Solution of Euler Cauchy Equation

nth order LDE (Euler Cauchy Equation)

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X, \cancel{X \neq 0}.$$

or $$a_0 x^n y^n + a_1 x^{n-1} y^{n-1} + \cdots + a_{n-1} x y' + a_n y = X.$$

where $a_0, a_1, \cdots, a_n$ are constants.

Method

Step 1 Let $x = e^z$ or $z = \log x$.

Step 2 Replace $x\dfrac{dy}{dx} = D_1 y$ where $D_1 = \dfrac{d}{dz}$

$$x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y$$

In general $$x^n\frac{d^ny}{dx^n} = D_1(D_1-1)\ ---\ (D_1-(n-1))y.$$

Step 3 Use previous methods to find ~~P.I.~~ G.S.

Step 4 Replace $z = \log x$ i.e. $x = e^z$.

Q Find the general sol of $2x^2y'' + 3xy' - 3y = x^3$

Sol: Let $x = e^z$ i.e. $z = \log x$

$$x\frac{dy}{dx} = \cancel{z}\, D_1 y, \quad x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y.$$

$2x^2y'' + 3xy' - 3y = x^3$ becomes

$$2D_1(D_1-1) + 3D_1y + 3y = e^{3z}$$

$$(2(D_1^2 - D_1) + 3D_1 + 3)y = e^{3z}$$

$$(2D_1^2 + D_1 + 3)\,y = e^{3z}$$

C.F.

$$2m^2 + m + 3 = 0$$

$$2m^2 + 3m - 2m - 3 = 0$$

$$m(2m+3) - 1(2m+3) = 0$$

$$(m-1)(2m+3) = 0$$

$m = 1, -\frac{3}{2}$

$$y_c(z) = C_1 e^{z} + C_2 e^{-3/2 z} = C_1(x) + C_2(x)^{-3/2}$$

$$= C_1 x + \frac{C_2}{x\sqrt{x}}.$$

ator

PI $\Rightarrow y_p(z) = \frac{1}{2D_1^2 + D_1 - 3} e^{3z}$

$$= \frac{1}{2(3)^2 + 3 - 3} e^{3z} = \frac{e^{3z}}{(9)2}$$

$$y_p(x) = \frac{x^3}{18}.$$

$$y(x) = C_1 x + \frac{C_2}{x\sqrt{x}} + \frac{x^3}{18}.$$

Ex Find G.S. of $x^2 y'' + 5xy' + 3y = \ln(x)$, $x > 0$.

Sol Let $x = e^z$ or $z = \ln(x)$

$D_1(D_1 - 1)y + 5D_1 y + 3y = z$

$\Rightarrow (D_1^2 - D_1 + 5D_1 + 3)y = z$

$\Rightarrow (D_1^2 + 4D_1 + 3)y = z$

C.F. $m^2 + 4m + 3 = 0$

$m^2 + 3m + m + 3 = 0$

$(m+1)(m+3) = 0$

$m = -1, -3$

$y_c(z) = C_1 e^{-z} + C_2 e^{-3z}$

$= C_1 x^{-1} + C_2 x^{-3}$

$= \frac{C_1}{x} + \frac{C_2}{x^3}$

PI $y_p(z) = \frac{1}{D_1^2 + 4D_1 + 3} z$

$= \frac{1}{3}\left(1 + \frac{D_1^2}{3} + \frac{4}{3}D_1\right)^{-1} z$

$= \frac{1}{3}\left[1 - \frac{D_1^2}{3} - \frac{4}{3}D_1 + \dots\right] z$

$= \frac{1}{3}\left[z - \frac{4}{3}\right] \Rightarrow y_p(x) = \frac{1}{3}\left(\ln(x) - \frac{4}{3}\right)$

$= \frac{1}{3}\ln(x) - \frac{4}{9}$

$$y(x) = \frac{C_1}{x} + \frac{C_2}{x^3} + \frac{1}{3}\ln(x) - \frac{4}{9}.$$

Ex Find G.S. of $x^2y'' - 5xy' + 13y = 30x^2$.

