Chapter 9

Fourier Series, Fourier Integrals and Fourier Transforms

9.1 Introduction

In Chapter 7 (section 7.2.4), we have defined the orthogonal and orthonormal functions. For example, the functions $\cos mx$ and $\sin mx$ are orthogonal over the interval $[-\pi, \pi]$. The orthonormal set of functions corresponding to $\cos mx$ are given by (Example 7.7)

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos mx}{\sqrt{\pi}}, \ldots$$

An important application discussed in section 7.2.4 was the series expansion of (suitable) functions in terms of a complete set of orthogonal or orthonormal functions. The expansion of a continuous function $f(x)$, having continuous derivatives over the interval $[-1, 1]$ in terms of Legendre polynomials was discussed in section 7.2.6. This series was called Fourier-Legendre series. The Fourier-Bessel series was discussed in section 7.4.2. A series expansion in terms of the trigonometric functions $\cos mx$ and $\sin mx$ is called a Fourier series. Many functions including some discontinuous periodic functions can be expanded in a Fourier series. Therefore, Fourier series solution method is a powerful tool in solving some ordinary and partial differential equations.

9.2 Fourier Series

Let $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ be an orthogonal set of functions, orthogonal with respect to a weight function $W(x) > 0$, on an interval $[a, b]$. Let $f(x)$ be a continuous function defined on the same interval $[a, b]$. Then, $f(x)$ can be expanded in an infinite series of the form (see section 7.2.4)

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots. \tag{9.1}$$

The coefficients c_i, $i = 0, 1, 2, \ldots$ are given by

$$c_i = \left[\int_a^b f(x)\, W(x)\phi_i(x)\,dx\right] \Big/ \| \phi_i(x)) \|^2, \quad i = 0, 1, 2, \ldots \tag{9.2}$$

where $$\| \phi_i (x) \|^2 = \int_a^b W(x)\phi_i^2 (x)dx.$$

Consider now, the set of orthogonal functions

$$\left\{1, \cos\left(\frac{\pi x}{l}\right), \cos\left(\frac{2\pi x}{l}\right), \ldots, \sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right), \ldots\right\} \tag{9.3}$$

which are orthogonal on the interval $[-l, l]$ with respect to the weight function $W(x) = 1$. These functions have the following properties

$$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)dx = 0 \tag{9.4}$$

$$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) dx = \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = 0, \; m \neq n, \tag{9.5}$$

$$\int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = 0, \text{ for all } m \text{ and } n, \tag{9.6}$$

$$\int_{-l}^{l} \cos^2\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \sin^2\left(\frac{m\pi x}{l}\right)dx = l, \tag{9.7}$$

where m and n are integers.

Now, let $f(x)$ be a periodic function of period $2l$ defined on $[-l, l]$, that is $f(x+2l) = f(x)$ and assume that it can be expanded in an orthogonal series in terms of the trigonometric functions. We shall discuss later in this section, the conditions under which such an expansion is possible. Let the series be written as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \ldots\right] + \left[b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \ldots\right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]. \end{aligned} \tag{9.8}$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2 \ldots$ can be determined by using the orthogonal properties of the trigonometric functions given in Eqs. (9.4) to (9.7). Integrating Eq. (9.8) term by term on the interval $[-l, l]$, we obtain

$$\int_{-l}^{l} f(x)dx = \frac{a_0}{2}\int_{-l}^{l} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)dx + b_n \int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)dx\right] = la_0$$

since $\cos(n\pi x/l)$ and $\sin(n\pi x/l)$ are orthogonal with respect to $W(x) = 1$, on $[-l, l]$. Therefore,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx.$$

Now, multiply both sides of Eq. (9.8) by $\cos(m\pi x/l)$ and integrate term by term on the interval $[-l, l]$. We obtain

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{a_0}{2}\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx\right.$$

$$\left. + b_n\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right].$$

Using the orthogonal properties given in Eqs. (9.4) to (9.7), we get

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = la_m.$$

Therefore,
$$a_m = \frac{1}{l}\int_{-l}^{1} f(x)\cos\left(\frac{m\pi x}{l}\right)dx.$$

Multiplying both sides of Eq. (9.8) by sin $(m\pi x/l)$ and integrating term by term on the interval $[-l, l]$, we obtain

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \frac{a_0}{2}\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx\right.$$

$$\left. + b_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right].$$

Using the orthogonal properties given in Eqs. (9.4) to (9.7), we get

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = lb_m.$$

Therefore,
$$b_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx.$$

It can be observed that the expressions for a_0 and a_m can be combined as a single expression. It is to obtain this simplicity of notation that $a_0/2$ is used in Eq. (9.8). This does not mean that the value of a_0 can be obtained after evaluating a_n and setting $n = 0$ in this expression.

The orthogonal series for $f(x)$ given in Eq. (9.8) is called the *Fourier series*. The coefficients a_0, a_n, b_n are called the *Fourier coefficients* on $[-l, l]$. The expressions for the coefficients

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, \tag{9.9}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \tag{9.10}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{9.11}$$

are called the *Euler formulas*.

If the period of the function is 2π, that is $f(x)$ is defined on $[-\pi, \pi]$, then the Euler formulas are simplified as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{9.12}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \tag{9.13}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{9.14}$$

From the definition of definite integrals, we have that if $f(x)$ *is continuous or piecewise continuous (continuous except for a finite number of finite jumps) then the integrals given in Eqs. (9.9) to (9.14) exist and* $f(x)$ *can be expanded as Fourier series.*

Example 9.1 Find the Fourier series expansion of the periodic function

$$f(x) = x, \quad -\pi \le x \le \pi, \quad f(x + 2\pi) = f(x).$$

Solution The Fourier coefficients are obtained as follows.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0, \qquad (x \text{ is an odd function on } [-\pi, \pi])$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0, \qquad (x\cos nx \text{ is an odd function on } [-\pi, \pi])$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx, \quad (x\sin nx \text{ is an even function on } [-\pi, \pi])$$

$$= 2\pi\left[-x\left(\frac{\cos nx}{n}\right)+\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{2}{n}(-1)^{n+1}$$

Therefore, the Fourier expansion of the given function on $[-\pi, \pi]$ is given by

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right].$$

Example 9.2 Find the Fourier series expansion of the following periodic function with period 2π

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ 0, & 0 \le x < \pi, \end{cases} \quad f(x+2\pi) = f(x).$$

Solution The graph of $f(x)$ is given in Fig. 9.1. The Fourier coefficients are obtained as follows.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\,dx = \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} = \frac{\pi}{2}.$$

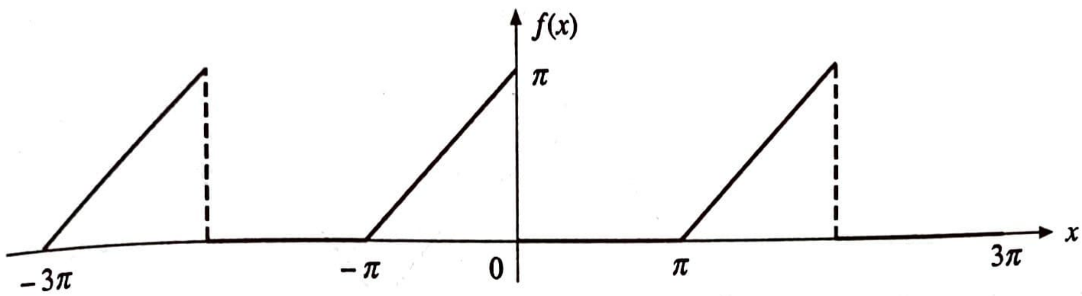

Fig. 9.1. Example 9.2.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi \cos nx dx + \int_{-\pi}^{0} x \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\sin nx}{n}\right) + \left\{x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1 - \cos n\pi)\right] = \frac{1}{\pi n^2}[1 - (-1)^n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 2/(\pi n^2), & \text{for } n \text{ odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\sin nx\, dx = \frac{1}{\pi}\left[(\pi + x)\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\right] = -\frac{1}{n}.$$

