CONVEX: $\to \emptyset, \mathbb{R}^n, \{p\}$ ARE CONVEX
INTERIOR, CLOSURE ARE CONVEX

NOTE THAT CONVEXITY ONLY MAKES SENSE AS A SUBSET OF $\mathbb{R}^n$
INTERIOR WORKS DIFFERENTLY DEPENDING ON THE ENCLOSING SET

$\partial C = C \setminus \mathring{C}$, $\dim \le \dim(C) - 1$ (BOUNDARY HAS AT LEAST ONE DIMENSION LESS THAN THE SET)
SET OF COMMON LIMITS OF SEQUENCES BETWEEN C, $\mathbb{R}^n \setminus C$

$co(\text{OPEN}) = \text{OPEN}$
$co(\text{COMPACT}) = \text{COMPACT}$
$\overline{\text{BOUNDED}} = \text{COMPACT}$

$\begin{pmatrix} C \subseteq V \text{ CONVEX} \\ p \in C \end{pmatrix}$ p IS AN EXTREME $\iff$ CANNOT BE WRITTEN AS CONVEX COMBINATION OF TWO DISTINCT POINTS $\iff$ $p \notin co(C \setminus \{p\})$

$[a,b]$ IS CONVEX INTERVAL BETWEEN a, b VECTORS

CARATHEODORY THEOREM: $S \subseteq \mathbb{R}^n$

$co(S) = \{c \mid \exists x_1 \ldots x_{n+1} \in S \text{ st. } c = \sum_{i=1}^{\le n+1} \lambda_i x_i \text{ FOR } \lambda_i \ge 0, \sum \lambda_i = 1\}$ — AT MOST (EXTREMAL), CONVEX COMBIN.

TO EXPRESS A POINT IN $co(S)$ YOU CAN FIND, AS NEEDED, AT MOST $n+1$ POINTS IN S ST $p = \sum \lambda_i x_i$

CONVEXITY IS INDEP. OF BASIS OF V.S

C NONEMPTY CLOSED CONVEX $\iff$ $\forall p \in \partial C$ $\exists$ SUPPORTING HYPERPLANE PASSING THROUGH p

JENSEN: f CONVEX
$\|\alpha\| = 1 \,/\, \alpha \in S^{n-1}$
$f(\alpha \cdot x) \le \alpha \cdot f(x)$

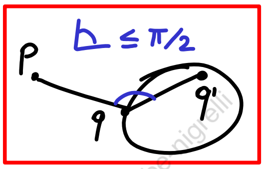
$S \ne \emptyset$, CLOSED $\Rightarrow$ NEAREST POINT PROJECTION EXISTS
$\oplus$ CONVEX $\Rightarrow$ POINT IS UNIQUE

$\mathbb{R} \times \{0\}$ HAS NO EXTREMES (NOT BOUNDED)

MINKOWSKI (EFFICIENT DERIVATION OF CONVEX SET): C CONVEX, COMPACT IFF $C = co(\text{EXTREMES})$

$f: C \to \mathbb{R}$ CONVEX $\iff$ $\forall a, b \in C, \lambda \in [0,1]$
$f(\alpha a + (1-\alpha) b) \le \alpha f(a) + (1-\alpha) f(b)$
IF $f \in C^0$ IT'S ENOUGH TO CHECK $\alpha = 1/2$
$\iff$ IF $f \in C^1$, f' INCREASING $(\nabla f(a) - \nabla f(b)) \cdot (a-b) \ge 0$
OR $f(x) \ge f(x_0) + f'(x_0)(x - x_0)$ } ABOVE THE TANGENT
$\iff$ IF $f \in C^2$, $f'' \ge 0 \,/\, \nabla^2 f(x) \ge 0$

$x = p(y) \implies \forall x' \in C$ PROJECTION $(x-y) \cdot (x'-x) \ge 0$
(THE ANGLE CAN BE AT MOST $\pi/2$)



