Date 26/09/2023

MTH-302 | UNIT-4

Special Continuous Distribution

→ Normal Distribution and M.G.F

→ Normal approximation of Binomial Distribution

→ Exponential Distribution and M.G.F

→ Gamma Distribution and M.G.F

Normal Distribution

$X \longrightarrow$ Normal variate

$X \sim N(\mu, \sigma^2)$

$X \downarrow$ Normal Variate

Standard Normal Variate

$$Z = \frac{X - \mu}{\sigma}$$

Mean of z

$$E(z) = E\left[\frac{x-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma} E[x-\mu]$$

$$= \frac{1}{\sigma}\left[E[x] - E[\mu]\right]$$

$$= \frac{1}{\sigma}[\mu - \mu]$$

$$= \frac{1}{\sigma} \times 0$$

$$= 0$$

$$V(ax) = a^2 V(x)$$

$$V(z) = V\left[\frac{x-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma^2} V[x-\mu]$$

$$= \frac{1}{\sigma^2} V(x)$$

$$= \frac{1}{\sigma^2} \times \sigma^2 = 1$$

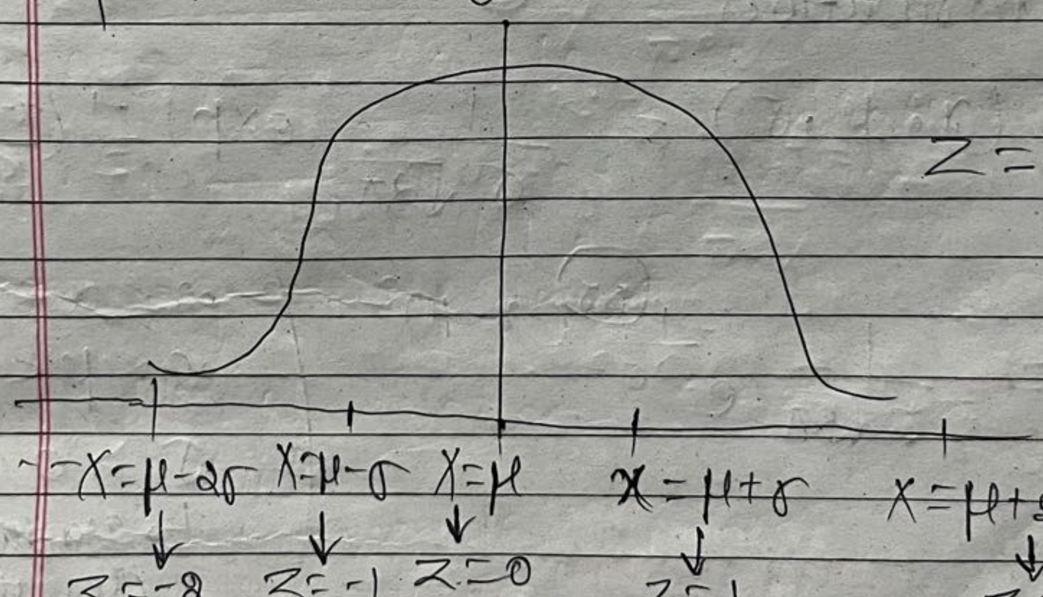
$$X \sim N(\mu, \sigma^2)$$

$$Z \sim N(0, 1)$$

Standard Normal variate

Remark: while dealing with the numericals of Normal distribution we will utilise Standard Normal variate because the Normal table is defined for Standard Normal variate

# properties of Normal Probability Curve




$$Z = \frac{X-\mu}{\sigma}$$

- The Normal probability curve is bell Shaped.

- The Normal probability curve is Symmetric about the line. $X = \mu$.

