$ IS CONVEX: USING $p^{(i)}, q^{(j)} \in S$

$$a = \sum_{i=1}^{k} \alpha_i p^{(i)} \quad \left(\sum_{i=1}^{k} \alpha_i = 1,\ \forall i\ \alpha_i \geq 0\right)$$

$$b = \sum_{i=1}^{\ell} \beta_j p^{(j)} \quad \left(\sum_{j=1}^{\ell} \beta_j = 1,\ \forall j\ \beta_j \geq 0\right)$$

$\rightarrow$ BY DEF OF CONVEX COMBINATION

SHOW $\forall \lambda \in (0,1),\ (1-\lambda)a + \lambda b \in \tilde{C}$ ($\tilde{C}$ IS CONVEX)

$$(1-\lambda)a + \lambda b = \sum_{i=1}^{k} (1-\lambda)\alpha_i p^{(i)} + \sum_{i=1}^{\ell} \lambda \beta_j p^{(j)}$$

THIS IS A CONVEX COMBINATION ($\Sigma .. = 1 | \dots \geq 0$)

- $\tilde{C}$ IS THE SMALLEST CONVEX SET CONTAINING S:
  $\forall C$ CONVEX, $S \subseteq C \Rightarrow \tilde{C} \subseteq C$

EQUIVALENTLY, C CONTAINS ALL CONVEX COMBINATIONS OF POINTS IN S (AKA $\tilde{C}$)

BY INDUCTION, ANY CONVEX COMBINATION OF k POINTS IN S IS IN C:

($k=1$) TRIVIAL $\rightarrow S \subseteq C$

($k=2$) $[p^{(1)}, p^{(2)}] \subseteq C$

($k+1$) $p = \sum_{i=1}^{k+1} \lambda_i p^{(i)}$ | $\lambda_i > 0$ OTHERWISE IT REVERTS TO k POINTS AND BY ASSUMPTION ✓

$$\lambda_{k+1} = 1 - \sum_{i=1}^{k} \lambda_i < 1$$

CONVEX COMBINATION OF k POINTS $\left(\begin{matrix}\dots \geq 0, \\ \Sigma \dots = 1\end{matrix}\right)$

$$p = (1-\lambda_{k+1}) \sum_{i=1}^{k} \left(\frac{\lambda_i}{1-\lambda_{k+1}} p^{(i)}\right) + \lambda_{k+1} p^{(k+1)}$$

$\left(\dots\right) = p' \in C$

REDUCES TO $k=2$ CASE

4.6

$S \subseteq \mathbb{R}^n$ NONEMPTY, CLOSED, $P \in \mathbb{R}^n \rightarrow$ A NEAREST POINT EXISTS

$$\delta := \inf \{ \|q - p\| \mid q \in S \}$$

BY INF DEF,

$\forall i \in \mathbb{N}, \exists q_i \in S$ ST. $\|q_i - p\| < \delta + 2^{-i}$

$\|q_i - p\| \rightarrow \delta$

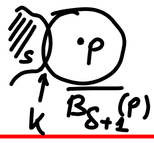
BOUNDED + CLOSED $\downarrow$ CREATE A COMPACT SET:

$\forall i \quad (q_i)_i \in \bar{B}_{\delta+1}(p) \rightarrow K := S \cap \bar{B}_{\delta+1}(p)$

$(q_i)_j \in K \rightarrow$ A CONVERG. SUBSEQUENCE EXISTS ST $q_{i_j} \rightarrow q \in K \leftarrow$ COMPACTNESS

$|q - p| = \lim_{j \to \infty} |q_{i_j} - p| = \delta$

SINCE $q \in K \subseteq S$, q IS NEAREST



4.7

$C \subseteq \mathbb{R}^n$ NONEMPTY, CLOSED, CONVEX, $P \in \mathbb{R}^n \Rightarrow$ NEAREST POINT IS UNIQUE

PROOF:

ASSUME TWO POINTS $q \neq q'$ EXIST.

