On Definitions of Bandwidth and Response Time

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***Abstract*—Centroidal definitions of bandwidth W and response time T of a linear system are proposed. These are easily calculated by solution of Lyapunov equations, and agree quite well with intuitive definitions for low order systems, as demonstrated by several examples.**

I. Introduction

Qualitative measures of performance of dynamic systems are useful for establishing standards and comparing systems. Among the qualitative performance standards are *bandwidth* and *response time*. Several definitions of these performance measures, are the "3 dB" or "half-power" bandwidth, and the "rise-time". The former is defined as the frequency at which the attenuation of the system, relative to the response at dc, is 3 dB. The latter is defined as the time it takes for the step-response of the system to reach to within $1/e$th of its steady-state value. For a first-order linear system having the transfer function

$$\mathsf{H}(s) = \frac{1}{\tau s + 1}$$

and impulse response

$$h(t) = e^{-t/\tau}$$

the bandwidth is $1/\tau$ and the rise time is τ, so that the product of bandwidth and rise-time is unity. The reciprocal relationship between 3-dB bandwidth and rise-time is empirically observed to obtain for linear systems in general, but no general relationship between them has been determined. Another shortcoming of these definitions is that they are not readily calculated from the matrices of the state-space representation of the system, and generally must be determined by numerical methods or simulation.

The definitions of bandwidth and response time proposed here overcome these limitations.

II. Definitions

In signal processing, the time-duration T and frequency-spread W of a signal $x(t)$ are often defined by

$$T^2 = \frac{\int_{-\infty}^{\infty} t^2 x^2(t)dt}{\int_{-\infty}^{\infty} x^2(t)dt} \tag{1}$$

$$W^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \tag{2}$$

where $X(j\omega)$ is the Fourier transform of of $x(t)$. The Fourier transform of the derivative of $x(t)$ is

$$\mathcal{F}\{\dot{x}(t)\} = j\omega X(j\omega)$$

By use of Parseval's theorem,

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

the frequency spread can be expressed in the time domain:

$$W^2 = \frac{\int_{-\infty}^{\infty} (\dot{x}(t))^2 dt}{\int_{-\infty}^{\infty} x^2(t)dt} \tag{3}$$

In this note we propose adopting these definitions for the bandwidth of a single-input, single-output (SISO) linear system having an impulse response $h(t)$. Since, as generally assumed, $h(t) = 0$, for $t < 0$, the integrals in the definitions start at $t = 0$. Thus we offer the following definitions of *bandwidth* W and *response time* T:

$$W^2 = \frac{\int_0^{\infty} (\dot{h}(t))^2 dt}{\int_0^{\infty} h^2(t)dt} \tag{4}$$

$$T^2 = \frac{\int_0^{\infty} t^2 h^2(t)dt}{\int_0^{\infty} h^2(t)dt} \tag{5}$$

More generally, when the system has one input, but multiple outputs (SIMO), we can define the set of response times and bandwidth from the input to each output, or obtain a scalar measure of the system response time and bandwidth by replacing $h^2(t)$ in the definitions by $h'(t)h(t)$. Similarly, for a multiple-input, single-output (MISO) system, we can obtain a scalar measure of of system bandwidth and response time by replacing $h^2(t)$ by $h(t)h'(t)$. For a single-input. single-output system, we can write either $h^2(t) = h'(t)h(t)$ or $h^2(t) = h(t)h'(t)$, which, as shown below, results in different state-space formulas.

III. State-space formulas

The definitions (4), (5) apply to any linear, tine-invariant system. But, for linear, time-invariant systems represented in state-space notation, computationally more convenient formulas apply. For the state-space representation

$$\dot{x} = Ax + Bu \tag{6}$$

$$y = Cx \tag{7}$$

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the impulse response (matrix) is given by

$$h(t) = Ce^{At}B \tag{8}$$

and

$$\dot{h}(t) = CAe^{At}B \tag{9}$$

Hence, for a SIMO system, on using $h(t)h'(t)$ in (4,5) we obtain

$$W^2 = \frac{CAPA'C'}{CPC'} \tag{10}$$

$$T^2 = \frac{CP_2C'}{CPC'} \tag{11}$$

with

$$P = \int_0^\infty e^{At}BB'e^{A't}dt \tag{12}$$

$$P_2 = \int_0^\infty t^2 e^{At}BB'e^{A't}dt \tag{13}$$

P recognized as the *controllability Grammian* of the system.

Similarly, for an MISO system, on using $h'(t)h(t)$ in (5),(4) we obtain

$$W^2 = \frac{B'A'MAB}{B'MB} \tag{14}$$

$$T^2 = \frac{B'M_2B}{B'MB} \tag{15}$$

where

$$M = \int_0^\infty e^{A't}C'Ce^{At}dt \tag{16}$$

$$M_2 = \int_0^\infty t^2 e^{A't}C'Ce^{At}dt \tag{17}$$

M recognized as the *observability Grammian* of the system.

