How to get Normal approximation of Binomial distribution

Unit - 5 $\xrightarrow{\text{practice from}}$ Refrence book

Point estimation and Central limit theorem

- Unbiased estimator
- Consistent estimator
- Efficient and sufficient Estimator
- Likelihood function and Maximum likelihood estimation
- The Central limit theorem [without proof]

The estimation theory was given by Prof. R.A. fisher around 1930. ~~[illegible]~~
Let us consider X is a R.V with the probability density function $F(x, \theta)$, where θ is an unknown parameter taken from the parameter space Θ

$$\{F(x, \theta) : \theta \in \Theta\}$$

parameter space is just a collection of the parameters.

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Normal Distribution:

$$\Theta = \left\{ (\mu, \sigma^2) : -\infty < \mu < \infty, \; 0 < \sigma < \infty \right\}$$

Unbiased Estimator

gamma θ → θ unknown parameter

→ $\gamma(\theta)$ → Function of unknown parameter

An estimator $T_n = T(x_1, \ldots, x_n)$ is said to be an unbiased estimation of θ or $\gamma(\theta)$ If $E(T_n) = \theta$ or $E(T_n) = \gamma(\theta)$

Q. Let $x_1, \ldots, x_n$ is a Random Sample from the population $N(\mu, 1)$. Show that $t = \frac{1}{n} \sum_{i=1}^{n} x_i^2$ is an unbiased estimator of $\mu^2 + 1$

→ Given:

x_i is Sample $\{i = 1, \ldots, n\}$

from $N(\mu, 1)$

$E(x_i) = \mu$

$V(x_i) = 1$

$V(\theta) = E(\theta^2) - \{E(\theta)\}^2$

Aim:

$E(t) = \mu^2 + 1$

proof:

$E(t) = E\left[\frac{1}{n} \sum_{i=1}^{n} x_i^2\right]$

$= \frac{1}{n} E\left[\sum_{i=1}^{n} x_i^2\right]$

$= \frac{1}{n} \sum_{i=1}^{n} E(x_i^2)$

$= \frac{1}{n} \sum_{i=1}^{n} (\mu^2 + 1)$

$= \frac{1}{\cancel{n}} \cancel{n} (\mu^2 + 1)$

$= \boxed{\mu^2 + 1}$

$V(x_i) = E(x_i^2) - \{E(x_i)\}^2$

$1 = E(x_i^2) - \mu^2$

$E(x_i^2) = \mu^2 + 1$

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Unbiased Estimator

Q. Let X is a normal Random variable with mean $= \mu$ and variance $= \sigma^2$

(i) Show that the sample mean $\bar{x}$ is unbiased estimator of population mean

(ii) Show that

Sample variance $\boxed{S^2} \xrightarrow[\text{estimator}]{\text{unbiased}} \boxed{\sigma^2}$ → population variance

(iii) Show that the s^2 is not unbiased estimator of σ^2

Sample variance $\boxed{s^2} \xrightarrow[\text{estimator}]{\text{not unbiased}} \sigma^2$

(i) Sample mean $\boxed{\bar{x}} \xrightarrow[\text{estimator}]{\text{unbiased}} \boxed{\mu}$ → population mean

→ Given:

mean $= \mu$ variance $= \sigma^2$

$x_i \in N(\mu, \sigma^2)$

$E(x_i) = \mu$

$V(x_i) = \sigma^2$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Aim:

1. $E[\bar{x}] = \mu$
2. $E[s^2] = \sigma^2$
3. $E(S^2) \neq \sigma^2$

(1) LHS

$E[\bar{x}]$

$= E\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]$ Given $E(x_i) = \mu$

