4.1 Why Learn Calculus

- Calculus is widely used in computer science in the following

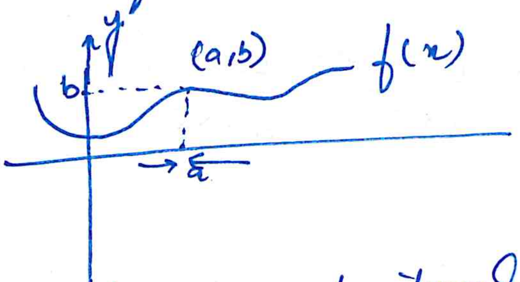
1. Analysis of Algorithms :- Limits, Taylor Series
2. Probability & Statistics :- Expectation → Integration
3. Numerical Optimization :- ML, AI, Robotics → differentiations (Maximum, Minimum)
4. Simulation :- Any physical system, Games → Physics → Calculus.

   ↓ Weather, Nuclear, Turbine, etc

4.2. Limits : An Introduction

→ limit at a point $x = a$ for a function $f(x)$ is written as

$$\lim_{x \to a} f(x).$$



- As x approaches "a" what value does $f(x)$ approach, it need not be equal to $f(a)$.

for example $f(x) = \dfrac{\sin x}{x}$. $f(0) = \dfrac{\sin 0}{0} = \dfrac{0}{0}$ not defined

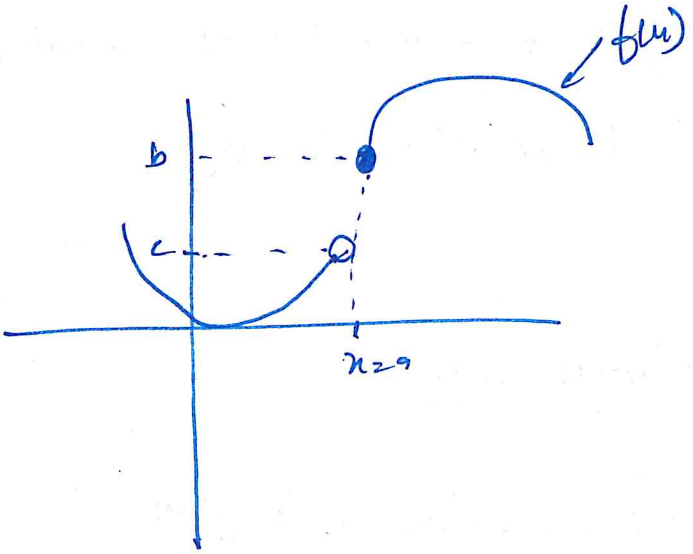
but $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$

Mathematical defination of limits

$$\lim_{x \to x_0} f(x) = L$$

if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

One Side Limits

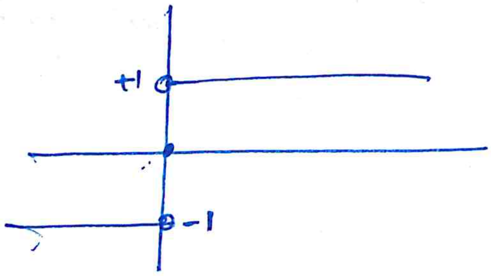


$$\lim_{x \to a^+} f(x) = b$$

$$\lim_{x \to a^-} f(x) = c$$

If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, the limit does not exist @ $x = a$.

eg $f(x) = \text{sign}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$



$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -1.$$

$\therefore \lim_{x \to 0}$ Does not exist.

Why limits are important

In computer science Analysis of Algorithms makes use of limits

$$f(n) = o(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

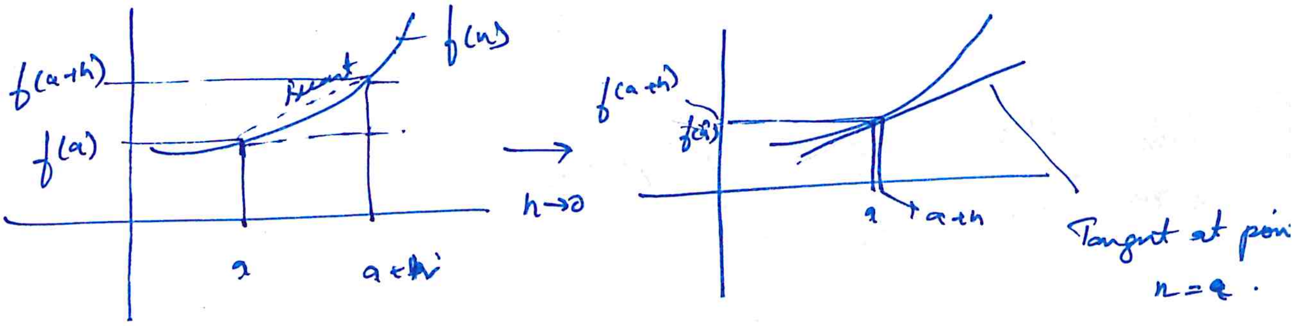
$\uparrow$

little oh notation.

- Many other areas in engineering like Mech, Civil, Physics and also C.S. make use of differentiation which can be defined with the help of limits

$$\left(\frac{df}{dx}\right)_{x=a} = f'(a) = \lim_{\substack{h \to 0 \\ x=a}} \frac{f(x+h) - f(x)}{h}$$

→ Derivative of any function at a point "a" is the slope of the tangent to that function at that point. As $h \to 0$ the secant becomes or reduces to tangent.



↓

slope of secant $= \frac{f(a+h) - f(a)}{h}$.

∴ slope of tangent is given by $\frac{f(a+h) - f(a)}{h}$.

4.3 Limits: Properties of Limits & Indeterminate Forms.

Properties of limits

If $\lim_{x \to a} f(x) = M$

$\lim_{x \to a} g(x) = N$

If the above limits exist, then

① $\lim_{x \to a} (f(x) + g(x)) = M + N.$

② $\lim_{x \to a} (f(x) - g(x)) = M - N$

③ $\lim_{x \to a} f(x) \cdot g(x) = M \cdot N$

④ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{M}{N}$ if $N \neq 0$

⑤ $\lim_{x \to a} f(x)^k = M^k$ (if $M, k > 0$).

⑥ $\lim_{x \to a} c = c$ (if c is a constant).

(7) $\lim_{x \to a} x = a$

(8) Polynomials : $f(x)$, $g(x)$, let $g(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$$

INDETERMINATE FORMS

(1) If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for example $\lim_{x \to 0} \frac{—}{x}$ is of $\frac{0}{0}$ form

then these are called as Quoitent Indeterminate forms. (Can be solved using L'Hopitals Rule).

or $\lim_{x \to 0} \frac{x \cdot \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ = Does not exist.

(2) $0 \cdot \infty$ form or $0 \cdot -\infty$ form. Known as product indeterminate form.

example $\lim_{x \to 0^+} \underset{\to 0}{x} \; \underset{\to -\infty}{\log(x)}$

This form can also be formed using L'Hopitals rule.

(3) $\infty - \infty$ (Subtraction Indeterminate Form)

ex. $\lim_{x \to \infty} \left(\underset{\infty}{\sqrt{x^2+3x+5}} - \underset{\infty}{x} \right)$

(4) Exponential Indeterminate Form.

$0^0, \infty^0, 1^\infty$

Non Indeterminate Forms

(1) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

(2) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ · Does Not Exist.

(3) $\frac{1}{\infty}$ or $\frac{0}{\infty} = 0$.

(4) $\infty \cdot \infty = \infty$.

(5) $0^\infty = 0$

(6) $\infty^\infty = \infty$

(7) $0^{-\infty} = \infty$

(8) $\infty^{-\infty} = \frac{1}{\infty^\infty} = 0$.

4.4 LIMITS – SOLVING LIMITS – I

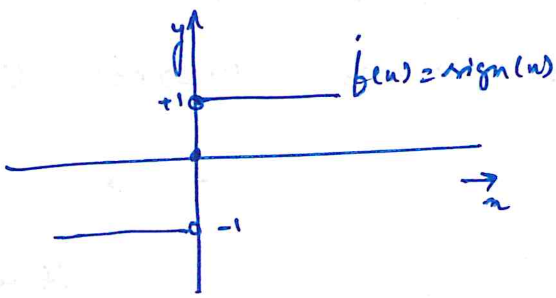
(i) Substitution Method

(a) $\lim\limits_{x \to -1} x^2 = (x^2 \text{ at } x = -1) = 1.$

NOTE * :- Substitution method works only when the function for which the limit is computed is continuous.

(b) $f(x) = \text{sign}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\lim\limits_{x \to 0} \text{sign}(x) = 0$ X incorrect



$\lim\limits_{x \to 0^+} f(x) = 1$

$\therefore$ limit D.N.E (Does not exist).

$\lim\limits_{x \to 0^-} f(x) = -1$

(E) $\lim_{x \to \infty} \frac{1}{x+2}$ as $x \to \infty$, $x+2 \to \infty$.

$$\frac{1}{\lim_{x \to \infty} (x+2)} = \frac{1}{\infty} = 0$$

(2) Tabular & Approximation Method

(a) $\lim_{x \to 2} \frac{(x-2)}{(x^2-4)}$ $\frac{0}{0}$ form

x	1.9	1.99	1.9999	2.0	2.0001	2.01
$f(x)$	0.2564	0.2506	0.25001	Undefined	0.24999	0.2494

$\longrightarrow$ 0.25 $\longleftarrow$

It approaches 0.25.

Note Not always $f(a) = \lim_{x \to a} f(x)$, we need to evalute the limit on both sides and then we can say that limit exists and value can be decided by observing the limits from both the sides if they come as close as possible as the value is approaching a in limit $x \to a$.

3 Factoring Method

(a) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ $\quad \dfrac{0}{0}$ form

$$\lim_{x \to 3} \frac{(\cancel{x-3})(x+3)}{(\cancel{x-3})} = x + 3 .$$

$$\lim_{x \to 3} (x+3) = \underline{\underline{6}}$$

(b) $\lim\limits_{x \to 5} \dfrac{x^2 - x - 20}{x - 5} = \dfrac{(\cancel{x-5})(x+4)}{(\cancel{x-5})}$

$$= \lim_{x \to 5} (x+4)$$

$$= \underline{\underline{9}}$$

(c) $\lim\limits_{h \to 0} \dfrac{(h+4)^2 - 16}{h} = \dfrac{h^2 + \cancel{16} + 8h - \cancel{16}}{h} = \lim\limits_{h \to 0} \dfrac{h^2 + 8h}{h}$

$$= \lim_{h \to 0} \frac{\cancel{h}(h+8)}{\cancel{h}} \Rightarrow \underline{\underline{8}}$$

(d) $\lim_{n\to\infty}\left(\sqrt{n^2+4n}-n\right)$ $\infty-\infty$ form.

