

Paths and Connectivity

Zeph Grunschlag

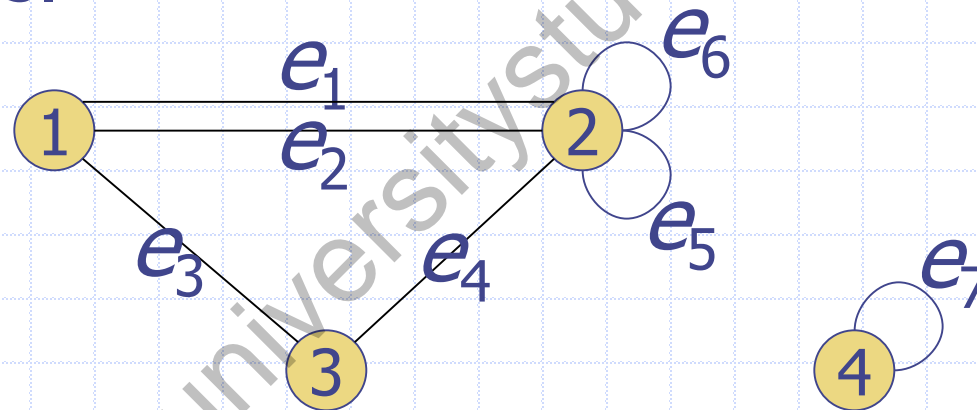
Agenda

- ◆ Paths
- ◆ Connectivity

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Paths

A path in a graph is a continuous way of getting from one vertex to another by using a sequence of edges.

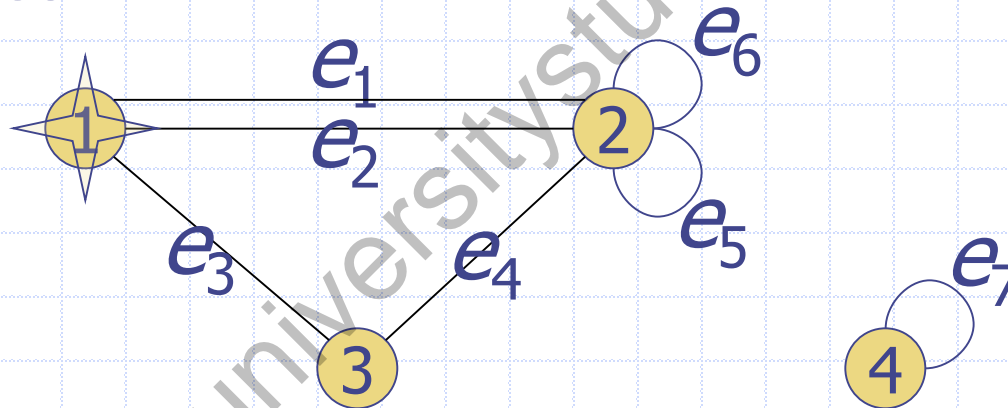


EG: could get from 1 to 3 circuitously as follows:

$1 - e_1 \rightarrow 2 - e_1 \rightarrow 1 - e_3 \rightarrow 3 - e_4 \rightarrow 2 - e_6 \rightarrow 2 - e_5 \rightarrow 2 - e_4 \rightarrow 3$

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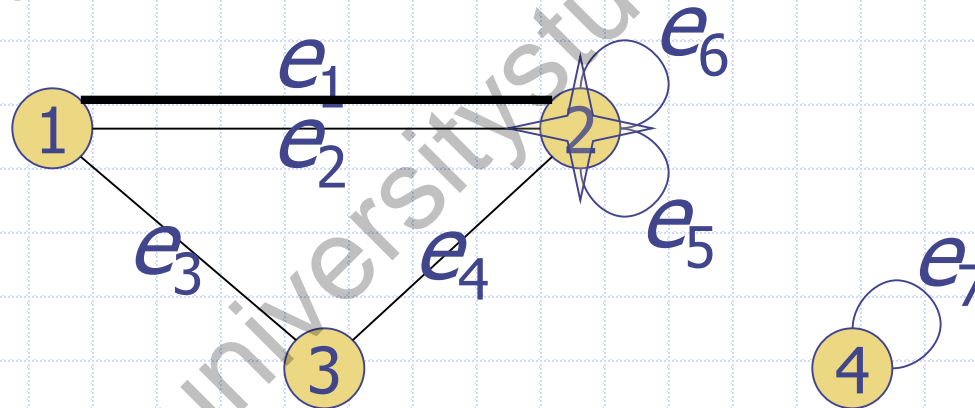


