Lecture-5

Topic : Eigen Values and Eigen Vectors

Learning obj. out comes

1. Properties of Eigen Values and eigen vectors
2. Use of properties of eigen values in various types of matrices.
3. Cayley Hamilton Theorem

Properties of Eigen Values

1. Product of Eigen Values is always equal to the determinant of the matrix.

$A = [a_{ij}]_{3\times3}$. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

$\Rightarrow$ $\det A = |A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

Sum of diagonal elements

2. Sum of eigen values is always equal to the trace of the matrix. i.e. $\text{Trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$

3. $\alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3$ are the eigen values of αA.

Eigen values of $A \rightarrow \lambda_1, \lambda_2, \lambda_3$

Eigen values of $\alpha A \rightarrow \alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3$, α is any scalar.

(4) $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}$ are eigenvalues of A^{-1}.

(5) $\lambda_1^{m}, \lambda_2^{m}, \lambda_3^{m}$ are eigenvalues of A^{m}.

(6) A and A^{T} have the same eigen values.

(7) For a real matrix A, if $\lambda = \alpha + i\beta$ is an eigen value, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also its eigen values. This result does not hold if A is a complex matrix.

(8) A matrix is singular iff it has a zero eigenvalue.

(9) The eigenvalues of an upper or lower triangular matrix are the elements on the main diagonal.

(10) If λ is an eigen value of A, then ~~prove~~ ~~that~~ the eigen value of $B = a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$.

① Find the eigen values and the corresponding eigen vectors of

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Show that (a) Sum of the eigen values is always equal to the trace of matrix A.

(b) Product of the eigen values is always equal to the determinant of matrix A.

Sol :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda + 2)(\lambda - 5) = 0$$

$\Rightarrow \lambda = 5, -2$. Say $\lambda_1 = 5$, $\lambda_2 = -2$

(a) $\text{Trace}(A) = 1 + 2 = 3$.

$\lambda_1 + \lambda_2 = 5 - 2 = 3$.

(b) $|A| = 2 - 12 = -10$

$\lambda_1 \lambda_2 = -10$.

(2) Find the eigen values and corresponding eigen vectors of

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Show that

(a) Sum of eigen values is equal to the trace of matrix A.

(b) Product of eigen values is equal to the determinant of matrix A.

Sol :-

$$|A - \lambda I| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 1-\lambda & +1 \\ -1 & 1-\lambda \end{bmatrix}\right| = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0$$

$$\Rightarrow \lambda^2 + 1 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\cancel{\Rightarrow \lambda(\lambda - 2) = 0}$$

$$\cancel{\Rightarrow \lambda = 0, 2}.$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\lambda_1 = 1+i, \ \lambda_2 = 1-i$$

(a) $\text{Trace}(A) = 2$

$$\lambda_1 + \lambda_2 = 1 + i + 1 - i = 2$$

(b) $|A| = 1 + 1 = 2$

$$\lambda_1 \lambda_2 = (1+i)(1-i)$$

$$= 1 - i + i - i^2$$

$$= 1 + 1 = 2.$$

③ ~~find~~

Eigen vector corresponding to $\lambda_1 = 1+i$

$$(A-(1+i)I)X=0$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + iR_1$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

$$ix_1 = x_2$$

Let $x_1 = \alpha$, $x_2 = i\alpha$

$$X = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Let $\alpha = 1$, then $X = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigen vector corresponding to $\lambda_1 = 1+i$.

Eigen vector corresponding to $\lambda_2 = 1-i$

$R_2 \to R_2 - iR_1$

$$[A-(1-i)I]X=0$$

$$\Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$ix_1 + x_2 = 0 \Rightarrow x_2 = -ix_1$

Let $x_1 = \alpha$

$x_2 = -i\alpha$

$$X = \alpha \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 1-i$.

(3) Let a 4x4 matrix A have eigen values $(1, -1, 2, -2)$. Find the value of determinant and trace of matrix $B = 2A + A^{-1} - I$.

