Chapter 2
Lagrangian Modeling

The basic laws of physics are used to model every system whether it is electrical, mechanical, hydraulic, or any other energy domain. In mechanics, Newton's laws of motion provide key concepts to model-related physical phenomenon. The Lagrangian formulation of modeling derives from the basic work–energy principle and Newton's laws of motion. The basic law states that the force acting on a body is directly proportional to its acceleration equated with constant mass.

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} \tag{2.1}$$

In this equation, force is directly proportional to the derivative of velocity v or double derivative of displacement x with respect to time. If a body moves from point A to point B then the work done by a body is a dot product of force and displaced path integrated from point A to B.

$$W = \int_A^B F \cdot dx \tag{2.2}$$

$$W = \int_A^B m\frac{d^2x}{dt^2} \cdot dx = \int_A^B m\ddot{x} \cdot dx \tag{2.3}$$

We now derive the equation as

$$\ddot{x}dx = \frac{d\dot{x}}{dt}dx = d\dot{x}\,\frac{dx}{dt} = \dot{x}\,d\dot{x} \tag{2.4}$$

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So Eq. (2.3) formulates as

$$W = \int_A^B m\dot{x}\, d\dot{x} = m\left(\frac{\dot{x}^2}{2}\right)\Bigg|_A^B = \frac{m}{2}\left(\dot{x}_B^2 - \dot{x}_A^2\right) \tag{2.5}$$

$$W = K_B - K_A = \Delta K \tag{2.6}$$

It is important to know that here we are discussing motion in a conservative field where work done is independent of path and depends upon the difference between the initial and final value of kinetic energy K. We note that work done is represented as change in K between two points. By the law of conversation of energy and work–energy principle we know that a change in kinetic energy should also change the potential energy U, as the total energy of the system remains constant. So a change in kinetic and change in potential energies should have a net effect of zero as

$$\Delta K + \Delta U = 0 \tag{2.7}$$

So

$$\Delta U = -W = -\int_A^B F \cdot dx \tag{2.8}$$

Using the fundamental theorem of calculus for antiderivatives, we express Eq. (2.8) as

$$F = -\frac{dU}{dx} \tag{2.9}$$

This shows forces required to change the potential energy, whereas the force required to change the kinetic energy is determined from the definition of K.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \tag{2.10}$$

So

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} \tag{2.11}$$

Differentiating Eq. (2.11) gives a force of kinetic energy

$$F = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = ma \tag{2.12}$$

As sum of forces equal to zero so

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{dU}{dx} = 0 \tag{2.13}$$

Now we define a Lagrangian function L as the difference of kinetic and potential energies

$$L = K - U \tag{2.14}$$

We observe that kinetic energy K is a function of velocity $\dot{x}$, and potential energy U is a function of displacement x. Accordingly the Lagrangian function L has two terms as a function of velocity $\dot{x}$ and displacement x independently. As we know that

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(\frac{\partial K}{\partial \dot{x}}\right) \text{ and } \frac{\partial L}{\partial x} = \frac{\partial U}{\partial x} \tag{2.15}$$

Lagrangian function L now represents Eq. (2.13) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{2.16}$$

This is Lagrangian formulation for a system with motion only in single axis.

2.1 Modeling in Three Axes

Now consider a motion in three axes and every equation now to be represented in three dimensional vector notations.

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} \tag{2.17}$$

The force $\vec{F}$ has three components, F_x, F_y, and F_z in x, y and z axes respectively. The potential energy U is a scalar field in three dimensions and the negative gradient of this field provides a force vector in three axes. Now Eq. (2.9) represents as

$$\vec{F} = -\nabla\, U(x, y, z) = -\left[\frac{\partial U}{\partial x} \quad \frac{\partial U}{\partial y} \quad \frac{\partial U}{\partial z}\right]^T \tag{2.18}$$