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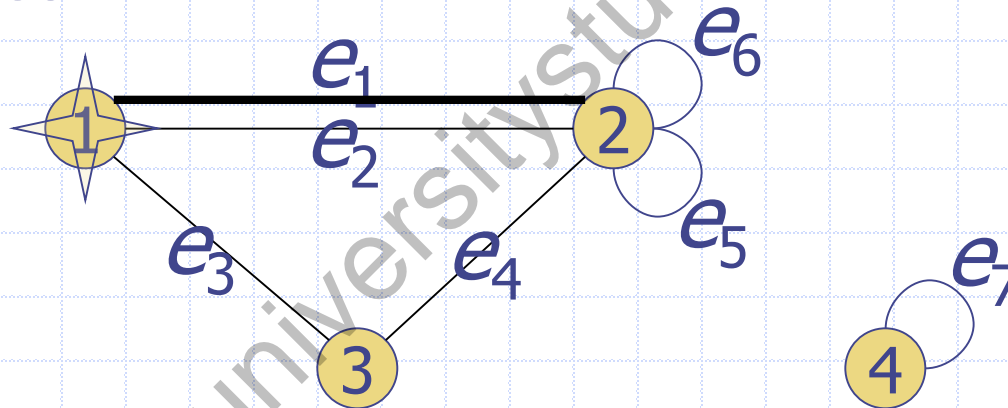


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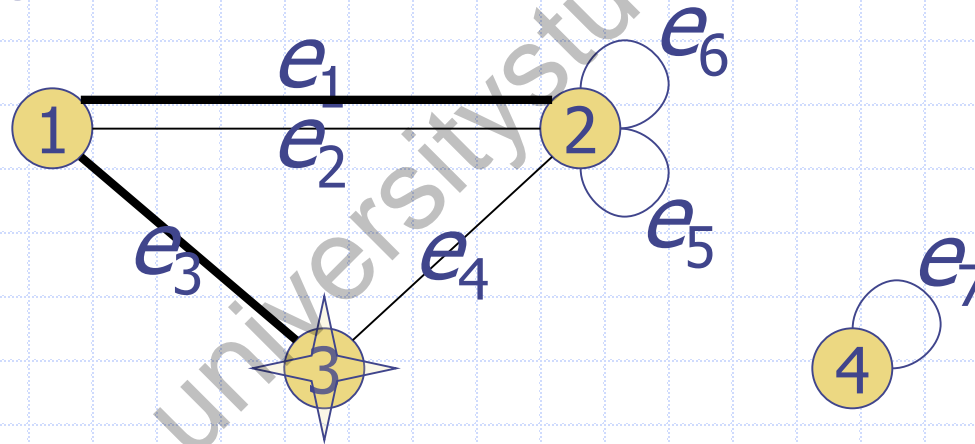


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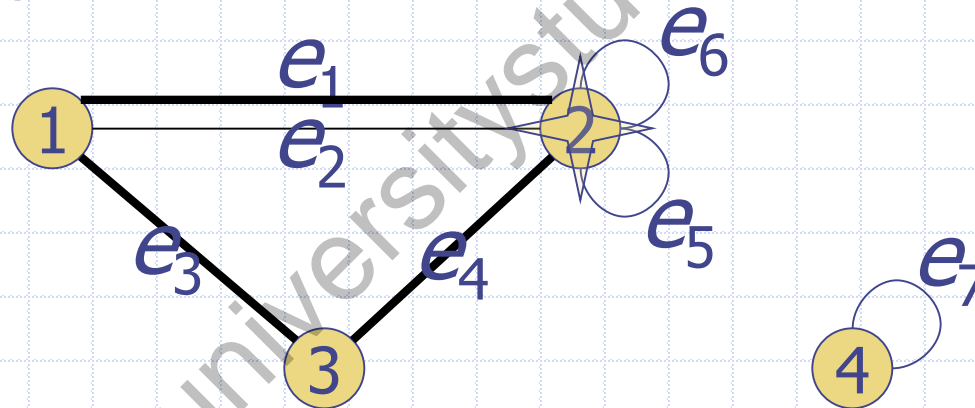


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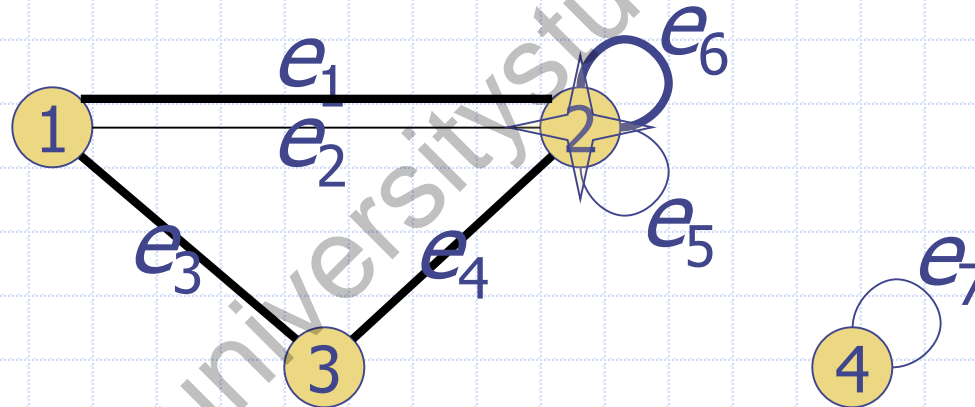