Therefore, the Fourier series expansion is given by

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\{1 - (-1)^n\}\cos nx - \frac{1}{n}\sin nx\right]$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right] - \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \ldots\right].$$

Example 9.3 Find the Fourier series expansion of the following periodic function of period 4

$$f(x) = \begin{cases} 2 + x, & -2 \le x \le 0, \\ 2 - x, & 0 < x \le 2, \end{cases} \quad f(x + 4) = f(x).$$

Solution The graph of $f(x)$ is given in Fig. 9.2. The function is defined on [–2, 2]. Using Eqs. (9.9) to (9.11), we obtain the Fourier coefficients as follows.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\left[\int_{-2}^{0} (2 + x)dx + \int_{0}^{2} (2 - x)dx\right]$$

$$= \frac{1}{2}\left[\left\{2x + \frac{x^2}{2}\right\}_{-2}^{0} + \left\{2x - \frac{x^2}{2}\right\}_{0}^{2}\right]$$

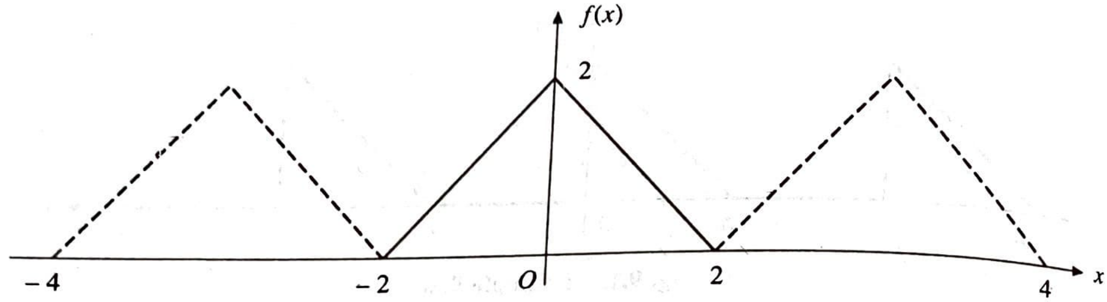

Fig. 9.2. Example 9.3.

$$= \frac{1}{2} [0 - (-4 + 2) + (4 - 2) - 0] = 2.$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} (2 + x) \cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} (2 - x) \cos\left(\frac{n\pi x}{2}\right) dx\right]$$

$$= \frac{1}{2}\left[\left\{(2 + x)\frac{\sin (n\pi x/2)}{(n\pi/2)} + \frac{\cos (n\pi x/2)}{(n\pi/2)^2}\right\}_{-2}^{0}\right.$$

$$\left. + \left\{(2 - x)\frac{\sin (n\pi x/2)}{(n\pi/2)} - \frac{\cos (n\pi x/2)}{(n\pi/2)^2}\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\left\{\frac{1}{(n\pi/2)^2} - \frac{\cos (n\pi)}{(n\pi/2)^2}\right\} + \left\{\frac{1}{(n\pi/2)^2} - \frac{\cos (n\pi)}{(n\pi/2)^2}\right\}\right]$$

$$= \frac{4}{n^2\pi^2}[1 - (-1)^n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 8/(n^2\pi^2), & \text{for } n \text{ odd.} \end{cases}$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} (2 + x) \sin\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} (2 - x) \sin\left(\frac{n\pi x}{2}\right) dx\right]$$

$$= \frac{1}{2}\left[\left\{-(2 + x)\frac{\cos (n\pi x/2)}{(n\pi/2)} + \frac{\sin (n\pi x/2)}{(n\pi/2)^2}\right\}_{-2}^{0}\right.$$

$$\left. + \left\{-(2 - x)\frac{\cos (n\pi x/2)}{(n\pi/2)} - \frac{\sin (n\pi x/2)}{(n\pi/2)^2}\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[-\frac{2}{(n\pi/2)} + \frac{2}{(n\pi/2)}\right] = 0.$$

Therefore, the Fourier series expansion is given by

$$f(x) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left[(2n-1)\frac{\pi x}{2}\right].$$

9.2.1 Fourier Series Expansions of Even and Odd Functions

Let $f(x)$ be a function defined on $[-l, l]$. Then, $f(x)$ is an even function on $[-l, l]$ if

$$f(-x) = f(x), \quad -l \le x \le l. \tag{9.15}$$

The function $f(x)$ is odd if

$$f(-x) = -f(x), \quad -l \le x \le l. \tag{9.16}$$

For example, x^{2n}, $\cos(n\pi x/l)$ are even functions on $[-l, l]$, since

$$f(-x) = (-x)^{2n} = [(-x)^2]^n = x^{2n} = f(x),$$

and
$$f(-x) = \cos(-n\pi x/l) = \cos(n\pi x/l) = f(x).$$

Similarly, x^{2n+1}, $\sin(n\pi x/l)$ are odd functions on $[-l, l]$, since

$$f(-x) = (-x)^{2n+1} = (-1)^{2n+1}\, x^{2n+1} = -x^{2n+1} = -f(x)$$

and
$$f(-x) = \sin(-n\pi x/l) = -\sin(n\pi x/l) = -f(x).$$

Graphs of even functions $|x|$, x^2 and a typical cosine like function are given in Figs. 9.3. a, b, c.

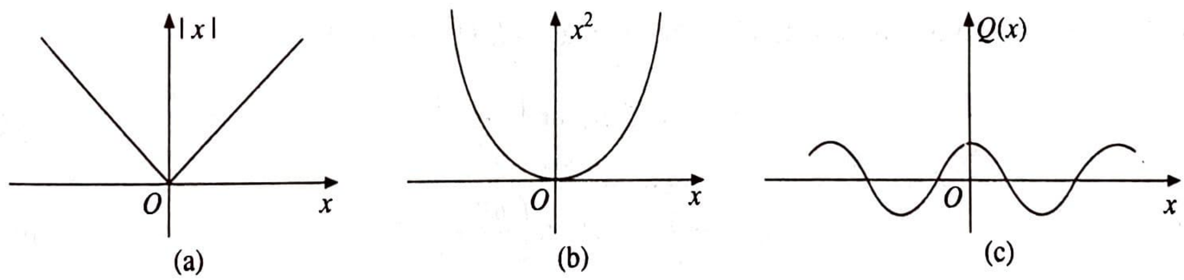

Fig. 9.3. Graphs of even functions.

Graphs of odd functions x, x^3 and a typical sine like function are given in Figs. 9.4. a, b, c.

