Linear Differential Equations

Let y be dependent variable and x be the independent variable. We denote the derivatives as

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''' \text{ etc.}$$

$\frac{dy}{dx}$ ↓ rate of change of y w.r.t. x.

Derivatives of higher order represents rate of rates.

Differential Equation :- A differential equation contains derivatives of various orders and the variables.

Ex - (i) $y' = 6x^2$

(ii) $y'' + 16y = 2x$.

Linear Differential Equation

A linear ordinary differential equation of order n, is written as

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots a_{n-1}(x)\frac{dy}{dx} + a_n(x)\, y = r(x)$$

or $\; a_0(x)\, y^n(x) + a_1(x)\, y^{n-1}(x) + \cdots\cdots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = r(x),$

where y is dependent variable and x is independent variable and $a_0(x) \neq 0$.

If $r(x) = 0$, then it is called a homogeneous equation, otherwise it is called a non-homogeneous equation.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0$$

is a second order homogeneous equation

and $$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \quad a_0(x) \neq 0$$

is a second order non-homogeneous equation.

If $a_i(x)$, $i = 0, 1, 2$ are constants, then the equations are linear second order constant coefficient equations.

Ex

$$\left.\begin{array}{l} y'' + 4y' + 3y = x^2 e^x \\ y'' + 2y' + y = \sin x \end{array}\right] \text{Constant coefficients}$$

$$\left.\begin{array}{l} x^2 y'' + xy' + (x^2 - 4)y = 0 \\ (1 - x^2)y'' - 2xy' + 20y = 0 \end{array}\right] \text{variable coefficients}$$

(1) $y'' - a^2 y = 0 \rightarrow$ Constant coefficient

(2) $y' = \frac{y}{x} \Rightarrow xy' = y \rightarrow$ variable coefficient

(3) $y''' + 3y'' + 6y' + 12y = x^2 \rightarrow$ Constant coefficient

(4) $x^3 y''' + 9x^2 y' + 18xy' + 6y = 0 \rightarrow$ Variable coefficient

(5) $(1-x)y'' + xy' - y = 0 \rightarrow$ Variable coefficient

(6) $y'' - (1 + x^2)y = 0 \rightarrow$ Variable coefficient.

Solutions of Linear Differential Equations

Th^m If the functions $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the ~~problem~~ linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n (y) = r(x) \quad -(1)$$

$$\cancel{y(x_0) = c_1,\ y'(x_0) = c_2, \ldots, y^{n-1}(x_0) = c_n}$$

~~where $x_0 \in I$ and $c_1, c_2, \ldots, c_n$ are n known constants.~~

Also, If the conditions of the th^m are satisfied, then the diff. eq (1) is said to be normal on I.

Ex :- Find the intervals on which the following differential equations are normal.

(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, n is an integer.

Sol :- Here $a_0(x) = 1-x^2$, $a_1(x) = -2x$, $a_2(x) = n(n+1)$.

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) \neq 0$ for all x in $(-\infty, \infty)$ except at the points $-1, 1$.

Hence, the diff. eq is normal on every subinterval I of the open intervals $(-\infty, -1), (-1, 1), (1, \infty)$.

(b) $x^2 y'' + xy' + (n^2 - x^2)y = 0$, n is real.

Sol :- Here $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = n^2 - x^2$

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also $a_0(x) \neq 0$ for all $x \in (-\infty, \infty)$ except $x = 0$.
Hence, the diff. eq is normal on every subinterval I of the open intervals $(-\infty, 0)$, $(0, \infty)$.

Find the intervals on which the following eqs are normals

(7) $y' = \frac{3y}{x} \Rightarrow xy' - 3y = 0$

$a_0(x) = x, \; a_1(x) = -3.$

Now a_0 and a_1 are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) = x \neq 0$ for all x in $(-\infty, \infty)$ except $x = 0$.

$\therefore$ The diff eq is normal on every subinterval of the open interval $(-\infty, 0)$ and $(0, \infty)$.

(8) $(1+x^2)y'' + 2xy' + y = 0$

Sol: $a_0(x) = 1+x^2$, $a_1(x) = 2x$, $a_2(x) = 1$

Now, $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) = 1+x^2 \neq 0$ everywhere in $(-\infty, \infty)$. ~~except~~

Hence, the diff eq is normal on every subinterval of $(-\infty, \infty)$.

(9) $x^2y'' - 4xy' + 6y = x$.

Sol: $a_0(x) = x^2$, $a_1(x) = -4x$, $a_2(x) = 6$, $r(x) = x$.

