ILP

→NP-HARD (YOU CAN EMBED 3SAT IN ILP)

→WEAK DUALITY HOLDS BUT NOT STRONG

TU

A MATRIX IS TU IFF DET(ANY SUBMATRIX) = 0, 1, −1 $\iff$ $\forall b \in \mathbb{Z}^n$ $P = \{x \mid Ax \le b, x \ge 0\}$ IS INTEGER

TU PRESERVED BY: $\begin{pmatrix} A \\ I \end{pmatrix}, \begin{pmatrix} A \\ -I \\ I \end{pmatrix}, \begin{pmatrix} -I \\ A^T \\ A^T \end{pmatrix}$ → PROVE DIRECTLY BY EXPANDING USING LAPLACE FORMULA

P IS INTEGER POLYHEDRON $\iff$ ALL VERTICES ARE INTEGER

INCIDENCE MATRIX OF A GRAPH: $A \in \mathbb{R}^{|V| \times |E|}$

$$A_{v,e} = \begin{cases} 1 & v \in e \\ 0 & / \end{cases} \rightarrow A = \left[\overbrace{\begin{bmatrix} \\ \\ \end{bmatrix} \text{SUM OVER} = 2}^{|E|}\right]\Big\}|V|$$

BIPARTITE GRAPH = YOU CAN PARTITION VERTICES IN TWO SO NO VERTICES ARE ADJACENT

GRAPH IS BIPARTITE ⇒ TU INCIDENCE MATRIX

MATCHING: SELECT EDGES WITHOUT TOUCHING THE SAME VERTEX MORE THAN ONCE

- SIZE → # EDGES USED
- max/min → ACCOUNTS FOR EDGE WEIGHT
- PERFECT → TOUCHES ALL VERTICES (MAY NOT EXIST)

max-CARD. MATCHING: SELECT EDGES ST. EACH VERTEX IS TOUCHED ONCE

$$\max \sum x_e$$

↖ 0/1 FOR WHETHER YOU SELECT EDGE x_e

$$\forall v \in V \quad \sum_{e \in v} x_e \le 1$$

(SELECT ONE EDGE PER VERTEX)

$$\forall e \in E \begin{cases} x_e \ge 0 \\ x_e \in \mathbb{Z} \end{cases}$$

PERFECT MATCHING = COVERS ALL VERTICES (NOT ALWAYS POSSIBLE)

$\xleftrightarrow{\text{DUAL}}$

STRONG DUALITY HOLDS IN THIS FOR BIPARTITE GRAPHS

KÖNIG'S MATCHING THEOREM

MIN CARD VERTEX COVER: SELECT VERTICES ST. ANY EDGE IS TOUCHED AT LEAST ONCE

$$\min \sum y_v$$

$$\text{ST. } \forall u,v \in E \quad y_u + y_v \ge 1$$

(YOU SELECT AT LEAST/MOST ONE OF TWO)

$$\forall v \in V \quad y_v \ge 0$$

$$y_v \in \mathbb{N}$$

REPRESENT A SELECTION OF EDGES

$x^F \in \mathbb{R}^E, \quad F \subseteq E$ (PARTITION OF EDGES)

$$x^F_e = \begin{cases} 1 & \text{IF } e \in F \\ 0 & / \end{cases}$$

(THIS IS SOMETHING YOU CAN EVALUATE WITH THE INCIDENCE MATRIX)

→

IN BIPARTITE GRAPHS, MATCHING POLYTOPES HAVE A SHORT EXPRESSION

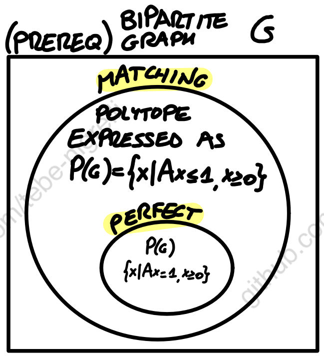
MATCHING POLYTOPE = CONV$\left\{ \chi^M \,\middle|\, M \text{ IS A MATCHING IN } G \right\}$

