

Multiple Integrals

The integral $\iint_R f(x,y) dx dy$ or $\iiint_T f(x,y,z) dx dy dz$, where f is continuous at all points inside and on the boundary of the region R or T , is called multiple integrals.

Double Integrals

$$I = \iint_R f(x,y) dx dy \text{ or } \iint_R f(x,y) dy dx \text{ or } \iint_R f(x,y) dA.$$

Evaluation of double Integral

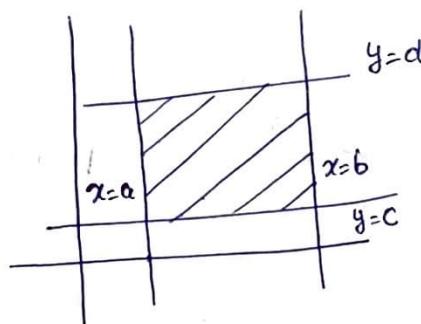
Case 1 Double integrals over rectangles (Limits of both of variables are constant)

If $f(x,y)$ is defined on a planar region (Rectangular Region) given by $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$.

The double integral of $f(x,y)$ over R is

given by

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) dx dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) dy dx$$



Note :- 1. When limits of both the variables are constant.

$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, the order of integration does not make any difference.

2. In case of constant limits,

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) dx dy = \int_{y=c}^{y=d} g(y) dy \int_{x=a}^{x=b} f(x) dx.$$

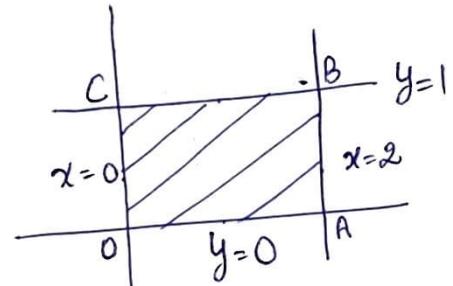
Q.1 The value of integral $\int_0^1 \int_{x=0}^2 e^x dx dy$ is

- (a) $e^2 - 1$ (b) $e - 1$ (c) $e^2 + 1$ (d) e^2 .

Sol:

$$\begin{aligned} \int_{y=0}^1 \left[\int_{x=0}^2 e^x dx \right] dy &= \int_0^1 [e^x]_0^2 dy = \int_0^1 (e^2 - 1) dy \\ &= \left[y(e^2 - 1) \right]_0^1 \\ &= e^2 - 1. \end{aligned}$$

or $\int_{x=0}^2 e^x dx \int_{y=0}^1 dy = e^2 - 1.$



Q.1 Sketch the region of integration and evaluate the double integral

$$I = \iint_R (x+y+1) dx dy \text{ over } R = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 0\}$$

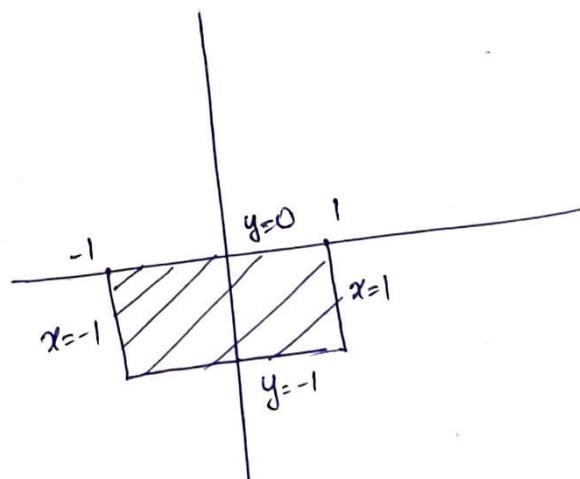
Sol: $I = \iint_{-1}^0 \int_{-1}^1 (x+y+1) dx dy$

$$= \int_{-1}^0 \left[\frac{x^2}{2} + xy + x \right]_{-1}^1 dy$$

$$= \int_{-1}^0 \left(\frac{1}{2} + y + 1 - \frac{1}{2} - y + 1 \right) dy$$

$$= \int_{-1}^0 (2y + 2) dy = \left[\frac{2y^2}{2} + 2y \right]_{-1}^0$$

$$= 0 - 1 + 2(0 + 1) \\ = 1.$$



$$\begin{aligned}
 \text{or } I &= \int_{-1}^1 \int_{-1}^0 (x+y+1) dy dx = \int_{-1}^1 \left[xy + \frac{y^2}{2} + y \right]_{-1}^0 dx \\
 &= \int_{-1}^1 0 - \left(-x + \frac{1}{2} - 1 \right) dx \\
 &= \int_{-1}^1 \left(x + \frac{1}{2} \right) dx = \left[\frac{x^2}{2} + \frac{x}{2} \right]_{-1}^1 \\
 &= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \\
 &= 1.
 \end{aligned}$$

Case 2: Let the region R be expressed in the form

$$R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are integrable functions, such that $g_1(x) \leq g_2(x)$ for all x in $[a, b]$.