Now, a_0, a_1, a_2 and $r(x)$ are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) \neq 0$ everywhere in $(-\infty, \infty)$ except $x = 0$.

Hence, the diff. eq is normal on every subinterval on $(-\infty, 0)$ and $(0, \infty)$.

(10) $y'' + 3y' + \sqrt{x}y = \sin x$

Sol: $a_0(x) = 1$, $a_1(x) = 3$, $a_2(x) = \sqrt{x}$, $r(x) = \sin x$.

Now $a_0(x)$, $a_1(x)$, $a_2(x)$, $r(x)$ are continuous everywhere in $[0, \infty)$.

Also, $a_0(x) \neq 0$ everywhere in $[0, \infty)$.

Hence, the diff eq is normal on every subinterval on $[0, \infty)$.

These sols does not contradict th^m 5.1 as the point $x=0$, where eq is not normal is included in the intervals.

So, th^m 5.1 is not applicable.

Linear Combination of functions

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$, where $c_1, c_2, \ldots, c_n$ are constants is called a linear combination of the given functions.

Th^m If $y_1(x), y_2(x), \ldots, y_m(x)$ are m solutions of the linear homogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \text{ on } I, \quad -(1)$$

then the linear combination of the solutions

$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_m y_m(x)$, where $c_1, c_2, \ldots, c_m$ are constants is also a sol of eq (1) on I.

Remark : The above th^m does not hold for a non-homogeneous equation or a nonlinear equation.

Verify that the given functions are solution of the associated differential equation. Verify also that a linear combination of these functions is also a solution.

(18) $1, x, e^x; \; y''' - y'' = 0.$

For $y_1 = 1$

$y_1' = y_1'' = y_1''' = 0.$

$y_1''' - y_1'' = 0 - 0 = 0.$

For $y_2 = x$, $y_2' = 1,\ y_2'' = 0,\ y_2''' = 0.$

$y_2''' - y_2'' = 0 - 0 = 0.$

For $y_3 = e^x$, $y_3' = e^x,\ y_3'' = e^x,\ y_3''' = e^x$

$y_3''' - y_3'' = e^x - e^x = 0.$

Substituting $y = c_1 \cdot 1 + c_2 \cdot x + c_3 e^x$

$= c_1 y_1 + c_2 y_2 + c_3 y_3$

$y''' - y'' = (c_1 y_1 + c_2 y_2 + c_3 y_3)''' - (c_1 y_1 + c_2 y_2 + c_3 y_3)''$

$= c_1 y_1''' + c_2 y_2''' + c_3 y_3''' - c_1 y_1'' - c_2 y_2'' - c_3 y_3''$

$= c_1(y_1''' - y_1'') + c_2(y_2''' - y_2'') + c_3(y_3''' - y_3'')$

$= c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 = 0.$

(19) $e^x, e^{-2x}; y''+y'-2y=0.$

or $y_1=e^x,\ y_1'=e^x,\ y_1''=e^x$

$\therefore y_1''+y_1'-2y = e^x+e^x-2e^x = 2e^x-2e^x=0.$

For $y_2=e^{-2x},\ y_2'=-2e^{-2x},\ y_2''=4e^{-2x}.$

$y_2''+y_2'-2y_2 = 4e^{-2x}+(-2e^{-2x})-2e^{-2x}=0.$

Substituting $y = c_1e^x+c_2e^{-2x} = c_1y_1+c_2y_2$

$y''+y'-2y = (c_1y_1+c_2y_2)''+(c_1y_1+c_2y_2)'-2(c_1y_1+c_2y_2)$

$= c_1y_1''+c_2y_2''+c_1y_1'+c_2y_2'-2c_1y_1-2c_2y_2$

$= c_1(y_1''+y_1'-2y_1)+c_2(y_2''+y_2'-2y_2)$

$= c_1\cdot 0+c_2\cdot 0=0.$

(20) $e^{-x}\cos 2x, e^{-x}\sin 2x;\ y''+2y'+5y=0.$

For $y_1 = e^{-x}\cos 2x \Rightarrow y_1' = -e^{-x}\cos 2x+e^{-x}(-2\sin 2x)$

$= -e^{-x}\cos 2x-2e^{-x}\sin 2x$

$y_1'' = e^{-x}\cos 2x+2e^{-x}\sin 2x+2e^{-x}\sin 2x$
$-4e^{-x}\cos 2x$

$= -3e^{-x}\cos 2x+4e^{-x}\sin 2x.$

$y_1''+2y_1'+5y = (-3e^{-x}\cos 2x+4e^{-x}\sin 2x)+2(-e^{-x}\cos 2x-2e^{-x}\sin 2x)$
$+5e^{-x}\cos 2x$

$=0.$

Wronskian

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions.

