1. Complex Numbers

$z = x + iy \in \mathbb{C}$

$|z| = \sqrt{x^2+y^2} \rightarrow \begin{cases} |Re(z)| \le |z| \\ |Im(z)| \le |z| \end{cases}$

$|z_1 + z_2| \le |z_1| + |z_2|$

$\bar{z} = a - bi$

$z^{-1} = \frac{\bar{z}}{|z|^2} \rightarrow$ INVERSE EXISTS AND IS UNIQUE

$z = re^{i\vartheta} = r(\cos(\vartheta) + i \cdot \sin(\vartheta))$

$\rightarrow r = |z|$

$\rightarrow \vartheta = \arctan(\frac{x}{y})$

$\bar{z} = re^{i(-\vartheta)}$

$z^{-1} = \frac{1}{r} e^{-i\vartheta}$

$z^n = r^n(\cos(n\vartheta) + i \cdot \sin(n\vartheta))$

$z \cdot \ldots \leftrightarrow A \ldots$, USING

$A = |z_0| \begin{bmatrix} \cos(\vartheta_0) & -\sin(\vartheta) \\ \sin(\vartheta_0) & \cos(\vartheta_0) \end{bmatrix}$

Roots of Unity

$\begin{cases} z_1, z_2 \text{ ROOTS} \Rightarrow z_1 \cdot z_2 \text{ ROOT} \\ z \text{ ROOT} \Rightarrow z^{-1} \text{ ROOT} \end{cases}$

$k \in \{0, \ldots, n-1\}$

$\sqrt[n]{z} = \sqrt[n]{r} \cdot e^{i(2\pi \frac{k}{n})}$

Lim-Sup

$\limsup_{n\to\infty} := \lim_{k\to\infty} \sup_{n \ge k} a_n$

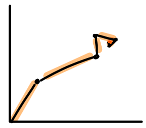
Convergence

$\lim_{n\to\infty} |z_n - z| = 0$ IFF $\begin{cases} Re(z_n) \to Re(z) \\ Im(z_n) \to Im(z) \end{cases}$

Absolute Convergence

GIVEN $\{a_n\}_{\mathbb{N}} \subset \mathbb{C}$,

$\sum_{n=0}^{\infty} |a_n| < \infty$



ABS. CONVERGENT MEANS THE TOTAL LENGTH OF THE 'ARM' IS FIXED

$+ \; + \; + \ldots \in \mathbb{R}$

ABSOLUTE CONVERGENCE IMPLIES CONVERGENCE:

$|a| + |b| \le |a + bi|$ FOR EACH a_n,

SO $\sum |Re| + \sum |Im| \le \sum |a_n|$.

THEN, IN $\mathbb{R}$, ABS. CONV. $\Rightarrow$ CONV.

Power Series

$\begin{cases} P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \; z \in \mathbb{C} \\ \text{DOMAIN}: \{z : P(z) \text{ CONVERGES TO } \mathbb{C}\} \end{cases}$

Geometric Series

$(1-z)(1 + z + z^2 + \ldots + z^N) = 1 - z^{N+1}$

HENCE $\frac{1 - z^{N+1}}{1 - z} = \sum_{n=0}^{N} z^n$

IF $|z| < 1 \rightarrow \frac{1}{1-z} = \sum_{n=0}^{N} z^n$

CONVERGENCE RADIUS $(r \in [0, \infty))$ $\rightarrow \begin{cases} \frac{1}{r} := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \\ \text{USE } \frac{1}{0} = \infty, \; \frac{1}{\infty} = 0 \end{cases}$

- ABSOLUTE CONVERGENCE IF $|z - z_0| < r$
- NOT CONVERGENT IF $|z - z_0| > r$

$e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\lim_{n\to\infty} \sup \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0$ SO IT CONVERGES IN ALL $\mathbb{C}$

$e^{a+bi} = e^a \cdot e^{bi} = e^a(\cos(b) + i \cdot \sin(b))$

$\cos(\vartheta) = \sum_{j=0} (-1)^j \frac{\vartheta^{2j}}{(2j)!}$

$\sin(\vartheta) = \sum_{j=0} (-1)^j \frac{\vartheta^{2j+1}}{(2j+1)!}$ ODD NUMBERS INSTEAD

$\cos(\vartheta) = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}$ $\quad$ $\sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$

2. Measure Theory

σ-ALGEBRA: $\mathcal{A} \in 2^X$

- $\phi \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
- $\{A_k\}_{\mathbb{N}} \in \mathcal{A} \Rightarrow \bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$ $\Rightarrow$ CLOSED IN COUNTABLE INTERSECTIONS

