SOLVED EXAMPLES

Example 1 Find the value of $\vec{\nabla} r^n$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution $\vec{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2}$

where $r = (x^2 + y^2 + z^2)^{1/2}$

$$\vec{\nabla} r^n = \left\{\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2x\right\}\hat{i} + \left\{\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2y\right\}\hat{j} + \left\{\frac{n}{2}(x^2 + y^2 + z^2)2z\right\}\hat{k}$$

$$= n(x^2 + y^2 + z^2)^{\left(\frac{n}{2}-1\right)}\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$$

$$= n r^{n-2}\vec{r}$$

Example 2 Prove that $\vec{\nabla}\left[\frac{1}{r^n}\right] = -\frac{n}{r^{n+2}}\vec{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

Solution $\vec{\nabla}\left(\frac{1}{r^n}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^{-n}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{\frac{-n}{2}}$$

$$= \left\{\frac{-n}{2}(x^2 + y^2 + z^2)^{\frac{-n}{2}-1} 2x\right\}\hat{i} + \left\{\frac{-n}{2}(x^2 + y^2 + z^2)^{\frac{-n}{2}-1} 2y\right\}\hat{j} + \left\{\frac{-n}{2}(x^2 + y^2 + z^2)^{\frac{-n}{2}-1} 2z\right\}\hat{k}$$

$$= -n(x^2 + y^2 + z^2)^{-\left(\frac{n+2}{2}\right)}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{-n}{r^{n+2}}\vec{r}$$

Hence proved.

Example 3 If $\phi = x^{3/2} + y^{3/2} + z^{3/2}$, find $\vec{\nabla}\phi$.

Solution $$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^{3/2} + y^{3/2} + z^{3/2})$$
$$= \frac{3}{2}(\hat{i}x^{1/2} + \hat{j}y^{1/2} + \hat{k}z^{1/2})$$

Example 4 If $\phi(x, y, z) = 3x^2y - yz^2$, find grad ϕ at point $(1, 2, -1)$.

Solution $$\text{grad}\,\phi = \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial x}\right)(3x^2y - yz^2)$$
$$= \hat{i}(6xy) + \hat{j}(3x^2 - z^2) + \hat{k}(-2yz)$$

Value of grad ϕ at point $(1, 2, -1) = 12\hat{i} + 2\hat{j} + 4\hat{k}$

Example 5 If ϕ is a scalar field and $\vec{A}$ is vector field find the value of div $(\phi\vec{A})$.

Solution $$\text{div}(\phi\vec{A}) = \vec{\nabla}\cdot(\phi\vec{A})$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial x}\right)\cdot(\phi A_x\hat{i} + \phi A_y\hat{j} + \phi A_z\hat{k})$$
$$= \frac{\partial}{\partial x}(\phi A_x) + \frac{\partial}{\partial y}(\phi A_y) + \frac{\partial}{\partial z}(\phi A_z)$$
$$= \frac{\partial\phi}{\partial x}A_x + \phi\frac{\partial A_x}{\partial x} + \frac{\partial\phi}{\partial y}A_y + \phi\frac{\partial A_y}{\partial y} + \frac{\partial\phi}{\partial z}A_z + \phi\frac{\partial A_z}{\partial z}$$
$$= \left(\frac{\partial\phi}{\partial x}A_x + \frac{\partial\phi}{\partial y}A_y + \frac{\partial\phi}{\partial z}A_z\right) + \phi\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$
$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) + \phi\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$
$$\text{div}(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot A)$$

Example 6 Prove that $\vec{\nabla}\cdot(\vec{A} + \vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$ where A and B are differentiable vector functions.

Solution $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z$

and $(\vec{A} + \vec{B}) = \hat{i}(A_x + B_x) + \hat{j}(A_y + B_y) + \hat{k}(A_z + B_z)$ (i)

Taking divergence on both sides of Eq. (i), we have

$$\vec{\nabla}\cdot(\vec{A} + \vec{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}\right]$$
$$= \frac{\partial}{\partial x}(A_x + B_x) + \frac{\partial}{\partial y}(A_y + B_y) + \frac{\partial}{\partial z}(A_z + B_z)$$
$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)$$
$$\vec{\nabla}\cdot(\vec{A} + \vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$$

Example 7 Prove that $\vec{A}=3y^2z^2\hat{i}+3x^2z^2\hat{j}+3x^2y^2\hat{k}$ is a solenoidal vector.

Solution $\vec{A}=3y^2z^2\hat{i}+3x^2z^2\hat{j}+3x^2y^2\hat{k}$

$$\vec{\nabla}\cdot\vec{A}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(3y^2z^2\hat{i}+3x^2z^2\hat{j}+3x^2y^2\hat{k})$$
$$=\frac{\partial}{\partial x}(3y^2z^2)+\frac{\partial}{\partial y}(3x^2z^2)+\frac{\partial}{\partial z}(3x^2y^2)$$
$$=\mathbf{0}$$

As the divergence of a vector field $\vec{A}$ is zero, the vector $\vec{A}$ is solenoidal.

