LULEÅ UNIVERSITY OF TECHNOLOGY
Dept. of Computer Science and Electrical Engineering

Course	E7021E
Date	2009-10-20
Time	15:00-19:00

Exam in:	**Measurement Technology & Uncertainty Analysis**
Attending teacher:	Johan Carlson (070-580 82 52)
Problems:	5 (5 points per problem)
Tools allowed:	BETA (Mathematics Handbook), Physics handbook, Language dictionary, calculator Text books: Principles of Measurement Systems, by John Bentley Introduction to Empirical Model Building and Parameter Estimation, by Johan E. Carlson

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1. A displacement sensor has an input range of 0.0 to 3.0 cm and a supply voltage $V_s = 0.5$ V. Results from a calibration experiment are given in the table below

Displacement x (cm)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Output voltage (mV)	0.0	16.5	32.0	44.0	51.5	55.5	58.0

(a) Calculate the maximum non-linearity as a percentage of the full-scale deflection (f.s.d.), assuming the steady state sensitivity is calculated as in the book, i.e. (2p)

$$K = \frac{O_{MAX} - O_{MIN}}{I_{MAX} - I_{MIN}}$$

Solution: The slope of the straight line is

$$K = \frac{58.0 - 0.0}{3.0 - 0.0} \text{ mV/cm} \approx 19.3 \text{ mV/cm}$$

The intersection is (page 10, below Eq. 2.2)

$$a = O_{MIN} - K \cdot I_{MIN} = 0.0 - \frac{58.0}{3.0} 0.0 = 0.0$$

So, the ideal straight line has the equation

$$O_{IDEAL} = \frac{58}{3} I$$

The non-linearities, given by the outputs in the table are thus

$$N(I) = O(I) - KI,$$

which becomes

Displacement x (cm)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Output voltage (mV)	0.0	16.5	32.0	44.0	51.5	55.5	58.0
Ideal straight line	0	9.67	19.33	29.0	38.67	48.33	58.0
Non-linearity	0.0	6.83	12.67	15.0	12.83	7.17	0.0

We see that the maximum non-linearity is $\widehat{N} = 15.0$, which in terms of the full-scale deflection ($O = 58.0$) is

$$\frac{15}{58 - 0.0} \times 100 \approx 25.9\%$$

(b) The performance of the system can easily be improved by instead fitting the *optimal* straight line, using the principle of least-squares. Give the necessary equations for finding the slope K and the intersection a of the straight line using the data in the table above.
Note: You do not need to solve for the actual numerical values of K and a. (2p)
Solution: Fitting the straight line by the method of least squares means solving the following over-determined system of equations

$$\mathbf{O} = \mathbf{Ia},$$

where

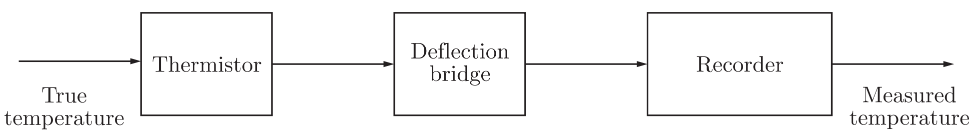
$$\mathbf{O} = \begin{bmatrix} 0 \\ 16.5 \\ 32 \\ 44 \\ 51.5 \\ 55.5 \\ 58 \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \\ 1 & 1 \\ 1 & 1.5 \\ 1 & 2 \\ 1 & 2.5 \\ 1 & 3 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} a \\ K \end{bmatrix}.$$

This gives the least-squares estimate of a and K as

$$\widehat{\mathbf{a}} = \begin{bmatrix} \widehat{a} \\ \widehat{K} \end{bmatrix} = \left(\mathbf{I}^T\mathbf{I}\right)^{-1}\mathbf{I}^T\mathbf{O} \approx \begin{bmatrix} 7.696 \\ 19.393 \end{bmatrix}$$

(c) Explain why the method in (b) gives a better average result.
Solution: Since the principle of least-squares minimizes the sum of squares of the errors over the entire range, the result is better than for the straight line fit in (a), where only two of the calibration points are used. (1p)

2. A temperature measurement system consists of the following elements



where θ is the true temperature and θ_M is the measured temperature (in Kelvin). The model equations and the corresponding uncertainties are given in the table below.

