Unit 4 Fourier Series

MTH-174

Sequence

A list of elements with a specific pattern.

$1, 3, 5, 7, \ldots$

Eg: **Arithmetic progression (A.P.)**

A seq of numbers such that the difference between two consecutive terms is a constant.

$2, 4, 6, 8, \ldots$ $\quad d = 2$

Geometric Progression (G.P.)

A seq where each succeeding term is produced by multiplying each proceding term by a fixed number, called common ratio.

$2, 4, 8, 16, \ldots$ $\quad r = 2$

Series

If $a_1, a_2, \ldots, a_n, \ldots$ be an infinite seq, then

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

or $\sum_{n=1}^{\infty} a_n$ is called a series.

Fourier Series

A series whose each and every term is either of sine or cosine, is called a fourier series.

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots +$$
$$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

We express a function $f(x)$ in the form of a series of sines and cosines.

Applications

1. Any sound wave can be expressed in terms of sine and cosine waves (fourier series), which is ideal for transmission in communication system.
2. Used in compressing any wave form.
3. Used in signal processing, modulation and demodulation of voice signals.

Euler's Formulae

The fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

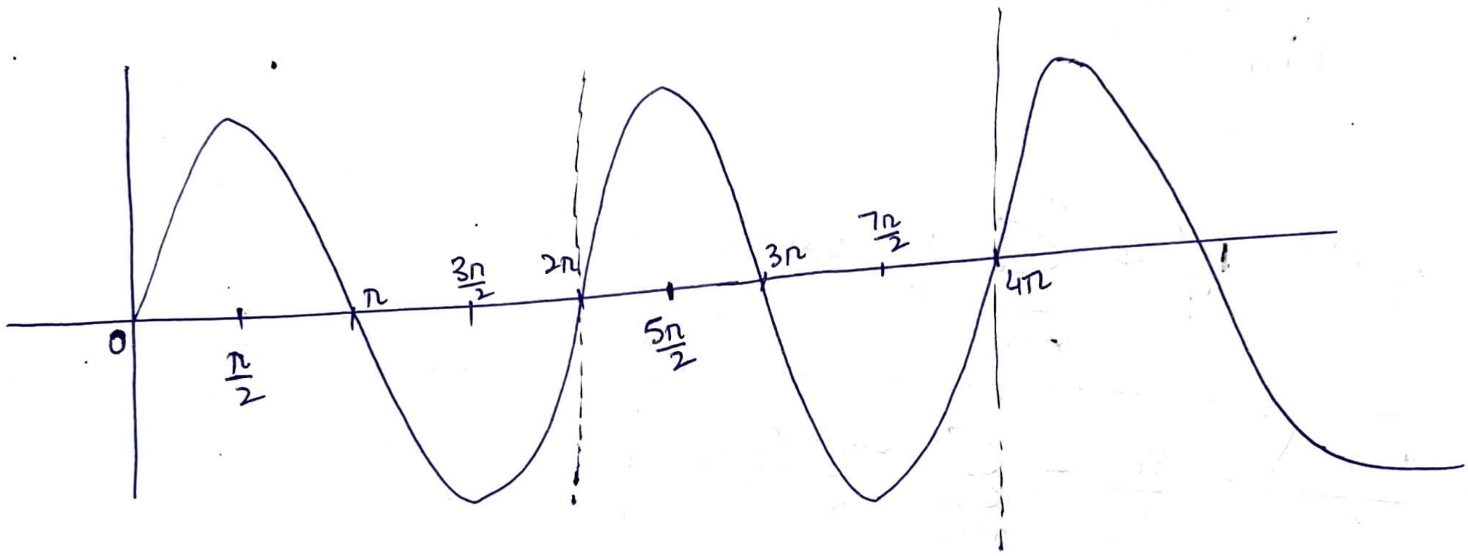
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \rightarrow \text{Euler's formulae}$$

