Lecture -2

Topic : Rank of a Matrix and Elementary operations of Matrices.

Learning outcomes :

① Definition of Rank of a matrix.
② Calculation of Rank using determinants.
③ Calculation of Rank using elementary operations.

Minor

If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the minor of A of order r.

Obviously, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

Rank Def. (Minor/Determinant based)

A matrix is said to be of rank r when

(i) It has at least one non-zero minor of order r,

and (ii) Every minor of order higher than r vanishes.

The rank of a matrix is the largest order of any non-vanishing minor of the matrix.

Notation :- $\rho(A)$ or $r(A)$.

$\rho \rightarrow$ rho

Note:- If A is a matrix of order $m \times n$ that is $[A]_{m\times n}$, $r(A) \le \min\{m,n\}$.

rectangular if $m \neq n$

If A is square matrix, then $r(A) \le n$.
$[A]_{n\times n}$

Eg :- ①

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 \end{bmatrix}$$

Order of A = 4×4.

$\rho(A) \le 4$.

$|A| = 0.$ $\Rightarrow \rho(A) \neq 4$

$\Rightarrow \rho(A) < 4$

Consider a 3×3 minor,

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}, \text{ Its det} = -1(6-0) = -6 \neq 0$$

$\therefore \rho(A) = 3.$

(2) (T) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Order of $A = 3 \times 4$

$\therefore \rho(A) \leq \min\{3,4\} = 3.$

$\Rightarrow \rho(A) \leq 3.$

Consider any minor of order 3.

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{vmatrix} = 1(0-49) - 2(0-56) + 3(35-48)$$

$$= -49 + 112 - 39$$

$$= 112 - 88$$

$$= 24 \neq 0.$$

$\therefore \rho(A) = 3.$

(3) (T)

$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

Order of $A = 3 \times 4$

$\rho(A) \leq \min\{3,4\} = 3.$

$\rho(A) \leq 3.$

Consider the third order minors,

~~R_1R_2~~ ① C_1, C_2, C_3, ② C_1, C_2, C_4 ③ C_1, C_3, C_4, ④ C_2, C_3, C_4

① $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 1(12-6) - 2(6-3) - 1(12-12)$

$= 6 - 6 - 0 = 0.$

② $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 1(-28+12) - 2(-14+6) + 3(12-12)$

$= -16 + 16 + 0$

$= 0$

③ $\begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 1(-7+6) + 1(-14+6) + 3(6-3)$

$= -1 - 8 + 9 = 0.$

④ $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 2(-7+6) + 1(-28+12) + 3(12-6)$

$\cancel{= -2 - 12 + 1}$

$= -2 - 16 + 18 = 0.$

Since all third order minors vanishes, $\rho(A) \neq 3$.

$\therefore \rho(A) < 3.$

Consider the second order minors,

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2 + 4 = 6 \neq 0.$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2$.

Elementary Transformations of a matrix

① Interchange of any two rows or columns.

$R_i \leftrightarrow R_j$, $C_i \leftrightarrow C_j$.

② Multiplication of each element of a row or column by any non-zero scalar k.

$R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$.

③ Addition to the elements of any row or column the same scalar multiples of corresponding elements of any other row or column.

$R_i \rightarrow R_i + kR_j$, $C_i \rightarrow C_i + kC_j$

Equivalent Matrices

Two matrices A and B are said to be equivalent if one is obtained from the ~~other~~ another by a sequence of elementary transformations.

We write it as $A \sim B$ or $B \sim A$.

Echelon Form of a matrix

A matrix of order $m \times n$ is said to be in echelon form if

~~① Every row of A~~

① Zero rows, if any exists, they should be below the ~~no~~ non-zero row.

Zero row → all elements are zero.

Non-zero row → If at least one element is zero.

② The number of zeroes before the first non-zero element in a row is less than the number of such zeroes in the next row.

Eg

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note → ① Only row operations are allowed.

② Reduce to upper triangular matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & b \end{bmatrix}$$

Rank of a matrix

Definition (Echelon form based)

The number of non-zero rows ~~or columns~~ in the Echelon form of a given matrix.

Eg ① Find the rank of the matrix ~~using~~ by reducing it to Echelon form.

Sol:

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix} \qquad \rho(A) \leq 3.$$

$R_1 \leftrightarrow R_3$.

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$.

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_2 \to \frac{1}{7} R_2$

$R_3 \to \frac{1}{9} R_3$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

① All the zero rows are below non zero rows.

② No. of zeros before first non zero entry in a row is less tha[n] no. of zeroes in the next row.

Thus, this is the echelon form of A.

Rank $= \rho(A) =$ Number of non-zero rows $= 2$.

② ① Find the ~~mat~~ rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \qquad \rho(A) \leq 3.$$

Sol → $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, ~~$R_3 \to 3R_1$~~. $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2.$

③ Find the rank of matrix A.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

$\rho(A) \leq \min\{3,4\} = 3.$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$\rho(A) = 3.$

④ Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{bmatrix}$$

$\rho(A) \leq 3.$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore \rho(A) = 3.$

⑤ Find the rank of the matrix

$$A = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$$

Determinant method $\rho(A) \leq 3$.

$|A| = 5(16-16) - 3(8-8) - 0 = 0$

$\therefore \rho(A) < 3$.

Consider a second order minor,

$$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10-3 = 7 \neq 0.$$

$\therefore \rho(A) = 2$.

Echelon form

$$A = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 + 2R_1$.

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & -7 & 20 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$.

6) Find rank of A.

(I)

$$A = \begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$$

Determinant method

$\rho(A) \leq 3$

$|A| = 0 + 1(10+6) + 5(2-4)$

$= 16 - 10 = 6 \neq 0$.

$\therefore \rho(A) = 3$.

Echelon form

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 2 & 4 & -6 \\ 0 & -1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & -16 \\ 0 & -1 & 5 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -8 \\ 0 & -1 & 5 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -8 \\ 0 & 0 & -3 \end{bmatrix}$$

$\therefore \rho(A) = 3.$

(7) For what values of k the matrix

$$\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

has rank 3.