LULEÅ UNIVERSITY OF TECHNOLOGY
Dept. of Computer Science and Electrical Engineering
EISLAB

Course	SME101
Date	2005-12-20
Time	9.00–13.00

Exam in: **Measurement Systems**
Teacher: Johan Carlson, Ext. (ankn.). 2517
Problems: 5 (5 points per problem)
Tools allowed: BETA (Mathematics Handbook), Physics handbook,
Language dictionary, calculator
Text book: Principles of Measurement Systems, by John Bentley

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Solutions

1. A thermocouple sensor has an electromotive force (e.m.f.) in μV.

$$E(T) = 38.74T + 3.319 \times 10^{-2}T^2 + 2.071 \times 10^{-4}T^3 - 2.195 \times 10^{-6}T^4,$$

for the range 0 to 400 °C. For $T = 0$ °C, $E(T) = 0$ μV and for $T = 400$ °C, $E(T) = 20\,869$ μV.

(a) Calculate the expression for the ideal straight line relationship, $E(T) = K \cdot T$.
Solution: According to the text book, the solution should be...

$$E(400) = 20869 \times 10^{-6} \text{ V} = 400 \cdot K$$
$$\Longrightarrow K = \frac{20869}{400} \text{ V/°C} \approx 52.17 \ \mu\text{V/°C}.$$

This gives the equation for the ideal straight line as:

$$E_L(T) = 52.17 \cdot T$$

(in μV).
However, the example in the book is wrong, so a better solution would be based on

$$E(400) = -22131 \times 10^{-6} \text{ V},$$

which gives

$$K = \frac{-22131}{400} = -55.328 \ \mu\text{V/°C}$$

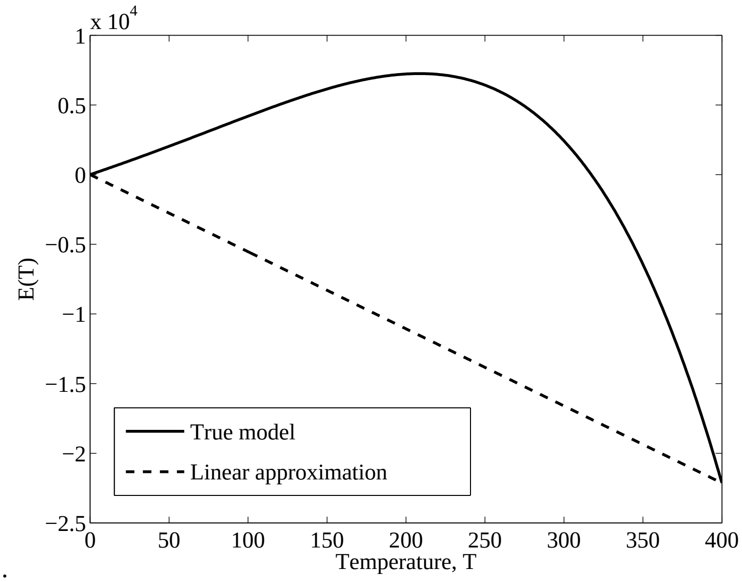
(b) Determine the sensitivity of the sensor.
Solution: Sensitivity is defined as $\frac{dE(T)}{dT}$, which gives

$$\frac{dE(T)}{dT} = \frac{d}{dT}\left(38.74T + 3.319 \times 10^{-2}T^2 + 2.071 \times 10^{-4}T^3 - 2.195 \times 10^{-6}T^4\right) =$$
$$\approx 38.74 + .06638T + 6.213 \times 10^{-4}T^2 - 8.78 \times 10^{-6}T^3,$$

which is temperature dependent.

(c) Determine the maximum non-linearity of the system, as function of the full-scale (400 °C).

Solution: The non-linearity is given by $E(T) - E_L(T)$, where $E_L(T)$ is the ideal straight line from exercise (1a),



i.e.

$$N(T) = E(T) - E_L(T) =$$
$$= 38.74T + 3.319 \times 10^{-2}T^2 + 2.071 \times 10^{-4}T^3 - 2.195 \times 10^{-6}T^4 + 55.328 \cdot T$$

The derivative w.r.t. T is then

$$\frac{d}{dT}\left(38.74T + 3.319 \times 10^{-2}T^2 + 2.071 \times 10^{-4}T^3 - 2.195 \times 10^{-6}T^4 + 52.17 \cdot T\right)$$
$$= 90.91 + .06638T + 6.213 \times 10^{-4}T^2 - 8.78 \times 10^{-6}T^3,$$