a_0, a_n, b_n are also called fourier coefficients / constants

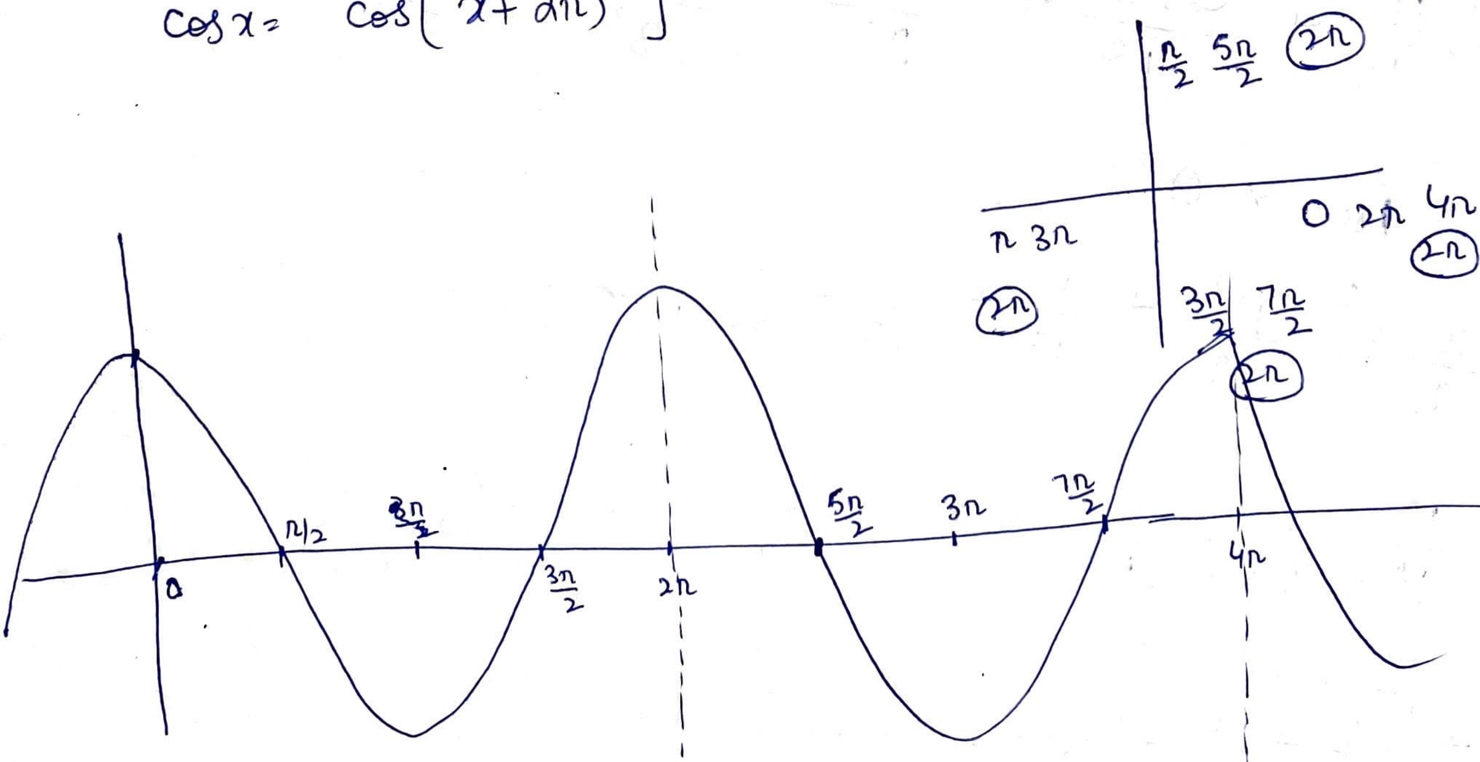
Periodic function

A function that repeats its values at regular intervals is called a periodic function.

i.e. if $f(x) = f(x+\alpha)$ for all x, then $f(x)$ is called a periodic function having period α.



$$\left.\begin{aligned} \sin(x) &= \sin(x+2\pi) \\ \cos x &= \cos(x+2\pi) \end{aligned}\right]$$ Periodic function with period 2π.



Q-1 Find the fourier series expansion of the periodic function $f(x) = x$, $-\pi \le x \le \pi$, $f(x+2\pi) = f(x)$.

Sol:- The fourier series for $f(x) = x$ in the interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - \pi^2\right) = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\left[x\,\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1\cdot\frac{\sin nx}{n}\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left(\sin n\pi + \sin n\pi\right)\right] - \frac{1}{\pi}\left[-\frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0+0\right] + \frac{1}{\pi n^2}\left(\cos n\pi - \cos n\pi\right)$$

$$= 0.$$

$$\boxed{a_n = 0}$$

~~box~~ or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - \int (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\cdot 0 + \frac{1}{n^2}\left(\cos n\pi - \cos n\pi\right)\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{(-\pi)\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n\right]$$

$$= \frac{2\pi}{n\pi}(-1)^{n+1} \qquad \frac{2}{n}(-1)^{n+1}$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}} \qquad \frac{-2}{n}(-1)^n$$

Thus,

$$f(x) = 0 + \sum_{n=1}^{\infty} 0 \cos nx + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1}\sin nx.$$

$$\Rightarrow x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Important formulae

(1) $\int 1\, dx = x + c$

(2) $\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$

(3) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

(4) $\int \sin ax\, dx = -\frac{\cos ax}{a} + c$

(5) $\int \cos ax\, dx = +\frac{\sin ax}{a} + c$

(6) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx) + c$

(7) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b\sin bx) + c$

(8) Integration by parts

$$\int I\, II\, dx = I\int II\, dx - \int\left(\frac{d}{dx} I\right)\left(\int II\, dx\right) dx$$

$$\boxed{\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots}$$

I → Inverse
L → Log
A → Algebraic
T → Trignometric
E → Exponential.

