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MATHEMATICS

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TENSORS

- MULTIPLE PARTICLES
- INTERACTING SPINS
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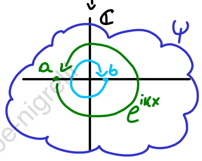
$|a+bi| = \sqrt{a^2+b^2}$

GENERIC SOLUTION OF QM

$\to \Psi = a\cdot\cos(qx) + b\cdot\sin(qx)$, $a,b\in\mathbb{C}$, $q\in\mathbb{R}$ } LOCALISED PARTICLE (0 AT ENDS)

$\to \Psi = a e^{qx} + b e^{-qx}$, $a,b\in\mathbb{C}$, $q\in\mathbb{R}$ } TO IMPOSE $\lim_{x\to\pm\infty}\Psi(x)=0$

$\to \Psi = a e^{ikx} + b e^{-ikx}$, $a,b\in\mathbb{R}$, $k\in\mathbb{C}$ } BOUNDED NEAR a,b



$Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$ = "PLANE WAVE"

- $\int|\Psi|^2dx$ NON-INTEGRABLE ⇒ MUST BE BOUNDED
- SOLUTION OF $V(x)=0$
- $E = \frac{\hbar^2 k^2}{2m} = \frac{(\hbar k)^2}{2m} = \frac{p^2}{2m}$

WAVES

$A(x,t) = A_0\cos(kx-\omega t)$

$\begin{cases}\lambda = \frac{2\pi}{k} & \text{(PEAK TO PEAK } \Delta x \text{ DISTANCE)}\\ T = \frac{2\pi}{\omega} & \text{(PEAK TO PEAK } \Delta t \text{ DISTANCE)}\end{cases}$

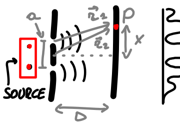
$\boxed{v = \omega/k = \lambda/T}$

$(\mathbb{R}^n)$

$A(\vec{x},t) = A_0\cos(\vec{k}\cdot\vec{r}-\omega t)$

$\vec{k}$ = PROPAGATION DIRECTION $\quad \lambda = \frac{2\pi}{|\vec{k}|}$

DOUBLE SLIT



LIGHT ACTS THIS WAY AT SCALES SMALL ENOUGH

BUT ALSO ELECTRONS, THOUGH THEY MUST PASS THROUGH A SINGLE HOLE SINCE CLOSING A SLIT CAUSES INTENSITY TO HALVE

YOU CAN USE COMPLEX NUMBERS TO EXPRESS THE 2D WAVE

$\begin{cases}A_0 e^{i(\vec{k}_1\cdot\vec{r}-\omega t)}\\ A_0 e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}\end{cases}$

$A_{sum} = A_0 e^{-i\omega t}\left(e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2}\right)$

$A^2_{sum} = |A_0(1+e^{i\vec{k}(\vec{r}_2-\vec{r}_1)})e^{i\vec{k}\vec{r}_1}|^2$ (NORM = 1)

$= A_0^2|1+e^{i\delta}|^2$

$= A_0^2|1+\cos(\delta)+i\sin(\delta)|^2$

$= 2A_0(1+\cos(\delta))$

CHANGE VARIABLES $\left(\frac{D}{a} >> 1\right)$*

$S_1 = \binom{0}{a/2}$

$S_2 = \binom{0}{-a/2}$

$P = \binom{D}{x}$

$\delta := \|P-S_1\| - \|P-S_2\|k$*

$\approx x\frac{a}{D}k$

$\delta = kx\frac{a}{D} \to \boxed{x = \frac{\lambda D}{a}}$

DE BROGLIE

$\boxed{\lambda = \frac{h}{m|\vec{v}|}}$ | PLANCK CONSTANT $h \cong 6.6\cdot10^{-34}\ Js$ → CONVERTS PARTICLES TO WAVE BEHAVIOUR

λ IS OF ORDER 10^0 Å (ARMSTRONG)

LARGE OBJECTS HAVE SMALL λ → NO WAVE BEHAVIOUR

$E = \sqrt{m^2c^4+p^2c} \approx mc^2 + \frac{p^2}{2m}$ } TAYLOR EXPAND $|p|<<1$

$\hbar = \frac{h}{2\pi}$ = REDUCED PLANCK CONSTANT

AS $\hbar\to0$, QUANTUM SYSTEM REVERTS TO CLASSICAL BEHAVIOUR

PHOTONS - PHOTOELECTRIC EFFECT *

THE MATH WORKS IF LIGHT IS DISCRETE, IN PACKETS:

$Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}$ ASSOCIATE TO AN EM WAVE A PARTICLE

$\begin{cases}\vec{P} = \hbar\vec{k}\\ E = h\nu = \hbar\omega \leftarrow \boxed{\hbar = \frac{h}{2\pi}}\end{cases}$

IN THE NIGHT SKY, THE INTENSITY OF A STAR

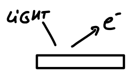
$I = 1.6\cdot10^{-10}\ Wm^{-2}$

SEEN BY EYE $(10^{-3}m = r) \to A = \pi r^2$

IA = POWER → $P\cdot1s = n\cdot\varepsilon$

$n \approx 3$ PHOTONS

PHOTOELECTRIC EFFECT *



- INTENSITY ⊥ e^- ENERGY
- e^- ENERGY DEPENDS ON f (LIGHT)

THE ENERGY OF e^- IS MEASURED BY STOPPING IT WITH A FIELD:

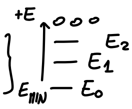
$\boxed{\varepsilon_{PHOTON} = h\nu} \to K = \varepsilon_{PH} - E_0 \quad \frac{E_0}{h}$

$= h(\nu-\nu_0)$

LIGHT COMES IN A SPECTRUM, WHILE e^- ARE DISCONTIN. OUT (IN BANDS)

$V = \frac{h}{e}(\nu-\nu_0)$ } VOLT

ENERGY IS ABSORBED IN $\Delta E = E_n - E_p = h\nu_{n\to p}$ (DISCRETE)



SCHRÖDINGER EQUATION

$\Psi: \mathbb{R}^n\times\mathbb{R}\to\mathbb{C}$ (SPACE, TIME) } DESCRIBES A SINGLE PARTICLE

$\rho(\vec{x},t) = |\Psi(\vec{x},t)|^2$

$= \mathbb{P}$(PARTICLE IN $\vec{x}$ AT t)

$\{\Psi\}$ ST. $\forall t \int|\Psi|^2 d\vec{x} = 1$

IS A VECTOR SPACE ON $\mathbb{C}$, BUT YOU HAVE TO NORMALISE TO ENSURE ADDITION KEEPS $|\Psi|=1$

$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m}\Delta\Psi + V(\vec{r})\Psi(\vec{r},t)$

LAPLACIAN: $\Delta\Psi = \sum_{m=1}^{d}\frac{\partial^2\Psi}{\partial x_m^2}$

$= \left[-\frac{\hbar}{2m}\Delta + V(\vec{r})\right]\Psi$

$\hat{H}$ (HAMILTONIAN) LINEAR OPERATOR

NOTE $\hat{H}\Psi$ = ANOTHER Ψ

$V(\vec{r}) = 0$, 1d

$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi$

SOLUTION GIVES $P(\vec{r}) = c^2 = \frac{1}{V}$,

WHERE VOLUME OF SPACE $= V$

NOTE, YOU MUST NORMALISE AS $f(t)$

$\Psi(\vec{r},t) = ce^{i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar}-\omega t\right)}$ } $\frac{\partial\Psi}{\partial x} = \frac{i}{\hbar}P_x\Psi$