$= \frac{1}{n}\left[E(x_1) + \cdots + E(x_n)\right]$

$= \frac{1}{n}\left[\mu + \cdots + \mu\right]$

$= \frac{1}{\cancel{n}} \times \cancel{n}\mu = \mu$

(2) LHS

$E[s^2]$

$= E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$

$$= \frac{1}{(n-1)}\left[E\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n}\left(x_i^2-2x_i\bar{x}+(\bar{x})^2\right)\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n}x_i^2-2\left(\sum_{i=1}^{n}x_i\right)\bar{x}+\sum_{i=1}^{n}(\bar{x})^2\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n}x_i^2-2(n\bar{x})\cdot\bar{x}+n(\bar{x})^2\right]$$

$$= \frac{1}{n-1}E\left[\sum_{i=1}^{n}x_i^2-n(\bar{x})^2\right]$$

$$= \frac{1}{n-1}\left\{\underbrace{E\left(\sum_{i=1}^{n}x_i^2\right)}_{P_1}-\underbrace{E\left[n(\bar{x})^2\right]}_{P_2}\right\}$$

$$E(s^2) = \frac{1}{n-1}\left[\boxed{\sum_{i=1}^{n}E(x_i^2)}-\boxed{nE(\bar{x})^2}\right] \text{ — eq}^n\ (1)$$

$$P_1 = \sum_{i=1}^{n}E(x_i^2)$$

$$E(X^2) = V(X)+[E(X)]^2$$

$$E(x_i^2) = V(x_i)+[E(x_i)]^2$$

$$= \sigma^2+\mu^2$$

$$= \sum_{i=1}^{n}(\sigma^2+\mu^2)$$

$$= n(\sigma^2+\mu^2)$$

$$T_2 = E[\bar{x}]^2$$

$$= V(\bar{x}) + [E(\bar{x})]^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

$$= E(s^2) = \frac{1}{n-1}\left\{n(\sigma^2+\mu^2) - \frac{\sigma^2}{n} + \mu^2\right\}$$

$$= \frac{1}{n-1}\left[n\sigma^2 + n\mu^2 - \frac{\sigma^2}{n} + \mu^2\right]$$

$$= \frac{1}{n-1}\left[\frac{n^2\sigma^2 + n^2\mu^2 - \sigma^2 + n\mu^2}{n}\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 - \sigma^2\right]$$

$$= \frac{\cancel{(n-1)}\,\sigma^2}{\cancel{(n-1)}} = \sigma^2$$

$$E(s^2) = \sigma^2$$

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MCQ

Q If $x_i \in N(\mu, \sigma^2)$ then which of the following is true about $\bar{x}$?

→ $E(x_i) = \mu$

$V(x_i) = \sigma^2$

$E(\bar{x}) = E\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]$

$= \frac{1}{n}(n\mu) = \mu$

$V(\bar{x}) = V\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]$

$= \frac{1}{\cancel{n^2}} \sigma^2 (\cancel{n})$

$= \frac{\sigma^2}{n}$

$E(\bar{x}) = \mu, \; V(\bar{x}) = \frac{\sigma^2}{n}$

$\Rightarrow (\bar{x}) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(3) Aim:

$E(s^2) \neq \sigma^2$

LHS $= E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$

$= \frac{1}{n}\left\{E\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)\right\}$

$= \frac{1}{n}E\left\{\sum_{i=1}^{n}(x_i^2-2x_i\bar{x}+(\bar{x})^2)\right\}$

$= \frac{1}{n}E\left\{\sum_{i=1}^{n}x_i^2-2\left(\sum_{i=1}^{n}x_i\right)\bar{x}+\sum_{i=1}^{n}(\bar{x})^2\right\}$

$= \frac{1}{n}E\left\{\sum_{i=1}^{n}x_i^2-2(n\bar{x})\cdot\bar{x}+n(\bar{x})^2\right\}$

$= \frac{1}{n}E\left\{\sum_{i=1}^{n}x_i^2-n(\bar{x})^2\right\}$

$= \frac{1}{n}\left\{E\left\{\sum_{i=1}^{n}x_i^2\right\}-E\left(n(\bar{x})^2\right)\right\}$

$E(s^2) = \frac{1}{n}\left[n(\sigma^2+\mu^2)-\frac{\sigma^2}{n}+\mu^2\right]$

$= \frac{(n-1)}{n}\sigma^2$

$E(s^2) \neq \sigma^2$ proved

~~Q~~ Condition for Consistency

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1) Consistent estimator: An estimator $\hat{\theta} = \hat{\theta}(x_1, ----, x_n)$ is known as a consistent estimator of population parameter θ if this estimator $\hat{\theta}$ approaches closer and closer to the population parameter when the sample size is approaching to infinite.

