CH4

PREIMAGE
$f^{-1}(J) = \{x \in X : f(x) \in J\} = \{f \in J\}$

EXTENDED FUNCTION
$f : X \to \overline{\mathbb{R}}$
$\overline{\mathbb{R}} := [-\infty, \infty]$

CHARACTERISTIC FUNCTION
$\chi_A(x) := \begin{cases} 1 & x \in A \\ 0 & \text{ELSE} \end{cases}$
ASSUMING $A \in \mathcal{A}$
$\int_X \chi_A(x)\, d\mu = \mu(A)$

SIMPLE FUNCTION
$\varphi(x) = \sum_{k=1}^{m} \alpha_k \chi_{A_k}(x)$
$m \in \mathbb{N}$, $\{A_i\} \subset \mathcal{A}$, $\{\alpha_i\} \subset \mathbb{R}$
- HAS A FINITE RANGE
- ARE MEASURABLE

(UNIQUE) CANONICAL REPRESENTATION: $\{A_i\}$ PAIRWISE DISJOINT, $\{\alpha_i\}$ PAIRWISE DIFFERENT

THE CANONICAL REPRESENTATION HAS A_k DISJOINT, $\alpha_i \neq \alpha_j$, $\neq 0$

$\mathcal{A}$-MEASURABLE
$\forall a \in \mathbb{R}$
$\{f > a\} \in \mathcal{A}$

$\chi_A(x)$ FITS:
$\begin{cases} \emptyset & \text{IF } 1 < a \\ A & \text{IF } 0 \le a \le 1 \\ X & \text{IF } a < 0 \end{cases}$

$\Longrightarrow$
$\in \mathcal{A}$
$\{f \ge a\} = \bigcap_{k=1}^{\infty} \{f > a - \frac{1}{k}\}$
$\{f \ge \infty\} = \bigcap_{k=1}^{\infty} \{f > k\}$
$f^{-1}(\text{BOREL SET}) \in \mathcal{A}$

$\mathcal{A} \subset 2^X$ SIGMA ALGEBRA

MEASURE
$\mu : \mathcal{A} \to [0, \infty]$ ST.
$\mu(A \cup B) = \mu(A) + \mu(B)$
FOR A, B DISJOINT

$f_k \to f$
- POINTWISE:
$\forall x \in X, \quad \lim_{k\to\infty} f_k(x) = f(x)$
- UNIFORMLY: (STRONGER)
$\lim_{k\to\infty} \sup_{x\in X} |f_k(x) - f(x)| = 0$

$f \in C^0 \Rightarrow f$ IS $\mathcal{H}^n$ MEASURABLE

PROOF:
$f^{-1}((a, \infty]) = $ OPEN SET IN $\mathbb{R}$ TOPOLOGY
IS OPEN BY CONTINUITY

STABLE UNDER:
→ POINTWISE LIMIT:
$\{f_k\} \to f \Rightarrow f$ $\mathcal{A}$-MEAS.
$\mathcal{A}$-MEAS.
→ SUM/PRODUCT (WHEN DEFINED)
→ COMPOSITION WITH C^0 (INVERSION PRESERVES OPENNESS)

f IS $\mathcal{A}$-MEASURABLE
$\Updownarrow$
f IS POINTWISE LIMIT OF SIMPLE FUNCTIONS

YOU CAN USE SIMPLE FUNCTIONS TO APPROX. $\mathcal{A}$-MEASUR. FUNCTIONS

f $\mathcal{A}$-MEASURABLE $\Rightarrow \exists \{f_k\}$ SIMPLE POINTWISE CONVERGENT
$\hookrightarrow \oplus$ BOUNDED $\Rightarrow \exists \{f_k\}$ SIMPLE, UNIFORMLY CONVERGENT
(2x MONOTONE EXIST: INCREASING, DECREASING)

$\begin{cases} \limsup_{k\to\infty} f_k(x) = \inf_{k\in\mathbb{N}} \sup_{m\ge k} f_m(x) \\ \liminf_{k\to\infty} f_k(x) = \sup_{k\in\mathbb{N}} \inf_{m\ge k} f_m(x) \end{cases}$