[limits of x are constant and limits of y are variable.]

$$I = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx.$$

Case 3 let the region R be defined as

$$R = \{(x, y) : h_1(y) \leq x \leq h_2(y), c \leq y \leq d\},$$

where $h_1(y)$ and $h_2(y)$ are integrable functions, such that $h_1(y) \leq h_2(y)$ for all y in $[c, d]$.

[limits of y are constant and limits of x are variable.]

$$I = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy.$$

Q: Evaluate $\int_{x=0}^2 \int_{y=0}^{x/2} e^{x^2} dy dx$

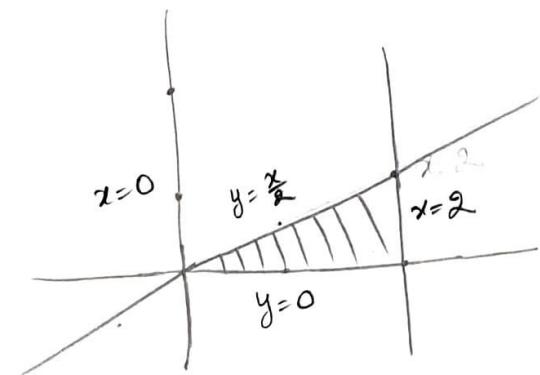
Sol: $\int_0^2 \left[\int_0^{x/2} e^{x^2} dy \right] dx$

$$= \int_0^2 \left[y e^{x^2} \right]_0^{x/2} dx$$

$$= \int_0^2 \frac{x}{2} e^{x^2} dx = \frac{1}{2} \int_0^{2^2} x e^{x^2} dx$$

$$= \frac{1}{4} \int_0^4 e^t dt$$

$$= \frac{1}{4} \left[e^t \right]_0^4 = \frac{1}{4} [e^4 - 1]$$



let $t = e^{x^2}$
 $dt = 2x dx$

$$\Rightarrow x dx = \frac{dt}{2}$$

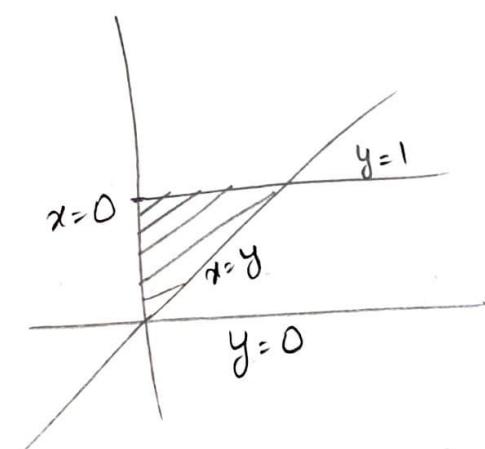
when $x=0, t=0$
 $x=2, t=4$

Q: Evaluate $\int_{y=0}^1 \int_{x=0}^y (x^2 + y^2) dx dy$

Sol: $\int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_0^y dy$

$$= \int_0^1 \left(\frac{y^3}{3} + y^3 \right) dy$$

$$= \frac{4}{3} \int_0^1 y^3 dy = \frac{4}{3} \left[\frac{y^4}{4} \right]_0^1 = \frac{1}{3}$$



(3)

Q: Evaluate $\iint_R f(x,y) dx dy$,

where $f(x,y) = 3y^3 e^{xy}$, $R: 0 \leq x \leq y^2$, $0 \leq y \leq 1$.

$$\text{Sol: } \int_0^1 \int_0^{y^2} 3y^3 e^{xy} dx dy$$

$$= \int_0^1 \left[3y^3 \frac{e^{xy}}{y} \right]_0^{y^2} dy$$

$$= \int_0^1 [3y^2 e^{xy}]_0^{y^2} dy = \int_0^1 3y^2 (e^{y^3} - 1) dy$$

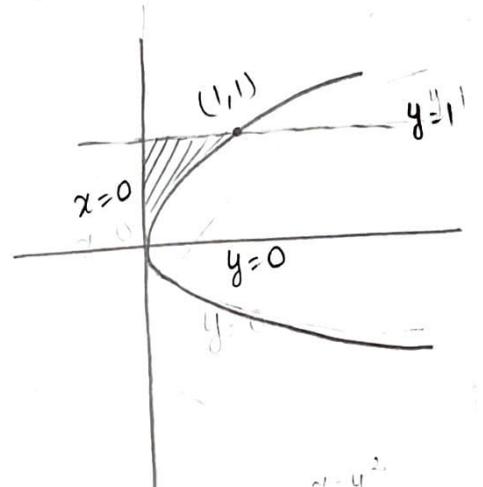
$$= 3 \int_0^1 y^2 e^{y^3} dy - 3 \int_0^1 y^2 dy$$

$$= \left[\int_0^1 e^t dt \right] - 3 \left[\frac{y^3}{3} \right]_0^1$$

$$= [e^t]_0^1 - 1$$

$$= e - 1 - 1$$

$$= e - 2.$$



$$y = y^2 \\ x = 1, \text{ when } y = 1$$

$$\text{let } y^3 = t$$

$$3y^2 dy = dt$$

$$\text{when } t=0, y=0 \\ y=1, t=1$$

$$y=1, t=1$$

Properties of double Integral

① If $f(x,y)$ and $g(x,y)$ are integrable functions, then

$$\iint_R [f(x,y) \pm g(x,y)] dx dy = \iint_R f(x,y) dx dy \pm \iint_R g(x,y) dx dy$$