The kinetic energy K in three axes is given as

$$K = \frac{1}{2}m\left\|\vec{\dot{x}}\right\|^2 = \frac{1}{2}m\left(\vec{\dot{x}}^T \cdot \vec{\dot{x}}\right) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{2.19}$$

This provides the three components of a force through kinetic energy Eq. (2.19) as

$$\vec{F} = \frac{d}{dt}\left(\begin{bmatrix} \frac{\partial K}{\partial \dot{x}} & \frac{\partial K}{\partial \dot{y}} & \frac{\partial K}{\partial \dot{z}} \end{bmatrix}^T\right) \tag{2.20}$$

By equating forces we can write Eq. (2.13) for three axes as

$$\frac{d}{dt}\left(\begin{bmatrix} \frac{\partial K}{\partial \dot{x}} & \frac{\partial K}{\partial \dot{y}} & \frac{\partial K}{\partial \dot{z}} \end{bmatrix}^T\right) - \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{bmatrix}^T = 0 \tag{2.21}$$

This gives us Lagrangian formulation of rigid body (particle) for each axis in Cartesian coordinates as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0 \end{aligned} \tag{2.22}$$

2.2 Modeling in Cartesian and Spherical/Polar Coordinates

In the above section, we developed a modeling scheme in only a Cartesian coordinate system but often the model is represented in spherical coordinates in 3D space or with polar coordinates in 2D space. The translation and rotational motion also become part of a model in dynamical equations. Now, we discuss the method in 3D space with spherical coordinates, which can also be used in polar coordinates by eliminating an axis. We know that

$$\begin{aligned} x &= f(r, \theta, \varphi, t) \\ y &= f(r, \theta, \varphi, t) \\ z &= f(r, \theta, \varphi, t) \end{aligned} \tag{2.23}$$

We define our system in Cartesian coordinates by using the transformation in Eq. (2.23). Now let us consider that we want to generalize the concept of spherical, polar, or cylindrical coordinates transformed into Cartesian coordinates by using a generalized set of variables instead of r, θ, φ. We now define a generalized vector $\vec{q} = [q_1 \quad q_2 \quad q_3]^T$ and the position in Cartesian coordinates with a vector $\vec{p} = [x \quad y \quad z]^T = [p_1 \quad p_2 \quad p_3]^T$.

$$\begin{bmatrix} p_1 = f(q_1, q_2, q_3, t) \\ p_2 = f(q_1, q_2, q_3, t) \\ p_3 = f(q_1, q_2, q_3, t) \end{bmatrix} = \vec{p} = f\left(\vec{q}, t\right) \tag{2.24}$$

The functions in Eq. (2.24) follow the chain rule for respective derivatives because x, y, z are differentiable with respect to time as well as with respect to each generalized coordinates. Now we follow the convention that $\dot{x} = \frac{dx}{dt}$; a dot on top of the variables represents its differentiation with respect to time for both Cartesian and generalized coordinates. The change with respect to time in Cartesian coordinates is defined as

$$\begin{aligned} \dot{p}_1 &= \frac{\partial p_1}{\partial q_1}\dot{q}_1 + \frac{\partial p_1}{\partial q_2}\dot{q}_2 + \frac{\partial p_1}{\partial q_3}\dot{q}_3 \\ \dot{p}_2 &= \frac{\partial p_2}{\partial q_1}\dot{q}_1 + \frac{\partial p_2}{\partial q_2}\dot{q}_2 + \frac{\partial p_2}{\partial q_3}\dot{q}_3 \\ \dot{p}_3 &= \frac{\partial p_3}{\partial q_1}\dot{q}_1 + \frac{\partial p_3}{\partial q_2}\dot{q}_2 + \frac{\partial p_3}{\partial q_3}\dot{q}_3 \end{aligned} \tag{2.25}$$

Now consider a generalized notation for any element p_i (i.e. x, y, z) of a position vector $\vec{p}$ with respect to time derivates and generalized coordinates

$$\dot{p}_i = \sum_{j=1}^{3}\left(\frac{\partial p_i}{\partial q_j}\dot{q}_j\right) \tag{2.26}$$