YOU CAN PROVE PROPERTIES THROUGH $\lim_{k\to\infty} \int f_k d\mu = \int f d\mu$

LEBESGUE INTEGRAL MAKES NO ASSUMPTION OF CONTINUITY / OTHER REGULARITY

$f : X \to [0, \infty]$
$\int_X f d\mu := \sup \left\{ \int_X \varphi d\mu : \varphi \le f \right\}$
$f : X \to \mathbb{R}$, $\int_X |f| d\mu < \infty$ → INTEGRABILITY ASSUMPTION
$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu$

CH5

LEBESGUE INTEGRAL
1. NONNEGATIVE, SIMPLE
2. NONNEGATIVE, MEASURABLE
3. INTEGRABLE $\left(\int_X |f| d\mu < \infty\right)$

THE VALUE OF THE INTEGRAL IS INDEPENDENT OF REPRESENTATION

- LINEARITY
- MONOTONICITY
- UNIFORM BOUND:
$\int_X \varphi d\mu \le \left(\sup_{x\in X} \varphi(x)\right) \mu(\{\varphi > 0\})$

POSITIVE FUNCTION $f : X \to [0, \infty]$
$\int_X f d\mu < \infty \Rightarrow \mu(\{f = \infty\}) = 0$
MARKOV INEQUALITY:
$\mu(\{f \ge \lambda\}) \le \frac{1}{\lambda} \int_X f d\mu \quad \lambda > 0$
$\int_X f d\mu = 0 \Rightarrow \mu(\{f > 0\}) = 0$

$\left| \int_X f d\mu \right| \le \int_X |f| d\mu$
$\int_X |f| d\mu \le \sup_{x\in X} |f(x)| \cdot \mu(\{|f| > 0\})$

THE LEBESGUE INTEGRAL IS INVARIANT TO MODIFICATIONS OVER A SET WITH MEASURE ZERO:
$\mu.a.e. \iff \mu(\{\text{THAT THING HAPPENS}\}) = 0$

$f : [a,b] \to \mathbb{R}$ BOUNDED,
f IS RIEMANN INTEGRABLE $\iff$ CONTINUOUS $\mu.a.e$
LEBESGUE INTEGRABLE IS A SUPERSET OF RIEMANN INT.
WHEN IT EXISTS, THE RIEMANN INTEGRAL IS EQUAL

FATOU'S LEMMA
$\{f_k\} : X \to [0, \infty]$
$\liminf_{k\to\infty} \int_X f_k d\mu \ge \int_X \liminf_{k\to\infty} f_k d\mu$

$\begin{cases} f_+ := \max\{f, 0\} \\ f_- := \max\{-f, 0\} \end{cases}$
$|f| = f_+ + f_-$

$\Rightarrow$ f INTEGRABLE: $\int_X |f| d\mu < \infty$

$\Rightarrow$ THE RESTRICTION $f \mapsto f\chi_E$ IS ALSO INTEGRABLE:
$\int_E f d\mu = \int_X f \chi_E d\mu$

$\mu.a.e$ INDEPENDENT
- $f = g \Rightarrow \int f d\mu = \int g d\mu$
- $f \le g \Rightarrow \int f d\mu \le \int g d\mu$
- $|f| \le \lambda \Rightarrow \left| \int_X f d\mu \right| \le \lambda \mu(\{|f| > 0\})$

f ACTUALLY STANDS FOR EQUIV. CLASS
$[f] = \{g : f = g\ \mu.a.e.\}$

MONOTONE CONVERGENCE
$f_k : X \to [0, \infty]$ SEQUENCE MEASURABLE, $f_k \le f_{k+1}$
THEN
$\lim_{k\to\infty} \int_X f_k d\mu = \int_X \lim_{k\to\infty} f_k d\mu$

f MEASURABLE,
$\nu(A) = \int_A f d\mu$
IS MEASURE

RIGHT CONTINUITY OF THE DEFINITE LEBESGUE INTEGRAL

DOMINATED CONVERGENCE
$\{f_k\} : X \to \overline{\mathbb{R}}$ MEASURABLE
$f_k \to f$ POINTWISE $\mu.a.e.$
$|f_k| \le g\ \mu.a.e.$ $\exists g$ MEASURABLE, NONNEGATIVE, INTEGRABLE
$\Downarrow$
$\lim_{k\to\infty} \int_X f_k d\mu = \int_X f d\mu$