Sol :- Eigen values of $A = 1, -1, 2, -2$

Eigen values of $A^{-1} = \left(1, -1, \frac{1}{2}, \frac{-1}{2}\right)$

Eigenvalues of $I = (1, 1, 1, 1)$

Eigen values of $B = 2(1, -1, 2, -2) + \left(1, -1, \frac{1}{2}, \frac{-1}{2}\right)$

$- (1, 1, 1, 1)$

$= \left(2+1-1, -2-1-1, 4+\frac{1}{2}-1, -4-\frac{1}{2}-1\right)$

$= \left(2, -4, \frac{7}{2}, \frac{-11}{2}\right)$

$|B| = (2)(-4)\left(\frac{7}{2}\right)\left(\frac{-11}{2}\right) = 154$

Trace $(B) = 2-4+\frac{7}{2}-\frac{11}{2} = \frac{4-8+7-11}{2} = \frac{-8}{2} = -4.$

Topic-: Cayley Hamilton Theorem

① Verification of Cayley Hamilton Theorem

② Using Cayley Hamilton Theorem to find the inverse of a matrix.

Thm Every square matrix satisfies its own characteristic equation.

Let A be 3X3 matrix.

The ch. eq is $|A-\lambda I|=0$

$$\Rightarrow \lambda^3+a\lambda^2+b\lambda+C=0$$

By Cayley Hamilton thm, $A^3+aA^2+bA+CI=0$.

Finding inverse using Cayley Hamilton Theorem

$$A^3+aA^2+bA+CI=0$$

$$\Rightarrow CI=-(A^3+aA^2+bA)$$

$$\Rightarrow I=\frac{-1}{C}(A^3+aA^2+bA)$$

Pre multiplying by A^{-1},

$$A^{-1}=\frac{-1}{C}A^{-1}(A^3+aA^2+bA)$$

$$=\frac{-1}{C}(A^2+aA+bI)$$

① Verify Cayley Hamilton Theorem for matrix $A=\begin{bmatrix}1&4\\3&2\end{bmatrix}$. If possible, find A^{-1}.

Sol: The ch. eq is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)(2-\lambda)-12=0$$

$$\Rightarrow 2-\lambda-2\lambda+\lambda^2-12=0$$

$$\Rightarrow \lambda^2-3\lambda-10=0$$

$$\cancel{\Rightarrow \lambda^2-5\lambda+2\lambda-10=0}$$

$$\cancel{\Rightarrow (\lambda-5)(\lambda+2)=0}$$

By Cayley Hamilton Thm,

$$A^2-3A-10I=0.$$

$$A^2=\begin{bmatrix}1&4\\3&2\end{bmatrix}\begin{bmatrix}1&4\\3&2\end{bmatrix}=\begin{bmatrix}13&12\\9&16\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}13&12\\9&16\end{bmatrix}-3\begin{bmatrix}1&4\\3&2\end{bmatrix}-10\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Hence, Cayley Hamilton Theorem is verified.

To find A^{-1},

$$A^2 - 3A - 10I = 0$$

$$\Rightarrow A^2 - 3A = 10I$$

$$\Rightarrow I = \frac{1}{10}(A^2 - 3A)$$

$$\Rightarrow A^{-1}I = \frac{1}{10}A^{-1}(A^2 - 3A)$$

$$\Rightarrow A^{-1} = \frac{1}{10}(A - 3I).$$

$$= \frac{1}{10}\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{10}\left(\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}\right)$$

$$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

(2.) If $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, then use Cayley Hamilton Theorem to find the matrix represented by A^5.