(G IS BIPART) $= \{x \in \mathbb{R}^E \mid Ax \le \underline{1}, x \ge 0\}$

↓

PERFECT MATCHING = CONV$\left\{ \chi^M \,\middle|\, M \text{ IS A P. MATCHING IN } G \right\}$

$= \{x \in \mathbb{R}^E \mid Ax = \underline{1}, x \ge 0\}$



(BIPARTITE GRAPH ASSUMPTION) ⊕ MAX INDEP. SET = THE LP RELAXATION HAS INTEGER OPTIMUM (AKA A^T IS TU MATRIX)

MAX CARD INDEP SET = LARGEST SET OF NON-ADJACENT VERTICES

$$\max \mathbb{I} y \leftarrow \mathbb{R}^V$$
$$\text{ST. } A^T y \leq \mathbb{I}$$
$$y \geq 0$$
$$y \in \{0,1\}$$

INDEP SET POLYTOPE $= \text{CONV}\{x^I \mid I \subseteq V$ IS INDEP. SET OF $G\}$

⊕

(BIPARTITE GRAPH ASSUMPTION)

=

$$\left\{ y \mid A^T y \leq 1, \; y \geq 0 \right\}$$

(WITHOUT BIPARTITE ASSUMPTION) → NO SIMPLE REPRESENTATION

$$P_{MATCH}(G) \subseteq \{x \in \mathbb{R}^E \mid Ax \leq \mathbb{I}, x \geq 0\}$$

$$P_{INDEP}(G) \subseteq \{x \in \mathbb{R}^V \mid A^T x \leq \mathbb{I}, x \geq 0\}$$

INCIDENCE MATRIX

DIRECTED GRAPH ⟹ TU MATRIX SO INTEGER POLYT.

$M \in M \in \mathbb{R}^{|V| \times |E|}$

$$M_{e,v} = \begin{cases} 1 & e \text{ LEAVES } v \\ -1 & e \text{ ENTERS } v \\ 0 & / \end{cases}$$

"FLOW CONFIGURATION"

$f \in \mathbb{R}^E$

PROPER NOTATION

$\delta(v) = \{e \mid e \text{ TOUCHES } v\}$

ADJACENT EDGES TO A CERTAIN NODE

CIRCULATION

$\forall v \in V$

$$\sum_{e_{IN}} f_e = \sum_{e_{OUT}} f_e$$

(THERE IS FLOW CONSERV.)

⇕

$Mf = \underline{0}$ (FOR EACH NODE)

≠

FLOW CONSERVATION HOLDS EXCEPT FOR s SOURCE t SINK

MAX FLOW PROBLEM

$$\max \mathbb{I}^T x$$
$M'x = \underline{0}$ (FLOW CONSERV.)
$x \leq c$ (CAPACITY CONSTRAINT)
$x \geq 0$

(MIN CUT)

$$\min k^T x$$
$\forall$ PATH $\sum x_e \leq 1$
$x_e \in \{0,1\}$

(M' BEING M EXCEPT 2 ROWS PRODUCING s,t)

↳ YOU DON'T CARE ABOUT FLOW CONSERVATION

INTEGRALITY GAP: (= 1 IF THE GRAPH IS BIPARTITE, ≤ 2 IN GENERAL)

$$\max c^T x$$
$$\text{ST. } Ax \leq b$$
$$x \in \mathbb{N}$$

→ x^I OPTIMAL ILP, x^* OPTIMAL OF RELAXATION → $\dfrac{c \cdot x^I}{c^T x^*}$ (YOU INVERT FRACTION IN MINIMISATION EXAMPLES)

MIN COST FLOW

$$\min k^T x$$
ST. $M'x = 0$
$x \leq c$ (CAPACITY CONSTRAINT)
$w^T x \geq \alpha$ (AT LEAST FLOW α)
$x \geq 0$

ROW THAT CORRESPONDS TO THE SOURCE

IF $c \in \mathbb{Z}_+^E$

⇓

INTEGER OPTIMAL SOLUTION EXISTS

APPROXIMATION RATIO (r):

$OPT(I)$ = BEST OBJECTIVE
$ALG(I)$ = SOLUTION

→ $\dfrac{OPT(I)}{ALG(I)} \leq r$ (YOU INVERT FRACTION IN MINIMISATION EXAMPLES)