NOTE THAT f BEING MEASURABLE IS A PREREQUISITE TO INTEGRABILITY

CH6

FOR INTEGRABILITY YOU JUST CHECK $\int_X |f|^p d\mu < \infty$

L^p NORM
$\|f\|_{L^p(X,\mu)} := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}, \quad p \in (0, \infty)$
$\|f\|_{L^\infty(X,\mu)} := \inf\{\lambda \in [0,\infty] : |f(x)| \le \lambda\ \mu.a.e.\}$

- $|f(x)| \le \|f\|_\infty\ \mu.a.e.\ \forall x \in X$
- $\|f\|_\infty = \sup_x |f(x)|$

- $\|\lambda f\|_{L^p} = \lambda \|f\|_{L^p}, \ \lambda > 0$
- $\|f - g\|_{L^p} = 0 \Rightarrow f = g\ \mu.a.e.$

THE L^p EQUIVALENCE CLASS IS ALWAYS THAT OF $\mu.a.e$

$(p, q) \in [1, \infty]^2$ CONJUGATED:
$\frac{1}{p} + \frac{1}{q} = 1$, USING $\frac{1}{\infty} = 0$

HÖLDER INEQUALITY
f, g MEASURABLE
$\int_X |f| \cdot |g| d\mu \le \|f\|_{L^p} \cdot \|g\|_{L^q}$

MINKOWSKI INEQUALITY
$p \in [1, \infty]$, $\|f\|_{L^p}, \|g\|_{L^p} < \infty$
$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$

TRIANGLE INEQUALITY HOLDS ONLY IF $p \in [1, \infty]$

L^p-SPACE IS A VECTOR SPACE OF EQUIVALENCE CLASSES
$L^p(X, \mu) := \{[f] : \|f\|_{L^p} < \infty\}$ ← ASSUMING $p \in [1, \infty]$

CHECK THAT ALL OPERATIONS ARE ALWAYS INDEPENDENT OF CLASS REPRESENTATIVE

CONVERGENCE IN L^p
$\lim_{k\to\infty} \|f_k - f\|_{L^p} = 0$
$(p = \infty) \Rightarrow$ UNIFORM CONVERGENCE $\mu.a.e$

AS p INCREASES, MORE IMPORTANCE IS GIVEN TO SPIKES

$f_k \to f$ IN L^p,
$\exists k_m \uparrow \infty$ SUBSEQUENCE ST. $f_{k_m} \to f$ POINTWISE $\mu.a.e$

SINCE $\left| \int_X f d\mu \right| \le \int_X |f| d\mu$, IT MAKES SENSE IN THE L^p NORM TO USE THE MODULUS, AS IT IMPOSES A STRONGER REGULARITY ON f BEING FINITE: BOTH f^+ AND f^- MUST BE INTEGRABLE

YOU CAN CHECK IF A SPACE IS HILBERT BY VERIFYING THE POLARISATION IDENTITY

HILBERT SPACE:
YOU CAN OBTAIN NORM FROM DOT BECAUSE THEY ARE TIGHTLY RELATED

CH7

METRIC/NORMED SPACE
$\downarrow \oplus$ CAUCHY SEQUENCES CONVERGE
COMPLETE SPACE
$\downarrow \oplus$ VS. STRUCTURE
BANACH SPACE
$\downarrow \oplus$ INNER PRODUCT RELATED TO NORM
HILBERT SPACE

CAUCHY SEQUENCE:
$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}$ ST.
$\forall k_1, k_2 \ge k_0$
$\|v_{k_1} - v_{k_2}\| \le \varepsilon$

COMPLETE SPACE:
ALL CAUCHY SEQ. CONVERGE IN THE SPACE
- FINITE DIM, NORMED IS ALWAYS COMPLETE
- $p \in [1, \infty]$ $L^p(X, \mu)$ IS COMPLETE

HILBERT SPACE
BANACH SPACE WITH NATURALLY INDUCED INNER PRODUCT
(AKA. YOU CAN DEFINE ANGLES AND ALIGNMENT)

