# Solution of Linear differential equation – Operator Method

Operators :- $\frac{d}{dx}, \frac{d^2}{dx^2}, ---, \frac{d^n}{dx^n}, ---$

For sake of convenience, the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, ---, \frac{d^n}{dx^n}$ are denoted by $D, D^2, D^3, ---, D^n$.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \quad -(2)$$

$\Rightarrow D^2y + 5Dy + 6y = 0$

$\Rightarrow (D^2 + 5D + 6)y = 0.$

## Solution of second order Homogeneous Linear Equations (Complementary function)

Eq in operator form :- $(D^2+5D+6)y=0 \Rightarrow f(D)y=0$

Auxiliary eq :- $m^2 + 5m + 6 = 0$ (Replace D by m to get A.E.)

$(m+2)(m+3) = 0. \Rightarrow m = -2, -3$

Solution of eq (2) dependes on the nature of the roots.

Case I The G.S. is $y = c_1e^{-2x} + c_2e^{-3x}$.

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0. \rightarrow$$ 2nd order homogeneous linear differential equation with constant coefficients.

<u>Eq in operator form</u>

$$(a_0D^2 + a_1D + a_2)y = 0.$$

A.E. is $a_0m^2 + a_1m + a_2 = 0.$ — (3)

<u>Case 1</u> The roots are real and distinct, say, $m_1, m_2$ are the roots of the equation (3).

The general solution is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x},$$ where $c_1$ and $c_2$ are constants.

<u>Ex</u> → $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$

In operator form, $(D^2 - D - 6)y = 0.$

A.E. is $m^2 - m - 6 = 0$

$\Rightarrow m^2 - 3m + 2m - 6 = 0$

$\Rightarrow m(m-3) + 2(m-3) = 0$

$\Rightarrow (m+2)(m-3) = 0.$

$\Rightarrow m = -2, 3.$

The general sol is $y(x) = c_1e^{-2x} + c_2e^{3x}.$

Ex $4y'' - 8y' + 3y = 0, \; y(0) = 1, \; y'(0) = 3.$

Eq in operator form:- $(4D^2 - 8D + 3)y = 0.$

A.E. is $4m^2 - 8m + 3 = 0$

$\Rightarrow 4m^2 - 6m - 2m + 3 = 0$

$\Rightarrow 2m(2m-3) - 1(2m-3) = 0$

$\Rightarrow (2m-1)(2m-3) = 0$

$\Rightarrow m = \frac{1}{2}, \frac{3}{2}$

The general sol is $y = C_1 e^{\frac{1}{2}x} + C_2 e^{\frac{3}{2}x}$, $y' = \frac{1}{2} C_1 e^{\frac{1}{2}x} + \frac{3}{2} C_2 e^{\frac{3}{2}x}$

$y(0) = 1 \Rightarrow C_1 e^0 + C_2 e^0 = 1$

$\Rightarrow C_1 + C_2 = 1.$

$y'(0) = 3 \Rightarrow \frac{1}{2} C_1 + \frac{3}{2} C_2 = 3$

$\Rightarrow C_1 + 3C_2 = 6$

$2C_2 = 5 \Rightarrow C_2 = \frac{5}{2}$

$C_1 = 1 - \frac{5}{2} = -\frac{3}{2}$

$\therefore$ The sol ~~is~~ of the given problem is

$y(x) = -\frac{3}{2} e^{\frac{1}{2}x} + \frac{5}{2} e^{\frac{3}{2}x}.$

## Case 2  $m^2 + a_1 m + a_2 = 0$. — (3)

Roots are real and equal, say, $m, m$ are the roots of the eq (3).

The G.S. is $y = (C_1 + x C_2) e^{mx}$.

Ex  $4y'' + 4y' + y = 0$.

$(4D^2 + 4D + 1) y = 0$ (Eq in operator form)

$\Rightarrow 4m^2 + 4m + 1 = 0$  (A.E.)

$\Rightarrow 4m^2 + 2m + 2m + 1 = 0$

$\Rightarrow 2m(2m+1) + 1(2m+1) = 0$

$\Rightarrow (2m+1)(2m+1) = 0$

$\Rightarrow m = \frac{-1}{2}, \frac{-1}{2}$.

The general sol. is $y(x) = \cancel{C_1} (C_1 + x C_2) e^{\frac{-1}{2}x}$

Ex  $y'' + 6y' + 9y = 0$, $y(0) = 2$, $y'(0) = 3$.

Eq in operator form, $(D^2 + 6D + 9) y = 0$

A.E. is  $m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$\Rightarrow (m+3)(m+3) = 0$

$\Rightarrow m = -3, -3$.

The general sol is $y(x) = (C_1 + x C_2) e^{-3x}$

$y'(x) = C_2 e^{-3x} + (-3)(C_1 + x C_2) e^{-3x}$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(0) = 3 \Rightarrow c_2 - 3(c_1) = 3$

$\Rightarrow c_2 - 3(2) = 3$

$\Rightarrow c_2 = 9.$

$y(x) = (2 + 9x)e^{-3x}$ is the sol of given problem.

<u>Case 3</u> $\quad m^2 + a_1 m + a_2 = 0.$

When the roots are complex, say, ~~p+iq, p-iq~~ $\alpha + i\beta, \alpha - i\beta$.

The general sol is

~~$y(x) = c_1 e^{px}[c_1 \cos p$~~

$$y(x) = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x].$$

<u>Ex</u> $\quad y'' + 2y' + 2y = 0.$

Eq in operator method, $(D^2 + 2D + 2)y = 0$

A.E. is $\quad m^2 + 2m + 2 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → Quadratic Formula

$$\Rightarrow m = \frac{-2 \pm \sqrt{4 - 4(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The general solution is

$$y(x) = e^{-x}[c_1 \cos x + c_2 \sin x].$$

Ayush:- Beta will always be positive

complex no. $= a + ib$, $\quad a \to \alpha$, $\quad b \to \beta$

$-1+i, -1-i$: $\quad \alpha = -1$, $\beta = 1$; $\quad \alpha = -1$, ~~$\beta = -1$~~

$\alpha \pm \beta i$: $\quad \alpha$, $\beta$

$-1 \pm i$: $\quad \alpha = -1$, $\beta = 1$

Ex $y'' + 4y' + 13y = 0, \; y(0) = 0, \; y'(0) = 1.$

$(D^2 + 4D + 13)y = 0$

$\Rightarrow m^2 + 4m + 13 = 0$

$\Rightarrow m = \dfrac{-4 \pm \sqrt{16 - 4(13)}}{2} = \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$

$y(x) = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x]$.

$y'(x) = e^{-2x}[C_1(-3\sin 3x) + C_2(3\cos 3x)]$

$= e^{-2x}[-3C_1 \sin 3x + 3C_2 \cos 3x]$

$y(0) = 0 \Rightarrow C_1 = 0.$

$y'(0) = 1 \Rightarrow 3C_2 = 1 \Rightarrow C_2 = \frac{1}{3}$

$\therefore y(x) = e^{-2x}\left[\frac{1}{3} \sin 3x\right] = \frac{1}{3} e^{-2x} \sin 3x.$