$$\begin{cases} \delta := \|q - p\| = \|q' - p\| \rightarrow \text{SAME DIST.} \\ (\forall a \in C)\ \|a - p\| \geq \delta \rightarrow \text{LEAST DIST.} \end{cases}$$

BY CONTRADICTION, $\bar{q} := \frac{1}{2}q + \frac{1}{2}q'$ (EXISTS IN S BY CONVEXITY)

HAS $\|\bar{q} - p\| < \delta$ BUT $\bar{q} \in C$

USE

$$\begin{cases} v := q - p \\ v' := q' - p \end{cases} \rightarrow \bar{q} - p = \frac{1}{2}v + \frac{1}{2}v'$$

$$\|\bar{q} - p\|^2 = \frac{1}{4}\|v\|^2 + \frac{1}{2} v \cdot v' + \frac{1}{4}\|v'\|^2 = \frac{\delta^2}{2} + \frac{1}{2} v \cdot v'$$

SINCE $\|v\| = \|v'\| = \delta$,

IF $v \cdot v < \delta^2$, $\|q - p\|^2 < \delta^2$

$\|\bar{q} - p\|^2 < \delta^2$, OR $v \cdot v' \leq \delta^2$

(CAUCHY SCHWARTZ) $v \cdot v' \leq \|v\| \cdot \|v'\| = \delta^2$

EQUALITY ONLY IF $v' = k \cdot v$ ($k \geq 0$)

BUT $\|v\| = \|v'\|$ AND $v \neq v'$ SO

THE INEQUALITY CANNOT BE TRUE

5.1

$C \subseteq \mathbb{R}^n$ NONEMPTY CLOSED CONVEX, $p \in \mathbb{R}^n \longrightarrow$ NEAREST POINT PROJECTION $q := \pi_C(q) \in C$ HAS:

$$(\forall q' \in C) \quad (q-p)\cdot(q'-q) \geq 0$$

PROOF:

$$\begin{pmatrix}\text{NEAREST}\\ \text{POINT}\\ \text{DEF}\end{pmatrix} \quad \underbrace{\|q-p\|^2}_{\text{NEAREST POINT}} \leq \underbrace{\|q'-p\|^2}_{\text{OTHER POINTS}}$$

FOR $\varepsilon \in (0,1]$, $q_\varepsilon := (1-\varepsilon)q + \varepsilon q'$

$q_\varepsilon \in [q, q']$, $q_\varepsilon \in C$

$$\|q_\varepsilon - p\|^2 = \|(q-p)+\varepsilon(q'-q)\|^2 \quad \begin{pmatrix}\text{BY}\\ \text{CONSTRUCTION}\end{pmatrix} \quad \begin{pmatrix}\text{NEAREST}\\ \text{POINT}\\ \text{DEF}\end{pmatrix}$$

$$= \|q-p\|^2 + 2\varepsilon(q-p)\cdot(q'-q) + \varepsilon^2\|q'-q\|^2 \geq \|q-p\|^2$$

$$\cancel{2}\varepsilon(q-p)\cdot(q'-q) + \frac{\varepsilon^{\cancel{2}}}{2}\|q'-q\|^2 \geq 0$$

$$(q-p)\cdot(q'-q) + \frac{\varepsilon}{2}\|q'-q\|^2 \geq 0$$

AS $\varepsilon \to 0$, THE CLAIM FOLLOWS.

ARGUMENT TO FIND H:

WTF: H CLOSED HALF-SPACE ST.

