→ DO THE TAYLOR EXPANSION:

$$\begin{cases} u' = f(u) \\ u(0) = u_0 \end{cases} \rightarrow u(t) \simeq u_0 + f(u(t))\,t + f'(u(t))\cdot u'(t)\frac{t^2}{2}$$

→ $u'' + u' = \sin t$ } ANSWER INVOLVES $t\sin t$ OR $t\cos t$

→ MATRIX EXPONENTIAL → $(\{\lambda_i\}, \{v_i\}, \overset{B}{[v_i]}, \overset{P}{[\lambda_i]}, P^{-1}, Pe^{Bt}P^{-1})$

- DIAGONAL: $e^A = e^{PDtP^{-1}} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{d_1} & 0 \\ 0 & e^{d_2}\end{bmatrix}P^{-1}$
- JORDAN $J = D + N$ (NILPOTENT), $Pe^{Dt}(I + Nt + N^2\frac{t^2}{2} + \ldots)P^{-1}$ (ALL TERMS AFTER SOME m ARE 0)

→ CHECK f IS A CONTRACTION → $|F'(x)| < \alpha$ (THE DERIVATIVE NEEDS NOT C^0)

→ CHECK THAT IT'S A DISTANCE:

L-NORM IS NORM IF $p \geq 1$
$(\|\cdot\|_L : \mathbb{R}^n \to \mathbb{R}$ IS $\sum^n (|a_i|^L)^{\frac{1}{L}})$

$$\begin{cases} d(x,x) = 0 \\ d(x,y) = d(y,z) \\ d(x,z) + d(z,y) \geq d(x,y) \end{cases}$$

→ CHECK THAT IT'S A NORM

$$\begin{cases} \|x\| = 0 \text{ IFF } x = \underline{0} \\ \|kx\| = |k|\|x\| \\ \|x-y\| \leq \|x-z\| + \|y-z\| \end{cases}$$

GET NORM FROM DISTANCE: $d(x,0) = \|x\|$

GET DISTANCE FROM NORM: $d(x,y) = \|x-y\|$

→ DRAW THE BALL $(c,R) = (\text{CENTER}, \text{RADIUS})$ WITH THIS NORM:
WRITE $d(c,\underline{x}) = R$ → SOLVE FOR $\underline{x}$ AND DRAW IT

→ FIND FIRST INTEGRAL $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \rightarrow \frac{y'}{x'} = \frac{g(x,y)}{f(x,y)}$ } SEPARABLE, INTEGRATE → USE $+c$ TO WRITE $c(x,y)$. THIS IS THE FIRST INTEGRAL

→ ISOCLINES: $x' = 0$ OR $y' = 0$

→ EQUILIBRIUM POINT → INTERSECTION OF ISOCLINES

→ BEHAVIOUR OF EQUILIBRIUM
(x_1, y_1) → COMPUTE JACOBIAN FOR $\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \begin{pmatrix} f(x) \\ g(y)\end{pmatrix}$ → EVALUATE THIS MATRIX AT (x_1, y_1) TO GET A MATRIX $\mathbb{R}^{m \times n}$ → SOLVE THE MATRIX FOR THE EIGENVALUES: λ S.T. $\det(A - \lambda I) = 0$

LINEARISATION IN 2D SYSTEMS

DRAW IN COMBINATION WITH THE ISOCLINES

ALGEBRAIC MULTIPLICITY — GEOMETRIC MULTIPLICITY $m = 1$

- $(n=1)$ → REAL, $\neq 0$ → PROPER NODE
  - REAL, OPPOSITE SIGN → SADDLE
  - COMPLEX, $Re(\lambda) \neq 0$ → FOCUS
  - PURELY IMAGINARY → CENTER
- $(n=2)$ → REAL, $\neq 0$
  - $(m=1)$: IMPROPER NODE
  - $(m=2)$: STAR