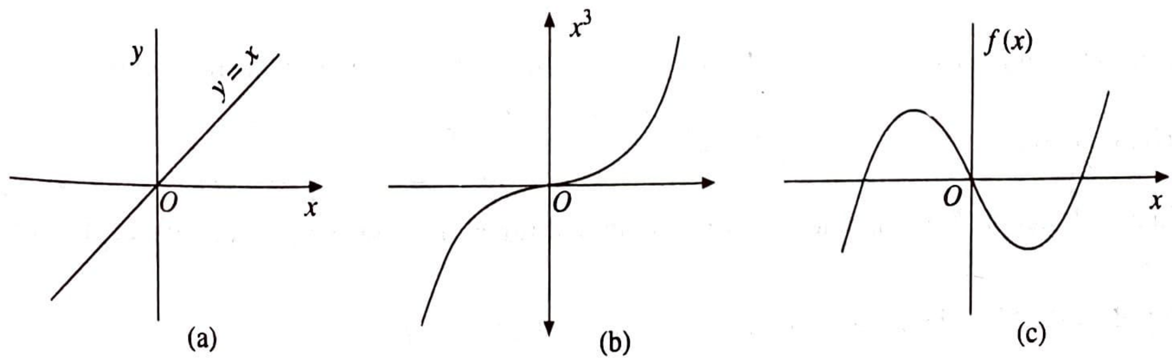

Fig. 9.4. Graphs of odd functions.

If $f(x)$ is an even function on $[-l, l]$, then we have

$$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx. \tag{9.17}$$

If $f(x)$ is an odd function on $[-l, l]$, then we have

$$\int_{-l}^{l} f(x)dx = 0. \tag{9.18}$$

The following results can be easily proved from the definition.

(even function) (even function) = even function

(even function) (odd function) = odd function

(odd function) (odd function) = even function.

Therefore, from the definition, we have that

if $f(x)$ is even then $f(x)\cos(n\pi x/l)$ is even and $f(x)\sin(n\pi x/l)$ is odd,

if $f(x)$ is odd then $f(x)\cos(n\pi x/l)$ is odd and $f(x)\sin(n\pi x/l)$ is even.

Hence, if $f(x)$ is an even function on $[-l, l]$, then we have the following Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{9.19}$$

where
$$a_0 = \frac{2}{l}\int_0^l f(x)dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \tag{9.20}$$

The Fourier series of an odd function on the interval $[-l, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{9.21}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{9.22}$$

The series given in Eq. (9.19) is called the *Fourier cosine series* and the series given in Eq. (9.21) is called the *Fourier sine series*.

Consider Example 9.1. The function $f(x) = x, -\pi \le x \le \pi$ is an odd function and we obtain a sine series.

The function $f(x) = x^2, -l \le x \le l$ is an even function and we obtain a cosine series.

The function
$$f(x) = \begin{cases} 2 + x, & -2 \le x \le 0, \\ 2 - x, & 0 < x \le 2 \end{cases}$$

defined in Example 9.3 is neither an even nor an odd function. However, we obtained only a cosine series.

Example 9.4 Find the Fourier series expansion of the function

$$f(x) = x^2, \quad -2 \le x \le 2.$$

Solution The given function $f(x) = x^2$ is an even function. Therefore,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \int_0^2 x^2 dx = \frac{8}{3}.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x^2 \frac{\sin(n\pi x/2)}{(n\pi/2)}\right]_0^2 - 2\int_0^2 x\frac{\sin(n\pi x/2)}{(n\pi/2)}dx$$

$$= -\frac{4}{n\pi}\left[-x\frac{\cos(n\pi x/2)}{(n\pi/2)} + \frac{\sin(n\pi x/2)}{(n\pi/2)^2}\right]_0^2 = \frac{16}{n^2\pi^2}\cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$$

Therefore, the Fourier series is given by

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

9.2.2 Convergence of Fourier Series

Many functions including some discontinuous periodic functions can be expanded in a Fourier series. In this section, we shall discuss the conditions under which Fourier series expansion is possible and also find the function to which the series converges.

Let $f(x)$ be piecewise continuous on $[-l, l]$, that is,

(i) $f(x)$ is defined and continuous for all x in $(-l, l)$ except, may be, at a finite number of points in $(-l, l)$.

(ii) At any point $x_0 \in (-l, l)$, where $f(x)$ is not continuous, both the one-sided limits $\lim\limits_{x \to x_0^-} f(x)$ and $\lim\limits_{x \to x_0^+} f(x)$ exist and are finite, that is, the discontinuities are jump discontinuities (Fig. 9.5).

(iii) The one-sided limits $\lim\limits_{x \to -l^+} f(x)$ and $\lim\limits_{x \to l^-}$ exist and are finite.

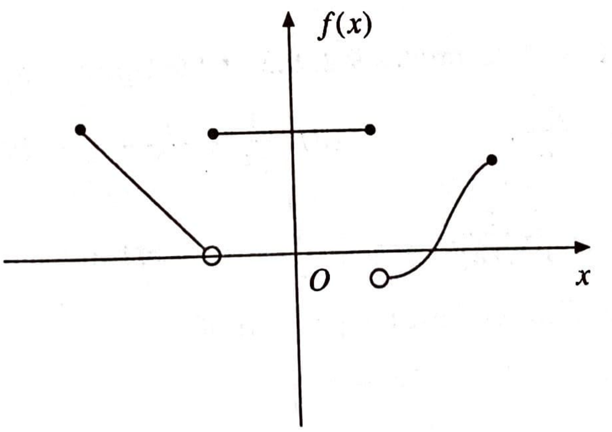

Fig. 9.5. Typical piecewise continuous function.

If both $f(x)$ and $f'(x)$ are piecewise continuous then the function $f(x)$ is also called *piecewise smooth.*

We now state the convergence theorem.

Theorem 9.1 Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $[-l, l]$. Then, the Fourier series of $f(x)$ on this interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to

$$\frac{1}{2}[f(x+)+f(x-)]$$

where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.

The proof of the theorem is omitted (see Problem 37, Exercise 9.1, for the proof of Theorem 9.1 for the particular case when $f(x)$, $f'(x)$ and $f''(x)$ are continuous).

Remark 1

At both the end points of the interval $[-l, l]$, the Fourier series converges to

$$\frac{1}{2}[f(-l+)+f(l-)]$$

The series converges to the same number at l and $-l$, since the Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

has the same value

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(-1)^n a_n$$

at both the end points, $x = l$ and $-l$.

Remark 2

Let the sum upto j terms of the Fourier series be denoted by

$$S_j=\frac{a_0}{2}+\sum_{n=1}^{j}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right],\quad j=1,2,3,\ldots.$$

Then, partial sums S_j give successive approximations to $f(x)$, that is, the approximations represent $f(x)$ closer and closer as j increases.

Example 9.5 Using the results of Examples 9.2, 9.3 and 9.4 prove the following

(i) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$, (ii) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\ldots=\frac{\pi^2}{6}$,

(iii) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$.

