Method of Variation of Parameters

(1) We can apply this method to find the solution of LDE with variable ~~Const~~ coefficients.

Ex $x^2y'' + xy' - y = x, \; x \neq 0.$

(2) We can apply this method to find the sol of LDE with constant coefficients, when we can't find the P.I. using 5 special cases.

Ex $y'' + 16y = 32 \sec 2x$

(3) We can also apply this method to find the sol of LDE with constant coefficients, when we can find the P.I. using any one of the five special cases.

Ex: $y'' + 3y' + 2y = 2e^x$.

Method Consider $a_0(x)y'' + a_1(x)y' + a_2(x)y = X, \; a_0(x) \neq 0.$

We can use this method, when C.F. is known or two L.I. solutions are given.

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

or $y_1(x)$ and $y_2(x)$ are two LI solutions.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

Let $g(x) = \frac{X}{a_0(x)}$

P.I. is

$$y_p(x) = -y_1 \int \frac{g(x)\, y_2}{W}\, dx + y_2 \int \frac{g(x)\, y_1}{W}\, dx.$$

① It is given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are two LI solutions of $x^2 y'' + xy' - y = x$, $x \neq 0$. Find G.S.

Sol: $y_1 = x$, $y_2 = \frac{1}{x}$

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix} = x\left(\frac{-1}{x^2}\right) - \frac{1}{x} = \frac{-1}{x} - \frac{1}{x} = \frac{-2}{x},\ x \neq 0.$$

$$g(x) = \frac{x}{x^2} = \frac{1}{x}$$

$$y_p(x) = -x \int \frac{\frac{1}{x} \cdot \frac{1}{x}}{\frac{-2}{x}}\, dx + \frac{1}{x} \int \frac{\frac{1}{x} \cdot x}{\frac{-2}{x}}\, dx$$

$$= -x \int \frac{-1}{2x}\, dx + \frac{1}{x} \int \left(-\frac{x}{2}\right) dx$$

$$= \frac{x}{2} \int \frac{dx}{x} - \frac{1}{2x} \int x\, dx$$

$$= \frac{x}{2} \log x - \frac{1}{2x}\left(\frac{x^2}{2}\right)$$

$$= \frac{x}{2} \log x - \frac{x}{4}$$

$\therefore$ The sol of $x^2y'' + xy' - y = 0$.

If y_1, y_2 are solutions, then $c_1y_1 + c_2y_2$ is also a sol.

$$\therefore \; y_c(x) = c_1 x + c_2 \frac{1}{x}$$

$$y(x) = c_1 x + c_2 \frac{1}{x} + \frac{x}{2}\log x - \frac{x}{4}.$$

$$= \left(c_1 - \frac{1}{4}\right)x + c_2 \frac{1}{x} + \frac{x}{2}\log x$$

$$= c^* x + c_2 \frac{1}{x} + \frac{x}{2}\log x.$$

Ex 5.4

(13) $x^2y'' + xy' - y = x^3$, $y_1 = x$, $y_2 = \frac{1}{x}$

Sol:- Here $y_1 = x$, $y_2 = \frac{1}{x}$, $g(x) = \frac{x^3}{x^2} = x$

$$y_p(x) = -x\int \frac{x\frac{1}{x}}{\frac{-2}{x}}dx + \frac{1}{x}\int \frac{x \cdot x\, dx}{\frac{-2}{x}}$$

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix} = \frac{-1}{x} \frac{-1}{x} = \frac{-2}{x} \neq 0$$

$$= -x\int\left(-\frac{x}{2}\right)dx + \frac{1}{x}\int x^2 \cdot \left(\frac{-x}{2}\right)dx$$

$$= +\frac{x}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2x}\int x^3\, dx$$

$$= \frac{x^3}{4} - \frac{x^4}{8x} = \frac{x^3}{4} - \frac{x^3}{8} = \frac{x^3}{8}$$

G.S. is

$$y(x) = C_1 x + C_2 \frac{1}{x} + \frac{x^3}{8}.$$

(14) $x^2y'' + xy' - 4y = x^2 \log|x|$, $y_1 = x^2$, $y_2 = \frac{1}{x^2}$.