Here we use the conjugate

$$\lim_{n\to\infty}\frac{\left(\sqrt{n^2+4n}-n\right)}{1}\frac{\left(\sqrt{n^2+4n}+n\right)}{\left(\sqrt{n^2+4n}+n\right)}$$

$$= \lim_{n\to\infty}\frac{\cancel{n^2}+4n-\cancel{n^2}}{\sqrt{n^2+4n}+n}. = \lim_{n\to\infty}\frac{4\cancel{n}}{\cancel{n}\left(\sqrt{1+4/n}+1\right)}$$

$$= \frac{4}{\sqrt{1+0}+1} = \frac{4}{2} = \underline{\underline{2}}$$

(e) $\lim_{n\to 8}\frac{\sqrt[3]{n}-2}{n-8}$ $\frac{0}{0}$ form.

$$= \lim_{n\to 8}\frac{\cancel{\sqrt[3]{n}-2}}{\cancel{(\sqrt[3]{n}-2)}\left(\sqrt[3]{n^2}+2\sqrt[3]{n}+4\right)}$$

Here $(n-8)$ is factorized as $\left(\sqrt[3]{n}-2\right)\left(\sqrt[3]{n^2}+2\sqrt[3]{n}+4\right)$

as. $(a^3-b^3) = (a-b)(a^2+ab+b^2)$

$$\lim_{n\to 8}\frac{1}{\left(\sqrt[3]{n^2}+2\sqrt[3]{n}+4\right)} = \frac{1}{4+4+4} = \underline{\frac{1}{12}}$$

(f) $\lim_{h \to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$ $\quad \frac{0}{0}$ form.

$\Rightarrow \lim_{h \to 0} \dfrac{\dfrac{x-(x+h)}{(x+h)x}}{h} = \lim_{h \to 0} \dfrac{\dfrac{-\cancel{h}}{(x+h)x}}{\cancel{h}} \Rightarrow \lim_{h \to 0} \dfrac{-1}{(x+h)x}$

$\Rightarrow \dfrac{-1}{x^2}$

4.5. Solving Limits-2

(4) Rationalization Method

- In this method we use the conjugate entity

(a) $\lim_{x \to 0} \dfrac{\sqrt{1+x}-1}{x}$

Conjugate $(\sqrt{1+x}-1)$ is $(\sqrt{1+x}+1)$ multiply and divide by its conjugate

$\lim_{x \to 0} \dfrac{\sqrt{1+x}-1}{x} \times \dfrac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$

$$\Rightarrow \lim_{x\to 0} \frac{\cancel{1} + x - \cancel{1}}{x(\sqrt{1+x}+1)}$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

(b) $\frac{d}{dx} f(x)$; $f(x) = \sqrt{x} = x^{1/2}$

$$\hookrightarrow \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \; \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\Rightarrow \lim_{h\to 0} \frac{(\cancel{x}+\cancel{h}) - \cancel{x}}{\cancel{h} \times \sqrt{x+h} + \sqrt{x}}$$

$$\Rightarrow \frac{1}{2\sqrt{x}}$$

(c) $\lim\limits_{x \to 0} \dfrac{\sec x - 1}{x^2}$

$$= \frac{\sec x - 1}{x^2} \frac{(\sec x + 1)}{(\sec x + 1)}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sec^2 x - 1}{x^2 (\sec x + 1)}$$

$$\Rightarrow \lim_{x \to 0} \frac{\tan^2 x}{x^2 (\sec x + 1)} = \lim_{x \to 0} \frac{\sin^2 x}{x^2 \cos^2 x (\sec x + 1)}$$

$$\Rightarrow \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \left(\frac{1}{\cos x}\right)^2 \cdot \frac{1}{\sec x + 1}$$

As $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

$$\Rightarrow (1)(1)\left(\frac{1}{1+1}\right)$$

$$\Rightarrow \underline{\underline{1/2}}$$

(5) Expansions Method

(a) $\lim_{x \to 0} \frac{(e^x - 1)}{x}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

$$= \lim_{x \to 0} \frac{\left[\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] - \cancel{1}}{x}$$

$$= \lim_{x \to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\cdots$$

$$= \underline{\underline{1}}$$

(b) $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^3}$

expansion for $\sin x$

$$= \lim_{x \to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \cancel{x} - \cancel{x^2}}{x^3}$$

$$\lim_{x \to 0} \frac{\cancel{x^3}\left(-\frac{1}{3!} + \frac{1}{2!}\right) + \cancel{x^4}^{\,x}(\text{—}) + \cancel{x^5}^{\,x^2}(\text{—}) \cdots\cdots}{\cancel{x^3}}$$

$= \cdot -\frac{1}{3!} + \frac{1}{2!}$

$\Rightarrow \frac{1}{2} - \frac{1}{6}$

$\frac{3-1}{6} = \frac{2}{6} = \underline{\underline{\frac{1}{3}}}$

4.6 L'Hôpital's Rule

L'Hopital's Rule – If we have two functions such that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (OR) $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

then if ① $\lim_{x \to a} \frac{f(x)}{g(x)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty}$ or $\frac{-\infty}{\infty}$ form.

i.e. $\boxed{\pm\frac{\infty}{\infty}}$

and ② $g'(x) \neq 0$ and $f'(x)$ and $g'(x)$ exists, and

③ $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(a) $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \underset{x \to 0}{\mathrm{lt}} \frac{\cos x}{1} = \underline{1}$$

(L Hopitals rule)

(b) $$\lim_{x \to 0} \frac{\sqrt{x+9} - 3}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{x+9}}}{1} = \frac{1}{2.3} = \frac{1}{6}.$$

Reference link for standard formulae for derivatives of standard functions https://en.wikipedia.org/wiki/Differentiation_rules

(c) $$\lim_{x \to 0} \frac{e^x - 1}{x^2 + x} - \frac{0}{0} \text{ form}.$$

$$\Rightarrow \underset{x \to 0}{\mathrm{lt}} \frac{e^x}{2x+1} = \frac{1}{1} = 1$$

(d) $$\lim_{x \to \infty} x^n e^{-x} \quad (n=2) \; . = \frac{x^n}{e^x} = \frac{\infty}{\infty} \text{ form}.$$

$$\Rightarrow \lim_{x \to \infty} \frac{n x^{n-1}}{e^x} \; . \; \frac{\infty}{\infty} \text{ form again}$$

$$\Rightarrow \underset{x \to \infty}{\mathrm{lt}} \frac{n(n-1)\, x^0}{e^x} = \frac{2.1}{e^x} \Rightarrow \underline{\underline{0}}$$

(e) $\lim_{x\to 0^+} x \ln x$ $\quad 0.\infty$ form.

$\Rightarrow \lim_{x\to} \frac{\ln x}{\frac{1}{x}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$\Rightarrow \lim_{x\to 0^+} \frac{\frac{1}{\cancel{x}}}{-\frac{1}{x\cancel{x}}} = \lim_{x\to 0^+} -x = \underline{\underline{0}}$

(f) $\lim_{x\to 0} \frac{1-\cos x^2}{x^4}$

$= \lim_{x\to 0} \frac{-(\cancel{2x})(-\sin x^2)}{\cancel{4}\,x^{\cancel{3}\,2}}$

2

$\Rightarrow \lim_{x^2\to 0} \frac{\sin x^2}{x^2}$

$y = x^2$

$\lim_{y\to 0} \frac{\sin y}{y} = \underline{\underline{1}}$

Proof for L'Hopitals rule can be found at https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule#General_proof.

OR Google for "L Hopitals rule Wiki"

Special Case

(1) $\lim_{x \to \infty} \frac{x + \sin x}{x}$ = $\lim_{x \to \infty} \frac{1 + \cos x}{1}$ → oscilates in between [0, 2].

$\frac{\infty}{\infty}$ form

∴ This limit does not exist

(2) $\lim_{x \to \infty} \sin x$ = Does not exist as $\sin x$ oscilates in between [-1, 1].

(3) $\lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$ $\frac{\infty}{\infty}$ form.

Applying L'hopitals rule.

$\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\frac{\infty}{\infty}$

Applying L'hopitals rule again.

we get back. $\lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Applying change of variables

$$y = e^x$$

$x \longrightarrow \infty \quad e^x \longrightarrow \infty \quad y \longrightarrow \infty.$

$$\lim_{y \to \infty} \frac{y + \frac{1}{y}}{y - \frac{1}{y}} = \lim_{y \to \infty} \frac{1 + y^{-2}}{1 - y^{-2}} = \frac{1}{1} = 1$$

4.7 STANDARD LIMITS

Reference link : https://en.wikipedia.org/wiki/List_of_limits

If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} g(x) = L_2$ then

1. $\lim_{x \to c} |f(x) \pm g(x)| = L_1 \pm L_2$

2. $\lim_{x \to c} |f(x) \cdot g(x)| = L_1 \times L_2.$

3. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$

4. $\lim_{x \to c} f(x)^n = L_1^n$, if n is a positive integer.

5. $\lim_{x \to c} f(x)^{\frac{1}{n}} = (L_1)^{\frac{1}{n}}$.

6. $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm \infty$ (L'Hopital's Rule).

Polynomials and functions of the form x^a

1. $\lim_{x \to c} a = a$.

2. $\lim_{x \to c} x = c$.

3. $\lim_{x \to c} x^n = c^n$

4. $\lim_{x \to \infty} \frac{x}{a} = \begin{cases} \infty & \text{if } a > 0 \\ \text{does not exist}, & a = 0 \\ -\infty, & a < 0. \end{cases}$

5. If $p(x)$ is a polynomial then by the continuity of polynomials

$$\lim_{x \to c} p(x) = p(c)$$

Functions of the form x^a.

1. $$\lim_{x \to c} x^a = c^a$$

2. $$\lim_{x \to \infty} x^a = \begin{cases} \infty, & a > 0 \\ 1, & a = 0 \\ 0, & a < 0 \end{cases}$$

3. $$\lim_{x \to c} x^{1/a} = \lim_{x \to \infty} \sqrt[a]{x} = \infty \text{ for any } a > 0.$$

4. $$\lim_{x \to 0^+} x^{-n} = \lim_{x \to 0^+} \frac{1}{x^n} = +\infty$$

5. $$\lim_{x \to 0^-} x^{-n} = \lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even.} \end{cases}$$

6. $$\lim_{x \to \infty} a x^{-1} = \lim_{x \to \infty} \frac{a}{x} = 0 \text{ for any real } a.$$

Exponential Functions

Functions of the form $a^{g(x)}$.

