ZERO-SUM GAMES
- PAYOFF MATRIX
- $A = -A^T$ (UTILITIES)
- MAXIMISE THE MIN POSSIBLE REWARD: $\max_x \min_y y^T A x$

→ LINEAR PROGRAM: $\max_x \min_y y^T A x$ SUB $\sum x_i = 1$, $\sum y_j = 1$, $x_i, y_j \geq 0$

$\Rightarrow \max \left(\min_i e_i^T A x \right)$ (e_i CANONICAL BASIS) SUB $\sum x_i = 1$, $x_i \geq 0$

$\Rightarrow \max v$ (v LOWER BOUND) SUB $v \leq e_i^T A x$, $\sum x_i = 1$, $x_i \geq 0$

TURNING A GAME INTO AN LP ⇓

$\Rightarrow \max \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$ SUB $\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \begin{matrix} \leq 0 \\ = 1 \end{matrix}$ OR $\max \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$ SUB $\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{pmatrix} y \\ v \end{pmatrix} \begin{matrix} \geq 0 \\ = 1 \end{matrix}$

NASH EQUILIBRIUM
PAIR OF i,j STRATEGIES, ONE BEST RESPONSE TO THE OTHER
- PURE: m_{ij} MIN IN ROW, MAX IN COLUMNS
- MIXED: $\mathbb{P}$ OVER STRATEGIES (ALWAYS EXISTS)

⇓

MIN-MAX THM: $\exists x^*, y^*$ PROBABILITY VECTORS ST.
$\max_x y^{*T} A x = \min_y y^T A x^*$

THIS NOTATION IS EQUIVALENT TO $E_{x,y}(u)$ OR $\sum_{x,y} u \cdot P_x P_y$

- THE ATTACK/DEFENSE IS TO MAXIMISE/MINIMISE EXPECTED LOSS

INTUITION FOR COMBINATIONS

DIFFER BY THE EXTENSION TO 0 — DIFFER IN BOUNDEDNESS

	LINEAR	AFFINE	CONVEX
$\{x\}$	LINE PASSING BY x, 0	$\{x\}$	$\{x\}$
$\{1,0\}$	UNBOUNDED LINE PASSING THROUGH 1, 0	UNBOUNDED LINE PASSING THROUGH 1, 0	LINE SEGMENT FROM 0 TO 1
$\{x,y,z\}$	PLANE OF x,y,z, EXTENDED TO ORIGIN	PLANE PASSING BY x,y,z	TRIANGLE WITH x,y,z VERTICES

NOTE
ADDING ZERO TO A SET MAINTAINS ITS LINEAR COMBINATION

GEOMETRY

LINEAR COMBINATION $\sum x_i \alpha_i = x$ —USE $X = \{x_i\}$→ LINEAR SUBSPACE L

$\cup$|

AFFINE COMBINATION $\sum \alpha_i = 1$ —USE $X = \{x_i\}$→ AFFINE SPACE $x + L$ OR AFF.HULL(X)

$\cup$|

CONVEX COMBINATION $\forall i \; \alpha_i \geq 0$ —USE $X = \{x_i\}$→ CONVEX SET $\mathrm{conv}(X)$

A HALF-SPACE IS CONVEX, NOT AFFINE $\{x \mid a^T x \leq b\}$

INTERSECTION OF HYPERPLANES CONTAINING X

- IF $x, y : c^T \ldots \leq b$ ALSO FOR $z \in \mathrm{conv}(\{x,y\})$
- IF $\{x_i\}$ FINITE, $\mathrm{conv}(\{x_i\})$ COMPACT

- EMPTY SET $\dim(\emptyset) = -1$
- SINGLETON $\dim(\{x\}) = 0$
- AFFINE SPACE → $\dim(x + L) = \dim(L)$
  ↳ $\dim\left(\{x \in \mathbb{R}^d \mid Ax = b\}\right) = d - \mathrm{RANK}(A)$ (EIGEN-SPACE) (ROUCHÉ CAPELLI THM)
- HYPERPLANE $\dim\left(\{x \in \mathbb{R}^d \mid a^T x = b\}\right) = d - 1$

DEPENDENCE OF A SET $\{x_i\}$
- AFFINE: $\exists \{\alpha_i\}$ ST. $\sum \alpha_i x_i = 0$, $\sum \alpha_i = 0$ (NOT ALL $\alpha_i = 0$) → ! $\{x_i\}$ CAN BE AFF. INDEP EVEN IF $x_1 = 0$; $d+1$ AFF. INDEP. VECTORS $\subseteq \mathbb{R}^d$
  ↳ IFF $(x_1 - x_n) \ldots (x_{n-1} - x_n)$ ARE LIN. INDEPENDENT
- LINEAR: $\exists \{\alpha_i\}$ ST. $\sum \alpha_i x_i = 0$