Sol:- $g(x) = \log|x|$

$$W = \begin{vmatrix} x^2 & \frac{1}{x^2} \\ 2x & \frac{-2}{x^3} \end{vmatrix} = \frac{-2}{x} - \frac{2}{x} = \frac{-4}{x}, x \neq 0.$$

$$\cancel{y_p(x)} = -x^2 \cancel{\int \frac{\frac{-4}{x} \frac{1}{x^2}}{\log}}$$

$$y_p(x) = -x^2 \int \frac{\log|x| \frac{1}{x^2}}{\frac{-4}{x}} dx + \frac{1}{x^2} \int \frac{\log|x| \, x^2}{\frac{-4}{x}} dx$$

$$= \frac{-x^2}{1} \int \frac{1}{x^2} \log|x| \times \left(\frac{-x}{4}\right) dx +$$

$$\frac{1}{x^2} \int x^2 \log|x| \times \left(\frac{-x}{4}\right) dx$$

$$= \frac{-x^2}{-4} \int \frac{1}{x} \log|x| \, dx - \frac{1}{4x^2} \int x^3 \log|x| \, dx$$

Let $\log x = t$ $\Rightarrow$ $\int \frac{1}{x} \log x \, dx \int t \, dt = \frac{t^2}{2} = \frac{(\log x)^2}{2}$

$\frac{1}{x} dx = dt$

$$\int x^3 \log|x| \, dx = \log|x| \int x^3 dx - \int \frac{1}{x} \left(\frac{x^4}{4}\right) dx = \frac{x^4}{4} \log|x| - \frac{1}{4}\left(\frac{x^4}{4}\right)$$

[Integration by parts : $\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx\right) dx$]

$$= \frac{x^4}{4}\log|x| - \frac{x^4}{16}$$

ecial

$$y_p(x) = \frac{x^2}{4}\frac{(\log x)^2}{2} - \frac{1}{4x^2}\left(\frac{x^4}{4}\log|x| - \frac{x^4}{16}\right)$$

$$= \frac{x^2(\log x)^2}{8} - \frac{x^2\log|x|}{16} + \frac{x^2}{64}$$

$$y(x) = c_1x^2 + \frac{c_2}{x^2} + \frac{x^2(\log x)^2}{8} - \frac{x^2\log|x|}{16} + \frac{x^2}{64}$$

Ex $y'' + 16y = 32\sec 2x$

A.E. $m^2 + 16 = 0$

$m = \pm 4i$

$y_c(x) = C_1\cos 4x + C_2 \sin 4x$

Let $y_1(x) = \cos 4x$, $y_2(x) = \sin 4x$, $g(x) = 32\sec 2x$

$$W = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4$$

$$y_p(x) = -\cos 4x \int \frac{32\sec 2x \sin 4x}{4}\,dx + \sin 4x \int \frac{32\sec 2x \cos 4x}{4}\,dx$$

$$= -8\cos 4x \int \frac{2\sin 2x \cos 2x}{\cos 2x}\,dx + 8\sin 4x \int \frac{(2\cos^2 2x - 1)}{\cos 2x}\,dx$$

$[\sin 2\theta = 2\sin\theta\cos\theta, \ \cos 2\theta = 2\cos^2\theta - 1]$

$$= -8\cos 4x \int 2\sin 2x\,dx + 8\sin 4x \int (2\cos 2x - \sec 2x)\,dx$$

$$= -16\cos 4x \cdot \left(-\frac{\cos 2x}{2}\right) + 8\sin 4x \left[\frac{2\sin 2x}{2} - \frac{\log|\sec 2x + \tan 2x|}{2}\right]$$

$$= 8\cos 2x\cos 4x + 8\sin 2x \sin 4x - 4\sin 4x \log|\sec 2x + \tan 2x|$$

$$= \frac{8}{2}[\cos 6x + \cos 2x] + \frac{8}{2}[\cos 2x - \cos 6x] - 4\sin 4x \log|\sec 2x + \tan 2x|$$

① $\sin(a+b) = \sin a \cos b + \cos a \sin b$

② $\sin(a-b) = \sin a \cos b - \cos a \sin b$

③ $\cos(a+b) = \cancel{\sin}\ \cos a \cos b - \sin a \sin b$

④ $\cos(a-b) = \cos a \cos b + \sin a \sin b$.

