ASSUME THE ROOTS OF $p(z)$ ARE REAL

$\downarrow$

FOR EACH ROOT α_ℓ, $\{e^{\alpha_\ell t}, t e^{\alpha_\ell t}, \dots, t^{m_\ell - 1} \cdot e^{\alpha_\ell t}\}$

· EACH SOLVES THE HOMOGENEOUS ODE, AND TOGETHER THEY ARE A BASIS OF THE SOLUTION SPACE.

PROOF

A) EACH SOLVES THE ODE

- CONSTRUCT A 1:1 CORRESPONDANCE FROM A CHARACTERISTIC POLYNOMIAL TO THE LINEAR OPERATOR THAT DETERMINES u $a(\dots) = \sum_{j=0}^{n} \mu_j \mu_{\dots}^{j}$

$$\begin{cases} \alpha(z) = \sum_{j=0}^{\ell} \mu_j z^j, & \text{(CHARACTERISTIC POLYNOMIAL)} \\ a(D) := \sum_{j=0}^{\ell} \mu_j \frac{d^j}{dt^j} & \text{(LINEAR OPERATOR)} \end{cases}$$

- $p(z) = (z-\alpha_1)^{k_1} \cdots (z-\alpha_n)^{k_n}$

$\Updownarrow \quad \overset{!}{=} \text{(NON-}\alpha_\ell\text{ COMPONENTS)} \cdot (\alpha_\ell \text{ COMPONENTS)}$

$p(D)u = 0$ (u IS A FUNCTION)

$\overset{!}{=} q(D)[r(D)u]$ (SPLIT LIKE A POLYNOMIAL)

- TO CHECK IF $t^j e^{\alpha t}$ IS A SOLUTION, CHECK $r(D)$ MAPS u TO ZERO

THIS M REPRESENTS ITERATION

- THIS IS EQUIVALENT TO: $(D-\alpha)^M [h(t) e^{\alpha t}] = 0$ FOR $h(t)$ POLYNOMIAL OF DEGREE $< M$
  - $(M=1)$ $(D-\alpha)e^{\alpha t} = \frac{d}{dt} e^{\alpha t} - \alpha e^{\alpha t} = 0$
  - $(M \geq 2)$ WITH $(D-\alpha)^M = (D-\alpha)^{M-1} \circ (D-\alpha)$,

$(D-\alpha)^M [h(t) e^{\alpha t}] = (D-\alpha)^{M-1} [h'(t) e^{\alpha t}]$

$h'(t)$ HAS $\text{DEG}(h(t)) - 1$ SO THE EXPRESSION VANISHES IF AN INNER TERM VANISHES

B) THEY ARE INDEPENDENT

FOR ANY EIGENVALUE γ,
$1 \leq m_\gamma \leq M_\gamma$

- FOR γ BEING AN EIGENVALUE, THERE IS v EIGENVECTOR $\Longrightarrow$ EIGENSPACE HAS DIM ≥ 1
$\text{KER}(A - \gamma I) \neq \{0\}$
$m_\gamma \geq 1$

- $f(v) = Av$

- TAKE A BASIS OF v (EIGENSPACE) AND EXPAND IT TO V

- $P = [v_1 \dots v_n] \to A = PBP^{-1}$

- $B = \left(\begin{array}{c|c} \begin{matrix} \gamma & \dots & 0 \\ 0 & \dots & \gamma \end{matrix} & *1 \\ \hline 0 & *2 \end{array}\right) \to (zI - B) = \left(\begin{array}{c|c} (z-\gamma)I & *1 \\ \hline 0 & zI_{n-m} - *2 \end{array}\right)$

- $\text{DET}(zI - B) \underset{\text{(BLOCK MATRIX)}}{=} (z-\gamma)^m \, \text{DET}(zI_{n-m} - *2)$

