Simultaneous differential equations using operator method

Ex Find the solution of the system of equations

$$\frac{dy_1}{dt} + 2\frac{dy_2}{dt} - 2y_1 - y_2 = e^{2t} \quad - (1)$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0. \quad - (2)$$

Sol : Equations in operator form.

$$Dy_1 + 2Dy_2 - 2y_1 - y_2 = e^{2t}$$

$$\Rightarrow (D-2)y_1 + (2D-1)y_2 = e^{2t} \quad - (3)$$

and $\quad Dy_2 + y_1 - 2y_2 = 0$

$$\Rightarrow y_1 + (D-2)y_2 = 0. \quad - (4)$$

We will eliminate one of the dependent variables.

Operating with D-2 on eq (4), we get

$$(D-2)y_1 + (D-2)^2 y_2 = 0 \quad - (5)$$

Subtracting eq (3) from (5), we get

$$\begin{array}{rcl} \cancel{(D-2)y_1} + (D-2)^2 y_2 & = & 0 \\ \cancel{(D-2)y_1} + (2D-1)y_2 & = & e^{2t} \\ - \quad\quad - & & - \\ \hline (D-2)^2 y_2 - (2D-1)y_2 & = & -e^{2t}. \end{array}$$

$$[(D-2)^2 - (2D-1)]y_2 = -e^{2t}$$

$$\Rightarrow [D^2+4-\cancel{4}D-2D+1]y_2 = -e^{2t}$$

$$\cancel{\Rightarrow [D^2-4D+5]y_2 = -e^{2t}}$$

$$\Rightarrow (D^2-6D+5)y_2 = -e^{2t}$$

$$m^2 - 6m + 5 = \cancel{e}\, 0$$

$$m^2 - 5m - m + 5 = 0$$

$$m(m-5) - 1(m-5) = 0$$

$$(m-1)(m-5) = 0$$

$$m = 1, 5.$$

$$(y_2)_c = c_1e^t + c_2e^{5t}$$

$$(y_2)_p = \frac{1}{D^2-6D+5}(-e^{2t})$$

$$= -\frac{1}{4-12+5}e^{2t}$$

$$= \frac{-1}{-3}e^{2t} = \frac{e^{2t}}{3}$$

$$y_2(t) = c_1e^t + c_2e^{5t} + \frac{e^{2t}}{3}.$$

from ②, $\quad y_1 = 2y_2 - \frac{dy_2}{dt}$

$$= 2\left(c_1e^t + c_2e^{5t} + \frac{e^{2t}}{3}\right) - \left(c_1e^t + 5c_2e^{5t} + \frac{2}{3}e^{2t}\right)$$

$$= 2c_1e^t - c_1e^t + 2c_2e^{5t} - 5c_2e^{5t} + \frac{2}{3}e^{2t} - \frac{2}{3}e^{2t}$$

$$y_1(t) = c_1e^t - 3c_2e^{5t}$$

Ex
$$(2D-4)y_1 + (3D+5)y_2 = 3t+2 \quad -①$$
$$(D-2)y_1 + (D+1)y_2 = t \quad -②$$

Sol → Multiplying eq ② by 2.
$$(2D-4)y_1 + (2D+2)y_2 = 2t \quad -③$$

Subtracting ③ from eq ①.
$$
\begin{array}{r}
\cancel{(2D-4)y_1} + (2D+2)y_2 = 2t \\
\overset{+}{-}\ \cancel{(2D-4)y_1} \overset{+}{-} (3D+5)y_2 = \overset{3t+2}{-} \\
\hline
(-D-3)y_2 = -t-2
\end{array}
$$

$$(D+3)y_2 = t+2$$
$$m+3=0$$
$$m=-3$$
$$\therefore (y_2)c = c_1 e^{-3t}$$

$$(y_2)p = \frac{1}{D+3}\, t+2$$
$$= \frac{1}{3}\left(1+\frac{D}{3}\right)^{-1}(t+2)$$
$$= \frac{1}{3}\left[1-\frac{D}{3}+\frac{D^2}{9}--\right](t+2)$$
$$= \frac{1}{3}\left[t+2-\frac{1}{3}(1)+0\right] = \frac{1}{3}\left[t+2-\frac{1}{3}\right] = \frac{t}{3}+\frac{5}{9}.$$