	Thermistor	Bridge	Recorder
Model equations	$R_\theta = K_1 \exp\left(\frac{\beta}{\theta}\right)$	$V_O = V_S\left(\frac{1}{1+\frac{3.3}{R_\theta}} - a_1\right)$	$\theta_M = K_2 V_O + a_2$
Mean values	$\overline{K}_1 = 5 \times 10^{-4}$ kΩ	$\overline{V}_S = -3.00$ V	$\overline{K}_2 = 50.0$ K/V
	$\overline{\beta} = 3 \times 10^3$ K	$\overline{a}_1 = 0.77$	$\overline{a}_2 = 300$ K
Standard deviations	$\sigma_{K_1} = 0.5 \times 10^{-4}$	$\sigma_{V_S} = 0.03$	$\sigma_{K_2} = 0$
	$\sigma_\beta = 0$	$\sigma_{a_1} = 0.01$	$\sigma_{a_2} = 3.0$

(a) Calculate the mean output $\overline{\theta}_M$ and the mean error $\overline{E} = \overline{\theta} - \overline{\theta}_M$ for an input temperature of 320 K. (2p)

Solution: Mean output (at 320 K):

$$\overline{R}_\theta = \overline{K}_1 \exp\left(\frac{\overline{\beta}}{\theta}\right) = 5 \times 10^{-4} \exp\left(\frac{3 \times 10^3}{320}\right) \approx 5.895 \text{ k}\Omega$$

$$\overline{V}_O = V_S\left(\frac{1}{1+\frac{3.3}{R_\theta}} - \overline{a}_1\right) = -3\left(\frac{1}{1+\frac{3.3}{5.895}} - 0.77\right) \text{ V} \approx 0.3867 \text{ V}$$

$$\overline{\theta}_M = \overline{K}_2 \overline{V}_O + \overline{a}_2 = 50.0 \cdot 0.3867 + 300 \text{ K} \approx 319.34 \text{ K}$$

The mean error is then

$$\overline{E} = \overline{\theta} - \overline{\theta}_M = 320 - 319.34 \text{ K } \approx 0.66 \text{ K}$$

(b) Calculate the standard deviation of the output error $\overline{E}$ for an input of 320 K. (3p)

Solution: The standard deviation of the error is the same as the standard deviation of the output, since the standard deviation only depends on the uncertainty of the elements, not the mean values of them. Doing the error propagation calculations we get that

$$\sigma_{R_\theta}^2 = \left(\frac{\partial R_\theta}{\partial K_1}\right)^2 \sigma_{K_1}^2 = \exp\left(\frac{\beta}{\theta}\right)^2 \sigma_{K_1}^2 = \exp\left(\frac{3 \times 10^3}{320}\right)^2 \left(0.5 \times 10^{-4}\right)^2 \approx 0.3475$$

$$\sigma_{V_O}^2 = \left(\frac{\partial V_O}{\partial}\right)^2 \sigma_{R_\theta}^2 + \left(\frac{\partial V_O}{\partial V_s}\right)^2 \sigma_{V_s}^2 + \left(\frac{\partial V_O}{\partial a_1}\right)^2 \sigma_{a_1}^2 =$$

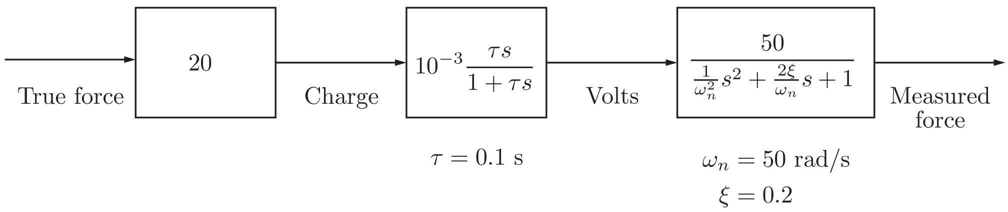
$$= \left(\frac{330 V_s}{(10R_\theta + 33)^2}\right)^2 \sigma_{R_\theta}^2 + \left(\frac{1}{1+\frac{3.3}{R_\theta}} - a_1\right)^2 \sigma_{V_s}^2 + (-V_s)^2 \sigma_{a_1}^2 \approx 0.00257$$

$$\sigma^2_{\theta_M} = \left(\frac{\partial\theta_M}{\partial a_2}\right)^2 \sigma^2_{a_2} + \left(\frac{\partial\theta_M}{\partial V_O}\right)^2 \sigma^2_{V_O} \approx 15.4265$$

This gives the standard deviation of the error as

$$\sigma_{\theta_M} = \sqrt{\sigma^2_{\theta_M}} \approx 3.93 \text{ K}$$

3. A force measurement system consisting of a piezoelectric crystal, charge amplifier and recorder is shown in the figure below.