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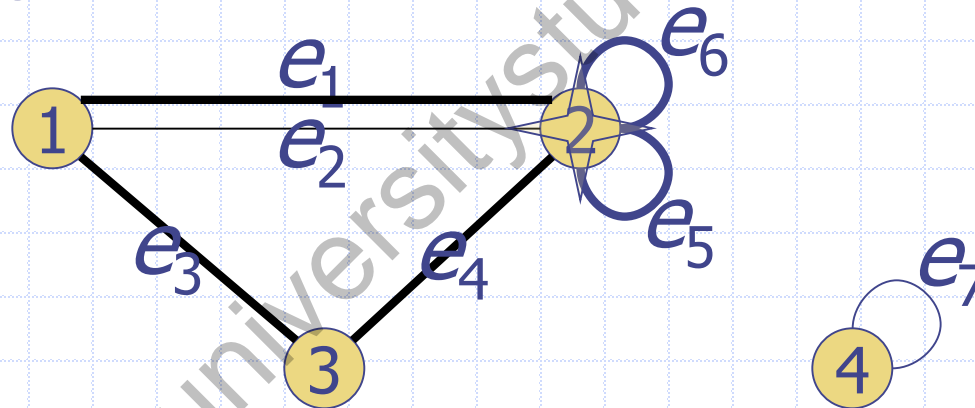


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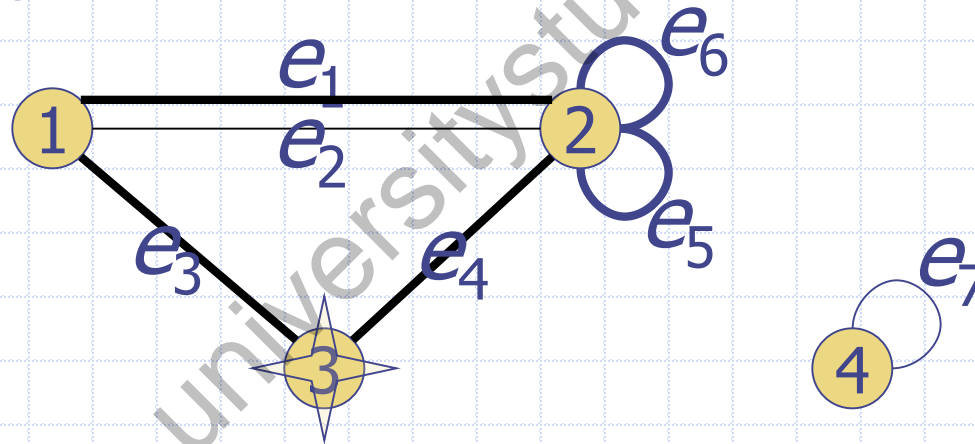


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Paths

DEF: A ***path*** of length n in an undirected graph is a sequence of n edges e_1, e_2, \dots, e_n such that each consecutive pair e_i, e_{i+1} share a common vertex. In a *simple graph*, one may instead define a path of length n as a sequence of $n+1$ vertices $v_0, v_1, v_2, \dots, v_n$ such that each consecutive pair v_i, v_{i+1} are adjacent. Paths of length 0 are also allowed according to this definition.

Q: Why does the second definition work for simple graphs?

Paths

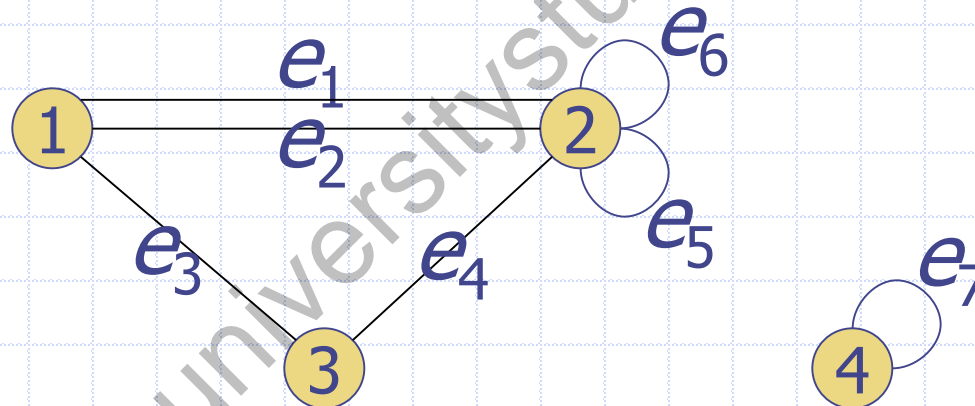
A: For simple graphs, any edge is unique between vertices so listing the vertices gives us the edge-sequence as well.

