Solve $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 4 \end{array}\right]$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$.

$$= \begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 = \rho(A|B) <$ No of unknowns.

$\Rightarrow$ The system has infinitely many solutions.

$$x - 2y + z + 2w = 1$$

$$3y - 2z - w = 1 \Rightarrow 3y = 1 + 2z + w$$

$$z = k_1,\ w = k_2 \qquad \Rightarrow y = \frac{1 + 2k_1 + k_2}{3}$$

$$x = 1 + \frac{2 + 4k_1 + 2k_2}{3} - k_1 - 2k_2$$

$$= \frac{3 + 2 + 4k_1 + 2k_2 - 3k_1 - 6k_2}{3}$$

$$= \frac{5 + k_1 - 4k_2}{3}.$$

$$\therefore (x, y, z, w) = \left(\frac{5 + k_1 - 4k_2}{3}, \frac{1 + 2k_1 + k_2}{3}, k_1, k_2 \right).$$

$$\begin{bmatrix} 1 & \cancel{2}1 & -2 & -1 \end{bmatrix}$$

Discuss for what values of λ, μ the equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$ have

(i) No sol

(ii) A unique sol

(iii) Infinite number of solutions

$$AX=B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

1 ~~No sol~~ $\Rightarrow$ Case (i) $\lambda \neq 3$, $\rho(A) = \rho(A|B) = 3 =$ No of unknows.

$\Rightarrow$ Unique sol if $\lambda \neq 3$ and for any value of μ.

Case (ii) If $\lambda = 3$, $\mu \neq 10$.

$\rho(A) = 2$
$\rho(A|B) = 3$

No sol.

Case (iii) Let $\lambda = 3$, $\mu = 10$.

$\rho(A) = \rho(A|B) = 2 <$ No of unknowns.

$\Rightarrow$ Infinitely many solutions.