Setting to zero and solving we get the maximum nonlinearity at $T = 256.98$ °C, which gives the maximum non-linearity as percentage of the full-scale:

$$\frac{E(256.98) - E_L(256.98)}{E(400)} = \times 100\% = -91.8\%$$

2. Two strain gauges are bonded onto a cantilever as shown in the figure below. Given that the gauges are placed halfway along the cantilever and the cantilever is subject to a downward force F. Use the tabulated data below to:

Cantilever data		*Strain gauge data*	
Length	$l = 25$ cm	Gauge factor	$G = 2.1$
Width	$w = 6$ cm	Unstrained resistance	$R_0 = 120\ \Omega$
Thickness	$t = 3$ mm		
Young's modulus	$E = 70 \times 10^9$ Pa		

(a) Calculate the resistance of each strain gauge for $F = 0.5\ N$ and $F = 10\ N$.
Solution:

$$\Delta R = GR_0 e,$$

where (from [8.57])

$$e = \frac{6\,(l - x)}{wt^2 E} F,$$

and $x = l/2 = 12.5 \times 10^{-2}$ m. That is, with all the numbers put in

$$e = \frac{6\,(12.5 \times 10^{-2})}{6 \times 10^{-2} \cdot (3 \times 10^{-3})^2 \cdot 70 \times 10^9} \cdot F \approx 1.984\,1 \times 10^{-5} F.$$

So, for $F = 0.5$ N, we get

$$\begin{aligned} R_1 &= R_0\,(1 + G \cdot e) = 120\,\bigl(1 + 2.1 \cdot 1.984\,1 \times 10^{-5} \cdot 0.5\bigr) = 120.002\,5\ \Omega \\ R_2 &= R_0\,(1 - G \cdot e) = 119.997\,5\ \Omega. \end{aligned}$$

Similarly, for $F = 10$ N, we get

$$\begin{aligned} R_1 &= 120\,\bigl(1 + 2.1 \cdot 1.984\,1 \times 10^{-5} \cdot 10\bigr) = 120.05\ \Omega \\ R_2 &= 120\,\bigl(1 - 2.1 \cdot 1.984\,1 \times 10^{-5} \cdot 10\bigr) = 119.95\ \Omega \end{aligned}$$

(b) Design a resistive deflection bridge suitable for force measurements in the interval in (a). Motivate your design choices clearly.
Solution: See text book or lab.

3. A resistive sensing element with resistance R_p is connected to a deflection bridge according to the figure below. The supply voltage was measured 10 times and stored in the vector

$$\mathbf{V}_s = \left[\begin{array}{cccccccccc} 5.02 & 4.99 & 5.0 & 4.98 & 5.01 & 4.99 & 5.01 & 5.0 & 5.01 & 4.98 \end{array}\right]^T.$$

Assume that $\sigma_{R_p} = 0.1$ and $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 0.05$, and that $\overline{R}_1 = \overline{R}_2 = \overline{R}_3 = 200$ Ω.

(a) Estimate the mean $\overline{V}_s$ and the standard deviation σ_{V_s} of the supply voltage.
Solution: The mean is given by

$$\overline{V}_s = \frac{1}{N} \sum_{n=1}^{N} V_s\,[n] = \frac{1}{10} \sum_{n=1}^{10} V_s\,[n] = 4.990$$

The standard deviation is given by

$$\sigma_{V_s} = \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} \left(V_s\,[n] - \overline{V}_s\right)^2} \approx 0.0137$$

Note that the book uses $\frac{1}{N}$ even for standard deviation. This is WRONG, especially for small N.

(b) Derive an expression for the total variance, σ_E^2 of E, as a function of the resistance R_p, where

$$E = V_s\left(\frac{R_1}{R_1+R_p} - \frac{R_2}{R_2+R_3}\right).$$

Solution: The total variance is given by

$$\sigma_E^2 = \left(\frac{\partial E}{\partial V_s}\right)^2 \sigma_{V_s}^2 + \left(\frac{\partial E}{\partial R_1}\right)^2 \sigma_{R_2}^2 + \left(\frac{\partial E}{\partial R_2}\right)^2 \sigma_{R_2}^2 + \left(\frac{\partial E}{\partial R_3}\right)^2 \sigma_{R_3}^2 + \left(\frac{\partial E}{\partial R_p}\right)^2 \sigma_{R_p}^2$$