Solution In Example 9.2, the Fourier series expansion of

$$f(x)=\begin{cases}\pi+x, & -\pi<x<0\\ 0, & 0\le x<\pi\end{cases}$$

was obtained as

$$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right]-\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\ldots\right].$$

At the point $x = 0$, $f(x)$ is discontinuous. Therefore, the series converges to

$$\frac{1}{2}[f(0-)+f(0+)] = \frac{1}{2}[\pi + 0] = \frac{\pi}{2}.$$

Setting $x = 0$, we obtain

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \text{ or } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

In Example 9.3, the Fourier series expansion of

$$f(x) = \begin{cases} 2 + x, & -2 \le x \le 0 \\ 2 - x, & 0 < x \le 2 \end{cases}$$

was obtained as

$$f(x) = 1 + \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \ldots\right].$$

At the point $x = 0$, the given function is continuous. Therefore, the series converges to $f(0) = 2$. Hence,

$$2 = 1 + \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \text{ or } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

In Example 9.4, the Fourier series expansion of $f(x) = x^2$, $-2 \le x \le 2$ was obtained as

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

At the end points of the interval $[-2, 2]$, the series converges to

$$\frac{1}{2}[f(-2+) + f(2-)] = \frac{1}{2}[4 + 4] = 4.$$

Substituting $x = 2$, we obtain

$$4 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Hence,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{16}\left(4 - \frac{4}{3}\right) = \frac{\pi^2}{6}.$$

which is the result given in (ii).

At $x = 0$, the given function is continuous. The series converges to $f(0) = 0$. Therefore,

$$0 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{4}{3} - \frac{16}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

or

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

which is the result given in (iii).

Theorem 9.2 (Term by term differentiation of Fourier series) Let $f(x)$ be continuous and $f'(x)$ be piecewise continuous on $[-l, l]$. Let $f(-l) = f(l)$. Then, at every $x \in (-l, l)$ where $f''(x)$ exists

$$f'(x) = \frac{\pi}{l}\sum_{n=1}^{\infty} n\left[b_n\cos\left(\frac{n\pi x}{l}\right) - a_n\sin\left(\frac{n\pi x}{l}\right)\right]. \tag{9.23}$$

Note that the right hand side of Eq. (9.23) is the series obtained by differentiating the Fourier series given in Eq. (9.8), term by term.

Theorem 9.3 (Term by term integration of Fourier series) Let $f(x)$ be piecewise continuous on $(-l, l)$ and its Fourier series be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right].$$

Then, for any $x \in (-l, l)$ we have

$$\int_{-l}^{x} f(x^*) dx^* = \frac{a_0}{2}(x + l) + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[a_n \sin\left(\frac{n\pi x}{l}\right) + b_n \left\{ \cos n\pi - \cos\left(\frac{n\pi x}{l}\right) \right\} \right]. \quad (9.24)$$

Note that the right hand side of Eq. (9.24) is the series obtained by integrating the Fourier series term by term.

Remark 3

The result of Theorem 9.3 holds even if the Fourier series does not converge to the function.

Gibbs phenomenon To discuss the Gibbs phenomenon, let us consider the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0, \\ -1, & 0 \le x < \pi. \end{cases}$$

The graph of $f(x)$ is given in Fig. 9.6.

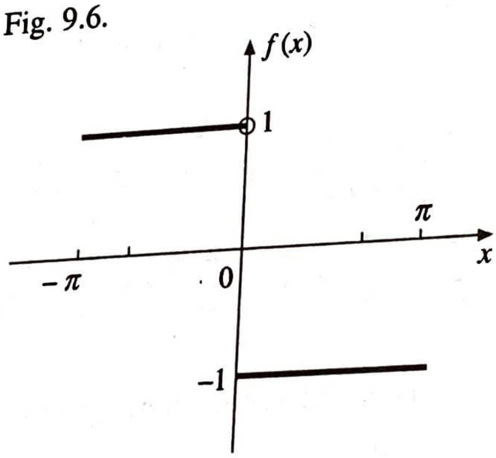

Fig. 9.6. Graph of $f(x)$.

The function is odd. Therefore, we have a sine series. We have

$$b_n = \frac{2}{\pi} \int_0^{\pi} -\sin nx\,dx = \frac{2}{n\pi}[\cos n\pi - 1] = \frac{2}{n\pi}[(-1)^n - 1] = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4}{n\pi}, & n \text{ odd}. \end{cases}$$

Therefore,

$$f(x) = -\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right].$$

Denote the partial sums of the series as

$$S_1 = -\frac{4}{\pi}\sin x,\quad S_2 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right),\quad S_3 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right),\ \ldots$$

The typical plots of S_1, S_2, S_3, S_{14} are given in Figs. 9.7 to 9.10.

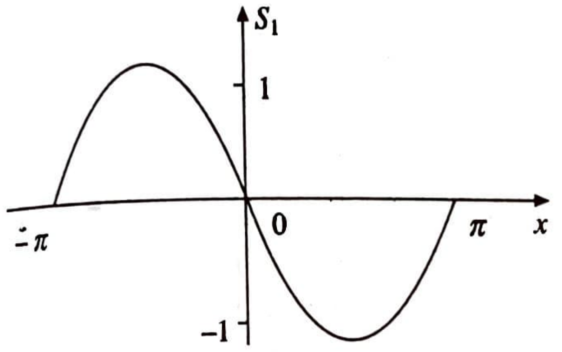

Fig. 9.7. Typical plot of $S_1(x)$.

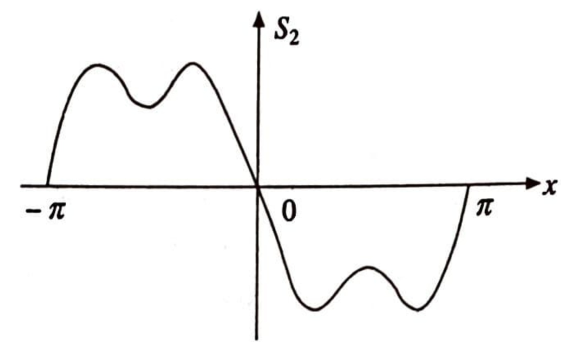

Fig. 9.8. Typical plot of $S_2(x)$.

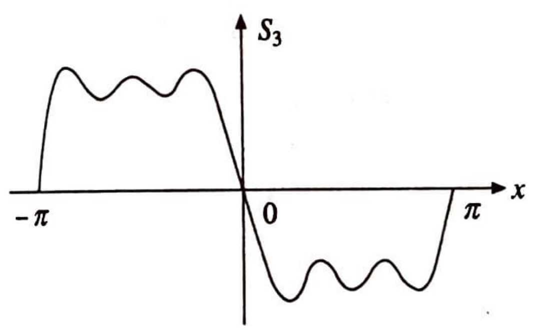

Fig. 9.9. Typical plot of $S_3(x)$.

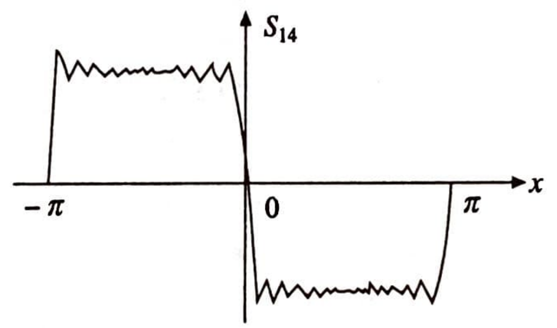

Fig. 9.10. Typical plot of $S_{14}(x)$.

It can be observed that the graph of $S_{14}(x)$ displays spikes near the discontinuities at $x = -\pi$, 0 and π. This oscillatory behaviour of the partial sums S_n for large n, about the true value near a point of discontinuity does not smoothen out even for very large n. This behaviour of the Fourier series near a point of discontinuity is called the *Gibbs phenomenon*.