DEF: A ***simple path*** contains no duplicate edges (though duplicate vertices are allowed). A ***cycle*** (or ***circuit***) is a path which starts and ends at the same vertex.

Note: Simple paths need not be in simple graphs. E.g., may contain loops.

Paths

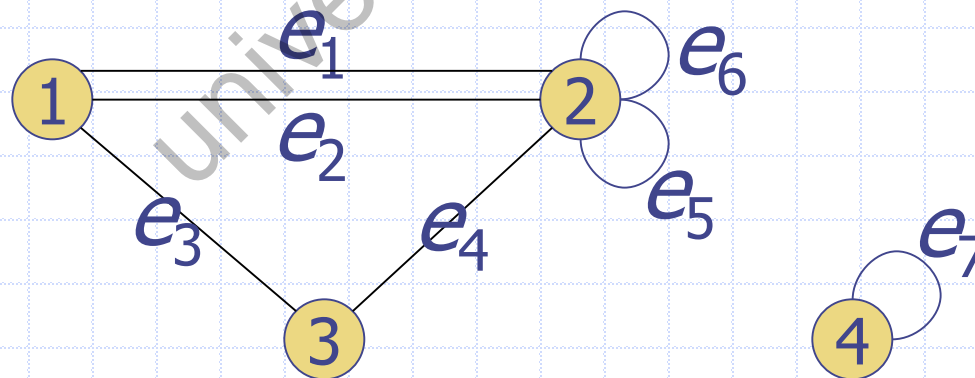
Q: Find a longest possible simple path in the following graph:



Paths

A: The following path from 1 to 2 is a maximal simple path because

- ◆ simple: each of its edges appears exactly once
- ◆ maximal: because it contains every edge except the unreachable edge e_7



The maximal path: $e_1, e_5, e_6, e_2, e_3, e_4$

Paths in Directed Graphs

One can define paths for directed graphs by insisting that the target of each edge in the path is the source of the next edge:

DEF: A **path** of length n in a directed graph is a sequence of n edges e_1, e_2, \dots, e_n such that the target of e_i is the source e_{i+1} for each i . In a *digraph*, one may instead define a path of length n as a sequence of $n+1$ vertices $v_0, v_1, v_2, \dots, v_n$ such that for each consecutive pair v_i, v_{i+1} there is an edge from v_i to v_{i+1} .

Paths in Directed Graphs

Q: Consider digraph adjacency matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. Find a path from 1 to 4.
2. Is there a path from 4 to 1?

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $1 \rightarrow 3$

2. There's no path from 4 to 1. From 4 must go to 2, from 2 must stay at 2 or return to 4. In other words 2 and 4 are *disconnected* from 1.

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $1 \rightarrow 3 \rightarrow 2$.

2. There's no path from 4 to 1. From 4 must go to 2, from 2 must stay at 2 or return to 4. In other words 2 and 4 are *disconnected* from 1.

Paths in Directed Graphs

A:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$.

2. There's no path from 4 to 1. From 4 must go to 2, from 2 must stay at 2 or return to 4. In other words 2 and 4 are *disconnected* from 1.

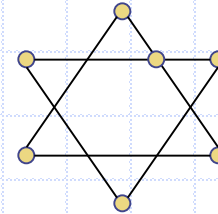
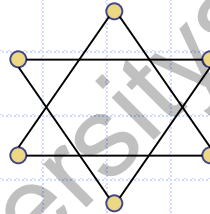
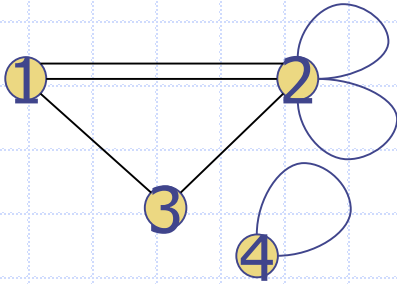
Connectivity

DEF: Let G be a pseudograph. Let u and v be vertices. u and v are ***connected to each other*** if there is a path in G which starts at u and ends at v . G is said to be ***connected*** if all vertices are connected to each other.

1. Note: Any vertex is automatically connected to itself via the empty path.
2. Note: A suitable definition for directed graphs will follow later.