MORGAN'S LAW

$(A \cap B)^c = A^c \cup B^c$

MEASURE ON $(X, \mathcal{A})$

$\mu : \mathcal{A} \to [0, \infty]$

$\oplus$ σ-ADDITIVE: $\{A_k\}_{\mathbb{N}} \subset \mathcal{A}$ DISJOINT

$\mu(\bigcup_{k\in\mathbb{N}} A_k) = \sum_{k=0}^{\infty} \mu(A_k)$

σ-ADDITIVITY $\oplus$ $\exists A \in \mathcal{A}$ ST. $\mu(A) < \infty$

$\Downarrow$

$\mu(A) = \mu(A \cup \phi) = \mu(A) + \mu(\phi)$

$\Downarrow$

$\mu(\phi) = 0$

OTHER MEASURES

- $\mu(A) = |A|$
- $\mu(x) = \delta_x(A) = \mathbb{1}(x \in A)$

- MONOTONICITY: $A \subset B \Rightarrow \mu(A) \le \mu(B)$
- DIFFERENCE: $\mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1)$
- SUBADDITIVITY: $\{A_k\}_{\mathbb{N}}$ HAS $\mu(\bigcup A_k) \le \sum_{k}^{\infty} \mu(A_k)$
- UPWARD MONOTONE CONVERGENCE: $\{A_k\}_{\mathbb{N}}$ NESTED EXPANDING SETS $\Rightarrow \begin{cases} A = \bigcup A_k \\ \lim_{k\to\infty} \mu(A_k) = \mu(A) \end{cases}$
- DOWNWARD MONOTONE CONVERGENCE: $\{A_k\}_{\mathbb{N}}$ NESTED SHRINKING SETS WITH ONE $\mu(A_k) < \infty$ $\Rightarrow \begin{cases} A = \bigcup A_k \\ \lim_{k\to\infty} \mu(A_k) = \mu(A) \end{cases}$

IN MEASURES, σ-SUBADDITIVE IS TRUE AS A CONSEQUENCE OF σ-ADDITIVE ON $\{A_k\}$ DISJOINT

σ-SUBADDITIVE FOR DISJOINT IS A WEAKENING OF THE NOTION OF MEASURE

WHEN DEFINING A MEASURE, YOU WANT σ-ADDITIVITY ON DISJOINT SETS

3. Box Measure

$B = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$

$|B| = (b_1 - a_1) \cdots (b_n - a_n)$

- TRANSLATION INVARIANCE: $|x + B| = |B|$
- n-HOMOGENEITY: $|\lambda B| = \lambda^n |B|$
- FINITE SUB-ADDITIVITY: $B_0 \subset (B_1 \cup \ldots \cup B_m)$ HAS $|B_0| \le |B_1| + \ldots + |B_m|$

Lebesgue Outer Measure

$\mathcal{L}^n_* : 2^{\mathbb{R}^n} \to [0, \infty]$, MORE GENERAL THAN FOR A MEASURE

σ-SUBADDITIVE (FOR $\{A_k\}$ GENERAL)

$\mathcal{L}^n_*(A) := \inf \left\{ \sum_{B \in R} |B| : |A| \subset \bigcup_{B \in R} B \right\}$

- BOUNDED $\Rightarrow \mathcal{L}^n_*(A) < \infty$
- TRANSLATION INVARIANT
- n-HOMOGENOUS
- $A \subset \mathbb{R}^n$ COUNTABLE $\Rightarrow \mathcal{L}^n_*(A) = 0$
- $\mathcal{L}^n_*(B) = |B|$, $\mathcal{L}^n_*(\partial B) = 0$

$\rightarrow$ USING THE VITALI SET, $\mathcal{L}^n_*$ σ-ADD. LEADS TO CONTRADICTIONS

$\uparrow$

$\mathcal{L}^n_*$ MUST BE DEFINED OVER A SMALLER SET

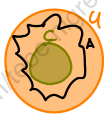
Lebesgue Measurable

$A \subset \mathbb{R}^n$ ST. $\forall \varepsilon > 0$,

$\exists U$ OPEN, $\exists C$ CLOSED ST.

$\rightarrow C \subset E \subset U$

$\rightarrow \mathcal{L}^n_*(U \setminus C) \le \varepsilon$



- BOXES ARE MEASURABLE
- $\{A_k\}_{\mathbb{N}}$

$\mathcal{M}^n$ IS A SIGMA ALGEBRA (σ-ADD. OVER MEASURABLE)

$\mathcal{L}^n := \mathcal{L}^n_* \big|_{\mathcal{M}^n} : \mathcal{M}^n \to [0, \infty]$

σ-SUBADD. FOR GENERAL SETS

$(\mathcal{L}^n_*, 2^{\mathbb{R}^n})$

σ-SUBADD. ON DISJOINT SETS

σ-ADD. ON DISJOINT SETS

MEASURES: DIRAC, $(\#, 2^{\mathbb{N}})$, BOX, $(\mathcal{L}^n_*, \mathcal{M}^n)$