Exercise 9.1

In the following problems, find the Fourier series of the given function on the given interval.

1. $f(x) = \begin{cases} k, & -\pi < x < 0, \\ 0, & 0 \le x < \pi. \end{cases}$

2. $f(x) = \begin{cases} \pi, & -\pi < x < 0, \\ \pi - x, & 0 \le x < \pi. \end{cases}$

3. $f(x) = \begin{cases} 0, & -\pi < x < -\pi/2, \\ k, & -\pi/2 \le x \le \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$

4. $f(x) = \begin{cases} -k, & -\pi < x < 0, \\ k, & 0 \le x < \pi. \end{cases}$

5. $f(x) = \begin{cases} 2, & -\pi < x < 0, \\ 4, & 0 \le x < \pi. \end{cases}$

6. $f(x) = \begin{cases} -(\pi + x), & -\pi < x < 0, \\ -(\pi - x), & 0 \le x < \pi. \end{cases}$

7. $f(x) = 1 - |x|, -\pi < x < \pi.$

8. $f(x) = x^2, -\pi < x < \pi.$

9. $f(x) = x^3, -\pi < x < \pi$.

10. $f(x) = \begin{cases} x^2, & -\pi < x < 0, \\ -x^2, & 0 \le x < \pi. \end{cases}$

11. $f(x) = \cos x - \sin(x/2), -\pi < x < \pi$.

12. $f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ \sin x, & 0 \le x \le \pi. \end{cases}$

13. $f(x) = \begin{cases} \cos x, & -\pi < x \le 0, \\ \frac{1}{\pi}(\pi - x), & 0 < x < \pi, \end{cases}$

14. $f(x) = x, -2 < x < 2$.

15. $f(x) = x^2, -3 < x < 3$.

16. $f(x) = |x|, -1 < x < 1$.

17. $f(x) = 1 + |x|, -3 < x < 3$.

18. $f(x) = \sinh(\pi x), -1 < x < 1$.

19. $f(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 0, & 0 < x \le 1. \end{cases}$

20. $f(x) = \begin{cases} -x, & -2 < x < 0, \\ x^2, & 0 \le x < 2. \end{cases}$

21. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ x, & 0 \le x < 1. \end{cases}$

22. $f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ \cos(\pi x/2), & -1 \le x \le 1, \\ 0, & 1 \le x \le 2. \end{cases}$

23. $f(x) = \begin{cases} 0, & -1 < x \le 0, \\ -2x, & 0 \le x < 1. \end{cases}$

24. $f(x) = \begin{cases} -(x+2), & -2 \le x < 0, \\ x - 2, & 0 \le x \le 2. \end{cases}$

25. $f(x) = \begin{cases} 0, & -2 < x < -1, \\ 1, & -1 \le x \le 1, \\ 0, & 1 < x < 2. \end{cases}$

26. $f(x) = 9 - x^2, -3 \le x \le 3$.

27. Let $f(x)$ and $g(x)$ be two integrable functions on the interval $[-l, l]$. It is given that $f(x) = g(x)$ for all x in this interval except at one point $x = c$. Are the Fourier coefficients of these functions different?

28. Find the Fourier series expansion of the function

$$f(x) = \pi + x, -\pi < x < \pi.$$

Hence, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

29. Find the Fourier series expansion of the following periodic function of period 2π

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x^2, & 0 \le x < \pi. \end{cases}$$

Hence, show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots, \text{ and } \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$$

30. Manipulate the two series in problem 29 to show that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

31. Use the result of Problem 12 to show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \ldots.$$

32. A periodic function of period 2π is defined as

$$f(x) = \begin{cases} 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 \le x < 3\pi/2. \end{cases}$$

Obtain the Fourier series expansion of $f(x)$ and hence, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

33. A periodic function of period 2 is defined as

$$f(x) = 1 + x, \; -1 < x < 1.$$

Obtain the Fourier series expansion of $f(x)$ and hence, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

34. Obtain the Fourier series expansion of the periodic function

$$f(x) = e^x, \; -\pi < x < \pi, \; f(x + 2\pi) = f(x).$$

Hence, find the sum of the series

$$\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots + \frac{(-1)^n}{1+n^2} + \ldots.$$

35. Obtain the Fourier series expansion of the following periodic function of period 4

$$f(x) = 4 - x^2, \; -2 \le x \le 2.$$

Hence, show that $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots.$

36. Show that $f(x) = |x|, -2 \le x \le 2$ satisfies the hypothesis of Theorem 9.2. Obtain the Fourier series expansion of $f(x)$ and hence find the series expansion of $f'(x)$. Verify that this series is same as the Fourier series expansion of $f'(x) = \begin{cases} -1, & \text{for} \quad -2 < x < 0 \\ 1, & \text{for} \quad 0 < x < 2. \end{cases}$

37. Prove the convergence (uniform convergence) of the Fourier series of a periodic function $f(x)$ defined over $[-\pi, \pi]$, when $f(x), f'(x)$ and $f''(x)$ are continuous over $[-\pi, \pi]$.

38. Prove that

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

39. Express the Fourier series given in Eq. (9.8) in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$$

where α_n is the phase angle defined by $\alpha_n = \tan^{-1}(b_n/a_n)$ and $A_n = \sqrt{a_n^2 + b_n^2}$.

40. Express the Fourier series given in Eq. (9.8) in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$$

where β_n is the phase angle $\beta_n = \tan^{-1}(a_n/b_n)$. Compare this result with the result given in Problem 39.

9.3 Fourier Half-Range Series

Suppose that a function $f(x)$ is defined on some finite interval. It may also be the case that a periodic function $f(x)$ of period $2l$ is defined only on a half-interval $[0, l]$. It is possible to extend the definition of $f(x)$ to the other half $[-l, 0]$ of the interval $[-l, l]$ so that $f(x)$ is either an even or an odd function. In the first case, we call it an even periodic extension of $f(x)$ and in the second case, we call it an odd periodic extension of $f(x)$. If $f(x)$ is given and an even periodic extension is done then $f(x)$ is an even function in $[-l, l]$. Hence, $f(x)$ has a Fourier cosine series. If $f(x)$ is given and an *odd periodic extension is* done then $f(x)$ is an odd function in $[-l, l]$. Hence, $f(x)$ has now a Fourier sine series. Therefore, if a function $f(x)$ is defined only on a half-interval $[0, l]$, then it is possible to obtain a Fourier cosine or a Fourier sine series expansion depending on the requirements of a particular problem, by suitable periodic extensions. We have the following results.

Theorem 9.4 **(Fourier cosine series)** Let $f(x)$ be piecewise continuous on $[0, l]$. Then, the Fourier cosine series expansion of $f(x)$ on the half-range interval $[0, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{9.25}$$

where
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

The convergence Theorem 9.1 can be extended as follows.

If $x \in [0, l]$ and $f(x)$ has left and right hand derivatives at x, then at x, the Fourier cosine series converges to $[f(x+) + f(x-)]/2$. At a point of continuity, the Fourier cosine series converges to $f(x)$.

If $\lim_{x\to 0^+} f'(x)$ exists, then at $x = 0$, the series converges to $f(0+)$.