- $C \subseteq H$
- $p \notin H$

$q = \pi_C(p)$

$v := q - p$

- $C \subseteq H$:

$\forall q' \in C$

$v\cdot(q'-q) \geq 0$

$v\cdot q' \geq v\cdot q$

$\beta := v\cdot q$

$H := \{ x \in \mathbb{R}^n \mid v\cdot x \geq \beta \}$

- $p \notin H$: $v\cdot p < \beta$

$(q-p)\cdot p < (q-p)\cdot q$

$0 < (q-p)\cdot(q-p)$

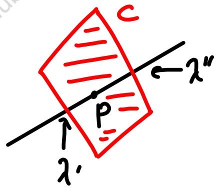
TRUE FOR $p \neq q$

6.3

(MINKOWSKI) $C \subseteq \mathbb{R}^n$ NONEMPTY COMPACT CONVEX $\Rightarrow C = co\begin{Bmatrix}\text{EXTREME}\\ \text{VALUES}\end{Bmatrix}$

PROOF

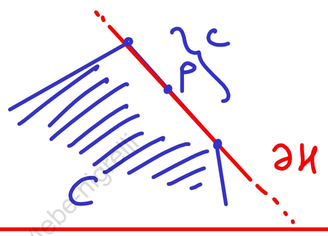
- $C \neq \emptyset$ BY ASSUMPTION.
- $\dim(C) = \dim(\text{AFF.SPAN}(C))$, REASON BY INDUCTION ON DIMENSION
- $\begin{pmatrix} d=0 \text{ OR} \\ d=1 \end{pmatrix}$ TRIVIALLY TRUE
- CONSIDER $d \geq 2$
- C HAS NONEMPTY INTERIOR (CONVEXITY, CLOSEDNESS)
- FIX $p \in C$, p IS A CONVEX COMBINATION OF SOME EXTREME POINTS IN C (CLAIM)



$\to p \in \partial C$: TAKE A SUPPORTING HYPERPLANE ∂H, THE EXTREME POINTS OF $C' := C \cap \partial H$ ARE THE SAME AS THE ONES OF C CONTAINED IN C'

$\text{AFF.SPAN}(C') \subseteq \partial H \Rightarrow \dim(\text{AFF.SPAN}(C')) \leq n-1$ $\leftarrow$ max dim of HYPERPLANE IN $\mathbb{R}^n$

$p \in C'$ MUST BE IN THE CONVEX HULL OF THE SET OF EXTREME POINTS OF C'! THESE ARE ALSO EXTREME IN C.



$\to p \notin \partial C$, $p \in C \Rightarrow p \in \mathring{C}$ (GENERIC ELEMENT)

FOR $w \in \mathbb{R}^n \setminus \{0\}$, $L := \{ p + \lambda w \mid \lambda \in \mathbb{R} \}$ (LINE IN THE SET)

CLAIM: $\exists \lambda' < 0, \lambda'' > 0$ ST. $\begin{pmatrix}\text{AT TWO DIRECTIONS THERE'S}\\ \text{POINTS IN BOUNDARY}\end{pmatrix}$

$\begin{cases} p' := p + \lambda' w \\ p'' := p + \lambda'' w \end{cases} \in \partial C$ *

NONEMPTY: $p \in \mathring{C}$, CLOSED, BOUNDED (C IS)

$\lambda'' := \sup\{\lambda > 0 : p + \lambda w \in C\} \to 0 < \lambda'' < \infty$

$\exists \lambda_k \to \lambda''$ SEQUENCE WITH LIMIT $p'' \in C$ (CLOSED)

(+) DEF OF $\lambda'' \in \partial C$ (SUP): $\forall \varepsilon > 0$, $p + \lambda'' w + \varepsilon w \notin C$ *

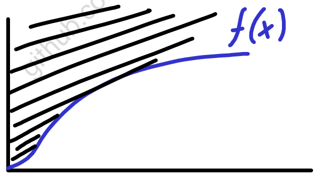
- SAME FOR p', SO $p \in [p', p'']$

$$p = \frac{\lambda''}{\lambda''-\lambda'}(p+\lambda' v) - \frac{\lambda'}{\lambda''-\lambda}(p+\lambda'' v) = \frac{\lambda''}{|\lambda|+\lambda''}p' + \frac{|\lambda'|}{|\lambda'|+\lambda''}p''$$

7.4

$S \subseteq \mathbb{R}^n$, $f: S \to \mathbb{R}^n$
NONEMPTY

$\boxed{S \text{ CONVEX}, \; f: S \to \mathbb{R} \text{ CONVEX} \iff \text{CONVEX } epi(f) = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in S, t \geq f(x) \right\}}$



