Topic: Eigen Values and Eigen Vectors

Learning Outcomes

1. Characteristic Equation
2. Eigen Values/ Characteristic Values / Latent values / Spectral Values
3. Eigen Vectors/ Latent Vectors

Let $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ → distance from x-axis → distance from y-axis

Scalar → "mag"
Vector → "mag + direction"
↓
a line segment with direction.

$$Au = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8-8 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

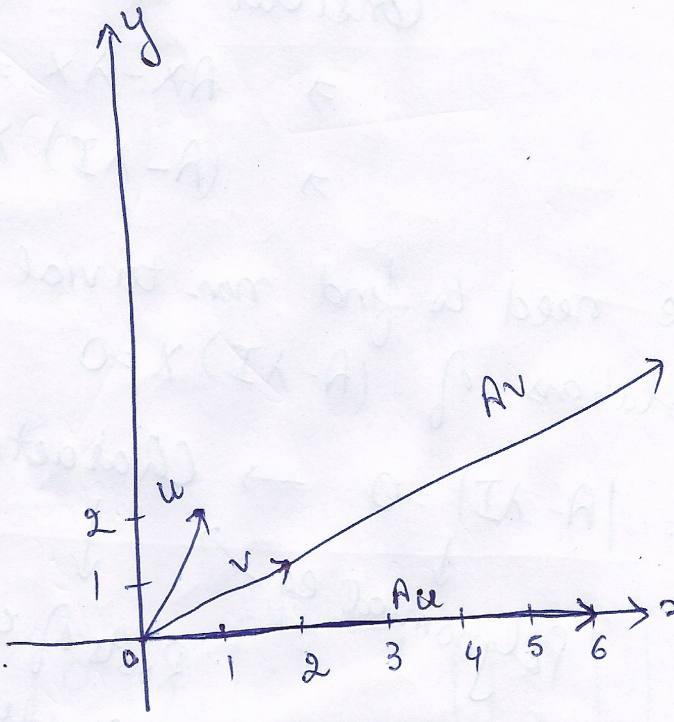
$$Av = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16-4 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 6v$$

$\Rightarrow Av = 6v.$

- No relation between u and Au

$\Rightarrow$ v and Av are parallel i.e. their direction is same.

Only the magnitude varies.



When we multiply A with v, it does not change its direction, only the magnitude is increased. Such vectors are called eigen vectors.

$Av = 6v \rightarrow$ Eigen Vector

$\downarrow$

Eigen Value

Def :- Let A be any square matrix (real or complex). A scalar λ is called ~~the~~ an eigenvalue of A if there exists a non-zero vector X such that

$AX = \lambda X$.

$\uparrow$ Eigen Vector $\quad \searrow$ Eigen Value

$\rightarrow$ X=0 always satisfies $AX = \lambda X$, We have to find $X (\neq 0)$ vectors or non-zero solution of $AX = \lambda X$

The eigenvector X is an eigenvector associated with an eigenvalue λ.

How to find an eigen value and eigen vector

Consider $AX = \lambda X$

$\Rightarrow AX - \lambda X = 0 \nearrow$ Two unknowns λ and X.

$\Rightarrow (A - \lambda I) X = 0 \rightarrow$ Homogeneous system of equations.

$\downarrow$

$|A| \neq 0 \rightarrow$ zero/trivial sol.

We need to find non-trivial solutions of $(A - \lambda I) X = 0$

$\therefore |A - \lambda I| = 0 \rightarrow$ Characteristic Eq.

$\downarrow$ Polynomial eq $\nearrow$

$\downarrow$ Roots of characteristic equations gives us eigen values λ.

<u>Remarks</u> ① Roots of Characteristic eq are eigenvalues, Some times called characteristic values/roots.

② Eigenvectors of A can be determined by solving the homogeneous system of equations $(A-\lambda I)x=0$ for each eigenvalue λ.

① $A=\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$

Ch. eq: $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)(2-\lambda)-12=0$$

$$\Rightarrow 2-\lambda-2\lambda+\lambda^2-12=0$$

$$\Rightarrow \lambda^2-3\lambda-10=0$$

$$\Rightarrow \lambda^2-5\lambda+2\lambda-10=0$$

$$\Rightarrow \lambda(\lambda-5)+2(\lambda-5)=0$$

$$\Rightarrow (\lambda+2)(\lambda-5)=0.$$

$$\Rightarrow \lambda=5,-2$$

Say, $\lambda_1=5,\ \lambda_2=-2.$

<u>Eigen vector corresponding to $\lambda_1=5$</u>

$$(A-5I)X=0$$

$$\Rightarrow \begin{bmatrix}-4 & 4\\ 3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \sim \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\Rightarrow x_1 = x_2$.

Let $x_2 = \alpha$. (free variable)

Then $x_1 = \alpha$

$$\therefore X = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For different values of α, we get different multiples of eigen vector X.

If take $\alpha = 1$, $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the corresponding eigen vector of $\lambda_1 = 5$.

Remark :- ① Any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also an eigen vector of ~~$\lambda_1=5$~~ corresponding to $\lambda_1 = 5$.

② We can say that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is generator of sol.

Note :- There can be more than one eigen vectors for one eigen value.

Eigen vector corresponding to $\lambda_2 = -2$

$$(A + 2I)X = 0$$

$$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\cancel{3x_1 + 3x_2}$ $3x_1 + 4x_2 = 0$

$\Rightarrow 3x_1 = -4x_2$

$\Rightarrow x_1 = -\frac{4}{3}x_2$

Let $x_2 = \alpha$, $x_1 = -\frac{4}{3}\alpha$

$$X = \alpha\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$$

Let $\alpha = 3$, $X = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_2 = -2$.