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x).$$

Thm If the coefficients $a_0(x), a_1(x), \ldots a_n(x)$ in the linear homogeneous equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \quad a_0 \neq 0 \quad \text{-(1)}$$

are continuous on I and $y_1(x), \ldots, y_n(x)$ are n sols of this eq, then

(i) $W(x) = W(y_1, y_2, \ldots, y_n) \neq 0$ for all $x \in I$,
$\Leftrightarrow$ $y_1(x), y_2(x), \ldots, y_n(x)$ are LI on I.

(ii) $W(x_0) = 0$ where $x_0 \in I$ is any fixed point, implies $W(x) = 0$ for all x in I and the functions $y_1(x), y_2(x), \ldots y_n(x)$ are LD.

Ex 5.6 Show that the functions x, x^2, x^3 are LI on any interval I, not containing zero.

Sol :-

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - x^2(6x) + x^3(2)$$

$$= 6x^3 - 6x^3 + 2x^3$$

$$= 2x^3.$$

Therefore, $W(x) \neq 0$ on any interval not containing zero. Hence, the functions are LI in $(-\infty, 0), (0, \infty)$.

Ex 5.7 Show that the functions $1, \sin x, \cos x$ are LI.

Sol:
$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= -\cos^2 x - \sin^2 x = -1 \neq 0.$$

Hence, the given functions are LI on any interval I.

Ex 5.8 Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$, on any interval I.

Sol: $y_1(x) = e^x, y_1' = e^x, y_1'' = e^x, y_1''' = e^x$

$y_1''' - 6y_1'' + 11y_1' - 6y = e^x - 6e^x + 11e^x - 6e^x = 0.$

$y_2(x) = e^{2x}, y_2'(x) = 2e^{2x}, y_2'' = 4e^{2x}, y_2''' = 8e^{2x}.$

$y_2''' - 6y_2'' + 11y' - 6y = 8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x} = 0.$

$y_3(x) = e^{3x}, y_3'(x) = 3e^{3x}, y_3''(x) = 9e^{3x}, y_3'''(x) = 27e^{3x}.$

$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x}$
$= 0.$

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^x & e^{2x} \\ 1 & 2e^x & 3e^{2x} \\ 1 & 4e^x & 9e^{2x} \end{vmatrix}$$

$$= e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}(18-12-9+3+4-2)$$

$$= 2e^{6x} \neq 0.$$

$\therefore$ Solutions are LI and they form a set of fundamental solutions on any interval I.

Solution of Linear differential equation – Operator Method

Operators : $\frac{d}{dx}, \frac{d^2}{dx^2}, ---, \frac{d^n}{dx^n}, ---$

For sake of convenience, the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, ---, \frac{d^n}{dx^n}$ are denoted by $D, D^2, D^3, ---, D^n$.

$$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = 0 \quad -(2)$$

$\Rightarrow D^2y + 5Dy + 6y = 0$

$\Rightarrow (D^2 + 5D + 6)y = 0.$

Solution of second order Homogeneous Linear Equations (Complementary function)

Eq in operator form : $(D^2+5D+6)y = 0 \Rightarrow f(D)y = 0$

Auxiliary eq : $m^2 + 5m + 6 = 0$ (Replace D by m to get A.E.)

$(m+2)(m+3) = 0. \Rightarrow m = -2, -3$

Solution of eq (2) depends on the nature of the roots.

Case I: The G.S. is $y = c_1e^{-2x} + c_2e^{-3x}$

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0.$$ → 2nd order homogeneous linear differential equation with constant coefficients.

Eq in operator form

$$(a_0D^2 + a_1D + a_2)y = 0.$$

A.E. is $a_0m^2 + a_1m + a_2 = 0.$ — (3)

Case I The roots are real and distinct, say, m_1, m_2 are the roots of the equation (3).

The general solution is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x},$$ where c_1 and c_2 are constants.

Ex → $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

In operator form, $(D^2 - D - 6)y = 0.$

A.E. is $m^2 - m - 6 = 0$

$\Rightarrow m^2 - 3m + 2m - 6 = 0$

$\Rightarrow m(m-3) + 2(m-3) = 0$

$\Rightarrow (m+2)(m-3) = 0.$

$\Rightarrow m = -2, 3.$

The general sol is $y(x) = c_1e^{-2x} + c_2e^{3x}.$

Ex $4y'' - 8y' + 3y = 0$, $y(0) = 1$, $y'(0) = 3$.