2. $\iint_R k f(x,y) dx dy = k \iint_R f(x,y) dx dy$, where k is any real constant.

3. When $f(x,y)$ is integrable, then $|f(x,y)|$ is also integrable,

$$\left| \iint_R f(x,y) dx dy \right| \leq \iint_R |f(x,y)| dx dy.$$

4. If $0 < f(x,y) \leq g(x,y)$ for all (x,y) in R , then

$$\iint_R f(x,y) dx dy \leq \iint_R g(x,y) dx dy.$$

5. If $f(x,y) \geq 0$ for all (x,y) in R , then

$$\iint_R f(x,y) dx dy \geq 0.$$

Applications of double Integrals

1. If $f(x,y)=1$, then $\iint_R dx dy$ gives the area A of region R .

Ex: If R is the rectangle bounded by lines $x=a$, $x=b$, $y=c$, $y=d$,

then
$$A = \iint_{c}^{d} \int_{a}^{b} dx dy = (b-a)(d-c)$$

gives the area of rectangle.

(4)

2. If $z = f(x, y)$ is a surface, then

$$\iint_R z \, dx \, dy \text{ or } \iint_R f(x, y) \, dx \, dy$$

gives the volume of the region beneath the surface $z = f(x, y)$ and above the xy plane.

Ex

Change of order of integration

$$a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \xrightarrow{\text{change of order}} c \leq y \leq d, h_1(y) \leq x \leq h_2(y).$$

i.e.

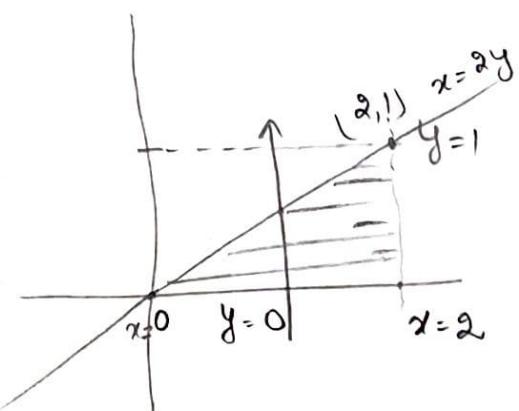
$$\int_{x=a}^{x=b} \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x, y) \, dy \right] dx \xrightarrow{\text{change of order}} \int_{y=c}^{y=d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x, y) \, dx \right] dy$$

Q-1 Evaluate the double integral $\iint_R e^{x^2} \, dx \, dy$, where the region R is given by

$$R: 2y \leq x \leq 2, 0 \leq y \leq 1.$$

Sol:

$$I = \int_0^2 \int_{2y}^2 e^{x^2} \, dx \, dy$$



Since, the integral cannot be evaluated by integrating first w.r.t x . So, we try to evaluate it by integrating it first w.r.t y .

Note: To find the limits of y , consider an imaginary line entering the region R vertically in the increasing direction of y . Check where it enters R and leaves R . That will give lower limit (enter R) and upper limit (exit R) of y .

After the change of order of integration, we have

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=\frac{x^2}{2}} e^{x^2} dy dx$$

$$= \int_0^2 \left[y e^{x^2} \right]_0^{\frac{x^2}{2}} dx$$

$$= \frac{1}{2} \int_0^2 \frac{x}{2} e^{x^2} dx$$

$$= \frac{1}{2} \int_0^2 x e^{x^2} dx$$

$$= \frac{1}{2} \int_0^4 e^t \frac{dt}{2}$$

$$= \frac{1}{4} [e^4 - 1].$$

Let $x^2 = t$
 $2x dx = dt$
 $x dx = \frac{dt}{2}$
when $t=0, x=0$
 $x=2, t=4$

Q:

Evaluate $\int_{x=0}^2 \int_{y=0}^{x^2/2} \frac{x}{\sqrt{x^2+y^2+1}} dy dx$.

