Derivatives of composite functions

Let $z = f(x,y)$ be a function of two independent variables x and y. Suppose that x and y are themselves functions of some independent variables t, say, $x = g(t)$,
$y = h(t)$.

Then, $z = f[g(t), h(t)]$ is a composite function of independent variable t.

Ex $\quad z = x^2 + y^2, \quad x = \frac{t^2-1}{t}, \quad y = \frac{t}{t^2+1}.$

Chain Rule

$$\begin{tikzcd}
& z \arrow[dl, dash, "\frac{\partial z}{\partial x}"'] \arrow[dr, dash, "\frac{\partial z}{\partial y}"] & \\
x \arrow[dr, dash, "\frac{dx}{dt}"'] & & y \arrow[dl, dash, "\frac{dy}{dt}"] \\
& t &
\end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\boxed{\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}}$$

Q:- Find $\frac{dz}{dt}$ if $z = x^2 + y^2$, $x = \frac{t^2-1}{t}$, $y = \frac{t}{t^2+1}$ at $t=1$.

Sol:- $z = x^2 + y^2$, $x = \frac{t^2-1}{t}$, $y = \frac{t}{t^2+1}$

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y, \quad \frac{dx}{dt} = \frac{t(2t) - (t^2-1).1}{t^2} = \frac{t^2+1}{t},$$

$$\frac{dy}{dt} = \frac{(t^2+1).1 - t(2t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x)\frac{(t^2+1)}{t} + 2y\left(\frac{1-t^2}{(t^2+1)^2}\right).$$

$$= \frac{2x(t^2+1)}{t} + \frac{2y(1-t^2)}{(t^2+1)^2}$$

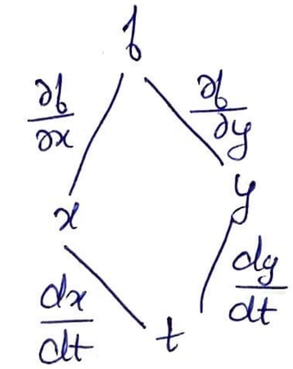
At $t=1$, $x=0$, $y=\frac{1}{2}$

$$\left.\frac{dz}{dt}\right|_{t=1} = 0 + \frac{2y(1-1)}{(1+1)^2} = 0.$$

$$\boxed{\left.\frac{dz}{dt}\right|_{t=1} = 0}$$

Q: Find $\frac{df}{dt}$ if $f(x,y) = x\cos y + e^x \sin y$, $x = t^2+1$, $y = t^3+t$ at $t=0$.

Sol: $f = x\cos y + e^x \sin y$, $x = t^2+1$, $y = t^3+t$



$$\frac{\partial f}{\partial x} = \cos y + e^x \sin y$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x \cos y$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2+1$$

$$\frac{df}{dt} = (\cos y + e^x \sin y)\,2t + \left(-x\sin y + e^x \cos y\right)(3t^2+1)$$

$$= 2t(\cos y + e^x \sin y) + (3t^2+1)(e^x \cos y - x\sin y)$$

At $t=0$, $x=1$, $y=0$

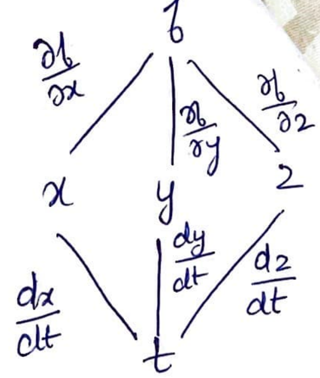
$$\left.\frac{df}{dt}\right|_{t=0} = 0 + 1(e) = e$$

$$\boxed{\left.\frac{df}{dt}\right|_{t=0} = e}$$

Q: Find $\frac{df}{dt}$ at $t=0$, when

$f(x,y,z) = x^3 + xz^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Sol :- $f = x^3 + xz^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$



$$\frac{\partial f}{\partial x} = 3x^2 + z^2 + yz$$

$$\frac{\partial f}{\partial y} = 3y^2 + xz$$

$$\frac{\partial f}{\partial z} = 2xz + xy$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -\sin t, \quad \frac{dz}{dt} = 3t^2.$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (3x^2+z^2+yz)\, e^t + (3y^2+xz)(-\sin t) + (2xz+xy)(3t^2)$$