Calculate the system output and the corresponding dynamic error when the force input signal is (5p)

$$F(t) = 50\left[\sin 10t + \frac{1}{3}\sin 30t + \frac{1}{5}\sin 50t\right].$$

Solution: The overall system transfer function is (the product of the three transfer functions in the figure)

$$\begin{aligned}
G(s) &= 20\left(\frac{10^{-3}\tau}{1+\tau s}\right)\left(\frac{50}{\frac{1}{\omega_n^2}s^2 + \frac{2\xi}{\omega_n}s + 1}\right) = \big/\text{Inserting the numbers}\big/ = \\
&= 20\left(\frac{10^{-3}\cdot 0.1}{1+0.1s}\right)\left(\frac{50}{\frac{1}{50^2}s^2 + \frac{2\cdot 0.2}{50}s + 1}\right) = \\
&= \frac{0.1}{(1.0+0.1s)(.0004s^2 + .008s + 1.0)}
\end{aligned}$$

Now, replacing s with $j\omega$ where ω is the angular frequency of the input, we can compute the output by the principle, sinusoid in gives sinusoid out, i.e. the output signal is given by

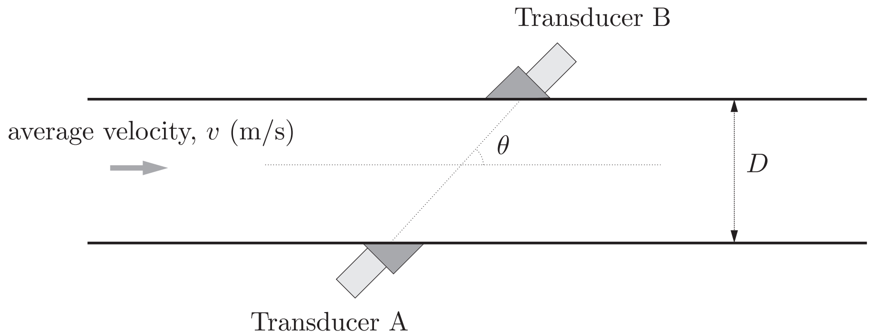
$$|G(j\omega)|\sin(\omega t + \measuredangle G(j\omega)),$$

where $|G(j\omega)|$ is the absolute value of the transfer function and $\measuredangle G(j\omega)$ is the phase of the transfer function. Doing this for each of the sinusoids in the input signal we obtain

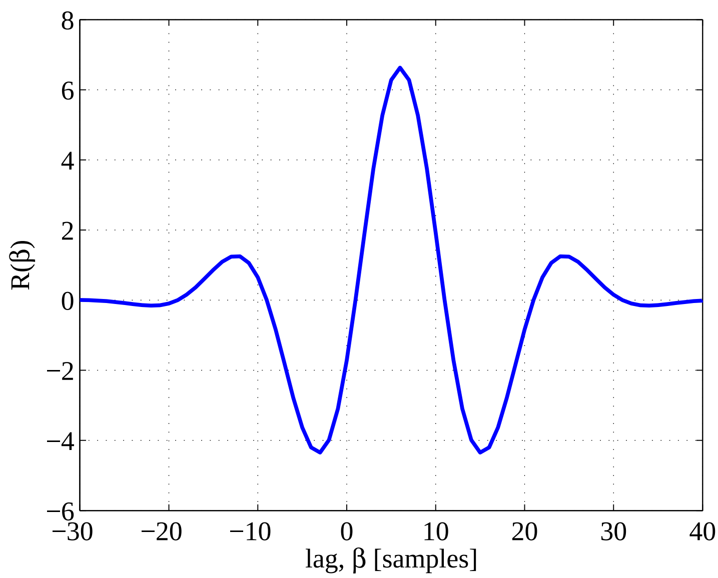
$$\begin{aligned}
|G(j\omega)| &= \left|\frac{0.1}{(1.0+0.1j\omega)\left(.0004(j\omega)^2 + .008j\omega + 1.0\right)}\right| = \\
&= \frac{0.1}{\left|(1.0+0.1j\omega)\left(.0004(j\omega)^2 + .008j\omega + 1.0\right)\right|}
\end{aligned}$$