# Normal distribution → Continuous in Nature



Date ___/___/___

Page ______

- In the Normal distribution mean, median and mode all will coincide

* $P[-1 \le z \le 1]$

  $= 2 \cdot P[0 \le z \le 1]$

# $P[-1 \le z \le 0]$

  $= P[0 \le z \le 1]$ → Symmetric

# Probability density function of Normal Distribution

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

or

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

↳ Standard deviation

# M.G.F of Normal Distribution

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \underbrace{f(x)}_{p.d.f} \, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \left[ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \right] dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \, e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \, dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{x-\mu}{\sigma}$$

$$x - \mu = \sigma z$$

$$x = \mu + \sigma z$$

$$\boxed{dx = \sigma dz}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t[\mu+\sigma z]} \cdot e^{-\frac{1}{2\cancel{\sigma^2}}\cancel{\sigma^2} z^2} dx$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} e^{\sigma t z} \cdot e^{-\frac{1}{2}z^2} (\cancel{\sigma} dz)$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2 + \sigma t z} \, dz$$

$$\zeta - \sigma t = u$$
$$d\zeta = du$$

Date ___ Page ___

$$e^{\mu t}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t}{2}}$$

$$= e^{\mu t}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\zeta^2 - 2t\sigma\zeta + \sigma^2 t^2 - \sigma^2 t^2\right]}\,d\zeta$$

$$= e^{\mu t}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(\zeta - \sigma t)^2 - \sigma^2 t^2\right]}\,d\zeta$$

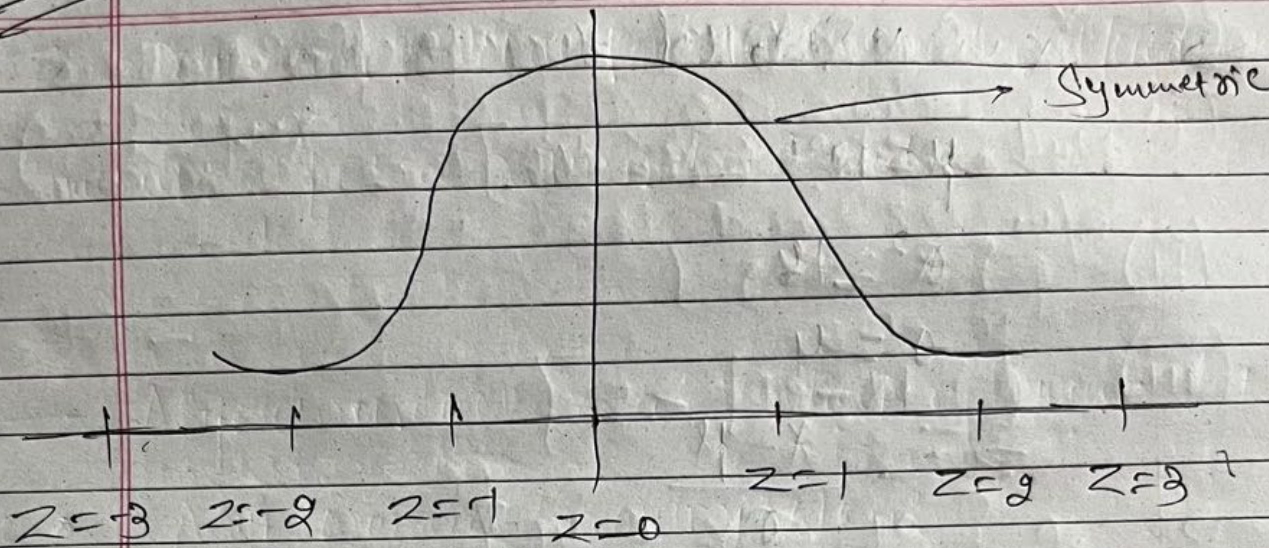
$$= e^{\mu t}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\zeta - \sigma t)^2}\cdot e^{\frac{1}{2}\sigma^2 t^2}\,d\zeta$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\zeta - \sigma t)^2}\,d\zeta$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\cdot du$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Date 28/09/2028 | MTH-202 |




$$P[-1 \le Z \le 1] = 0.6826$$

$$P[-2 \le Z \le 1] = 0.9544$$

$$P[-3 \le Z \le 3] = 0.9974$$

$$P[-1 \le Z \le 0] = P[0 \le Z \le 1] = \frac{1}{2}[0.6826]$$

$$P[-2 \le Z \le 0] = P[0 \le Z \le 2] = \frac{1}{2}[0.9544]$$

$$P[-3 \le Z \le 0] = P[0 \le Z \le 3] = \frac{1}{2}[0.9574]$$

Q Let $X$ is Normally Distributed with the mean of $X = 12$ and S.d of $X = 4$. find