$\vec{P}$ DIRECTION

$\begin{cases}i\hbar\frac{\partial\Psi}{\partial t} = i\hbar(-i\omega)\Psi = \hbar\omega\Psi\\ -\frac{\hbar^2}{2m}\Delta\Psi = \left(-\frac{\hbar}{2m}\right)\left(-\frac{1}{\hbar}\right)(P_x^2+P_y^2+P_z^2)\Psi = \frac{P^2}{2m}\Psi\end{cases}$

HENCE $\frac{\vec{P}^2}{2m}\Psi = \hbar\omega\Psi \to \boxed{\omega = \frac{\vec{P}^2}{2m\hbar}}$

$\mathcal{F}\{WAVE\} = ?$

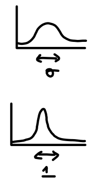
$(t=0)$

$\Psi(\vec{r},0) = \int\frac{d^3p}{(2\pi\hbar)^{3/2}}\varphi(\vec{p})e^{i\frac{(\vec{p}\cdot\vec{r})}{\hbar}}$

NOTE $f(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$

$\mathcal{F}^{-1}\uparrow\downarrow\mathcal{F}$

$\hat{f}(k) = \frac{e^{-\frac{k^2}{(\frac{2}{\sigma^2})}}}{\sqrt{2\pi}}$



KNOWING $x \equiv$ CONST, $\mathbb{P}(\vec{x}) = \delta(x)$ (DIRAC) } → $\delta(x)$ IS THE GAUSSIAN $\lim_{\sigma\to0}$

$\mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}$

PLANCHEREL THM: $f \xrightarrow{\mathcal{F}} \hat{f}$ IMPLIES $\int|f|^2dx = \int|\hat{f}|^2dk$

HENCE $\int|\Psi|^2dx = 1$ MEANS $\int|\Psi|^2dp = 1$, $\mathbb{P}(\text{MOMENTUM}) = |\varphi(p)|^2$

REMEMBER THERE IS AN EQUIVALENCE CLASS OF Ψ SOLUTIONS: $\Psi e^{i\theta}\sim\Psi$ (UP TO PHASE CHANGE)

GAUSSIAN WAVE PACKET (1D)

SUPERPOS. OF ∞ # PARTICLES

$\Psi(x,t=0) = c\int dk\exp\left(ikx - \frac{a^2}{4}(k-k_0)^2\right)$

$= \frac{c\sqrt{4\pi}}{a}\exp\left(ik_0x - \frac{x^2}{a^2}\right)$

$\boxed{\int e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}}}$

$|\Psi|$ HAS $\sigma = a\sqrt{2}$

ADD TIME EVOLUTION $(e^{-\omega t})$

$\Psi(x,t=0) = c\int dk\exp\left(ikx - \frac{a^2}{4}(k-k_0)^2 - i\frac{\hbar k^2}{2m\hbar}t\right)$

} MOVES AND SPREADS

$v = \frac{P}{m}$

GENERAL PROCEDURE TO SOLVE PROBLEMS

NONEMPTY ENVIRONMENT

$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi}$ ($V(x)$ CONST. IN TIME)

→ LOOK FOR EIGENSTATES Ψ_a

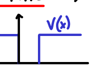
ST. $\hat{H}\Psi_a = E_a\Psi_a$ (E_a IS THE ENERGY OF THE STATE)

→ $\boxed{\Psi = \sum_a c_a\Psi_a}$ IS THE SPACE OF SOLUTIONS TO THE EQUATION (BY $\frac{\partial}{\partial t}$, $\hat{H}$ LINEARITY) (c_a NORMALISED)

→ $\boxed{\Psi(x,t) = \sum_a e^{-i\frac{E_at}{\hbar}}\Psi_a}$ } MORE E, HIGHER PHASE

INFINITE WELL POTENTIAL (1D)

$V(x) = \begin{cases}\infty & x<0\\ 0 & 0<x<L\\ \infty & L<x\end{cases}$



$(\Psi(0) = \Psi(L) = 0 \leftarrow$ PARTICLE CANNOT ESCAPE$)$

$\boxed{\Psi(x) = a\cos(kx) + b\sin(kx)}$ $\boxed{a,b\in\mathbb{C},\ k\in\mathbb{R}}$

$\Psi'(x) = -ak\sin(kx) + bk\cos(kx)$

$\Psi''(x) = -k^2\Psi(x)$ } SIMPLE LAPLACIAN FROM $d=1$

→ SINCE $\Psi(0) = 0 \to \boxed{a=0 \text{ SO NO } \cos(\cdot)}$

→ $\Psi(L) = b\sin(kL)$

$k = \frac{\pi}{L}n$, $(n\in\mathbb{N}) \to E_n = \frac{\hbar}{2m}\frac{\pi^2}{L^2}n^2$

NOTE E_n IS DISCRETE. $(\propto n^2)$

HEISENBERG UNCERTAINTY

TAKE $\Psi(x)$, $\varphi(p)$, CENTERED SO... $\begin{cases}\langle x\rangle = 0,\\ \langle p\rangle = 0\end{cases}$

$\Delta x\Delta p \geq \frac{\hbar}{2}$ (THROUGH COMPUTATION) $\Delta x := \sigma_x$ (ST. DEV)

← BEST BOUND IS FROM THE GAUSSIAN

SOME POINTS:

- $a_L \in \mathbb{C}$ MUST BE NORMALISED

$\int|\Psi|^2dx = |a|^2\int_0^L\sin^2\left(\frac{\pi x}{L}n\right)dx = 1$

$a = e^{i\theta}\cdot c^{-1}$ (EQUIVALENCE)

- E IS OF ORDER? $(n=1)$

$\Delta x\sim L \to \Delta p\sim\frac{\hbar}{L} \to \frac{(\Delta p)^2}{2m}\sim\frac{\hbar}{mL^2}$

- ≠ CLASSICAL PHYSICS $E_{MIN} = 0$, $E\in\mathbb{R}$, PARTICLE IS LOCALISED (TOT)
- IN $V(x)$, IF $V(x)$ DECREASES, E_{KIN} MUST INCREASE AND PARTICLE

NOTE, THE SPACE OF SOLUTIONS TO THE SCHRÖDINGER EQUATION IS SPANNED (1d) BY $\cos(kx)$, $\sin(kx)$, USING COMPLEX COEFFICIENTS

$\begin{cases}\cos(x) = \frac{e^{ix}+e^{-ix}}{2}\\ \sin(x) = \frac{e^{ix}-e^{-ix}}{2}\end{cases}$

SOLVING $V(x) = f(x)$

1. WRITE $\Psi(x)$ GENERIC
2. GO CASE BY CASE: $V(x)$ VALUE, E VALUE → IF THE PARTICLE HAS $E < V(x)$
3. IMPOSE CONTINUITY AT BOUNDARIES $C^0 \ldots C^\infty$ AS NEEDED

$P(x,t) = \|\Psi(x,t)\|^2$

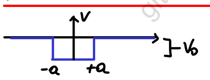
$= \left\|\sum c_i|\phi_i(0)\rangle e^{i\frac{E_it}{\hbar}}\right\|^2$ } YOU CAN FACTOR OUT A PHASE $\|e^{i\ldots t}\| = 1$