$(\mathbb{R}^n, L^2)$
$(\ell^2(\mathbb{R}), L^2)$: SQUARE-SUMMABLE SERIES

HILBERT BASIS (COUNTABLE SET)
→ $\dim < \infty$: $\exists$ ORTHONORMAL $\{e_i\}_1^n$
$v = \sum \langle v, e_i \rangle e_i$

PARSEVAL:
$\|v\|^2 = \sum_{k=i}^{n} \langle v, v_k \rangle^2$
CONSEQUENCE OF HILBERT STRUCTURE
NORM IS BASIS INDEPENDENT*

→ $\dim = \mathbb{N}$
$\mathcal{F} \subset \mathcal{H}$ COUNTABLE: ORTHO. + DENSE
- $\langle v, w \rangle = 0$, $v \ne w$, $v, w \in \mathcal{F}$
- $\|v\| = 1$, $v \in \mathcal{F}$
- $\forall v \in \mathcal{F}$, $\exists (v_k)_{k\in\mathbb{N}} \subset \text{SPAN}(\mathcal{F})$
ST. $v_k \to v$

AKA. $\mathcal{H}$ IS THE CLOSURE OF $\mathcal{F}$

HERMITIAN INNER PRODUCT
$\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$
- $v \mapsto \langle v, w \rangle$ ($\mathbb{C}$-LINEAR IN w)
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$
- $\langle v, v \rangle \ge 0$ ($=0$ IFF $v = 0$)
$\|v\|^2 \in \mathbb{R}$, INDUCES A NORM

HILBERT BASIS THM
$(\mathcal{H}, \langle\cdot,\cdot\rangle)$ WITH $\mathcal{F}$
$\forall v \in \mathcal{H}$, $v_k := \langle v, e_k \rangle$
- $\sum_{k=0}^{K} v_k e_k \xrightarrow{K\to\infty} v$
- $\|v\|^2 = \sum_{k=0}^{\infty} |v_k|^2$

COMPLEX HILBERT SPACE
$(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$
$\Downarrow$
HILBERT BASIS THM HOLDS

$L^2(X, \mu)$ IS ALSO HILBERT SPACE: (NOT ONLY BANACH)
$\langle f, g \rangle = \int_X f \cdot g\, d\mu$ → NORM

CAUCHY SCHWARTZ
$\langle f, g \rangle \le \|v\| \cdot \|w\|$

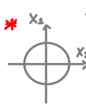
POLARISATION IDENTITY
$\langle v, w \rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 \right)$

$p = 2$ IS QUITE SPECIAL BECAUSE THE SPACE IS HILBERT

X, $\mathcal{A}$ σ-ALG ON X,
$\mu : \mathcal{A} \to [0, \infty]$ MEASURE
$f : X \to \mathbb{C}$
$f(x) = u(x) + iv(x)$
MEASURABLE IFF u, v MEAS.

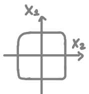
THE EXTENSION IS NATURAL:
$\int_X f d\mu = \int_X u\, d\mu + i \int_X v\, d\mu \in \mathbb{C}$
- $\mathbb{C}$-LINEAR
- $\left| \int_X f d\mu \right| \le \int_X |f| d\mu$

$\left\{ [f] \ \middle|\ f : X \to \mathbb{C} \text{ MEASURABLE}, \int_X |f| d\mu < \infty \right\}$ INTEGRABLE



* IN $\mathcal{H} = \mathbb{R}^2$, PARSEVAL SAYS $\|x\|$ IS PARAMETRISATION INDEPENDENT, AKA RADIUS IS CONSTANT IF YOU ROTATE THE VECTOR

TAKE $p = 10$



THE RADIUS CHANGES IF YOU ROTATE THE VECTOR

CH8

Fourier Decomposition

$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx}$ | $\hat{f}(k)$ being a coefficient

$\left\{ \frac{e^{ikx}}{\sqrt{2\pi}} : k\in\mathbb{Z} \right\}$ → Hilbert basis of $L^2([-\pi,\pi], \mathcal{L}^1, \mathbb{C})$