1. $$\lim_{x \to c} e^x = e^c$$

2. $$\lim_{n \to \infty} a^{x} = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & 0 < a < 1 \end{cases}$$

3. $$\lim_{x \to \infty} a^{-x} = \begin{cases} 0, & a > 1 \\ 1, & a = 1 \\ \infty, & 0 < a < 1 \end{cases}$$

4. $$\lim_{n \to \infty} \sqrt[n]{a} = \lim_{n \to \infty} a^{1/n} = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ \text{does not exist}, & a < 0. \end{cases}$$

- Functions of the form $f(n)^{g(n)}$

1. $$\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} (n)^{1/n} = 1$$

- Functions of the form $f(n)^{g(n)}$

1. $$\lim_{n \to \infty} \left(\frac{n}{n+k} \right)^{n} = \frac{1}{e^{k}}.$$

2. $(1+x)^{\frac{1}{x}} = e$

3. $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$

4. $\lim\limits_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$

5. $\lim\limits_{x \to \infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$.

6. $\lim\limits_{x \to 0} \left(1 + a(e^{-x} - 1)\right)^{-\frac{1}{x}} = e^{a}$

Sums, Products & Composites

1. $\lim\limits_{x \to 0} x e^{-x} = 0$

2. $\lim\limits_{x \to \infty} x e^{-x} = 0$

3. $\lim\limits_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \ln a \quad \forall\, a > 0.$

Logarithmic Functions

$\lim_{x \to c} \ln x = \ln c$, due to the continuity of $\ln x$.

1. $\lim_{x \to 0^+} \log x = -\infty$

2. $\lim_{x \to \infty} \log x = \infty$

3. $\lim_{x \to 1} \frac{\ln(x)}{x-1} = 1$

4. $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$

5. $\lim_{x \to 0} \frac{-\ln(1+a(e^{-x}-1))}{x} = a$ (By using L'Hopitals rule).

6. $\lim_{x \to 0} x \ln x = 0$

7. $\lim_{x \to \infty} \frac{\ln x}{x} = 0.$

Logarithms to arbitary bases

For $a > 1$

1. $\lim\limits_{x \to 0^+} \log_a x = -\infty$

2. $\lim\limits_{x \to \infty} \log_a x = \infty$

For $a < 1$

3. $\lim\limits_{x \to 0^+} \log_a x = \infty$

4 $\lim\limits_{x \to \infty} \log_a x = -\infty$

Trigonometric Functions

If x is expressed in radains

1. $\lim\limits_{x \to a} \sin x = \sin a$.

2. $\lim\limits_{x \to a} \cos x = \cos a$.

As $\sin x$ and $\cos x$ are continuous functions.

3. $\lim\limits_{x \to 0} \dfrac{\sin x}{ax} = 1 \quad (a \neq 0)$

4. $\lim\limits_{x \to 0} \dfrac{\sin ax}{x} = a \quad (a \neq 0)$

3. $\lim_{x \to 0} \dfrac{\sin ax}{bx} = \dfrac{a}{b} \quad (a, b \neq 0)$

4. $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = 1$

5. $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

6. $\lim_{x \to 0} \dfrac{1 - \cos x}{\cos^2 x} = \dfrac{1}{2}.$

4. $\lim_{x \to n^{\pm}} \tan\left(\pi x + \frac{\pi}{2}\right) = \mp \infty$ for integer n.

5. $\lim_{n \to \infty} \underbrace{\sin \sin \sin \cdot \sin(x_0)}_{n \text{ times}} = 0$ where x_0 is an arbitrary real number.

6. $\lim_{n \to \infty} \cos \cos \cdots \cos(x_0) = d$, where d is Dottie number, x_0 is any real number.

4.8 Solved Problems

Q1) $\lim_{x \to 0} \dfrac{x(e^x - 1) + 2(\cos x - 1)}{x(1 - \cos x)} = \_\_\_?$

$$\Rightarrow \quad \frac{0(1-1)+2(1-1)}{0(1-1)} = \frac{0}{0} \text{ form.}$$

Applying L'Hospitals rule.

$$\Rightarrow \lim_{x\to 0} \frac{(e^x-1)+x(e^x)+2(-\sin x)}{(1-\cos x)+x\sin x} \quad \frac{0}{0} \text{ form.}$$

$\Rightarrow$ Applying L' Hospitals rule again

$$\lim_{x\to 0} \frac{xe^x+e^x+e^x-2\cos x}{x\cos x+\sin x+\sin x} \quad \frac{0}{0} \text{ form}$$

Applying L' Hospitals rule

$$\lim_{x\to 0} \frac{xe^x+e^x+e^x+e^x+2\sin x}{x\sin x+\cos x+\cos x+\cos x} = \frac{3}{3} = 1$$

Q2 $\lim_{x \to \infty} \dfrac{x - \sin x}{x + \cos x} = \_\_\_\_$.

A. 1 B. -1 C. ∞ D = $-\infty$

÷ Numerator and Denominator by x .

$$\lim_{x \to \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} \Rightarrow \frac{1-0}{1+0} \Rightarrow 1 .$$

Q3 What is the value of $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{2n}$

A. 0

B. $= e^{-2}$

C. $e^{-1/2}$

D. 1

solution

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{2n}$$

1^{∞} Indeterminate form .

$\lim_{x \to c} f(x)^{g(x)}$ is 1^{∞} form we can solve using $\underline{e^{\lim_{x \to a} g(x) \times (f(x) - 1)}}$

$$e^{\lim_{n\to\infty} 2n\left(1-\frac{1}{n}-1\right)} \Rightarrow \underline{\underline{e^{-2}}}$$

Q 4

$$\lim_{x\to\infty} x^{\frac{1}{x}}.$$

A. ∞

B. 0

C. 1

D. Not defined

solution

let $m = \lim_{x\to\infty} x^{\frac{1}{x}}$

$$\log m = \lim_{x\to\infty} \frac{1}{x}\log x$$

$$\log m = \lim_{x\to\infty} \frac{\log x}{x}.$$

As we know $\log n$ grows very slow when compared to x

therefore $\lim\limits_{n\to\infty} \dfrac{\log x}{n} = 0$.

$\log m = 0$

$\Rightarrow m = 1$

$\therefore$ C is the correct option

5.

$\lim\limits_{n\to\infty} (1+n^2)^{e^{-n}}$

A $= 0$

B. $1/2$

C 1

D ∞

let us take

$y = \lim\limits_{n\to\infty} (1+n^2)^{e^{-n}}$

$\log y = \lim\limits_{n\to\infty} e^{-n} \log(1+n^2)$

$$\log y = \lim_{x \to \infty} \frac{\log(1+x^2)}{e^x}$$

= We know that e^x is an exponential function and it grows very fast when compared to polynomial function $(1+x^2)$

$$\therefore \lim_{x \to \infty} \frac{\log(1+x^2)}{e^x} = 0$$

$\log y = 0$

$y = 1$ c is the correct option.

6 $\lim_{x \to 4} \frac{\sin(x-4)}{(x-4)} = \_\_\_\_$

take $y = (x-4)$

$$= \lim_{y \to 0} \frac{\sin y}{y} = 1$$

Q7 The value of the limit

$$\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}.$$

A. is 0

B. is -1

C. is 1

D. does not exist

Soln.

$$\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \frac{0}{0} \text{ form } \quad \text{L'Hopital's rule}$$

$$= \lim_{x \to 1} \frac{7x^6 - 10x^4}{3x^2 - 6x} = \frac{7 - 10}{3 - 6} \Rightarrow \frac{-3}{-3} = \underline{1}$$

4.9 Continuity: An Introduction

Defination 1 :- A function $f(x)$ is continious at $x = a$ if and only if

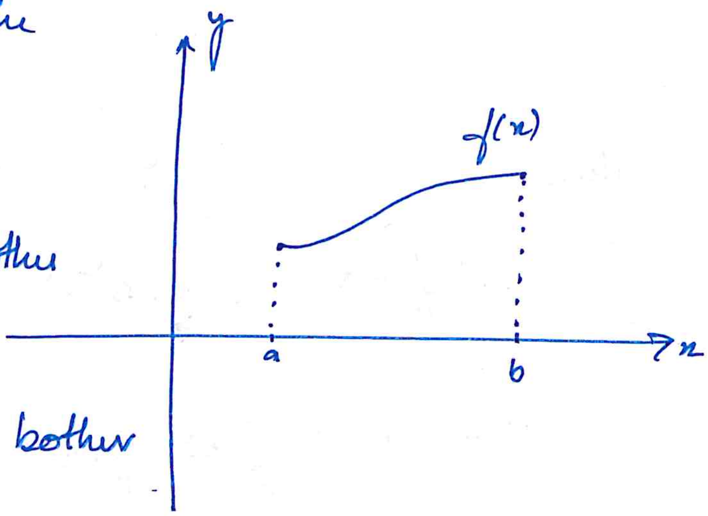
1. $f(a)$ exists

2. $\lim_{x \to a} f(a)$ exists (Which means $\lim_{x\to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$)

3. $\lim_{x \to a} f(x) = f(a)$.

Defination 2. A function $f(x)$ is continous [a, b] if $f(x)$ is continious at all points in the given interval. At the

We do not have to bother at the extreme points $x = a, b$.

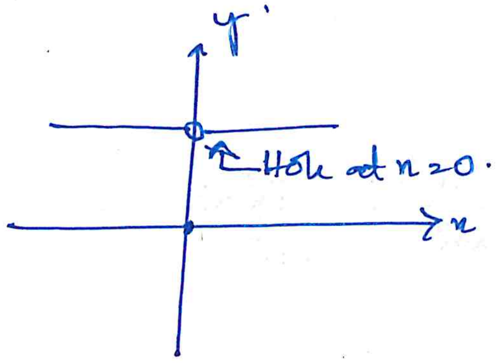


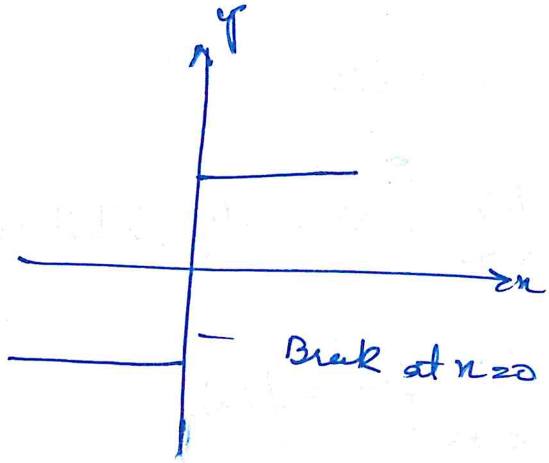
At $x = a$ we don't have to bother about the left hand limit and at $x = b$ we do not have to bother about the right hand limit.

i.e we do not need to worry about $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$.

