Conditions for a Fourier Expansion

Dirichlet's Conditions

Any function $f(x)$ can be expressed in a Fourier series

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where a_0, a_n, b_n are

Constants, provided,

(i) $f(x)$ is periodic, single valued and finite.
(ii) $f(x)$ has a finite number of discontinuities in any period.
(iii) $f(x)$ has at the most a finite number of maxima and minima in any one period.

Ex 9.1

(34) Obtain the Fourier series expansion of the periodic function $f(x) = e^x, -\pi < x < \pi, f(x+2\pi) = f(x)$. Hence find the sum of the series $\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \cdots \frac{(-1)^n}{1+n^2} + \cdots$

Sol:- $f(x) = e^x, -\pi < x < \pi$

The Fourier expansion of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$

$$\boxed{a_0 = \frac{2\sinh\pi}{\pi}}$$

$$\boxed{\sinh ax = \frac{e^{ax} - e^{-ax}}{2}, \quad \cosh ax = \frac{e^{ax} + e^{-ax}}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$$\boxed{\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + c}$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(\cos n\pi + n\sin n\pi\right) - \frac{e^{-\pi}}{n^2+1}\left(\cancel{\sin}\cos n\pi - n\sin n\pi\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}(-1)^n - \frac{e^{-\pi}}{n^2+1}(-1)^n\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2+1}\left(e^{\pi} - e^{-\pi}\right)\right]$$

x $$\boxed{a_n = \frac{2(-1)^n \sinh \pi}{\pi(n^2+1)}}$$

$$\boxed{\sinh ax = \frac{e^{ax} - e^{-ax}}{2}}$$

$\boxed{x\mathcal{E}}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$\boxed{\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + c}$$

$$= \frac{1}{\pi}\left[\frac{e^x}{n^2+1}\left(\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}\left(-n(-1)^n\right) - \frac{e^{-\pi}}{n^2+1}\left(-n(-1)^n\right)\right]$$

$$b_n = \frac{2}{\pi}\left(\frac{-n(-1)^n}{n^2+1}\right)\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$\boxed{b_n = \frac{-2n(-1)^n}{\pi(n^2+1)} \sinh\pi}$$

$$\boxed{e^x = \frac{\sinh\pi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}\sinh\pi\cos nx - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n \sinh\pi}{n^2+1}\sin nx}$$

$$\Rightarrow e^x = \frac{\sinh\pi}{\pi} + \frac{2}{\pi}\sinh\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx - \frac{2}{\pi}\sinh\pi\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx$$

$$= \frac{\sinh\pi}{\pi} + \frac{2}{\pi}\sinh\pi\left[\frac{-1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x - \frac{1}{3^2+1}\cos 3x + \frac{1}{4^2+1}\cos 4x + ----\right]$$

$$-\frac{2}{\pi}\left[\frac{-1}{1^2+1}\sin x + \frac{1}{2^2+1}(2)\sin 2x - \frac{3}{3^2+1}\sin 3x + \_\_\right]\sinh\pi$$

Put $x=0$.

$$e^0 = \frac{\sinh\pi}{\pi} + \frac{2}{\pi}\sinh\pi\left[\frac{-1}{2} + \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} --\right]$$

$$\Rightarrow 1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\left[\frac{-1}{2} + \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - - -\right]$$

$$\Rightarrow \left(1 - \frac{\sinh \pi}{\pi}\right) \times \frac{\pi}{2\sinh \pi} = \frac{-1}{2} + \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - - - -$$

$$\Rightarrow \left(\frac{\pi - \sinh \pi}{\cancel{\pi}}\right) \frac{\cancel{\pi}}{2\sinh \pi} + \frac{1}{2} = \frac{1}{1+2^2} + \frac{1}{1+3^2} + \frac{1}{1+4^2} - - - -$$

$$\Rightarrow \frac{\pi - \cancel{\sinh \pi} + \cancel{\sinh \pi}}{2\sinh \pi} = \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - - - -$$

$$\Rightarrow \boxed{\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - - - - = \frac{\pi}{2\sinh \pi}}$$

x

$x.$

Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$. ①

Sol :- The required Fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x\, dx$$

$$= \frac{1}{\pi} \left[x(-\cos x) - (1)(-\sin x) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-x \cos x + \sin x \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-2\pi(1) + 0 + 0 - 0 \right]$$

$$= -\frac{2\pi}{\pi} \Rightarrow \boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx\, dx$$

$$\boxed{2 \cos A \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2} \left[\sin(n+1)x - \sin(n-1)x \right] dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin[(n+1)x]\,dx - \frac{1}{2\pi}\int_0^{2\pi} x \sin[(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[x\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[-\frac{x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 2(n+1)\pi}{n+1} + \frac{\sin 2(n+1)\pi}{(n+1)^2} + 0 - 0\right]$$

$$-\frac{1}{2\pi}\left[-\frac{2\pi\cos 2(n-1)\pi}{n-1} + \frac{\sin 2(n-1)\pi}{(n-1)^2}\right]$$

$\cos n\pi = (-1)^n$

$\cos 2(n+1)\pi = (-1)^{2n+2}$

$= [(-1)^2]^n(-1)^2$

$= 1$

$\cos 2(n-1)\pi = (-1)^{2n-2}$

$= 1$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1} + 0\right] - \frac{1}{2\pi}\left[\frac{-2\pi}{n-1} + 0\right]$$

$$= \frac{1}{\cancel{2\pi}}\left(\frac{-\cancel{2\pi}}{n+1}\right) + \frac{1}{\cancel{2\pi}}\left(\frac{\cancel{2\pi}}{n-1}\right)$$

$$= \frac{-1}{n+1} + \frac{1}{n-1} = \frac{-n+1+n+1}{n^2-1} = \frac{2}{n^2-1}$$

$$a_n = \frac{2}{n^2-1}, \quad n \neq 1.$$

When $n=1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\frac{-2\pi}{2}\right] = \frac{-1}{2} \quad \cancel{a_1} \quad \cancel{a_1 = 1}$$

$$\boxed{a_1 = \frac{-1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x(\sin nx\sin x)\,dx$$

$$\boxed{2\sin A\sin B = \cos(A-B)-\cos(A+B)}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(n-1)x-\cos(n+1)x\right)dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\cos[(n-1)x]\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos[(n+1)x]\,dx$$

$$=\frac{1}{2\pi}\left[x\frac{\sin(n-1)}{n-1}+(1)\frac{\cos(n-1)}{(n-1)^2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\frac{\sin(n+1)}{n+1}+\frac{\cos(n+1)}{(n+1)^2}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2}-\frac{1}{(n-1)^2}\right]-\frac{1}{2\pi}\left[\frac{\cos 2(n+1)\pi}{(n+1)^2}-\frac{1}{(n+1)^2}\right]$$

$$b_n = 0,\ n \neq 1.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}(4\pi^2) - \frac{1}{2\pi}\left[0 + \frac{1}{4} - \frac{1}{4}\right]$$

$$= \pi$$

$$\boxed{b_1 = \pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$$

$$x\sin x = -1 + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x$$

$$\boxed{x\sin x = -1 - \frac{\cos x}{2} + \pi\sin x + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx}$$