($\Rightarrow$) KNOWING f CONVEX, SHOW EPIG. IS CONVEX

FIX SOME $\begin{cases} x, x' \in S \\ \lambda \in [0,1] \end{cases}$ ST. $\begin{cases} f(x) \leq t \\ f(x') \leq t' \end{cases}$ MULTIPLY BOTH SIDES BY $(1-\lambda), \lambda$

$(1-\lambda)x + \lambda x' \in S$ CONVEXITY OF f

$f((1-\lambda)x + \lambda x') \leq (1-\lambda)f(x) + \lambda f(x') \leq (1-\lambda)t + \lambda t'$

$\hookrightarrow ((1-\lambda)x + \lambda x', (1-\lambda)t + \lambda t') \in epi(f)$

THIS IS $(1-\lambda)(x,t) + \lambda(x',t')$

($\Leftarrow$) $epi(f)$ IS CONVEX

- S IS THE IMAGE OF $epi(f)$ THROUGH $(x,t) \mapsto (x)$ (LINEAR)
  SO S SET IS CONVEX

LINEAR $g(x)$ PRESERVE THE CONVEXITY OF SETS

- FOR $x, x' \in S$, $\lambda \in [0,1]$
  $(x, f(x)), (x', f(x')) \in epi(f)$ $\rightarrow$ CONVEXITY
  $((1-\lambda)x + \lambda x', (1-\lambda)f(x) + \lambda f(x')) \in epi(f)$
  BY EPIG. DEFINITION $\rightarrow$ $f((1-\lambda)x + \lambda x') \leq (1-\lambda)f(x) + \lambda f(x')$

8.1

$\boxed{f: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \text{ IS LOWER-SEMICONT} \iff epi(f) \text{ IS CLOSED IN } \mathbb{R}^n \times \mathbb{R} \; \binom{\text{DISCARDING}}{t=\infty}}$

f LOWER SEMICONTINUOUS IF $f(x_0) \leq \liminf_{i \to \infty} f(x)$

($\Rightarrow$) f IS L.SEMICONT. SHOW THAT IF $\exists$ $(x^{(i)}, t^{(i)}) \in epi(f)$ ST. CONVERGES $(x^{(i)}, t^{(i)}) \mapsto (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$

SINCE $x^{(i)} \mapsto x_0 \oplus f(x^{(i)}) \leq t^{(i)}$ (BY $epi(f)$ DEFINITION)

$f(x_0) \leq \liminf_{i\to\infty} f(x^{(i)}) \leq \liminf_{i\to\infty} t^{(i)} = \lim_{i\to\infty} t^{(i)} = t_0$

EQUIVALENTLY, $(x_0, t_0) \in epi(f)$
SO $epi(f)$ IS CLOSED.

($\Leftarrow$) $epi(f)$ IS CLOSED $\qquad (x_0, m) \in epi(f)$

$x^{(i)} \to x_0$

$m := \liminf_{i\to\infty} f(x^{(i)}) \cup \{\pm\infty\}$ WTS $f(x_0) \leq m$

- BY CONTR., IF $f(x_0) > m$, THE INTERVAL
  * $(m, f(x_0))$ HAS SOME α INSIDE OF IT.
  $\rightarrow$ (WLOG $m = \lim_{i\to\infty} f(x^i)$ FOR SOME SEQUENCE) $\rightarrow$ $f(x^i)$ DECREASES TOWARDS ITS INF: m
- EVENTUALLY, $f(x^i) \leq \alpha$ BY INF DEFINITION.
  EQUIVALENTLY $(x^i, \alpha) \in epi(f)$ [CLOSED SET DEF]
  SINCE $epi(f)$ IS CLOSED, DEFINE SEQ.
  $(x^i, \alpha) \to (x_0, \alpha) \in epi(f)$ } BOTH POINTS ARE IN epi
  MEANING $f(x_0) \leq \alpha$ * (CONTRADICTION)
  HENCE $f(x_0) \leq \liminf_{i\to\infty} f(x^{(i)})$
  FOR ALL CONVERGING SUBSEQUENCES