$$\measuredangle G(j\omega) = \tan^{-1} 0.1 - \tan^{-1}\left(\frac{\mathrm{Re}\left\{(1.0+0.1j\omega)\left(.0004\,(j\omega)^2+.008j\omega+1.0\right)\right\}}{\mathrm{Im}\left\{(1.0+0.1j\omega)\left(.0004\,(j\omega)^2+.008j\omega+1.0\right)\right\}}\right)$$
$$= -\tan^{-1}\left(\frac{\mathrm{Re}\left\{(1.0+0.1j\omega)\left(.0004\,(j\omega)^2+.008j\omega+1.0\right)\right\}}{\mathrm{Im}\left\{(1.0+0.1j\omega)\left(.0004\,(j\omega)^2+.008j\omega+1.0\right)\right\}}\right)$$

4. An ultrasonic transit-time flowmeter is used to measure the average flow velocity v (m/s) through a pipe.



A short ultrasonic pulse is first transmitted from Transducer B to Transducer A and then from Transducer A to Transducer B. The cross-correlation function $R(\beta)$ between the two pulses is shown in the figure below.



The flow velocity is given by

$$v = \frac{\Delta T \cdot c^2}{2D\cot\theta},$$

where θ is the angle between the transducers and the center axis of the flow, c is the speed of sound through the fluid, and D is the pipe diameter.

Assuming that $\theta = \pi/6$ rad, $c = 1480$ m/s, $D = 10$ cm, and the sampling frequency of the A/D converter used to measure the pulses is $f_s = 10$ MHz:

(a) Determine the average flow velocity, v. (2p)

Solution: We know that the cross-correlation has its maximum at the lag corresponding to the time delay between the pulses. From the figure we see that $\Delta T_{samp} = 6$ samples. This corresponds to a time delay of

$$\Delta T = \frac{\Delta T_{samp}}{f_s} = \frac{6}{10^7} \text{ seconds } = 6 \times 10^{-7} \text{ s} = 0.6\ \mu\text{s}.$$

Using this together with the numbers given in the problem, we obtain the average flow velocity as

$$v = \frac{\Delta T \cdot c^2}{2D \cot\theta} = \frac{6 \times 10^{-7} \cdot 1480^2}{2 \cdot 0.1 \cdot \cot(\pi/6)} \text{ m/s} \approx 3.79 \text{ m/s}$$

(b) In reality, the speed of sound, c, is temperature dependent. Assuming it can be modeled as a second-order polynomial function of temperature, make the necessary modifications to the equation for the flow velocity. (1p)

Solution: Assuming that the speed of sound can be modeled as

$$c(T) = a_0 + a_1 T + a_2 T^2,$$

the expression for the average flow velocity becomes

$$v = \frac{\Delta T \cdot \left(a_0 + a_1 T + a_2 T^2\right)^2}{2D \cot\theta}$$

(c) Assuming there is some uncertainty regarding the mounting of the transducer, modeled as $\sigma_\theta = 0.01$ rad, what is the effect on the overall uncertainty of the measured flow velocity?

Note: You do not need to estimate the total uncertainty, only the contribution of the transducer angle uncertainty. (2p)

Solution: The sensitivity of the velocity measurement to the angle of the transducer is given by

$$\left(\frac{\partial v}{\partial \theta}\right)^2 \sigma_\theta^2 = \left(\frac{\partial}{\partial \theta} \frac{\Delta T \cdot c^2}{2D \cot\theta}\right)^2 \sigma_\theta^2 =$$

$$= \left(\frac{\Delta T c^2 \left(1 + \cot^2\theta\right)}{2D \cot^2\theta}\right)^2 \sigma_\theta^2$$