The partial derivatives are:

$$\begin{aligned}
\left(\frac{\partial E}{\partial V_s}\right)^2 &= \left(\frac{R_1}{R_1+R_p} - \frac{R_2}{R_2+R_3}\right)^2 \\
\left(\frac{\partial E}{\partial R_1}\right)^2 &= \left(V_s\frac{R_p}{(R_1+R_p)^2}\right)^2 \\
\left(\frac{\partial E}{\partial R_2}\right)^2 &= \left(-V_s\frac{R_3}{(R_2+R_3)^2}\right)^2 \\
\left(\frac{\partial E}{\partial R_3}\right)^2 &= \left(V_s\frac{R_2}{(R_2+R_3)^2}\right)^2 \\
\left(\frac{\partial E}{\partial R_p}\right)^2 &= \left(-V_s\frac{R_1}{(R_1+R_p)^2}\right)^2.
\end{aligned}$$

(c) Looking at a $2\sigma_E$, how much of the total variation does the supply voltage variation account for, when $\overline{R}_p = 150\ \Omega$?

Solution: Putting in numerical values, we get

$$\begin{aligned}
\sigma_E^2 = &\left(\frac{200}{200+150} - \frac{200}{200+200}\right)^2 (0.0137)^2 + \left(4.99\frac{150}{(200+150)^2}\right)^2 0.0025+ \\
&+\left(-4.99\frac{200}{(200+200)^2}\right)^2 0.0025 + \left(4.99\frac{200}{(200+200)^2}\right)^2 0.0025+ \\
&+\left(-4.99\frac{200}{(200+150)^2}\right)^2 0.01 \\
&\approx 1.252\,1\times 10^{-6} \\
&\implies 2\sigma_E = 2\sqrt{1.252\,1\times 10^{-6}} \approx 2.2380\times 10^{-3}
\end{aligned}$$

This corresponds to a 95 % confidence interval for the output, E. The contribution from V_s is

$$\left(\frac{\partial E}{\partial V_s}\right)^2 \sigma_{V_s}^2 = \left(\frac{R_1}{R_1+R_p} - \frac{R_2}{R_2+R_3}\right)^2 \sigma_{V_s}^2$$
$$= \left(\frac{200}{200+150} - \frac{200}{200+200}\right)^2 (0.0137)^2$$
$$\approx 1.6294 \times 10^{-3}$$

As a percentage of the $2\sigma_E$ interval, this is

$$\frac{2\sqrt{\left(\frac{\partial E}{\partial V_s}\right)^2 \sigma_{V_s}^2}}{2\sigma_E} = \frac{2 \cdot \sqrt{9.5760 \times 10^{-7}}}{2.2380 \times 10^{-3}} \times 100\ \% \approx 87.45\ \%.$$

4. A temperature measurement system is given in the figure below, with transfer functions of the individual elements as in the figure.

   (a) Calculate the dynamic error $E(T)$ of the system. Which element in the system is the dominant cause of this error?

   Solution: We want to know the dynamic error, $E(t)$, when the system input is a unit step, $u(t)$. This was unclear in the problem description, but was clarified during the exam. From [4.45] in the text book we have the system Laplace transform

$$\Delta\overline{T}_M(s) = \frac{1}{s}\frac{1}{1+10s}\frac{1}{(1+10^{-4}s)}\frac{1}{(1+1/200s)^2} =$$
$$= \frac{1}{s} - \frac{A}{s+0.1} - \frac{B}{s+10^4} + \frac{Cs+D}{(s+200)^2}.$$

   The error is thus

$$\overline{E}(s) = \Delta\overline{T}_M(s) - \Delta\overline{T}_T(s) = \frac{1}{s} - \frac{A}{s+0.1} - \frac{B}{s+10^4} + \frac{Cs+D}{(s+200)^2} - \frac{1}{s} =$$
$$= -\frac{A}{s+0.1} - \frac{B}{s+10^4} + \frac{Cs+D}{(s+200)^2},$$

   which gives

$$E(t) = -\left[Ae^{-0.1t} + Be^{-10^4 t} + Ee^{-200t}(1+200t)\right].$$

   We see that the B and E terms decay rapidly to zero, while the first term takes about 50 s to decay. Thus, the first block is dominant when it comes to the *dynamic error*.

(b) Assume that the input temperature is varying as

$$T_T(t) = 5\sin(10^{-2}t) + 2\sin(50t).$$

What is the output of the system? Explain this behavior.