⑤ $\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

⑥ $\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

⑦ $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$.

$$= 4\cos 6x + 4\cos 2x + 4\cos 2x - 4\cos 6x - 4\sin 4x \log|\sec 2x + \tan 2x|$$

$$y_p(x) = 8\cos 2x - 4\sin 4x \log|\sec 2x + \tan 2x|$$

$$y(x) = c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x \log|\sec 2x + \tan 2x|$$

Ex :- 5.4

⑤ $y'' + y = \operatorname{cosec} x$.

C.F. $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

Let $y_1(x) = \cos x$, $y_2(x) = \sin x$, $g(x) = \operatorname{cosec} x$.

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

$$y_p(x) = -\cos x \int \frac{\sin x \operatorname{cosec} x}{1}\,dx + \sin x \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$$

$$= -\cos x \int dx + \sin x \int \cot x\,dx$$

$$= -x\cos x + \sin x \ln|\sin x|$$

$$y(x) = \cancel{C_1 x +} \; C_1\cos x + C_2 \sin x - x\cos x + \sin x \ln|\sin x|.$$

(10) $y'' + 4y' + 4y = e^{-2x}\sin x.$

<u>C.F</u> $m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m = -2, -2$

$$y_c(x) = (C_1 + xC_2)e^{-2x} = C_1e^{-2x} + C_2 x e^{-2x}$$

Let $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$, $g(x) = e^{-2x}\sin x$

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & -2xe^{-2x} + e^{-2x} \end{vmatrix} = e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$$

$$= e^{-4x}$$

$$y_p(x) = -e^{-2x} \int \frac{xe^{-2x} \cdot e^{-2x}\sin x}{e^{-4x}}\,dx + xe^{-2x} \int \frac{e^{-2x}\, e^{-2x}\sin x}{e^{-4x}}\,dx$$

$$= -e^{-2x} \int x\sin x\,dx + xe^{-2x} \int \sin x\,dx$$

$$= -e^{-2x}\left[x(-\cos x) - \int -\cos x\,dx\right] + xe^{-2x}(-\cos x)$$

$$= -e^{-2x}\left[-x\cos x + \sin x\right] - xe^{-2x}\cos x$$

$= xe^{-2x}\cos x - e^{-2x}\sin x - xe^{-2x}\cos x$

$= -e^{-2x}\sin x.$

$y(x) = (C_1 + xC_2)e^{-2x} - e^{-2x}\sin x.$

(11) $y'' + 6y' + 9y = \dfrac{e^{-3x}}{x}$

C.F. $m^2 + 6m + 3^2 = 0$

$(m+3)^2 = 0$

$m = -3, -3.$

$y_c(x) = (C_1 + xC_2)e^{-3x}$

(operator method)

P.I. $y_p(x) = \dfrac{1}{D^2+6D+9} e^{-3x}/x = \dfrac{1}{(D+3)^2} e^{-3x}/x$

$= e^{-3x} \dfrac{1}{(D-3+3)^2}\left(\dfrac{1}{x}\right)$

$= e^{-3x} \dfrac{1}{D^2}\left(\dfrac{1}{x}\right)$

$= e^{-3x} \dfrac{1}{D}(\ln|x|)$

$= e^{-3x}\, x(\ln|x| - 1)$

$= xe^{-3x}(\ln|x| - 1)$

$y(x) = (C_1 + xC_2)e^{-3x} + xe^{-3x}(\ln|x| - 1)$

Variation of Parameters

$$y_p(x) = -y_1\int \frac{y_2\, g(x)}{W}\,dx + y_2\int \frac{y_1\, g(x)}{W}\,dx$$

$$y_1(x) = e^{-3x},\quad y_2(x) = xe^{-3x},\quad g(x) = \frac{e^{-3x}}{x}$$

$$W = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{vmatrix} = e^{-6x} - 3xe^{-6x} + 3xe^{-6x}$$

$$= e^{-6x}$$

$$y_p(x) = -e^{-3x}\int \frac{xe^{-3x}\,\frac{e^{-3x}}{x}}{e^{-6x}}\,dx + xe^{-3x}\int \frac{e^{-3x}\cdot\frac{e^{-3x}}{x}}{e^{-6x}}\,dx$$

$$= -e^{-3x}\int dx + xe^{-3x}\int \frac{dx}{x}$$

$$= -xe^{-3x} + xe^{-3x}(\ln|x|)$$

$$= (\ln|x| - 1)\,xe^{-3x}$$

$$y(x) = (C_1 + xC_2)e^{-3x} + xe^{-3x}(\ln|x| - 1).$$