FACTORISATION THEOREM

$X \sim$ DISCRETE

$V = V(X)$ SUFFICIENT $\iff \exists h, g_\vartheta$ ST $\forall x, \vartheta$
$P_\vartheta(x) = g_\vartheta(V(x))\,h(x)$

($\Rightarrow$) V IS SUFFICIENT

$$P_\vartheta(x) = P_\vartheta(X=x) \overset{!}{=} P_\vartheta(X=x, V=v) = P(X=x \mid V=v)\,P_\vartheta(V=v) \overset{!}{=} h(x)\cdot g_\vartheta(V(x))$$

$P(X=x \mid V=v)$ DOES NOT DEPEND ON ϑ BECAUSE V IS SUFF

($\Leftarrow$) $\exists\, h(x), g_\vartheta(V(x))$ ST. $P_\vartheta(x) = g_\vartheta(V(x))\,h(x)$

WTS: $P(X=x \mid V=v)$ DOES NOT DEP. ON ϑ

TAKE x_0 ST. $V(x_0) = v$

$$P_\vartheta(X=x_0 \mid V=v) = \frac{P_\vartheta(X=x_0, V=v)}{P_\vartheta(V=v)} = \frac{P_\vartheta(X=x_0)}{\sum_{x:V(x)=v} P_\vartheta(X=x)} = \frac{\cancel{g_\vartheta(V(x))}\cdot h(x_0)}{\sum \cancel{g_\vartheta(V(x))}\cdot h(x)} = \frac{h(x_0)}{\sum_{x:V(x)=v} h(x)}$$

V IS $f(X)$

VALUE IS THE SAME

NO ϑ DEPEND.

RAO-BLACKWELL

$V = V(X)$ SUFF.
T ESTIMATOR FOR $g(\vartheta)$
$\exists T^* = T^*(V)$ ST.
$E_\vartheta(T) = E_\vartheta(T^*)$,
$V_\vartheta(T^*) \le V_\vartheta(T)$
($\Rightarrow$ MSE WILL ALSO BE SMALLER)

$X \sim$ DISCRETE
$T^* = E(T \mid V)$

1) $E_\vartheta(T) \overset{?}{=} E_\vartheta(T)$:

$$E_\vartheta T^* = E_\vartheta[E_\vartheta(T \mid V)] = E_\vartheta[T] \quad \text{(TOWER RULE)}$$

2) $V_\vartheta T^* \overset{?}{\le} V_\vartheta T$:

$$E_\vartheta TT^* = \sum_v E[TT^* \mid V=v]\,P_\vartheta(V=v) \quad \text{(CONDITION BY } V\text{)}$$
$$= \sum_v T^*(v)\,E[T \mid V=v]\,P_\vartheta(V=v) \quad \text{(EXTRACT } T^* \text{ FROM INDEP.)}$$
$$= \sum T^*(v)^2\,P_\vartheta(V=v) \quad \text{(AGAIN)}$$
$$\overset{!}{=} E_\vartheta[(T^*)^2]$$

USED TO SHOW $=0$

$$E_\vartheta T^2 = E_\vartheta(T-T^*)^2 + 2E_\vartheta(T-T^*)T^* + E_\vartheta(T^*)^2 = E_\vartheta(T-T^*)^2 + 0 + E_\vartheta(T^*)^2 \ge E_\vartheta[T^{*2}]$$

T, T^* HAVE SAME EXPECTATION,
$V(T) = E(T^2) - E(T)^2$ IMPLIES
$V_\vartheta T^* \le V_\vartheta T$

LEHMAN-SCHEFFE

$V = V(X)$ SUFFICIENT, COMPLETE
$T = T(V(X))$ UNBIASED ESTIMATOR FOR $g(\vartheta)$
$\Rightarrow T$ IS UMVU

1) RAO-BLACKWELL: $\forall S$ ESTIMATOR $S(V)$ OF $g(\vartheta)$ FOR $g(\vartheta)$ ST. $E_\vartheta(S) = g(\vartheta)$,
USE SUFFICIENCY OF $V \rightarrow$ BY $V = V(X)$ SUFFICIENT,
$\exists S^* = S^*(V)$ FOR $g(\vartheta)$, USING ONLY V,
$E_\vartheta(S^*) = E(S) = g(\vartheta)$ AND $V(S^*) \le V(S)$

FOR ALL ESTIMATORS THERE IS A BEST ONE

2) $E_\vartheta(S^* - T) = E_\vartheta(S^*) - E_\vartheta(T) = 0 \rightarrow$ THE BEST ESTIMATOR IS UNIQUE

BY V BEING COMPLETE: $P_\vartheta(S^* = T) = 1$ $\quad \forall S$ EST. FOR $g(\vartheta)$
$\Rightarrow \underbrace{S^* = T \text{ A.S.}}_{\text{UNIQUENESS}}$ AND $V_\vartheta(T) \le V_\vartheta(S)$

