# Method of Undetermined Coefficients

In the cases when right hand side $X$ is of a special form containing

(i) Exponentials
(ii) Polynomials
(iii) Cosine and Sine functions
(iv) Sums or product of these functions.

then we use the method of undetermined coefficients to find P.I.

**Method:-** We choose a particular integral depending on the form of $X$.

| Cases | P.I. |
|---|---|
| 1) $X = e^{ax}$ | $y_p(x) = Ce^{ax}$, $c$ – constant |
| 2) $X = x^m$ | $y_p(x) = C_0x^m + C_1x^{m-1} + \dots + C_{m-1}x + C_m$ <br> $C_0, C_1, \dots, C_m$ are constants |
| 3) $X = e^{ax}\sin bx$ or $e^{ax}\cos bx$ | $y_p(x) = e^{ax}(C_1\cos bx + C_2\sin bx)$ |
| 4) $X = \cos\beta x$ or $\sin\beta x$ | $y_p(x) = C_1\cos\beta x + C_2\sin\beta x$ |
| 5) $X = e^{ax}x^m$ | $y_p(x) = e^{ax}(C_0x^m + C_1x^{m-1} + \dots + C_m)$ |

**Note:** If any term of the trial solution appears in C.F., we multiply the trial solution by $x^m$, $m$ represents the number of times the term is repeated in C.F.

Ex: $y'' - 2y' - 3y = 6e^{-x} - 8e^{x} \rightarrow (1)$

Sol$^n$: $X = 6e^{-x} - 8e^{x}$

C.F. $y'' - 2y' - 3y = 0$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = -1, 3$

$y_c(x) = C_1e^{-x} + C_2e^{3x}$

P.I. Let $y_p(x) = axe^{-x} + be^{x}$

$y' = a(-xe^{-x} + e^{-x}) + be^{x}$

$= -axe^{-x} + ae^{-x} + be^{x}$

$y'' = -a(-xe^{-x} + e^{-x}) - ae^{-x} + be^{x}$

$= axe^{-x} - ae^{-x} - ae^{-x} + be^{x}$

$= axe^{-x} - 2ae^{-x} + be^{x}$

From ①

$axe^{-x} - 2ae^{-x} + be^{x} + 2axe^{-x} - 2ae^{-x} - 2be^{x} - 3axe^{-x} - 3b$
$= 6e^{-x} - 8e^{x}$

$\Rightarrow -4ae^{-x} - 4be^{x} = 6e^{-x} - 8e^{x}$

Equate the coefficients of $e^{-x}$ & $e^{x}$, we have.

$-4a = 6 \Rightarrow a = -\frac{6}{4} = -\frac{3}{2}$

$-4b = -8 \Rightarrow b = 2$

$\therefore \; y_p(x) = -\frac{3}{2} xe^{-x} + 2e^{x}.$

$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{3}{2} xe^{-x} + 2e^{x}.$

Q: $y'' + y = 32x^3 \longrightarrow ①$

$m^2 + 1 = 0$

$m = \pm i$

$y_c(x) = C_1 \cos x + C_2 \sin x$

Let $y_p(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$

$y' = 3a_0 x^2 + 2a_1 x + a_2$

$y'' = 6a_0 x + 2a_1$

From ①, $6a_0 x + 2a_1 + a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 32x^3$

$\Rightarrow (6a_0 + a_2)x + a_1 x^2 + a_0 x^3 + (2a_1 + a_3) = 32x^3$

Compare coeff of $x^3$ : $a_0 = 32$

$x^2$ : $a_1 = 0$

$x$ : $6a_0 + a_2 = 0 \Rightarrow a_2 = -6(32)$
$= -192$

$1$ (constant) : $2a_1 + a_3 = 0$
$\Rightarrow a_3 = 0.$

$\therefore \; y_p(x) = 32x^3 - 192x$

$\therefore \; y(x) = C_1 \cos x + C_2 \sin x + 32x(x^2 - 6)$ Ans.

Practice of $y_p(x)$:-

Q: $y'' + 9y = \cos 3x \longrightarrow ①$

$y_c(x) = C_1 \cos 3x + C_2 \sin 3x$

then we will assume $y_p(x)$ as:

$y_p(x) = x(a \cos 3x + b \sin 3x)$

Substitute $y_p$ in ① and find 'a' & 'b'.

Q: $y'' + 4y' + 4y = 12e^{-2x} \longrightarrow ②$

$y_c(x) = (C_1 + xC_2)e^{-2x}$

then we will assume $y_p(x)$ as

$y_p(x) = ax^2 e^{-2x}$ ; Find 'a' for final solution

Q: $y'' - 4y' + 13y = 12e^{2x} \sin 3x$

$y_c(x) = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$

then we will assume $y_p(x)$ as:

$y_p(x) = xe^{2x}(a \cos 3x + b \sin 3x)$

find a and b for final solution

Q: $y''' - 2y'' - 5y' + 6y = 18e^{x}$

$y_c(x) = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{+3x}$

then we will assume $y_p(x)$ as:

$y_p(x) = axe^{x}$

find a for final solution.

$m^3 - 2m^2 - 5m + 6 = 0$

$(m-1)(m^2 - m - 6) = 0$

$(m-1)(m^2 - 3m + 2m - 6) = 0$

$(m-1)(m+2)(m-3) = 0$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Q: $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x} \longrightarrow ③$

$y_c(x) = (C_1 + C_2 x + C_3 x^2)e^{2x}$

then we will assume $y_p(x)$ as:

$y_p(x) = ax^3 e^{2x} + be^{-x}$

substitute $y_p(x)$ in ③ & find 'a' & 'b'.

$m^3 - 6m^2 + 12m - 8 = 0$

$(m-2)^3 = 0$

$$\begin{array}{c|cccc} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline 2 & 1 & -4 & 4 & 0 \\ & & 2 & -4 & \\ \hline & 1 & -2 & 0 & \end{array}$$

Solutions

① $a=0,\ b=\frac{1}{6}$. $y_p(x)=\frac{1}{6}x\sin 3x$, $y(x)=C_1\cos 3x+C_2\sin 3x+\frac{x}{6}\sin 3x$.

② $a=6$, $y_p(x)=6x^2e^{-2x}$, $y(x)=(C_1+xC_2)e^{-2x}+6x^2e^{-2x}$

③ $a=-2,\ b=0$, $y_p(x)=-2xe^{2x}\cos 3x$

$y(x)=e^{2x}\left[C_1\cos 3x+C_2\sin 3x-2x\cos 3x\right]$.

④ $a=-3$, $y_p(x)=-3xe^{x}$, $y(x)=C_1e^{x}+C_2e^{-2x}+C_3e^{3x}-3xe^{x}$.

⑤ $a=2,\ b=-1$, $y_p(x)=2x^3e^{2x}-e^{-x}$, $y(x)=(C_1+xC_2+x^2C_3)e^{2x}+2x^3e^{2x}-e^{-x}$.