At $t = 0$, $x = 1$, $y = 1$, $z = 0$

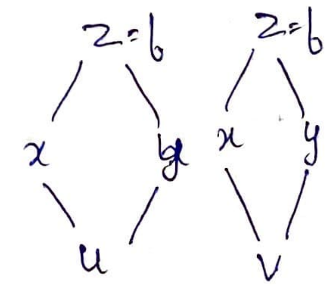
$$\left.\frac{df}{dt}\right|_{t=0} = (3+0+0)\cdot 1 + (3+0)(0) + (0+1)(0)$$

$$= 3$$

Thus, $\boxed{\left.\frac{df}{dt}\right|_{t=0} = 3}$

Q:- If $z = f(x,y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$$



Sol :- $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x} \cdot \left(2e^{2u}\right) + \frac{\partial f}{\partial y}\left(-2e^{2u}\right)$$

$$\frac{\partial f}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y} \quad - (1)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}\left(-2e^{-2v}\right) + \frac{\partial f}{\partial y}\left(2e^{2v}\right)$$

$$\frac{\partial f}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y} \quad - (2)$$

(1) − (2) gives,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left(e^{2u} + e^{-2v}\right)\frac{\partial f}{\partial x} - 2\left(e^{-2u} + e^{2v}\right)\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}$$

Thus, $$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Derivative of Implicit functions

A function $f(x,y)=c$ is called an Implicit function.

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}} \quad \text{and} \quad \boxed{\frac{dx}{dy} = -\frac{f_y}{f_x}}$$

If $f(x,y,z)=c$, then

$$\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}, \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}, \quad \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \text{ etc.}$$

Q:- Using Implicit differentiation, find $\frac{dy}{dx}$, when $x^y + y^x = \alpha$, α is any Constant, $x>0, y>0$.

Sol:- Let $f = x^y + y^x = \alpha \Rightarrow f_x = yx^{y-1} + y^x \ln y$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \qquad f_y = x^y \ln x + xy^{x-1}$$

$$= -\left[\frac{yx^{y-1} + y^x \ln y}{x^y \ln x + xy^{x-1}}\right]$$

Q:- Find $\frac{dy}{dx}$ when $\cot^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$

Sol:- Let $f = \cot^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$

$$f_x = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{-y}{x^2+y^2}$$

$$f_y = \frac{-1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) + 3y^2 = \frac{x}{x^2+y^2} + 3y^2$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{\frac{y}{x^2+y^2}}{\frac{x+3y^2(x^2+y^2)}{x^2+y^2}} = \frac{y}{x+3y^2(x^2+y^2)}$$

Q:- $\left(\frac{\partial z}{\partial x}\right)_y$ and $\left(\frac{\partial z}{\partial y}\right)_x$, when $\cos xy + \cos yz + \cos zx = 1$

Sol:- Let $f = \cos xy + \cos yz + \cos zx = 1$

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z}, \quad \left(\frac{\partial z}{\partial y}\right)_x = -\frac{f_y}{f_z}$$

$f_x = -y\sin xy - z\sin zx$, $f_y = -x\sin xy - z\sin yz$

$f_z = -y\sin yz - x\sin zx$

$$\left(\frac{\partial z}{\partial x}\right)_y = -\left[\frac{y\sin xy + z\sin zx}{y\sin yz + x\sin zx}\right]$$

$$\left(\frac{\partial z}{\partial y}\right)_x = -\left[\frac{x\sin xy + z\sin yz}{y\sin yz + x\sin zx}\right]$$

$\underline{Q:}$ $y\left(\frac{\partial x}{\partial y}\right)_z + z\left(\frac{\partial x}{\partial z}\right)_y$, when $f\left(\frac{z}{y}, \frac{x}{y}\right) = 0$

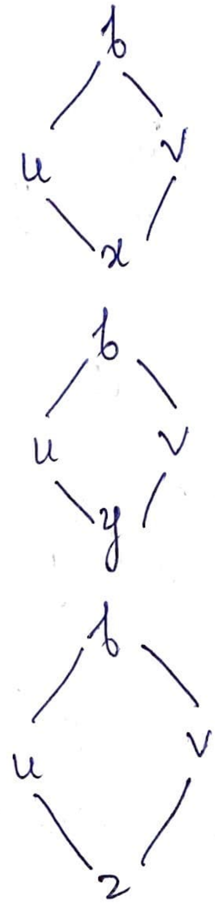
$\underline{Sol:}$ $\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}$, $\frac{\partial x}{\partial z} = -\frac{f_z}{f_x}$

$$f\left(\frac{z}{y}, \frac{x}{y}\right) = 0$$

$f.$ $\cancel{f'\left(\frac{z}{y}, \frac{x}{y}\right) \cdot \frac{1}{y}}$

Let $u = \frac{z}{y}$, $v = \frac{x}{y}$

$\Rightarrow f(u, v) = 0.$



$f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x}$

$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}\left(\frac{1}{y}\right)$

$f_x = \frac{1}{y} \frac{\partial f}{\partial v}$

$f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y}$

$= \frac{\partial f}{\partial u}\left(\frac{-z}{y^2}\right) + \frac{\partial f}{\partial v}\left(\frac{-x}{y^2}\right)$

