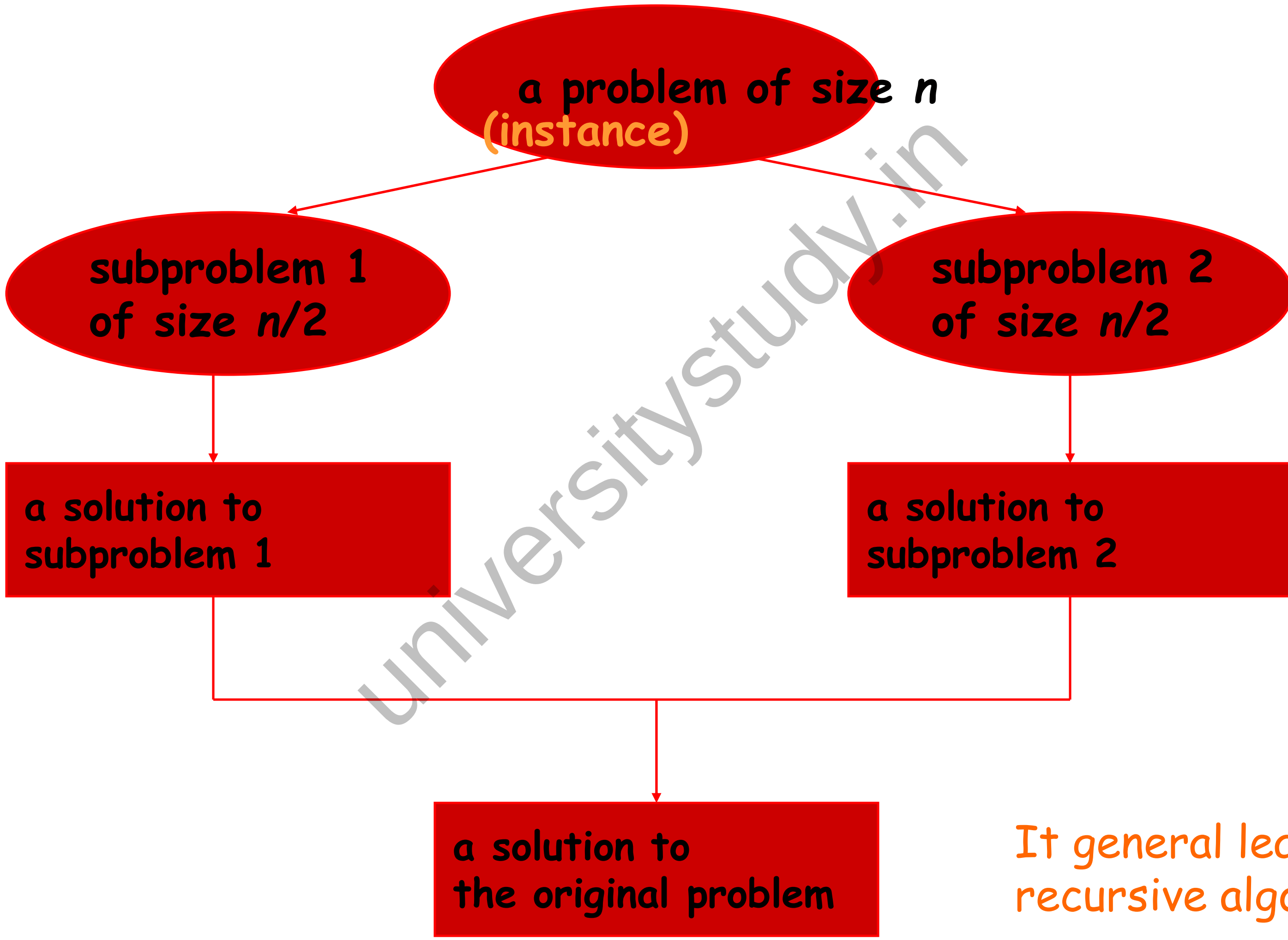
CSE408
Closest pair & Convex Hull Problem, Insertion Sort

Lecture #13

The most-well known algorithm design strategy:

1. Divide instance of problem into two or more smaller instances
2. Solve smaller instances recursively
3. Obtain solution to original (larger) instance by combining these solutions

Divide-and-Conquer Technique (cont.)