STANDARD ⟺ EQUATIONAL

STANDARD	EQUATIONAL
$\max c^T x$	$\max c^T x$
SUB $Ax \leq b$, $x \geq 0$	SUB $\begin{bmatrix} A & \mathbb{1} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = b$, $x \geq 0$
LP	(NOTE y IS A VECTOR)

Caratheodory

("Basis" of an affine d-dim. set)

$X \subseteq \mathbb{R}^d$, $\dim(X) = d$, X

$\text{conv hull}(X) = \left\{ \sum_{i=1}^{d+1} \alpha_i x_i \mid x_i \in X, \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$

The convex hull can be written as a combination of at most $d+1$ extreme points

Separation Theorem

→ $C, D \subseteq \mathbb{R}^d$ (nonempty, convex, closed, disjoint)

→ C bounded

There exists a hyperplane strongly separating C, D:

$$\begin{cases} C \subseteq \{x \mid a^T x < b\} \\ D \subseteq \{x \mid a^T x > b\} \end{cases}$$

half-spaces (open sets)

Cones

Generalisation of convex: includes 0 and all points on the ray from 0 to any point on the set and beyond

nonempty $C \subseteq \mathbb{R}^n$

$\forall x, y \in C$, $\forall \lambda, \mu \geq 0$

$\lambda x + \mu x \in C$

- X finite ⇒ $\text{cone}(X)$ is closed
- X lin. indep ⇒ cone is called "primitive" ⇒ cone is a closed set
- A finitely generated cone is the union of finitely many primitive cones

Farkas' Lemma

Exactly one of the following is true:

- $b \in \text{cone}(\{a_1 \dots a_n\})$
- $\exists$ hyperplane $\{x \mid y^T x = 0\}$ with
  1. $\text{cone}(\{a_i\})$ on one side
  2. b strictly on the other side

Equivalently,

- $\exists x \in \mathbb{R}^n$ st. $Ax = b$, $x \geq 0$
- $\exists y \in \mathbb{R}^m$ st. $\begin{cases} y^T A \geq 0^T \\ y^T b < 0 \end{cases}$

TFAE for $\delta = \sup\{c^T x \mid Ax \leq b\}$

- $\forall x \in \mathbb{R}^n$, $Ax \leq b \rightarrow c^T x \leq \delta$
- $\forall y \in \mathbb{R}^m$, $y \geq 0 \rightarrow A^T y = c$, $b^T y \leq \delta$

Duality

Primal → max $c^T x$ st. $Ax \leq b$

Dual → min $b^T y$ st. $\begin{cases} A^T y = c \\ y \geq 0 \end{cases}$

Note that primal and dual exist in two different spaces, so they are geometrically unrelated

Weak LP duality → $c^T x \leq b^T y$

(The relation between x, y feasible sets is in their objectives)

→ P or D unbounded ⇓ D or P infeasible

Strong LP duality → you have 4 cases for P, D

- both feasible
- unbounded, infeasible
- both infeasible

The dual of the dual is the primal again

Complementary slackness conditions

For x, y feasible solutions to the dual

optimal ⟺ $y^T(b - Ax) = 0$

Polyhedra

Minkowski-Weyl

- Polytope ⟺ $\exists$ finite $V \subseteq \mathbb{R}^n$ st $P = \text{conv}(V)$
- Polyhedron ⟺ $\exists$ finite $V, Y \subseteq \mathbb{R}^n$ st. $P = \text{conv}(V) + \text{cone}(Y)$

Face $= P \cap$ supporting hyperplane → not any hyperplane: must $P \subseteq$ half-space

$= \{x \in P \mid c^T x = \delta\}$

- get c from $c^T x \leq \delta$ $\forall x \in P$
- get c from A (subsistem)

You can define a face in two subsistems A' with $\text{Rank}(A') = \text{Rank}(A) - 1$, and the inequalities $\begin{cases} a_1^T x \leq b_1 \\ a_2^T x \leq b_2 \end{cases}$

st. face $= \{x \mid A'x = b', a_1^T x \leq b_1, a_2^T x \leq b_2\}$

Minimal larger face

- $F_1, F_2 \rightarrow \exists F$ st $F_1 \subseteq F$, $F_2 \subseteq F$

Face F is contained in all faces containing F_1, F_2

Use the extremal vertices:

$\{v \in P \mid v \notin \text{conv}(P \setminus \{x\})\}$