Eq in operator form: $(4D^2 - 8D + 3)y = 0$.

A.E. is $4m^2 - 8m + 3 = 0$

$\Rightarrow 4m^2 - 6m - 2m + 3 = 0$

$\Rightarrow 2m(2m-3) - 1(2m-3) = 0$

$\Rightarrow (2m-1)(2m-3) = 0$

$\Rightarrow m = \frac{1}{2}, \frac{3}{2}$

The general sol is $y = c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}$, $y' = \frac{1}{2} c_1 e^{\frac{1}{2}x} + \frac{3}{2} c_2 e^{\frac{3}{2}x}$

$y(0) = 1 \Rightarrow c_1 e^0 + c_2 e^0 = 1$

$\Rightarrow c_1 + c_2 = 1.$

$y'(0) = 3 \Rightarrow \frac{1}{2} c_1 + \frac{3}{2} c_2 = 3$

$\Rightarrow c_1 + 3c_2 = 6$

$2c_2 = 5 \Rightarrow c_2 = \frac{5}{2}$

$c_1 = 1 - \frac{5}{2} = -\frac{3}{2}$

$\therefore$ The sol ~~is~~ of the given problem is

$y(x) = -\frac{3}{2} e^{\frac{1}{2}x} + \frac{5}{2} e^{\frac{3}{2}x}$.

Case 2 $\quad m^2 + a_1 m + a_2 = 0$. — (3)

Roots are real and equal, say, m, m are the roots of the eq (3).

The G.S. is $y = (C_1 + x C_2) e^{mx}$.

Ex $\quad 4y'' + 4y' + y = 0$.

$(4D^2 + 4D + 1) y = 0$ (Eq in operator form)

$\Rightarrow 4m^2 + 4m + 1 = 0$ (A.E.)

$\Rightarrow 4m^2 + 2m + 2m + 1 = 0$

$\Rightarrow 2m(2m+1) + 1(2m+1) = 0$

$\Rightarrow (2m+1)(2m+1) = 0$

$\Rightarrow m = \frac{-1}{2}, \frac{-1}{2}$.

The general sol. is $y(x) = \cancel{C_1 e} \; (C_1 + x C_2) e^{-\frac{1}{2}x}$

Ex $\quad y'' + 6y' + 9y = 0, \; y(0) = 2, \; y'(0) = 3$.

Eq in operator form, $(D^2 + 6D + 9) y = 0$

A.E. is $\quad m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$\Rightarrow (m+3)(m+3) = 0$

$\Rightarrow m = -3, -3$.

The general sol is $y(x) = (C_1 + x C_2) e^{-3x}$

$y'(x) = C_2 e^{-3x} + (-3)(C_1 + x C_2) e^{-3x}$

$y(0) = 2 \Rightarrow C_1 = 2$

$y'(0) = 3 \Rightarrow C_2 - 3(C_1) = 3$

$\Rightarrow C_2 - 3(2) = 3$

$\Rightarrow C_2 = 9.$

$y(x) = (2 + 9x)e^{-3x}$ is the sol of given problem.

<u>Case 3</u> $\quad m^2 + a_1 m + a_2 = 0.$

When the roots are complex, say, $\alpha + i\beta, \alpha - i\beta$ ~~p+iq, p-iq~~.

The general sol is

~~$y(x) = C_1 e^{px}[C_1 \cos p$~~

$$y(x) = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x].$$

<u>Ex</u> $\quad y'' + 2y' + 2y = 0.$

Eq in operator method, $(D^2 + 2D + 2)y = 0$

A.E is $\quad m^2 + 2m + 2 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4 - 4(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The general solution is

$$y(x) = e^{-x}[C_1 \cos x + C_2 \sin x].$$

Ex $y'' + 4y' + 13y = 0, \; y(0) = 0, \; y'(0) = 1.$

$$(D^2 + 4D + 13)y = 0$$

$$\Rightarrow m^2 + 4m + 13 = 0$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$y(x) = e^{-2x}[C_1 \cos 3x + C_2 \sin 3x].$$

$$y'(x) = e^{-2x}[C_1(-3\sin 3x) + C_2(3\cos 3x)]$$

$$= e^{-2x}[-3C_1 \sin 3x + 3C_2 \cos 3x]$$

$y(0) = 0 \Rightarrow \quad C_1 = 0.$

$y'(0) = 1 \Rightarrow \; 3C_2 = 1 \Rightarrow C_2 = \frac{1}{3}$

$$\therefore y(x) = e^{-2x}\left[\frac{1}{3}\sin 3x\right] = \frac{1}{3}e^{-2x}\sin 3x.$$

Solution of Higher order Homogeneous linear equations with Constant coefficients.