- $\begin{pmatrix} A = PBP^{-1} \\ zI = P(zI)P^{-1} \end{pmatrix} \to \text{DET}(zI - A) = \text{DET}(P(zI)P^{-1}) = \text{DET}(P)\,\text{DET}(zI-B)\,\text{DET}(P^{-1})$
$\overset{!}{=} \text{DET}(zI - B)$ (A, B HAVE SAME POLYNOMIAL)

- $(z-\gamma)^m$ DIVIDES $\text{DET}(zI - A)$, $M_\gamma \geq m_\gamma$

- B, B^T HAVE THE SAME RANK

- LET $\{v^{(1)}, \dots, v^{(\ell)}\}$ BE A BASIS OF ROW SPACE $\leftarrow$ AKA. BASIS OF IMAGE(B^T)
- $\underset{\text{ROWS}}{B^{(i)}} = \sum_{k=1}^{\ell} c_{ik} v^{(k)}$ (EACH ROW IS A LINEAR COMBINATION OF THE BASES)

- YOU CAN WRITE $B \Rightarrow b_{ij} = \sum_{k=1}^{\ell} c_{ik} v_j^{(k)}$

- FREEZING j, EACH COEFFICIENT OF b_{ij} IS A LINEAR COMBINATION OF c_{ik} AND THE SAME $v_j^{(k)}$ } YOU EXTRACT THE VECTOR

- EACH COLUMN B IS IN THE SPAN OF
$\begin{pmatrix} c_{1,1} \\ \dots \\ c_{n,1} \end{pmatrix} \dots \begin{pmatrix} c_{1,\ell} \\ \dots \\ c_{n,\ell} \end{pmatrix}$

- THE LINEAR SPAN OF THE COLUMNS IS AT MOST $\text{RANK}(B^T)$
$\text{RANK}(B) \leq \text{RANK}(B^T)$

- OTHER ARGUMENT
$\text{RANK}(B) = \text{RANK}(B^T)$

A COMPLETE METRIC SPACE IS ONE WHERE EVERY CAUCHY SEQUENCE HAS A LIMIT IN THE SPACE

- A DIAGONALISABLE IFF $\forall i \; m_i = M_i$

$(\Rightarrow)$ - DIAGONALISABLE CAN FIND A BASIS OF EIGENVECTORS ST. $Av_j = \lambda_j v_j$ FOR λ_i ROOTS OF $p(z)$

- n_γ # INDICES SO $\gamma_j = \gamma$ (ALGEBRAIC MULTIPLICITY OF γ_j)
- $n_\gamma = m_\gamma \leq M_\gamma$ (m DIMENSION OF THE EIGENSPACE)

DIMENSION OF THE EIGENSPACE OVER THE REALS

- v_j FORM A BASIS SO $\sum_\gamma n_\gamma = n$
- $n = \sum_h n_\gamma = \sum_\gamma m_\gamma \leq \sum_\gamma M_\gamma \underset{=}{\leq} n$ (SUM OF MULTIPLICITIES IS AT MOST THE DEGREE)
- THIS MUST BE AN EQUALITY

$(\Leftarrow)$ - IF $\forall \gamma \; m_\gamma = M_\gamma$, FIND A BASIS OF THE EIGENSPACE $\text{KER}(A - \lambda I)$

- THERE ARE n VECTORS $\longrightarrow$ SHOW LINEARLY INDEPENDENT
- IF A COMBINATION IS DEPENDENT, $\sum_\gamma v_\gamma = 0$ (FOR SOME VECTORS) BUT THE SUM OF EIGENVECTORS FROM DIFFERENT γ CANNOT VANISH

THE SUM OF EIGENVECTORS FROM DISTINCT EIGENVALUES CANNOT VANISH

(THIS SIMPLY MEANS THAT EIGENSPACES ARE INDEPENDENT, NOT ORTHOGONAL)