Q Let x_i is taken from a Normal population then prove that the sample mean is the consistent estimator of popul^n mean

$\Rightarrow$ Give:

$x_i \in N(\mu, \sigma^2)$

$E(x_i) = \mu$

$V(x_i) = \sigma^2$

Aim:

$\bar{x} \xrightarrow[\text{estimator}]{\text{consistent}} \mu$

(1) $E(\bar{x}) = \mu \quad n \to \infty$

(2) $V[\bar{x}] = 0 \quad n \to \infty$

(1) $E[\bar{x}] = E\left[\frac{x_1 + ---- + x_n}{n}\right]$

$= \frac{1}{\cancel{n}}\{\cancel{n}\mu\} = \mu$

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Sufficient Condition for Consistency

(1) $E(\hat{\theta}) = \theta, \ n \to \infty$

(2) $V(\hat{\theta}) = 0, \ n \to \infty$

Eg:-

(2) $V(\bar{x}) = 0 \quad n \to \infty$

$$= V\left[\frac{x_1 + \cdots + x_n}{n}\right]$$

$$= \frac{1}{\cancel{n^2}}(\cancel{n}\sigma^2)$$

$$= \frac{\sigma^2}{n}, \quad n \to \infty$$

$$= 0$$

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Efficient Estimator

Let $\hat{\theta}_1 = \hat{\theta}_1(x_1 \dots x_n)$ and $\hat{\theta}_2 = \hat{\theta}_2(x_1 \dots x_n)$ unknown are two unbiased estimator of ~~the~~ parameter θ then the $\hat{\theta}_1$ is more efficient then $\hat{\theta}_2$ if ~~variance~~ $\text{var}(\hat{\theta}_1) < \text{var}(\hat{\theta}_2)$

$$\left.\begin{array}{l} \hat{\theta}_1 = \hat{\theta}_1(x_1, \dots, x_n) \\ \hat{\theta}_2 = \hat{\theta}_2(x_1, \dots x_n) \end{array}\right\} '\theta'$$

Q. Consider the following estimators from a Normal population with mean $= \mu$ and variance $= \sigma$ is here

$$\hat{\theta}_1 = \frac{x_1 + \dots + x_{10}}{10}$$

$$\hat{\theta}_2 = \frac{x_1 + x_5 + x_{10}}{3}$$

which of the following estimators is efficient

Aim ①

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$$\rightarrow E(\hat{\theta}_1) = E\left[\frac{x_1 + \cdots + x_{10}}{10}\right] = \frac{1}{10}(10\mu) = \mu$$

$$E(\hat{\theta}_2) = E\left(\frac{x_1 + x_5 + x_{10}}{3}\right)$$

$$\Rightarrow \frac{1}{3}(3\mu) = \mu$$

Aim 2

$$\mathrm{var}(\hat{\theta}_1) = V\left(\frac{x_1 + \cdots + x_{10}}{10}\right)$$

$$= \frac{1}{10 \times \cancel{10}}(\cancel{10} \times \sigma^2)$$

$$\Rightarrow \frac{\sigma^2}{10} = 0.1\sigma^2$$

$$\mathrm{var}(\hat{\theta}_2) = V\left(\frac{x_1 + x_5 + x_{10}}{3}\right)$$

$$= \frac{1}{\cancel{9}\,3}(\cancel{3}\sigma^2)$$

$$= 0.3\sigma^2$$

Since

$$V(\hat{\theta}_1) < \mathrm{var}(\hat{\theta}_2)$$

$\therefore$ It means $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$.

Q. Suppose that a sample is taken from Normal population with mean μ and var. σ^2 alongwith the following details. Check the unbiasness of both estimators and then check the efficiency.