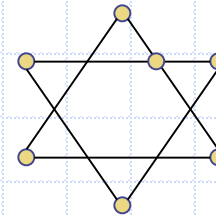
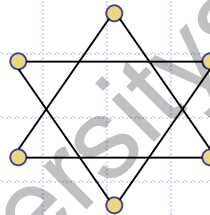
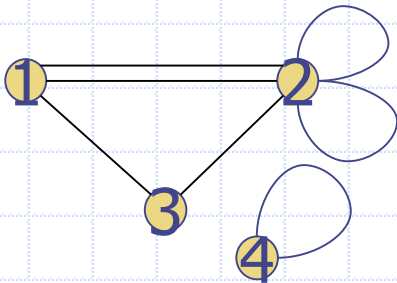
Connectivity

Q: Which of the following graphs are connected?



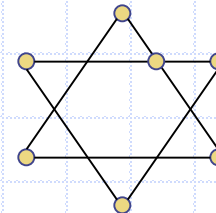
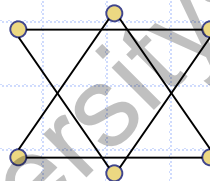
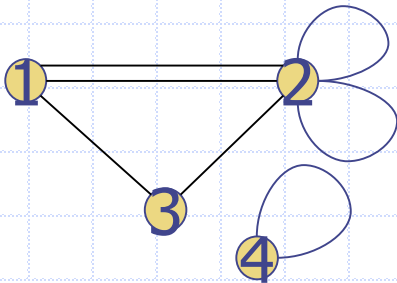
Connectivity

A: First and second are disconnected.
Last is connected.



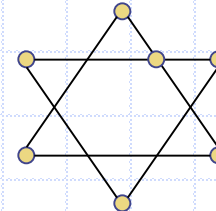
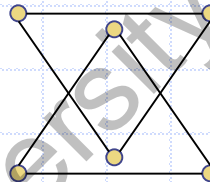
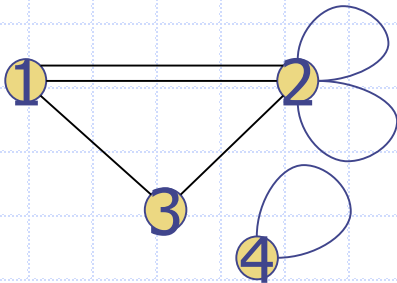
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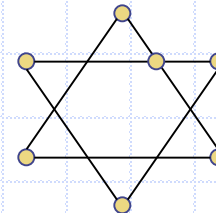
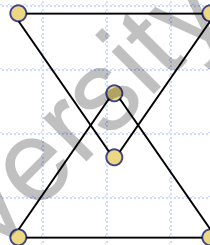
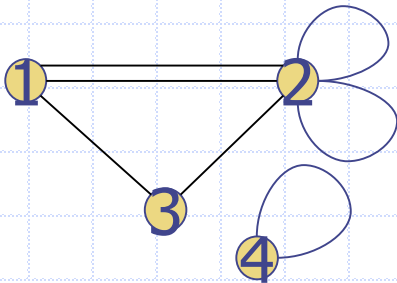
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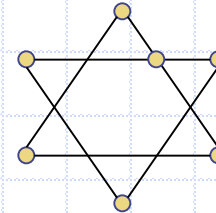
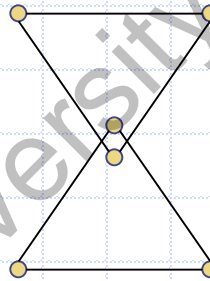
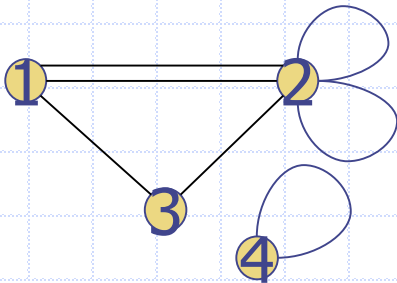
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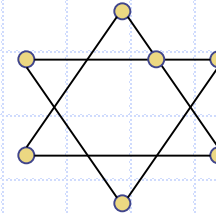
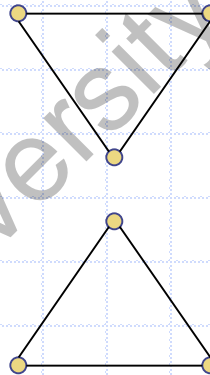
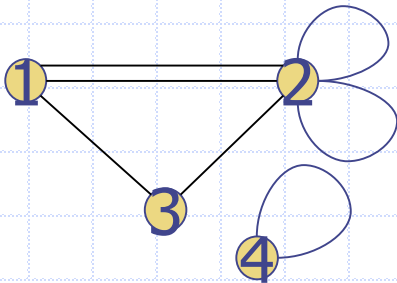
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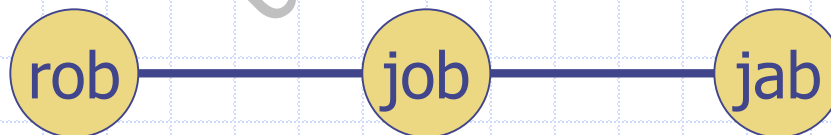
English Connectivity Puzzle

Can define a puzzling graph G as follows:

$V = \{3\text{-letter English words}\}$

E : two words are connected if can get one word from the other by changing a single letter.