SIMPLIFY

$\langle\hat{H}(t)\rangle = \sum_n|C_n(t)|^2E_n$

$\langle\hat{X}(t)\rangle = \int_0^L xP(x,t)dx$

FINITE SQUARE WELL



1. EIGENSTATES: $(E<V_0)$

$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$

$(x<-a) \to V(x) = V_0$

$\psi'' = \frac{2m}{\hbar^2}(V_0-E)\psi = q^2\psi$

$\psi = a'e^{qx} + \cancel{be^{-qx}}$ (e^{-x} DIVERGES AS $x\to\infty$)

$(x>a)$ $\psi = be^{-qx}$

$(-a<x<a)$ $\psi'' = -\frac{2m}{\hbar}E\psi \to \psi'' = -k\psi$

$\psi(x) = c\sin(kx) + d\cos(kx)$

SOLVE FOR a', b, c, d

$(\psi(x) = \psi(-x)) \to \cancel{\sin(kx)}$

$$\psi(x) = \begin{cases} a'e^{qx} & x<-a \\ d\cos(kx) & -a<x<a \\ be^{-qx} & a<x \end{cases}$$

IMPOSE C^1 OF ψ $(\psi, \psi', \psi'' \in C)$

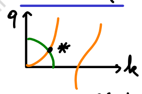
$\psi(a^+) = \psi(a^-) \to a'e^{-qa} = b\cos(ka)$

$\psi'(a^+) = \psi'(a^-) \to -a'qe^{-qa} = -bk\sin(ka)$ DIVIDE BY

$\boxed{q = k\tan(ka)}$

$\psi''(a^+) = \psi''(a^-) \to k^2 = \frac{2mE}{\hbar^2}(V_0-E)$

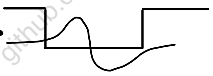
$\boxed{k^2+q^2 = \frac{2mEV_0}{\hbar^2}}$



$\#SOL = f(V_0)$

(q, k, E) ARE A SOLUTION

$(\psi(x) = -\psi(-x)) \to$



IF $E>V_0$ THE PARTICLE CAN FREELY ENTER/EXIT THE WELL. $\int|\psi|^2dx$ DIVERGES

↓

YOU USE A WAVEPACKET, BUT THEN ENERGY BECOMES A CONTINUUM

$\hat A = |\phi_1\rangle\langle\phi_2|$, IMPOSE HERMITIAN

$\hat A = (|\phi_1\rangle\langle\phi_2|)^{T*} = (\langle\phi_2^T|\phi_1^T\rangle)^*$

$\langle x|\hat A|\psi\rangle^* = [\langle x|(|\phi_1\rangle\langle\phi_2|\psi\rangle)]^*$

(LINEARITY) $= \langle x|\phi_1\rangle^*\langle\phi_2|\psi\rangle^*$

$= (\langle\psi|\phi_2\rangle\langle\phi_1|x\rangle)^*$ $\boxed{(ab)^* = a^*b^*}$

$= \langle x|\hat A^+|\psi\rangle$

$\langle\phi|(\hat A|\psi\rangle) = \sum_i\phi_i^*\sum_j A_{ij}\psi_j$

$= \sum_{ij}\phi_i^*A_{ij}\psi_j$

$\boxed{(a+b) = (a^*+b^*)^*}$ $= (\sum_{ij}\psi_j A_{ij}^*\phi_i)^*$

$= (\sum_{ij}\psi_j(A^+)_{ij}\phi_j)^*$

$= [\langle\psi|(A^+|\phi\rangle)]^*$

ADJOINT

$A^+ := (A^*)^T$

$(A^+)^+ = A$

$(A+B)^T = A^+ + B^+$

$(AB)^+ = B^+A^+$

NOTE

$c\in\mathbb{C}$, $c^+ = c^*$.

$|\psi\rangle^+ = \langle\psi|$

$A: \xi\to\xi$ (AKA. MATRIX)

$\boxed{\xi\mapsto A\xi}$ $\boxed{A|\psi\rangle := \int dy\, A(x,y)\psi(y)}$

$\boxed{\langle\phi_n|\phi_n\rangle} = 1 \to \hat A = |\phi_n\rangle\langle\phi_n|$

$\hat A^+ = \hat A$ (BY DEF)

$\hat A^2 = (|\phi_n\rangle\langle\phi_n|)^2$

$= |\phi_n\rangle(\langle\phi_n|\phi_n\rangle)\langle\phi_n|$

$= |\phi_n\rangle\langle\phi_n| = \hat A$ } PROJECTION MATRIX, IDEMPOTENT

$\hat B := \alpha_n\hat A \Rightarrow (\hat B^T)^* = \hat B^* = \hat B$ IMPLIES $\alpha_n\in\mathbb{R}$

(YOU CAN EXTEND HERMITIAN + $|\phi\rangle$ DEFINED)

SUPPOSE $\{|\phi_n\rangle\}$ ORTHONORMAL BASIS

$(n=1)$ $\hat A := |\phi_1\rangle\langle\phi_1|$, $|\psi\rangle := \lambda|\phi_1\rangle$

$\hat A|\psi\rangle = |\phi_1\rangle\langle\phi_1|\lambda\phi_1\rangle = \lambda|\phi_1\rangle\underbrace{\cancel{\langle\phi_1|\phi_1\rangle}}_{=1}$

$(n=2)$ $\hat A := |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$, $|\psi\rangle := \lambda|\phi_1\rangle + \mu|\phi_2\rangle$

$\hat A = \mathbb{1}$ (ALL OTHER ϕ GO TO ZERO)

$\int|\psi|^2dx = \int\psi^*\psi\, dx = 1$

UNITARY OPERATORS

$\langle\psi|\phi\rangle = \langle\hat U\psi|\hat U\phi\rangle$ *

$= \langle\psi|\hat U^+\hat U|\phi\rangle$

$\boxed{\hat U^+ = U^{-1}}$ $= \langle\psi|\phi\rangle$

$|\omega\rangle$ ST $|\omega|^2 = 1$

$\hat U|\omega\rangle = \lambda|\omega\rangle$

$\langle\hat U\omega|\hat U\omega\rangle = \langle\lambda\omega|\lambda\omega\rangle$

$\hat U$ IS A ROTATION $= \lambda^*\lambda\cancel{\langle\omega|\omega\rangle}$

$= |\lambda|^2 = 1$ *

$P(E_i) = |\langle E_i|\psi\rangle|^2$ ⊕ $\langle\psi|\phi\rangle = \langle\hat U\psi|\hat U\phi\rangle$ } $P(E_i) = |\langle\hat U\psi|\hat U\phi\rangle|^2$

HENCE $\hat U$ REPRESENTS ALL THE TRANSFORMATIONS THAT WOULD PRESERVE PROBABILITY

LAGRANGIAN

$\mathcal{L}(t, x(t), \dot x(t))$

$\to \frac12 m\dot x^2 - V(x)$ } KINETIC, POTENTIAL

PRINCIPLE OF STATIONARY ACTION

ACTION INTEGRAL

$S[\mathcal{L}] = \int_{t_i}^{t_f} dt\, \mathcal{L}$

SOME $x(t)$ PRODUCE $\mathcal{L} = S[\mathcal{L}]$ } $x^*(t)$ GIVES THE CORRECT PATH

$\boxed{\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot x}}$ → EULER-LAGRANGE EQUATION

↳ PRODUCES $\vec F = m\vec a$

$\boxed{\frac{\partial\mathcal{L}}{\partial x} = \frac{dp}{dt}}$ $\boxed{\frac{\partial\mathcal{L}}{\partial p} = \frac{dx}{dt}}$