$\rightarrow$
$$\hat{\theta}_1 = \frac{x_1 + \cdots + x_7}{7}$$

$$\hat{\theta}_2 = \frac{2x_1 - x_6 + x_4}{2}$$

Aim (1)

$$E(\hat{\theta}_1) = E\left(\frac{x_1 + \cdots + x_7}{7}\right)$$

$$= \frac{1}{\cancel{7}}(\cancel{7}\mu) = \mu$$

$$E(\hat{\theta}_2) = E\left(\frac{2x_1 - x_6 + x_4}{2}\right)$$

$$= \frac{1}{\cancel{2}}(\cancel{2}\mu - \cancel{\mu} + \cancel{\mu})$$

$$= \mu$$

Aim 2:

$$var(\hat{\theta}_1) = v\left(\frac{x_1 + \cdots + x_7}{7}\right)$$

$$= \frac{1}{\cancel{49}\,7} \times (\cancel{7}\sigma^2)$$

$$= 0.14\cancel{0}\cancel{0}\sigma^2$$

$$var(\hat{\theta}_2) = v\left(\frac{2x_1 - x_6 + x_4}{2}\right)$$

$$= \frac{1}{4}\left(\cancel{4\sigma^2 + \sigma^2 + \sigma^2}\right)$$

$$4\sigma^2 + \sigma^2 + \sigma^2$$

$$\Rightarrow \frac{\cancel{6}^{3}\sigma^2}{\cancel{4}\,2} = \frac{3}{2}\sigma^2$$

$$\Rightarrow 1.5\sigma^2$$

$\cancel{var(\hat{\theta}_2) < var(\hat{\theta}_1)}$

$$var(\hat{\theta}_1) < var(\hat{\theta}_2)$$

It means $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$.

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Q Suppose that 3 random sample of size 20, 10 and 8 are drawn from a population with variance σ^2. If the respective sample variance is S_1^2, S_2^2 and S_3^2 are the unbiased estimators of σ^2, then prove that

$$\frac{20 S_1^2 + 10 S_2^2 + 8 S_3^2}{38}$$ is also an

unbiased estimator of σ^2

Given:

$\rightarrow$ $S_1^2 = 20$

$S_2^2 = 10$

$S_3^3 = 8$

$$S = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$E(S_1^2) = \mu$

$E(S_2^2) = \mu$

$E(S_3^2) = \mu$

Aim:

$$\frac{20 S_1^2 + 10 S_2^2 + 8 S_3^2}{38} \xrightarrow{\text{Unbiased}} \sigma^2$$

Likelihood function and Max^m likelihood estimator

Likelihood function: Suppose $x_1 \ldots x_n$ is a random sample from a population with unknown parameter 'θ' alongwith the density function $f(x, \theta)$ then the likelihood function is denoted as L and alternatively it is known as the joint density function.

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

Likelihood eq^n :-

$$\boxed{\frac{\partial L}{\partial \theta} = 0}, \quad \frac{\partial^2 L}{\partial \theta^2} < 0$$

$$\boxed{\frac{\partial L}{\partial \theta_i} = 0}$$

$L > 0$

$\frac{1}{L}$ exists

$$\frac{1}{L} \cdot \frac{\partial L}{\partial \theta_i} = \frac{1}{L} \cdot 0$$

$$\frac{1}{L}\frac{\partial L}{\partial \theta_i} = 0$$

Final form of Likelihood eq^n

$$\boxed{\frac{\partial}{\partial \theta_i}(\log L) = 0}$$

↓

Unknown Parameter

$\hat{\theta}$ $\boxed{M.L.E}$ → Max^m Likelihood estimator

Flowchart for the process of M.L.E

$\Pi = \pi$

Step ①

Create the likelihood function

$$L = \prod_{i=1}^{n} f(x_i;\theta)$$

Step ②

Create Likelihood Equations

$$\frac{\partial(\log L)}{\partial \theta_i} = 0$$

$$\frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{\partial L}{\partial \theta_2} = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial \theta_n} = 0$$

Step ③

Find the estimated value of θ.