For a deviation of $2\sigma_\theta = 0.02$ rad, this would mean (for the example in (a)), that the uncertainty is increased by

$$\Delta\sigma_v^2 = \left(\frac{6\times 10^{-7}\cdot 1480^2\left(1+\cot^2\left(\pi/6+0.02\right)\right)}{2\cdot 0.1\cdot \cot^2\left(\pi/6+0.02\right)}\right)^2 (0.01)^2 \approx 8.05\times 10^{-3},$$

meaning that the total standard deviation is increased by

$$\Delta\sigma_v \approx 8.97\times 10^{-2}\text{ m/s}$$

5. We have a differential pressure flowmeter setup designed for incompressible fluids. The volume flow rate is given by

$$Q = \frac{A_2}{\sqrt{1-\left(\frac{A_2}{A_1}\right)^2}}\sqrt{\frac{2\left(P_1-P_2\right)}{\rho}},$$

where A_1 and A_2 are the cross-sectional areas of the pipe where the pressures P_1 and P_2 are measured, respectively. The problem now is that we do not know the fluid density. Assuming we can control the volume flow rate, Q to within some uncertainty, i.e. the flow rate can be said to be normally distributed as $N\left(Q,\sigma_Q\right)$, derive an expression for the least-squares estimator of the flow rate, using the Gauss-Newton linearization method, that based on the calibration measurements also estimates the unknown fluid density ρ. (5p)
Solution: Given that the cross-sectional areas are constant, and only the pressures change, let's assume that the volume flow rate we are actually able to set is

$$y_m = Q_m\left(P_1,P_2;\rho\right) + e_m,$$

where

$$e_m \sim N(0,\sigma_Q).$$

We then have the problem of finding an estimate of the flow rate, $\widehat{Q}$ so that the sum of squares of the errors is minimized. For repeated calibration measurements, this can be written on vector form as

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = f\left(\mathbf{X};\rho\right) + \mathbf{e} = \begin{bmatrix} Q_1\left(\mathbf{X}_1;\rho\right) \\ Q_2\left(\mathbf{X}_2;\rho\right) \\ \vdots \\ Q_M\left(\mathbf{X}_M;\rho\right) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_M \end{bmatrix},$$

where

$$\mathbf{X} = \left[\begin{array}{cc} P_{1,m} & P_{2,m} \end{array}\right]^T.$$

The subscript n denotes pressures measured for experiment m, where $m = 1,2,\ldots,M$. Now, assuming we have an initial guess of the density, let's say $\widehat{\rho} = \rho_0$. We can then use the Gauss-Newton linearization method to find the estimate of ρ that miminizes the sum of squares of the errors

$$J = \left(\mathbf{y} - \widehat{\mathbf{Q}}\right)^T \left(\mathbf{y} - \widehat{\mathbf{Q}}\right).$$

The iteration is given by (see text book)

$$\widehat{\rho}_{i+1} = \widehat{\rho}_i + \left(\mathbf{H}^T\left(\widehat{\rho}_i\right)\mathbf{H}\left(\widehat{\rho}_i\right)\right)^{-1}\mathbf{H}^T\left(\widehat{\rho}_i\right)\left(\mathbf{y} - \widehat{\mathbf{Q}}\left(\widehat{\rho}_i\right)\right),$$

where $\mathbf{H}(\rho)$ is the gradient of the model with respect to the unknown parameter ρ, i.e.

$$\mathbf{H}(\rho) = \frac{\partial \mathbf{Q}(\mathbf{X};\rho)}{\partial \rho} = \begin{bmatrix} h_1 \\ h_2 \\ \\ h_M \end{bmatrix},$$

where

$$\begin{aligned} h_m &= \frac{d}{d\rho}\left(\frac{A_2}{\sqrt{1-\left(\frac{A_2}{A_1}\right)^2}}\sqrt{\frac{2\left(P_{1,m}-P_{2,m}\right)}{\rho}}\right) = \\ &= \frac{d}{d\rho}K\sqrt{\frac{L_m}{\rho}} = -\frac{L_m}{2\rho^2}\frac{K}{\sqrt{\frac{L_m}{\rho}}}, \end{aligned}$$

where

$$\begin{aligned} K &= \frac{A_2}{\sqrt{1-\left(\frac{A_2}{A_1}\right)^2}} \\ L_m &= 2\left(P_{1,m}-P_{2,m}\right) \end{aligned}$$

So, for each element in the gradient vector, the only things that vary are the measured pressures.