- ASSUME A SUM CAN VANISH (CONTRADICTION)
- SELECT THE MINIMUM k, $k>1$ BECAUSE v_j CANNOT BE $= 0$
- $0 = \sum v_i = \sum A v_i = \sum \gamma_i v_i$
- SUBTRACT $\gamma_k (\sum v_i) = 0$ (THIS IS A NEW TERM THAT REMOVES THE LAST TERM $\gamma_k v_k$)
- THE LAST TERM IS REMOVED SO ONE CAN FIND A SMALLER SET
- THIS CONTRADICTS THE MINIMALITY

LIPSCHITZ PERTURBATION OF THE IDENTITY ARE CONTINUOUS

$f(x) = x + g(x)$, $\|g(x)-g(y)\| \le \alpha \cdot \|x-y\|$, $\alpha \in [0,1)$

FOR $U \subseteq \mathbb{R}^n$ OPEN, $f(U)$ IS OPEN

- U IS OPEN $\rightarrow \forall x \in U \ \exists r>0$ ST $B_r(x) \subseteq U$
- TO SHOW $f(U)$ IS OPEN, $B_{r'}(f(x)) \subseteq f(B_r(x))$ } A NEIGHBOURHOOD OF THE MAPPING CONTAINS A MAPPING OF THE NEIGHBOURHOOD
- PICK $r' = (1-\alpha) r$, ASSUMING WLOG $x_0 = 0$, $f(x_0) = 0$
- $\forall y \in B_{(1-\alpha)r}(x)$: $x + g(x) = y$ $\quad F(x) := y - g(x) \rightarrow$ FIND A FIXED POINT
- $\overline{B_s(0)}$ TAKEN AS THE DOMAIN OF F } AN OPEN BALL WOULDN'T HAVE BEEN COMPLETE
- APPLY BANACH FIXED POINT THM?

$F : \overline{B_s(0)} \rightarrow \overline{B_s(0)}$ } CHECK APPLICABILITY

$$|F(x)| = |y - g(x)| \le |y| + g(x) \le |y| + |g(x) - g(0)| \le |y| + \alpha |x - 0| \le |y| + \alpha x$$

- $y \in B_{(1-\alpha)r}(0)$, $|y| \le (1-\alpha) r$

PICK s: $\frac{|y|}{1-\alpha} < s < r$

$|y| \le s(1-\alpha) \Rightarrow |y| + \alpha s \le s$

HENCE $F(x) \in \overline{B_s(0)}$

- F IS A CONTRACTION: $\longrightarrow$ YOU CAN APPLY BANACH THM

$$|F(x) - F(x')| = |y - g(x) - y + g(x')| = |g(x) - g(x')| \le \alpha |x - x'|$$

$f(U)$ IS SHRUNK INTO THE BALL

BANACH FIXED POINT THEOREM

(X,d) COMPLETE, $F : X \rightarrow X$ CONTRACTION : $\exists \alpha \in [0,1)$

$d(F(x), F(y)) \le \alpha \cdot d(x,y) \quad \forall x,y \in X$

THERE IS A UNIQUE $x = F(x)$

EXISTENCE

- FOR $x_0 \in X$, (x_n) IS $x_{n+1} := F(x_n)$
- $d(x_{n+2}, x_{n+1}) = d(F(x_{n+1}), F(x_n)) \le \alpha \cdot d(x_{n+1}, x_n) \ \forall n$
- $d(x_{k+1}, x_k) \le \alpha^k d(x_1, x_0) = \alpha^k \cdot C$ (AT n DEPTH, SUCCESSIVE TERMS ARE $d(\dots) \le \alpha^n d(0,1)$)
- $\ell > n$ $\quad d(x_\ell, x_n) \le C \sum_{k=n}^{\ell-1} \alpha^k = C \cdot \left(\frac{\alpha^\ell - \alpha^n}{1-\alpha} \right)$
- AS $\ell, n \rightarrow \infty$, $d(x_\ell, x_n) \rightarrow 0$ (CAUCHY SEQUENCE)
- $x_\ell, x_n \longrightarrow x$
- SINCE F IS CONTINUOUS, $\lim F(x_k) = \lim x_k$

