Fourier Series – Change of Interval

Let $f(x)$ be a periodic function of period $2l$. The Fourier series expansion of $f(x)$ on the interval $[\alpha, \alpha+2l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where Euler's coefficients a_0, a_n, b_n are given by

$$\left.\begin{aligned}
a_0 &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx \\
a_n &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\
b_n &= \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx
\end{aligned}\right\} \text{ Euler's formulae}$$

Remark:- If the interval is $[\alpha, \alpha+2\pi]$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + \sum_{n=1}^{\infty} b_n \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx.$$

Even Function

Let $f(x)$ be defined on $[-l,l]$.
Then $f(x)$ is said to be even function if

$$f(-x) = f(x), \quad -l \le x \le l.$$

Ex:- $f(x) = x^2$, $f(-x) = (-x)^2 = x^2 \Rightarrow f(x) = f(-x)$

$f(x) = x^4$

$f(x) = k$

$f(x) = \cos x$

Odd function

Let $f(x)$ be defined on $[-l,l]$.
Then $f(x)$ is said to be odd function if

$$f(-x) = -f(x), \quad -l \le x \le l.$$

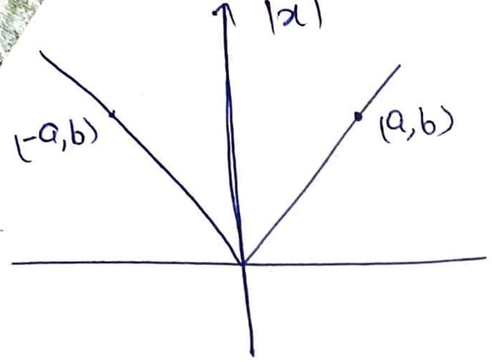
Ex:- $f(x) = x$

$f(x) = x^3$

$f(x) = \sin x$

$f(x) = \tan x$

Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around origin.

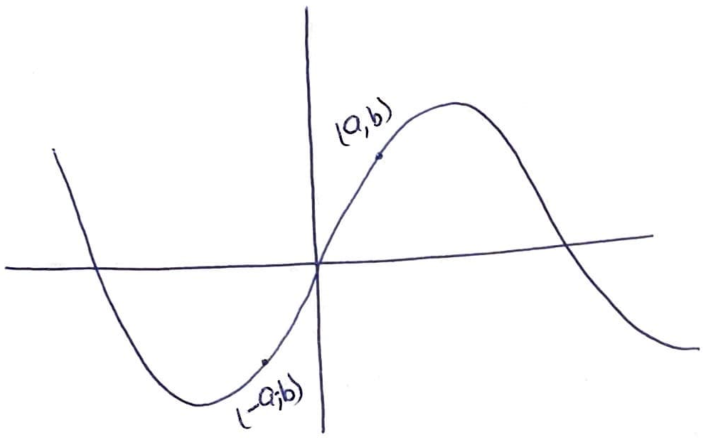


$|x|$ is symmetrical about y-axis.

↓

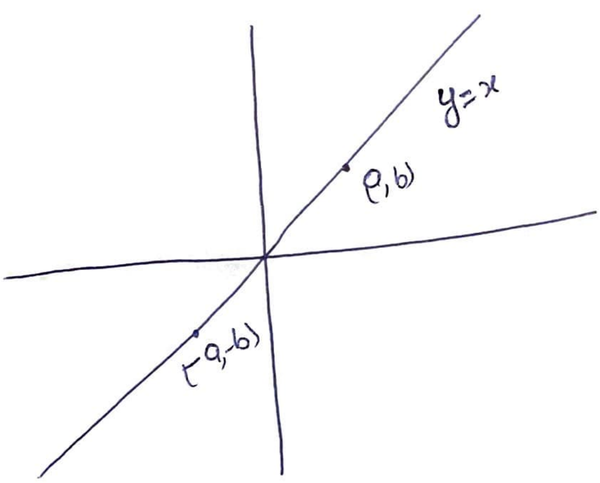
(a,b) belongs to figure

$\Rightarrow$ $(-a,b)$ also belong to figure.

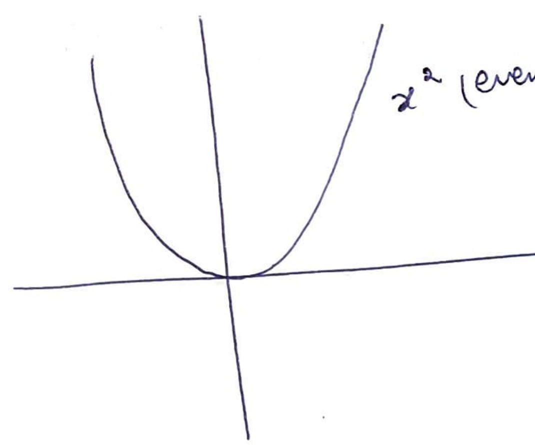


$\sin x$ (odd)

$\sin x$ is symmetrical around origin.



$y=x$ (odd)



x^2 (even)

Case 1

When $f(x)$ is an even function.

$\boxed{b_n = 0}$

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

Case 2 When $f(x)$ is an odd function.

$$\boxed{a_0 = a_n = 0}$$

$$\int_{-l}^{l} f(x)\,dx = 0.$$

Remark :- ① (Even fn) (Even fn) = Even fn.

Ex:- $x^2 \cos x \Rightarrow$ even fn.