$\boxed{\frac{\partial\mathcal{L}}{\partial t} = -\frac{dE}{dt}}$ $\boxed{\frac{\partial\mathcal{L}}{\partial\vartheta} = \frac{dL}{dt}}$

HILBERT SPACE (∞ DIM. VECTOR SPACE)

- VECTOR SPACE WITH AN INNER PRODUCT MAKING THE SPACE COMPLETE:

CAUCHY SEQUENCE ⇒ CONVERGENT SEQUENCE

KET: $|\psi\rangle$ ← VECTOR THAT REPRESENTS A PARTICLE

$|\psi\rangle$ = SUM OF POSSIBLE MEASUREMENTS

= COULD ALSO BE ∞ LINEAR COMB

$|\psi\rangle = \int dx\, c(x)|x\rangle$ ← STATE, COEFFICIENT

$|\phi\rangle + |\psi\rangle = |\phi+\psi\rangle$ (SAME NOTATION)

MEAN MOMENTUM: $\langle\hat p(t)\rangle = \int\psi^*\left(-i\hbar\frac{d\psi}{dt}\right)dx$

QM. FORMALISM

$|\psi\rangle$ IS THE NOTATION FOR VECTOR $(= \vec\psi, \underline{\psi})$

- $\mathbb{C}$ IS THE FIELD OF THE VECTOR SPACE
- $|\phi\rangle \mapsto \langle\phi|$ IS AN OPERATOR ST. $\langle\phi| = (|\phi\rangle)^{T*}$ } TRANSPOSE, CONJUGATE

$|\psi\rangle\in\mathcal{H}$, $\langle\psi|\in\mathcal{H}^*$ (DUAL)

PROPERTIES

$\langle\phi|a\psi\rangle = a\langle\phi|\psi\rangle$ (LINEAR)

$\langle a\phi|\psi\rangle = a^*\langle\phi|\psi\rangle$ (ANTILIN.)

$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^*$

$|\psi\rangle = \sum_i c_i|\psi_i\rangle \to \langle\psi| = \sum_i c_i^*\langle\psi_i|$

- $\langle\phi|: \xi\to\mathbb{R}$ SCALAR PRODUCT:

$\langle\phi|\psi\rangle := \int\phi^*\psi\, dx$ OR $\sum_i\phi_i^*\psi_i$

(INFINITE DIMENSIONAL) (FINITE DIM)

DAGGER NOTATION → $a^\dagger = (a^T)^*$

$|\psi\rangle = \int dx\,\psi(x)|x\rangle = \int dp\,\phi(p)|p\rangle = \sum_i c_i|E_i\rangle$ } ALL EQUIVALENT BASIS EXPANSION

$\langle E_i|\psi\rangle = c_i$

$P_\psi(E_i) = |c_i|^2 = |\langle E_i|\psi\rangle|^2 = \langle\psi|E_i|\psi\rangle$ } REQUIRES NORMALISATION

THE MATRICES ARE GENERALLY HERMITIAN

OBSERVABLE

$E(\dots)$ LINEAR ⇒ CAN BE REPR BY MATRIX

$E_\psi[x] = \langle\psi|\hat x|\psi\rangle = \int\psi^*x\psi\, dx = \int|\psi|^2x\, dx$

EG: $\psi(x)\leftrightarrow\varphi(p)$ RELATION } $\frac{\partial\psi}{\partial x} = \int\frac{dx}{\sqrt{2\pi}}e^{-ikx}\frac{\partial\varphi}{\partial x} = i\hbar\varphi(p)$

$\langle p\rangle = \langle\psi|\frac{\hbar}{i}\frac{\partial}{\partial x}|\psi\rangle = \hat A|\psi\rangle$ } ψ IS THE x IN $f(x)$

COMMUTATOR

$\hat A, \hat B$ COMMUTE IF $\hat A\hat B = \hat B\hat A$

$[\hat A, \hat B] = \hat A\hat B - \hat B\hat A$ } QUANTIFIES HOW MUCH TWO OPERATORS DON'T COMMUTE

TIME EVOLUTION

$\hat U(t)|\psi\rangle = |\psi_t\rangle$

- $\hat U(0) = \mathbb{1}$
- $\hat U^{-1}$ EXISTS
- $\langle\hat U(t)\psi|\hat U(t)\psi\rangle = \langle\psi|\psi\rangle = 1$ (PROBABILITY IS CONSERVED)

$\frac{d}{dt}|\psi\rangle = \dot{\hat U}(0)|\psi\rangle$

$\dot{\hat U}(0)^+ = -\dot{\hat U}(0) \to (i\dot{\hat U}(0))^+ = i\dot{\hat U}(0)$

(ANTIHERMITIAN) (HERMITIAN)

$\dot{\hat U}(0) = \frac{\hat H}{i}$

$i\frac{d}{dt}|\psi\rangle = \hat H|\psi\rangle$ ← ENERGY SHOULD GENERATE TIME EVOLUTION

$i\hbar\frac{d}{dt}|\psi\rangle = \hat H|\psi\rangle$

↑ ENSURES DIMENSIONAL ANALYSIS

PROPERTIES OF BASIS

- YOU WRITE $|\psi\rangle$ AS $\sum c(\cdot)|\phi\rangle$ BASIS
- BASIS IS ORTHONORMAL → $\langle\phi_i|\phi_j\rangle = \delta_{ij}$ (KRONECKER DELTA)

→ SELECT A COMPONENT: $\langle\phi_i|\psi\rangle = c_i$

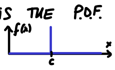
→ $\langle\phi|\psi\rangle^* = (\sum\langle a_i\phi_i|)(\sum\langle b_i\psi_i|)$

$= \sum\langle a_i\phi_i|b_i\psi_i\rangle$ } THE OTHER TERMS ARE $\delta_{ij}, i\ne j$

$= \sum a_i^*b_i\delta_{ii} = \sum a_i^*b_i$

$\delta(x)$ IS THE P.D.F. OF $x=c$:

$P(x) =$



$\mathbb{E}[f(x)] = f(c)$

$\delta(x) = \begin{cases}\infty & x=c \\ 0 & x\ne 0\end{cases}$

→ $\langle\phi|\psi\rangle^* = \left(\int dx\,\psi(x)\langle x|\right)\left(\int dy\,\phi(y)|y\rangle\right)$

$= \iint dx\,dy\,\psi(x)^*\phi(y)\langle x|y\rangle$

$= \iint dx\,dy\,\psi(x)^*\phi(y)\delta(x-y)$

$= \int dx\,\psi(x)^*\phi(x)$

$\langle\psi|\psi\rangle = |\psi|^2$ → WHEN COMPUTING $P(\text{STATE})$ ⟹ NORMALISE $\psi\mapsto\frac{\psi}{|\psi|^2}$

HERMITIAN MATRIX

TO GET VALUES OF MEASURABLES, USE A HERMITIAN OPERATOR

$\hat A^+ = \hat A$

$\boxed{(A^+) = A^*}$ $\boxed{A = (A^T)^*}$

$\boxed{\langle\psi|\hat A\phi\rangle = \langle\phi|\hat A|\psi\rangle}$

- EIGENVECTORS SPAN $\mathcal{H}$
- EIGENVALUES $\in\mathbb{R}$ } THEY REPRESENT ENERGY VALUES