$f(x) = \sum_{k\in\mathbb{Z}} \langle f, \frac{e^{ikx}}{\sqrt{2\pi}} \rangle \frac{e^{ikx}}{\sqrt{2\pi}}$ | using the L^2 inner product

$\sum_{k=-N}^{N} \hat{f}(k) e^{ikx} \to f$, $\in C^\infty$

- (L^p) Pointwise $\mathcal{L}^1$ a.e. up to taking a subsequence
- (L^2) Pointwise $\mathcal{L}^1$ a.e
- (C^1) uniformly

$f(x) \in L^1([-\pi,\pi])$

$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$

- $|\hat{f}(k)| \le \frac{1}{2\pi} \|f\|_{L^1}$

Parseval

$\|f\|^2_{L^2} = 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$

$f: [-\pi,\pi] \to \mathbb{R}$

$\overline{\hat{f}(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = \hat{f}(-k)$

(and. $\mathrm{Im}(\hat{f}(k))$ is an odd function) ← $\int \overline{a b} d\mu = \int \bar{a}\bar{b} d\mu$

Tricks:

→ replace $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

→ use Parseval to relate an infinite sum to an integral

$\hat{f}(k) = \int_{-\pi}^{\pi} f(x)\cos(kx)dx + i\int_{-\pi}^{\pi} f(x)\sin(kx)dx$ (cosine transform, sine transform)

→ You can also see the span of $\mathcal{F}$ as $f(x) = p(e^{it}) + q(e^{-it})$, p, q complex polynomials

Fourier and Derivatives

$\hat{f'}(k) = ik\hat{f}(k)$

$\hat{f^{(m)}}(k) = (ik)^m \hat{f}(k)$ } f regular enough

The smoother f, the faster coefficients decay:

1. $f \in L^2$ has $2\pi\sum|\hat{f}(k)|^2 = \|f\|_{L^2} < \infty$ so $\lim_{k\to\pm\infty} \hat{f}(k) = 0$ → Also holds for L^1 but not from Parseval's identity

2. $|\hat{f}(k)| \le \frac{C}{|k|^m}$ implies $\sum |k|^m |\hat{f}(k)| < \infty$ → It relates to the finiteness of infinite sums

n-variables

$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{ik\cdot x}$ | $x\in\mathbb{R}^n$

$\hat{f}(k) := \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} f(x) e^{-ik\cdot x} dx$ → Hilbert basis $\left\{ \frac{e^{ik\cdot x}}{(2\pi)^{n/2}} : k\in\mathbb{Z} \right\}$

$\|f\|_{L^2} = (2\pi)^n \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2$

$\widehat{\frac{\partial f}{\partial x_j}}(k) = ik_j \hat{f}(k)$

$\mathcal{A} \subset C(K, \mathbb{C})$ is an algebra: VS under product. Also,

→ $\mathcal{A}$ closed in conjugation: $f\in\mathcal{A} \Rightarrow \bar{f}\in\mathcal{A}$

→ $\mathcal{A}$ separates points: $\forall x,y\in K$, $\exists f\in\mathcal{A}$ st. $f(x)\ne f(y)$

Stone-Weierstrass:

- K compact metric space
- $\mathcal{A} \subset C(K,\mathbb{C})$ algebra,
- $\mathcal{A}$ contains constants, separates points, closed in conjugation

⇓

$\mathcal{A}$ is dense in $C(K,\mathbb{C})$

Density of C^0 in L^p

$\Omega \subset \mathbb{R}^n$ open, $1 \le p \le \infty$.

$\forall f \in L^p(\Omega, \mathcal{L}^n, \mathbb{C})$

$\exists (f_h)_{\mathbb{N}} \subset C(\mathbb{R}^n; \mathbb{C})$ st. $f_h|_\Omega \to f$ in L^p

For any $f\in L^p$, there is a sequence of C^0 ($/C^\infty$) functions converging to it

---

Expand the domain from $[-\pi,\pi]$ to $\mathbb{R}$

CH9

Fourier Transform

$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$

- $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$ } The max coeff. in Fourier is bounded by the max deviation from 0 of $f(x)$

$\mathcal{F}: L^1(\mathbb{R},\mathbb{C}) \to C_0(\mathbb{R},\mathbb{C})$

Fourier operator is 1-Lipschitz

If $f\in L^2$, then $\hat{f}$ may not be well-defined, with $f(x)e^{ikx} \notin L^2$ → If the function is L^2, you should check that the FT is integrable