(2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Ch. Eq.^{n}

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda = 1, 2, 3.$ Let $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$

Eigenvector Corresponding to $\lambda_1 = 1$.

$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{R_2 \Rightarrow}$ $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

Let $x_3 = \alpha \Rightarrow x_1 = -\alpha,\ x_2 = -\alpha$

$$X = \alpha\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $\alpha = 1 \Rightarrow X = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_1 = 1$.

Eigen vector corresponding to $\lambda_2 = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$-x_1 = 0 \Rightarrow x_1 = 0$

$x_3 = 0$

Let $x_2 = \alpha \to$ free variable.

$$X = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $\alpha = 1$, $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 2$.

Eigenvector corresponding to $\lambda_3 = 3$

$(A-3I)X = 0$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 = 0 \Rightarrow x_1 = 0$

$-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$

Let $x_3 = \alpha$

$$X = \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $\alpha = 1$, then $X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_3 = 3$.

Note:- ① Distinct eigen values have distinct eigenvectors.

② Eigenvectors corresponding to distinct eigenvalues are L.I.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \det = (-1)(1-0) = -1 \neq 0.$$

③ $$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Ch. Eq.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 2\right] - 2(0+1) + 2(0+2-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left[4+\lambda^2-4\lambda-2\right] - 2 + 4 - 2\lambda = 0$$

$$\Rightarrow 4+\lambda^2-4\lambda-2-4\lambda-\lambda^3+4\lambda^2+\cancel{2\lambda}+\cancel{8}^{2}-\cancel{2\lambda}=0$$

$$\Rightarrow -\lambda^3+5\lambda^2-8\lambda+4=0$$

$$\Rightarrow \lambda^3-5\lambda^2+8\lambda-4=0$$

$\lambda = 1$,

$$\begin{array}{r|l} & \lambda^2-4\lambda+4 \\ \lambda-1 & \lambda^3-5\lambda^2+8\lambda-4 \\ & \pm\lambda^3 \mp \lambda^2 \\ \hline & -4\lambda^2+8\lambda \\ & -4\lambda^2+4\lambda \\ & + \quad - \\ \hline & 4\lambda-4 \\ & 4\lambda-4 \\ \hline & \times \end{array}$$

$$(\lambda-1)(\lambda^2-4\lambda+4) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$$\Rightarrow \lambda = 1, 2, 2., \text{ Say, } \lambda_1 = 1, \lambda_2 = \lambda_3 = 2.$$

Eigen vector corresponding to $\lambda_1 = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\cancel{-x_1 + 2x_2 + x_3 = 0}$$

$$\cancel{-x_1 + 2x_2 + x_3 = 0}$$

$$\cancel{x_2 + x_3 = 0 \Rightarrow x_2 = -x_3}$$

$$\cancel{\text{Let } x_3 = \alpha \Rightarrow x_2 = -\alpha}$$

$$\cancel{x_1 = 2\alpha + \alpha \Rightarrow 3\alpha}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$ i.e. $x_1 = x_2$

$x_2 + x_3 = 0 \Rightarrow x_3 = -x_2$

Let $x_2 = \alpha$ (say)

$x_1 = \alpha, \quad x_3 = -\alpha$

$$X = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Let $\alpha = 1$, $X = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_1 = \cancel{2}\, 1$.

Eigen vector Corresponding to $\lambda_2 = \lambda_3 = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$x_3 = 0$$

Let $x_2 = \alpha$, $x_1 = 2\alpha$

$$X = \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Let $\alpha = 1$, $X = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_2 = \lambda_3 =$

(4) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 0] - 2[3(1-\lambda) - 0] + 3[0 + 2(1-\lambda)] = 0$

$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 6 + 6] = 0$

$\Rightarrow (1-\lambda)^3 = 0$

$\lambda = 1, 1, 1.$

Eigen vector Corresponding to $\lambda = 1$

$(A - I)X = 0$

$$\begin{bmatrix} 0 & 2 & 3 \\ 3 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

$2x_2 + 3x_3 = 0 \Rightarrow x_2 = -\frac{3}{2} x_3$

let $x_3 = \alpha$.

$$X = \alpha \begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix}$$

let $\alpha = 2$

$\Rightarrow X = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ is an eigen vector of $\lambda = 1, 1, 1$.

Note :- Repeated Eigen values may have same or distinct eigen vectors.

(5) $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)\left((3-\lambda)^2 - 4\right) = 0$

$\Rightarrow (5-\lambda)\left[9 + \lambda^2 - 6\lambda - 4\right] = 0$

$\Rightarrow (5-\lambda)\left[\lambda^2 - 6\lambda + 5\right] = 0$

$\Rightarrow (\lambda-5)(\lambda^2 - 5\lambda - \lambda + 5) = 0$

$\Rightarrow (\lambda-1)(\lambda-5)^2 = 0 \Rightarrow \lambda = 1, 5, 5$

$\lambda_1 = 1;\ \lambda_2 = \lambda_3 = 5.$

Eigenvector corresponding to $\lambda_1 = 1$

$(A-I)X = 0$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2$ Let $x_2 = \alpha \Rightarrow x_1 = \alpha$

$x_3 = 0$

$$X = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $\alpha = 1$, $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_1 = 1$.

Eigen vector corresponding to $\lambda_2 = \lambda_3 = 5$

$(A-5I)X = 0$

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$. Say, $x_2 = \alpha_1 \Rightarrow x_1 = -\alpha_1$

Say, $x_3 = \alpha_2$

$$X = \begin{bmatrix} -\alpha_1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So, corresponding to $\lambda_2 = \lambda_3 = 5$, we have two L.I. eigenvectors

$$\text{L.I.} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Note:</u> Repeated eigen values may have same or distinct eigen vectors.