Linear Differential Equations

Let y be dependent variable and x be the independent variable. We denote the derivatives as

$$\frac{dy}{dx} = y', \ \frac{d^2y}{dx^2} = y'', \ \frac{d^3y}{dx^3} = y''' \text{ etc.}$$

$\frac{dy}{dx}$ ↓ rate of change of y w.r.t. x.

Derivatives of higher order represents rate of rates.

Differential Equation :- A differential equation contains derivatives of various orders and the variables.

Ex:- (i) $y' = 6x^2$

(ii) $y'' + 16y = 2x$.

Linear Differential Equation

A linear ordinary differential equation of order n, is written as

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = r(x)$$

or $$a_0(x)\, y^n(x) + a_1(x)\, y^{n-1}(x) + \cdots\cdots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = r(x),$$

where y is dependent variable and x is independent variable and $a_0(x) \neq 0$.

If $r(x) = 0$, then it is called a homogeneous equation,

otherwise it is called a non-homogeneous equation.

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0, \quad a_0(x) \neq 0$$

is a second order homogeneous equation

And $$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = r(x), \quad a_0(x) \neq 0$$

is a second order non-homogeneous equation.

If $a_i(x)$, $i = 0, 1, 2$ are constants, then the equations are linear second order constant coefficient equations.

Ex

$$\left.\begin{aligned} y'' + 4y' + 3y &= x^2 e^x \\ y'' + 2y' + y &= \sin x \end{aligned}\right\} \text{Constant coefficients}$$

$$\left.\begin{aligned} x^2 y'' + xy' + (x^2 - 4)\,y &= 0 \\ (1 - x^2)\,y'' - 2xy' + 20y &= 0 \end{aligned}\right\} \text{variable coefficients}$$

(1) $y'' - a^2 y = 0 \rightarrow$ Constant coefficient

(2) $y' = \dfrac{y}{x} \Rightarrow xy' = y \rightarrow$ variable coefficient

(3) $y''' + 3y'' + 6y' + 12y = x^2 \rightarrow$ Constant coefficient

(4) $x^3 y''' + 9x^2 y' + 18xy' + 6y = 0 \rightarrow$ Variable coefficient

(5) $(1 - x)\,y'' + xy' - y = 0 \rightarrow$ Variable coefficient

(6) $y'' - (1 + x^2)\,y = 0 \rightarrow$ Variable coefficient.

Solutions of Linear Differential Equations

Th^m If the functions $a_0(x), a_1(x), \ldots a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n(y) = r(x) \quad -(1)$$

$$y(x_0) = c_1,\ y'(x_0) = c_2, \ldots, y^{n-1}(x_0) = c_n$$

where $x_0 \in I$ and $c_1, c_2, \ldots, c_n$ are n known constants.

If the conditions of the th^m are satisfied, then the diff. eq (1) is said to be normal on I.

Ex: Find the intervals on which the following differential equations are normal.

(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, n is an integer.

Sol: Here $a_0(x) = 1-x^2$, $a_1(x) = -2x$, $a_2(x) = n(n+1)$.

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) \neq 0$ for all x in $(-\infty, \infty)$ except at the points $-1, 1$.

Hence, the diff. eq is normal on every subinterval I of the open intervals $(-\infty, -1), (-1, 1), (1, \infty)$.

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0$, n is real.

Sol: Here $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = n^2 - x^2$

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also $a_0(x) \neq 0$ for all $x \in (-\infty, \infty)$ except $x = 0$.
Hence, the diff. eq is normal on every subinterval I of the open intervals $(-\infty, 0)$, $(0, \infty)$.

Linear Combination of functions

Let $f_1(x), f_2(x), \dots, f_n(x)$ be n functions. Then $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)$, where $c_1, c_2, \dots, c_n$ are constants is called a linear combination of the given functions.

<u>Thm</u> If $y_1(x), y_2(x), \dots, y_m(x)$ are m solutions of the linear homogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0 \text{ on } I, \quad -(1)$$

then the linear combination of the solutions

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_m y_m(x), \text{ where } c_1, c_2, \dots, c_m$$

are constants is also a sol of eq (1) on I.

<u>Remark</u>: The above thm does not hold for a non-homogeneous equation or a nonlinear equation.

Ex 5.2 Show that e^{-x}, e^{x} and their linear combinations $c_1e^{-x}+c_2e^{x}$ are solutions of the homogeneous equation $y''-y=0$.

Sol :- For $y_1=e^{-x}$, $y_1'=-e^{-x}$, $y_1''=e^{-x}$

$$y_1''-y_1=e^{-x}-e^{-x}=0.$$

For $y_2=e^{x}$, $y_2'=e^{x}$, $y_2''=e^{x}$

$$y_2''-y_2=e^{x}-e^{x}=0.$$

Hence e^{-x}, e^{x} are solutions of $y''-y=0$.