$M \subseteq \mathbb{R}^d$
$f: M \to \mathbb{R}^n$ is C^1 $\iff$ CAN BE EXTENDED IN C^1 WAY TO ANY OPEN NEIGHBOURHOOD OF $p \in M$ IN $\mathbb{R}^k$

OPEN SET IS RELATED TO PARAMETRISATION

$\forall p \in M$, $\exists U$ open, $p \in U$, $\tilde{f}: U \to \mathbb{R}$, C^1; $\tilde{f}|_{M \cap U} = f|_{M \cap U}$

YOU CAN EXTEND f NEAR ANY POINT, NOT ALL POINTS AT THE SAME TIME

$(\Rightarrow)$ ASSUME f ADMITS C^1 EXTENSION

- φ PARAMETR. OF M $\boxed{\varphi: V \to M,\ \varphi \in C^1}$ (MANIFOLD DEFINITION)
- FIX $\bar{x} \in V$, $p := \varphi(\bar{x})$ (GENERIC p) $\to$ $\boxed{\exists U \text{ OPEN SUBSET OF } M,\ p \in U \quad \exists \tilde{f}: U \to \mathbb{R}^n \quad \tilde{f}|_{M\cap U} = f|_{M \cap U} \quad \text{AND} \quad f \circ \varphi = \tilde{f} \circ \varphi \text{ FOR } \varphi^{-1}(U) \subseteq V}$
- $f \circ \varphi$ IS C^1 ON $\varphi^{-1}(U)$ BECAUSE IT IS THE COMPOSITION OF C^1 FUNCTIONS BETWEEN OPEN SETS

SHOW $f \in C^1$ $(\Leftarrow)$

ALL MANIFOLD DEFINITION

- $p \in M$, APPLY 4^{TH} MANIFOLD DEFINITION: MANIFOLD AS (LOCALLY) GRAPH OF C^1 FUNCTION $\oplus$ UP TO A COORDINATE PERMUTATION
- FIND OPEN $U = V \times W$ ($V \subseteq \mathbb{R}^k$, $W \subseteq \mathbb{R}^{n-k}$ OPEN), $p \in U$, ST. $M \cap U = \{(y, \tilde{\varphi}(y)) \mid y \in V\}$ FOR $\tilde{\varphi}: V \to W$, $\tilde{\varphi} \in C^1$
- TAKE $x \in U$, $\underbrace{\pi(x) = (x_1 \dots x_n)}_{\text{PERMUTATION } (C^1)}$, $\tilde{f}(x) := f(\pi(x), \tilde{\varphi} \circ \pi(x))$ (DEFINE $\tilde{f}$ ON MANIFOLD)
- $\tilde{f} = f \circ \varphi \circ \pi \in C^1$ (BY COMPOSITION)

COMPOSITION IS C^1 BECAUSE x IS ALWAYS RESTRICTED

- ON THE RESTRICTION $x \in M \cap U$:

$\begin{cases} x = (y, \tilde{\varphi}(y)) \\ y = \pi(x) \end{cases}$ $\quad \tilde{f}(x) = f(y, \tilde{\varphi}(y)) = f(x)$

T_pM IS k-DIM LINEAR SUBSPACE, $\forall p \in M$

NOTE: FOR BOTH YOU SHOW k-DIM VECTOR SPACES RELATE, $\gamma \in C^1$, $[0, \varepsilon] \subseteq U$, $[x_0, \varepsilon_v v] \subseteq V$

PROOF: BY MANIFOLD DEFINITION, $\forall p \in M$, $\exists U \subseteq \mathbb{R}^n$ OPEN, $p \in U$
$\exists h: U \to \mathbb{R}^{n-k}$, $h \in C^1$
$M \cap U = \{x \in U : h(x) = 0\}$