Sol:

$$\int_0^2 \int_0^2 \frac{x}{\sqrt{x^2+y^2+1}} dx dy$$

$$x = \sqrt{2}y$$

$$= \int_0^2 \int_0^{y+5} \frac{dt/2}{\sqrt{t}} dy$$

$$= \frac{1}{2} \int_0^2 \left[\frac{t^{1/2}}{1/2} \right]_{(y+1)^2}^{y^2+5} dy$$

$$= \frac{1}{2} \int_0^2 \left[\sqrt{y^2+5} - (y+1) \right] dy$$

$$= \int_0^2 \sqrt{y^2 + (\sqrt{5})^2} dy - \int_0^2 (y+1) dy$$

$$= \left[\frac{y}{2} \sqrt{y^2+5} + \frac{5}{2} \log|y+\sqrt{y^2+5}| \right]_0^2$$

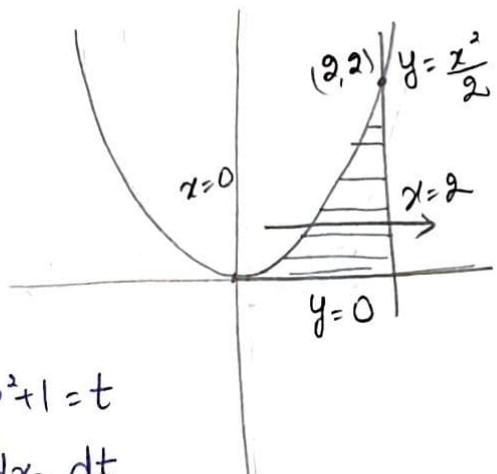
$$- \left[\frac{y^2}{2} + y \right]_0^2$$

$$= 3 + \frac{5}{2} \log|2+3) - \frac{5}{2} \log|\sqrt{5}| - 4$$

$$= \frac{5}{2} \log 5 - \frac{5}{2} \log \sqrt{5} - 1$$

$$= \frac{5}{2} \log 5 - \frac{5}{4} \log 5 - 1$$

$$= \frac{5}{4} \log 5 - 1.$$



$$x^2 + y^2 + 1 = t$$

$$2x dx = dt$$

$$\text{when } x = \sqrt{2}y$$

$$t = \frac{2y}{\sqrt{2}} + y^2 + 1$$

$$= (y+1)^2$$

$$\text{when } x=2, t = \frac{2}{\sqrt{2}} y^2 + 5$$

$$\int \sqrt{a^2+x^2} dx = \frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \log|x+\sqrt{a^2+x^2}| + C$$

Q: Evaluate the double integral $\iint_R xy \, dx \, dy$, where R is the region bounded by the x -axis, the line $y=2x$ and the parabola $y=\frac{x^2}{4a}$.

Sol:

$$\iint_R xy \, dx \, dy$$

$$R \quad 16a \sqrt{4ay}$$

$$= \iint_{\substack{R \\ 0 \leq y \leq 2 \\ 0 \leq x \leq \frac{y}{2}}} xy \, dx \, dy \text{ or } \iint_{\substack{R \\ 0 \leq x \leq \frac{y^2}{4a} \\ 0 \leq y \leq 8a}} xy \, dy \, dx$$

$$= \int_0^{16a} y \left[\frac{x^2}{2} \right]_{y/2}^{\sqrt{4ay}} dy \text{ or } \int_0^{8a} x \left[\frac{y^2}{2} \right]_{\frac{x^2}{4a}}^{\frac{8a}{2}} dx$$

$$= \int_0^{16a} \frac{y}{2} \left[4ay - \frac{y^3}{4} \right] dy \text{ or } \int_0^{8a} \frac{x}{2} \left[4x^2 - \frac{x^4}{16a^2} \right] dx$$

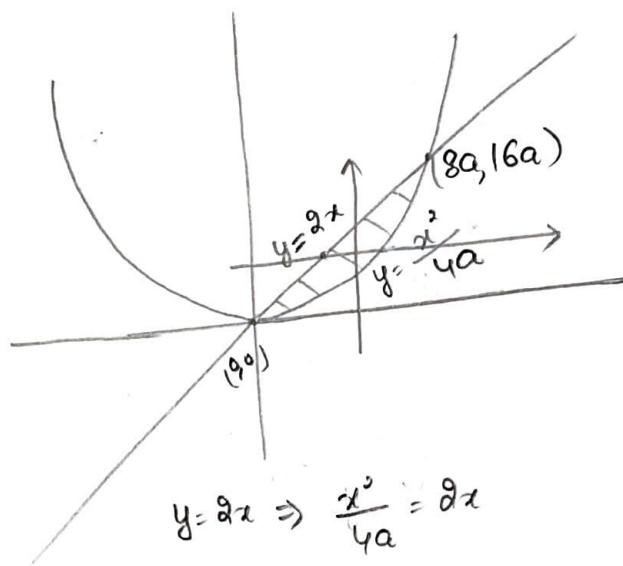
$$= \frac{1}{2} \int_0^{16a} \left[4ay^2 - \frac{y^4}{4} \right] dy \text{ or } \frac{1}{2} \int_0^{8a} \left(4x^3 - \frac{x^5}{16a^2} \right) dx$$

$$= \frac{1}{2} \left[4a \frac{y^3}{3} - \frac{y^5}{16} \right]_0^{16a} \text{ or } \frac{1}{2} \left[\frac{4x^4}{4} - \frac{x^6}{16a^2(6)} \right]_0^{8a}$$

$$= \frac{4a}{6} (16a)^3 - \frac{(16a)^5}{8^2 \cdot 16} \text{ or } \frac{1}{2} \left[(8a)^4 - \frac{(8a)^6}{16 \times 6a^2} \right]$$