If $\lim_{x\to l^-} f'(x)$ exists, then at $x = l$, the series converges to $f(l-)$.

We now, define the Fourier sine series.

Theorem 9.5 (Fourier sine series) Let $f(x)$ be piecewise continuous on $[0, l]$. Then, the Fourier sine series expansion of $f(x)$ on $[0, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{9.26}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

If $x \in [0, l]$ and $f(x)$ has left and right hand derivatives at x, then at x, the Fourier sine series converges to $[f(x+) + f(x-)]/2$. At both the end points $x = 0$ and l, the series converges to 0.

Example 9.6 Find the Fourier cosine and sine series of the function $f(x) = 1$, $0 \le x \le 2$.

Solution The Fourier coefficients for the cosine series are obtained as follows,

$$a_0 = \frac{2}{2}\int_0^2 1\cdot dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx = \left[\frac{\sin(n\pi x/2)}{(n\pi/2)}\right]_0^2 = 0.$$

Therefore, the Fourier cosine series is $f(x) = 1$ itself.

For the Fourier sine series, we have

$$b_n = \frac{2}{2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{\cos(n\pi x/2)}{(n\pi/2)}\right]_0^2 = \frac{2}{n\pi}[1 - \cos n\pi]$$

$$= \frac{2}{n\pi}[1-(-1)^n] = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd.} \end{cases}$$

Therefore, $$f(x) = \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \ldots\right].$$

Example 9.7 Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4. \end{cases}$$

Solution Note that $f(x)$ is to be extended as an even function. We have

$$a_0 = \frac{2}{4}\int_0^4 f(x)dx = \frac{1}{2}\left[\int_0^2 x^2 dx + \int_2^4 4\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{x^3}{3}\right)_0^2 + \{4x\}_2^4\right] = \frac{1}{2}\left[\frac{8}{3} + 8\right] = \frac{16}{3}.$$

$$a_n = \frac{1}{2}\left[\int_0^2 x^2\cos\left(\frac{n\pi x}{4}\right)dx + \int_2^4 4\cos\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{x^2\sin(n\pi x/4)}{(n\pi/4)} + \frac{2x\cos(n\pi x/4)}{(n\pi/4)^2} - \frac{2\sin(n\pi x/4)}{(n\pi/4)^3}\right\}_0^2\right.$$

$$\left. + 4\left\{\frac{\sin(n\pi x/4)}{(n\pi/4)}\right\}_2^4\right]$$

$$= \frac{1}{2}\left[\frac{4\sin(n\pi/2)}{(n\pi/4)} + \frac{4\cos(n\pi/2)}{(n\pi/4)^2} - \frac{2\sin(n\pi/2)}{(n\pi/4)^3} - \frac{4\sin(n\pi/2)}{(n\pi/4)}\right]$$

$$= \frac{32}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\right].$$

Therefore,

$$f(x) = \frac{8}{3} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{4}\right).$$

Exercise 9.2

In each of the following problems, write the Fourier cosine series and Fourier sine series for the function in the given interval (wherever they are possible).

1. $f(x) = k,\ 0 \le x \le 5.$

2. $f(x) = \cos x,\ 0 < x \le \pi/2.$

3. $f(x) = \begin{cases} -1, & 0 \le x \le 1, \\ 1, & 1 < x \le 2. \end{cases}$

4. $f(x) = \begin{cases} x, & 0 < x < 2, \\ 2, & 2 \le x < 4. \end{cases}$

5. $f(x) = x + x^2,\ 0 < x < 1.$

6. $f(x) = \begin{cases} x, & 0 < x < \pi/2, \\ \pi - x, & \pi/2 \le x < \pi. \end{cases}$

7. $f(x) = \begin{cases} \pi - x, & 0 < x < \pi, \\ 0, & \pi \le x < 2\pi. \end{cases}$

8. $f(x) = \begin{cases} 2, & 0 < x < 2, \\ 4 - x, & 2 \le x < 4. \end{cases}$

9. $f(x) = \begin{cases} x^2, & 0 \le x < 1, \\ 1, & 1 \le x < 2. \end{cases}$

10. $f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ -1, & 1 < x < 2, \\ 0, & 2 \le x < 3. \end{cases}$

11. $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1; & 1 \le x < 2, \\ 3 - x, & 2 \le x \le 3. \end{cases}$

12. $f(x) = e^{-x},\ 0 \le x \le 2.$

13. $f(x) = \cos 3x,\ 0 \le x \le \pi.$

14. $f(x) = 1 + x,\ 0 \le x \le 1.$

15. $f(x) = \sin 3x,\ 0 \le x \le \pi.$

Find the complex form of Fourier series of $f(x)$ on the given interval.

16. $f(x) = \begin{cases} 1, & -2 \le x < 1, \\ 0, & 1 \le x < 2. \end{cases}$

17. $f(x) = \begin{cases} x, & -\pi < x \le 0, \\ 0, & 0 < x < \pi. \end{cases}$

18. $f(x) = e^{-|x|},\ -2 < x < 2.$

Find the frequency spectrum of $f(x)$ in the following problems.

19. $f(x) = \begin{cases} 0, & -\pi/2 \le x < 0 \\ \sin x, & 0 \le x \le \pi/2. \end{cases}$

$f(x + \pi) = f(x).$

20. $f(x) = \begin{cases} -2\cos x, & -\pi/2 < x < 0 \\ 2\cos x, & 0 \le x < \pi/2. \end{cases}$

$f(x + \pi) = f(x).$

9.4 Fourier Integrals

In the previous sections, we have seen that if $f(x)$ is piecewise smooth on any interval $[-l, l]$ or $[0, l]$ then it can be represented by a Fourier series. If $f(x)$ is a periodic function (so that it is defined on the entire real line) then also $f(x)$ can be represented by a Fourier series. If $f(x)$ is not a periodic function then it cannot be represented by a Fourier series over the entire real line. However, we may be able to represent $f(x)$ in an integral form.

Let $f(x)$ have the following properties.

(P1) $f(x)$ is piecewise continuous on every interval $[-l, l]$.

37. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \; -\infty < x < \infty, t > 0.$

$u(x, 0) = e^{-4x^2}, -\infty < x < \infty.$

It is given that $\mathcal{F}[e^{-at^2}] = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/(4a)}$.

38. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; 0 < x < \infty, t > 0,$

$$u(x, 0) = \begin{cases} 1, & 0 < x \le l, \\ 0, & x > l. \end{cases}$$

$u(0, t) = 0, t > 0.$

39. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < \pi, y > 0.$

$u(0, y) = 0, u(\pi, y) = 0, y > 0.$

$u(x, 0) = \sin x, 0 < x < \pi.$

40. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, 0 < y < \pi,$

$u(x, 0) = e^{-2x} H(x),$

$u(x, \pi) = 0, -\infty < x < \infty.$

41. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, 0 < y < 1,$

$\frac{\partial u}{\partial y}(x, 0) = 0, u(x, 1) = e^{-2|x|}, -\infty < x < \infty.$

9.7 Answers and Hints

Exercise 9.1

In the following problems denote $p_n = 1 - \cos n\pi = 1 - (-1)^n$. The summations are all from $n = 1$ to ∞, except where it is specifically mentioned.