It is well known that the integrals appearing in the definitions (12) and (16) are solutions algebraic Lyapunov equations:

$$AP + PA' + BB' = 0$$
$$MA + A'M + C'C = 0$$

It is shown (Appendix 1) that P_2 and M_2 can also be be determined by solving Lyapunov equuations: namely

$$AP_1 + P_1A + P = 0$$
$$AP_2 + P_2A + 2P_1 = 0$$

or

$$M_1A + A'M_1 + M = 0$$
$$M_2A + A'M_2 + 2M_1 = 0$$

A Matlab implementation of the state-space calculation is given in Appendix 2.

IV. Properties

A. Invariance to change of state variables and scaling

Since the definitions of bandwidth and response time depend only on the impulse response of the system, which is independent of the choice of state variables that define the system, the definitions of these quantities are independent of any change of state variables.

Moreover, since the defining integrals contain $h^2(t)$ in both the numerator and the denominator, the definitions are invariant to a scalar multiple of the impulse response, or to a scalar multiple of the B matrix, or of the C matrix, or both.

B. Time scaling

If the impulse response $h(t)$ is scaled in time, so that the new impulse response is

$$h_1(t) = h(\omega t)$$

it is straightforward to show using the definitions (14) and (15) that the new response time T_1 and bandwidth are given by

$$T_1 = \omega T, \quad W_1 = W/\omega$$

where T and W are the response time and bandwidth of the original (unscaled) system.

Because of this property, there is no loss in generality in time-scaling the impulse response of a system such that its bandwidth is unity.

C. Uncertainty principle

By the use of the Schwarz inequality, it can be established [1] that

$$WT \geq 1/2$$

As shown in Example A below, the minimum WT product of 1/2 is achieved for a gaussian pulse, which is a well-known result in the signal processing field. The WT product might serve as a figure of merit, as well, for a feedback control system.

V. Examples

A. Gaussian pulse

The "gaussian" pulse, although not realizable by a system of finite order, has been of considerable theoretical interest. It is expressed as the familiar "bell curve"

$$h(t) = e^{-t^2}$$

We find that

$$\int_0^\infty h^2(t)dt = \int_0^\infty e^{-2t^2}dt = \sqrt{\pi/8}$$

$$\int_0^\infty \dot{h}^2(t)dt = \int_0^\infty 4te^{-2t^2}dt = \sqrt{\pi/8}$$

$$\int_0^\infty t^2h^2(t)dt = \int_0^\infty t^2e^{-2t^2}dt = \sqrt{\pi/32}$$

so that the bandwidth and response time are

$$W = 1, \quad T = \frac{1}{2}$$

Thus, in accordance with the uncertainty principle, the gaussian pulse has the shortest possible response time of all systems having the same bandwidth. In a sense the gaussian pulse is a paradigm of performance measure.

B. First-order system

A simple first-order system has the (normalized) impulse response

$$h(t) = e^{-t}$$

for which

$$\int_0^\infty h^2(t)dt = \int_0^\infty e^{-2t}dt = 1/2$$

$$\int_0^\infty \dot{h}^2(t)dt = \int_0^\infty e^{-2t}dt = 1/2$$

$$\int_0^\infty t^2h^2(t)dt = \int_0^\infty t^2e^{-2t}dt = 1/4$$

so that

$$W = 1, \quad T = \frac{1}{\sqrt{2}}$$

We observe that by this definition the response time of a first-order system is $\sqrt{2}$ times as large as that of the gaussian pulse. A graphical comparison of the gaussian pulse and the first-order system response would show that the former is more concentrated near the origin and a more rapidly vanishing tail, which accounts for its significantly shorter response time.

It is also noted that the response time of a system is often defined as the "time-constant", which is the time it takes for $h(t) = 1/e$, which, for a the first-order system with $W = 1$, and also the magnitude of slope of $h(t)$ at $t = 0$. happens to be 1.0. So the response time defined here is somewhat less than the time constant of the system.

C. Second-order system

The "standard" form of the transfer function of a generic second-order system is frequently given [2] as

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2}$$

for ω is normalized to 1, a state space representation of the system having this transfer function has the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Using these matrices and performing the calculations for W and T as given by (10) and (11), we find the bandwidth and response time to be

$$W = 1, \quad T = \sqrt{\frac{4\zeta^4 + \zeta^2 + 1}{2\zeta^3 + \zeta}}$$

A plot the expression for the response time is shown in Figure 1. It is observed that the minimum response time results when the damping factor ζ is about 0.7. These results for the bandwidth and response time are consistent with accepted practice in which the a damping factor of about 0.7 is generally regarded as a reasonable compromise between speed of response and low overshoot.