CHECK THE FIRST INTEGRAL TO BE SURE

IF LINEARISATION IS A CENTER, IT IS NOT GUARANTEED THAT THE SYSTEM HAS A CENTER

$$\begin{cases} x' = f(x,y)\,h(x,y) \\ y' = g(x,y)\,h(x,y) \end{cases}$$

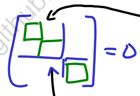
THE ZEROES OF $h(x,y)$ REPRESENT A CONTINUUM OF STEADY STATES:
- LINEARISATION WON'T REVEAL BEHAVIOUR
- $\lambda_i = 0$ APPEARS AS AN EIGENVALUE

$$f(x,y) \approx f(0,0) + \begin{pmatrix} f_x(0,0) \\ f_y(0,0)\end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} f_{xx}(0,0) \\ f_{yy}(0,0)\end{pmatrix}\cdot\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

IF $y = u(t)$ USE THE CHAIN RULE

→ TURN TO JORDAN FORM

1. FIND EIGENVALUES → ARRANGE IN A ZERO MATRIX THE EIGENVALUE IN ORDER OF INCREASING SIZE
2. FOR EACH EIGENVALUE → FIND ALL THE EIGENVECTORS $(A - \lambda_i)v_1 = \underline{0}$
3. YOU SHOULD HAVE AS MANY GENERALISED EIGENVECTORS AS THE ALGEBRAIC MULTIPLICITY → FOR EACH EIGENVECTOR, FIND $(A - \lambda)v_{n+1} = v_n$ UNTIL YOU CAN'T
4. SORT THE CHAINS
5. ADD THE TERMS:



② SIZE OF BOXES FROM LENGTH OF GEN. EIGENVECTORS → PLACE 1'S IN BOXES, ABOVE THE DIAGONAL

① SIZE OF BLOCKS FROM ALGEBRAIC MUL.

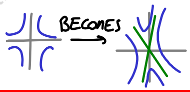
→ CHECK THAT UNIFORM CONVERGENCE IS SATISFIED (DRAWING) → INTUITION: NOT SATISFIED IF THE FUNCTION CONVERGES WITH AN INFINITESIMALLY THIN SPIKE OF CONSTANT HEIGHT

→ DRAW THE VELOCITY FIELD
1. FIND THE SIGN OF x' BY REGION
2. DRAW AN ARROW TOWARDS $+\infty$ IF $x' > 0$ OR TOWARDS $-\infty$ IF $x' < 0$ (DRAW ON THE ISOCLINES)
3. DO SAME FOR y'
4. COMBINE THE ARROWS

THE HAMILTONIAN H(POSITION, MOMENTUM) REPRESENTS TOTAL ENERGY IN THE SYSTEM

→ DETERMINE IF TRAJECTORIES ARE BOUNDED
$\|(x,y)\| \to \infty \Rightarrow H(x,y) \to \infty$ ALL TRAJECTORIES ARE BOUNDED ⇒ ALL TRAJECTORIES ARE BOUNDED EXCEPT STEADY STATES

→ DRAW A TRAJECTORY USING EIGENVECTORS AS AXES:
1. FIND THE BEHAVIOUR FROM LINEARISING
2. DRAW THE EIGENVECTORS
3. STRETCH THE TRAJECTORIES TO THE AXES



→ DEFINITION OF LYAPUNOV FUNCTION / CHECK THAT IT'S LYAPUNOV

THE HAMILTONIAN IS A LYAPUNOV FUNCTION.

- $u' = f(u)$
- $f: \Omega \to \mathbb{R}^n$
- $f \in C^1$
- EQUILIBRIUM POINT p_0

$\Phi: B_r(p_0) \to [0, \infty)$
FOR $B_r(p_0) \subseteq \Omega$ (BALL)
$\Phi(p_0) = 0$,
$\Phi(p) > 0 \;\forall p \in B_r(p_0)$
$\nabla\Phi(p)\cdot f(p) \leq 0 \;\forall p \in B_r(p_0)$

IF STRICT, ASYMPTOTICALLY STABLE

CLASSIFY THE POINTS:
- UNSTABLE: ANY EIG. VALUE > 0
- LYAPUNOV STABLE:
  - STABLE/ATTRACTOR: ALL EIGENVALUES ARE NEGATIVE