For $y=c_1e^{-x}+c_2e^{x}=c_1y_1+c_2y_2$

$$\cancel{y'=-c_1e^{-x}+c_2e^{x}}$$

$$\cancel{y''=c_1e^{-x}+c_2e^{x}}$$

$$\cancel{y''-y=c_1e^{-x}+c_2e^{x}-c_1e^{-x}-c_2e^{x}=0.}$$

$$y''-y=(c_1e^{-x}+c_2e^{x})''-(c_1e^{-x}+c_2e^{x})$$

$$=(c_1y_1+c_2y_2)''-(c_1y_1+c_2y_2)$$

$$=c_1y_1''+c_2y_2''-c_1y_1-c_2y_2$$

$$\cancel{=c_1e^{-x}+c_2e^{x}-c_1}$$

$$=c_1(y_1''-y_1)+c_2(y_2''-y_2)$$

$$=c_1\cdot 0+c_2\cdot 0$$

$$=0.$$

Verify that the given functions are solution of the associated differential equation. Verify also that a linear combination of these functions is also a solution.

(18) $1, x, e^x;\ y''' - y'' = 0.$

For $y_1 = 1$

$y_1' = y_1'' = y_1''' = 0.$

$y_1''' - y_1'' = 0 - 0 = 0.$

For $y_2 = x$, $y_2' = 1,\ y_2'' = 0,\ y_2''' = 0.$

$y_2''' - y_2'' = 0 - 0 = 0.$

For $y_3 = e^x$, $y_3' = e^x,\ y_3'' = e^x,\ y_3''' = e^x$

$y_3''' - y_3'' = e^x - e^x = 0.$

Substituting $y = c_1 \cdot 1 + c_2 \cdot x + c_3 e^x$
$= c_1 y_1 + c_2 y_2 + c_3 y_3$

$y''' - y'' = (c_1 y_1 + c_2 y_2 + c_3 y_3)''' - (c_1 y_1 + c_2 y_2 + c_3 y_3)''$

$= c_1 y_1''' + c_2 y_2''' + c_3 y_3''' - c_1 y_1'' - c_2 y_2'' - c_3 y_3''$

$= c_1 (y_1''' - y_1'') + c_2 (y_2''' - y_2'') + c_3 (y_3''' - y_3'')$

$= c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 = 0.$

(19) $e^{x}, e^{-2x}; \; y''+y'-2y=0.$

For $y_1=e^{x}, \; y_1'=e^{x}, \; y_1''=e^{x}$

$$y_1''+y_1'-2y_1=e^{x}+e^{x}-2e^{x}=2e^{x}-2e^{x}=0.$$

For $y_2=e^{-2x}, \; y_2'=-2e^{-2x}, \; y_2''=4e^{-2x}.$

$$y_2''+y_2'-2y_2=4e^{-2x}+(-2e^{-2x})-2e^{-2x}=0.$$

Substituting $y=c_1e^{x}+c_2e^{-2x}=c_1y_1+c_2y_2$

$$y''+y'-2y=(c_1y_1+c_2y_2)''+(c_1y_1+c_2y_2)'-2(c_1y_1+c_2y_2)$$

$$=c_1y_1''+c_2y_2''+c_1y_1'+c_2y_2'-2c_1y_1-2c_2y_2$$

$$=c_1(y_1''+y_1'-2y_1)+c_2(y_2''+y_2'-2y_2)$$

$$=c_1\cdot 0+c_2\cdot 0=0.$$

(20) $e^{-x}\cos 2x, \; e^{-x}\sin 2x; \; y''+2y'+5y=0.$

For $y_1=e^{-x}\cos 2x \Rightarrow y_1'=-e^{-x}\cos 2x+e^{-x}(-2\sin 2x)$

$$=-e^{-x}\cos 2x-2e^{-x}\sin 2x$$

$$y_1''=e^{-x}\cos 2x+2e^{-x}\sin 2x+2e^{-x}\sin 2x-4e^{-x}\cos 2x$$

$$=-3e^{-x}\cos 2x+4e^{-x}\sin 2x.$$

$$y_1''+2y_1'+5y_1=(-3e^{-x}\cos 2x+4e^{-x}\sin 2x)+2(-e^{-x}\cos 2x-2e^{-x}\sin 2x)+5e^{-x}\cos 2x$$

$$=0.$$

$y_2 = e^{-x} \sin 2x$

$y_2' = -e^{-x} \sin 2x + 2e^{-x} \cancel{\sin 2x} \cos 2x$

$y_2'' = e^{-x} \sin 2x - 2e^{-x} \cos 2x - 2e^{-x} \cos 2x - 4e^{-x} \sin 2x$

$= -3e^{-x} \sin 2x - 4e^{-x} \cos 2x$

$y_2'' + 2y_2' + 5y = -3e^{-x} \sin 2x - 4e^{-x} \cos 2x - 2e^{-x} \sin 2x + 4e^{-x} \cos 2x$