1. $\frac{k}{2} - \frac{k}{\pi} \Sigma \left[\frac{1}{n} p_n \sin(nx) \right].$

2. $\frac{3\pi}{4} + \Sigma \left[\frac{1}{\pi n^2} p_n \cos(nx) + \frac{1}{n} \cos(n\pi) \sin(nx) \right].$

3. $\frac{k}{2} + \frac{2k}{\pi} \Sigma \left[\frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos(nx) \right].$

4. $\frac{2k}{\pi} \Sigma \left[\frac{1}{n} p_n \sin(nx) \right].$

5. $3 + \frac{2}{\pi} \Sigma \left[\frac{1}{n} p_n \sin(nx) \right].$

6. $-\frac{\pi}{2} - \frac{2}{\pi} \Sigma \left[\frac{1}{n^2} p_n \cos(nx) \right].$

7. $\frac{1}{2}(2 - \pi) + \frac{2}{\pi} \Sigma \left[\frac{1}{n^2} p_n \cos(nx) \right].$

8. $\frac{\pi^2}{3} + 4 \Sigma \left[\frac{1}{n^2} \cos(n\pi) \cos(nx) \right].$

9. $\frac{2}{\pi} \Sigma \left[\left(\frac{6\pi}{n^3} - \frac{\pi^3}{n} \right) \cos(n\pi) \sin(nx) \right].$

10. $\frac{2}{\pi} \Sigma \left[\frac{\pi^2}{n} \cos n\pi + \frac{2}{n^3} p_n \right] \sin(nx).$

11. $\cos x + \frac{8}{\pi} \sum_{n=2}^{\infty} \left[\frac{n \cos(n\pi)}{(4n^2 - 1)} \sin(nx) \right].$

12. $\frac{1}{\pi} + \frac{1}{2} \sin x + \frac{1}{\pi} \sum_{n=2}^{\infty} \left[\frac{(-1)^{n-1} - 1}{(n^2 - 1)} \cos nx \right].$

13. $\frac{1}{4} + \frac{1}{2\pi^2}(4 + \pi^2) \cos x + \frac{1}{\pi} \sin x$

$$+ \frac{1}{\pi^2} \sum_{n=2}^{\infty} \left[\frac{1}{n^2} p_n \cos(nx) + \left\{ \frac{n\pi}{(n^2 - 1)} ((-1)^{n-1} - 1) + \frac{\pi}{n} \right\} \sin(nx) \right]$$

14. $\frac{4}{\pi}\Sigma\left[\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)\right]$.

15. $3+\frac{36}{\pi^2}\Sigma\left[\frac{1}{n^2}\cos(n\pi)\cos\left(\frac{n\pi x}{3}\right)\right]$.

16. $\frac{1}{2}-\frac{2}{\pi^2}\Sigma\left[\frac{1}{n^2}p_n\cos(n\pi x)\right]$.

17. $\frac{5}{2}-\frac{6}{\pi^2}\Sigma\left[\frac{1}{n^2}p_n\cos\left(\frac{n\pi x}{3}\right)\right]$.

18. $\frac{2}{\pi}\sinh\pi\,\Sigma\left[\frac{n(-1)^{n+1}}{1+n^2}\sin(n\pi x)\right]$.

19. $\frac{1}{4}+\Sigma\left[\frac{1}{n^2\pi^2}p_n\cos(n\pi x)-\frac{1}{n\pi}\sin(n\pi x)\right]$.

20. $\frac{7}{6}+\Sigma\left[\frac{2}{n^2\pi^2}(5\cos(n\pi)-1)\cos\left(\frac{n\pi x}{2}\right)-\left\{\frac{2}{n\pi}\cos(n\pi)+\frac{8p_n}{(n\pi)^3}\right\}\sin\left(\frac{n\pi x}{2}\right)\right]$.

21. $-\frac{1}{4}+\Sigma\left[-\frac{1}{(n\pi)^2}p_n\cos(n\pi x)+\frac{1}{n\pi}(1-2\cos(n\pi))\sin(n\pi x)\right]$.

22. $\frac{1}{\pi}+\frac{1}{2}\cos x-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1}{(n^2-1)}\cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]$.

23. $-\frac{1}{2}+2\Sigma\left[\frac{1}{(n\pi)^2}p_n\cos(n\pi x)+\frac{1}{n\pi}\cos(n\pi)\sin(n\pi x)\right]$.

24. $-1-\frac{4}{\pi^2}\Sigma\frac{1}{n^2}p_n\cos\left(\frac{n\pi x}{2}\right)$.

25. $\frac{1}{2}+\frac{2}{\pi}\Sigma\left[\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]$.

26. $6-\frac{36}{\pi^2}\Sigma\left[\frac{1}{n^2}\cos(n\pi)\cos\left(\frac{n\pi x}{3}\right)\right]$.

27. They are same.

28. $\pi+2\Sigma\left[\frac{(-1)^{n+1}}{n}\sin(nx)\right]$. Set $x=\frac{\pi}{2}$ (point of continuity).

29. $\frac{\pi^2}{6}+\Sigma\left[\frac{2}{n^2}(-1)^n\cos(nx)+\left\{\frac{\pi}{n}(-1)^{n+1}-\frac{2}{\pi n^3}p_n\right\}\sin(nx)\right]$.
Set $x=0$, (point of continuity). Set $x=\pi$ (point of discontinuity).

30. Add the two series.

31. Set $x=\pi/2$, (point of continuity) and re-arrange the terms.

32. $\frac{1}{2}+\frac{2}{\pi}\Sigma\left[\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(nx)\right]$. Set $x=0$.

33. $1-\frac{2}{\pi}\Sigma\left[\frac{1}{n}\cos(n\pi)\sin(n\pi x)\right]$. Set $x=1/2$.

34. $\frac{1}{\pi}\sinh\pi+\frac{2\sinh\pi}{\pi}\Sigma\left[\frac{(-1)^n}{n^2+1}\{\cos(nx)-n\sin(nx)\}\right]$. Set $x=0$. $\pi/[2\sinh\pi]$.

35. $\frac{8}{3}-\frac{16}{\pi^2}\Sigma\left[\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)\right]$. Set $x=0$.

36. $1-\frac{4}{\pi^2}\Sigma\left[\frac{1}{n^2}p_n\cos\left(\frac{n\pi x}{2}\right)\right]$, $\frac{2}{\pi}\Sigma\left[\frac{1}{n}p_n\sin\left(\frac{n\pi x}{2}\right)\right]$.

37. Integrate a_n, b_n (Eqs. (9.13), (9.14)), by parts. Using the periodicity and continuity of $f'(x)$, we obtain

$$a_n = -\frac{1}{\pi n^2}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx,\ b_n = -\frac{1}{\pi n^2}\int_{-\pi}^{\pi} f''(x)\sin nx\,dx.$$

Since $f''(x)$ is continuous, $|f''(x)| < L$ for some real constant L., We get

$$|a_n| < \frac{1}{\pi n^2}\int_{-\pi}^{\pi} L\,dx = \frac{2L}{n^2} \text{ and } |b_n| < \frac{2L}{n^2}, \text{ for all } n.$$

Hence, the absolute value of each term of the Fourier series is less than or equal to the corresponding term of the series

$$|a_0| + 2L\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \ldots\right) = |a_0| + 4L\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

Hence, the Fourier series is convergent and by Weirstrass test it is also uniformly convergent.