$\hat A|\phi_\alpha\rangle = \lambda_\alpha|\phi_\alpha\rangle$

$\begin{cases}\langle\phi_\alpha|\hat A|\phi_\alpha\rangle = \lambda_\alpha \\ \langle\phi_\alpha|\hat A|\phi_\alpha\rangle^* = \lambda_\alpha^*\end{cases}$ → $\boxed{\lambda_\alpha = \lambda_\alpha^*,\ \lambda_\alpha\in\mathbb{R}}$

- $\lambda_\alpha\ne\lambda_\beta \Rightarrow \langle\phi_\alpha|\phi_\beta\rangle = 0$

SPECTRAL THEOREM

→ EIGENDECOMPOSITION OF OPERATORS: (λ_α EIGENVALUES)

- λ_α NON-DEGENERATE (EACH λ_α HAS ONE $|\psi_\alpha\rangle$) → $\{|\psi_\alpha\rangle : \alpha\}$ FORM A BASIS
- λ_α HAS n_α DEGENERACY → ∃ BASIS ST. ($|\psi_{\alpha,r}\rangle$ EIGENVECTORS) $\hat A = \sum_\alpha\lambda_\alpha\sum_r|\psi_{\alpha r}\rangle\langle\psi_{\alpha r}|$

REMEMBER THAT IN N-DIM,

$\mathbb{1} = \sum_n^N|e_n\rangle\langle e_n|$

FOR $|\psi\rangle$, $\langle e_n|\psi\rangle$ = LENGTH OF $|\psi\rangle$ IN $|e_n\rangle$

$|e_n\rangle\alpha$ = SCALING OF $|e_n\rangle$ BY α

NxN HERMITIAN MATRIX: } HAS REAL, POSITIVE EIGENVALUES

$\hat A = \sum_{n=1}^N\lambda_n|\psi_n\rangle\langle\psi_n|$

$A_{np} = \langle e_n|\hat A|e_p\rangle = \sum_{k=1}^N\lambda_k U_{nk}U_{kp}^*$

USING $U_{np} = \langle e_p|\psi_n\rangle$ ST. $U^+ = U^{-1}$

REMEMBER, IN 1d, $\hat p^2 = \psi''$

↑ KINETIC ENERGY

→ "SQUARING" ACTUALLY MEANS COMPOSING THE OPERATION WITH ITSELF, TWICE, IN THIS CONTEXT

Energy Operator

$\hat{H} = \left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]$

You can show directly $\langle\psi|\hat{H}|\phi\rangle = \langle\phi|\hat{H}|\psi\rangle^*$, since

$\psi(x) \mapsto V(x)\psi(x)$

$V(\hat{x})|\psi\rangle = V(x)\psi(x)$ (matrix notation)

(1D) $\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$\langle\phi|\hat{P}_x|\psi\rangle = \int dx\, \phi^* \frac{\hbar}{i}\frac{\partial\psi}{\partial x}$

(Product rule) $= \frac{\hbar}{i}\int dx\, \psi \frac{\partial\phi^*}{\partial x}$

$\langle\psi|\hat{P}_x|\phi\rangle^* = \left(\frac{\hbar}{i}\int dx\, \psi^* \frac{\partial\phi}{\partial x}\right)^*$

i stays the same despite the complex conjugate because of $\frac{\partial\phi}{\partial x}$ chain rule

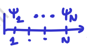
Matrix Derivations:

You can discretise space and solve ψ as $N \to \infty$

- N-dim space
- $|\psi\rangle = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix}$ for some basis of $\mathcal{H}$ Hilbert
- Approximate $\hat{P}_x$ discretely:

$\boxed{\hat{P}_x := \frac{\hbar}{i}\frac{\partial\psi}{\partial x}}$ and $\frac{\partial\psi}{\partial x} \approx \frac{\psi(x_{n+1}) - \psi(x_{n-1})}{2\delta}$

- There are N dimensions in the space
- δ is the distance between any two x_n, x_{n+1}:



View ψ as $f: \mathbb{N} \to \mathbb{C}$

- Momentum can be written as

$\hat{P}_x = \begin{pmatrix} 0 & \frac{\hbar}{2i\delta} & & 0 \\ -\frac{\hbar}{2i\delta} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\hbar}{2i\delta} \\ 0 & & -\frac{\hbar}{2i\delta} & 0 \end{pmatrix}$ Hermitian

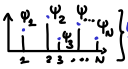
$\hat{P}_x$ is mostly zeroes because it only needs $\psi_i(x_{n+1}), \psi_i(x_{n-1})$ for each ψ_i in ψ

- Kinetic energy: $\frac{\hat{P}_x^2}{2m} = \frac{1}{2m}P_x P_x$

Shows that there is not just one way to solve ψ: ODE or matrix

A Hilbert space can be finite dim in the sense of:

A) $|\psi\rangle \cong \mathbb{C}^N$ AKA



discrete

B) $|\psi\rangle = \sum_i^N c_i|\phi_i\rangle$, $|\phi_i\rangle$ basis vectors

Time Evolution Operator

$|\psi(t)\rangle = U(t)|\psi(0)\rangle$

- Must preserve norm:

$\langle\psi(t)|\psi(t)\rangle = |\psi(t)|^2 = 1$

$= \langle U(t)\psi(0)|U(t)\psi(0)\rangle$

$(U^\dagger = U^{-1})$ $= \langle\psi(0)|U^\dagger(t)U(t)|\psi(0)\rangle$

$= \langle\psi(0)|\psi(0)\rangle = 1$

$\begin{cases} \frac{d|\psi(t)\rangle}{dt} = \frac{1}{i\hbar}\hat{H}|\psi(t)\rangle \\ \frac{d\langle\psi(t)|}{dt} = \frac{-1}{i\hbar}\langle\psi(t)|\hat{H} \end{cases}$

$\frac{d(\langle\psi(t)|\psi(t)\rangle)}{dt} = \langle\psi(t)|\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle + \langle\psi(t)|-\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle = 0$

$|\psi(t)\rangle = e^{\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$ (compare to $\frac{d\vec{a}}{dt} = B\vec{a}$, $\vec{a}(t) = \vec{a}e^{Bt}$)

($\hat{H}$ Hermitian hence diagonalisable)

$|\psi\rangle = \sum_n c_n|\phi_n\rangle$ with eigenvectors

$\hat{H}|\psi\rangle = \sum_n E_n c_n|\phi_n\rangle$ hence

$|\psi(t)\rangle = \sum_n c_n e^{\frac{-iE_n t}{\hbar}}|\phi_n\rangle$

Free Particle - Time Evolution

$\hat{H} = \frac{\hat{p}^2}{2m} \to \hat{H}\psi = E\psi$ (1d)

$-\frac{\hbar^2}{2m}\psi'' = E\psi$

Using $\begin{cases}\psi_k(x) = e^{ikx} \\ \hat{H}|\psi_k\rangle = \frac{\hbar^2 k^2}{2m}|\psi_k\rangle\end{cases}$

$\psi(x) = \sum_k c_k\psi_k(x) = \int dk\, c(k)e^{ikx}$ (infinite dim. case)

$\psi(t) = \sum_k c_k e^{i(kx - \frac{E_k t}{\hbar})}$ time dependency

Different E_k in Σ $\Rightarrow$ different phases $\Rightarrow$ different $\mathbb{P}(E_k)$

$\psi(t) = \sum_k c_k e^{\frac{iE_k}{\hbar}t}|\phi_k\rangle$ $\Big\}$ E_k eigenvalue, $|\phi_k\rangle$ eigenvec.