$C_c = \{\text{compactly supported } f\}$

$\forall f\in L^1(\mathbb{R},\mathbb{C})$, $\forall\varepsilon>0$, $\exists g\in C^\infty_c(\mathbb{R},\mathbb{C})$ st. $\|f-g\|_{L^1} \le \varepsilon$

Isometries (preserve $\int f d\mu$, hence $\|f\|_{L^1}$)

- $\tau_h f := f(x-h)$ → translation
- $\sigma_\delta f := \frac{1}{|\delta|} f(x/\delta)$ → dilation
- $\widehat{\tau_h f} = e^{-i\xi h}\hat{f}(\xi)$
- $\widehat{e^{ixh} f} = \tau_h \hat{f}(\xi)$
- $\widehat{\sigma_\delta f} = \hat{f}(\xi\delta)$

} Prove as an exercise

Examples

- $e^{-|x|} \leftrightarrow \frac{2}{1+\xi^2}$
- $\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \leftrightarrow e^{-\frac{\xi^2}{2}}$
- You should try to write $f(x)$ as a sum of e^{-ikx} to get the coefficients directly

Convolution

$(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$

→ $\|f*g\|_{L^1} \le \|f\|_{L^1} \cdot \|g\|_{L^1}$

→ $(\frac{1}{p} + \frac{1}{q} = 1)$

$\|f*g\|_{L^\infty} \le \|f\|_{L^p} \cdot \|g\|_{L^q}$

$f\in L^1$, $g\in C^\infty$ → $f*g \in C^\infty$ (Also if compactly supported)

$\widehat{f*g} = \hat{f}(\xi)\cdot\hat{g}(\xi)$

Approximation of identity

$\varphi: \mathbb{R}\to[0,\infty)$ st. $\int_{\mathbb{R}} \varphi(x)dx = 1$

$\sigma_\delta\varphi = \frac{1}{\delta}\cdot\varphi(\frac{x}{\delta})$ As $\delta\to 0$, mass goes to 0

$f_\delta(x) = (f*\sigma_\delta\varphi)(x)$

$f_\delta \to f$ in L^1 as $\delta\to 0$

If you take the limit you get the Dirac distribution, with $\mathbb{E}_{\delta_c}(f(x)) = f(c)$

Adjoint

$\mathcal{F}^T = \mathcal{F}^{-1}$

CH10

Fourier Anti-Transform

$\mathcal{F}^* f = \check{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi) e^{i\xi x} d\xi$

$\mathcal{F}^* f(\xi) = \mathcal{F}f(-\xi)$

Pick a convention for the normalisation:

A. $\hat{f} = \int_{\mathbb{R}} f e^{-i\xi x} dx$, $\check{f} = \frac{1}{2\pi}\int_{\mathbb{R}} f e^{i\xi x} d\xi$ → (in the course) $\|\hat{f}\|^2_{L^2} = 2\pi\|f\|_{L^2}$

B. $\hat{f} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f e^{-i\xi x} dx$, $\check{f} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f e^{i\xi x} d\xi$ → Results in $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$

Inner Product

$f, g \in L^1\cap L^2$, $\mathcal{F}f \in L^2$ ⇒ $\langle \mathcal{F}f, g\rangle = \langle f, \mathcal{F}^* g\rangle$ — $\mathcal{F}$ is adjoint operator

$\mathcal{F}\mathcal{F}^* f = \mathcal{F}^*\mathcal{F}f = 2\pi f$ → assuming $f\in L^1\cap L^2$

$\|\mathcal{F}f\|^2_{L^2} = 2\pi\|f\|^2_{L^2}$

$\langle\hat{f},\hat{g}\rangle = \langle\mathcal{F}f,\mathcal{F}g\rangle = 2\pi\langle f,g\rangle$ } $\mathcal{F}$ acts like a change of basis

$\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ ← In the Fourier transform, $e^{i\xi x} \to e^{-i\xi x}$

Operator Properties

$\mathcal{F}: L^1(\mathbb{R},\mathbb{C}) \to C_0(\mathbb{R},\mathbb{C})$ injective.