SOLUTIONS OF THE CHARACTERISTIC POLYNOMIAL — NUMBER OF REPETITIONS
- REAL: $\{e^{\alpha_i t}, te^{\alpha_i t}, \ldots, t^{n-1}e^{\alpha_i t}\}$
- COMPLEX
  - (SINGLE) $\{\cos(Im(\alpha_i)t), \sin(Im(\alpha_i)t)\}$
  - (REPEATED) $\{e^{Re(\alpha_i)t}\cos(Im(\alpha_i)t), te^{Re(\alpha_i)t}\cos(Im(\alpha_i)t), \ldots\}$

YOU ADD HIGHER ORDER t^i.__

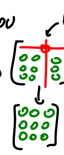
LAPLACE EXPANSION

COMPUTE THE DET OF A MATRIX (ONLY A SQUARE MATRIX HAS DETERMINANT):

MATRIX INDEX NOTATION (SAME AS IN THE JACOBIAN)

$$\begin{pmatrix} (1,1) & \cdots & (1,n) \\ \vdots & \ddots & \vdots \\ (m,1) & \cdots & (m,n) \end{pmatrix} \Rightarrow \text{YOU WRITE } \begin{matrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \end{matrix}$$

1. IDENTIFY A COLUMN/ROW WITH A LOT OF ZEROES
$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_m}{\partial x_n} & \cdots \end{bmatrix}$$
2. $\det = \sum^n (-1)^{j+i} \ell_{i,j} m_{i,j}$ — COFACTOR MATRIX: WHAT YOU GET BY REMOVING ROW AND COLUMN — $\ell_{i,j}$
PICKING i^{th} OR j^{th} ROW/COLUMN
3. ITERATE RECURSIVELY TO FIND THE DET OF THE COFACTORS



→ COMPLETE METRIC SPACE:
ALL CAUCHY SEQUENCE CONVERGE
$\|x_k - x_\ell\| \to 0$ AS $k, \ell \to 0$
SHOULD IMPLY $x_\ell \to$ LIMIT IF $x_k \to 0$

→ YOU DISPROVE THAT (X, d) IS COMPLETE BY SHOWING THAT A CAUCHY SEQUENCE DOES NOT CONVERGE
FIND $x_k \to \underline{0} \Rightarrow d(x_k, 0) \to 0$ — SEQUENCE OF 0 CONVERGING TO ZERO

FIND A NON-CONVERGING CAUCHY SEQUENCE:
- LOOK FOR A DISCONTINUITY IN d
- FIND TWO SEQUENCES WITH THE SAME LIMIT (THEY MUST CONVERGE AT ≠ RATES)
- SHOW THAT $d(x_k, x_\ell) \not\to 0$

→ FIND THE GENERAL SOLUTION:
- $\sum \mu_i u^{(i)}(t) = \varphi(t)$
  1. SOLVE $\varphi(t) \equiv 0$ CASE
  2. FIND A BASIS FOR $\varphi(t)$
  3. SUBSTITUTE AS $u(t)$, $a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots$ INTO (*) AND SOLVE FOR ALL a_i

OBTAIN V.S. OF SOLUTIONS: $u(t) = Au_1(t) + Bu_2(t) + \ldots$

EG FOR $\sin t$ $\{\sin t, t\sin t, \ldots$
$e^t, te^t, \ldots$
$\ldots$

MATRIX NOT INVERTIBLE ⟺ $\ker \neq \{0\}$ ⟺ NOT DIAGON.