(LINEAR) INVERTIBLE IN ITS IMAGE $\to$ $Dh(p)$ IS SURJECTIVE:
$\ker(Dh(p))$ IS k-DIM BY RANK-NULLITY

$\forall \gamma: [0, \varepsilon] \to M$, $\gamma(0) = p$, $\gamma \in C^1$

$h \circ \gamma(t) \equiv 0$

$\frac{d}{dt}(h \circ \gamma(t)) \equiv 0$

BY DEFINITION OF T_pM $\{ Dh(p) \cdot \gamma'(0) = 0$

$\Downarrow$

$T_pM \subseteq \ker(Dh(p))$

PARAMETRISATION

BY THIRD MANIFOLD DEFINITION, $\exists \varphi: V \to U$, $\varphi \in C^1$,
$\varphi(V) = M \cap U$, $\boxed{V \subseteq \mathbb{R}^k \to \text{OPEN},\ U \subseteq \mathbb{R}^n,\ p \in U}$

$D\varphi(x_0)$ INJECTIVE FOR $\varphi(x_0) = p$

$\forall v \in V$, $\gamma_v(t) := \varphi(x_0 + tv)$ FOR $t \in [0, \varepsilon)$

WITH ε DEPENDING ON v ST. $[x_0, x_0 + \varepsilon] \subseteq V$

$\gamma'(0) \in T_pM$ SINCE $\gamma(0) = p$
$\gamma([0, \varepsilon]) \subseteq \varphi(V) \in M$

(CHAIN RULE) $\forall v \in \mathbb{R}^k$
$D\varphi(x_0)[v] = \gamma'(0) \in T_pM$
$\Downarrow$
$\mathrm{Im}(D\varphi(x_0)) \subseteq T_pM \subseteq \ker(Dh(p))$

$D\varphi(x_0)$ IS k-DIM BECAUSE $D\varphi(x_0)$ IS INJECTIVE

T_pM IS CONTAINED BY AND CONTAINS A V.S OF $\dim = k$, SO:
$\mathrm{Im}(D\varphi(x_0)) = T_pM = \ker(Dh(p))$

$\forall p \in M$, $\forall \tilde{f}$ LOCAL C^1 EXTENSION OF f NEAR p $(\tilde{f}: U \to \mathbb{R}^n)$

$Df(p)[v] = D\tilde{f}(p)[v]$

USUALLY THE C^1 EXTENSION IS JUST f BUT DEFINED BEYOND THE CONSTRAINT

PROOF:
- FOR $f: M \to N$, $f \in C^1 \to$ EXPANSION NEAR p EXISTS (BY C^1 DEF)
- $\forall v \in T_pM$, $\gamma: [0, \varepsilon] \to M$, $\gamma \in C^1$ ST. $\begin{cases} \gamma(0) = p \\ \gamma'(0) = v \end{cases}$ (PICK A PARAMETR.)
- FOR ε SMALL ENOUGH, $\gamma(t) \in U$ SO $\tilde{f} \circ \gamma$ IS DEFINED

$(f \circ \gamma)'(0) = (\tilde{f} \circ \gamma)'(0)$ (CHAIN RULE)
$= D\tilde{f}(\gamma(0))[\gamma'(0)]$
$= D\tilde{f}(p)[v]$

- $Df(p)$ HAS NO γ DEPENDENCY (ONLY v)
- $Df(p)$ IS OBTAINED RESTRICTING $D\tilde{f}(p)$ TO LINEAR SUBSPACE T_pM

x_0 CONSTRAINED MINIMUM OR MAXIMUM FOR $f: \mathbb{R}^n \to \mathbb{R}$

$x_0 \in U$ SUB $h_1(x_0) = \dots = h_\ell(x_0)$ $\in C^1$

$\nabla h_1(x_0), \dots \nabla h_\ell(x_0)$ $\boxed{\text{LINEARLY INDEP.}}$

$(\Rightarrow \ell \leq n)$

$\Rightarrow$

LAGRANGE MULTIPLIERS

$\exists \mu_1 \dots \mu_\ell \in \mathbb{R}$ ST.