~~$$= \frac{4a^2}{6} \cdot \frac{256}{8} 16a^3 - \frac{1}{8} (4096)a^4$$~~

~~$$= 8192$$~~



$$y=2x \Rightarrow \frac{x^2}{4a} = 2x$$

$$\Rightarrow x^2 = 8ax$$

$$\Rightarrow x^2 - 8ax = 0$$

$$\Rightarrow x(x - 8a) = 0$$

$$\Rightarrow x=0, x=8a$$

$$y=0, y=16a$$

$$\begin{aligned}
 &= \frac{4a}{3}(2048)a^3 - 2048a^4 \quad \text{or} \quad 2048a^4 - \frac{8192}{2 \cdot 3} a^4 \\
 &= \frac{8192}{3} a^4 - 2048a^4 \quad \text{or} \quad 2048a^4 - \frac{4096}{3} a^4 \\
 &= \frac{2048}{3} a^4 \quad \text{or} \quad \frac{2048}{3} a^4.
 \end{aligned}$$

Note :- ① To find the limits of x , consider an imaginary line entering the region R horizontally in the increasing direction of x . Check where it enters R (lower limit) and leaves R (upper limit).

② To find the limits of y , consider an imaginary line entering the region R vertically in the increasing direction of y . Check where it enters R (lower limit) and leaves R (upper limit).

Area using double integrals

If $f(x,y)=1$, then the integral

$A = \iint_R dx dy$ is called the area of the region R .

Q: Find the area bounded by the curves $y=x^2$, $y=4-x^2$.

Sol:

$$y = x^2, y = 4 - x^2$$

$$\Rightarrow 4 - x^2 = x^2$$

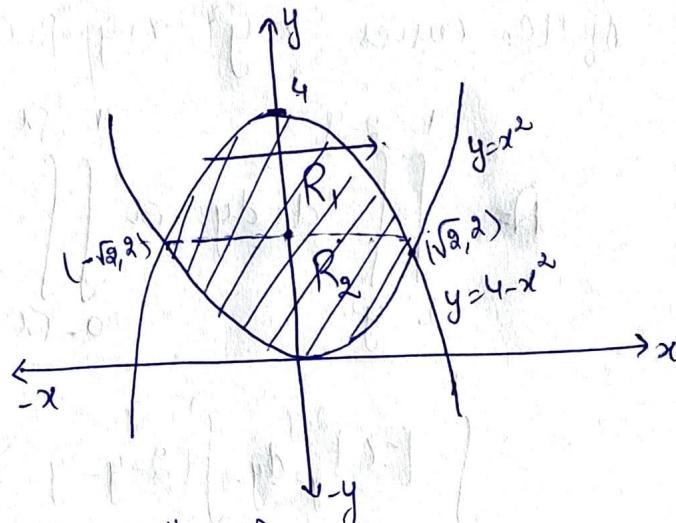
$$\Rightarrow 2x^2 = 4$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \sqrt{2}, -\sqrt{2}$$

$$y = 2, 2$$

So, $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ are points of intersection of parabolas.



$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^{4-x^2} 1 \cdot dy dx$$

$$\text{or } \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dy dx + \int_{-\sqrt{y}}^{\sqrt{y}} dx dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} [y]_{x^2}^{4-x^2} dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4-x^2-x^2) dx = \left[4x - \frac{2x^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} + 4\sqrt{2} - \frac{4\sqrt{2}}{3}$$

$$= \frac{16\sqrt{2}}{3}$$

①

Q: Find the area bounded by the curves $x=y^2$, $x+y-2=0$. Fig 1, 2nd

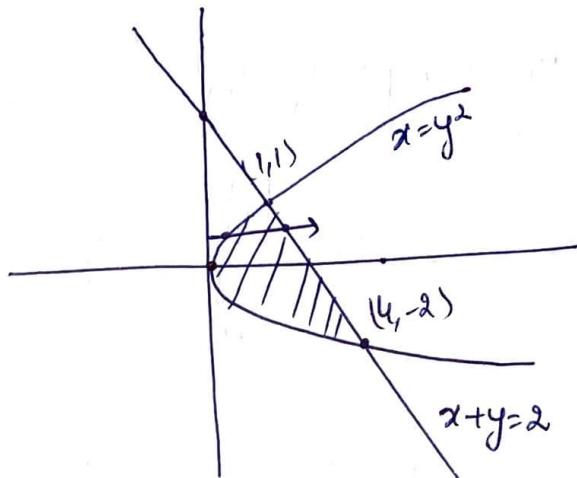
Sol: $x=y^2$, $x+y-2=0$

$$y^2+y-2=0$$

$$(y-1)(y+2)=0$$

$$y=1, -2$$

$$x=1, 4$$



So, (1, 1) and (4, -2) are the points of intersection
of the curves $x=y^2$, $x+y-2=0$.