$\hat{\theta}$

↓

M.L.E

Q2 In a Random Sampling from a Normal population $N(\mu, \sigma^2)$, find the max^m Likelihood estimator of

(1) μ, when σ^2 is known

(2) σ^2, when μ is known

(3) Find simultaneous estimations of μ and σ^2

→ p.d.f Normal function

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

Likelihood function

$$L = \prod_{i=1}^{n} f(x_i;\mu,\sigma)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x_i-\mu)^2}$$

$$L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{\sum_{i=1}^{n} \frac{-1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

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$$\Rightarrow \log L = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n + \left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$\Rightarrow \log L = n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\log L = -n\log\left(\sigma\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -n\left\{\log\sigma + \log\sqrt{2\pi}\right\} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -n\log\sigma + n\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -n\log\sigma^{2/2} - n\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\boxed{\log L}$

$$= \frac{-n}{2}\log\sigma^2 - n\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Aim ①:

To find M.L.E of μ:

$$\frac{\partial}{\partial\theta}\log L = 0$$

$$\frac{\partial}{\partial\mu}\log L = 0$$

$$= \frac{\partial}{\partial \mu}\left\{-\frac{n}{2}\log\sigma^2 - n\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} = 0$$

$$= 0 - 0 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu)(-1) = 0$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$= \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$= \sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\mu$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \mu$$

$$= \bar{x} = \hat{\mu}$$

Sample mean

σ^{-2} $-2\sigma^{-3}$

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Q Find M.L.E of σ^2 where μ is known

→ Aim:

To find MLE of σ^2:

$$\frac{\partial}{\partial \theta} \log L = 0$$

$$\frac{\partial}{\partial \sigma^2} \log L = 0$$

$$\Rightarrow \frac{\partial}{\partial \sigma^2}\left[\frac{-n}{2}\log \sigma^2 - n\log\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] = 0$$

$$= \frac{-n}{2}\left(\frac{1}{\sigma^2}\right) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\left(\frac{-1}{\sigma^4}\right) = 0$$

$$\Rightarrow \frac{1}{\cancel{2}}\sum_{i=1}^{n}(x_i-\mu)^2\left[\frac{1}{\sigma^4}\right] = \frac{n}{\cancel{2}\sigma^2}$$

$$= \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

↓

Sample variance (s^2)

(3) $\frac{\partial}{\partial \mu}(\log L) = 0$ and $\frac{\partial}{\partial \sigma^2}(\log L) = 0$

$\hat{\mu} = \bar{x}$ $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$

Sample mean

Sample variance

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Sufficiency and Sufficient estimator

An estimator $\hat{\theta} = \hat{\theta}(x_1, ---, x_n)$ is said to be sufficient estimator of unknown parameter 'θ' if this estimator $\hat{\theta}$ contains all the required information of the unknown parameter 'θ'

Otherwise suppose 'X' $\Rightarrow$ Random variable with density function $F(x, \theta)$ where θ is the unknown parameter then $\hat{\theta} = \hat{\theta}(x_1, ---, x_n)$ is said to be a sufficient statistic if the conditional probability that is mass function or conditional density does not depend upon θ for any value of x

In other words the observed value of $\hat{\theta}$ is said to be a sufficient statistic if it contains all the information about the parameter θ which the data x contain.

Factorisation Theorem

The necessary and sufficient condition for a distribution to have a sufficient statistic is given by factorisation theorem due to Neymann.

Statement:

$T = t(x)$ is sufficient for θ if and only if the joint distribution function L can be expressed in the following form

$$\boxed{L = g_\theta(t_1, x)\, h(x)}$$

where $g_\theta[t_1, x]$ depends upon θ and x through $t(x)$ and $h(x)$ is independent of θ

Remark:

In variance property of sufficient estimator

$T \xrightarrow[\text{Estimator}]{\text{Sufficient}} \theta$ and

ψ is a one-one function then

$$\psi(T) \xrightarrow[\text{Estimator}]{\text{Sufficient}} \psi(\theta)$$

Fisher - Neymann Criterian:

A statistic $t_1 = t_1(x_1, \ldots x_n)$ is a sufficient estimator of θ iff

L [Joint density function] Can be expressed as:

$$L = \prod_{i=1}^{n} F(x_i; \theta)$$

$$\boxed{L = g_1(t_1, \theta) \times k(x_1, \ldots, x_n)}$$

Q. Let $(x_1, \ldots x_n)$ is a random sample from population with density function

p.d.f $F(x, \theta) = \theta . x^{\theta - 1}$; $0 < x < 1$, $\theta > 0$

then show that

$$t_1 = \prod_{i=1}^{n} x_i \text{ is sufficient for } \theta.$$

$\Rightarrow$ $f(x, \theta) = \theta \, x^{\theta - 1}$

$F(x_i, \theta) = \theta (x_i)^{\theta - 1}$

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \left\{ \theta (x_i)^{\theta - 1} \right\}$$

$$= \prod_{i=1}^{n} \left(\frac{\theta \cdot (x_i)^{\theta}}{x_i} \right)$$

$$= \left(\prod_{i=1}^{n} \theta \right) \left(\prod_{i=1}^{n} (x_i)^{\theta} \right) \cdot \left(\frac{1}{\prod_{i=1}^{n} x_i} \right)$$

$$= \left\{ (\theta)^n \times \{ x_1^{\theta} \cdots x_n^{\theta} \} \times \left[\frac{1}{x_1 \cdots x_n} \right] \right.$$

$$= g[t_1, \theta] \cdot h(x_1, \cdots x_n)$$

Some miscellaneous Question of estimator

Q Show that $\frac{\sum x_i [\sum x_i - 1]}{n(n-1)}$

is an unbiased estimator of θ^2, for the sample $x_1, \ldots x_n$ which is drawn on the Random variable X which takes the values 1 and 0 along with the probabilities θ and $(1-\theta)$.

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→ **Given:**

$$T = x_1 + \cdots + x_n \sim B(n, \theta)$$

Binomial variate

→ $E(T) = n\theta$

→ $V(T) = n\theta(1-\theta)$

Aim:

$$\frac{T[T-1]}{n(n-1)} \xrightarrow[\text{estimator}]{\text{unbiased}} \theta^2$$

$$E\left[\frac{T(T-1)}{n(n-1)}\right] = \theta^2$$

L.H.S

$$E\left\{\frac{T(T-1)}{n(n-1)}\right\}$$

$$= \frac{1}{n(n-1)}\left\{E(T^2) - E(T)\right\}$$

$V(T) = E(T^2) - \{E(T)\}^2$

$E(T^2) = V(T) + \{E(T)\}^2$

$$= \frac{1}{n(n-1)}\left[V(T) + \{E(T)\}^2 - E(T)\right]$$

$$= \frac{1}{n(n-1)}\left\{\cancel{n\theta} - n\theta^2 + n^2\theta^2 - \cancel{n\theta}\right\}$$

$$= \frac{1}{\cancel{n(n-1)}} \cancel{n(n-1)} \sigma^2$$

$$= \sigma^2$$

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The Central Limit theorem

Statement: If $\bar{X}$ is the mean of a random sample of size 'n' taken from a popl^n with mean μ and variance σ^2 where the value of variance is finite, then the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}, \quad as \ n \to \infty,$$

is the standard Normal distribution ~~and~~ $N(0, 1)$

Q An electrical firm manufatures the light bulbs that have a length of life that is approximately Normaly distributed with mean = 800 hrs. and S.D = 40 hrs. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hrs.

→ $n = 16$ bulbs

mean $= 800$ hours

S.D $= 40$ hours.

$n = 16$ bulbs $\xrightarrow[\text{life}]{\text{average}}$ less than 775 hrs.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 800}{40/\sqrt{16}}$$

$$Z = \frac{\bar{X} - 800}{10} = \frac{775 - 800}{10} = -2.5$$

Aim: $P[\bar{X} < 775 \text{ hrs.}]$

$P[Z < -2.5]$

$= 0.5 - P[-2.5 < Z < 0]$

$= 0.5 - P[0 < Z < 2.5]$

$= 0.0062$