(i) $P[X \ge 20]$

(ii) $P[X \le 20]$

(iii) $P[0 \le X \le 12]$

$\rightarrow$ $X \sim N[12, 16]$

$\mu = 12$

$\sigma^2 = 16$

$\sigma = 4$

$$z = \frac{X - \mu}{\sigma}$$

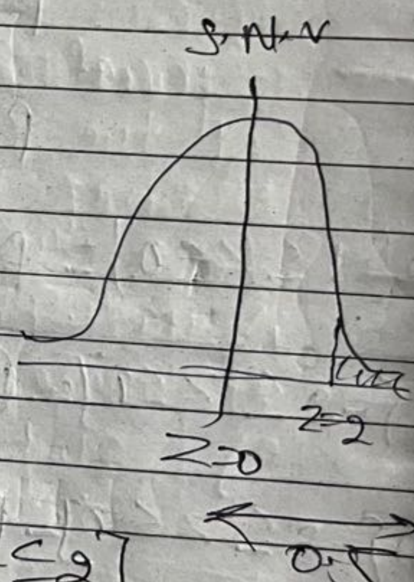
$$Z = \frac{X - 12}{4}$$

(i) $P[X \geq 20]$

$$P\left[\frac{X-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}\right]$$

$$P\left[Z \geq \frac{20-12}{4}\right]$$

$$P[Z \geq 2]$$



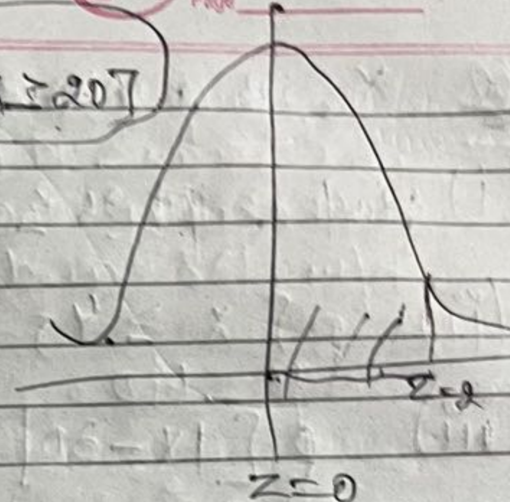

$$P[Z \geq 2] = 0.5 - P[0 \leq Z \leq 2]$$

$$= 0.5 - \frac{1}{2}[0.9544]$$

$$= 0.5 - 0.4772$$

$$= \underline{0.0228}$$

(ii) $P[X \le 20] \to 1 - P[X \ge 20]$




$\Rightarrow P\left[\frac{X-\mu}{\sigma} \le \frac{20-\mu}{\sigma}\right]$

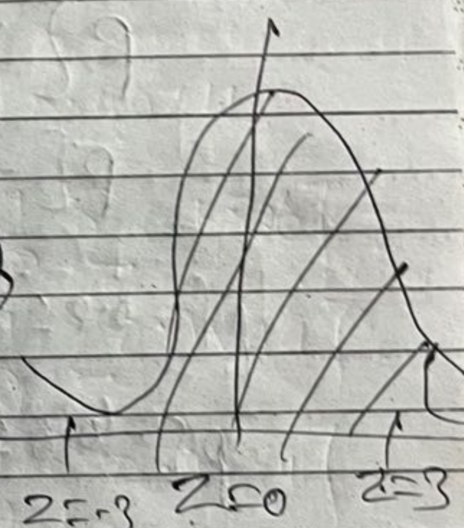
$P\left[Z \le \frac{20-12}{4}\right]$

$P[Z \le 2]$

$P[Z \le 2] = P[0 \le Z \le 2]$

$= \underline{0.9544}$

(iii) $P[0 \le X \le 12]$




$P\left[0 \le \frac{X-\mu}{\sigma} \le \frac{12-12}{4}\right]$

$P[0 \le Z \le 0]$

$\cancel{0.9544}$ $0$

$= P\left[\frac{0-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{12-\mu}{4}\right]$

