Functions of two variables

Consider the function of two variables

$$z = f(x,y)$$

Let $x, y \in \mathbb{R} \Rightarrow (x,y) \in \mathbb{R}^2$.

If to each point $(x,y) \in \mathbb{R}^2$, there corresponds a real value z according to some rule $f(x,y)$, then $f(x,y)$ is called a real valued function of two variables x and y.

x, y are independent variables and z is dependent variable.

Domain :- The set of points (x,y) in the xy-plane for which $f(x,y)$ is defined is called the domain of the function and denoted by D.

Range :- The collection of corresponding values of z, is called the range.

Ex :- (1) $z = \sqrt{1 - x^2 - y^2}$

$D = 1 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 1$, ~~Range is the set of all positive real numbers~~. Range $= [0, 1]$.

(2) $z = \dfrac{1}{x^2 - y^2}$

$$D = \{ (x,y) \in \mathbb{R}^2 : x^2 - y^2 \neq 0 \}$$
$$= \{ (x,y) \in \mathbb{R}^2 : y \neq \pm x \}$$

Range is ~~the~~ $\mathbb{R} - \{0\}$

(3) $z = \log(x+y)$

$$D = \{ (x,y) \in \mathbb{R}^2 : x + y > 0 \}$$

Range is $\mathbb{R}$.

Bdd function

A fn $f(x,y)$ defined in some domain D in $\mathbb{R}^2$ is bdd if there exists a real finite positive number M s.t $|f(x,y)| \leq M$ for all $(x,y) \in D$.

$\log(0.1) = -1$

Limits

Let $z = f(x,y)$ be a function of two variables defined in domain D.

Let $P(x_0, y_0)$ be any point in Domain D.

Then, the real number L is called the limit of the function $f(x,y)$ as $(x,y) \to (x_0, y_0)$ if for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$ s.t.

s.t = such that

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

or

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta, \; 0 < |y - y_0| < \delta.$$

Symbolically, $\lim\limits_{(x,y) \to (x_0, y_0)} f(x,y) = L$.

Note:

1. If $\lim\limits_{(x,y) \to (x_0, y_0)} f(x,y) = L$ exists, it is unique.
2. The limit is path independent.
3. If the limit depends on path, then limit does not exist.

Ex: Find the value of (i) $\lim\limits_{(x,y) \to (2,1)} (3x + 4y)$ (ii) $\lim\limits_{(x,y) \to (1,1)} x^2 + 2y$

Sol: (i) $\lim\limits_{(x,y) \to (2,1)} 3x + 4y = 3(2) + 4(1) = 6 + 4 = 10.$

(ii) $\lim\limits_{(x,y) \to (1,1)} x^2 + 2y = 1 + 2 = 3.$

Ex: Determine the following limits, if they exist:

(i) $\lim_{(x,y)\to(0,0)} \left[\dfrac{x+y}{x^2+y^2+1}\right]$

(ii) $\lim_{(x,y)\to(0,1)} \left[\dfrac{(y-1)\tan^2 x}{x^2(y^2-1)}\right]$

(iii) $\lim_{(x,y)\to(0,0)} \left[y + x\cos\left(\dfrac{1}{y}\right)\right]$

Sol: (i) $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y^2+1} = \dfrac{0+0}{0+0+1} = \dfrac{0}{1} = 0.$

since, the limit value = 0, not N.D. we can say the limit is defined at point (0,0) so, limit exists

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

just remeber

(ii) $\lim_{(x,y)\to(0,1)} \left[\dfrac{(y-1)(\tan^2 x)}{x^2(y^2-1)}\right]$

if we put (0,1) in this limit then we get N.D so, will try another method

$$= \lim_{x\to 0} \frac{\tan^2 x}{x^2} \cdot \lim_{y\to 1} \frac{y-1}{y^2-1}$$

$$= \lim_{x\to 0} \left[\frac{\tan x}{x}\right]^2 \lim_{y\to 1} \frac{\cancel{(y-1)}}{\cancel{(y-1)}(y+1)}$$

$\because \lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \dfrac{\sin x}{x} \lim_{x\to 0} \dfrac{1}{\cos x}$

$= 1 \cdot \dfrac{1}{1} = 1$

sin 0 = 0
0/0=N.D!!

$$= (1)^2 \frac{1}{1+1} = \frac{1}{2}$$

(iii) $\lim_{(x,y)\to(0,0)} y + x\cos\left(\dfrac{1}{y}\right)$

$= \cancel{\lim}\ 0 + 0 = 0.$

$\left|\cos \dfrac{1}{y}\right| \le 1$

bdd fn.

$f, g \to$ funs.

$|f| \to$ bdd fn; $g \to 0$

$\lim_{x\to c} fg = 0.$

Note: The limit does not exist if it is not finite, path dependent or unique.

Ex: Show that the following limits do not exist.