S CONVEX, $p(y)$ IS LIPSCHITZ $\to$ AN UPPER BOUND CONSTANT LIMITS RATE OF CHANGE OF NEAREST POINT

CLOSED IFF $\forall p$, S HAS $\pi(p)$

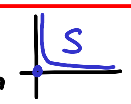
S CLOSED, CONVEX $\iff$ S INTERSECTION OF TANGENT HALF SPACES

f CONVEX $\Rightarrow$ DOMAIN RESTRICTION IS STILL CONVEX
$\Rightarrow f(x) \ge f(x_0) + \nabla f(x_0) \cdot (\underline{x} - x_0)$

BUT $co(\text{COMPACT}) = \text{COMPACT}$

THE SMALLEST CLOSED SET CONTAINING THE CONVEX HULL IS THE CLOSURE OF co

$\overline{\text{CONVEX}} = \text{CONVEX}$
$\mathring{\text{CONVEX}} = \text{CONVEX}$
$co(\text{CLOSED}) \ne \text{CLOSED}$



CONVEXITY IN HIGHER DIMENSIONS: $\nabla^2 f$ POS. SEMIDEF: $v^T \nabla f^2 v \ge 0$ (SYMMETRIC $n \times n$)

SEE INTERVAL OF C ST. $\frac{|f(x) - f(x_0)|}{|x - x_0|} \ge c$

$\partial f(x_0) := \{\underline{w} \mid f(x) \ge f(x_0) + \underline{w} \cdot (x - x_0) \;\; \forall x \in C\}$ $\leftarrow$ SET OF SLOPES ST. AFFINE f IS BELOW THE CONVEX FUNCTION

EPIGRAPH: $epi(f) = \left\{ \binom{x}{t} \in V \times \mathbb{R} : x \in C, t \ge f(x) \right\}$

$f(a) = \infty \implies \partial f(a) = \emptyset$ (BY CONVENTION)

f CONVEX $\iff$ EPIGRAPH CONVEX

$f: C \to \mathbb{R}$ CONVEX, $x_0 \in C$
$\to w \in \partial f(x_0)$
$\to \exists r > 0 : f(x) \ge f(x_0) + w \cdot (x - x_0) \;\; \forall x \in C \cap B_r(x_0)$
$\to f(x) \ge f(x_0) + w \cdot (x - x_0) + o(|x - x_0|)$

COMPOSITION WITH INCREASING, CONVEX f

CONVEXITY IS PRESERVED:
- COMPOSITION WITH CVX, STRICTLY INCREASING
- CONICAL COMBINATIONS
- MAX OF CONVEX

$e^{\text{CONVEX}} = \text{CONVEX}$

f CONVEX $\Rightarrow$ LOCAL MINIMA $=$ GLOBAL

LOCAL CHARACT. OF CONVEX

f CONVEX GLOBALLY $\iff$ $\exists$ OPEN BALL NEAR EVERY POINT ST. $C \cap B_r(x)$ IS CONVEX

C CONVEX, NONEMPTY $\subseteq \mathbb{R}^n$
$f: C \to \mathbb{R}$ CONVEX
f IS LOCALLY LIPSCHITZ IN INTERIOR OF C

$\partial f(x_0)$ OF f CONVEX | x_0 MIN $\Rightarrow \partial f(x_0) = 0$
$\to$ CLOSED, CONVEX (CAN BE $\emptyset$)
$\to x_0 \in \mathring{C} \Rightarrow \partial f(x_0)$ NONEMPTY, COMPACT
$\partial f(x_0)$ SINGLETON $\iff$ f IS C^1 AT x_0
$\partial f(x_0) = \{\nabla f(x_0)\}$

f UNDEFINED, $f(x_0) = \infty$
$\Downarrow$
$\partial f(x_0) = \emptyset$

f CONVEX IFF ABOVE ALL OF ITS SUBTANGENTS

$\begin{pmatrix} C \text{ CLOSED} \\ \text{CONVEX} \end{pmatrix} \iff$ CONNECTED, LOCALLY CONVEX