The change with respect to generalized coordinates is given as

$$\delta p_i = \sum_{j=1}^{3}\left(\frac{\partial p_i}{\partial q_j}\delta q_j\right) \tag{2.27}$$

We observe that

$$\frac{d}{dt}\left(\dot{p}_i \frac{\partial p_i}{\partial q_j}\right) = \ddot{p}_i \frac{\partial p_i}{\partial q_j} + \dot{p}_i \frac{d}{dt}\left(\frac{\partial p_i}{\partial q_j}\right) \tag{2.28}$$

Solving for the second derivative of position coordinate, we get

$$\ddot{p}_i \frac{\partial p_i}{\partial q_j} = \frac{d}{dt}\left(\dot{p}_i \frac{\partial p_i}{\partial q_j}\right) - \dot{p}_i \frac{d}{dt}\left(\frac{\partial p_i}{\partial q_j}\right) \tag{2.29}$$

Differentiating Eq. (2.26) with respect to q_j where $j = \{1, 2, 3\}$

$$\frac{\partial \dot{p}_i}{\partial \dot{q}_j} = \frac{\partial p_i}{\partial q_j} \tag{2.30}$$

As p_i is a function of q_j then the partial derivatives with respect to time are given as

$$\frac{d}{dt}\left(\frac{\partial p_i}{\partial q_j}\right) = \sum_{k=1}^{3} \frac{\partial}{\partial q_k}\left(\frac{\partial p_i}{\partial q_j}\right)\dot{q}_k \tag{2.31}$$

We know that in complex or real fields, second-order cross partial derivatives are equal for any position variable with respect to other generalized coordinates

$$\frac{\partial}{\partial q_j}\left(\frac{\partial p_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{\partial p_i}{\partial q_j}\right) \tag{2.32}$$

Now taking partial derivative of Eq. (2.26) with respect to q_j we get

$$\frac{\partial \dot{p}_i}{\partial q_j} = \sum_{k=1}^{3} \frac{\partial}{\partial q_j}\left(\frac{\partial p_i}{\partial q_k}\right)\dot{q}_k = \sum_{k=1}^{3} \frac{\partial}{\partial q_k}\left(\frac{\partial p_i}{\partial q_j}\right)\dot{q}_k \tag{2.33}$$

By comparing Eqs. (2.31) and (2.33) we get

$$\frac{d}{dt}\left(\frac{\partial p_i}{\partial q_j}\right) = \frac{\partial \dot{p}_i}{\partial q_j} \tag{2.34}$$

Now substituting Eqs. (2.30) and (2.34) in Eq. (2.29)

$$\ddot{p}_i \frac{\partial p_i}{\partial q_j} = \frac{d}{dt}\left(\dot{p}_i \frac{\partial \dot{p}_i}{\partial \dot{q}_j}\right) - \dot{p}_i \frac{\partial \dot{p}_i}{\partial q_j} \tag{2.35}$$

Now we write Eq. (2.35) as

$$\ddot{p}_i \frac{\partial p_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\dot{p}_i{}^2}{2}\right)\right) - \frac{\partial}{\partial q_j}\left(\left(\frac{\dot{p}_i{}^2}{2}\right)\right) \tag{2.36}$$

2.3 Work and Energy Formulation

We know that change in work done is the dot product of force applied and corresponding change in position given as

$$\delta W = \vec{F} \cdot \delta\vec{p} \tag{2.37}$$

Equivalently, force vector $\vec{F}$ has three components F_{P_1}, F_{P_2}, and F_{P_3}, such as

$$\delta W = \sum_{i=1}^{3} F_{P_i} \delta_{P_i} \tag{2.38}$$

Each component of force is expressed with Newton's law as

$$F_{P_i} = m\ddot{p}_i \tag{2.39}$$

Equating δW in Eq. (2.38) we have

$$\delta W = \sum_{i=1}^{3} m\ddot{p}_i \delta_{P_i} = \sum_{i=1}^{3} F_{P_i} \delta_{P_i} \tag{2.40}$$

