Partial Derivatives

If $y = f(x)$, y is dependent variable and x is independent variable.
We differentiate y w.r.t. x and denote it as $\frac{dy}{dx}$.

Consider $z = f(x, y)$.
Here, z is dependent variable and x, y are independent variables.

Partial derivative The derivative of a function of two or more variables w.r.t. independent variable, keeping all the other variables as constant is called as partial derivative.

If $z = f(x, y)$

We can partially differentiate z w.r.t. x or y and it is denoted as $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$, resp.

Standard Notations

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= f_x = p \\ \frac{\partial f}{\partial y} &= f_y = q \end{aligned}\right\}$$ First order partial derivatives

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= f_{xx} = r \\ \frac{\partial^2 f}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial y \partial x} &= f_{xy} \text{ or } f_{yx} = s \\ \frac{\partial^2 f}{\partial y^2} &= f_{yy} = t \end{aligned}\right\}$$ Second order partial derivatives

becoz f.xy & f.yx are equal for continuous functions

Ex: Find the first order partial derivatives of the following functions:

(i) $f(x,y) = x^4 - x^2y^2 + y^4$ at point $(-1,1)$.

Sol: $f_x = \frac{\partial f}{\partial x} = 4x^3 - 2xy^2$

$f_x\Big|_{(-1,1)} = 4(-1)^3 - 2(-1)(1)^2 = -4 + 2 = -2$

$f_y = \frac{\partial f}{\partial y} = -2x^2y + 4y^3$.

$f_y\Big|_{(-1,1)} = -2(-1)^2(1) + 4(+1)^3 = -2 + 4 = 2$

(ii) $f(x,y) = \log\left(\frac{x}{y}\right)$ at point $(2,3)$.

Sol: $f_x = \frac{\partial f}{\partial x} = \frac{1}{\frac{x}{y}} \cdot \left(\frac{1}{y}\right) = \frac{y}{x} \cdot \frac{1}{y} = \frac{1}{x}$

$f_x\Big|_{(2,3)} = \frac{1}{2}$

$f_y = \frac{1}{\left(\frac{x}{y}\right)} \cdot x\left(\frac{-1}{y^2}\right) = \frac{-x}{y^2} \cdot \frac{y}{x} = \frac{-1}{y}$

$f_y\Big]_{(2,3)} = \frac{-1}{3}$.

(iii) $f(x,y) = x^2 e^{y/x}$ at point $(4,2)$

Sol: $f_x = \frac{\partial f}{\partial x} = x^2 e^{y/x}\left(\frac{-1}{x^2}\right)y + 2xe^{y/x}$

$= -ye^{y/x} + 2xe^{y/x}$

$= (2x-y)e^{y/x}$

$$f_x\Big]_{(4,2)} = [2(4)-2]e^{2/4} = 6e^{1/2} = 6\sqrt{e}$$

$$f_y = \frac{\partial f}{\partial y} = x^2 e^{y/x} \cdot \frac{1}{x} = x e^{y/x}$$

$$f_y\Big]_{(4,2)} = 4e^{1/2} = 4\sqrt{2}.$$

(iv) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ at point $(6,7)$.

Sol :-

$$f_x = \frac{\partial f}{\partial x} = \frac{\sqrt{x^2+y^2}\cdot 1 - x\,\frac{1}{2\sqrt{x^2+y^2}}\,2x}{x^2+y^2}$$

$$= \frac{\cancel{x^2}+y^2-\cancel{x^2}}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$f_x\Big]_{(6,7)} = \frac{49}{(36+49)^{3/2}} = \frac{49}{(85)^{3/2}}$$

$$f_y = \frac{\partial f}{\partial y} = x\left[\frac{0-\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\,\cancel{2}y}{x^2+y^2}\right]$$

$$= \frac{-xy}{(x^2+y^2)^{3/2}}$$

$$f_y\Big]_{(6,7)} = \frac{-42}{(85)^{3/2}}$$

(V) $f(x,y,z) = (xy)^{\sin z}$ at $(3, 5, \pi/2)$

$$f_x = \frac{\partial f}{\partial x} = (\sin z)\,(xy)^{\sin z - 1} \cdot y$$

$$f_x\Big|_{(3,5,\pi/2)} = \left(\sin\frac{\pi}{2}\right)(3\cdot 5)^{\sin\frac{\pi}{2}-1}\,(5)$$

$$= 1\,(15)^0 \cdot 5 = 5.$$

$$f_y = \frac{\partial f}{\partial y} = (\sin z)\,(xy)^{\sin z - 1} \cdot x$$

$$f_y\Big|_{(3,5,\pi/2)} = 1\,(3\cdot 5)^{1-1} \cdot 3 = 3$$

$$f_z = \frac{\partial f}{\partial z} = (xy)^{\sin z}\log(xy) \cdot \cos z \qquad \frac{d}{dx}(a^x) = a^x \log a$$

$$f_z\Big|_{(3,5,\pi/2)} = (15)^1 \log(15)\cos\frac{\pi}{2}$$

$$= 0.$$

Q:- If $z = f(ax+by)$, then show that $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0.$

Sol:- $z = f(ax+by)$

$$\frac{\partial z}{\partial x} = f'(ax+by).a \quad , \quad \frac{\partial z}{\partial y} = f'(ax+by)\cdot b$$

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = ab\,f'(ax+by) - abf'(ax+by) = 0.$$

Hence proved.