MIN-COST NON BIPARTITE PERFECT MATCH

$$\min \sum_{e \in E} c_e x_e$$

$\forall v \in V \quad x(\delta(v)) = 1$ (SELECT ONE EDGE PER VERTEX)

$\forall U \subseteq V, |U| \geq 3 \quad x(\delta(U)) \geq 1$

$\forall e \in E \quad x_e \geq 0$

MATROID

$(X, \mathcal{I})$ — $\mathcal{I} \subseteq 2^X$ (SET OF SUBSETS OF X)

GIVES THE ELEMENTS

1. $\emptyset \in \mathcal{I}$
2. $Y \in \mathcal{I}, Z \subseteq Y \Rightarrow Z \in \mathcal{I}$ (CONTAINS ALL SUBSETS OF ITS ELEMENTS)
3. $Y, Z \in \mathcal{I}, |Y| < |Z| \Rightarrow \exists x \in Z \setminus Y :$ $Y \cup \{x\} \in \mathcal{I}$

SUBMODULAR FUNCTIONS

X FINITE SET

$f : 2^X \to \mathbb{R}$

$\forall A, B \subseteq X \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

- LINEAR → $f(A) = \sum_{i \in A} w(i)$ [ALSO $-f(A)$]
- BUDGET → $f(A) = \min\{\sum_{i \in A} w(i), B\}$ FOR $B \geq 0$
- CUT IN GRAPH → $c : 2^E \to \mathbb{R}_+$
  $f : 2^V \to \mathbb{R}_+$
  $f(S) = c(\delta(S))$

$G = (V, E)$ THEN $(E, \mathcal{I} \subseteq 2^E)$ IS A MATROID

$w : X \to \mathbb{R}$, FIND $Y \in \mathcal{I}$

ST. $w(Y) = \sum_{y \in Y} w(y)$ IS MAX

↓

PICK THE GREATEST Y

(IF GREEDY FINDS Y^*, $\forall w$ FUNCTION $\Rightarrow (X, \mathcal{I})$ IS MATROID)

BRANCH AND BOUND

$\max c^T x$

ST. $Ax \leq b$

$x \in \mathbb{Z}^n$

1. SOLVE RELAXATION
2. SELECT i ST. $x_i^* \notin \mathbb{Z}$
   CREATE TWO BRANCHES:
   $\begin{cases} x_i \leq \lfloor x_i^* \rfloor \\ x_i \geq \lfloor x_i^* \rfloor + 1 \end{cases}$ → A. INFEASIBLE
   B. INTEGER OPT.
   C. RELAXATION IS SUBOPT. WRT. INTEGER } STOP BRANCHES

(USE DFS HEURISTICS)

$Ax \leq b$ CAN BE TDI WITH b NON-INTEGER

||

RATIONAL INEQUALITY A, b → $Ax \leq b$ TDI SYSTEM

MEANING THE DUAL HAS INTEGRAL SOLUTIONS:

$\min\{b^T y \mid A^T y = c, y \geq 0\}$

$\forall c \in \mathbb{Z}^n$

A TU MATRIX

$\oplus$

b INTEGRAL

$\Downarrow$

$Ax \leq b$ TDI SYSTEM $\oplus$ A RATIONAL

||

POLYTOPE IS INTEGRAL

TDI ⇒ DUAL SOLUTION IMPLIES BOUNDED PRIMAL OPTIMUM

$\Downarrow$

POLYTOPE, NOT POLYHEDRON

YOU CAN WRITE POLYHEDRON AS TDI SYSTEM, A INTEGRAL $Ax \leq b$ TDI → AS A RESULT b INTEGRAL $\Updownarrow$ P INTEGRAL

SPECTRAL NORM

$\|A\| = \max \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$

- GRADIENT DESCENT → $x_{t+1} = x_t - \gamma \cdot \nabla f(x_t)$
- PROJECTED → $y_{t+1} = x_t - \gamma \nabla f(x_t)$
  $x_{t+1} = \Pi_X(y_{t+1}) = \underset{x \in X}{\operatorname{argmin}} \|x - y_{t+1}\|^2$
- SUBGRADIENT → $(g_t \in \partial f(x_t))$ $x_{t+1} = x_t - \gamma_t g_t$