One small subgraph of G is:



Q: Is "fun" connected to "car" ?

English Connectivity Puzzle

A: Yes: $\text{fun} \rightarrow \text{fan} \rightarrow \text{far} \rightarrow \text{car}$

Or: $\text{fun} \rightarrow \text{fin} \rightarrow \text{bin} \rightarrow \text{ban} \rightarrow \text{bar} \rightarrow \text{car}$

Extra Credit: Write a little program that can find 3 letter words which are not connected to each other.

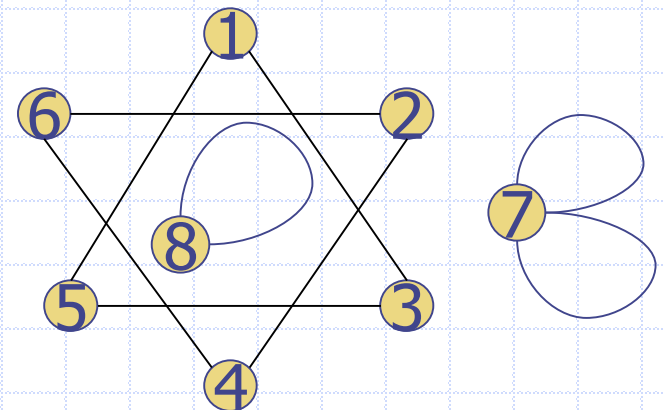
How many *connected components* are there, in the sense defined on next slide.

[Payam Ahdut's solution.](#)

Connected Components

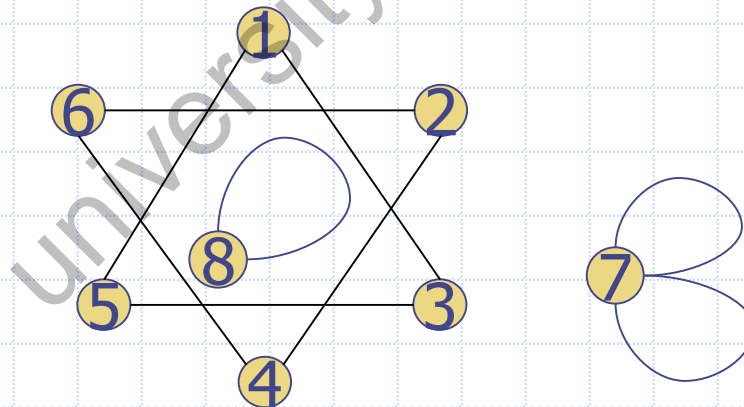
DEF: A ***connected component*** (or just ***component***) in a graph G is a set of vertices such that all vertices in the set are connected to each other and every possible connected vertex is included.

Q: What are the connected components of the following graph?



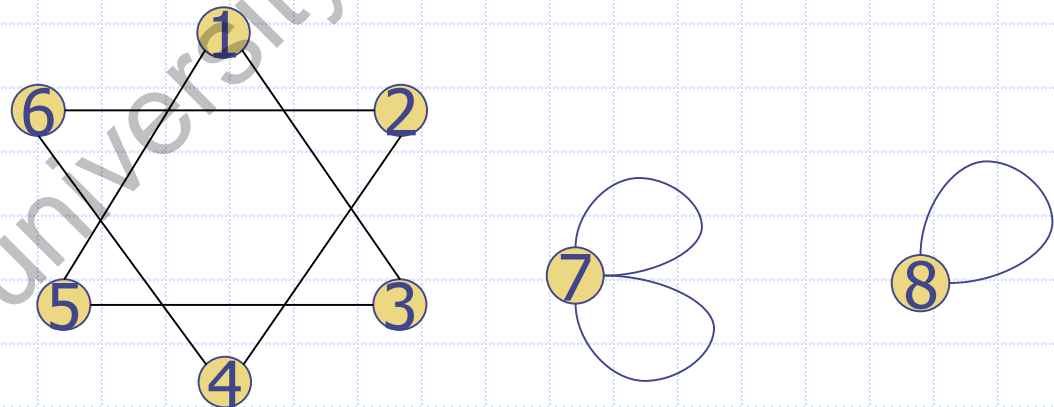
Connected Components

A: The components are $\{1,3,5\}$, $\{2,4,6\}$, $\{7\}$ and $\{8\}$ as one can see visually by pulling components apart:



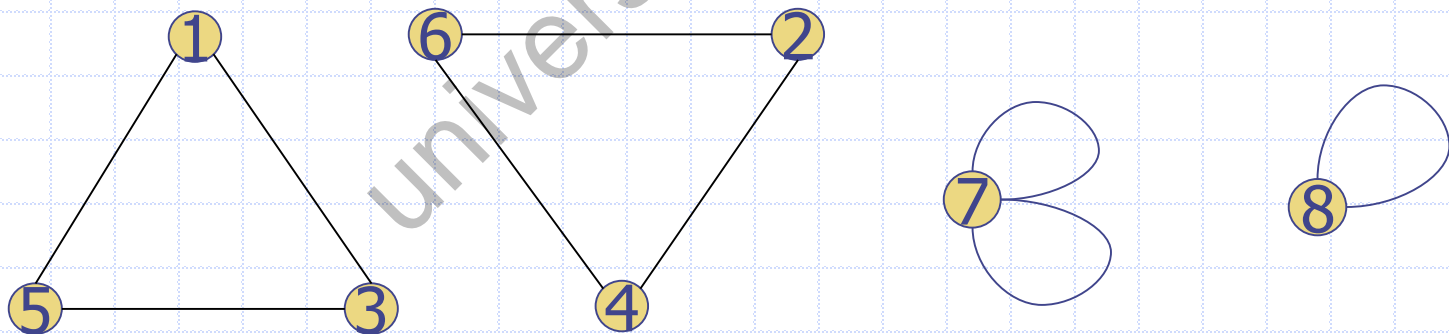
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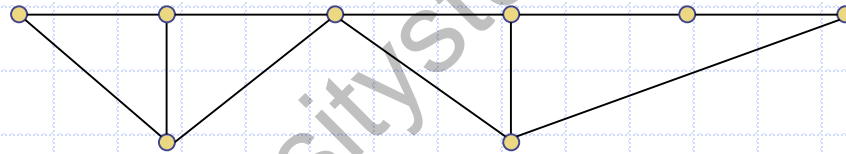


N-Connectivity

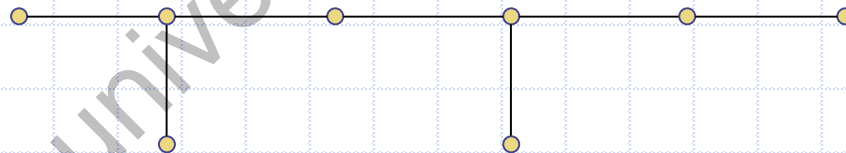
Not all connected graphs are created equal!

Q: Rate following graphs in terms of their design value for computer networks:

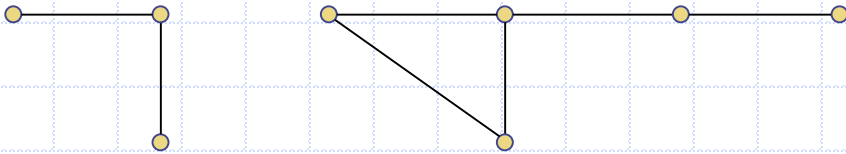
1)



2)



3)



4)



N-Connectivity

A: Want all computers to be connected, even if 1 computer goes down:

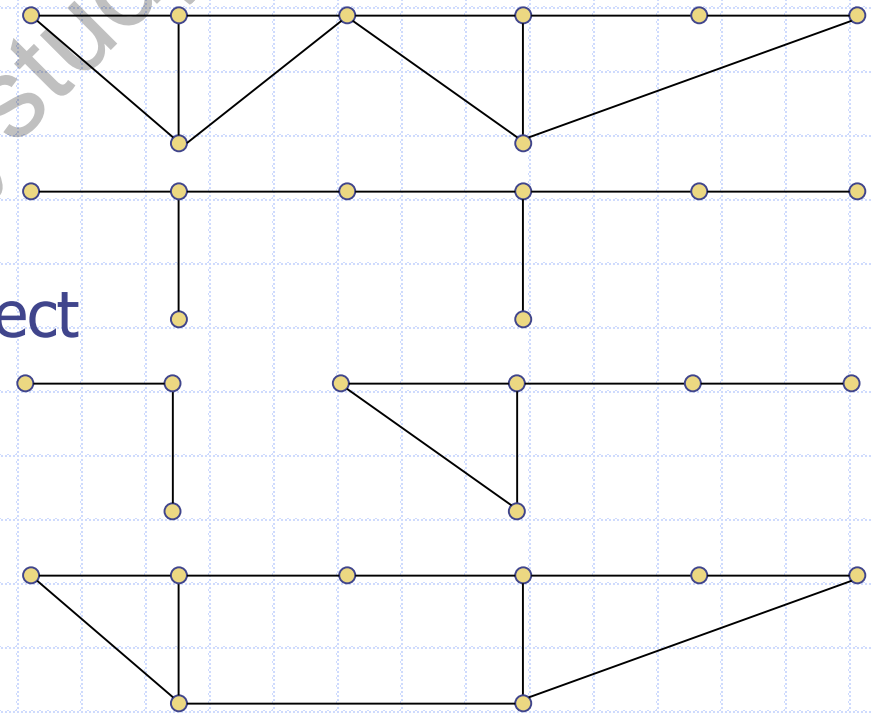
1) 2nd best. However, there's a weak link— "cut vertex"