Properties of Continuities.

- Intuitively :- The graph for the function does not have no breaks and no holes.





→ If f and g are continous functions $f : \mathbb{R} \to \mathbb{R}$.

$g : \mathbb{R} \to \mathbb{R}$.

then

1. $f+g = f(x)+g(x)$
2. $f-g = f(x)-g(x)$
3. $f*g = f(x) \times g(x)$
4. $f/g = \dfrac{f(x)}{g(x)}$ if $g(x) \neq 0$.
5. $f \circ g = f(g(x))$

Are all continous functions.

(a)

$$f(x) = \begin{cases} 2x+1 & x<3 \\ 3x-2 & x \geq 3 \end{cases} \quad \forall x \in \mathbb{R}$$

Note :- All polynomials are continous in nature.

$2x+1$ and $3x-2$ are continous we need to check for $x=3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} 2x+1 = 7.$$

$$\lim_{x\to 3^+} f(x) = 3x-2 = 7.$$

$\therefore \lim_{x\to 3} f(x) = 7.$ $\therefore f(3) = 7.$ $\therefore$ The fn $f(x)$ is continous $\forall x \in \mathbb{R}$.

(b)

$$f(x) = \begin{cases} x+1 & x<2 \\ x^2 & x=2 \\ 2x-1 & x>2. \end{cases} \quad x \in \mathbb{R}$$

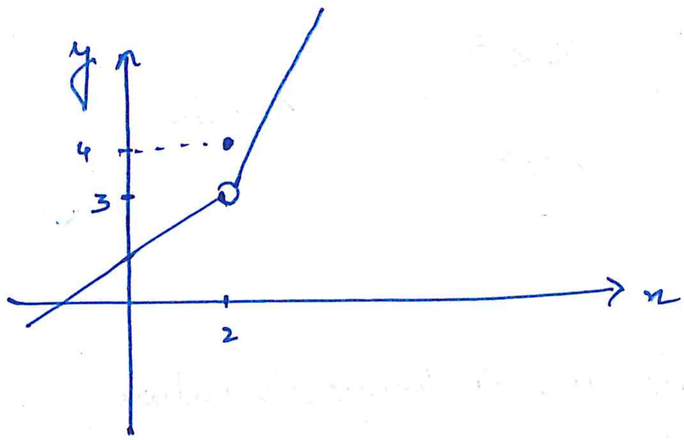
A $x=2$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1 = 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-1. = 3$$

$\lim_{x\to 2} f(x) = 3$

$f(2) = 4$, $\therefore \lim_{x\to 2} f(x) \neq f(2)$



The graph looks like this. there is a hole at $x=2$.

(c)

$$f(x) = \begin{cases} -\cos(x) & x < 0 \\ e^x - 2 & x \geq 0 \end{cases} \qquad x \in \mathbb{R}$$

$\cos x$ is a continuous fn also $e^x - 2$ is a continuous function,

we need to check @ $x=0$ how $f(x)$ is i.e. continous or discontinous.

$$\lim_{x \to 0^-} f(x) = -1$$

$$\lim_{x \to 0^+} f(x) = -1$$

$$\therefore \lim_{x \to 0} f(x) = -1.$$

$$f(0) = -1$$

(d) Modulus function, Absolute value function.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

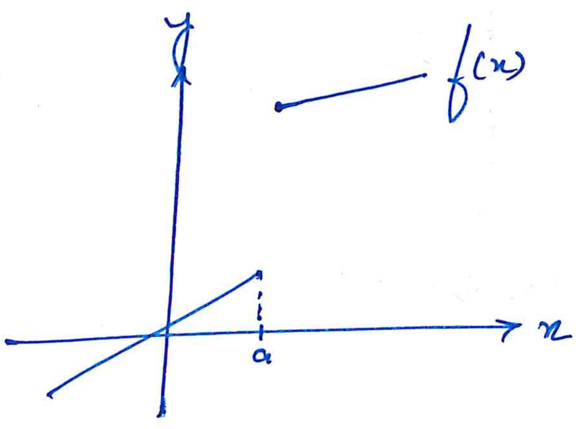
$$\left.\begin{aligned} \lim_{x \to 0^+} f(x) &= 0 \\ \lim_{x \to 0^-} f(x) &= 0 \end{aligned}\right\} \lim_{x \to 0} f(x) = 0.$$

$f(0) = 0.$

$\therefore$ The fn. is continous at $x = 0$.

4.10 Discontinuities.

(1) Jump Discontinuity.

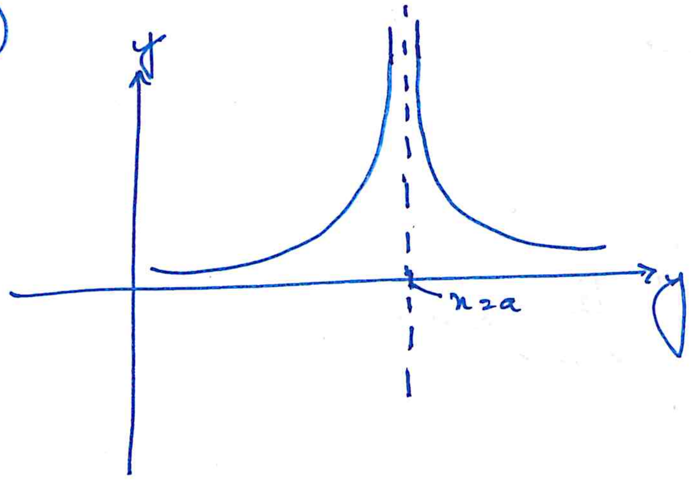


$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x).$$

RHL LHL

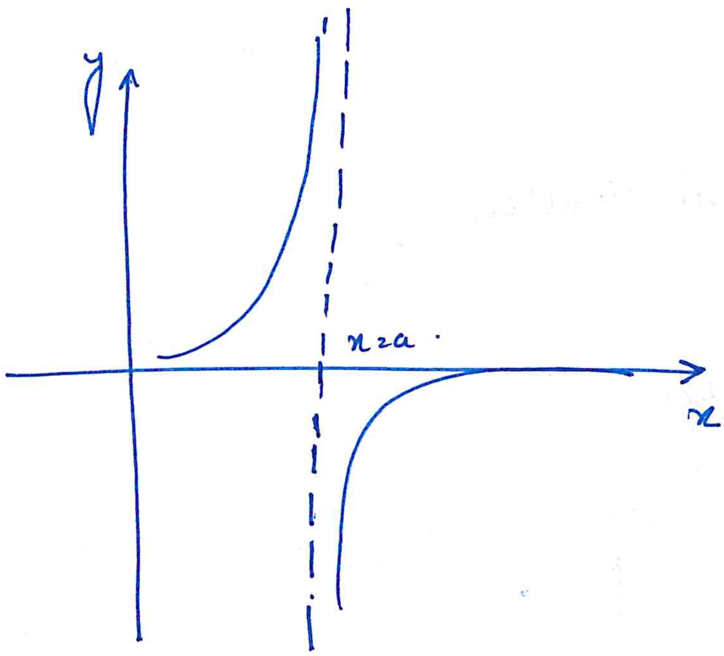
② Infinite discontinuity   3 types

①



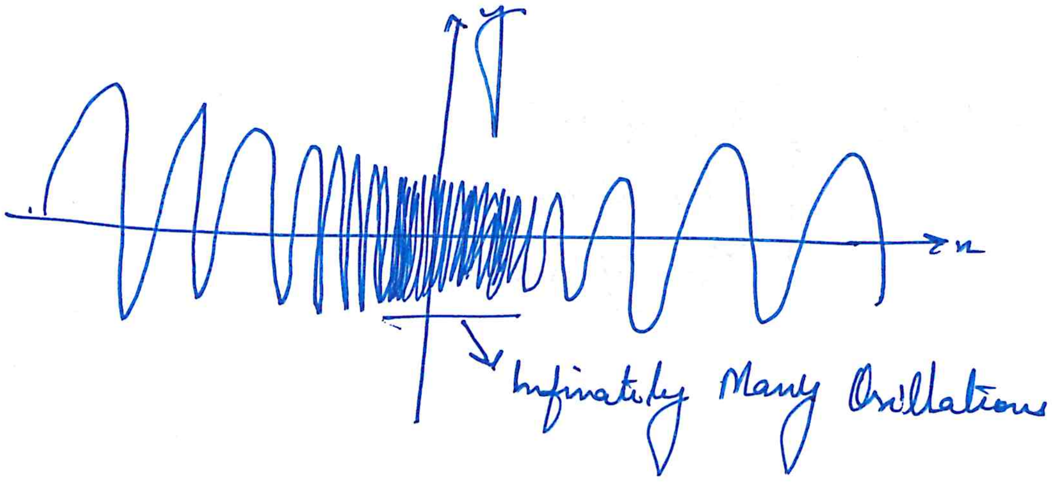
f(a) is not finite

②



f(a) is not finite

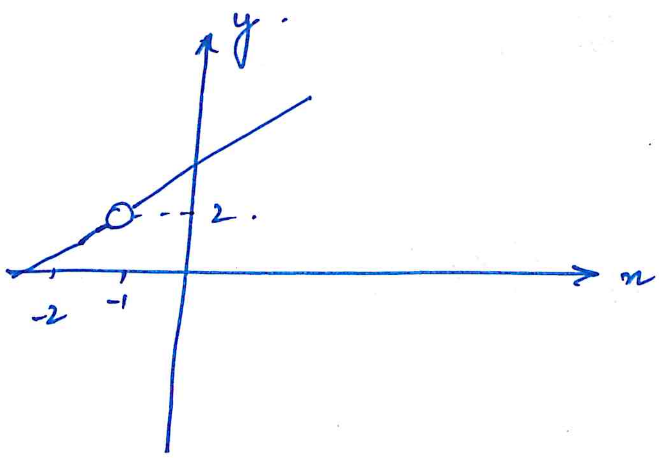
③ $f(x) = \sin\left(\frac{1}{x}\right)$ Infinitely Many Oscillations at $x = 0$.