Consider $a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0.$

Auxiliary or Characteristic eq is

$$a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0. \quad -(4)$$

Case 1 The roots of (4) are real and distinct, say, $m_1, m_2, \ldots m_n$.

Then, general solution is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

Ex $3y''' - 2y'' - 3y' + 2y = 0.$

Eq in operator form, $(3D^3 - 2D^2 - 3D + 2)y = 0.$

A.E. is $3m^3 - 2m^2 - 3m + 2 = 0.$

$\Rightarrow m^2(3m-2) - 1(3m-2) = 0$

$\Rightarrow (m^2-1)(3m-2) = 0$

$\Rightarrow (m-1)(m+1)(3m-2) = 0$

$\Rightarrow m = 1, -1, \frac{2}{3}.$

$2 \to 1, -1, 2, -2$

$3 \to 1, -1, 3, -3$

Roots $= 1, -1, \frac{1}{3}, \frac{-1}{3}, 2, -2, \frac{2}{3}, \frac{-2}{3}.$

1	3	-2	-3	2
		3	1	-2
-1	3	1	-2	0
		-3	2	
	3	-2	0	

The sol is

$$y = c_1e^{x} + c_2e^{-x} + c_3e^{\frac{2}{3}x}.$$

Ex Solve the IVP

$4y''' - 4y'' - 9y' + 9y = 0, \; y(0) = 1, y'(0) = 0, y''(0) = 0.$

Eq in operator form,

$$(4D^3 - 4D^2 - 9D + 9)y = 0$$

A.E. is $4m^3 - 4m^2 - 9m + 9 = 0$

$$4m^2(m-1) - 9(m-1) = 0$$

$$\Rightarrow (4m^2 - 9)(m-1) = 0$$

$$\Rightarrow ((2m)^2 - 3^2)(m-1) = 0$$

$$\Rightarrow (2m-3)(2m+3)(m-1) = 0$$

$$\Rightarrow m = 1, \frac{3}{2}, -\frac{3}{2}.$$

The general sol is $y = c_1 e^x + c_2 e^{\frac{3}{2}x} + c_3 e^{-\frac{3}{2}x}$

$$y' = c_1 e^x + \frac{3}{2} c_2 e^{\frac{3}{2}x} - \frac{3}{2} c_3 e^{-\frac{3}{2}x}$$

$$y'' = c_1 e^x + \frac{9}{4} c_2 e^{\frac{3}{2}x} + \frac{9}{4} c_3 e^{-\frac{3}{2}x}$$

$y(0) = 1 \Rightarrow c_1 + c_2 + c_3 = 1$ — (1)

$y'(0) = 0 \Rightarrow c_1 + \frac{3}{2} c_2 - \frac{3}{2} c_3 = 0 \Rightarrow 2c_1 + 3c_2 - 3c_3 = 0$ — (2)

$y''(0) = 0 \Rightarrow c_1 + \frac{9}{4} c_2 + \frac{9}{4} c_3 = 0 \Rightarrow 4c_1 + 9c_2 + 9c_3 = 0$ — (3)

① $\times 3$ $\quad 3C_1 + 3C_2 + 3C_3 = 3$

$\quad 2C_1 + 3C_2 - 3C_3 = 0$

$\Rightarrow 5C_1 + 6C_2 = 3$

① $\times 9 \Rightarrow 9C_1 + 9C_2 + 9C_3 = 9$

$\quad 4C_1 + 9C_2 + 9C_3 = 0$

$\Rightarrow 5C_1 = 9 \Rightarrow \boxed{C_1 = \frac{9}{5}}$

$\Rightarrow \cancel{5} \cdot \frac{9}{\cancel{5}} + 6C_2 = 3$

$\Rightarrow 6C_2 = -6 \Rightarrow \boxed{C_2 = -1}$

$C_1 + C_2 + C_3 = 1 \Rightarrow \frac{9}{5} - 1 + C_3 = 1$

$\Rightarrow C_3 = 1 - \frac{4}{5} = \frac{1}{5} \quad \boxed{C_3 = \frac{1}{5}}$

$$y = \frac{9}{5}e^{x} - e^{\frac{3}{2}x} + \frac{1}{5}e^{\frac{-3}{2}x}$$

<u>Case 2</u> When two roots are real and equal, say,

~~$m_1, m_1, m_2, m_3, ---$~~

(Two roots are equal)