$A \pm B$ IS *
$\bigcup_{a \in A} a \pm B = \bigcup_{b \in B} b \pm A$
USEFUL FOR CLOSURE / CONVEXITY CHECKS

f CONVEX, $x_0 \in \mathring{C}$, $S \subseteq \mathring{C}$ (POINTS ST. f DIFF.)
$(x_i)_i$ ON S : $x_i \to x_0$, $\nabla f(x_i) \to v$
$\partial f(x_0) = co\{\text{LIMIT VECTORS}\}$ * — USUALLY LEFT AND RIGHT LIM

- $\partial(\alpha f) = \alpha \partial f(x) \mid \alpha > 0$
- $\partial(\sum f_i) = \sum \partial f_i$ (MINKOWSKI SUM)
- $\min\{f_i\}$, $\max\{f_i\}$ $\to$ SELECTING "ACTIVE" FUNCTIONS $\to co(\bigcup \partial f_i)$ *

$\max\{f \text{ convex}\} = \text{convex}$
BUT $\min\{f\}$ IS ON A CASE BY CASE BASIS
$\downarrow$
CHECK THE CONVEXITY OF $\min\{f_i\}$

LOWER SEMICONTINUOUS $f:\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ $f(x) \le \liminf_{i\to\infty} f(x^i)$ $\iff$ $\mathrm{EPI}(f) := \{(x,t)\in\mathbb{R}^n\times\mathbb{R} : t \ge f(x)\}$ IS CLOSED IN $\mathbb{R}^n\times\mathbb{R}$ ($t=\infty$ DISCARDED)

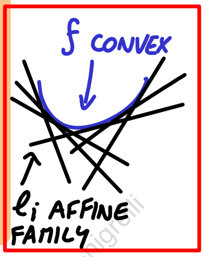
YOU NEED LOW. SEMICONT. FOR LEGENDRE TRANSFORM

L.SEMIC. CAN HAVE JUMPS, BUT THE LOWER EXTREME IS DEFINED

⊕ CONVEX $\iff$ CAN BE WRITTEN AS SUP OF A FAMILY OF AFFINE FUNCTIONS $\{\ell_i : \mathbb{R}^n \to \mathbb{R}\}$ $f(x) = \sup_{i\in I}[\ell_i(x)]$

CONVEX ON OPEN SET $\subseteq$ LOWER SEMICONTINUOUS

NOT CONVEX $\Downarrow$ NOT LOWER SEMICONT.



$f^*:\mathbb{R}^n \to [-\infty,\infty]$ REQUIRES LOW. SEMICONT

$f^*(\xi) = \sup_{x\in\mathbb{R}^n}[\xi\cdot x - f(x)]$

f^* OF CONVEX IS CONVEX

$f^{**}(x) = f(x)$

$f(x) + f^*(\xi) \ge x\cdot\xi$ (YOUNG'S INEQUALITY)

YOU CAN CHECK $g = f^*$ IF $g^* = f$

NOTIONS OF TOPOLOGY

S COMPACT $\Rightarrow$ ALL SEQUENCES IN S HAVE CONVERGING SUBSEQUENCES IN S

- S OPEN $\to$ $\forall p\in S, \exists r$ ST. $B_r(p)\subseteq S$
- S CLOSED $\to$ $\mathbb{R}^n\setminus S$ IS OPEN $\to$ FOR $(p^{(i)})_{i\in\mathbb{N}}$ CONVERGING, $\lim_{i\to\infty} p^{(i)} \in S$

$\emptyset$, $\mathbb{R}^n$ ARE OPEN AND CLOSED

- INTERIOR $\to$ LARGEST OPEN SUBSET $(\mathring{S})$
- CLOSURE $\to$ SMALLEST CLOSED SUPERSET $(\bar{S})$