Q:- If $z = \log\left[\frac{x^2-y^2}{x^2+y^2}\right]$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

Sol:- $z = \log(x^2-y^2) - \log(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2-y^2}\,2x - \frac{1}{x^2+y^2}\,2x$$

$$= \frac{2x}{x^2-y^2} - \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{x^2-y^2} - \frac{2y}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2}{x^2-y^2} - \frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2-y^2} - \frac{2y^2}{x^2+y^2}$$

$$= 2\left[\frac{x^2-y^2}{x^2-y^2}\right] - 2\left[\frac{x^2+y^2}{x^2+y^2}\right]$$

$$= 2-2$$

$$= 0.$$

Q:- If $w = \sqrt{x^2+y^2+z^2}$, $x = u\cos v$, $y = u\sin v$, $z = uv$, then show that $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

Sol:- $w = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2v^2} = \sqrt{u^2+u^2v^2} = u\sqrt{1+v^2}$

$$\frac{\partial w}{\partial u} = \sqrt{1+v^2}, \quad \frac{\partial w}{\partial v} = u\,\frac{2v}{2\sqrt{1+v^2}} = \frac{uv}{\sqrt{1+v^2}}$$

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = u\sqrt{1+v^2} - \frac{uv^2}{\sqrt{1+v^2}}$$

$$= \frac{u(1+v^2) - uv^2}{\sqrt{1+v^2}}$$

$$= \frac{u}{\sqrt{1+v^2}}$$

Q- If $z = y + f(u)$, $u = \frac{x}{y}$, then show that

$$u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

Sol:-

$$z = y + f\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$

$$\frac{\partial z}{\partial y} = 1 + f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{u}{y} f'\left(\frac{x}{y}\right) - \frac{x}{y^2} f'\left(\frac{x}{y}\right) + 1$$

$$= \cancel{\frac{x}{y^2} f'\left(\frac{x}{y}\right)} - \cancel{\frac{x}{y^2} f'\left(\frac{x}{y}\right)} + 1$$

$$= 1.$$

Q- If $z = f\left(\frac{ax}{by}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

Sol-

$$z = f\left(\frac{ax}{by}\right)$$

$$\frac{\partial z}{\partial x} = f'\left(\frac{ax}{by}\right)\cdot\frac{a}{by}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{ax}{by}\right)\cdot\left(\frac{-ax}{by^2}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{ax}{by} f'\left(\frac{ax}{by}\right) - \frac{ax}{by} f'\left(\frac{ax}{by}\right)$$

$$= 0.$$

Partial derivatives

$z = f(x,y)$

$\left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$ = Partial derivative of f wrt x at (a,b)

$$= \lim_{h\to 0} \frac{f(a+h,\, b) - f(a,b)}{h}$$

$\left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$ = Partial derivative of f w.r.t. y at (a,b)

$$= \lim_{k\to 0} \frac{f(a,\, b+k) - f(a,b)}{k}$$

Q:- Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & x+y \neq 0 \\ 0, & x+y = 0 \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives f_x and f_y do not exist at $(0,0)$.

Sol:- T.P.: $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x+y}\right) = f(0,0)$

$$\left|(x+y)\sin\left(\frac{1}{x+y}\right) - 0\right| = \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right| = |x+y|\left|\sin\frac{1}{x+y}\right|$$

$\leq |x+y|$ $\quad\quad\quad\quad$ $[\because |\sin\theta| \leq 1.]$

$\leq |x|+|y|$

If we take $|x|<\delta$, $|y|<\delta$

$\Rightarrow \left| (x+y)\sin\left(\frac{1}{x+y}\right) - 0 \right| < 2\delta \leq \epsilon$ if $2\delta \leq \epsilon$.

$\therefore \left| (x+y)\sin\left(\frac{1}{x+y}\right) - 0 \right| < \epsilon$ whenever $0<|x-0|<\delta$, $0<|y-0|<\delta$.

Thus, $\lim\limits_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x+y}\right) = f(0,0)$.

$\therefore$ $f(x,y)$ is continuous at $(0,0)$

$f_x|_{(0,0)} = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin\left(\frac{1}{h}\right)-0}{h}$

$= \lim\limits_{h\to 0} \sin\left(\frac{1}{h}\right)$, which does not exist.

$f_y|_{(0,0)} = \lim\limits_{k\to 0} \dfrac{f(0,0+k)-f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{\cancel{k}\sin\frac{1}{k}}{\cancel{k}}$, which does not exist

Hence, f_x and f_y does not exist at $(0,0)$.

Q-1 Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

Sol: $\displaystyle \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2}$

Choose the path $y = mx$, so that $y \to 0$ as $x \to 0$.

$$= \lim_{x\to 0} \frac{mx^2}{x^2+2m^2x^2} = \frac{m}{1+2m^2}$$, which depends on m.