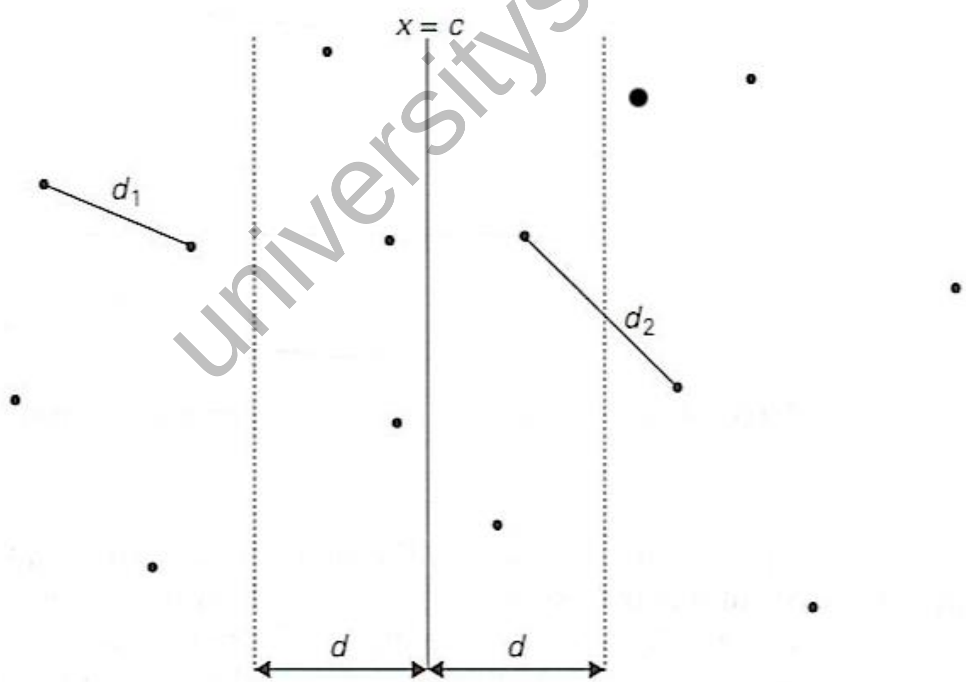


It general leads to a recursive algorithm!

Closest-Pair Problem by Divide-and-Conquer

Step 0 Sort the points by x (list one) and then by y (list two).

Step 1 Divide the points given into two subsets S_1 and S_2 by a vertical line $x = c$ so that half the points lie to the left or on the line and half the points lie to the right or on the line.



Closest Pair by Divide-and-Conquer (cont.)

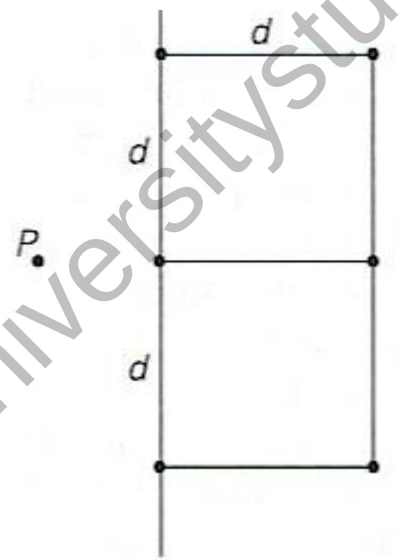
Step 2 Find recursively the closest pairs for the left and right subsets.

Step 3 Set $d = \min\{d_1, d_2\}$
We can limit our attention to the points in the symmetric vertical strip of width $2d$ as possible closest pair. Let C_1 and C_2 be the subsets of points in the left subset S_1 and of the right subset S_2, respectively, that lie in this vertical strip. The points in C_1 and C_2 are stored in increasing order of their y coordinates, taken from the second list.

Step 4 For every point $P(x,y)$ in C_1, we inspect points in C_2 that may be closer to P than d. There can be no more than 6 such points (because $d \le d_2$)!

Closest Pair by Divide-and-Conquer: Worst Case

The worst case scenario is depicted below:



Efficiency of the Closest-Pair Algorithm

Running time of the algorithm (without sorting) is:

$$T(n) = 2T(n/2) + M(n), \quad \text{where } M(n) \in \Theta(n)$$

By the Master Theorem (with $a = 2$, $b = 2$, $d = 1$)

$$T(n) \in \Theta(n \log n)$$

So the total time is $\Theta(n \log n)$.

Convex hull: smallest convex set that includes given points. An O(n^3) bruteforce time is given in Levitin, Ch 3.

