① For trivial sol

$$|A| \neq 0$$

$$2k^2 + 2k - 12 \neq 0$$

$\Rightarrow k \neq 2$ and $k \neq -3$

$$k = \frac{-2 \pm \sqrt{4 - 4(2)(-12)}}{2(2)}$$

$= 2, -3$

② For non-trivial sol

$$|A| = 0$$

$\Rightarrow k = 2$ or $k = -3$.

Solution of the linear system of equations by using Gauss elimination Method (Rank of the matrix).

Non homogeneous system:-

Given the linear system $Ax = B$ and the augmented matrix $[A|B]$.

↓ matrix obtained by attaching the elements of B as the last column in the coefficient matrix A.

① If $\rho(A) = \rho(A|B) \Rightarrow$ The system of equations $AX = B$ is Consistent. (Sol exists)

Case 1 :- If $\rho(A) = \rho(A|B) =$ ~~No of rows in x~~, No. of unknowns

Then the system has a Unique solution.

Case 2 If $\rho(A) = \rho(A|B) <$ No ~~of rows in x~~, of unknowns

then the system has ∞ many solutions.

② If $\rho(A) < \rho(A|B) \Rightarrow$ The system is inconsistent.

$\Rightarrow$ No sol.

Homogeneous System

$$AX = 0.$$

The system is always consistent.
i.e. $x = y = z = 0$ is always the solution of the system, known as zero solution.

Non trivial solution

If $\rho(A|B) = \rho(A) = r < n \rightarrow$ No. of unknowns, then system will have an infinite number of non-zero (non-trivial) solutions.

① $x + y + 2z = 4$
$2x - y + 3z = 9$
$3x - y - z = 2$

The system can be written as

$$AX = B.$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

The augmented matrix $[A|B]$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

Convert it to the upper triangular matrix.

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{array}\right]$$

$R_3 \to 3R_3 - 4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{array}\right]$$

$\rho(A) = 3, \ \rho(A|B) = 3$

= No of unknowns

$\Rightarrow$ The system is consistent and has a unique sol.

Using the back substitution method, we get,

$$-17z = -34$$

$$\boxed{z = 2}$$

$\cancel{-3x =} \quad \cancel{-3y0 - z}$

$$-3y - z = 1$$

$$-3y = 1 + z = 3$$

$$\boxed{y = -1}$$

$$x + y + 2z = 4$$

$$x - 1 + 4 = 4$$

$$\boxed{x = 1}$$

$x = 1, \ y = -1, \ z = 2$ is the solution.

(2) Solve $x+2y-z=3$

$2x+2y=4$

$x+3y-2z=4$

The system can be written as

$AX=B$

$\rho(A)=2$

$\cancel{\rho(A|B)}$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 2 & 0 & 4 \\ 1 & 3 & -2 & 4 \end{array}\right]$$

$R_2 \to R_2-2R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 2 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$R_2 \to -\frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$\rho(A|B)=2$

Since $\rho(A)=\rho(A|B)=2<$ No. of unkno

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ The system is consiste

and have ∞ many solutions.

By back substitution,

$$y - z = 1$$

$$x + 2y - z = 3$$

$$\boxed{y = 1 + z}$$

$$x = 3 + z - 2 - 2z = 1 - z$$

$$\boxed{x = 1 - z}$$

$x = 1 - z$, $y = 1 + z$, ~~where z is arbitrary.~~

Let $z = k$
$x = 1-k$, $y = 1+k$, $z = k$.
For different values of k, we have ~~different~~ infinite values of x, y, z

(3) Solve

$$x + 2y - 3z = 1$$
$$2x + 4y - 6z = 1$$
$$3 + 6y - 9z = 1$$

$$AX = B$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -6 & 1 \\ 3 & 6 & -9 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 1$, $\rho(A|B) = 2$

$\rho(A) < \rho(A|B) \Rightarrow$ The system is inconsistent.

There is no solution.

④ Solve

$$x+2y+z=1$$
$$2x+2y \quad =1$$
$$x+3y+z=1$$

$AX=B$

$$[A|B]=\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \end{array}\right)$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$ $\qquad$ $R_3 \to \frac{-1}{2} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$\rho(A) = 3$

$\rho(A|B) = 3$

$\Rightarrow \rho(A) = \rho(A|B) = 3 =$ No of unknowns

$\Rightarrow$ The system is consistent and have a unique solution.

By back substitution,

$z = \frac{1}{2}$

$y = 0$

$x + 2y + z = 1$

$x + \frac{1}{2} = 1 \Rightarrow x = \frac{1}{2}$

(5) $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

$AX = 0$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$ $\qquad$ $R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2,\ \rho(A|B) = 2$

$\rho(A) = \rho(A|B) = 2 <$ No of unknowns $= 3$.

$\Rightarrow$ We have infinite number of solutions.

$$-7y + 8z = 0 \Rightarrow y = \frac{8}{7}z$$

$$x + 3y - 2z = 0$$

$$x = -\frac{24}{7}z + 2z = -\frac{10}{7}z$$

Let $z = k$, $x = -\frac{10}{7}k$, $y = \frac{8}{7}k$, $z = k$.

For different values of k, we get infinite number of values of x, y, z.

Linear system of Equations

- **Homogeneous** → (Sol always exists)
  $AX = 0$
  - **Cramer's Rule**
    - $|A| \neq 0$
      Trivial sol
      $x = y = z = 0$
    - $|A| = 0$
      Non trivial
      i.e. Infinitely many solutions
  - **Gauss Elimination**
    - $x = y = z = 0$
      (zero sol) always exists
    - $\rho(A|B) = \rho(A) <$ No of unknowns
      Then non trivial i.e. infinitely many sol.
- **Non-Homogeneous**
  $AX = B$
  - **Cramer's Rule**
  - **Gauss Elimination**

$AX = B$

- **Cramer's Rule**
  - **Consistent**
    - Unique sol
      If $D \neq 0$
      $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$
    - Infinitely many sol
      $D = 0$ and $D_x = D_y = D_z = 0$
  - **Inconsistent**
    $D = 0$ but D_x, D_y, D_z not all zero.
- **Gauss Elimination** → Inconsistent
  $\rho(A) < \rho(A|B)$
  - **Consistent**
    - Unique sol
      $\rho(A) = \rho(A|B)$ = No of unknowns
    - Infinitely many sol
      $\rho(A) = \rho(A|B) <$ No of unknowns