YOU CAN ADD REDUNDANT CONSTRAINTS TO LP FORMULATIONS TO REDUCE THE SIZE OF THE POLYHEDRON

P POLYHEDRON REPRESENTED WITH $\{x \mid Ax \leq b\}$

TDI: A INTEGRAL
b RATIONAL

$P' = \{Ax \leq \lfloor b \rfloor\}$

AND $P' \subseteq P$

ONLINE OPTIMISATION

- ROUNDS:
  → PICK x_t ACTION
  → LOSS $f_t(x_t)$
- MINIMISE REGRET
  $R_T = \sum_{t=1}^{T} (f_t(x_t) - f(x^*))$

STOCHASTIC	STOCHASTIC SUBGRADIENT	STOCHASTIC MINIBATCH
$i \sim \text{UNIF}(1..n)$	$g_t \in \partial f_i(x_t)$	$i_1 \dots i_m \sim \text{UNIF}(1..n)$
$f(x) = \frac{1}{n} \sum^{n} f_i(x)$	$x_{t+1} = x_t - \gamma_t g_t$	$\tilde{g}_t = E_m(g_t^j)$
$x_{t+1} = x_t - \gamma_t \nabla f_i(x_t)$		$x_{t+1} = x_t - \gamma \tilde{g}_t$

TSP

$\min c^T x$

$0 \leq x_e \leq 1 \quad \forall e \in \binom{V}{2}$

$x_e(\delta(v)) = 2 \quad \forall v \in V$

$\sum_{e \in \binom{X}{2}} x_e \leq |X| - 1$ ← FOR ALL NONEMPTY SMALLER SUBSETS OF VERTICES

ONLINE GD

$y_{t+1} = x_t - \gamma \ell_t$ — TELLS HOW TO CHANGE EACH EXPERT'S ACTIONS USING LOSS

$x_{t+1} = \Pi_X(y_{t+1})$

$R_T \leq RB\sqrt{T}$

FOLLOW THE LEADER (FTL)

$i_t = \underset{i}{\operatorname{argmin}} \left\{ \sum_{j=1}^{t-1} \ell_j \cdot e_i \right\}$

(KEEP TRACK RECORD FOR EACH, PICK THE BEST)

REGULARISED / USES PROB. DISTRIBUTION

$x_0 = \underset{x \in X}{\operatorname{argmin}} \phi(x)$

$i_t = \underset{i}{\operatorname{argmin}} \left\{ \sum_{j=1}^{t-1} \ell_j \cdot x + \frac{1}{\gamma} \phi(x) \right\}$

ϕ CONVEX | x AS ACTION MEANS TAKING A DISTRIBUTION OVER THE CHOICES

IF ℓ_2 REGULARISED, EQUIVALENT TO:

$y_{t+1} = y_t - \gamma \nabla f_t(x_t)$

$x_{t+1} = \Pi_X(y_{t+1})$

THIS IS LAZY BECAUSE y IS UPDATED FROM y, IT'S LIKE x_t FOLLOWS y_t FROM INSIDE $X \subset \mathbb{R}^d$, LIKE A DOG INSIDE AN ENCLOSURE, TRYING TO GET AS CLOSE AS POSSIBLE TO THE MAILMAN

DUAL NORM → $\|x_*\| = \max\{z^T x \mid \|z\| \leq 1\}$

OF NORM $\ell_p \leftrightarrow \ell_{\frac{p}{p-1}}$

ENTROPY REGULARISATION

$\phi(x) = \sum x_i \log(x_i)$

1-STRONG CONVEX WITH $\|\cdot\|_1$ ON Δ_n = PROBABILITY SIMPLEX

$R_T \leq 2RB\sqrt{2T}$ →

HEDGE

$w_1 = \mathbb{1}$

$w_{t+1}(j) = w_t(j) \cdot e^{-\gamma \ell_t(j)}$ (UPDATE WEIGHTS)

$x_{t+1} = \frac{w_{t+1}}{\|w_t\|_1}$ (NEW $\mathbb{P}$ FOR x PICKED)