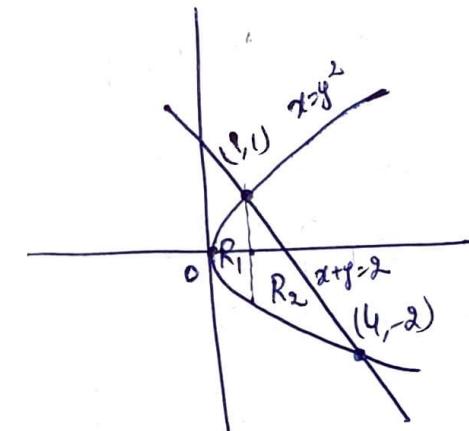
$$A = \iint_{R_1} dx dy \text{ or } \iint_{R_2} dy dx$$

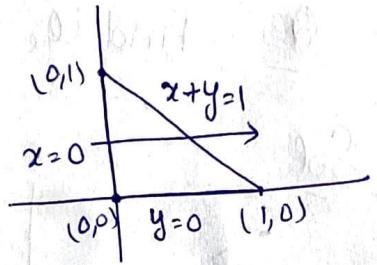
$$= \int_{-2}^1 \left[x \right]_{y^2}^{2-y} dy = \int_{-2}^1 (2-y-y^2) dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^1$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + \frac{4}{2} - \frac{8}{3}$$

$$= 8 - \frac{4}{3} - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \frac{9}{2}$$





$$0 \leq x \leq 1-y, 0 \leq y \leq 1.$$

$$0 \leq y \leq 1-x, 0 \leq x \leq 1.$$

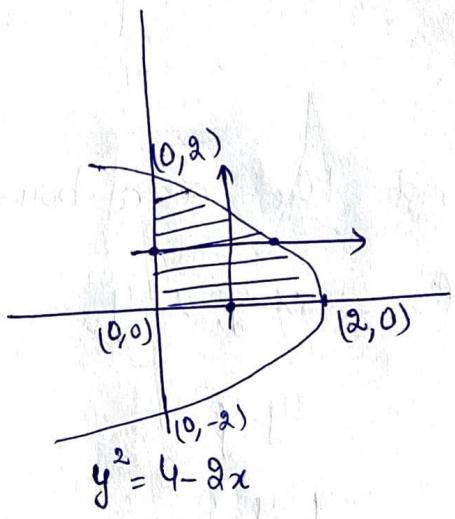
Q: Find the area bounded by the curves $y^2 = 4 - 2x, x \geq 0, y \geq 0$

Sol:

$$A = \int_0^2 \int_0^{2-\frac{y^2}{2}} dx dy$$

$$= \int_0^2 \left(2 - \frac{y^2}{2}\right) dy$$

$$= \left[2y - \frac{y^3}{6}\right]_0^2 = 4 - \frac{8}{6} = 4 - \frac{4}{3} = \frac{8}{3}.$$



$$\begin{aligned} 2x &= 4 - y^2 \\ x &= 2 - \frac{y^2}{2} \end{aligned}$$

$$\text{or } A = \int_0^2 \int_0^{\sqrt{4-2x}} dy dx = \frac{8}{3}.$$

Q: The area of the region R bounded by triangle with the vertices $(0,0), (1,0), (0,1)$ is

- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{3}$.

Find the area bounded by the curves $x^2 = y^3$, $x = y$.