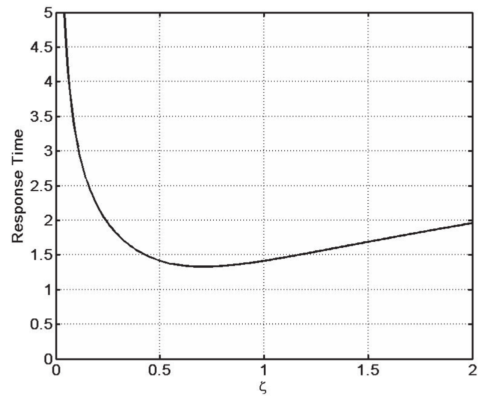

Fig. 1. Response time of generic second-order system

VI. Conclusions

The centroidal definitions of bandwidth and response time proposed in Section II comport quite well with the empirical definitions in customary use and they have several advantages:

- They can be readily computed using the matrices A, B, and C of a state space representation of a system. Hence, they can be computed in the design stage of a control system, for example, without the need to create a Bode plot to determine the bandwidth, or a simulation to determine the response time.
- The uncertainty principle $WT \geq 1/2$ can be the basis of a figure of merit for comparing systems: there may be some justification in claiming that the better of two systems is the one with a smaller TW product.

It would be of interest to examine the bandwidth and response time of composite systems: systems in tandem (series), systems in parallel, and feedback systems.

Acknowledgments

The author is grateful to Dr. Zoran Gajic, of the Department of Electrical and Computer Engineering at Rutgers University, co-author of an important text [3] on the Lyapunov equation. He provided the derivation given below in response to an email inquiry.

The subject of bandwidth of a system represented in state-space form was a topic of discussion between the author and the late Dr. Robert W. Bass, and was one of the motivations of the present note.

References

[1] M. Vetterli and J.Kovacevic, "Wavelets and Subband Coding", Prentice Hall PTR, Englewood Cliffs, NJ, 1995.

[2] C.L. Phillips and R.D. Harbor, "Feedback Control Systems," Prentice Hall, Uppersaddle River, NJ, 2000.

[3] Z. Gajic and M. Qureshi, "Lyapunov Matrix Equation in System Stability and Control," Academic Press, Mathematics in Science and Engineering Series, San Diego, 1995.

APPENDIX 1 – LYAPUNOV EQUATIONS FOR INTEGRALS IN DEFINITIONS.

The derivations of this section were provided by Dr. Zoran Gajic of Rutgers University, to whom the author is most grateful.

Consider

$$\frac{d}{dt}(te^{At}Qe^{A't} = e^{At}Qe^{A't} + tAe^{At}Qe^{A't} + te^{At}Qe^{A't}A' \quad (*)$$

by the chain rule.

Integrate both sides of (*) between 0 and ∞ to obtain

$$\begin{aligned}\int_0^\infty \frac{d}{dt}(te^{At}Qe^{A't}dt &= \int_0^\infty e^{At}Qe^{A't}dt \\ &+ A\int_0^\infty te^{At}Qe^{A't}dt + \int_0^\infty te^{At}Qe^{A't}dtA' \\ &= P + AP_1 + P_1A' \qquad (**)\end{aligned}$$

where

$$P = \int_0^\infty e^{At}Qe^{A't}dt$$

$$P_1 = \int_0^\infty te^{At}Qe^{A't}dt$$

The left-hand side of (**), by the fundamental theorem of calculus is

$$te^{At}Qe^{A't}|_0^\infty = 0$$

(assuming A is a stability matrix, of course). Thus we have

$$AP_1 + P_1A' + P = 0$$

It is well-known that

$$AP + PA' + Q = 0$$

Finally, consider

$$\frac{d}{dt}(t^2e^{At}Qe^{A't}) = 2te^{At}Qe^{A't} + t^2Ae^{At}Qe^{A't} + t^2e^{At}Qe^{A't}A' \quad (*)$$

Proceeding as before we obtain

$$AP_2 + P_2A' + 2P_1 = 0$$

where

$$P_2 = \int_0^\infty t^2e^{At}Qe^{A't}dt$$

APPENDIX 2 – MATLAB FUNCTION TO CALCULATE BANDWIDTH AND RESPONSE TIME

```
function [T,W]=respband(A,B,C)
P=lyap(A,B*B');
P1=lyap(A,P);
P2=lyap(A,2*P1);
numer=C*P2*C';
denom=C*P*C';
T=sqrt(numer/denom);
top=C*A*P*A'*C';
W=sqrt(top/denom);
end
```