$y_2(t) = C_1 e^{-3t} + \frac{1}{9}(3t+5).$

Substituting this value in (2), we get

$(D-2)y_1 + Dy_2 + y_2 = t$

$\Rightarrow (D-2)y_1 + \left(-3C_1 e^{-3t} + \frac{1}{9}(3)\right) + C_1 e^{-3t} + \frac{1}{9}(3t+5) = t$

$\Rightarrow (D-2)y_1 - 3C_1 e^{-3t} + C_1 e^{-3t} + \frac{1}{3} + \frac{1}{3}t + \frac{5}{9} = t$

$\Rightarrow (D-2)y_1 - 2C_1 e^{-3t} + \frac{t}{3} + \frac{8}{9} = t$

$\Rightarrow (D-2)y_1 = 2C_1 e^{-3t} + \frac{2}{3}t - \frac{8}{9}$

$(y_1)_c = C_2 e^{2t}.$

$(y_1)_p = \frac{1}{D-2} 2C_1 e^{-3t} + \frac{1}{D-2}\left(\frac{2}{3}t\right) - \frac{1}{D-2}\left(\frac{8}{9}\right)e^{0t}$

$= \frac{-2C_1 e^{-3t}}{5} + \frac{1}{(-2)}\left(1-\frac{D}{2}\right)^{-1}\left(\frac{2}{3}\right)t - \frac{8}{9(-2)}$

$= -\frac{2}{5}C_1 e^{-3t} - \frac{1}{3}\left(1+\frac{D}{2} \cdots\right](t) + \frac{4}{9}$

$= -\frac{2}{5}C_1 e^{-3t} - \frac{1}{3}\left(t+\frac{1}{2}\right) + \frac{4}{9}$

$= -\frac{2}{5}C_1 e^{-3t} - \frac{t}{3} - \frac{1}{6} + \frac{4}{9}$

$= -\frac{2}{5}C_1 e^{-3t} + \frac{1}{18}\left(-6t-3+8\right) = -\frac{2}{5}C_1 e^{-3t} + \frac{5-6t)}{18}.$

$$y_1(t) = C_2 e^{2t} - \frac{2}{5} C_1 e^{-3t} + \frac{(5-6t)}{18}.$$

Ex Find the solution of

$$(3D+1)y_1 + 3Dy_2 = 3t+1 \quad - (1)$$

$$(D-3)y_1 + Dy_2 = 2t. \quad - (2)$$

Sol: Multiplying eq (2) by 3. and subtracting from (1), we get

$$\begin{aligned} &(3D+1)y_1 + \cancel{3D y_2} = 3t+1 \\ + \; &(3D-9)y_1 + \cancel{3D y_2} = 6t \\ &- \qquad\quad - \qquad\quad - \end{aligned}$$

$$10y_1 = -3t+1$$

$$y_1 = \frac{1}{10}(1-3t)$$

$$y_1(t) = \frac{1}{10}(1-3t)$$

$$(D-3)y_1 + Dy_2 = 2t$$

$$\Rightarrow Dy_1 - 3y_1 + Dy_2 = 2t$$

$$\Rightarrow \frac{1}{10}(-3) - \frac{3}{10}(1-3t) + Dy_2 = 2t$$

$$\Rightarrow \frac{-3}{10} - \frac{3}{10} + \frac{9}{10}t + Dy_2 = 2t$$

$$\Rightarrow Dy_2 = \frac{6}{10} + 2t - \frac{9}{10}t = \frac{6}{10} + \frac{11}{10}t$$

$$Dy_2 = \frac{11}{10}t + \frac{6}{10}$$

$$y_2(t) = \frac{11}{20}t^2 + \frac{6}{10}t + C_1.$$