Solution: Sinusoidal input gives sinusoidal output, with output

$$|G(j\omega_0)|\sin(\omega_0 t + \arg G(j\omega_0)).$$

In this case, $|G(j\omega)|$, is

$$|G(j\omega)| = \left|\frac{1}{j\omega}\frac{1}{1+10j\omega}\frac{1}{(1+10^{-4}j\omega)}\frac{1}{(1+1/200j\omega)^2}\right|$$

With $\omega = 10^{-2}$, this is

$$= \left|\frac{1}{j10^{-2}}\frac{1}{1+10j10^{-2}}\frac{1}{(1+10^{-4}j10^{-2})}\frac{1}{(1+1/200j10^{-2})^2}\right| \approx 80$$

and with $\omega = 50$ this is

$$\left|\frac{1}{i50}\frac{1}{1+10i50}\frac{1}{(1+10^{-4}i50)}\frac{1}{(1+1/200i50)^2}\right| \approx 4\times 10^{-5}.$$

The same principle is used to calculate the argument. The important part here is, however to realize that the 50 Hz part will almost negligible in the output. This is because the system acts as a low-pass filter. Fast temperature fluctuations will not be captured using this system.

5. An ultrasonic Doppler flowmeter is used to measure the flowrate of particles suspended in a liquid. The setup is shown in the figure below. The transmitting transducer is transmitting a continuous sound wave at $f = 1$ MHz and the velocity of the particles is in the range of 5 to 20 m/s. The angle $\theta = 30°$, and the speed of sound in the fluid is $c = 1485$ m/s.

(a) Explain the working principle of this flowmeter and give an approximate expression for the flow velocity, v, in terms of the Doppler shift Δf. You may assume that only the frequency of the Doppler signal is used, not the amplitude.

Hint: You may also assume that v/c is small.

Solution: The motion will cause a frequency shift of the received signal, which is proportional to the velocity of the fluid. Using the assumptions given in the problem, we can use equation [16.58] in the text book, giving

$$\Delta f = \frac{2f}{c}(\cos\theta)\,v$$
$$\implies v = \frac{c\Delta f}{2f\cos\theta}.$$

(b) What is the minimum sampling frequency required for a analog-to-digital converter (ADC) recording Δf?

Solution: For a flow of 5 m/s, we have

$$\Delta f_5 = \frac{2 \times 10^6}{1485} \cos(30^\circ) \cdot 5 \approx 5831.8 \text{ Hz}$$

For a flow of 20 m/s we have

$$\frac{2 \times 10^6}{1485} \cos(30^\circ) \cdot 20 = 23327 \text{ Hz}$$

The sampling theorem states that we need to sample faster than twice the maxmimum frequency. This means we need to sample faster than $2 \cdot 23327 = 46654$ Hz, i.e. faster than 46.7 kHz.

(c) Mistakes when installing the flowmeter causes the angle to vary randomly around $\theta = 30^\circ$, so that the true angle is Gaussian distributed with mean $\overline{\theta} = 30^\circ$ and standard deviation $\sigma_\theta = 1^\circ$. Determine a 95 % confidence interval (corresponding to 2σ) for the flow velocity, v. What is the maximum error due to this misalignment as percentage of the maximum flow velocity?

Solution: From (a) we have

$$v = \frac{c\Delta f}{2f\cos\theta}.$$

The variance of the measured velocity is

$$\sigma_v^2 = \left(\frac{\partial v}{\theta}\right)^2 \sigma_\theta^2 = \left(\frac{c}{2}\frac{\Delta f}{f\cos^2\theta}\sin\theta\right)^2 \sigma_\theta^2$$

For the numerical values given in the problem, this gives

$$2\sigma_v \approx 0.001\Delta f.$$

The maximum error is at the highest velocity, and with an angular error at one extreme end of the interval. Given that the error is Gaussian, it is unlikely to have angular errors of more than $3\sigma_\theta$ or $4\sigma_\theta$. For 20 m/s and an angle of 34°, we will receive a Δf of

$$\Delta f = \frac{2f}{c}(\cos\theta)\, v = \frac{2 \times 10^6}{1485}(\cos 34^\circ) \cdot 20 = 22331 \text{ Hz},$$

and given that we believe the angle to be 30°, we will estimate the velocity according to

$$v = \frac{c\Delta f}{2f\cos\theta} = \frac{1485 \cdot 22331}{2 \times 10^6 \cos 30^\circ} \approx 19.15 \text{ m/s},$$

thus the error is

$$\text{error} \;= \frac{1485 \cdot 22331}{2 \times 10^6 \cos 30^\circ} - 20 \approx -0.85 \text{ m/s}$$

As a percentage of the full scale velocity this is

$$\frac{-0.85}{20} \times 100 \;\% = -4.\,25 \;\%$$