$= -\frac{z}{y^2} \frac{\partial f}{\partial u} - \frac{x}{y^2} \frac{\partial f}{\partial v}$

$f_z = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} = \frac{1}{y} \frac{\partial f}{\partial u} \Rightarrow f_z = \frac{1}{y} \frac{\partial f}{\partial u}$

$$\frac{\partial x}{\partial y} = \frac{\frac{z}{y^2}\frac{\partial f}{\partial u} + \frac{x}{y^2}\frac{\partial f}{\partial v}}{\frac{1}{y}\frac{\partial f}{\partial v}} = \frac{z}{y}\frac{f_u}{f_v} + \frac{x}{y}\frac{f_v}{f_v}$$

$$\frac{\partial x}{\partial y} = \frac{z}{y}\frac{f_u}{f_v} + \frac{x}{y}$$

$$\frac{\partial x}{\partial z} = \frac{-\frac{1}{y}\frac{\partial f}{\partial u}}{\frac{1}{y}\frac{\partial f}{\partial v}} = -\frac{f_u}{f_v}$$

Thus, $y\left(\frac{\partial x}{\partial y}\right)_z + z\left(\frac{\partial x}{\partial z}\right)_y = \cancel{z\frac{f_u}{f_v}} + x - \cancel{z\frac{f_u}{f_v}} = x$

Change of variables

Let $f(x,y)$ be a function of two independent variables x and y and x, y are functions of two new independent variables u and v given by $x = \phi(u,v)$, $y = \psi(u,v)$

Then,
$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f,y)}{\partial(u,v)}\right]$$

and

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f,x)}{\partial(u,v)}\right]$$

Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Note:- If $J=0$, then variables are functionally related. i.e. dependent on each other.

If $J \neq 0$, then variables are independent of any relation.

Q:- Check whether the variables are functionally related:

$u = x + 3z,\ v = x - y - z,\ w = y^2 + 16z^2 + 8yz.$

Sol:-

$$J = \cancel{\frac{\partial(x,y,z)}{\partial t}} \quad \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 1 & -1 & -1 \\ 0 & 2y+8z & 32z+8y \end{vmatrix}$$

$$= 1(-32z - 8y + 2y + 8z) - 0 + 3(2y + 8z)$$

$$= -32z - 8y + 2y + 8z + 6y + 24z$$

$$= 0$$

So, the variables are functionally related.

Ex If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Sol:

$$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y)}{\partial(r,\theta)}, \quad \frac{\partial f}{\partial y} = \frac{-1}{J}\frac{\partial(f,x)}{\partial(r,\theta)}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\boxed{J = r}$$

$$\frac{\partial f}{\partial x} = \frac{1}{r}\begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \frac{1}{r}\left[r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta}\right]$$

$$= \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \frac{1}{r}\begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \cos\theta & -r\sin\theta \end{vmatrix}$$

$$= -\sin\theta\frac{\partial f}{\partial r} - \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}$$

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Practice ① If $u = f(x, y, z)$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$

$$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)} = \frac{1}{r^2\sin\theta}\left[r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r} + r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial \theta} - r\sin\phi\frac{\partial f}{\partial \phi}\right]$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)} = \frac{-1}{r^2\sin\theta}\left[-r^2\sin^2\theta\sin\phi\frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi\frac{\partial f}{\partial \theta} - r\cos\phi\frac{\partial f}{\partial \phi}\right]$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)} = \frac{1}{r^2\sin\theta}\left[r^2\sin\theta\cos\theta\frac{\partial f}{\partial r} - r\sin^2\theta\frac{\partial f}{\partial \theta}\right]$$

Q: Show that the ~~functions~~ variables $u = x - y + z$, $v = x + y - z$, $w = x^2 + xz - xy$ are functionally related. Find the relationship between them.

$$w = x(x + z - y) = xu, \quad u + v = 2x \Rightarrow x = \frac{u+v}{2}$$

$$\boxed{2w = u(u+v)}$$