39. $f(x) = \frac{a_0}{2} + \Sigma\left[A_n\left\{\frac{a_n}{A_n}\cos\left(\frac{n\pi x}{l}\right) + \frac{b_n}{A_n}\sin\left(\frac{n\pi x}{l}\right)\right\}\right], A_n = \sqrt{a_n^2 + b_n^2}.$

Set $\cos\alpha_n = a_n/A_n$ and $\sin\alpha_n = b_n/A_n$.

40. In problem 39, set $\cos\beta_n = b_n/A_n$ and $\sin\beta_n = a_n/A_n$. $\beta_n = (\pi/2) - \alpha_n$.

Exercise 9.2

In the following problems denote $p_n = 1 - \cos n\pi = 1 - (-1)^n$. The summations are all from $n = 1$ to ∞, except where it is specifically mentioned.

1. $\frac{2k}{\pi}\Sigma\left[\frac{1}{n}p_n\sin(n\pi x/5)\right]$.

2. $\frac{2}{\pi} - \frac{4}{\pi}\Sigma\left[\frac{(-1)^n}{4n^2-1}\cos(2nx)\right]$, No sine series is possible.

3. $-\frac{4}{\pi}\Sigma\left[\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)\right], \frac{2}{\pi}\Sigma\left[\frac{1}{n}\left\{2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right\}\right]\sin\left(\frac{n\pi x}{2}\right)$.

4. $\frac{3}{2} + \frac{8}{\pi^2}\Sigma\left[\frac{1}{n^2}\left\{\cos\left(\frac{n\pi}{2}\right) - 1\right\}\cos\left(\frac{n\pi x}{4}\right)\right], \frac{4}{\pi}\Sigma\left[\frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi)\right]\sin\left(\frac{n\pi x}{4}\right)$.

5. $\frac{5}{6} + \frac{2}{\pi^2}\Sigma\left[\frac{1}{n^2}\{3\cos(n\pi) - 1\}\cos(n\pi x)\right], -4\Sigma\left[\frac{1}{n\pi}\cos(n\pi) + \frac{1}{(n\pi)^3}p_n\right]\sin(n\pi x)$.

6. $\frac{\pi}{4} + \frac{2}{\pi}\Sigma\left[\frac{1}{n^2}\left\{2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right\}\cos(nx)\right], \frac{4}{\pi}\Sigma\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(nx)\right]$.

7. $\frac{\pi}{4} + \frac{4}{\pi}\Sigma\left[\frac{1}{n^2}\left\{1 - \cos\left(\frac{n\pi}{2}\right)\right\}\cos\left(\frac{nx}{2}\right)\right], \frac{1}{\pi}\Sigma\left[\left\{\frac{2\pi}{n} - \frac{4}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin\left(\frac{nx}{2}\right)\right]$.

8. $\frac{3}{2} + \frac{8}{\pi^2}\Sigma\left[\frac{1}{n^2}\left\{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right\}\cos\left(\frac{n\pi x}{4}\right)\right], 4\Sigma\left[\left\{\frac{1}{n\pi} + \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin\left(\frac{n\pi x}{4}\right)\right]$.

9. $\frac{2}{3} + \frac{8}{\pi^2} \Sigma \left[\frac{1}{n^2} \left\{ \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right\} \cos\left(\frac{n\pi x}{2}\right) \right]$,

$\Sigma \left[\frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{16}{(n\pi)^3} \left\{ \cos\left(\frac{n\pi}{2}\right) - 1 \right\} - \frac{2}{n\pi} \cos(n\pi) \right] \sin\left(\frac{n\pi x}{2}\right)$.

10. $\frac{2}{\pi} \Sigma \left[\frac{1}{n} \left\{ 2 \sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{2n\pi}{3}\right) \right\} \cos\left(\frac{n\pi x}{3}\right) \right]$,

$\frac{2}{\pi} \Sigma \left[\frac{1}{n} \left\{ 1 - 2\cos\left(\frac{n\pi}{3}\right) + \cos\left(\frac{2n\pi}{3}\right) \right\} \sin\left(\frac{n\pi x}{3}\right) \right]$.

11. $\frac{2}{3} + \frac{6}{\pi^2} \Sigma \left[\frac{1}{n^2} \left\{ \cos\left(\frac{n\pi}{3}\right) - \cos(n\pi) + \cos\left(\frac{2n\pi}{3}\right) - 1 \right\} \cos\left(\frac{n\pi x}{3}\right) \right]$,

$\frac{6}{\pi^2} \Sigma \left[\frac{1}{n^2} \left\{ \sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right) \right\} \sin\left(\frac{n\pi x}{3}\right) \right]$.

12. $\frac{1}{2}(1 - e^{-2}) + 4 \Sigma \frac{1}{4 + n^2\pi^2} \left[\{1 - e^{-2}\cos(n\pi)\} \cos\left(\frac{n\pi x}{2}\right) \right]$,

$2\pi \Sigma \frac{n}{4 + n^2\pi^2} \left[\{1 - e^{-2}\cos(n\pi)\} \sin\left(\frac{n\pi x}{2}\right) \right]$.

13. The given function itself is the Fourier cosine series, $\frac{2}{\pi} \Sigma \left[\frac{n}{(n^2 - 9)} \{(-1)^n + 1\} \sin(nx) \right]$.

14. $\frac{3}{2} + \frac{2}{\pi^2} \Sigma \left[\frac{1}{n^2} \{\cos(n\pi) - 1\} \cos(n\pi x) \right]$, $\frac{2}{\pi} \Sigma \left[\frac{1}{n} \{1 - 2\cos(n\pi)\} \sin(n\pi x) \right]$.

15. $\frac{2}{3\pi} - \frac{6}{\pi} \Sigma \left[\frac{1}{(n^2 - 9)} \{(-1)^n + 1\} \cos(nx) \right]$.

16. $\frac{3}{4} + \frac{1}{2\pi i} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{n} (e^{in\pi} - e^{-in\pi/2}) e^{in\pi x/2} \right]$. 17. $-\frac{\pi}{4} + \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{n^2}(1 - e^{in\pi}) + \frac{i\pi}{n} e^{in\pi} \right] e^{inx}$.

18. $\sum_{n=-\infty}^{\infty} \frac{2}{4 + n^2\pi^2} [\{1 - (-1)^n e^{-2}\} e^{in\pi x/2}]$. 19. $\left[n\omega, \frac{\sqrt{1 + 4n^2}}{\pi(4n^2 - 1)} \right]$, $\omega = 2$, $c_0 = \frac{1}{\pi}$.

20. $\left[n\omega, \frac{8n}{\pi(4n^2 - 1)} \right]$, $\omega = 2$.

Exercise 9.3

1. $A(\omega) = 0$, $B(\omega) = \frac{2}{\omega}[1 - \cos(2\omega)]$.
2. $A(\omega) = [2\omega \sin(2\omega) + \cos(2\omega) - 1]/\omega^2$, $B(\omega) = [\sin(2\omega) - 2\omega\cos(2\omega)]/\omega^2$.
3. $A(\omega) = 1/(1 + \omega^2)$, $B(\omega) = \omega/(1 + \omega^2)$.
4. $A(\omega) = 2[(9\omega^2 - 2)\sin(3\omega) + 6\omega\cos(3\omega)]/\omega^3$, $B(\omega) = 0$.