BOUNDARY: $\partial S := \bar{S}\setminus\mathring{S}$ (NOTE: CLOSURE) / SET OF LIMIT POINTS OF A SET AND ITS COMPLEMENT

COMPACT SET: $S\subseteq\mathbb{R}^n$ IF $S\subseteq\bigcup_{i\in I} u_i$ $\Downarrow$ $\exists$ FINITE # OPEN SETS ST. $S\subseteq\bigcup_j u_j$

*

OPEN $\to$ UNION IS ALWAYS OPEN | (FINITE) INTERSECTION IS OPEN

CLOSED $\to$ INTERSECTION IS ALWAYS CLOSED | (FINITE) UNION IS OPEN

IN A VECTOR SPACE, DEFINING A BASIS CREATES A CORRESPONDENCE WITH $\mathbb{R}^n$ $\Downarrow$ (DEFINE OPEN/CLOSED BASE INDEPENDENT, BECAUSE A CHANGE IN BASIS IS AN INVERTIBLE CONTINUOUS f)

f CONVEX MAPS CONVEX SETS TO CONVEX SETS
COROLLARY: AFFINE

OTHER CASES VARY

$(\mathbb{R}^n)$ COMPACT $\iff$ CLOSED ⊕ BOUNDED

IN GENERAL, COMPACT $\iff$ ALL SEQUENCES HAVE A CONVERGING SUBSEQUENCE WITH LIMIT IN S
$\Downarrow$
CLOSED, BOUNDED

f CONTINUOUS
$\Downarrow$
f^{-1}(OPEN) IS OPEN
f^{-1}(CLOSED) IS CLOSED
f(COMPACT) = COMPACT

CHECK FOR EXISTENCE OF SOLUTIONS BEFORE APPLYING LEGENDRE MULT

WEIERSTRASS

$f: K\to\mathbb{R}$, $f\in C^0$, K COMPACT
HAS MAX, MIN

AFFINE SPAN:
SMALLEST AFFINE SUBSPACE CONTAINING S.

(SUPERSET OF CONVEX HULL): $\{\sum \alpha_i \cdot a_i \mid (a_i)_i \subseteq S, \sum \alpha_i = 1\}$ $\to$ DIMENSION OF AFF. SPAN: # OF LIN. INDEP. DIFFERENCE VECTORS IN SET S

f CONVEXITY IS STABLE UNDER DOMAIN RESTRICTION

HAHN-BANACH

(CONVEX $S\subseteq V$, $p\in V$) $p\notin \overline{co}(S\setminus\{p\})$ IFF $\exists$ CLOSED HALF-SPACE ST. $C\subseteq H$, $p\notin H$

STRONG SEPARATION BETWEEN SETS

STRONG

S_1, S_2 NONEMPTY, DISJOINT, CONVEX, CLOSED. x2 BOUNDED x1 $\Rightarrow$ $\exists v\in\mathbb{R}^n\setminus\{0\}$ $\exists \beta<\beta'\in\mathbb{R}$ ST. $\forall a\in S_1$ $\forall b\in S_2$ $v\cdot a \le \beta < \beta' \le v\cdot\beta$

WEAK

C_1, C_2 NONEMPTY, DISJOINT, CONVEX $\Rightarrow$ $\exists v\in\mathbb{R}^n\setminus\{0\}$ $\exists\beta\in\mathbb{R}$ ST. $\forall a\in C_1$ $\forall b\in C_2$ (SINGLE NUMBER) $v\cdot a \le \beta \le v\cdot\beta$

KKT CONDITIONS

CONVEX(CONVEX INCREASING) = CONVEX

$\sup\{f_i \text{ CONVEX}\}$ = CONVEX

$f:\mathbb{R}^n\to\mathbb{R}^n$ (CONVEX) $x_0\in\mathbb{R}^n$ IS GLOBAL $\iff$ $0\in\partial f(x_0)$