$m_1 = m,\ m_2 = m,\ m_3, --- , m_n$

The general sol is

$$y = (C_1 + xC_2)e^{mx} + C_3 e^{m_3 x} + --- + C_n e^{m_n x}$$

(Three roots are equal) $m_1 = m_2 = m_3 = m,\ m_4, ---, m_n$

The general sol is

$$y = (C_1 + xC_2 + x^2 C_3)e^{mx} + C_4 e^{m_4 x} + --- + C_n e^{m_n x}.$$

$2x^3 + 9x^2 + 13x + 6 = 0.$

$1, 2 \leftarrow$ factors $\rightarrow 1, 2, 3, 6$

Cubic eq's integer sols are somewhere in the this list :

$\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1, -1, 2, -2, 3, -3, 6, -6.$

~~Case~~ 3 <u>Ex.</u> $y''' - 3y' - 2y = 0$

Eq in operator form, $(D^3 - 3D - 2)y = 0$

A.E. is $m^3 - 3m - 2 = 0. \ -①$

int solutions of this eq are somewhere in the list : $1, -1, 2, -2.$

$m = 1,\quad 1 - 3 - 2 \neq 0$

$m = -1,\quad -1 + 3 - 2 = 0. \Rightarrow$ $m = -1$ is a root of ①.

$$(m+1)(m^2 - m - 2) = 0$$

$$\Rightarrow (m+1)(m^2 - 2m + m - 2) = 0$$

$$\Rightarrow (m+1)(m(m-2) + 1(m-2)) = 0$$

$$\Rightarrow (m+1)(m+1)(m-2) = 0$$

$$\Rightarrow m = -1, -1, 2.$$

-1	1	0	-3	-2
		-1	1	2
-1	1	-1	-2	0
		-1	2	
	1	-2	0	

$$
\begin{array}{r}
m^2 - m - 2 \\
m+1 \overline{)\, m^3 - 3m - 2} \\
\underline{\pm m^3 \pm m^2} \\
-m^2 - 3m \\
\underline{-m^2 - m} \\
+ \quad + \\
-2m - 2 \\
\underline{-2m - 2} \\
\times
\end{array}
$$

The general sol is

$$\cancel{y = C_1 e^{-x} + x C_2 e}$$

$$y = (C_1 + x C_2)e^{-x} + C_3 e^{2x}$$

<u>Ex</u> $\quad 8y''' - 12y'' + 6y' - y = 0$

Eq in operator form, $(8D^3 - 12D^2 + 6D - 1)y = 0$.

A.E. is $\quad 8m^3 - 12m^2 + 6m - 1 = 0$.

factors of 8 = 1, 2, 4

-1, 1, -2, 2, -4, 4.

Factors of 1 = 1, -1

factors of given eq =

$-1, 1, \frac{-1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{-1}{4}$.

$m = 1, \quad 8 - 12 + 6 - 1 \neq 0$

$m = -1, \quad -8 - 12 - 6 - 1 \neq 0$

$\cancel{m = 2, \quad 64 - 48 + 12 - 1}$

$m = \frac{-1}{2}$,

$$\begin{array}{r|rrrr}
\frac{-1}{2} & 8 & -12 & 6 & -1 \\
 & & -4 & 8 & -7 \\
\hline
 & 8 & -16 & 14 & \boxed{-8}
\end{array}$$

$$\begin{array}{r|rrrr}
\frac{-1}{2} & 8 & -12 & 6 & -1 \\
 & & -4 & -8 & 1 \\
\hline
\frac{1}{2} & 8 & -16 & -2 & \boxed{0} \\
 & & 4 & & \\
\hline
 & 8 & -12 & &
\end{array}$$

$m = \frac{1}{2}$

$$\begin{array}{r|rrrr}
\frac{1}{2} & 8 & -12 & 6 & -1 \\
 & & 4 & -4 & 1 \\
\hline
 & 8 & -8 & 2 & \boxed{0}
\end{array}$$

$$\left(m - \frac{1}{2}\right)(8m^2 - 8m + 2) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)(8m^2 - 4m - 4m + 2) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)\big(4m(2m-1) - 2(2m-1)\big) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)(4m - 2)(2m - 1) = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$$