② Removable Discontinuities:

$$f(x) = \frac{(x+3)(x+1)}{(x+1)}$$

At $x = 1$ Denominator $= 0$



At $x = -1$ if we define $f(-1)$ as 2 our function $f(x)$ will become continous.

Q1) Which of the following functions are continous at $x=3$?

A. $f(x) = \begin{cases} 2, & \text{if } x=3 \\ x-1, & \text{if } x>3 \\ \dfrac{x+3}{3} & \text{if } x<3. \end{cases}$

B. $f(x) = \begin{cases} 4, & \text{if } x=3 \\ 8-x, & \text{if } x\neq 3. \end{cases}$

C. $f(x) = \begin{cases} x+3 & \text{if } x<=3 \\ x-4 & \text{if } x>3. \end{cases}$

D. $f(x) = \begin{cases} \dfrac{1}{x^3-27} & \text{if } x\neq 3 \end{cases}$

Solution For continuous $LHL = RHL = f(3)$.

A. $f(3)=2$. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3} \dfrac{x+3}{3} = 2$; $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} x-1=2$.

$LHL = RHL = f(3)$ ∴ A is the correct option.

Q2

A function $f(x)$ is continuous in the interval $[0, 2]$.

It is known that $f(0) = f(2) = -1$ and $f(1) = 1$. Which of the following statements must be true?

A. There exists a y in the interval $(0,1)$ such that $f(y) = f(y+1)$

B. For every y in the interval $(0,1)$, $f(y) = f(2-y)$.

C. The maximum value of the function in the interval $(0,2)$ is 1.

D. There exist a y in the interval $(0,1)$ such that $f(y) = -f(2-y$

Solution.

By looking at the 1st option

Let us define a function $g(x) = f(y) - f(y+1)$

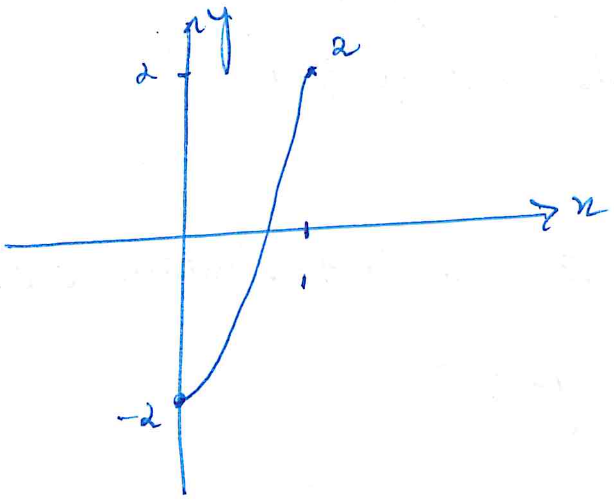
Since $f(x)$ is continuous in $[0,2]$ $g(x)$ is continuous in $(0,1)$.

as if f, g are continuous then $f-g$ is also a continuous function.

$g(0) = f(0) - f(1) = -1 - (1) = -2$

$g(1) = f(1) - f(2) = 1 - (-1) = 2.$

If we try to plot $g(x)$



It may be of different shape but it has to cross 0 and $g(x)$ will be $= 0$ at some point.

$$g(x) = 0 = f(y) - f(y+1) = 0$$

$f(y) = f(y+1)$ option 1 is correct.

(Q3) Let $f(x) = x^{-\frac{1}{3}}$ and A is the area at the region bounded by $f(x)$ and the x-axis when x varies from -1 to 1.

Which of the following statements is/are TRUE?

I. f is continous in $[-1, 1]$

II. f is not bonded in $[-1, 1]$

III. A is nonzero and finite

A. II only.

B. III only.

C. II and III only.

D. I, II and III.

Solution

I. $f(x) = x^{-1/3} = \frac{1}{\sqrt{x}}$

At $x=0$ the function is not defined.

$\therefore$ $f(x)$ is not continuous in $[-1,1]$.

II $f(x)$ is not bounded in $[-1,1]$.

As $x \to 0$ $\frac{1}{\sqrt{x}} \to \infty$ $\therefore$ $f(x)$ is not bounded.

Statement II is true.

III A is the area can be calculated by using definite integral $\int_{-1}^{1} f(x)\,dx$

which is finite. $\therefore$ option C is correct.

4.12 DIFFERENTIABILITY

Two ways we can discuss about differentiability

1. Mathematical (Rigourous)
2. Geometric way.

1. Mathematical Approach :-

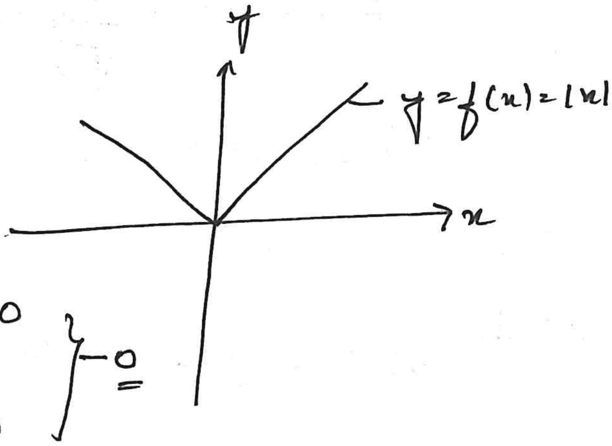
A function $f(x)$ is differentiable (if $f: \mathbb{R} \to \mathbb{R}$) at $x = a$

if $f'(a)$ exists i.e. $\lim\limits_{h \to 0} \frac{(f(a+h) - f(a))}{h}$ exists

Note :- 1. If f is differentiable at $x = a$ then f is continous at $x = 0$.

2. If f is continous at $x = a$ then f need not be differentiable at $x = a$.

Example $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



At $x = 0$

$$\lim_{x \to 0} f(x) = \left\{ \begin{array}{l} \lim\limits_{x \to 0^+} |x| = 0 \\ \lim\limits_{x \to 0^-} |x| = 0 \end{array} \right\} = \underline{\underline{0}}$$

limit exists @ $x=0$ $f(0) = 0$. and the fn is continous at $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

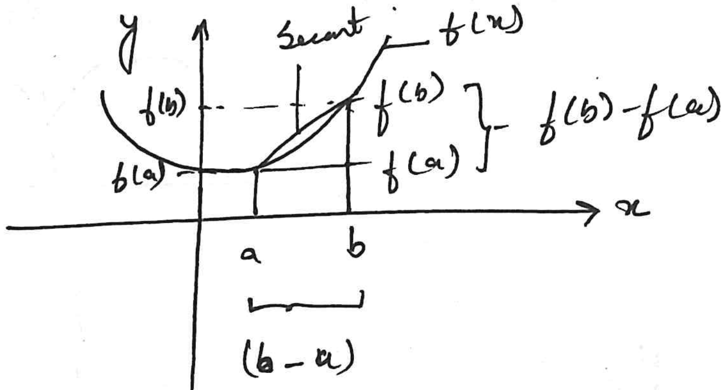
Left hand limit $\lim_{h \to 0^+} \frac{f(h)}{h} = \frac{h}{h} = 1$

Right hand limit $\lim_{h \to 0^-} \frac{-f(h)}{h} = \frac{-h}{h} = -1$

$\therefore$ LHL $\neq$ RHL. $\therefore$ limit does not exist at $x = 0$.

$\therefore f'(x)$ does not exist at $x = 0$ and $f(x)$ is not differentiable at $x = 0$.

Geometrically what does derivative mean?



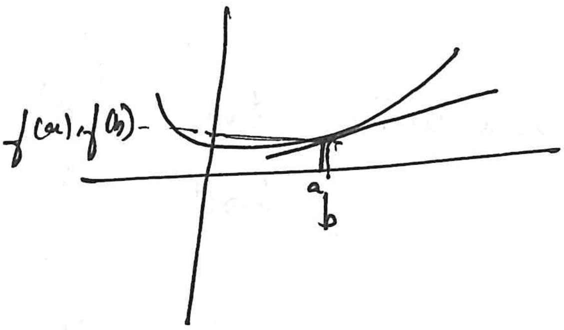
Slope of the secant $= \frac{f(b) - f(a)}{b - a}$ Let $h = b - a$ then.

$$= \frac{f(h+a) + f(a)}{h}$$

As point b approaches point a the secant reduces to a tangent

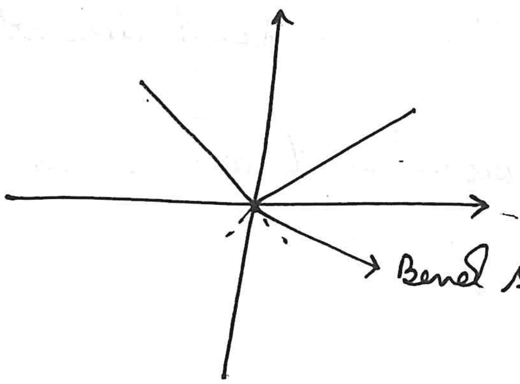
Secant $\longrightarrow$ tangent

$$\text{Slope of tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \underline{\underline{f'(a)}}$$

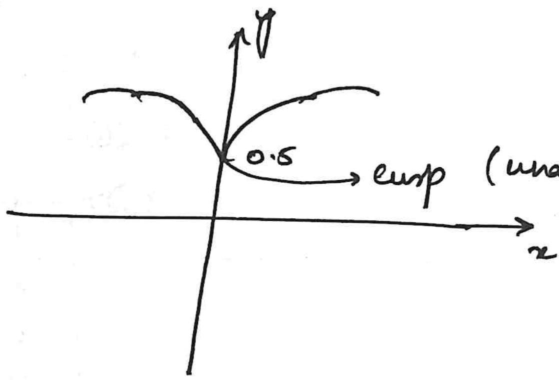


Non-differentiability

(a) $f(x) = |x|$

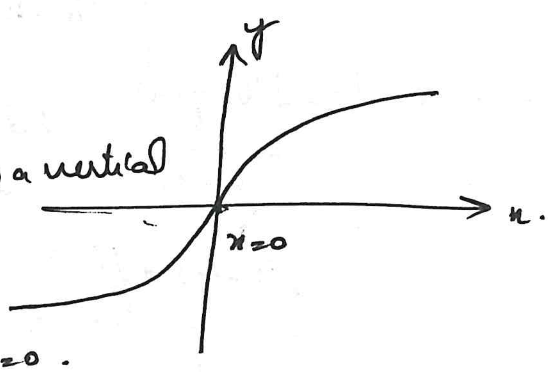


(b) $f(x) = \sqrt{|x|} + 0.5$



(c) $f(x) = \sqrt[3]{x}$

At $x=0$ the tangent is a vertical line, i.e undefined slope ∴ fn. is not differentiable at $x=0$.