Substituting Eq. (2.27) in Eq. (2.40) we get

$$\delta W = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} m\ddot{p}_i \frac{\partial_{P_i}}{\partial q_j}\right)\delta_{q_j} = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} F_{P_i} \frac{\partial_{P_i}}{\partial_{q_j}}\right)\delta_{q_j} \tag{2.41}$$

Let

$$F_{q_j} = \sum_{i=1}^{3} F_{P_i} \frac{\partial_{P_i}}{\partial_{q_j}} \tag{2.42}$$

Equation (2.41) becomes

$$\delta W = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} m\ddot{p}_i \frac{\partial_{P_i}}{\partial q_j}\right)\delta_{q_j} = \sum_{j=1}^{3} F_{q_j} \delta_{q_j} \tag{2.43}$$

Now by substituting Eq. (2.36) in Eq. (2.43) we get

$$\delta W = \sum_{j=1}^{3} m\left(\sum_{i=1}^{3}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right) - \frac{\partial}{\partial q_j}\left(\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right)\right)\delta_{q_j} = \sum_{j=1}^{3} F_{q_j}\delta_{q_j} \tag{2.44}$$

By definition of kinetic energy of a particle of mass m is given as

$$K = \frac{m}{2}\sum_{i=1}^{3}\dot{p}_i^2 \tag{2.45}$$

There are three Lagrange equations which generate from Eq. (2.44) for each axis $(j = 1, 2, 3)$ as

$$m\left(\sum_{i=1}^{3}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right) - \frac{\partial}{\partial q_j}\left(\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right)\right) = F_{q_j} \tag{2.46}$$

Each value of j represents a Lagrange equation in its axis. The purpose of keeping δW in Eq. (2.44) is that we need to equate change in energy through work done by other physical interpretations. The Lagrange equation deals with the kinetic energy of an object in motion in Eq. (2.44). Work done is also represented by change in potential energy as given in Eq. (2.8). There may be non-conservative forces that are acting on the body or internal energy of the system, but due to the law of conservation of energy these forces must balance each other. The negative gradient of potential energy provides forces acting upon the body which may be causing it to move or stop. So if we define potential energy U in Cartesian coordinates, forces in Cartesian coordinates are represented as

$$F_{P_i} = -\frac{\partial U}{\partial p_i} \tag{2.47}$$

Equation (2.42) represent as

$$F_{q_j} = -\sum_{i=1}^{3}\left(\frac{\partial U}{\partial p_i}\right)\frac{\partial_{P_i}}{\partial_{q_j}} = -\frac{\partial U}{\partial q_j} \tag{2.48}$$

Now we write Eq. (2.46) as a Lagrange equation in q_j coordinates

$$m\left(\sum_{i=1}^{3}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right) - \frac{\partial}{\partial q_j}\left(\left(\frac{\dot{p}_i^{\,2}}{2}\right)\right)\right) = -\frac{\partial U}{\partial q_j} \tag{2.49}$$

Again using the definition of kinetic energy from Eq. (2.45) to represent Eq. (2.49) in energy variables

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} = -\frac{\partial U}{\partial q_j} \tag{2.50}$$

Kinetic energy is a function of both position and velocity of a particle in generalized coordinate q_j and the potential energy is only a function of generalized position coordinates,

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \left(\frac{\partial K}{\partial \dot{q}_j}\right) \text{ and } \frac{\partial L}{\partial q_j} = \frac{\partial U}{\partial q_j} \tag{2.51}$$

Now using a Lagrange variable L from Eq. (2.14) in three axes system and using Eq. (2.51) in each coordinates we get the following equation for a conservative system in each q_j coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{2.52}$$