$\nabla f(x_0) + \mu_1 \nabla h_1(x_0) + \dots + \mu_\ell \nabla h_\ell(x_0) = 0$

[WITHOUT CONVEXITY IT BECOMES $0 \in \partial \dots$]

JOIN THE CONSTRAINTS INTO h.

$h := (h_1, \dots h_\ell) : U \to \mathbb{R}^\ell$

n INPUTS
ℓ OUTPUTS

$Dh(x_0)$ IS $\boxed{\ell \times n}$ MATRIX OF RANK ℓ:

ROWS ARE $\nabla h_i(x_0)^T$ SO LINEARLY INDEP. BY ASSUMP.

$\Rightarrow$ AT $x = x_0$, $\exists\, \ell \times \ell$ MINOR WITH $\det \neq 0$.

SINCE DET VARIES AS C^0, (SUM AND PRODUCT OF C^0 FUNCTIONS)

THE SAME HOLDS IN A NEIGHBOURHOOD OF P.

$\boxed{\text{SHRINKING } U,\ Dh(x) \text{ HAS FULL RANK } \forall x \in U \ (\text{RANK} = \ell)}$

USE CONSTRAINTS TO DEFINE MANIFOLD NEAR x_0

$M := \{x \in U : \underline{h}(x) = \underline{0}\}$

$\dim(M) = n - \ell$ (RANK-NULLITY THEOREM)

$T_{x_0}M = \operatorname{Ker} Dh(x_0) = \{v : \forall i = 1 \dots \ell \ \ \nabla h_i(x_0) \cdot v = 0\}$

$=$ ORTHOGONAL COMPLEMENT OF THE SPAN $\{\nabla h_i(x_0)\}$

ASSUME ($x_0 \in M$ IS A CRITICAL POINT) FOR $\tilde{f}|_{M \cap U} = f|_{M \cap U} \to \underline{Df(x_0) = 0}$

$\tilde{f}$: C^1 EXTENSION

IT IS ZERO BECAUSE IT CAN'T GO IN A NEAR DIRECTION IN THE NEIGHBOURHOOD TO INCREASE

DEFINED OVER MANIFOLD, YOU CAN GO NOWHERE TO INCREASE f

THIS IS DIFFERENT FOR THE EXTENSION

$(\forall v \in T_{x_0}M)$ $\nabla f(x_0) \cdot v = Df(x_0) \cdot [v]$

$= \underbrace{D\tilde{f}(x_0)[v] = 0}_{\text{BY DEFINITION}}$

THESE ARE VECTORS IN THE KERNEL OF $Dh(x_0)$

$\Rightarrow$ $\nabla f(x_0) \perp T_{x_0}M$

SO $\nabla f(x_0)$ IS IN THE BASIS OF $T_{x_0}M$, IE. $\{\nabla h_i(x_0)\}$

$\nabla f(x_0)$ CAN BE EXPRESSED IN TERMS OF $\nabla h_i(x_0)$

SPECTRAL THEOREM:

$f:\mathbb{R}^n\to\mathbb{R}$ LINEAR,

$f(x)=Ax$ WITH $A^T=A$,

$\exists\{v^{(1)}\dots v^{(n)}\}$ BASIS OF EIGENVECTORS SO f IS REPRESENTED BY DIAGONAL MATRIX

→ FOR σ SCALAR PRODUCT ON $\mathbb{R}^n$,

$\exists\{v^{(1)}\dots v^{(n)}\}$ ORTHONORMAL BASIS

$\forall i j,\ \sigma(v^{(i)},v^{(j)})=0 \quad \forall i\neq j$ → ALL VECTORS IN THIS BASIS ARE ORTHOGONAL

[σ IS A DIAGONAL MATRIX USING THIS BASIS]