4.13 DIFFERENTIABILITY

SOLVED PROBLEMS

Q1) The function $f(x) = |2 - 3x|$

A. is continous $x \in \mathbb{R}$ and differentiable $\forall x \in \mathbb{R}$.

B. is continous $x \in \mathbb{R}$ and differentiable $\forall x \in \mathbb{R}$ except $x = 3/2$

C. is continous $x \in \mathbb{R}$ and differentiable $\forall x \in \mathbb{R}$ except $x = 2/3$

D. is continous $x \in \mathbb{R}$ except $x = 3$ and differentiable $x \in \mathbb{R}$.

Soln.

$$f(x) = |2x - 3| = \begin{cases} (2 - 3x) & x \geq 2/3 \\ -(2 - 3x) & x < 2/3 \end{cases}$$

LHL at $x = 2/3$ $\quad \lim_{x \to 2/3^-} f(x) = -(2x - 3) = 0$.

R.HL @ $x = 2/3$ $\quad \lim_{x \to 2/3^+} f(x) = (2x - 3) = 0$.

$f(2/3) = 0$.

$\therefore$ The fn is continous @ $x = 2/3$

Right hand derivative at $x = 2/3$ $\quad \lim_{x \to 2/3^+} = +3$, LHD $\lim_{x \to 2/3^-} = +3$.

$\therefore$ fn is not differentiable at $x = 2/3$ $\therefore$ C is the correct option.

(Q2) Consider the function $y = f(x) = |x|$ in the interval $[-1, 1]$. In this interval, the function is

A. Continuous and differentiable

B. Continuous but not differentiable

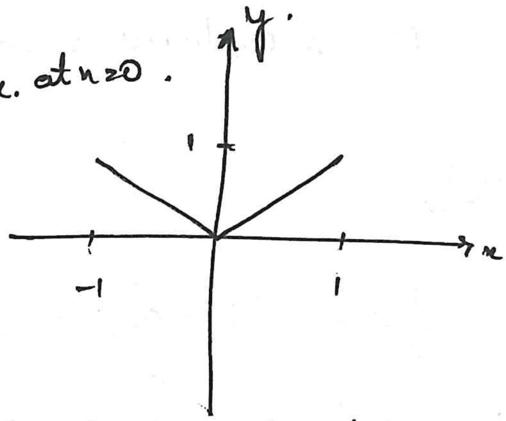
C. differentiable but not continuous.

D. neither continuous nor differentiable

Soln

in the interval $[-1, 1]$ $f(x) = |x|$ looks like the below plot

$|x|$ is continuous but not differentiable at $x=0$.



(Q3) Consider the following two statements about the function $f(x) = |x|$

P. $f(x)$ is continuous for $\forall x \in \mathbb{R}$.

Q $f(x)$ is differentiable for $\forall x \in \mathbb{R}$.

Which of the following is true?

A. P is true & Q is false.

B. P is false & Q is true.

C. Both P and Q are true

D. Both P and Q are false

Solution

We know that $|x|$ is continuous $\forall x \in \mathbb{R}$ but it is not differentiable at $x=0$. Therefore A is the correct option.

Q4

The function $f(x) = x \sin x$ satisfies the following equation

$f''(x) + f(x) + t \cos x = 0$. The value of t is ___.

Solution

$$f'(x) = \sin x + x \cos x$$

$$f''(x) = \cos x - x \sin x + \cos x$$

$$= 2\cos x - x \sin x.$$

$$f''(x) + \underbrace{x \sin x}_{f(x)} \underbrace{- 2\cos x}_{t} = 0$$

$$\underline{\underline{t = -2}}.$$

Q5

Let $f(x)$ be a polynomial and $g(x) = f'(x)$ be its derivative. If the degree of $(f(x) + f(-x))$ is 10 then the degree of $g(x) - g(-x)$ is ___.

Lets assume $f(x)$ is of degree 10

$f(x) = kx^{10}$ $\quad$ $f(-x) = k(-x)^{10} = kx^{10}$

$f(x) + f(-x) = 2kx^{10}$.

$g(x) = 10kx^9$

$g(-x) = 10k(-x)^9$

$= -10k(x)^9$

$g(x) - g(-x) = 10kx^9 - (-10k(x^9))$

$= 20kx^9$ is of degree 9

4.14 MEAN VALUE THEOREM:

- Mean Value theorems are used for.

    1. Proving other theorems :- 1. Taylor Series.

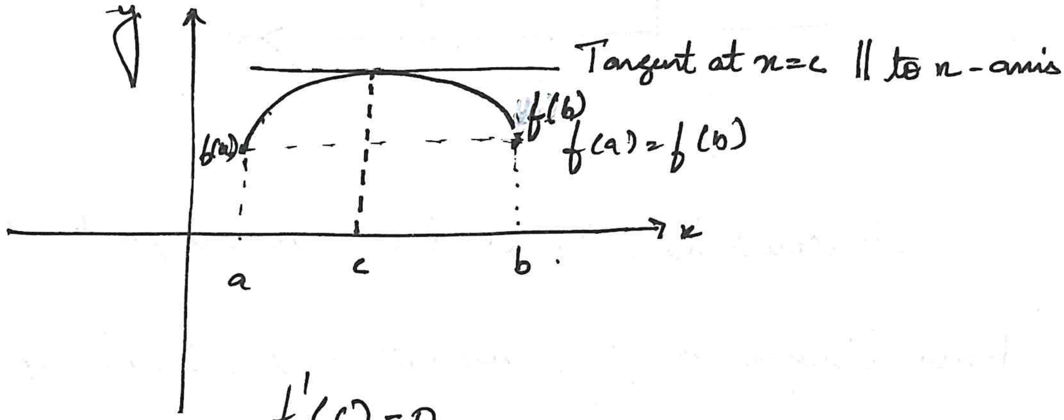
       2. L'Hopitals Rule.

    2. Real World Applications : example track speeding cars.

1. ROLLE'S THEOREM

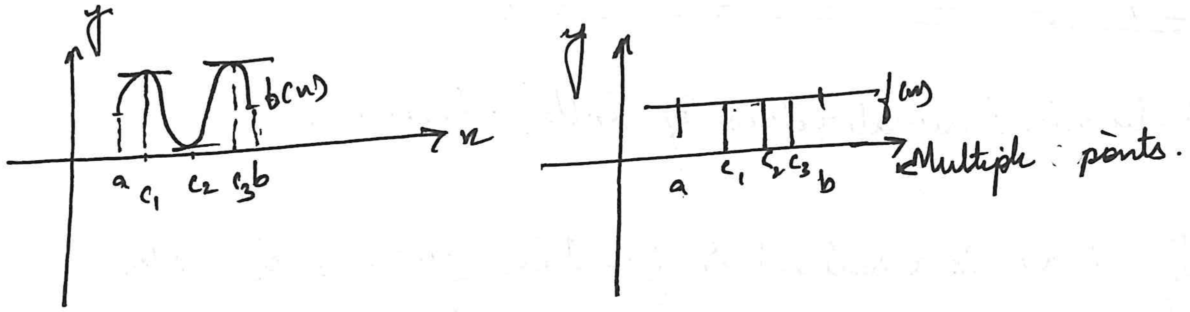
If $f(x)$ is a realvalued continous in $[a,b]$ and differentiable in (a,b) then $\exists\, c \in (a,b)$ such that $f'(c)=0$.

Geometrically

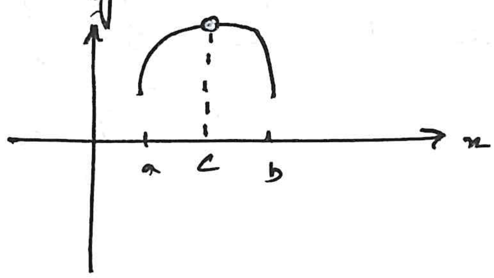


$f'(c)=0$

Note It could also be possible to have multiple such points having $f'(x)=0$.



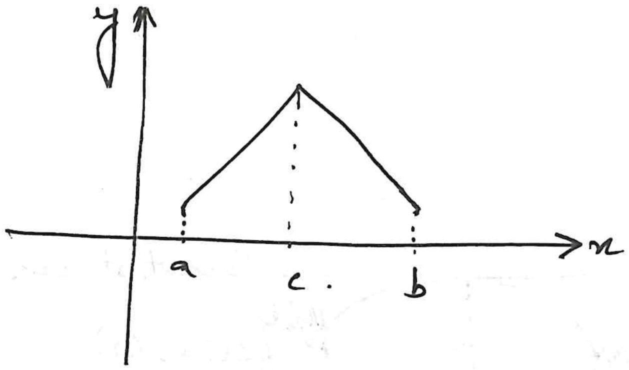
- What happens if the function is not continous?



There could be a hole at the point where $f'(x)$ is going to be $=0$, then we will have no such point c.

What happens when the function is not differentiable?

consider the following example

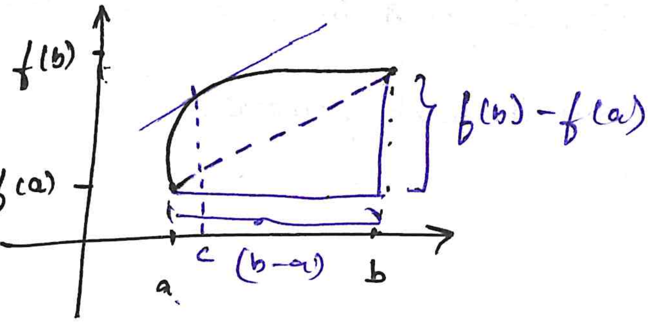


We could have the point $x = c$ as the point where we cannot draw a tangent or $f'(c)$ does not exist / is not defined.