Sol :- Let $x = e^z$ i.e. $z = \ln(x)$

$$(D_1(D_1-1) - 5D_1 + 13)y = 30e^{2z}$$

$$(D_1^2 - D_1 - 5D_1 + 13)y = 30e^{2z}$$

$$(D_1^2 - 6D_1 + 13)y = 30e^{2z}$$

C.F $y_c(z) = e^{3z}[C_1\cos 2z + C_2\sin 2z)$

$= x^3[C_1\cos 2(\ln(x)) + C_2\sin(2\ln(x))]$

$= \cancel{x^3\{C_1\cos 2\ln(x}$

$m^2 - 6m + 13 = 0$

$$m = \frac{6 \pm \sqrt{36-52}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i$$

PI $y_p(z) = \frac{1}{D_1^2 - 6D_1 + 13} 30e^{2z} = 30 \frac{1}{4-12+13} e^{2z}$

$$= \frac{30}{5}e^{2z} = 6e^{2z} = 6x^2$$

$$y(x) = x^3[C_1\cos(2\ln x) + C_2\sin(2\ln(x))] + 6x^2.$$

Ex $x^3y''' + 5x^2y'' + 5xy' + y = x^2 + \ln x$, $x > 0$

Sol :- Let $x = e^z$ or $z = \ln x$

$$(D_1(D_1-1)(D_1-2) + 5D_1(D_1-1) + 5D_1 + 1)y = x^2 + \ln x.$$

$$\Rightarrow [(D_1^2 - D_1)(D_1-2) + 5D_1^2 - 5D_1 + 5D_1 + 1]y = \cancel{x}\, e^{2z} + z^2.$$

$$[D_1^3 - 2D_1^2 - D_1^2 + 2D_1 + 5D_1^2 + 1]y = e^{2z} + z^2$$

$$\cancel{D_1^3 - 3D_1^2 +}$$

$$(D_1^3 + 2D_1^2 + 2D_1 + 1)\,y = e^{2z} + z^2$$

C.F. $m^3 + 2m^2 + 2m + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & 2 & 1 \\ & & -1 & -1 & -1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$(m+1)(m^2+m+1) = 0$

$m = -1,\ \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$

$m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{1-4}}{2}$

$$y_c(z) = c_1 e^{-z} + e^{-z/2}\left[c_1 \cos \frac{\sqrt{3}}{2} z + c_2 \sin \frac{\sqrt{3}}{2} z\right]$$

$$\cancel{c_1 e^{-z}}$$

$$y_c(x) = c_1 x^{-1} + (x)^{-1/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} \ln x\right)\right]$$

$$= \frac{c_1}{x} + \frac{1}{\sqrt{x}}\left[c_1 \cos\left(\frac{\sqrt{3} \ln x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3} \ln x}{2}\right)\right]$$

PI $\frac{1}{D^3 + 2D^2 + 2D + 1} e^{2z} + \frac{1}{D^3 + 2D^2 + 2D + 1} z^2$

$$= \frac{1}{8 + 8 + 4 + 1} e^{2z} + [1 + (D^3 + 2D^2 + 2D)]^{-1} z^2$$

$$= \frac{e^{2z}}{21} + [1 - (D^3 + 2D^2 + 2D) + \cdots] z^2$$

$$= \frac{e^{2z}}{21} + z - 2$$

$$y_p(z) = \frac{e^{2z}}{21} + z - 2$$

$$y_p(x) = \frac{x^2}{21} + \ln x - 2$$

$$y(x) = \frac{C_1}{x} + \frac{1}{\sqrt{x}}\left[C_1 \cos\cancel{x}\left(\frac{\sqrt{3}\ln x}{2}\right) + C_2 \sin\left(\frac{\sqrt{3}\ln x}{2}\right)\right] +$$

$$\frac{x^2}{21} + \ln x - 2.$$