FISHER INFORMATION

X, Y ⫫
$i_{(X,Y)} = i_X + i_Y$

$\vartheta \in \mathbb{R}$
$P_\vartheta(x,y) = P_\vartheta(x)\,q_\vartheta(y)$

$$i_\vartheta = \mathrm{Var}\left(\frac{\partial}{\partial\vartheta}\ln(P_\vartheta(x,y))\right) = \mathrm{Var}\left(\frac{\partial}{\partial\vartheta}[\ln(P_\vartheta(x)) + \ln(P_\vartheta(y))]\right)$$

X, Y IID SO $\mathrm{Var}(X+Y) = V(X) + V(Y)$

NEYMAN-PEARSON LEMMA:

$\exists c_{\alpha_0}$ ST. $P(L(\vartheta_0, \vartheta_1, X) \geq c_{\alpha_0}) = \alpha_0$
THEN THE TEST
WITH $K = \{x : L(\vartheta_0, \vartheta_1, X) \geq c_{\alpha_0}\}$
IS THE MOST POWERFUL AT α_0
FOR $H_0: \vartheta \in \Theta_0$
$H_1: \vartheta \in \Theta_1$

FIX $\alpha = \alpha_0$

ANY OTHER CONFIDENCE SET

BY ASSUMPTION, $\exists K'$ ST. $P_{\Theta_0}(X \in K') \leq \alpha_0$ } MORE POWERFUL TEST REGION

$\hookrightarrow P_{\vartheta_1}(X \in K') \leq P_{\vartheta_1}(X \in K)$ } MEANS THAT K IS MOST POWERFUL

$X \in K$, $\dfrac{P_{\vartheta_1}(x)}{P_{\vartheta_0}(x)} \geq c_{\alpha_0}$

$\Downarrow$

$P_{\vartheta_1}(x) \geq c_{\alpha_0} P_{\vartheta_0}(x)$

1) $\forall x: \ (I_{K'}(x) - I_K(x))\,(P_{\vartheta_1}(x) - c_{\alpha_0} P_{\vartheta_0}(x)) \leq 0$

DEPENDING ON THE PLACEMENT OF X, THE TEST BEHAVES IN ONE OF TWO WAYS

SAME TEST, DIFFERENT CRITICAL REGION

2) $P_{\vartheta_1}(X \in K') - P_{\vartheta_1}(X \in K) = \int I_{K'}(x) P_{\vartheta_1}(x)\,dx - \int I_K(x) P_{\vartheta_1}(x)\,dx$

$= \int (I_{K'}(x) - I_K(x))\, P_{\vartheta_1}(x)\,dx$ (SEE INEQUALITY)

$\leq c_{\alpha_0} \int (I_{K'}(x) - I_K(x))\, P_{\vartheta_0}(x)\,dx$

$= c_{\alpha_0} \left(P_{\vartheta_0}(X \in K') - P_{\vartheta_0}(X \in K)\right)$

$\leq c_{\alpha_0} (\alpha - \alpha_0) = 0$

REASON BY CONTRADICTION, EXPECT VALUE TO BE ≥ 0

BREAKING ASSUMPTION $\Rightarrow$ K IS MOST POWERFUL TEST

CRAMER-RAO LOWER BOUND

$\vartheta \longmapsto p_\vartheta(x)$ DIFFERENTIABLE

IN ANY UNBIASED ESTIMATOR FOR $g(\vartheta)$: $V_\vartheta(T) \geq \dfrac{g'(\vartheta)^2}{I_\vartheta}$

$g'(\vartheta) = E_\vartheta[T]$

$= \dfrac{\partial}{\partial\vartheta} \int T(x)\, p_\vartheta(x)\,dx$

$= \int T(x)\, \dot{p}_\vartheta(x)\,dx$

$= \int T(x)\, p_\vartheta(x)\, \dot{\ell}(x)\,dx$

$= E[T(x)\,\dot{\ell}(x)]$

(IF $X \sim$ DISCRETE, USE Σ INSTEAD)

$E_\vartheta[\dot{\ell}(x)] = 0$

"PHANTOM TERM"

$g'(\vartheta) = E_\vartheta[T\dot{\ell}(x)] - \underbrace{E_\vartheta(T)\, E_\vartheta[\dot{\ell}(x)]}$

$= \mathrm{cov}(T, \dot{\ell}(x))$

$g'(\vartheta)^2 = \mathrm{cov}(T, \dot{\ell}(x))^2$

$\leq V_\vartheta(T) \cdot V_\vartheta(\dot{\ell}(x))$ } CAUCHY SCHWARTZ

$= V_\vartheta(T) \cdot i_\vartheta$

$V_\vartheta(T) \geq \dfrac{g'(\vartheta)}{i_\vartheta}$