Example 8 Find the constant a, the the vector $\vec{A}=(x+3y)\hat{i}+(2y+3z)\hat{j}+(x+az)\hat{k}$ is a solenoidal vector.

Solution $\vec{A}=(x+3y)\hat{i}+(2y+3z)\hat{j}+(x+az)\hat{k}$

$$\text{div A}=\vec{\nabla}\cdot\vec{A}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(x+3y)\hat{i}+(2y+3z)\hat{j}+(x+az)\hat{k}\right]=0$$

or $$\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(2y+3z)+\frac{\partial}{\partial z}(x+az)=0$$

$1+2+a=0$

or $\mathbf{a=-3}$

Example 9 Calculate the value of $\vec{\nabla}\cdot(r^3\vec{r})$ where $\vec{r}=(\hat{i}x+\hat{j}y+\hat{k}z)$.

Solution Given $\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$

$$\vec{\nabla}\cdot(r^3\vec{r})=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left[(x^2+y^2+z^2)^{3/2}(\hat{i}x+\hat{j}y+\hat{k}z)\right]$$
$$=\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{3/2}\right]+\frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{3/2}\right]+\frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{3/2}\right]$$

Now $$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{3/2}\right]=(x^2+y^2+z^2)^{3/2}+x\frac{3}{2}2x[x^2+y^2+z^2]^{1/2}$$
$$=r^3+3x^2r$$

Similarly, $$\frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{3/2}\right]=r^3+3y^2r$$

and $$\frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{3/2}\right]=r^3+3z^2r$$

so $$\vec{\nabla}(r^3\vec{r})=3r^3+3(x^2+y^2+z^2)r$$
$$=3r^3+3r^2r=\mathbf{6r^3}$$

Example 10 Show that the vector field $\vec{A}=\frac{-2z^2y}{x^3}\hat{i}+\frac{z^2}{x^2}\hat{j}+\frac{2yz}{x^2}\hat{k}$ is irrotational.

Solution Given $\vec{A}=\frac{-2z^2y}{x^3}\hat{i}+\frac{z^2}{x^2}\hat{j}+\frac{2yz}{x^2}\hat{k}$

If a vector field $\vec{A}$ is irrotational, then

$$\text{Curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = 0$$

$$\text{Now } \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-2z^2y}{x^3} & \frac{z^2}{x^2} & \frac{2yz}{x^2} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{2yz}{x^2}\right) - \frac{\partial}{\partial z}\left(\frac{z^2}{x^2}\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{-2z^2y}{x^3}\right) - \frac{\partial}{\partial x}\left(\frac{2yz}{x^2}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{z^2}{x^2}\right) - \frac{\partial}{\partial y}\left(\frac{-2z^2y}{x^3}\right)\right]$$

$$= \hat{i}\left[\frac{2z}{x^2} - \frac{2z}{x^2}\right] + \hat{j}\left[\frac{-4yz}{x^3} + \frac{4yz}{x^3}\right] + \hat{k}\left[\frac{-2z^2}{x^3} + \frac{2z^2}{x^3}\right]$$

$$\vec{\nabla}\times\vec{A} = 0$$

Hence the curl of a vector field $\vec{A}$ is zero. So the vector field $\vec{A}$ is zero. So the vector field $\vec{A}$ is irrotational.

Example 11 Consider a vector field $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

(i) Is the field solenoidal?

(ii) Is the field irrotational?

Solution Given $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ (i)

Now $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$ (ii)

By using Eqs. (i) and (ii), We have

$A_x = x^2, A_y = y^2$ and $A_z = z^2$

$$\text{Now}\quad \vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z$$

$$\vec{\nabla}\cdot\vec{A} \neq 0$$

From the above, it is clear that divergence of vector field $\vec{A}$ is not equal to zero. Hence, this field is not solenoidal.

$$\text{(ii)}\quad \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$\therefore\quad \vec{\nabla}\times\vec{A} = 0$$

Since curl of the vector field $\vec{A}$ is zero, the field is irrotational.

Example 12 A vector field is given by $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k}$. Show that it is both irrotational and solenoidal.

Solution Given $\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Comparing it with $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

So $A_x = yz$, $A_y = xz$ and $A_z = xy$

Divergence of vector field $\vec{A}$

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial(yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z}$$

$$\therefore \quad \vec{\nabla}\cdot\vec{A} = 0$$

Hence vector field $\vec{A}$ is solenoidal.

$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial xy}{\partial y} - \frac{\partial xz}{\partial z}\right) + \hat{j}\left(\frac{\partial yz}{\partial z} - \frac{\partial xy}{\partial x}\right) + \hat{k}\left(\frac{\partial xz}{\partial x} - \frac{\partial yz}{\partial y}\right)$$

$$= \hat{i}(x - x) + \hat{j}(y - y) + \hat{k}(z - z)$$

$$\therefore \quad \vec{\nabla}\times\vec{A} = 0$$

Hence vector field $\vec{A}$ is irrotational.