$\hat{g} = \hat{f} \Rightarrow f = g$ $\mathcal{L}^1$-a.e.

$\mathcal{F}: L^1\cap L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$

$\mathcal{F}$ is an endomorphism

CH11

$\mathcal{F}$ of Derivative

$f\in C^1(\mathbb{R},\mathbb{C})$ has $f, f'\in L^1$

$\hat{f'}(\xi) = i\xi\hat{f}(\xi)$

↳ (if you add $f, f'\in L^2$) $\|\xi\hat{f}\|^2_{L^2} = \|\hat{f'}\|^2_{L^2} = 2\pi\|f'\|^2_{L^2}$

For the m-th derivative → $\hat{f^{(m)}} = (i\xi)^m\hat{f}(\xi)$

A more regular f ($\dots\in C^m$) corresponds to a more integrable $\hat{f}$ (m-moment is defined)

Derivative of $\hat{f}$

$f\in L^1(\mathbb{R},\mathbb{C})$, $xf(x)$ integrable → $\frac{d\hat{f}(\xi)}{d\xi} = -i\widehat{xf}(\xi)$

$f\in L^1(\mathbb{R},\mathbb{C})$, $x^m f(x)$ integrable → $\frac{d^m\hat{f}(\xi)}{d\xi^m} = \widehat{(-ix)^m f}(\xi)$

More regular $\hat{f}$, more integrable f

Both identities relate $\hat{f}$ through derivatives

Heat Equation

$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$, $u(0,x)$ = given

Solve:

$0 = \mathcal{F}(\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u)(\xi,t) = 0$

$= \frac{\partial}{\partial t}\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t)$

$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-t\xi^2}$

$2\pi u(x,t) = \mathcal{F}^*(\hat{u}(\xi,t))$

$= \mathcal{F}^*(\hat{u}_0(\xi)e^{-t\xi^2})(x)$

$u(x,t) = u_0 * P_t(x)$

$= u_0 * \sigma_{\sqrt{t}} P_1(x)$

Term	Series	Transform
$\hat{f}(k)$	$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$	$= \int_{\mathbb{R}} f(x)e^{-i\xi}dx$
$f(x)$	$= \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{-ikx}$	$= \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{i\xi x}dx$
$\|f\|^2_{L^2}$	$= 2\pi\sum_{k\in\mathbb{Z}} \|\hat{f}(k)\|^2$	$= 2\pi\|\hat{f}(\xi)\|^2_{L^2}$
$\hat{f^n}(k)$	$= (ik)^n\hat{f}(k)$	$= (i\xi)^m\hat{f}(\xi)$
$f\in C^m$	$\to \|\hat{f}(k)\| \le \frac{C}{\|k\|^m}$	$\left(\frac{d\hat{f}(\xi)}{d\xi} = \widehat{(-ix)^m f}(\xi)\right)$
$f\in C^1$	→ Fourier series converges uniformly	Series/transform are defined for both L^1, L^2
$f\in L^p$	→ Fourier series converges pointwise up to SSQ.	
$f\in L^1$	→ $\lim_{k\to\pm\infty}\hat{f}(k) = 0$	Parseval/Plancherel holds only in L^2
$f\in L^2$	→ Fourier series converges pointwise $\mathcal{L}^1$-a.e.	(ensures convergence in norm of sums)
	$\sum\|k\|^m\|\hat{f}(k)\| < \infty$ ⇓ $f\in C^m$	

Checking for what $p\in[1,\infty]$, $f\in L^p$

- $p=\infty$: look for a $\sup_x |f(x)|$ μ.a.e. If $f\in C^0$, $\lim_{x\to\pm\infty} f(x) = 0 \to f\in L^\infty$
- $p\in[1,\infty)$: find upper bound that converges (eg. $f = O(g)$) ⇒ $f\in L^p$; find lower bound that diverges ⇒ $f\notin L^p$

$\int_{-\infty}^{\infty} |f|^p dx = \int_{-\infty}^{-k} |f|^p dx + \int_{-k}^{k} |f|^p + \int_{k}^{\infty} |f|^p dx$

(Study behaviour one at a time)