Sol :- $x = y$, $x^2 = y^3$

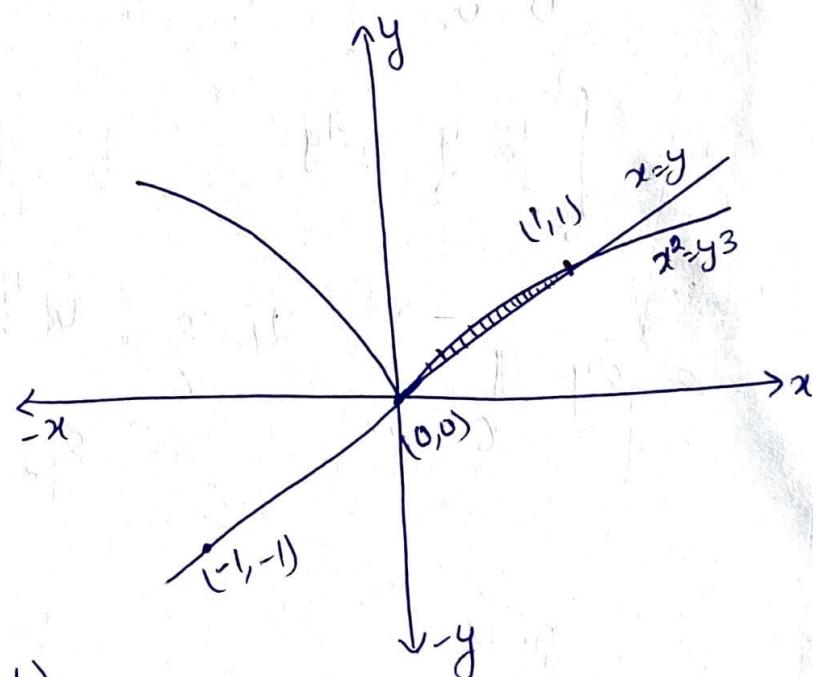
$$y = y^3$$

$$y^3 - y = 0$$

$$y(y^2 - 1) = 0$$

$$y = 0, 1, -1$$

$$x = 0, x = 1, -1$$



Points of intersection are $(0,0)$, $(1,1)$.

$$A = \int_0^1 \int_{y^{3/2}}^y dx dy \quad \text{or} \quad \int_0^1 \int_x^{x^{1/3}} dy dx$$

$$\begin{aligned} &= \int_0^1 [x]_{y^{3/2}}^y dy = \int_0^1 [y - y^{3/2}] dy = \left[\frac{y^2}{2} - \frac{2}{5} y^{5/2} \right]_0^1 \\ &= \frac{1}{2} - \frac{2}{5} = \frac{1}{10}. \end{aligned}$$

Volume by double Integrals

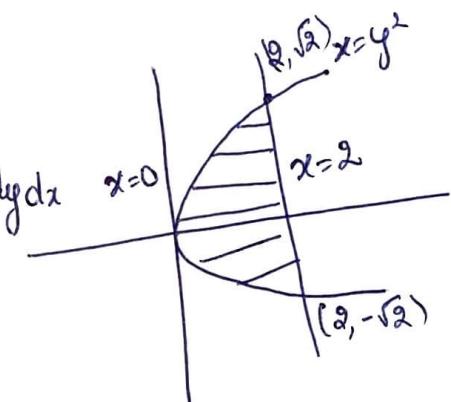
(3)

$I = \iint_R f(x,y) dA$ represents Volume of the region R bounded below by the surface $z = f(x,y)$ and above by the region R (in xy plane).

Q: Find the volume of the solid which is below the plane $z = 2x+3$ and above the xy -plane and bounded by $y^2 = x$, $x=0$, $x=2$.

Sol:

$$\text{Volume} = \iint_{y^2 \leq x} (2x+3) dx dy \quad \text{or} \quad \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} (2x+3) dy dx$$



$$= 2 \int_0^{\sqrt{2}} [x^2 + 3x]_{y^2}^2 dy$$

$$= 2 \int_0^{\sqrt{2}} (10 - y^4 - 3y^2) dy = 2 \left[10y - \frac{y^5}{5} - \frac{3y^3}{3} \right]_0^{\sqrt{2}}$$

$$= 2 \left\{ 10\sqrt{2} - \frac{4\sqrt{2}}{5} - 2\sqrt{2} \right\}$$

$$= 2 \left(\frac{36\sqrt{2}}{5} \right) = \frac{72\sqrt{2}}{5}$$

Triple Integral

Let $f(x, y, z)$ be a function defined over a closed and bounded region T in \mathbb{R}^3 . Then, triple integral of f over T is

$$I = \iiint_T f(x, y, z) dV$$

$$= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) dx dy dz$$

Application of triple Integral

If $f(x, y, z) = 1$, then the triple integral

$$V = \iiint_T dx dy dz$$

gives the volume of the region T .

Evaluation of Triple Integrals

1. If the limits of all three variables are constant, then order of integration does not make any difference.
2. We evaluate the triple integral by three successive integrations. If the region R can be described by

$$R = \left\{ \begin{array}{l} a \leq x \leq b, \\ g_1(x) \leq y \leq g_2(x) \\ h_1(x, y) \leq z \leq h_2(x, y) \end{array} \right\}$$

then we evaluate the triple integral as

$$\int_{a_1}^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z) dz dy dx.$$

Q.1 Evaluate $\int_0^a \int_0^b \int_0^c x^2 y^2 z^2 dx dy dz$.

Sol.: $I = \int_0^a \int_0^b y^2 z^2 \left(\frac{x^3}{3}\right)_0^c dy dz$
 $= \int_0^a \int_0^b y^2 z^2 \left(\frac{c^3}{3}\right) dy dz$
 $= \int_0^a \frac{c^3}{3} z^2 \left(\frac{b^3}{3}\right) dz = \frac{a^3 b^3 c^3}{3 \cdot 9} = \frac{a^3 b^3 c^3}{27}$

Q.1 If T: $0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq x+y$.