$= P\left[\frac{0-12}{4} \le Z \le \frac{12-12}{4}\right]$

$= P[-3 \le Z \le 0]$

$= P[0 \le Z \le 3] = \underline{0.9974}$

Q $X$ is a Normal variate $\mu = 30$, $\sigma = 5$

$X \sim (30, 25)$

(i) find $p[26 \le X \le 40]$

(ii) $p[X \ge 45]$

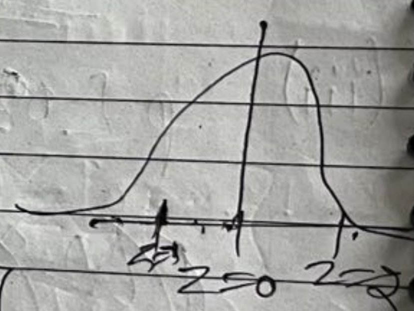
(iii) $p[|X - 30| > 5]$

$\rightarrow$ $p[26 \le X \le 40]$

$p\left\{\frac{26-30}{5} \le z \le \frac{40-30}{5}\right\}$

$p\left\{-\frac{4}{5} \le z \le 2\right\}$

$p[-0.8 \le z \le 2]$




$p[-0.8 \le z \le 0] + p\{0 \le z \le 2\}$

$p[-0.8 \le z \le 0] + \frac{1}{2}\{0.9544\}$

$p\{0 \le z \le 0.8\} + \frac{1}{2}[0.9544]$

It will take from tabular value $\leftarrow$ $0.2881 + \frac{1}{2} \times 0.9544$

$0.2881 + 0.4772$

$= \boxed{0.7653}$

5) 40(0.8

Date __/__/__

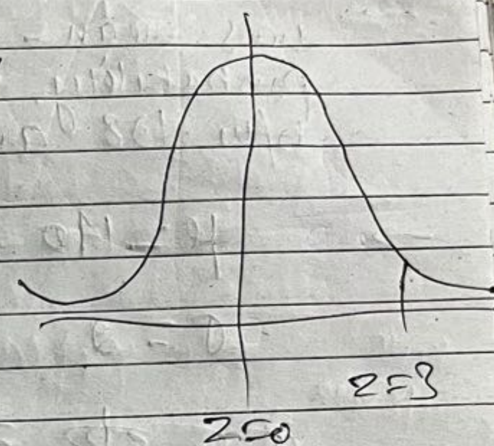
Page ____

(ii) $p[X \geq 45]$

$\rightarrow p\left[Z \geq \frac{45-30}{5}\right]$

$p\left[Z \geq \frac{15}{5}\right]$

$p[Z \geq 3]$




$= p[Z \geq 3] = 0.5 - p[0 \leq Z \leq 3]$

$= 0.5 - \frac{1}{2}[0.9974]$

$= 0.5 - 0.4987$

$= 0.0013$

(iii) $p\{|X-30| > 5\}$

$\rightarrow 1 - p[|X-30| \leq 5]$

$1 - p\left[-\frac{5}{5} \leq \frac{X-30}{5} \leq \frac{5}{5}\right]$

$1 - p[-1 \leq Z \leq 1]$

$1 - 0.6826$

$= 0.3174$

Date ___ Page ___

Q2 The thickness of a polyster film base has mean = 140 and S.D. = 2. Find the probability that the thickness tested in b/w 138 and 143