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

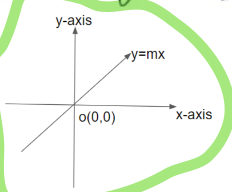
(ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+\sqrt{y}}{x^2+y}$

(iii) $\lim\limits_{(x,y)\to(0,1)} \tan^{-1}\left(\dfrac{y}{x}\right)$

(iv) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$

Sol: (i) Consider the path $y=mx$ such that ~~$x\to 0, y\to 0$~~ $y\to 0$ as $x\to 0$.

(A path passing through origin)



$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2}, \text{ which depends on } m.$$

For different values of m, we obtain different limits.

Hence, limit does not exist.

Alternative: Let $x = r\cos\theta$, $y = r\sin\theta$ ~~or~~ so that $x^2+y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{\cancel{r^2}\sin\theta\cos\theta}{\cancel{r^2}} = \sin\theta\cos\theta, \text{ which depends on } \theta.$$

Hence, limit does not exist.

(ii) Consider the path $y = mx^2$ s.t. $y \to 0$ as $x \to 0$

$$\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y} = \lim_{x\to 0} \frac{x+\sqrt{m}x}{x^2+mx^2}$$

$$= \lim_{x\to 0} \frac{(1+\sqrt{m})}{x(1+m)}$$

$$= \infty$$

(1+rt.m)/0(1+m)
= (1+rt.m)/0
= N.D/infty

Hence, the limit does not exist.

(iii)

$$\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$$

1/0
=infty

Thus, $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right) = \pi, -\frac{\pi}{2}$.

Since, limit is not unique. Hence, limit does not exist.

(iv) $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

Let $y = mx$ s.t. $y \to 0$ as $x \to 0$

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{\cancel{x}}{\sqrt{m^2x^2+x^2}} = \frac{1}{\sqrt{1+m^2}}$$, which depends on m.

Hence, limit does not exist.

Ex: Using δ-ϵ approach, show that

https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-limits-optional/v/proving-a-limit-using-epsilon-delta-definition

(i) $\lim_{(x,y)\to(2,1)} 3x+4y = 10$

(ii) $\lim_{(x,y)\to(1,1)} x^2+2y = 3$

(iii) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0.$

Sol: (i) $f(x,y) = 3x+4y$

T.P.: $|f(x,y)-10| < \epsilon$ whenever $0<|x-2|<\delta$, $0<|y-1|<\delta$.

$$
\begin{aligned}
|f(x,y)-10| &= |3x+4y-10| \\
&= |3x+4y-4-6| \\
&= |3(x-2)+4(y-1)| \\
&\le |3(x-2)| + |4(y-1)| \\
&\le 3|x-2| + 4|y-1| \\
&\le 3\delta + 4\delta = 7\delta
\end{aligned}
$$
, if we take $|x-2|<\delta$, $|y-1|<\delta$.

$\Rightarrow |f(x,y)-10| \le 7\delta$

Choose $7\delta < \epsilon$ i.e. $\delta < \epsilon/7$.

$\therefore$

$\Rightarrow |f(x,y)-10| \le 7\delta < \epsilon$

$\Rightarrow |f(x,y)-10| < \epsilon$ whenever $0<|x-2|<\delta$, $0<|y-1|<\delta$.

Hence, $\lim_{(x,y)\to(2,1)} 3x+4y = 10.$

(ii) $f(x,y) = x^2+2y$

T.P. : $|f(x,y)-3|<\epsilon$ whenever $0<|x-1|<\delta$, $0<|y-1|<\delta$.

$$|f(x,y)-3| = |x^2+2y-3|$$
$$= |(x-1+1)^2+2y-2-1|$$
$$= |(x-1)^2+1+2(x-1)+2(y-1)-1|$$
$$= |(x-1)^2+2(x-1)+2(y-1)|$$
$$\leq |x-1|^2+2|x-1|+2|y-1|$$

Take $|x-1|<\delta$, $|y-1|<\delta$

$\Rightarrow |f(x,y)-3| < \delta^2+2\delta+2\delta = \delta^2+4\delta.$

Choose $\delta^2+4\delta<\epsilon$ ~~ϵ~~

Thus, $|f(x,y)-3|<\epsilon$ whenever $0<|x-1|<\delta$, $0<|y-1|<\delta$.

Hence, $\lim\limits_{(x,y)\to(1,1)} f(x,y) = 3.$

(iii) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

T.P. : $|f(x,y)-0|<\epsilon$ whenever $0<\sqrt{(x-0)^2+(y-0)^2}<\delta$.

$$|f(x,y)-0| = \left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|$$
$$= \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{|xy|}{x^2+y^2}$$

$(x-y)^2 \geq 0$

$x^2+y^2-2xy \geq 0$

$x^2+y^2 \geq 2xy \Rightarrow xy \leq \frac{1}{2}(x^2+y^2)$

$$\leq \frac{1}{2} \frac{(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2} \sqrt{x^2+y^2}$$

Choose $\sqrt{x^2+y^2} < \delta$.

$\Rightarrow \quad |f(x,y) - 0| < \frac{\delta}{2}$

Take $\frac{\delta}{2} \leq \epsilon$

$\Rightarrow \quad |f(x,y) - 0| < \epsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$.

$\Rightarrow \quad \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

one more question

check notebook-piyush