REAL, SYMMETRIC MATRICES ARE ORTHOGONALLY DIAGONALISABLE

FOR $A\in\mathbb{R}^{m\times n}$, $P\in O(m)$, $Q\in O(n)$ ST. $A=PDQ$, D DIAGONAL $(m=n)$

$D_{ij}\geq 0$ ALWAYS

$\begin{pmatrix}D'\\0\end{pmatrix}$ D' DIAG. $n\times n$ $(m>n)$

$(D'\ 0)$ D' DIAG $m\times m$ $(m<n)$

$O(n)$ IS ROTATION MATRICES OF DEG n

PROOF:

IF ONE EIGENVECTOR CAN BE FOUND $\begin{pmatrix}w\in\mathbb{R}^n,\\ \lambda\in\mathbb{R}\end{pmatrix}$, WLOG $\|w\|=1$

THEN $\{w\}$ CAN BE COMPLETED TO AN ORTHONORMAL BASIS $\{w^{(i)}\,|\,i=1..\ell\}$ — ITERATED ?

P CHANGE OF BASIS MATRIX

$\tilde{A}:=P^{-1}AP$

$Pe_1=w^{(1)}=w$ SO

$\tilde{A}e_1=P^{-1}Aw=P^{-1}(\gamma w)=\gamma e_1$

$\tilde{A}$ → FIRST COL $\begin{pmatrix}\gamma\\0\\\dots\\0\end{pmatrix}$
→ SYMM. BY A BEING SYMM.

$\tilde{A}=\begin{bmatrix}\gamma & 0\\ 0 & B\end{bmatrix}$, B SYMM (SUBMATRIX) $(n-1)\times(n-1)$

BY INDUCTION ON SMALLER n VALUE

ASSUMING $Q^{-1}BQ$ DIAGONALISABLE FOR $Q\in O(n-1)$,

$$\underbrace{\begin{bmatrix}1&0\\0&Q\end{bmatrix}^{-1}P^{-1}AP\begin{bmatrix}1&0\\0&Q\end{bmatrix}}_{\in O(n)}=\begin{bmatrix}\gamma&0\\0&\underbrace{Q^{-1}BQ}_{\text{DIAGONAL}}\end{bmatrix}$$

SINCE THE MATRIX IS DIAGONAL,

$P\begin{bmatrix}1&0\\0&Q\end{bmatrix}$ USES THE DESIRED BASIS

FIND W EIGENVECTOR:

min GIVES $-k\cdot\|x\|^2$ FOR BIGGEST EIGENVALUE

$f(x):=x\cdot(Ax)$, FIND MIN ON S^{n-1} (SPHERE) — COVERS ALL DIRECTIONS

$f\in C^0$, S^{n-1} COMPACT $\Rightarrow$ MINIMUM EXISTS (WEIERSTRASS)

$h(w)=\|w\|^2-1=0$, $\nabla h(w)=2w\neq 0$ OVER THE MANIFOLD } SATISFIES L. MULTIPLIERS ASSUMPTION $(U:=\mathbb{R}^n)$

$\nabla f(w)+\mu\nabla h(w)=0$ FOR SOME $\mu\in\mathbb{R}$

$\nabla f(w)\cdot v=Df(w)\cdot v=\left(\lim_{\varepsilon\to 0}\frac{f(w+\varepsilon v)-f(w)}{\varepsilon}\right)$

$f(w+\varepsilon v)=(w+\varepsilon v)^TA(w+\varepsilon v)$

$=w^TAw+\varepsilon(w^TAv+v^TAw)+\varepsilon^2v^TAv$ — EQUAL BECAUSE A IS SYMMETRIC

$=f(w)+2\varepsilon\cdot w^TAv+\varepsilon^2f(v)$

$\nabla f(w)\cdot v=2v^TAw=2(v\cdot Aw)\Rightarrow\nabla f(w)=2Aw$

WITH $2Aw+2\mu w=0$ } → w IS AN EIGENVECTOR

(LAGRANGE GUARANTEES EXISTENCE OF EIGENV.)

FINALLY, FOR $w\neq 0$, $\gamma=-\mu$