MATRIX IS NILPOTENT OF DEG n ⇒ 0 IS THE ONLY EIGENVALUE, WITH ALG. MULT. $m = n$

- $y' + a(x)y = b(x)$
  1. MULTIPLY BOTH SIDES BY $e^{\int a(x)dx}$
  2. $\frac{d}{dy}\left(ye^{\int a(x)dx}\right) = b(x)e^{\int a(x)dx}$
  3. INTEGRATE $\binom{\text{ADD}}{+c}$, SOLVE $y = [\ldots]$

$\frac{dy}{dt} = y(1-y)$
1. $\frac{1}{y(1-y)}dy = dt$
2. PARTIAL FRACTION DECOMPOSITION AND INTEGRATE
REMEMBER: $\int \frac{1}{1-x}dx = -\ln(1-x) + c$
3. SOLVE FOR y

$$u''' = t^3(u+u') + t^2(u'+u'')$$

→ DECOUPLE HIGHER ORDER ODE: $\sum a_i u^{(i)} = 0$

1. WRITE $\begin{cases} v_1(t) = u(t) \\ v_2(t) = u'(t) \\ \ldots \end{cases}$
2. WRITE THE HIGHEST ORDER TERM AS A COMBINATION OF THE OTHERS
3. COMBINE ALL EQUATIONS

→ SOLVE $\begin{cases} u' = Au \\ u(0) = \underline{u_0} \end{cases} \rightarrow u(t) = u_0 e^{At}$

DECOUPLING A LINEAR SYSTEM: (DIAGONALISABLE ⇓ CAN BE FULLY DECOUPLED)

1. FIND EIGENVECTORS
2. SOLVE $v_1' = \gamma_1 v_1$ AS $e^{-\gamma_1 t} = v_1(t)$
3. WRITE v_2 IN TERMS OF v_1: IN THE REFERENCE FRAME v_2 IS kx^n
4. SOLVE FOR u, v $\begin{cases} v_i = hu + mv \\ v_l = au + bv \end{cases}$

→ FIND OUT THE BEHAVIOUR OF $u(t)$ IF $u(0) = u_0$:

1. PLOT THE SIGN OF $f(u)$
2. FOLLOW THE TRAJECTORY FROM u_0 TO ITS LIMIT

$$\begin{cases} v_1(t) = c_1 e^{-\gamma_1 t} \\ v_2(t) = c_2 e^{-\gamma_2 t} \end{cases}$$

$$\left(\frac{v_2(t)}{c_2}\right)^{\frac{1}{\gamma_2}} = \left(e^{-\gamma_2 t}\right)^{\frac{1}{\gamma_2}} = e^{-t} \quad \ldots$$

WRITE v_1 AS A FUNCTION OF v_2

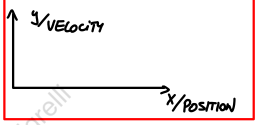
NEWTONIAN SYSTEM

- $V(x)$ IS GIVEN / x'' GIVEN
- $x'' = -V'(x)$

$\begin{cases} x' = y \\ y' = V'(x) \end{cases}$ STUDY THE LINEAR SYSTEM

↓

POTENTIAL CAN BE NEGATIVE, BUT E_k IS SUPPOSED TO BE POSITIVE



- LYAPUNOV : 0 AT CENTER POINT, DECREASING NONSTRICTLY OVER PATH, ≥0
  ↳ HAMILTONIAN: (PHYSICS CONNOTATION)
- FIRST INTEGRAL: CONSTANT OVER PATH

- HORIZONTAL (SLOPE) ISOCLINES → $y' = 0$
- VERTICAL (SLOPE) ISOCLINES → $x' = 0$

ISOCLINES AND EIGENVECTORS ARE RELATED BUT DON'T ENCODE ENOUGH INFORMATION ON THEIR OWN

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

- FIND $\{\lambda_i\}$
- FIND $u(t)$

→ COMPOSE THE SYSTEM INTO SECOND ORDER, SOLVE USING $u(t)$ FOR x, THEN OBTAIN y IN TERMS OF x

$$\begin{cases} x' = ax + by \rightarrow y = \frac{x'}{b} - \frac{ax}{b} \\ y' = cx + dy \end{cases}$$

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx+dy) \\ &= ax' + b\left[cx + d\left(\frac{x'}{b} - \frac{ax}{b}\right)\right] \\ &= ax' + bcx + dx' - adx \\ &= (a+d)x^2 + (bc-ad)x \end{aligned}$$

1. SOLVE FOR $x(t)$
2. GET $y(t)$

github.com/tebe-nigrelli