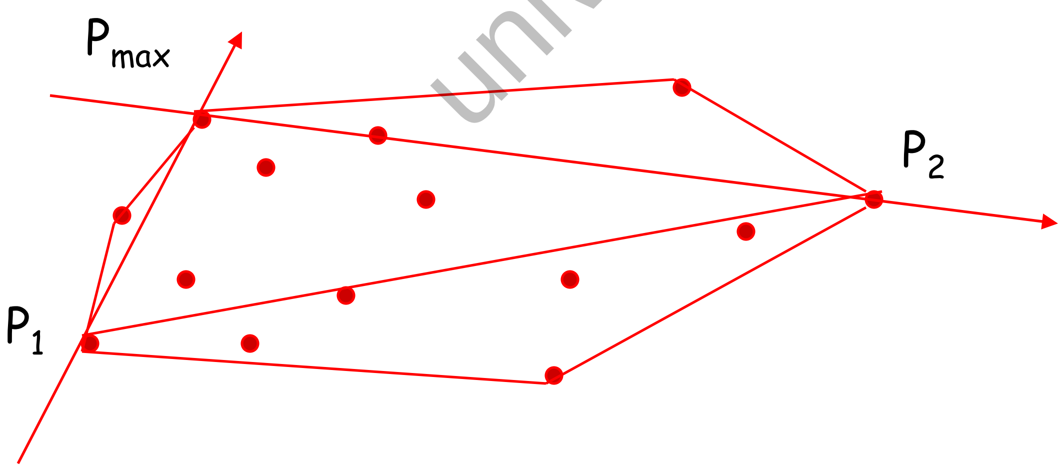
Assume points are sorted by x-coordinate values

Identify *extreme points* P_1 and P_2 (leftmost and rightmost)

Compute *upper hull* recursively:

- find point P_{max} that is farthest away from line P_1P_2
- compute the upper hull of the points to the left of line P_1P_{max}
- compute the upper hull of the points to the left of line $P_{max}P_2$

Compute *lower hull* in a similar manner



Efficiency of Convex hull Algorithm

Finding point farthest away from line P_1P_2 can be done in linear time

Time efficiency: $T(n) = T(x) + T(y) + T(z) + T(v) + O(n)$,

where $x + y + z + v \le n$.

- worst case: $\Theta(n^2)$ $T(n) = T(n-1) + O(n)$
- average case: $\Theta(n)$ (under reasonable assumptions about distribution of points given)

If points are not initially sorted by *x*-coordinate value, this can be accomplished in $O(n \log n)$ time

Several $O(n \log n)$ algorithms for convex hull are known

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	2.78	7.42	0.56	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 0: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	2.78	7.42	0.56	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 1: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	2.78	0.56	7.42	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 2: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	2.78	7.42	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 2: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	2.78	7.42	1.12	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 2: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	2.78	1.12	7.42	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 3: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	2.78	7.42	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 3: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	2.78	7.42	1.17	0.32	6.21	4.42	3.14	7.71

Iteration 3: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	2.78	1.17	7.42	0.32	6.21	4.42	3.14	7.71

Iteration 4: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	1.17	2.78	7.42	0.32	6.21	4.42	3.14	7.71

Iteration 4: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	1.17	2.78	7.42	0.32	6.21	4.42	3.14	7.71

Iteration 4: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	1.17	2.78	0.32	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	1.17	0.32	2.78	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 1.

Insertion Sort

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	1.12	0.32	1.17	2.78	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.56	0.32	1.12	1.17	2.78	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 3.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 4.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	7.42	6.21	4.42	3.14	7.71

Iteration 5: step 5.

Insertion Sort

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	6.21	7.42	4.42	3.14	7.71

Iteration 6: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	6.21	7.42	4.42	3.14	7.71

Iteration 6: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	6.21	4.42	7.42	3.14	7.71

Iteration 7: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	4.42	6.21	7.42	3.14	7.71

Iteration 7: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	4.42	6.21	7.42	3.14	7.71

Iteration 7: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	4.42	6.21	3.14	7.42	7.71

Iteration 8: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	4.42	3.14	6.21	7.42	7.71

Iteration 8: step 1.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	3.14	4.42	6.21	7.42	7.71

Iteration 8: step 2.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	3.14	4.42	6.21	7.42	7.71

Iteration 8: step 3.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	3.14	4.42	6.21	7.42	7.71

Iteration 9: step 0.

Iteration i. Repeatedly swap element i with the one to its left if smaller.

Property. After ith iteration, `a[0]` through `a[i]` contain first i+1 elements in ascending order.

Array index	0	1	2	3	4	5	6	7	8	9
Value	0.32	0.56	1.12	1.17	2.78	3.14	4.42	6.21	7.42	7.71

Iteration 10: DONE.