Decomposition $\to \sum \mathbb{P}_i = 1$

$A = \sum_n \alpha_n|\phi_n\rangle\langle\phi_n|$

$\mathbb{P}(\alpha_n) = |\langle\phi_n|\psi\rangle|^2$

$= \langle\phi_n|\psi\rangle^*\langle\phi_n|\psi\rangle$

$= \langle\psi|\phi_n\rangle\langle\phi_n|\psi\rangle$

$= \langle\psi|A_n|\psi\rangle$

$\sum_n \mathbb{P}(\alpha_n) = \langle\psi|\sum_n A_n|\psi\rangle$ ($\sum_n A_n = \mathbb{1}$)

$= \langle\psi|\psi\rangle = 1$

Wave Collapse

- Observables require a projection of $|\psi\rangle$:

$|\psi\rangle \mapsto c\,\mathbb{P}_{\alpha_n}|\psi\rangle$

- Recording position corresponds to collapsing the $\mathbb{P}(x)$ to being a Dirac. This is true because IRL the particle is absorbed $\Rightarrow$ at that point, by the distribution of its momentum is a uniform (most unknown)

$|\psi\rangle$ is a random variable that takes a value $\to$ you then evolve the value it received (in t) $\to$ you get a later $\mathbb{P}$ of values

Harmonic Oscillator

$V(x) = \frac{1}{2}kx^2$

(classic)

$m\ddot{x} = -\frac{\partial V}{\partial x} = -kx$

$x(t) = a\cos(\omega t + \varphi)$

$\omega = \sqrt{\frac{k}{m}}$

$\hat{H}|\psi\rangle = E|\psi\rangle$

$\hat{H} = \left[\frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2\right]$, $\frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\Delta\psi$

1D assumption $\to \Delta\psi = \frac{\partial^2}{\partial x^2}$

Simplifications $\longrightarrow -\frac{1}{2}\frac{d^2\phi}{dy^2} + \frac{1}{2}y^2\phi = \epsilon\phi$

$\to \epsilon = \frac{E}{\hbar\omega}$ reduced energy units

$\to a = \sqrt{\frac{\hbar}{m\omega}}$

$\to y = \frac{x}{a}$

$\to \phi(y) = \sqrt{a}\,\psi(x)$

$(\epsilon = \frac{1}{2})$ $\phi(y) = ce^{-y^2/2}$

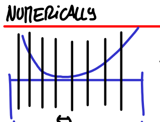
$\begin{cases}\phi' = -y\phi \\ \phi'' = -y + y^2\phi\end{cases}$

ϵ general case

Hermite polynomials form an orthogonal basis. $\{H_n(y) : n\}$

$\phi_n(y) = c_n H_n(y)e^{-y^2/2}$

Numerically



$\begin{cases} y_n = -L + n\delta \\ y_0 = -L, y_N = L \end{cases}$

$\delta = \frac{2L}{N}$

$\frac{\partial\phi}{\partial y} \approx \frac{\phi_{n+1} - \phi_n}{\delta}$

$\frac{\partial^2\phi}{\partial y^2} \approx \frac{\frac{\phi_{n+1} - \phi_n}{\delta} - \frac{\phi_n - \phi_{n-1}}{\delta}}{\delta}$

$\approx \frac{\phi(y_{n+1}) - 2\phi(y_n) + \phi(y_{n-1})}{\delta^2}$

$\frac{\partial^2}{\partial y^2} = \frac{1}{\delta^2}\begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2\end{pmatrix} = M$

$y_2 = \begin{pmatrix} y_0^2 & & 0 \\ & \ddots & \\ 0 & & y_N^2 \end{pmatrix}$

$[-\frac{1}{2}M + \frac{1}{2}y_2]\phi = \epsilon\phi$

Solve directly for ϕ

Ammonia Molecules

$\to$ can be oriented L or R

$|\psi\rangle = a|\psi_L\rangle + b|\psi_R\rangle$ $\Big\}$ consider state as superposition

Use $|\psi_R\rangle, |\psi_L\rangle$ as a basis: $\to$ $\begin{cases}|\psi_L\rangle = \begin{pmatrix}1\\0\end{pmatrix} \\ |\psi_R\rangle = \begin{pmatrix}0\\1\end{pmatrix}\end{cases}$ ①

Take a generic:

$\hat{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$, $A \in \mathbb{R}^+$ — what follows is an application of the spectral thm

Eigenvalues:

$\det(\hat{H} - \lambda) = (E_0 - \lambda)^2 - A^2 = 0$ $\boxed{\lambda = E_0 \pm A}$

$E_0 - \lambda = \pm A$

$\lambda = E - A$: $\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix} \to \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |\psi_+\rangle}$

$\lambda = E + A$: $\begin{pmatrix} -A & -A \\ -A & -A \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}1\\-1\end{pmatrix} \to \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |\psi_-\rangle}$ ②

You are given a $\hat{H}$ that describes ψ in the environment, so you rewrite its eigenbasis to get the likely measurements of the state.

Change of basis: $\longrightarrow$ $|\psi_\pm\rangle$ are the eigenstates, so write $|\psi_{LR}\rangle$ as a combination of eigenstates

$\begin{cases}|\psi_+\rangle = \frac{1}{\sqrt{2}}(|\psi_L\rangle - |\psi_R\rangle) \\ |\psi_-\rangle = \frac{1}{\sqrt{2}}(|\psi_L\rangle + |\psi_R\rangle)\end{cases}$

$|\psi(0)\rangle = |\psi_L\rangle \to$ find $\mathbb{P}_L(|\psi_{LR}\rangle = |\psi_R\rangle) = ?$

$P(L,t) = |\langle\psi_L|\psi(t)\rangle|^2$

$\begin{cases}|\psi(0)\rangle = |\psi_L\rangle = \frac{1}{\sqrt{2}}|\psi_+\rangle + \frac{1}{\sqrt{2}}|\psi_-\rangle \\ |\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\frac{(E_0 - A)t}{\hbar}}|\psi_+\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{(E_0 + A)t}{\hbar}}|\psi_-\rangle\end{cases}$

$\langle\psi_L| = \frac{1}{\sqrt{2}}\langle\psi_+| + \frac{1}{\sqrt{2}}\langle\psi_-|$

$\langle\psi_L|\psi_t\rangle = \frac{1}{2}e^{-i\frac{(E_0 - A)t}{\hbar}}\cancel{\langle\psi_+|\psi_+\rangle} + \frac{1}{2}e^{-i\frac{(E_0+A)t}{\hbar}}\cancel{\langle\psi_-|\psi_-\rangle}$

$= \frac{1}{2}\left(e^{-i\frac{(E_0 - A)t}{\hbar}} + e^{-i\frac{(E_0 + A)t}{\hbar}}\right)$

$= e^{\frac{-iE_0 t}{\hbar}}\frac{\left(e^{\frac{+iA}{\hbar}t} + e^{\frac{-iA}{\hbar}t}\right)}{2}$

$= e^{\frac{-iE_0 t}{\hbar}}\cos\left(\frac{At}{\hbar}\right)$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

$P(L,t) = \left|e^{\frac{-iE_0 t}{\hbar}}\cos\left(\frac{At}{\hbar}\right)\right|^2 = \cos^2\left(\frac{At}{\hbar}\right) = \frac{1 + \cos\left(\frac{2At}{\hbar}\right)}{2}$

Tensor Product

$\neq$ Cartesian product

"Multiply" more vector spaces to form an

$\begin{cases} m = \dim(V) \\ n = \dim(W)\end{cases} \Rightarrow \frac{m \times n}{\dim(V \otimes W)}$