The general sol is

$$y = (C_1 + xC_2 + x^2 C_3)e^{\frac{x}{2}}.$$

Ex :- $y''' + 3y'' - 4y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = \frac{1}{2}.$

$$(D^3 + 3D^2 - 4)y = 0. \Rightarrow (D^3 + 3D^2 - 4)y = 0$$

$$m^3 + 3m^2 - 4 = 0 \Rightarrow m^3 + 3m^2 - 4 = 0.$$

$4 = 1, 2, 4$

factors of $4 = -1, 1, 2, -2, 4, -4$

$$\cancel{\Rightarrow m(m^2 + 3m - 4) = 0}$$

$m = 1,\ 1 + 3 - 4 = 0$

1	1	3	0	-4
		1	4	4
	1	4	4	0

$$(m-1)(m^2 + 4m + 4) = 0$$

$$\Rightarrow (m-1)(m+2)^2 = 0.$$

$$\Rightarrow m = 1, -2, -2.$$

the G.S. is $y = C_1e^x + (C_2 + xC_3)e^{-2x}$

$$y' = C_1e^x + (C_2 + xC_3)(-2e^{-2x}) + C_3 e^{-2x}$$

$$= C_1e^x - 2C_2 e^{-2x} - 2xC_3e^{-2x} + C_3e^{-2x}$$

$$y'' = C_1e^x + 4C_2e^{-2x} - 2C_3(1 \cdot e^{-2x} + x(-2e^{-2x})) + C_3(-2e^{-2x})$$

$$\cancel{= C_1e^x + 4C_2e^{-2x} - 2C_3e^{-2x} + 4xC_3e^{-2x} - 2C_3e^{-2x}}$$

$$\cancel{= C_1e^x + 4C_2e^{-2x}}$$

$$y'' = C_1e^x + 4C_2e^{-2x} - 2C_3e^{-2x} + 4C_3xe^{-2x} - 2C_3e^{-2x}.$$

$$y'' = c_1 e^{x} + 4c_2 e^{-2x} - 4c_3 e^{-2x} + 4c_3 x e^{-2x}.$$

$y(0) = 1 \Rightarrow \quad c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow \quad c_1 - 2c_2 + c_3 = 0$

$y''(0) = \frac{1}{2} \Rightarrow \quad c_1 + 4c_2 - 4c_3 = \frac{1}{2}$

$$4c_1 - 8c_2 + 4c_3 = 0$$

$\Rightarrow 5c_1 - 4c_2 = \frac{1}{2}$

$5c_1 + 5c_2 = 5 \qquad \Rightarrow 9c_2 = 5 - \frac{1}{2} = \frac{9}{2}$

$\cancel{c_2 = 5 + \frac{1}{2} = \frac{11}{2}}. \qquad \boxed{c_2 = \frac{1}{2}}$

$c_1 = 1 - \frac{1}{2} \Rightarrow \boxed{c_1 = \frac{1}{2}}$

$\frac{1}{2} - \cancel{2} \cdot \frac{1}{\cancel{2}} + c_3 = 0$

$\boxed{c_3 = \frac{1}{2}}$

$$\therefore \quad y = \frac{1}{2} e^{x} + \left(\frac{1}{2} + \frac{x}{2}\right) e^{-2x}.$$

When one pair of roots is complex.

Case 3 (i) Roots are complex and non repeated.

say, $\alpha \pm i\beta, \alpha_3, ---, \alpha_n$.

The general sol is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{\alpha_3 x} + --- + c_n e^{\alpha_n x}.$$

(ii) Roots are complex and repeated. (When one pair of roots is equal and imaginary)

Say, $\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, ---, \alpha_n$.

The G.S. is

$$y = e^{\alpha x}\left[(c_1 + x c_2)\cos\beta x + (c_3 + x c_4)\sin\beta x\right] + c_5 e^{\alpha_5 x} + ---- + c_n e^{\alpha_n x}.$$

Ex $y^{iv} + 5y'' + 4y = 0$

$(D^4 + 5D^2 + 4)y = 0$

$m^4 + 5m^2 + 4 = 0$

$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$

$\Rightarrow m^2(m^2 + 4) + 1(m^2 + 4) = 0$

$\Rightarrow (m^2 + 1)(m^2 + 4) = 0$

$\Rightarrow m = \pm i, \pm 2i$

$m^2 = \frac{-5 \pm \sqrt{25 - 16}}{2}$

$= \frac{-5 \pm 3}{2} = -1, -4$

$m = \pm i, \pm 2i$.

$m^2 + 1 = 0$

$\Rightarrow m = \frac{-0 \pm \sqrt{0 - 4(1)}}{2}$

$= \pm i$

$y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$.