(9) $\sin n\pi = 0$ → $\sin 0 = 0$

→ $\sin \pi = 0$

→ $\sin 2\pi = 0$

$\vdots$

~~sin~~ ~~sin~~ 2

π to

$\sin (n+1)\pi = 0$

$\sin (n-1)\pi = 0$

(10) $\cos n\pi = (-1)^n$ → $\cos \pi = -1$

→ $\cos 2\pi = 1$

→ $\cos 3\pi = -1$

$\vdots$

$\cos (n+1)\pi = (-1)^{n+1}$

$\cos (n-1)\pi = (-1)^{n-1}$

(11) $\sin(-\theta) = -\sin\theta$

(12) $\cos(-\theta) = \cos\theta$

(13) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$

(14) $\cos(2n+1)\frac{\pi}{2} = 0.$

(15) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(16) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(17) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(18) $2\sin A \sin B = \cos(A-B) - \cos(A+B).$

Ex 9.1

(28) Find the fourier series expansion of the function

$f(x) = \cancel{x}\,\pi + x, \quad -\pi < x < \pi.$

Hence, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - - - .$$

Sol: The fourier series expansion for $f(x) = x + \pi$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\left[\cancel{\frac{\sin nx}{nx}}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi + x)\, dx = \frac{1}{\pi} \left[\pi x + \frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\pi^2 + \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} (2\pi^2) = 2\pi$$

π

$\frac{n\pi)}{n}$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi+x) \cos nx \, dx$$

to

$$= \frac{1}{\pi}\left[(\pi+x) \frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi+x}{n}\right) \sin nx + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n} \sin n\pi + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos n\pi}{n^2}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi+x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi+x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(\pi+x)}{n} \cos nx + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n} \cos n\pi + \frac{\sin n\pi}{n^2} + 0 + \frac{\sin n\pi}{n^2}\right]$$

$$= -\frac{2\pi}{\pi n}(-1)^n = \frac{-2}{n}(-1)^n$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

$$f(x) = \frac{2\pi}{2} + 0 + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$\boxed{\pi + x = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx}$$

$$\pi + x = \pi + 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} \cdots\right] \quad -(1)$$

Put $x = \frac{\pi}{2}$ in (1), we get

$$\pi + \frac{\pi}{2} = \pi + 2\left[\sin\frac{\pi}{2} - \frac{\sin \pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} - \frac{\sin 2\pi}{4} + \frac{\sin\frac{5\pi}{2}}{5} \cdots\right]$$

$$\Rightarrow \frac{3\pi}{2} = \pi + 2\left[1 - 0 - \frac{1}{3} - 0 + \frac{1}{5} \cdots\right]$$

$$= \pi + 2\left[1 - \frac{1}{3} + \frac{1}{5} \cdots\right]$$

$$\boxed{\sin(2n+1)\frac{\pi}{2} = (-1)^n}$$

$$\Rightarrow \frac{3\pi}{2} - \pi = 2\left[1 - \frac{1}{3} + \frac{1}{5} \cdots\right]$$

$$\Rightarrow \frac{\pi}{2} \cdot \frac{1}{2} = 1 - \frac{1}{3} + \frac{1}{5} \cdots$$

$$\Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} \cdots = \frac{\pi}{4}}$$

(1) Find the Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$.

Sol: $f(x)=x-x^2;\ -\pi<x<\pi.$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx=\frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{(-\pi^3)}{3}\right]$$

$$=\frac{1}{\pi}\left[\frac{-2\pi^3}{3}\right] \qquad \boxed{a_0=\frac{-2\pi^2}{3}}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$\cancel{=\frac{1}{\pi}\left\{\left[(x-x^2)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-(1-2x)\left(-\frac{\cos nx}{n}\right)+(-2)\left(-\frac{\sin nx}{n^2}\right)\right\}_{-\pi}^{\pi}}$$

$$\cancel{=\frac{1}{\pi}\left\{(\pi-\pi^2)\frac{\sin n\pi}{n}\right.}$$

$$=\frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right)-(1-2x)\left(-\frac{\cos nx}{n^2}\right)+(-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(\pi-\pi^2)\frac{\sin n\pi}{n}+(1-2\pi)\frac{\cos n\pi}{n^2}+2\frac{\sin n\pi}{n^3}-(-\pi-\pi^2)\frac{(-\sin n\pi)}{n}\right.$$
$$\left.+(1+2\pi)\left(-\frac{\cos n\pi}{n^2}\right)+\frac{2\sin n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[\frac{(1-2\pi)(-1)^n}{n^2}-\frac{(1+2\pi)(-1)^n}{n^2}\right]=\frac{(-1)^n}{n^2\pi}\left[1-2\pi-1+2\pi\right]$$

$$\boxed{a_n=\frac{-4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(+\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x^2-x)\frac{\cos nx}{n} + (1-2x)\frac{\sin nx}{n^2} - \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{\cos n\pi}{n} + (1-2\pi)\frac{\sin n\pi}{n^2} - \cancel{\frac{2\cos n\pi}{n^3}} - (\pi^2+\pi)\frac{\cos n\pi}{n}\right.$$
$$\left. + (1+2\pi)\frac{\sin n\pi}{n^2} + \cancel{\frac{2\cos n\pi}{n^3}}\right]$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi-\pi^2-\pi)\frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}(-2\pi)\frac{(-1)^n}{n}$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}}$$

$$f(x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$x-x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$
$$-2\left[\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right.$$

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$