$\mathcal{E}, \mathcal{F}$ Hilbert spaces: $\begin{cases}|e_n\rangle \text{ basis } \mathcal{E} \\ |f_n\rangle \text{ basis } \mathcal{F}\end{cases}$

$\zeta = \mathcal{E} \otimes \mathcal{F} = \left\{\sum_{n,m} c_{m,n}|e_n\rangle \otimes |f_n\rangle\right\}$

the basis expands ζ basis

$d_{\mathcal{E}\otimes\mathcal{F}} = d_{\mathcal{E}} \times d_{\mathcal{F}} \to$ distance

$\langle\phi_1 \otimes \phi_2|\psi_1 \otimes \psi_2\rangle$

$= \langle\phi_1|\psi_1\rangle_{\mathcal{H}_1} \cdot \langle\phi_2|\psi_2\rangle_{\mathcal{H}_2}$

$\hat{A}, \hat{B}$ Herm. $\Rightarrow \hat{A} \otimes \hat{B}$ Herm

$(\hat{A} \otimes \hat{B})(|\phi\rangle \otimes |\psi\rangle)$

$(\hat{A}|\phi\rangle) \otimes (\hat{B}|\psi\rangle)$

Multiple Particles

$(n=2)$ $\begin{cases}\hat{P}_1, \hat{P}_2 \\ \cdots\end{cases}$ hence

$\hat{H} = \left[\frac{\hat{P}_1}{2m_1} + U(\hat{n}_1)\right] \otimes \mathbb{1} + \mathbb{1} \otimes \left[\frac{\hat{P}_2}{2m_2} + U(\hat{n}_2)\right] + V(\hat{n}_1, \hat{n}_2)$

If zero, the system is decoupled

Solve each particle, tensor product is the solution

Interacting Spins

$|\psi\rangle = \alpha|++\rangle_z + \beta|+-\rangle_z + \gamma|-+\rangle_z + \delta|--\rangle_z$

$\mathbb{P}_{|+\rangle_z}(|\psi\rangle) = \|(|+\rangle\langle+| \otimes \mathbb{1})|\psi\rangle\|^2$

$= |\alpha|^2 + |\beta|^2$ After σ_z^a, do σ_z^b

$\mathbb{P}_{|-\rangle_z}(|\psi\rangle) = \|(\mathbb{1} \otimes (|-\rangle\langle-|))(\sigma_z^a|\psi\rangle)\|^2$

$= \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}$ caused by normalisation

Factorisable states $\Rightarrow$ classical behaviour

Otherwise, entangled $\left(\frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)\right)$ $\to$ $\begin{cases}\mathbb{P}(|++\rangle) = \frac{1}{2} \\ \mathbb{P}(|--\rangle) = \frac{1}{2} \\ P(\text{else}) = 0\end{cases}$

Spin

Spin $\to$ micros. property of quantum particles

- It's a vector $\cong \mathbb{R}^2$
- Stern-Garlach: "filter" that uses a field to split a beam of particles based on — same as polariser with light; measures spin

SG has 3 indep. directions. Mathematically, this are Hermitian, idempotent: $(A^2 = A$, aka projection matrix$)$

Eigenvalues are always ± 1: $a\hat{\sigma}$ has $\pm a$ $\to$

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Measuring on one axis collapses that state:

- SG is deterministic on the right axis:

$|+\rangle_z \xrightarrow{\sigma_z} \frac{1}{2}$ and

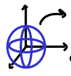
$|-\rangle_z \xrightarrow{\sigma_z} -\frac{1}{2}$

- SG gives a superposition otherwise:

$\alpha|+\rangle_z + \beta|-\rangle_z \xrightarrow{\sigma_x}$ $\begin{cases} -\frac{1}{2} & \mathbb{P} = |\beta|^2 \\ +\frac{1}{2} & \mathbb{P} = 1 - |\beta|^2\end{cases}$

Measurement gives α, β, up to $e^{i\theta}$ multiplication (you won't know phase)

$\alpha|+\rangle_z + \beta|-\rangle_z \xrightarrow{\sigma_z}$ $\begin{cases} -\frac{1}{2} & \mathbb{P} = 1/2 \\ \frac{1}{2} & \mathbb{P} = 1/2\end{cases}$



Spin is like a point on the surface of a sphere → you can describe any point as a convex comb, needing at most 2x extremes — Caratheodory theorem + normalisation

$\hat{H}$ matrix 2x2

$(A^\dagger = A)$

$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a, d \in \mathbb{R} \,|\, c^* = b$

$\hat{A} = \begin{pmatrix} a & b + ic \\ b - ic & d \end{pmatrix} \cong \mathbb{R}^4$

Space of 2x2 Hermitian is 4 dim as opposed to $8 \leftarrow \mathbb{C}^4$

$\hat{A} = \frac{a+d}{2}\mathbb{1} + \frac{a-d}{2}\sigma_z + b\sigma_x + c\sigma_y$

General way to measure quantum properties (3x SG)

In general, find basis, find $\hat{H}$ eigenvectors, express $|\psi(0)\rangle$ as sum ...

Whenever you want to measure something, you express it with $\hat{H}$ Hermit. to find $|\hat{\psi}\rangle$, the wave function of the measurement. To project $|\psi\rangle$ through $\hat{H}$, you express it as the sum of eigenvectors, weighted by the $\langle\phi_i|\psi\rangle$ values

General Procedure:

1. $|\psi\rangle = |+\rangle \ldots + |-\rangle \ldots$
2. Decompose $|+\rangle \ldots, |-\rangle \ldots$ into eigenvectors of $\hat{\sigma} \ldots$

$|+\rangle_z = (\langle+|+\rangle_z)|+\rangle_b + (\langle+|+\rangle_z)|+\rangle_b$

(you generally need the eigendecomp.)

3. Map the eigenvectors

$\mathbb{P}_{|-\rangle_x}(|+\rangle_x) = \|\langle-|+\rangle_x\|^2$

$= \|\langle-|(1 \cdot |+\rangle + 0 \cdot |-\rangle)\|^2$

$= 0$ hence $\langle-|+\rangle = 0$

$\hat{H}$ is Hermitian, so $\exists$ orthonorm. basis and $\langle\phi_i|\phi_j\rangle = \delta_{ij}$

$|\pm\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_x \pm |-\rangle_x)$

$|\pm\rangle_z = \frac{1}{\sqrt{2}}(|+\rangle_y \pm i|-\rangle_y)$

$|\pm\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_y \mp i|-\rangle_y)$

The identities are symmetric so $\cdots_x = \cdots_y$ is also true as in $\cdots_y = \cdots_x$

PROCEDURE

1. SPLIT $V(x)$ BY VALUE
2. $\begin{cases}\psi = a\cos(kx) + b\sin(kx)\\ a,b \in \mathbb{C},\ k \in \mathbb{R}\end{cases}$
3. SOLVE $\hat{H}\psi = E\psi$ FOR E, ψ IN EACH V VALUE
3. USE $\psi \in C^\infty$ TO STITCH TOGETHER THE FUNCTIONS AT EACH BOUNDARY: $\psi^{(i)}(c^-) = \psi^{(i)}(c^+)$

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1. WRITE $|\psi(0)\rangle$ IN E.BASIS OF: $\hat{H} = -\alpha\hat{\sigma}_x \longrightarrow \{(E_i, \phi_i)\}$
2. EXPAND $|\psi(t)\rangle = \mu|\phi_1\rangle e^{\frac{iE_1 t}{\hbar}} + \nu|\phi_2\rangle e^{\frac{iE_2 t}{\hbar}}$
   $\overset{\text{DE MOIVRE}}{=} \begin{pmatrix}\cos(kx)\\ \sin(kx)\end{pmatrix}$