REMEMBER TO CHECK THE MAXIMAL RANK CONDITION:
FOR $h(x,y,z)=0$, CHECK $\nabla_x h \perp\!\!\!\perp \nabla_y h \perp\!\!\!\perp \nabla_z h$ $\forall(x,y,z): h(x,y,z)=0$

$g_{1\ldots k}:\mathbb{R}^n\to\mathbb{R}$ (CONVEX)
$h_{1\ldots \ell}:\mathbb{R}^n\to\mathbb{R}$ (AFFINE)
⊕ SLATER / NONTRIVIALITY: $\exists x: \forall i\ g_i(x)<0$ } CHECK CONDITION

$\min f(x)$ (CONVEX) ST. $\forall_i\ g_i(x)\le 0$, $\forall_j\ h_j(x)=0$

x_0 IS OPTIMAL $\iff$ x_0 SATISFIES CONDITIONS, $\exists\lambda_{1\ldots k}\ge 0$, $\exists\mu_{1\ldots\ell}$ ST. CONICAL COMBINATION

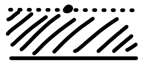
$0\in\partial f(x_0) + \underline{\lambda}\cdot\partial \underline{g}(x_0) + \underline{\mu}\cdot\partial\underline{h}(x_0)$

AND $\forall i$ $\lambda_i = 0$ OR $\partial g_i(x_0) = 0$ (DUALITY)

WITH C^1 ASSUMPTION ∂f BECOMES $\{\nabla f\}$

TYPES OF EXERCISES

CONVEX HULL:

$co(\{\_\_\cdot\_\_\}) \to$ 

(EMPTY SET OF EXTREMES)

C CONVEX CLOSED $\iff$ $C = \cap \partial H$ HALF SPACES ON ∂C

$\updownarrow$

f CONVEX $\iff$ $f = \sup\{\ell_i\}$ AFFINE

$C \subseteq \mathbb{R}^n$ CONVEX

$\to \forall a, b \in C,\ [a,b] \subseteq C$

$\to A \pm B$ OF CONVEX SETS IS CONVEX

f CONVEX $\iff$ $epi(f)$ CONVEX

SHOW f IS CONVEX:

$\to$ EPIGRAPH IS CONVEX

$\to$ CONVEXITY-PRESERVING TRANSFORMATIONS

$\to$ WRITE THE FAMILY OF AFFINE FUNCTIONS ST. f IS $\sup\{\ell_i\}$: $\left\{ \overbrace{f(x_0) + w\cdot(x-x_0)}^{\text{AFFINE}} \;\middle|\; \begin{array}{l} x_0 \in dom(f) \\ w \in \partial f(x_0) \end{array} \right\}$

$\to$ LIMIT CHARACTERISATION:

$\forall x, y,\ \lambda_k \mapsto 0$

$$f(x) \le \liminf_{i \to \infty} \left[\lambda_k f(x) + (1-\lambda_k) f(y) \right] \Big| \lambda \in [0,1]$$

$\to (\nabla f(x) - \nabla f(y)) \cdot (x-y) \ge 0$

$\to \forall a, b, c$

$$\frac{f(b)-f(a)}{b-a} \le \frac{f(c)-f(b)}{c-b}$$

COMPUTING f^*:

$$f^*(\xi) = \sup_{x \in dom(f)} [\xi \cdot x - f(x)]$$

1. $g(\xi, x) = \xi x - f(x)$
2. SOLVE $g_x = 0$,
3. SOLVE FOR ξ SUCH THAT $\forall x \in dom(f)$ $g_{xx} < 0$
4. FOR EVERY OTHER INTERVAL, FIND:
   $\max g(\xi)$ SUB TO $x \in dom(f)$ $\Big\}$ EG. IF $g_x > 0$, PICK x_{max}