$\rightarrow$ $\mu = 140$ $\qquad P\{138 \leq X \leq 143\}$

$\sigma = 2$

$z = \frac{X - \mu}{\sigma}$

$P\left\{\frac{138-140}{2} \leq z \leq \frac{143-140}{2}\right\}$

$P\{-1 \leq z \leq 1.5\}$

$P\{-1 \leq z \leq 0\} + P\{0 \leq z \leq 1.5\}$

$\frac{1}{2}[0.6826] + P[0 \leq z \leq 1.5]$

$\frac{1}{2}[0.6826] + 0.4332$

$0.3413 + 0.4332$

$\boxed{0.7745}$ Ans

Date __/__/__

Page ____

Q The time for a super glue to set can be treated as a R.V having a Normal distn with $\mu = 30$ sec. Find its S.D if the probability is 0.20 that it will take on a value greater than 39.2 sec

→ $\mu = 30$ sec

$\sigma = ?$

$P[X > 39.2] = 0.20$

$$z = \frac{X - \mu}{\sigma}$$

$$P\left[z > \frac{39.2 - 30}{\sigma}\right] = 0.20$$

0, 1, 1, 2, 3, 5, 8, 13, 21, ——

print (1)
(2)

Date 29/09/23.

MTH-202

Gamma Distribution

Suppose X is Random variable which follows the gamma distribution with the following density function

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^{\alpha} \cdot \Gamma\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{else} \end{cases}$$

Parameters

$\alpha > 0$

$\beta > 0$

Gamma function $\Gamma\alpha = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$

$$\frac{\Gamma\alpha}{t^{\alpha}} = \int_0^{\infty} x^{\alpha-1} e^{-tx} dx$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

$$\Gamma(\alpha+1) = \lfloor\underline{\alpha} \rightarrow \text{factorial } \alpha$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

# Moment Generating function of Gamma distribution

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx}\left[\frac{1}{\beta^\alpha \cdot \Gamma\alpha}\, x^{\alpha-1} e^{-x/\beta}\right] dx$$

$$= \frac{1}{\beta^\alpha \cdot \Gamma\alpha}\int_0^\infty e^{tx} \cdot x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta^\alpha \cdot \Gamma\alpha}\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}+tx}\,dx$$

$$= \frac{1}{\beta^\alpha \cdot \Gamma\alpha}\int_0^\infty x^{\alpha-1} e^{-\left[\frac{1}{\beta}-t\right]x}\,dx$$

$$= \frac{1}{\beta^{\alpha} \cdot \cancel{\Gamma\alpha}} \left[ \frac{\cancel{\Gamma\alpha}}{\left(\frac{1}{\beta} - t\right)^{\alpha}} \right]$$

$$= \frac{1}{(1-\beta t)^{\alpha}}$$

$$= (1-\beta t)^{-\alpha}$$

Mean and variance of Gamma distribution by utilising m.G.F

$$\mu_1' = \left[ \frac{d}{dt} M_X(t) \right]_{t=0}$$

$$= \left[ \frac{d}{dt} \{1 - \beta t\}^{-\alpha} \right]_{t=0}$$

$$= \left[ -\alpha (1-\beta t)^{-\alpha-1} (-\beta) \right]_{t=0}$$

$$= \left[ \alpha\beta (1-\beta t)^{-\alpha-1} \right]_{t=0}$$

$$= \alpha\beta$$

$$\mu_2' = \left[\frac{d^2}{dt^2} m_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt} m_x(t)\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\alpha\beta(1-\beta t)^{-\alpha-1}\right\}\right]_{t=0}$$

$$= \alpha\beta\left[\frac{d}{dt}(1-\beta t)^{-\alpha-1}\right]_{t=0}$$

$$= \alpha\beta\left[(-\alpha-1)(1-\beta t)^{-\alpha-2}\cdot(-\beta)\right]_{t=0}$$

$$= \alpha\beta\left[(\alpha+1)\beta\right]$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

Variance

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

$$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2}$$

$$= \alpha\beta^2$$

Date ____ Page ____

## Exponential distribution

We can obtain exponential distribution via the Gamma distribution. If in the Gamma distribution we will take $\alpha = 1$ then the density fun$^n$ of exponential distribution will be obtain

$$F(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^{\alpha}\cdot\Gamma\alpha} x^{\alpha-1} e^{-x/\beta} & , x>0 \\ 0 & , \text{else} \end{cases}$$