   (NOTE: $|+\rangle_- \to |\phi_i\rangle$ ST. $E_i = -1$)
3. $\mathbb{P}(|\psi(t)\rangle = |\phi_1\rangle) = \| \langle\phi_1|\psi(t)\rangle \|^2 = \cos^2(kx)$

PARTICLE IN A BOX

$$V(x) = \begin{cases}\infty & x \le 0\\ 0 & x \in (0,1)\\ \infty & 1 \le x\end{cases}$$

(USING $V(x) = 0$ AT $x \in (0, L)$)

$$\hat{H}|\psi\rangle = E|\psi\rangle$$
$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]|\psi\rangle = E|\psi\rangle$$

$$-\frac{\hbar^2}{2m}\psi'' + 0\psi = E\psi$$

$$|\psi\rangle = a\sin(kx) + b\cos(kx)$$

$$\begin{cases}|\psi(0)\rangle = 0\\ |\psi(L)\rangle = 0\end{cases} \to \begin{cases}b = 0\\ k = \frac{n\pi}{L},\ n \in \mathbb{N}^+\end{cases}$$

$$\psi'' = -k\psi$$

$$|\psi\rangle = a\sin\left(\frac{n\pi}{L}x\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$\int|\psi|^2 = 1$

$$\frac{1}{a^2} = \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx \quad \Big| \quad y = \frac{n\pi}{L}x,\ dx = \frac{L}{n\pi}x$$
$$= \frac{L}{n\pi}\int_0^1 \sin^2(y)\,dy \quad \Big| \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$
$$= \frac{L}{n\pi}\left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{n\pi} = \frac{L}{n\pi}\cdot\frac{n\pi}{2} = \frac{L}{2}$$
$$a = \sqrt{\frac{2}{L}}$$

$$\psi'' = -k^2\psi = -E\frac{2m}{\hbar}\psi$$

$$E_n = \frac{(k\hbar)^2}{2m} = \frac{(n\pi\hbar)^2}{2mL^2}$$

ENERGY IS QUANTISED

ONCE OBTAINING THE EIGENSTATES OF $\hat{H}$, THE SCHRÖD. EQUATION TELLS HOW TO EVOLVE THEM IN TIME:

$$|\psi(x,t)\rangle = \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\pi}{L}x\right)e^{\frac{iE_n t}{\hbar}}$$
$$= \begin{pmatrix}\sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{E_n t}{\hbar}\right)\\ \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{E_n t}{\hbar}\right)\end{pmatrix} = a + bi$$

$\|\psi(x,t)\| = a^2 + b^2$ (THERE IS CONSTR. AND DESTR. INTERFERENCE)

$$\hat{H} = -\alpha\sigma_x = -\alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}\lambda & -\alpha\\ -\alpha & \lambda\end{pmatrix} = \lambda^2 - \alpha^2$$
$$\lambda = \pm\alpha$$

FIND EIGENVALUES
↓
FIND EIGENSTATES

$\lvert+\rangle_x \to \hat{H}\lvert\phi\rangle = \alpha\lvert\phi\rangle$	$\lvert-\rangle_x \to \hat{H}\lvert\phi\rangle = -\alpha\lvert\phi\rangle$
$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$
$\begin{pmatrix}-d\\ -c\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$	$\begin{pmatrix}d\\ c\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$
$\lvert+\rangle_x = \begin{pmatrix}1\\ -1\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$	$\lvert+\rangle_x = \begin{pmatrix}1\\ 1\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$

$(t=0)$

$$|\psi(0)\rangle = |+\rangle_z = \begin{pmatrix}1\\ 0\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

NOTE THAT VECTOR NOTATION IS BASICALLY THE NATURAL NOTATION: THERE IS NO BASIS CHANGE

$$|\psi(t)\rangle = \frac{1}{2}\left(|+\rangle_x e^{\frac{i\alpha t}{\hbar}} + |-\rangle_x e^{\frac{-i\alpha t}{\hbar}}\right)$$

$\hat{\sigma}_x$ WAS NOT APPLIED YET:

$$= \frac{1}{2}\begin{pmatrix}2\cdot\cos\left(\frac{\alpha t}{\hbar}\right)\\ 2i\cdot\sin\left(\frac{\alpha t}{\hbar}\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\alpha t}{\hbar}\right)\\ i\sin\left(\frac{\alpha t}{\hbar}\right)\end{pmatrix}$$

*YOU FIRST EXPRESS $|\psi(0)\rangle$ IN THE NATURAL FORM OF THE SYSTEM, AKA. THE EIGENBASIS OF THE $\hat{H}$ HAMILT.
↓
YOU THEN PROJECT THIS VECTOR ONTO THE MEASUREMENT YOU NEED

$$\mathbb{P}(\hat{\sigma}_z|\psi(t)\rangle = |-\rangle_z) = \| \langle -|_z\psi(t)\rangle \|^2 = \sin^2\left(\frac{\alpha t}{\hbar}\right) \qquad |-\rangle_z = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\mathbb{P}(\hat{\sigma}_y|\psi(t)\rangle = |-\rangle_z) = \left\| \frac{1}{\sqrt{2}}(1; i)\begin{pmatrix}\cos\left(\frac{\alpha t}{\hbar}\right)\\ i\sin\left(\frac{\alpha t}{\hbar}\right)\end{pmatrix}\right\|^2 = \frac{1}{2}\left(\cos^2\left(\frac{\alpha t}{\hbar}\right) + \sin^2\left(\frac{\alpha t}{\hbar}\right)\right) = 1/2$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

YOU NEVER REALLY COMPUTE $\hat{\sigma}_x|\psi\rangle$: WRITE $|\psi\rangle = \sum c_i|\phi_i\rangle$ AND PROJECT THE VECTOR ONTO THE VECTOR YOU WANT TO MEASURE $\mathbb{P}$ OF.

A,B OPERATORS COMMUTE:
$AB = BA$ $(\Rightarrow [A,B] = 0)$
$\Downarrow$
EIGENBASES ARE $\neq$, SO MEASURING ONE WILL AFFECT THE OTHER

THIS EXPLAINS HOW AT $t=0$ THERE MAY BE A PURE STATE, WHILE AT $t=1$ THERE IS A SUPERPOS: $t \neq 0$ RESULTS IN A PHASE CHANGE

IN A 2 STATE SYSTEM:

$$E = E_1 + E_2 \longrightarrow \begin{cases}\varepsilon := E_1 - E_2\\ E_1 = \frac{1}{2}(E+\varepsilon)\\ E_2 = \frac{1}{2}(E-\varepsilon)\end{cases}$$

APPLY SUBSTIT. TO $e^{i\ldots t}$ TIME EVOLUTION, SO YOU CAN FACTOR OUT

$$e^{\frac{iE}{2\hbar}t}\left(A|a\rangle e^{\frac{i\varepsilon t}{\hbar}} + B|b\rangle e^{\frac{-i\varepsilon t}{\hbar}}\right)$$
$$e^{\frac{iE}{2\hbar}t}\left[(A|a\rangle + B|b\rangle)\cos\left(\frac{\varepsilon t}{\hbar}\right) + i(A|a\rangle - B|b\rangle)\sin\left(\frac{\varepsilon t}{\hbar}\right)\right]$$

THEY USUALLY SIMPLIFY