② Lagrange's Mean Value Theorem / MVT

- Extension / Generalization of Rolle's theorem.
- If $f(x)$ is a real valued function $f(x): [a,b] \to \mathbb{R}$
  1. Continous in $[a,b]$
  2. Differentiable in (a,b).

then MVT states that $\exists\, c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{(b-a)}$



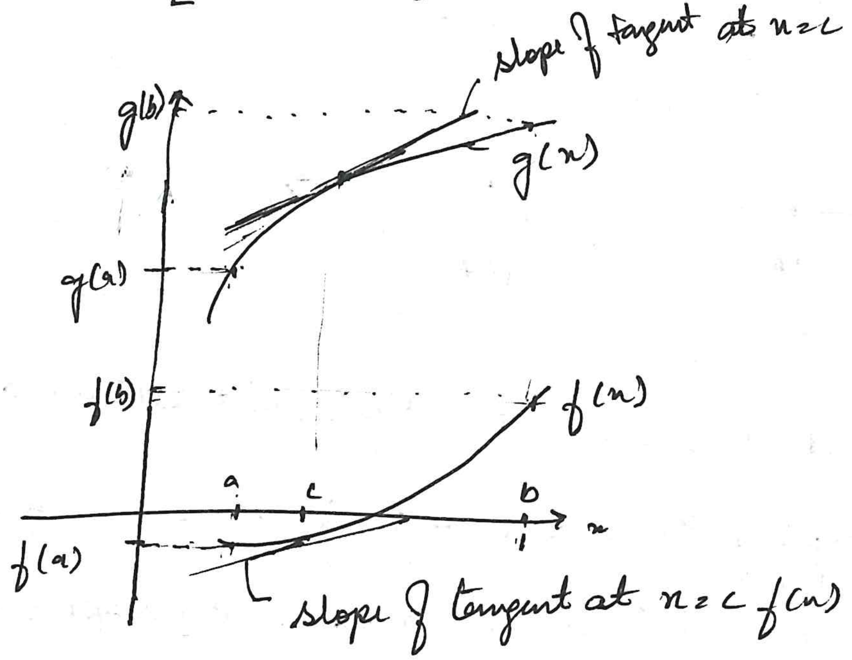
In other words slope of the tangent at $x = c$ = slope of the secant $(a, f(a))$, $(b, f(b))$.

(3) Extended Mean Value Theorem / Cauchy's MVT

- The L'Hopitals rule is a special case of the extended M.V.T.

- If f and g are real continuous in $[a, b]$ and differentiable in (a, b). then $\exists\, c \in (a, b)$ such that

$$\left[f(b) - f(a)\right] g'(c) = \left(g(b) - g(a)\right) f'(c).$$



Ratio of difference of the values of the functions is same as the ratio of the derivatives at a point c for the functions

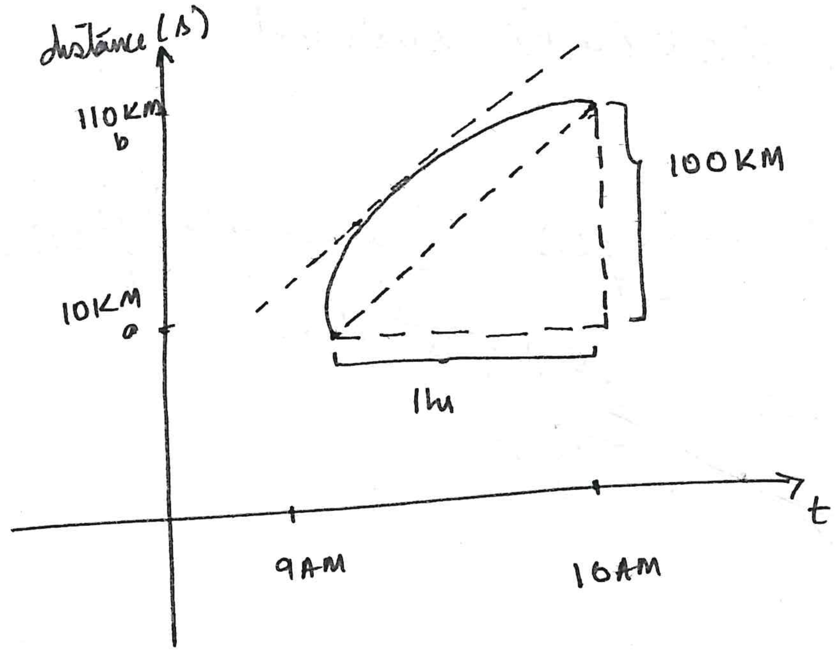
4.15 MVT : Real World Applications to speeding cars and fines

→ A lot of ideas in calculus have arrived from Physics/Mechanics.

→ Consider an example

→ A vehicle moves point A (City centre say) to point A and then it moves to point B.

- It reaches point A at 9AM and point B at 10AM.



→ According to MVT there exist atleast one point between a and b such that the slope of the secant connecting a and b is equal to slope of the curve at the point in between a and b (here the slope of the curve is the slope of the tangent at that point and it is given by the instantaneous speed at that point).

→ If we have a speed limit of 70 KM/H. if slope of the secant > 70 KM/H then we can say for sure the driver has overspeeded atleast once in between the points a and b, if we know the time of arrival at points a and b we can easily track the overspeeding vehicles.

4.16 MVT: SOLVED PROBLEMS

Q

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

Where $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $f'(\theta)$ denote the derivative of f with respect to θ. Which of the following statements are true?

I. There exists $\theta \in (\pi/6, \pi/3)$ such that $f'(\theta) = 0$

II. There exists $\theta \in (\pi/6, \pi/3)$ such that $f'(\theta) \neq 0$

A. I only.

B. II only.

C. Both I and II are true

D. Neither I nor II are true.

Substituting $\theta = \pi/6$. $f(\pi/6) = \begin{vmatrix} \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$.

As the first two rows are identical $f(\pi/6) = 0$.

for $\theta = \pi/3$ Also we will get R_2 and R_3 as identical and therefore $f(\pi/3) = 0$.

- $f(a) = f(b)$ and $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b)

  $\therefore$ A point exist such that $f'(x) = 0$. $\therefore$ statement I is true.

- $f(x)$ or $f(\theta)$ is not a constant function, therefore statement II is also true. Option C is true.

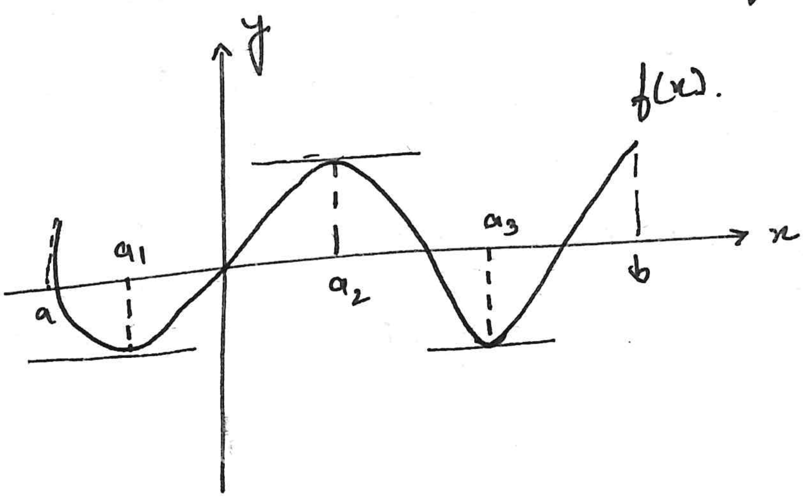
4.16 MAXIMA AND MINIMA : AN INTRODUCTION

→ Applications in areas of sciences and engineering, ML & AI.

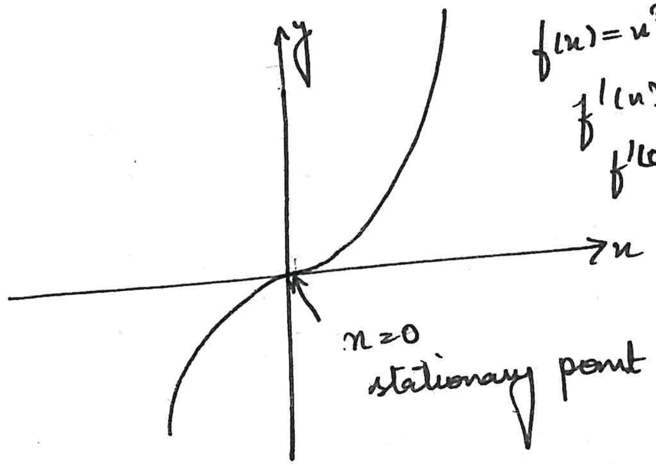
1. Stationary Point : If $f(x)$ is a function $f(x): \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ is a stationary point if $f'(a) = 0$ or in other words, tangent at $x = a$ is parallel to x-axis.

2. Critical Point : A point for which $f'(x) = 0$ or $f'(x)$ does not exist.

$f(x) : [a, b] \to \mathbb{R}$



a_1, a_2 and a_3 are stationary points.



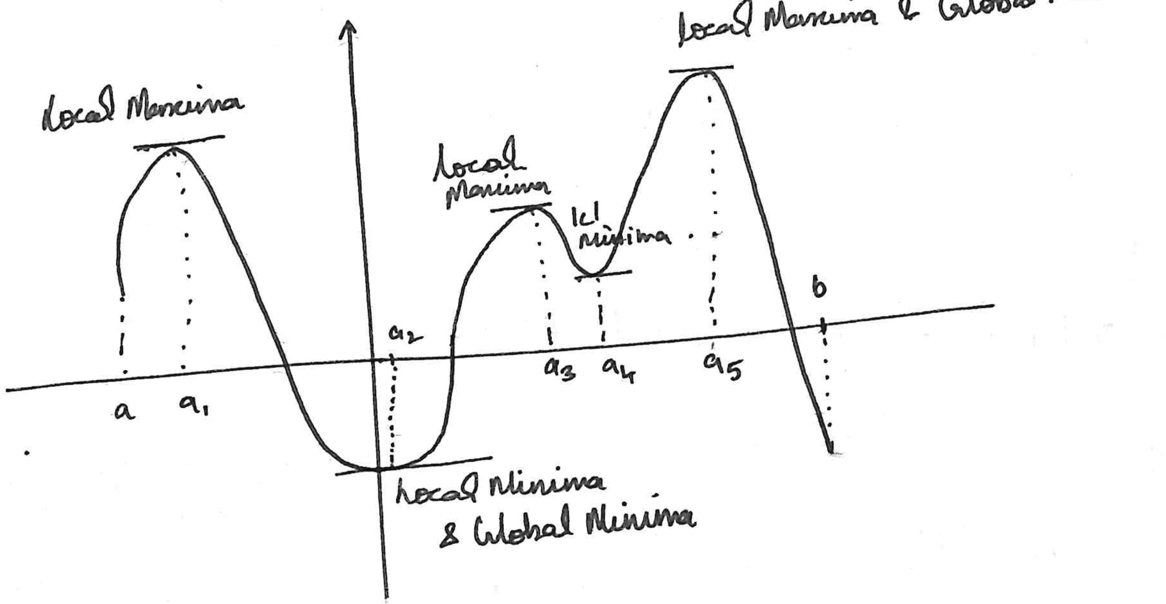
$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

- In the first graph at the stationary point the curve is changing from increasing to decreasing and vice versa but in case of the curve $y = x^3$ there is no such change it was increasing before and afterwards as well. The second curve shows $y = x^3$, at $x = 0$ it has a different type of stationary point.