In a conservative field Eq. (2.7) holds for change in both potential and kinetic energy, but in a non-conservative field the change is also due to non-conservative, external, or internal forces in each coordinate system. E_{nc} is a non-conservative energy due to all of these forces acting upon the body, so the energy equation becomes

$$\Delta K + \Delta U = E_{\mathrm{nc}} \tag{2.53}$$

In this case the Lagrange equation (2.52) does not satisfy the due non-conservative force acting upon a body. Let $F_{\mathrm{nc}q_j}$ is the sum of all non-conservative forces acting in q_j direction. The generalized Lagrange equation in generalized coordinates is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = F_{\mathrm{nc}q_j} \tag{2.54}$$

Depending upon the space in which coordinates are defined, the value of j can be changed. For a motion with two degrees of freedom there will be only two Lagrange equations but in a motion with three degrees of freedom there will be three Lagrange equations to describe the system. In certain cases, there is motion in both Cartesian coordinates and generalized coordinates; then accordingly a direction of motion in Cartesian is also treated as a variable q_j.

2.4 State Space Representation

The Lagrange equation itself is a nonlinear or linear state space representation. All non-conservative forces are treated as input to the system whether these are controllable or exogenous. The generalized coordinates and their first-order derivatives constitute the state vector.

Example 2.1: A Simple Pendulum The simple pendulum is a body with mass m attached at length l of massless string from the origin (or ground) to move between two end points as shown in Fig. 2.1. If no external force is applied then the movement will depend upon the starting point and eventually comes to end at mass exactly below the origin. In Cartesian coordinates x and y mass move in a semicircle, which can also be related through polar coordinates l and θ, which are fixed length of a string and angle from the origin respectively. There is only one degree of freedom, so there is only one generalized coordinate $q_1 = \theta$. Conventionally these simple examples are represented with θ instead of q_1. Now we define the position and velocity of Cartesian coordinate as a function of a generalized coordinate as

$$\begin{aligned} x(\theta) &= l\ \sin\theta \\ y(\theta) &= -l\ \cos\theta \end{aligned} \tag{2.55}$$

$$\begin{aligned} \dot{x}\left(\theta, \dot{\theta}\right) &= l\dot{\theta}\cos\theta \\ \dot{y}\left(\theta, \dot{\theta}\right) &= l\dot{\theta}\sin\theta \end{aligned} \tag{2.56}$$

The kinetic energy of the mass m in Cartesian coordinate is given as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}ml^2\dot{\theta}^2 \tag{2.57}$$

The potential energy U is due to gravitational pull and it is lowest when $\theta = 0$ or where kinetic energy is highest and vice versa when $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. So

$$U = mgl(1 - \cos\theta) \tag{2.58}$$

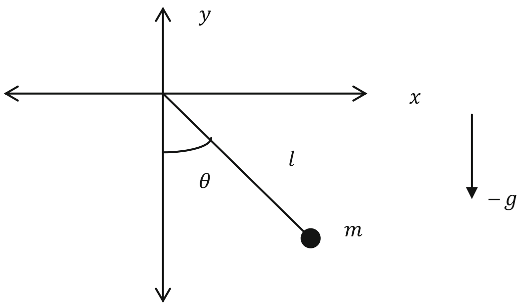


Fig. 2.1 A simple pendulum

The Lagrangian $L = K - U$ is given as

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta) \tag{2.59}$$

The Lagrange equation of the system derives as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{2.60}$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - mgl\sin\theta = 0 \tag{2.61a}$$

$$\left(ml^2\ddot{\theta}\right) + mgl\sin\theta = 0 \tag{2.61b}$$

The final equation with one degree of freedom now appears as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2.62}$$

State space formulation of the system represent Eq. (2.62) with definition of state variables given as $x_1 = \theta$ and $x_2 = \dot{\theta}$. We have state vector $\vec{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ and the nonlinear state space representation as