Ex

$$y^{iv} + 32y'' + 256y = 0$$

$$(D^4 + 32D^2 + 256)y = 0$$

$$m^4 + 32m^2 + 256 = 0$$

$$\Rightarrow (m^2)^2 + (16)^2 + 2(m^2)(16) = 0$$

$$\Rightarrow (m^2 + 16)^2 = 0$$

$$\Rightarrow m = \pm 4i, \pm 4i.$$

The G.S is

$$y = (C_1 + xC_2)\cos 4x + (C_3 + xC_4)\sin 4x.$$

Ex

$$y^{iv} + y'' = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = -1,\ y'''(0) = -1$$

$$(D^4 + D^2)y = 0$$

$$\Rightarrow m^4 + m^2 = 0$$

$$\Rightarrow m^2(m^2 + 1) = 0$$

$$\Rightarrow m = 0, 0, \pm i$$

$$y = C_1 + xC_2 + C_3\cos x + C_4\sin x$$

$$y' = C_2 + C_3(-\sin x) + C_4\cos x$$

$$= C_2 - C_3\sin x + C_4\cos x$$

$$y'' = -C_3\cos x - C_4\sin x$$

$$y''' = +C_3\sin x - C_4\cos x$$

$y(0)=1 \Rightarrow C_1 + C_3 = 1$

$y'(0)=2 \Rightarrow C_2 + C_4 = 2$

$y''(0)=-1 \Rightarrow -C_3 = -1 \Rightarrow C_3 = 1$

$y'''(0) = -1 \Rightarrow -C_4 = -1 \Rightarrow C_4 = 1$

$\cancel{C_1=1}, \; C_2 = 1, \; C_1 = 0.$

$\therefore y = x + \cos x + \sin x.$

<u>Ex</u>

$$\frac{d^3y}{dx^3} + y = 0$$

$$(D^3+1)y=0$$

$$a^3+b^3 = (a+b)(a^2+b^2-ab)$$

$$m^3+1=0$$

$$(m+1)(m^2+1-m)=0 \Rightarrow (m+1)(m^2-m+1)=0$$

$$m=-1, \; \frac{1\pm\sqrt{1-4}}{2} = -1, \; \frac{1\pm\sqrt{3}i}{2}$$

$$m = -1, \; \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\therefore y = C_1 e^{-x} + e^{\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right].$$

Ex 5.3

(29) Find a homogeneous LDE which has following particular ~~integral~~ solution.

$xe^{-x} + e^{2x}$.

Sol: $xe^{-x} \rightarrow$ Repeated roots. $m = -1, -1$

$e^{2x} \rightarrow$ Root is $m = 2$.

$$(m+1)^2 (m-2) = 0$$

$$(m^2 + 1 + 2m)(m-2) = 0$$

$$m^3 - \cancel{2m^2} + m - 2 + \cancel{2m^2} - 4m = 0$$

$$\Rightarrow m^3 - 3m - 2 = 0$$

$$\frac{d^3y}{dx^3} - 3\frac{dy}{dx} - 2y = 0.$$

Ex $(D^2 + D + 5)y = 3$

Sol: C.F. $m^2 + m + 5 = 0$

$$m = \frac{-1 \pm \sqrt{1-20}}{2}$$

$$= \frac{-1 \pm \sqrt{19}i}{2}$$

$$y(x) = e^{-x/2}\left[C_1 \cos \sqrt{19}x + C_2 \sin \sqrt{19}x\right].$$

P.I. $y_p(x) = \dfrac{1}{D^2+D+5} \, 3e^{0x}$

$= 3 \dfrac{1}{0+0+5} e^{0x}$

$= \dfrac{3}{5}.$

$y(x) = e^{-x/2}\left[C_1 \cos\sqrt{19}x + C_2 \sin\sqrt{19}x\right] + \dfrac{3}{5}.$

Ex $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

$m = 1, \quad 1 - 2 + 5 - 8 + 4 = 0.$

$$\begin{array}{c|ccccc}
1 & 1 & -2 & 5 & -8 & 4 \\
 & & 1 & -1 & 4 & -4 \\
\hline
 & 1 & -1 & 4 & -4 & \boxed{0}
\end{array}$$

$(m-1)(m^3 - m^2 + 4m - 4) = 0$

$(m-1)\left[m^2(m-1) + 4(m-1)\right] = 0$

$(m-1)(m^2+4)(m-1) = 0$

$m = 1, 1, \pm 2i$

$y(x) = (C_1 + xC_2)e^x + C_3 \cos 2x + C_4 \sin 2x.$