# Matrices

Complex number:- $z = a+ib$, $a, b \in \mathbb{R}$

$\bar{z} = a - ib$. $i = \sqrt{-1}$

## Conjugate of a matrix

Let $A = [a_{ij}]_{m\times n}$ be a complex matrix. Then the matrix obtained by replacing all the elements of A by their complex conjugate ~~is~~ is called the conjugate of the matrix and is denoted by $\bar{A} = [\bar{a}_{ij}]_{m\times n}$, where $\bar{a}_{ij}$ is the conjugate of $a_{ij}$.

Ex $A = \begin{bmatrix} 2 & 5+i \\ 3+i & 1 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 2 & 5-i \\ 3-i & 1 \end{bmatrix}$.

- $(\bar{A})^T$ is called the ~~is~~ conjugate transpose of A and is denoted by $A^\theta$.

## Hermitian Matrix

A square matrix A over the complex numbers is said to be hermitian if $A^\theta = A$. $a_{ij} = \bar{a}_{ji}$ for all $i, j$.

$A = \begin{bmatrix} 2 & 5+i \\ 5-i & 7 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 2 & 5-i \\ 5+i & 7 \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} 2 & 5+i \\ 5-i & 7 \end{bmatrix} = A$

$\Rightarrow A = A^\theta$

Remark :- ① The diagonal elements of a ~~skew~~ hermitian matrix are always real.

② Eigen values of a hermitian matrix are real.

## Skew Hermitian Matrix

A square matrix $A$ over the complex numbers is said to be skew hermitian if $A^{\theta} = -A$. or $a_{ij} = -\bar{a}_{ji} \ \forall\, i,j$.

$$A = \begin{bmatrix} 3i & 3i+5 \\ 3i-5 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -3i & -3i+5 \\ -3i-5 & 0 \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} -3i & -3i-5 \\ -3i+5 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 3i & 3i+5 \\ 3i-5 & 0 \end{bmatrix} = -A.$$

**Remark** ① The diagonal elements of a skew hermitian matrix are either purely imaginary or zero.

② The eigenvalues of a skew hermitian matrix are zero or purely imaginary.

## Symmetric & Skew symmetric

① Eigen values of a real symm matrix are real.

② Eigenvalues of a real skew symmetric matrix are either zero or purely imaginary.

**Thm** Every complex square matrix can be uniquely expressed as the sum of hermitian and skew hermitian matrix.

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta}).$$

$$A = \begin{bmatrix} 2-3i & -4+7i \\ 9i & -12 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2+3i & -4-7i \\ -9i & -12 \end{bmatrix} \Rightarrow A^{\theta} = \begin{bmatrix} 2+3i & -9i \\ -4-7i & -12 \end{bmatrix}$$

$$A + A^{\theta} = \begin{bmatrix} 4 & -4-2i \\ -4+2i & -24 \end{bmatrix} \Rightarrow \frac{1}{2}(A+A^{\theta}) = \begin{bmatrix} 2 & -2-i \\ -2+i & -12 \end{bmatrix}$$

$$A - A^{\theta} = \begin{bmatrix} -6i & -4+16i \\ 4+16i & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(A-A^{\theta}) = \begin{bmatrix} -3i & -2+8i \\ 2+8i & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-3i & -4+7i \\ 9i & -12 \end{bmatrix} = \begin{bmatrix} 2 & -2-i \\ -2+i & -12 \end{bmatrix} + \begin{bmatrix} -3i & -2+8i \\ 2+8i & 0 \end{bmatrix}$$

## Orthogonal Matrix

A square matrix A is called orthogonal if $AA^T = A^TA = I$.

$\Rightarrow A^{-1} = A^T$.

Ex

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & \frac{-4}{9} \\ \frac{4}{9} & \frac{-4}{9} & \frac{-7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$$

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad B^T = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \\ \frac{8}{9} & \frac{-4}{9} & \frac{1}{9} \\ \frac{-4}{9} & \frac{-7}{9} & \frac{4}{9} \end{bmatrix}$$

~~$BB^T$~~ =

$AA^T = I$.

Trick If you multiply one row with another, and ~~the multiplicati~~ if its 0, then matrix is orthogonal.

**Properties** ① Every orthogonal matrix is non-singular/invertible.

② If $A$ is orthogonal, then $A^{-1}$ and $A^T$ are also orthogonal.

③ Determinant of orthogonal matrix is $\pm 1$.

④ Eigenvalues of an orthogonal matrix are of unit modulus.
i.e. $|\lambda| = 1$.

## Unitary Matrix

A square matrix with complex entries is said to be unitary if $A^\theta A = AA^\theta = I$.

**Eg:**
$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \Rightarrow A^\theta = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$AA^\theta = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1+i)(1-i)+(1-i)(1+i) & (1+i)^2+(1-i)^2 \\ (1-i)^2+(1+i)^2 & (1-i)(1+i)+(1+i)(1-i) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1-\cancel{i}+\cancel{i}-i^2+1+\cancel{i}-\cancel{i}-i^2 & 1+i^2+\cancel{2i}+1+i^2-\cancel{2i} \\ 2-2 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

**Properties** ① Inverse of a unitary matrix is unitary.

② Inverse of orthogonal matrix is orthogonal.

③ Transpose of unitary | orthogonal matrix is unitary | orthogonal.

④ Product of two unitary (orthogonal) matrices is unitary (orthogonal).

⑤ The eigenvalues of a unitary matrix are of unit modulus. i.e. $|\lambda| = 1$.

⑥

## Normal Matrix

A square matrix is called normal if

$$AA^{\theta} = A^{\theta}A.$$

For real matrices, $AA^T = A^TA$.

$$A = \begin{bmatrix} 4i & -1+i \\ 1-i & 4i \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -4i & -1-i \\ 1+i & -4i \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} -4i & 1+i \\ -1-i & -4i \end{bmatrix}$$

$$AA^{\theta} = \begin{bmatrix} 4i & -1+i \\ 1-i & 4i \end{bmatrix}\begin{bmatrix} -4i & 1+i \\ -1-i & -4i \end{bmatrix} = \begin{bmatrix} 16+2 & 8i \\ -8i & 18 \end{bmatrix} = \begin{bmatrix} 18 & 8i \\ -8i & 18 \end{bmatrix}.$$

$$A^{\theta}A = \begin{bmatrix} -4i & 1+i \\ -1-i & -4i \end{bmatrix}\begin{bmatrix} 4i & -1+i \\ 1-i & 4i \end{bmatrix} = \begin{bmatrix} 18 & 8i \\ -8i & 18 \end{bmatrix}. \Rightarrow \cancel{A}$$

**Properties**:- ① A symm and a skew symm matrix both are normal.

② Orthogonal matrix is normal.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$AA^T = A^TA.$$

## Idempotent Matrix

A square matrix is said to be idempotent if $A^2 = A$.

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 16-12 & -4+3 \\ 48-36 & -12+9 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A.$$

## Involutary Matrix

A sq matrix ———— if $A^2 = I$.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

## Nilpotent

A sq matrix ———— if $A^k = 0$, where $k$ is least positive integer.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 4-4 & 8-8 \\ -2+2 & -4+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$