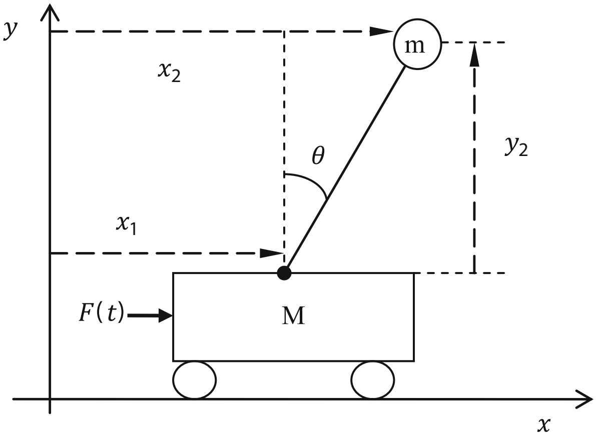
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned} \tag{2.63}$$

The system can be linearized at relaxed equilibrium point $\begin{bmatrix} \theta_e & \dot{\theta}_e \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$ with only one matrix $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$

Example 2.2: Inverted Pendulum on a Moving Cart A cart of mass M is moving by an applied force F acting along x-axis. An inverted pendulum of mass m is attached on the center of cart with a rod of length l (Fig. 2.2).

The motion of the cart is defined from origin and the angle is measured from vertical y-axis. The system can either be represented with horizontal and vertical variables of cart and bob or by using horizontal displacement of cart and angular position of bob. There are only two degrees of freedom; an angle can be measured from the horizontal but in order to be conversant with remaining literature we chose the angle θ measured from y-axis. So for 2-DoF system, there are two generalized coordinates $x_1 = x$, the cart's displacement and θ the angle of a bob from vertical. The position of bob (x_2, y_2) can be given in terms of x_1 and θ as

Fig. 2.2 Inverted pendulum on a moving cart



$$\begin{aligned} x_2 &= x + l \cdot \sin\theta \\ y_2 &= l \cdot \cos\theta \end{aligned} \tag{2.64}$$

The velocity components of bob are given as

$$\begin{aligned} \dot{x}_2 &= \dot{x} + l \cdot \dot{\theta} \cdot \cos\theta \\ \dot{y}_2 &= -l \cdot \dot{\theta} \cdot \sin\theta \end{aligned} \tag{2.65}$$

The kinetic energies of both masses are given as

$$K_1 = \frac{1}{2} M \dot{x}^2 \tag{2.66}$$

$$K_2 = \frac{1}{2} m \left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right) \tag{2.67a}$$

$$K_2 = \frac{1}{2} m \left[\left(\dot{x} + l \cdot \dot{\theta} \cdot \cos\theta\right)^2 + \left(-l \cdot \dot{\theta} \cdot \sin\theta\right)^2\right] \tag{2.67b}$$

$$K_2 = \frac{1}{2} m \left(\dot{x}^2 + 2l\dot{x}\dot{\theta} \cdot \cos\theta + l^2\dot{\theta}^2\right) \tag{2.67c}$$

So the total kinetic energy of the system from Eqs. (2.66) and (2.67c) is given as

$$K = K_1 + K_2 = \frac{1}{2}(M + m)\dot{x}^2 + ml\dot{x}\dot{\theta} \cdot \cos\theta + \frac{1}{2} ml^2\dot{\theta}^2 \tag{2.68}$$

The potential energy in a bob is given as

$$U = mgy_2 = mgl\cos\theta \tag{2.69}$$

The Lagrangian L is given as

$$L = K - U = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta} \cdot \cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{2.70}$$

we represent two equations of motion sing the variable L and two generalized coordinates (x, θ) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F \tag{2.71}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{2.72}$$