Evaluate $\iiint_T (2x-y-z) dx dy dz$.

Sol.: $I = \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z) dz dy dx$
 $= \int_0^1 \int_0^{x^2} \left[2xz - yz - \frac{z^2}{2} \right]_0^{x+y} dy dx$
 $= \int_0^1 \int_0^{x^2} \left[2x(x+y) - y(x+y) - \frac{(x+y)^2}{2} \right] dy dx$
 $= \int_0^1 \int_0^{x^2} \left[2x^2 + 2xy - xy - y^2 - \frac{x^2}{2} - \frac{y^2}{2} - \frac{2xy}{2} \right] dy dx$

$$\begin{aligned}
 &= \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) dy dx \\
 &= \int_0^1 \left[\frac{3}{2}x^2y - \frac{y^3}{2} \right]_0^{x^2} dx \\
 &= \int_0^1 \left(\frac{3}{2}x^4 - \frac{x^6}{2} \right) dx \\
 &= \frac{3}{2} \left[\frac{x^5}{5} \right]_0^1 - \frac{1}{2} \left[\frac{x^7}{7} \right]_0^1 \\
 &= \frac{3}{2} \cdot \frac{1}{5} - \frac{1}{2} \cdot \frac{1}{7} = \frac{1}{2} \left[\frac{3}{5} - \frac{1}{7} \right] = \frac{1}{2} \left[\frac{21-5}{35} \right] = \frac{1}{2} \left[\frac{16}{35} \right] = \frac{8}{35}.
 \end{aligned}$$

$$I = \frac{8}{35}.$$

Q: Evaluate $\int_0^1 \int_0^x \int_0^{x+y} dz dy dx$ (Volume)

Sol:

$$\begin{aligned}
 I &= \int_0^1 \int_0^x \int_0^{x+y} dz dy dx \\
 &= \int_0^1 \int_0^x (x+y) dy dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^x dx \\
 &= \int_0^1 \left(x^2 + \frac{x^2}{2} \right) dx = \frac{3}{2} \int_0^1 x^2 dx \\
 &= \frac{3}{2} \left(\frac{x^3}{3} \right)_0^1 = \frac{1}{2}.
 \end{aligned}$$

Q: The value of integral $\iiint_0^1 dz dy dx$ is (Volume)

- (a) 3 (b) 2 (c) 1 (d) $\frac{1}{3}$.

Sol: $I = \iiint_0^1 1 dz dy dx = \int_0^1 1 dx = 1.$

Q: Evaluate $\iiint_R \frac{dz dy dx}{(x+y+2+1)^3}$.

R is bounded by $x=0, y=0, z=0, x+y+z=1$.

Sol: $x+y+z=1 \Rightarrow z=1-x-y, z=0$

let $z=0, x+y=1 \Rightarrow y=1-x, y=0$

let $y, z=0, x=1, x=0$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+2+1)^{-3} dz dy dx$$

$$= \int_0^1 \int_0^x \left[\frac{(x+y+2+1)^{-2}}{-2} \right]_0^{1-x-y} dy dx$$

$$= \int_0^1 \int_0^{1-x} \left[x+y+1 - x-y+1 - \frac{(x+y+1)^{-2}}{-2} \right] dy dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} [x^2 - (x+y+1)^{-2}] dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{2^2} - (x+y+1)^{-2} \right] dy dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y - \frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1-x}{4} + (x+1-x+1)^{-1} - (1+x)^{-1} \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1} \right) dx$$

$$= -\frac{1}{2} \left[\frac{1}{4} \left(x - \frac{x^2}{2} \right) + \frac{1}{2} x - \log(x+1) \right]_0^1$$

$$= -\frac{1}{2} \left[\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} - \log 2 \right]$$

$$= -\frac{1}{2} \left[\frac{1}{8} + \frac{1}{2} - \log 2 \right]$$

$$= -\frac{1}{16} [5 - 8 \log 2] = \frac{8 \log 2 - 5}{16}$$

Volume using triple integrals

Q:- Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Sol:

$$x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 \Rightarrow z = \pm \sqrt{a^2 - x^2}$$

$$x^2 + y^2 = a^2 \Rightarrow y = \pm \sqrt{a^2 - x^2}$$

$$x = \pm a.$$

$$V = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dz dy dx$$

$$= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy dx$$

$$= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-y^2} dy dx$$

$$= 2 \int_{-a}^a \sqrt{a^2-x^2} \left(2\sqrt{a^2-x^2} \right) dx$$

$$= 4 \int_{-a}^a (a^2-x^2) dx = 4 \left[a^2x - \frac{x^3}{3} \right]_a^a$$

$$= 4 \left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right]$$

$$= 8 \left(\frac{2a^3}{3} \right)$$

$$= \frac{16a^3}{3}$$