Example 13 Given $\vec{A} = x^2y\hat{i} + (x - y)\hat{k}$. Find (i) $\vec{\nabla}\cdot\vec{A}$ and (ii) $\vec{\nabla}\times\vec{A}$.

Solution Given $\vec{A} = x^2y\hat{i} + (x - y)\hat{k}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$A_x = x^2y$, $A_y = 0$ and $A_z = (x - y)$

(i) $$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(x - y)}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A} = 2xy$$

(ii) $$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 0 & (x - y) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial(x - y)}{\partial y} - \frac{\partial(0)}{\partial z}\right) + \hat{j}\left(\frac{\partial x^2y}{\partial z} - \frac{\partial(x - y)}{\partial x}\right) + \hat{k}\left(\frac{\partial(0)}{\partial x} - \frac{\partial x^2y}{\partial z}\right)$$

$$= \hat{i}(-1) + \hat{j}(-1) - x^2\hat{k}$$

$$\vec{\nabla} \times \vec{A} = -\hat{i} - \hat{j} - x^2\hat{k}$$

Example 14 Check whether the electrostatic field represented by $\vec{E} = axy^2(y\hat{i} + x\hat{j})$ is conservative or not.

Solution If $\vec{\nabla} \times \vec{E}$ is zero, then the electrostatic field is conservative.

Given $\vec{E} = axy^3\hat{i} + ax^2y^2\hat{j}$

$$\vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z$$

So $E_x = axy^3$, $E_y = ax^2y^2$ and $E_z = 0$

$$\text{Then,} \quad \vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy^3 & ax^2y^2 & 0 \end{vmatrix} \neq 0$$

Hence $\vec{E}$ is nonconservative field.

Example 15 If 2000 flux through lines enter a given volume of space and 4000 lines diverge from it, find the total charge within the volume.

Solution Given $\phi_1 = 2000$ Vm and $\phi_2 = 4000$ Vm.

According to Gauss's theorem,

$$\phi = \frac{q}{\varepsilon_0}$$

Net flux emerging out of the surface, i.e.

$$\phi = \phi_2 - \phi_1 = 4000 - 2000 = 2000 \text{ Vm}$$

By using Eq. (i), we get

$$q = \varepsilon_0\phi = 8.85 \times 10^{-12} \times 2000$$

$$= \mathbf{1.77 \times 10^{-8}\ C}$$

Example 16 Find the total charge enclosed by a closed surface if number of lines entering is 20,000 and emerging out is 45000.

Solution Given $\phi_1 = 20{,}000$ Vm and $\phi_2 = 45{,}000$ Vm.

$\phi = \phi_2 - \phi_1$ = net flux emerging out the surface

$$\phi = 45000 - 20000$$

$$= 25{,}000 \text{ Vm}$$

According to Gauss's theorem

$$\phi = \frac{q}{\varepsilon_0} \text{ or } q = \varepsilon_0\phi$$

$$\text{or} \quad q = 8.85 \times 10^{-12} \times 25{,}000$$

$$= \mathbf{22.125 \times 10^{-8}\ C}$$

Example 17 A point charge of 13.5 Coulomb is enclosed at the centre of the cube of side 6.0 cm. Find the electric flux (i) through the whole volume and (ii) through one face of the cube.

Solution Given $q = 13.5\,\mu C = 13.5 \times 10^{-6} C$ and $a = 6.0\,cm$.

(i) According to Gauss's theorem, the total flux through the whole volume

$$\phi = \frac{q}{\varepsilon_0}$$

$$= \frac{13.5 \times 10^{-6}}{8.85 \times 10^{-12}}$$

$$= 1.525 \times 10^{6}\, Nm^2/C$$

Since a cube has 6 faces of equal area, the flux through one face of the cube would be

$$= \frac{1}{6}\frac{q}{\varepsilon_0} = \frac{1.525}{6} \times 10^{6}\ Nm^2/C$$

$$= \mathbf{2.54 \times 10^{5}\ Nm^2/C}$$

Example 18 A point charge of 11 Coulomb is located at the centre of a cube of side 5.0 cm. Calculate the electric flux through each surface.

Solution Given $q = 11\,C$ and $a = 5.0\,cm$

As a cube has six faces of equal area, so the flux through each surface of the cube is

$$= \frac{1}{6}\frac{q}{\varepsilon_0} = \frac{11}{6 \times 8.85 \times 10^{-12}}$$

$$= \mathbf{2.07 \times 10^{11} Nm^2 / C}$$

Example 19 A hollow metallic sphere of radius 0.1 m has 10^{-8} Coulomb of charge uniformly spread over it. Determine the electric field intensity (i) on the surface of the sphere (ii) at point 7 cm away from the centre and (iii) at point 0.5 m away from the centre.

Solution Given radius of the hollow sphere $(r) = 0.1\,m$ and charge on it $q = 10^{-8} C$.

Formula used for electric field intensity

$$E = \frac{1}{4\pi \epsilon_0}\frac{q}{r^2}$$

(i) Intensity on the surface of the sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$= \frac{1}{4 \times 3.14 \times 8.85 \times 10^{-12}} \times \frac{10^{-8}}{(0.1)^2}$$