IF IN THE LINEARISATION ALL EIGENVALUES HAVE A NEGATIVE REAL PART a_0 IS AN ASYMPTOTICALLY STABLE EQUILIBRIUM POINT FOR $u' = f(u)$

PROOF (ASSUMING REAL EIGENVALUES)

- $\{v_i\}$ IS A BASIS OF THE JORDAN FORM OF A.
- $\forall \varepsilon > 0$, ONE CAN OBTAIN $\tilde{J}$ WITH ε AT THE SUPERDIAGONALS

USE $\{\varepsilon^i v_i\}$ AS A BASIS:

$\rightarrow A v_j = \gamma v_{j-1} + v_j$ (FOR γ EIGENVALUE)

$\rightarrow A(\varepsilon^j v_j) = \gamma \varepsilon^j v_j + \varepsilon (\varepsilon^{j-1} v_{j-1})$

$\rightarrow$ TAKE AS A BASIS

- IF p_0 IS EQUILIBRIUM : $f(p_0) = 0 \oplus f(p) = A(p - p_0) + o(|p - p_0|)$
- FOR $u(t)$ STARTING NEAR p_0 : $\tilde{u}(t) = P^{-1}(u(t) - p_0)$

$$\tilde{u}' = P^{-1} u' = P^{-1} A(u - p_0) + o(|u - p_0|) = P^{-1}(P \tilde{J} P^{-1})(u - p_0) + o(|\tilde{u}|) = \tilde{J} \tilde{u} + o(|\tilde{u}|)$$

- SET $h(t) = |\tilde{u}(t)|^2$, λ IS THE MAX EIGENVALUE OF A
- $h' = 2 \tilde{u}' \cdot \tilde{u} = 2 (\tilde{J} \tilde{u}) \cdot \tilde{u} + o(|\tilde{u}|^2)$
- $J_{(i,j)}$ WITH $i \ne j$ IS ε AT $j = i+1$ AND ZERO ELSEWHERE
- $(\tilde{J} v) \cdot v = \sum_i \tilde{J}_{ii} v_i^2 + \sum_{i \ne j} \tilde{J}_{ij} v_i v_j \le \sum_i (-\lambda) v_i^2 + \sum_{i<n} \varepsilon |v_i| |v_{i+1}| \le -\lambda |v|^2 + \varepsilon |v|^2$
- $h' \le -2\lambda |\tilde{u}|^2 + 2\varepsilon |\tilde{u}|^2 + \underbrace{o(|\tilde{u}|^2)}_{\text{BOUNDED BY } \frac{\lambda}{4}|\tilde{u}|^2 \text{ FOR } |\tilde{u}|<r,\ r>0 \text{ (AKA. SMALL ENOUGH)}}$
- $\varepsilon = \frac{\lambda}{4} \Rightarrow h'(t) \le -\lambda h$ (FOR $h < r$)
- FOR $u(0) \approx p_0$, $|\tilde{u}(0)| < r$ AND $|\tilde{u}| < r \ \forall t > 0$ ST $t \in I$ (DOMAIN OF u)
- $\tau = \sup \{ t > 0 : |\tilde{u}(s)| < r \ \forall s \in [0,t) \}$ (TIMEFRAME WHERE THE TRAJECTORY IS INSIDE r)

$\rightarrow \forall t \in [0,t) \Rightarrow |\tilde{u}(t)| < r$ AND $h' \le 0 \Rightarrow h(t) \le h(0)$

BY NONEMPTY CONTINUITY, FOR $|\tilde{u}(s)| < r$

- $\tau \ge \sup I$, OTHERWISE $s > t$ AND $|\tilde{u}(s)| \ge r$ WITH $|\tilde{u}(\tau)| \ge r$ FOR SOME s
- BY $h' \le -\lambda h$, $0 \le h(t) \le e^{-\lambda t} h(0)$ SO $h(t) \rightarrow 0$ AS $t \rightarrow \infty$