$+ 5e^{-x} \sin 2x$

$= 0.$

Substituting $y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

$= c_1 y_1 + c_2 y_2$

$y'' + 2y' + 5y = (c_1 y_1 + c_2 y_2)'' + 2(c_1 y_1 + c_2 y_2)' + 5(c_1 y_1 + c_2 y_2)$

$= c_1 (y_1'' + 2y_1' + 5y_1) + c_2 (y_2'' + 2y_2' + 5y_2)$

$= c_1 . 0 + c_2 . 0$

$= 0.$

Linear Independence and Dependence

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then, these functions are said to be LI on some interval I (where they are defined) if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \quad -(1)$$

implies $c_1 = 0 = c_2 = \ldots = c_n$.

These functions are said to be LD on I, if eq (1) holds for $c_1, c_2, \ldots, c_n$ not all zero.

In this case, one or more functions can be expressed as a linear combination of the remaining functions.

For ex, if $c_1 \neq 0$, then

$$f_1(x) = \frac{-1}{c_1}\left(c_2 f_2(x) + \cdots + c_n f_n(x)\right).$$

Conversely, if any function $f_i(x)$ can be expressed as a linear combination of the functions $f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots f_n$, then the given set of functions are linearly dependent.

Wronskian

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions.

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x).$$

Th^m If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ in the linear homogeneous equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \quad a_0 \neq 0 \quad -(1)$$

are continuous on I and $y_1(x), \ldots, y_n(x)$ are n sols of this eq, then

(i) $W(x) = W(y_1, y_2, \ldots, y_n) \neq 0$ for all $x \in I$.

$\Leftrightarrow$ $y_1(x), y_2(x), \ldots, y_n(x)$ are LI on I.

(ii) $W(x_0) = 0$ where $x_0 \in I$ is any fixed point, implies $W(x) = 0$ for all x in I and the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are LD.

Th^m If the coefficients $a_0(x), a_1(x), \_\_\_, a_n(x), a_0(x) \neq 0$, in the linear homogeneous equation ① are continuous on I, then the eq ① has n linearly independent solutions. If $y_1(x), ---, y_n(x)$ are n linearly independent solutions, then the general solution is given by $y(x) = \cancel{c_1(x)}\ c_1y_1(x) + c_2y_2(x) + --- + c_ny_n(x)$, that is, their linear combination.

The n L.I. solutions $y_1(x), -, y_n(x)$ are also called the fundamental solutions of eq ①. This set of fundamental solutions forms a basis of the nth order linear homogeneous equation.

Ex 5.6 Show that the functions x, x^2, x^3 are LI on any interval I, not containing zero.

Sol :

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - x^2(6x) + x^3(2)$$
$$= 6x^3 - 6x^3 + 2x^3$$
$$= 2x^3.$$

Therefore, $W(x) \neq 0$ on any interval not containing zero. Hence, the functions are LI in $(-\infty, 0), (0, \infty)$.

Ex 5.7 Show that the functions $1, \sin x, \cos x$ are LI.

Sol :
$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= -\cos^2 x - \sin^2 x = -1 \neq 0.$$

Hence, the given functions are LI on any interval I.

Ex 5.8 Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$, on any interval I.

Sol : $y_1(x) = e^x$, $y_1' = e^x$, $y_1'' = e^x$, $y_1''' = e^x$

$y_1''' - 6y_1'' + 11y_1' - 6y = e^x - 6e^x + 11e^x - 6e^x = 0.$

$y_2(x) = e^{2x}$, $y_2'(x) = 2e^{2x}$, $y_2'' = 4e^{2x}$, $y_2''' = 8e^{2x}$.

$y_2''' - 6y_2'' + 11y' - 6y = 8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x} = 0.$

$y_3(x) = e^{3x}$, $y_3'(x) = 3e^{3x}$, $y_3''(x) = 9e^{3x}$, $y_3'''(x) = 27e^{3x}$.

$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x}$
$= 0.$

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^x & e^{2x} \\ 1 & 2e^x & 3e^{2x} \\ 1 & 4e^x & 9e^{2x} \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}(18-12-9+3+4-2)$$

$$= 2e^{6x} \neq 0.$$

$\therefore$ Solutions are LI and they form a set of fundamental solutions on any interval I.