$\alpha = 1$

exponential distribution

$$\boxed{F(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & , x>0 \\ 0 & , \text{else} \end{cases}}$$

M.G.F of exponential distribution

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \int_0^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx}\left[\frac{1}{\beta} e^{-x/\beta}\right] dx$$

$$= \frac{1}{\beta}\int_0^\infty e^{tx}\cdot e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\int_0^\infty e^{tx-\frac{x}{\beta}}\,dx$$

$$= \frac{1}{\beta}\int_0^\infty e^{\left(t-\frac{1}{\beta}\right)x}\,dx$$

$$\Rightarrow \frac{1}{\beta}\int_0^\infty e^{-\left(\frac{1}{\beta}-t\right)x}\,dx$$

$$= \frac{1}{\beta}\int_0^\infty x^{1-1}e^{-\left(\frac{1}{\beta}-t\right)x}\,dx$$

$$= \frac{1}{\beta}\left[\frac{\Gamma 1}{\left(\frac{1}{\beta}-t\right)}\right]$$

$$\Rightarrow \frac{1}{(1-\beta t)}$$

$$= (1-\beta t)^{-1}$$

Mean and variance

# $\mu_1' = \left[ \frac{d}{dt} M_X(t) \right]_{t=0}$

$= \left[ \frac{d}{dt} \{1-\beta t\}^{-1} \right]_{t=0}$

$= \left[ -1 (1-\beta t)^{-2} \cdot (-\beta) \right]_{t=0}$

$= \beta$

$\mu_2' = \left[ \frac{d^2}{dt^2} M_X(t) \right]_{t=0}$

$= \left[ \frac{d}{dt} \{ - (1-\beta t)^{-2} \cdot (-\beta) \} \right]_{t=0}$

$= \left[ 2 (1-\beta t)^{-3} \cdot (-\beta) \cdot (-\beta) \right]_{t=0}$

$= +2\beta^2$

variance

$= 2\beta^2 - \beta^2$

$= \beta^2$

Date / /
Page

Q Suppose that the telephone calls arriving at a particular switch board follow a poisson process with an average of 5 calls / minute

(1) what is the probability that upto a minute will arrives by the time 2 calls have come into the switch board

$\lambda = 5$ $\qquad \alpha = 2$ Calls

$\lambda = \frac{1}{\beta}$

$$\boxed{\beta = \frac{1}{5}}$$

$P[X \leq 1]$

$$= \int_0^1 \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int_0^1 \frac{1}{\left(\frac{1}{5}\right)^2 \Gamma 2} x^{2-1} e^{-x/\frac{1}{5}} dx$$

$\Rightarrow$ 0.96

Date 03/10/2023

MTH-302

Q Suppose that a system containing a certain type of component whose time, in years to the failure is given by T. The R.V T is modern nicely by the exponential Distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that atleast 2 are still functioning and the end of 8 yrs

$\to \quad \beta = 5$

$$P(T>8) = \int_8^\infty f(x)\,dx$$

$$= \int_8^\infty \left[\frac{1}{\beta} e^{-x/\beta}\right] dx = \frac{1}{5}\int_8^\infty e^{-x/5}\,dx$$

$\simeq$ 0.2 $\to$ probability of success $\to p$

$\Rightarrow$ $q = 0.8$ $\to$ Probability of failure

$$P[X\geq 2] = P(2) + P(3) + P(4) + P(5)$$

$\Rightarrow 0.2627$

Ans

$p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$= {}^{5}C_{x}\,(0.2)^{x}\,(0.8)^{5-x}$

$\Rightarrow P[x \geq 2] = P(2) + P(3) + P(4) + P(5)$

$= P(2) = {}^{5}C_{2}\,(0.2)^{2}\,(0.8)^{3}$

$= {}^{5}C_{2}\,(0.04)\,(0.512)$

$=$