Hence $\displaystyle \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$.

Now $$f_x\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(h)(0)}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$f_y\Big|_{(0,0)} = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Hence, f_x and f_y exist at $(0,0)$.

Differentiability of $z = f(x,y)$

A function $z = f(x,y)$ is said to be differentiable at a point (x_0, y_0) in domain D, if f has continuous first order partial derivatives ~~to~~ f_x and f_y.

Total Differential

For a function $z = f(x,y)$, total differential is given by

$$\boxed{dz = f_x\, dx + f_y\, dy}$$

Q: Find the total differential of the following functions,

(i) $z = \tan^{-1}\left(\frac{x}{y}\right)$, $(x,y) \neq (0,0)$ (ii) $u = \left(z + \frac{x}{z}\right)^{y}$, $z \neq 0$.

Sol: (i) $f_x = \frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$

$$f_y = \frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) = \frac{-x}{x^2+y^2}$$

$$dz = f_x\, dx + f_y\, dy$$

$$= \left(\frac{y}{x^2+y^2}\right) dx + \left(\frac{-x}{x^2+y^2}\right) dy$$

$$\Rightarrow dz = \frac{1}{x^2+y^2}\left[y\,dx - x\,dy\right]$$

(ii) $u = \left(xz + \frac{x}{z}\right)^{y}, \quad z \neq 0$

$$f_x = \frac{\partial u}{\partial x} = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$f_y = \frac{\partial u}{\partial y} = \left(xz + \frac{x}{z}\right)^{y} \cancel{\ln(y)} \ln\left(xz + \frac{x}{z}\right)$$

$$f_z = \frac{\partial u}{\partial z} = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right)$$

$$du = f_x\,dx + f_y\,dy + f_z\,dz$$

$$= y\left(z + \frac{1}{z}\right)\left(xz + \frac{x}{z}\right)^{y-1} dx + \left(xz + \frac{x}{z}\right)^{y} \ln\left(xz + \frac{x}{z}\right) dy$$

$$+ xy\left(1 - \frac{1}{z^2}\right)\left(xz + \frac{x}{z}\right)^{y-1} dz$$

Second order partial derivatives

Let $z = f(x,y)$ be a function of two variables and let its first order partial derivatives exist at all the points in the domain.

Second order partial derivatives are defined as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx} = \lim_{h\to 0}\left[\frac{f_x(x+h,y)-f_x(x,y)}{h}\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{yx} = \lim_{h\to 0}\left[\frac{f_x(x,y+h)-f_x(x,y)}{h}\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{xy} = \lim_{h\to 0}\left[\frac{f_y(x+h,y)-f_y(x,y)}{h}\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy} = \lim_{h\to 0}\left[\frac{f_y(x,y+h)-f_y(x,y)}{h}\right]$$

1. f_{xy} and f_{yx} are called mixed derivatives.
2. If f_{xy} and f_{yx} are continuous at a point $P(x,y)$, then at this point, $f_{xy} = f_{yx}$.

Q: Find all the second order partial derivatives of the function $f(x,y) = \ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, $(x,y) \neq (0,0)$.

Sol: $f_x = \dfrac{2x}{x^2+y^2} + \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{2x}{x^2+y^2} - \dfrac{y}{x^2+y^2} = \dfrac{2x-y}{x^2+y^2}$

$f_y = \dfrac{2y}{x^2+y^2} + \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\dfrac{1}{x} = \dfrac{2y}{x^2+y^2} + \dfrac{x}{x^2+y^2} = \dfrac{2y+x}{x^2+y^2}$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{x+2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - (x+2y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2-\cancel{2}\,4xy}{(x^2+y^2)^2} = \frac{-x^2+y^2-4xy}{(x^2+y^2)^2}$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(2)-(2x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} = \frac{-2x^2+2y^2+2xy}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1)-(2x-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2-4xy+2y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2-4xy}{(x^2+y^2)^2}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{x+2y}{x^2+y^2}\right) = \frac{(x^2+y^2)(2)-(x+2y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-2xy-4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

Q: For the function $f(x,y) = \begin{cases} \dfrac{xy(2x^2-3y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sol:

$$f_{xy}(0,0) = \frac{\partial}{\partial x}(f_y)$$

$$= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_{yx}(0,0) = \frac{\partial}{\partial y}(f_x)$$

$$= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

Now, $f_x(0,h) = \lim_{k\to 0} \dfrac{f(0+k,h) - f(0,h)}{k}$

$$= \lim_{k\to 0} \frac{\frac{\cancel{k}h(2k^2-3h^2)}{k^2+h^2} - 0}{\cancel{k}} = \frac{h(0-3h^2)}{h^2}$$

$$= \frac{-3h^3}{h^2} = -3h$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{\frac{h\cancel{k}(2h^2-3k^2)}{h^2+k^2} - 0}{\cancel{k}}$$

$$= \frac{2h^3 - 0}{h^2} = 2h$$

$$f_x(0,0) = \lim_{k\to 0} \frac{f(k,0) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{2h - 0}{h} = 2$$

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{-3h - 0}{h} = -3.$$

Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$.