Sol: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(5-\lambda) - 9 = 0$$

$$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 9 = 0$$

$$\Rightarrow \cancel{\lambda^2 - 7\lambda + 4 = 0}. \quad \lambda^2 - 7\lambda + 1 = 0.$$

By Cayley Hamilton Th^m,

$$A^2 - 7A + I = 0.$$

$$\Rightarrow A^2 = 7A - I$$

$$\Rightarrow A^4 = 49A^2 + I - 14A$$

$$= 49(7A - I) + I - 14A$$

$$= 343A - 49I + I - 14A$$

$$= 329A - 48I$$

$$\Rightarrow A^4 = \cancel{32A}\ 329A - 48I$$

$$\Rightarrow \cancel{A^5 = A(329A - 48I)}$$

$$= \cancel{329A^2 - 48A}$$

$$= \cancel{329(7}$$

$$A^5 = A^4 \cdot A = (329A - 48I)A$$

$$= 329A^2 - 48A$$

$$= 329(7A - I) - 48A$$

$$= 2303A - 329I - 48A$$

$$= 2255A - 329I$$

$$= 2255\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - 329\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4510 - 329 & 6765 \\ 6765 & 11275 - 329 \end{bmatrix}$$

$$= \begin{bmatrix} 4181 & 6765 \\ 6765 & 10946 \end{bmatrix}$$

(3) Verify Cayley Hamilton Theorem for matrix

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. If possible, find A^{-1}.

Sol-) The ch. eq is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0$$

$$\Rightarrow \lambda^2 + 1 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

By Cayley Hamilton Thm,

$$A^2 - 2A + 2I = 0.$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$A^2 - 2A + 2I = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

To find A^{-1}

$$2I = -(A^2 - 2A)$$

$$I = \frac{1}{2}(-A^2 + 2A)$$

$$A^{-1} = \frac{1}{2}(-A + 2I) = \frac{1}{2}\begin{bmatrix} -1+2 & -1 \\ +1 & -1+2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}$$

(4) Verify Cayley Hamilton Theorem for matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

If possible, find A^{-1}.

Sol: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(6-\lambda) - 6 = 0$$

$$\Rightarrow 6 - \lambda - 6\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda = 0.$$

By Cayley Hamilton, $A^2 - 7A = 0$.

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 42 \end{bmatrix}$$

$$A^2 - 7A = \begin{bmatrix} 7 & 14 \\ 21 & 42 \end{bmatrix} - \begin{bmatrix} 7 & 14 \\ 21 & 42 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence, Cayley Hamilton Theorem is verified.

To find A^{-1} $\quad A^2 - 7A = 0$

We cannot find A^{-1} as there is no constant term in characteristic equation.

$|A| = 0 \Rightarrow A$ is singular and A^{-1} does not exist.

$$\left[A^{-1} = \frac{\operatorname{adj} A}{|A|}\right]$$

Verify Cayley-Hamilton Theorem for matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}.$$ If possible, find A^{-1}.

Sol :- $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\cancel{\Rightarrow \lambda = 1, 2, 3}$$

$$\Rightarrow (1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 - \lambda^3 + 5\lambda^2 - 6\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

By Cayley Hamilton th^m,

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 5 \\ 8 & 0 & 9 \end{bmatrix}.$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 5 \\ 8 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 12 & 8 & 19 \\ 26 & 0 & 27 \end{bmatrix}$$

$A^3 - 6A^2 + 11A - 6I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 12 & 8 & 19 \\ 26 & 0 & 27 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 5 \\ 8 & 0 & 9 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6+11-6 & 0-0+0+0 & 0+0+0+0 \\ 12-12+0-0 & 8-24+22-6 & 19-30+11+0 \\ 26-48+22+0 & 0-0+0+0 & 27-54+33-6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

To find A^{-1}

$6I = A^3 - 6A^2 + 11A \Rightarrow I = \frac{1}{6}(A^3 - 6A^2 + 11A)$.

$\Rightarrow A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$

$$A^{-1} = \frac{1}{6}\left[\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 5 \\ 8 & 0 & 9 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{6}\begin{bmatrix} 1-6+11 & 0 & 0 \\ 2 & 4-12+11 & 5-6+0 \\ 8-12+0 & 0 & 9-18+11 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 2 & 3 & -1 \\ -4 & 0 & 2 \end{bmatrix}.$$

... the value of