Continuity of functions of two variables

A function $f(x,y)$ is said to be continuous at a point (x_0,y_0) if

1. $f(x,y)$ is defined at (x_0,y_0).
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

A fn $f(x)$ is cont at $x=a$, if (i) $f(x)$ ~~exists~~ is defined at $x=a$
(ii) $\lim\limits_{x\to a} f(x)$ exists.
(iii) $\lim\limits_{x\to a} f(x) = f(a)$

If any of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0,y_0).

A function is continuous if it is continuous at every point of its domain.

Ex Show that the following functions is continuous at the point $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{2x(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: ① $f(x,y)$ is defined at $(0,0)$.

T.P.: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

T.P. $|f($

whenever

$0<$

Consider $\left|\dfrac{2x(x^2-y^2)}{x^2+y^2}-0\right| = \left|\dfrac{2x(x^2-y^2)}{x^2+y^2}\right|$

$$= \left|\frac{2x^3-2xy^2}{x^2+y^2}\right|$$

$$= \left|\frac{2x^3}{x^2+y^2}-\frac{2xy^2}{x^2+y^2}\right|$$

$$\le \left|\frac{2x^3}{x^2+y^2}\right| + \left|\frac{2xy^2}{x^2+y^2}\right| = 2\left|\frac{x^3}{x^2+y^2}\right| + 2\left|\frac{xy^2}{x^2+y^2}\right|$$

$x^2 \le x^2+y^2$

$y^2 \le x^2+y^2$

$\le 2|x| + 2|x|$

$< 2\delta + 2\delta = 4\delta$, if we take $|x| < \delta$.

$\Rightarrow \dfrac{x^2}{x^2+y^2} \le 1$ and

$\dfrac{y^2}{x^2+y^2} \le 1.$

Choose $4\delta < \epsilon$.

Thus, $\left|\dfrac{2x(x^2-y^2)}{x^2+y^2}-0\right| < \epsilon$ whenever $0<|x-0|<\delta$, $0<|y-0|<\delta$.

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2x(x^2-y^2)}{x^2+y^2} = f(0,0).$$

Hence, $f(x,y)$ is continuous at $(0,0)$.

Ex :- Show that the following functions ~~are~~ is discontinuous at the given point

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the point $(0,0)$.

Sol-:

$$f(x,y) = \frac{x^2 - x\sqrt{y}}{x^2+y}$$

Choose the path $y = mx^2$ so that $y \to 0$ as $x \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2 - x\sqrt{mx^2}}{x^2 + mx^2} = \lim_{x\to 0} \frac{\cancel{x^2}(1-\sqrt{m})}{\cancel{x^2}(1+m)}$$

$$= \frac{1-\sqrt{m}}{1+m}, \text{ which depends on } m.$$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Thus, $f(x,y)$ is not continuous at $(0,0)$.

Ex Discuss the continuity of the following function

$$f(x,y) = \begin{cases} \dfrac{xy(x-y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

alternate method attached

Sol→ $f(x,y) = \dfrac{xy(x-y)}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy(x-y)}{x^2+y^2}$$

Along the path $y = mx$, we have

$$\lim_{x\to 0} \frac{x(mx)(x-mx)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{mx^3(1-m)}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{mx(1-m)}{1+m^2}$$

$$= 0.$$

T.P.:- If $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ i.e. $|f(x,y) - 0| < \epsilon$ whenever $0<|x-0|<\delta$, $0<|y-0|<\delta$.

then ~~Hence~~, $f(x,y)$ is Continuous at $(0,0)$.

Consider
$$\left|\frac{xy(x-y)}{x^2+y^2} - 0\right| = \left|\frac{xy(x-y)}{x^2+y^2}\right| = \left|\frac{x^2y - xy^2}{x^2+y^2}\right|$$

$$\le \left|\frac{x^2y}{x^2+y^2}\right| + \left|\frac{xy^2}{x^2+y^2}\right|$$

$$\le |y| + |x|$$

$x^2 \le x^2+y^2$
$y^2 \le x^2+y^2$

$< \delta+\delta = 2\delta < \epsilon$ if we take $|x|<\delta$, $|y|<\delta$.

If we choose $2\delta < \epsilon$

$\Rightarrow |f(x,y) - 0| < \epsilon$ whenever $0<|x-0|<\delta$, $0<|y-0|<\delta$.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Thus, $f(x,y)$ is Continuous at $(0,0)$.

OR

Alternative method

Let $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$, $\tan\theta = \frac{y}{x}$

and $r \to 0$ as ~~$(x,y)\to(0,0)$~~. ~~$x\to 0, y\to 0$~~. $(x,y)\to(0,0)$.

$[\,(x,y)\to(0,0)$ iff $r\to 0\,]$.

$$\lim_{(x,y)\to(0,0)} \frac{xy(x-y)}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos\sin\theta(-r\sin\theta + r\cos\theta)}{r^2[\cos^2\theta + \sin^2\theta]}$$

$$= \lim_{r\to 0} r(\cos\theta\sin\theta(\cos\theta - \sin\theta))$$

$$= 0.$$

$$\lim_{r\to 0} |f(r\cos\theta, r\sin\theta) - 0| = |r(\sin\theta\cos^2\theta - \cos\theta\sin^2\theta)|$$

$$= |r\sin\theta\cos^2\theta| + |r\cos\theta\sin^2\theta|$$

$\leq |r| + |r|$ as $|\sin\theta| < 1$, $|\cos\theta| < 1$.

$< 2\delta$ as we take $|r| < \delta$.

If we choose $2\delta < \epsilon$,

$\lim_{r\to 0} |f(r\cos\theta, r\sin\theta) - 0| < \epsilon$ whenever ~~$0<$~~ $|r| < \delta$.

$$\Rightarrow \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0$$

i.e. $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Q: Discuss the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y} , & (x,y) \neq (2,2) \\ 4 , & (x,y) = (2,2) \end{cases}$$

at the point (2,2).

Sol:
$$\lim_{(x,y)\to(2,2)} \frac{x^2+xy+x+y}{x+y} = \lim_{(x,y)\to(2,2)} \frac{x(x+y)+1(x+y)}{x+y}$$

$$= \lim_{(x,y)\to(2,2)} \frac{\cancel{(x+y)}(x+1)}{\cancel{x+y}}$$

$$= 3.$$

$$\lim_{(x,y)\to(2,2)} f(x,y) = 3 \neq 4 = f(2,2).$$

The function is not continuous i.e. discontinuous at (2,2)