From Eqs. (2.70), (2.71), and (2.72) are solved as

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta} \cdot \cos\theta \tag{2.73a}$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x} \cdot \cos\theta + ml^2\dot{\theta} \tag{2.73b}$$

$$\frac{\partial L}{\partial x} = 0 \tag{2.73c}$$

$$\frac{\partial L}{\partial \theta} = mgl\sin\theta \tag{2.74}$$

So equations of motion are

$$(M+m)\ddot{x} + ml\ddot{\theta} \cdot \cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{2.75}$$

$$ml\ddot{x} \cdot \cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta} - mgl\sin\theta = 0 \tag{2.76}$$

These are nonlinear equations for the system of a moving cart with an inverted pendulum. In order to make a state space system, first we need to solve variables with double derivatives simultaneously. Each generalized coordinate of the system is second-order so a total order of the system is 4. We have both $\ddot{x}$ and $\ddot{\theta}$, so we need to solve these equations simultaneously. Eqs. (2.75) and (2.76) are represented as

$$ml^2\ddot{\theta} = \left(\frac{l}{\cos\theta}\right)\left[F - (M+m)\ddot{x} + ml\dot{\theta}^2\sin\theta\right] \tag{2.77}$$

$$\ddot{x} = \dot{x}\dot{\theta}\tan\theta - \frac{l\ddot{\theta}}{\cos\theta} + g\tan\theta \tag{2.78}$$

Inserting value of $\ddot{x}$ from Eq. (2.78) in Eq. (2.75) and then inserting the value of $ml^2\ddot{\theta}$ from Eq. (2.77) in Eq. (2.76) simultaneously solve the equations for state space representation of a state vector $\begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$ as

$$\ddot{x} = \left[m\cos\theta - \left(\frac{M+m}{\cos\theta}\right)\right]^{-1}\left(m\dot{x}\dot{\theta}\,\sin\theta + mg\sin\theta + ml\dot{\theta}^2\tan\theta + \frac{F}{\cos\theta}\right) \tag{2.79}$$

$$\ddot{\theta} = \left(\frac{1}{l}\right) \times \left[-\frac{M}{\cos\theta} - \frac{m}{\cos\theta} + m\cos\theta\right]^{-1}\left(-(M+m)\left(\dot{x}\dot{\theta}\,\tan\theta + g\tan\theta\right) + ml\dot{\theta}^2\sin\theta + F\right) \tag{2.80}$$

Problems

P2.1 Consider spherical coordinates of mass m in motion

$$\begin{aligned} x &= r\cos\theta\sin\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\varphi \end{aligned}$$

Formulate Lagrangian equation of motion in three axis for three generalized coordinates and represent as state space model.

P2.2 A pendulum given in Example 2.1 is rotating with angular velocity ω in a circular path of radius r. The motion is under the influence of gravity with no external forces acting upon it. The Cartesian coordinates are given as

$$\begin{aligned} x(t) &= r\cos(\omega t) + l\cos\theta \\ y(t) &= -r\sin(\omega t) + l\sin\theta \end{aligned}$$

Find the state space representation of a system through Lagrangian formulation.

P2.3 Define a generalized coordinate system for the problem given in Example 2.2 and represent its nonlinear state space formulation. Use the symbolic math toolbox to obtain Jacobian matrices A and B given in Eq. (1.19) and Eq. (1.20) by linearizing at relaxed position.

P2.4 Nonlinear systems are also linearized by approximating nonlinear terms within a specified range on equilibrium point. The inverted pendulum on a cart can be linearized by approximating a small angle and negligible velocities at equilibrium point for trigonometric functions i.e., $\sin\theta \approx \theta$, $\cos\theta \approx 1$. Find a linear state space equation by these assumptions and obtain a state space representation of the inverted pendulum on a cart system.

P2.5 The position of bob of the inverted pendulum on a moving cart can be controlled by applying external torque τ at the hinge changing the Eq. (2.78) as

$$ml\ddot{x} \cdot \cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta} - mgl\sin\theta = \tau$$

Linearize the complete system by the assumption given in P2.4 and obtain state space representation to monitor all state variables at output independently.

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