② (Even fn) (Odd fn) = Odd fn.

$x \cos x \rightarrow$ odd fn. , $x^2 \sin x \rightarrow$ odd fn.

③ (Odd fn) (Odd fn) = odd fn.

$x \sin x \rightarrow$ odd fn.

$\underline{x}$ $f(x) = |x|, \ -1 \leq x \leq 1$

$f(-x) = |-x| = |x| = f(x)$

$\Rightarrow f(x)$ is an even function.

$\Rightarrow b_n = 0$ in the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$[\alpha, \alpha+2l] = [-1, 1]$

$l = \dfrac{\text{upper limit} - \text{lower limit}}{2}$

$\alpha = -1, \quad \alpha + 2l = 1$

$\Rightarrow 2l = 1 - \alpha = 1 + 1 = 2$

$\boxed{l = 1}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0. \end{cases}$

where $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} |x|\,dx$

$= 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = \frac{2}{2}[1-0]$

$= 1$

$\boxed{a_0 = 1}$

a_n ~~b_n~~ $= \int_{-1}^{1} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \int_{-1}^{1} |x| \cos n\pi x\,dx$

$= 2\int_0^1 x \cos n\pi x\,dx$

$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$$= 2\left[\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \cos n\pi x$$

$$\boxed{|x| = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left[(-1)^n - 1\right] \cos n\pi x}$$

Ex :- For $f(x) = 2x - x^2$ in $(0, 3)$, find the Fourier series and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi}{12}$.

Sol:-

The Fourier series expansion for $f(x) = 2x - x^2$ in $(0, 3)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$l = \frac{3-0}{2} = \frac{3}{2}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right).$$

where $a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_0^3$

$$= \frac{2}{3}\left[\frac{2}{2}(9) - \frac{1}{3}(27)\right]$$

$$= \frac{2}{3}\left[9-9\right] = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi} - (2-2x)\left(-\cos\frac{2n\pi x}{3}\right)\frac{9}{4n^2\pi^2}\right.$$

$$\left. + (-2)\left(-\sin\frac{2n\pi x}{3}\right)\frac{27}{8n^3\pi^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9(2-2x)}{4n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{1}{n^2\pi^2}\,\frac{2\cdot 9}{3\cdot 4}\left[(2-6)\cos 2n\pi - (\cos 0)\,2\right]$$

$$= \frac{3}{2n^2\pi^2}\left[-4-2\right] = \frac{3}{2n^2\pi^2}(-6)$$

$$= \frac{-9}{n^2\pi^2}$$

$$\boxed{a_n = -\frac{9}{n^2\pi^2}}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(-\cos\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi} - (2-2x)\left(-\sin\frac{2n\pi x}{3}\right)\cdot\frac{9}{4n^2\pi^2} + (-2)\left(\cos\frac{2n\pi x}{3}\right)\frac{27}{8n^3\pi^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3(2x-x^2)}{2n\pi}\cos\left(\frac{2n\pi x}{3}\right) + \frac{9(2-2x)}{4n^2\pi^2}\sin\left(\frac{2n\pi x}{3}\right) - \frac{54}{8n^3\pi^3}\cos\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}(6-9)(1) - \cancel{\frac{54}{8n^3\pi^3}(1)} + 0 + \cancel{\frac{54}{8n^3\pi^3}}\right]$$

$$= \frac{\cancel{2}}{\cancel{3}}\frac{(-\cancel{3})}{\cancel{2}n\pi}(-3) = \frac{3}{n\pi}$$

$$\boxed{b_n = \frac{2}{n\pi}}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

$$\boxed{2x - x^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)}$$

$$2x-x^2 = -\frac{9}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{2\pi x}{3}\right)+\frac{1}{2^2}\cos\left(\frac{4\pi x}{3}\right)+\frac{1}{3^2}\cos\left(\frac{6\pi x}{3}\right)\right.$$

$$\left.+\frac{1}{4^2}\cos\left(\frac{8\pi x}{3}\right)+\cdots\right]$$

$$+\frac{3}{\pi}\left[\frac{1}{1}\sin\left(\frac{2\pi x}{3}\right)+\frac{1}{2}\sin\left(\frac{4\pi x}{3}\right)+\frac{1}{3}\sin\left(\frac{6\pi x}{3}\right)\right.$$

$$\left.+\frac{1}{4}\sin\left(\frac{8\pi x}{3}\right)+\cdots\right]$$

Put $x = 3/2$

$$2\left(\frac{3}{2}\right)-\frac{9}{4} = -\frac{9}{\pi^2}\left[\frac{1}{1^2}\cos\pi+\frac{1}{2^2}\cos 2\pi+\frac{1}{3^2}\cos 3\pi+\right.$$

$$\left.\frac{1}{4^2}\cos 4\pi+\cdots\right]$$

$$+\frac{3}{\pi}\left[\sin\pi+\frac{1}{2}\sin 2\pi+\frac{1}{3}\sin 3\pi+\cdots\right]$$

$$3-\frac{9}{4} = -\frac{9}{\pi^2}\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots\right]+\frac{3}{\pi}(0)$$

$$\frac{3}{4} = \frac{9}{\pi^2}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right]$$

$$\Rightarrow \boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\cancel{3}}{4}\cdot\frac{\pi^2}{\cancel{9}_3} = \frac{\pi^2}{12}}$$