2) 3rd best. Connected but any computer can disconnect


3) Worst!

Already disconnected

4) Best! Network dies only with 2 bad computers




N-Connectivity

The network  is best because it can only become disconnected when 2 vertices are removed. In other words, it is 2-connected. Formally:

DEF: A connected simple graph with 3 or more vertices is **2-connected** if it remains connected when any vertex is removed. When the graph is not 2-connected, we call the disconnecting vertex a **cut vertex**.

Q: Why the condition on the number of vertices?

N -Connectivity

A: To avoid  being 2-connected.

There is also a notion of N -Connectivity where we require at least N vertices to be removed to disconnect the graph.

Connectivity in Directed Graphs

In directed graphs may be able to find a path from a to b but not from b to a . However, Connectivity was a symmetric concept for undirected graphs. So how to define directed Connectivity is non-obvious:

- 1) Should we ignore directions?
- 2) Should we insist that can get from a to b in actual digraph?
- 3) Should we insist that can get from a to b and that can get from b to a ?

Connectivity in Directed Graphs

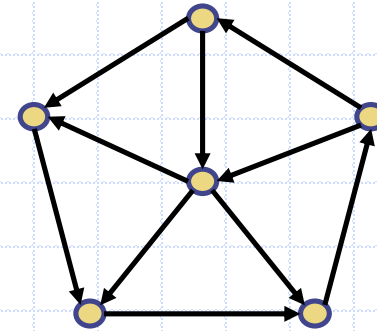
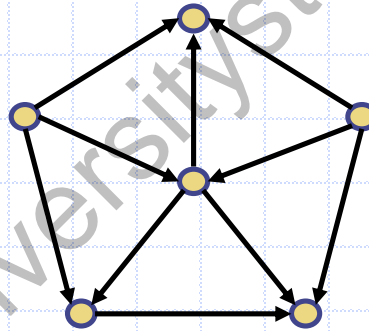
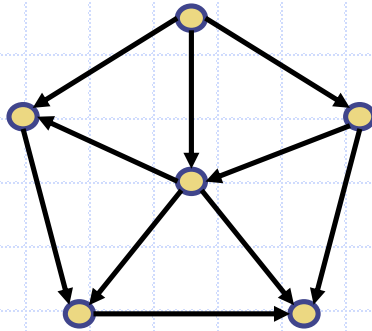
Resolution: Don't bother choosing which definition is better. Just define to separate concepts:

- 1) **Weakly connected**: can get from a to b in underlying undirected graph
- 2) **Semi-connected** (my terminology): can get from a to b OR from b to a in digraph
- 3) **Strongly connected**: can get from a to b AND from b to a in the digraph

DEF: A graph is **strongly** (resp. **semi**, resp. **weakly**) connected if every pair of vertices is connected in the same sense.

Connectivity in Directed Graphs

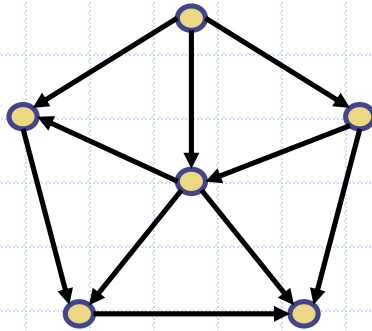
Q: Classify the connectivity of each graph.



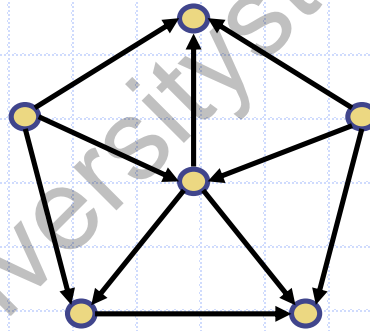
Connectivity in Directed Graphs

A:

semi



weak



strong