MAXIMA AND MINIMA



$f'(x) = 0$ at $x = a_1, a_2, a_3, a_4, a_5$ they are stationary points

<u>Local Maxima / Minima</u> Are those points for which the value of the function $f(x)$ is maximum / minimum in the local neighbourhood.

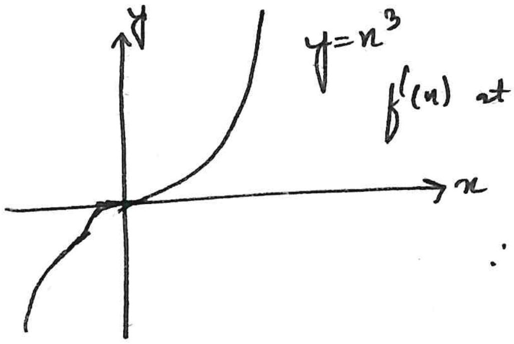
<u>Global Maxima / Minima</u> :- It is that point c such that $f(c) \geq f(x) \; \forall \; x \in [a, b]$ (global maxima)

$f(c) \leq f(x) \; \forall \; x \in [a, b]$ (global minima).

where $c \in [a, b]$

- All the Local, Global Minima, Maxima are stationary points.

<u>Exceptional Case Example.</u>



$y = x^3$

$f'(x)$ at $x = 0$ $f'(0) = 0$. but 0 is neither a maxima nor a minima.

$\therefore$ All stationary points need not be maxima/minima

Such points which are stationary but not max/min are known as SADDLE POINTS!

→ Those points (stationary points) where the curve is at one side of the tangent such points are local minima / maxima.

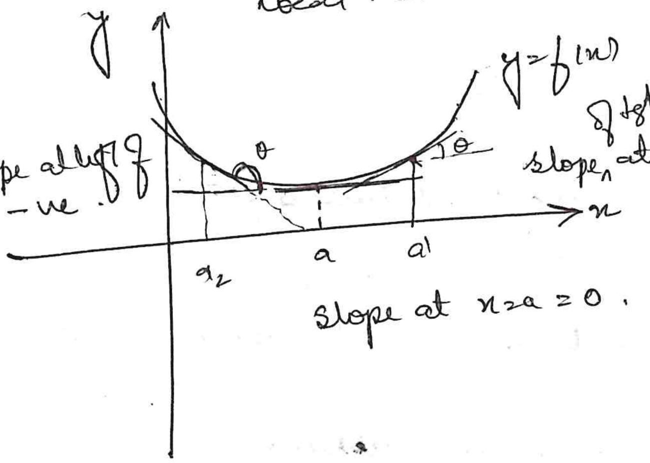
→ The stationary points where we have the curve on both sides of the tangent such points are the <u>Saddle Points</u>.

→ For some functions Maxima/Minima do not exist.

4.17 FINDING MAXIMA AND MINIMA

Geometric Intuition

Local Minima.



Slope at left of a −ve.

slope of tgt at a₁ rt of a +ve

Slope at $x = a = 0$.

f(x) is continuous and differentiable

$f'(a) = 0$, $a_2 < a$ → slope −ve

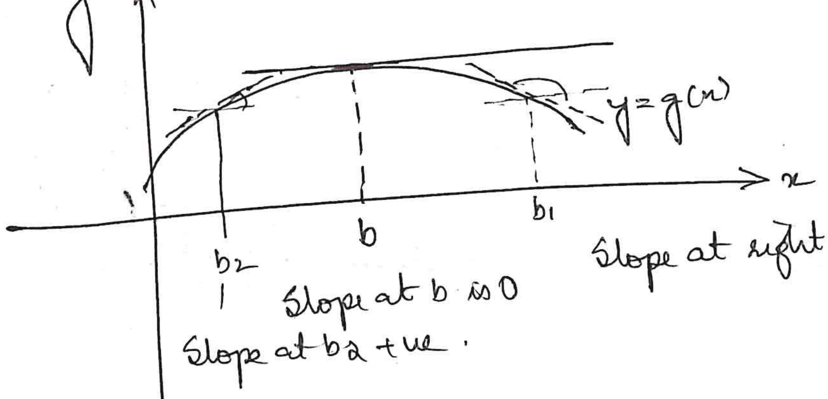
$a_1 > a$ → slope +ve

⇓

Slope is ↑sing (increasing)

$\underbrace{\quad}_{-ve} \overset{a}{\downarrow} 0 \underbrace{\quad}_{+ve}$

Local Maxima.



Slope at b is 0

Slope at b₂ +ve.

$g'(b) = 0$

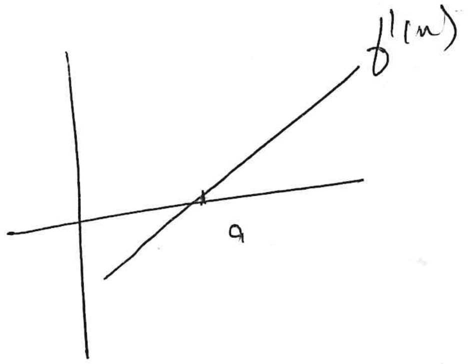
$b_1 < b$ slope +ve.

$b_1 > b$ slope −ve.

$\boxed{\text{Slope at right side of } b \text{ @ } b_1 - ve}$

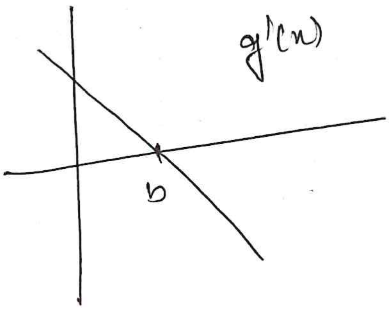
$\underbrace{\quad}_{+ve} \overset{b}{\downarrow} 0 \underbrace{\quad}_{-ve}$

Slope is ↓ decreasing.



If $f'(x)$ is an increasing function then at a $f(x)$ is minimum (local)

$f''(x) = \dfrac{d f'(x)}{dx} > 0$ at $x = a$



If $g'(x)$ is a decreasing function then is maximum (local) at $x = b$ the function $g(x)$ is maximum (local).

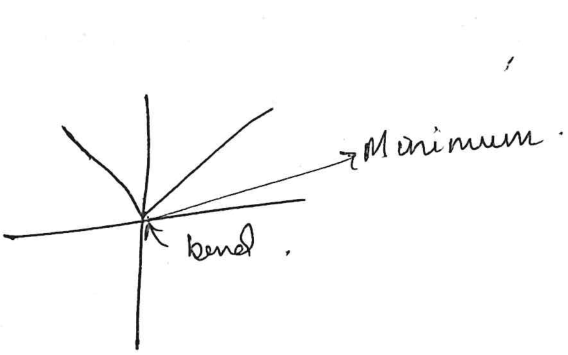
$g''(x) = \dfrac{d\, g'(x)}{dx} < 0$ at $x = b$.

- For local Minima / Maxima $\rightarrow f'(x) = 0$ $f''(x) = -ve/+ve$ respectively
- For Global Minima / Maxima $\rightarrow$ compute all local minima a_1, a_2, a_3

  if $f(a_1) < f(a_2) < f(a_3)$ then f has a global minima at $x = a_1$.
- Similarly we can also calculate Global Maxima using local maxima.

Some Examples

① $f(x) = |x|$



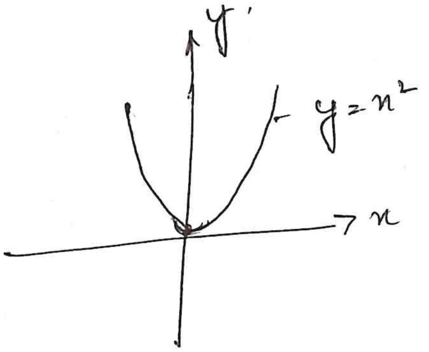
$f'(x)$ at $x = 0$ is not defined as it is not differentail at $x = 0$.

② $f(x) = x^2$

$f'(x) = 2x = 0$

$x = 0$

$f''(x) = 2 > 0$.



$f'(0) = 0$

$f''(0) > 0$

∴ It is local Minima.

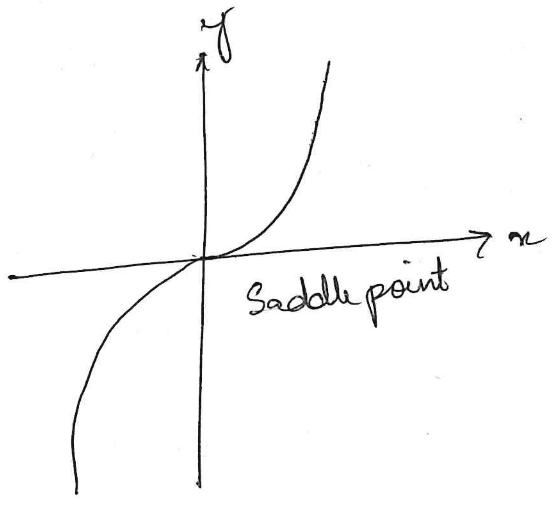
As it is the only minima therefore it is also the global minima.

③ $f(x) = x^3$.

$f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow 3x^2 = 0$

$\underline{\underline{x = 0}}$.



$f''(x) = 6x$

at $x = 0$

$f''(0) = 0$ $\not< 0$ neither maxima nor minima.

$\not> 0$

(4) $f(x) = \frac{x^3}{3} - x$

$f'(x) = \frac{3x}{3} - 1 = 0$

$x = \pm 1$

$f''(x) = 2x$

$f''(-1) = -2 \xrightarrow{<0}$ Maxima

$f''(1) = 2 > 0 \rightarrow$ Minima.

(5) $f(x) = 2\cos x - x$.

$f'(x) = -2\sin x - 1 = 0$.

$\sin x = -1/2$

There are infinitely many values of x for which $\sin x = -1/2$

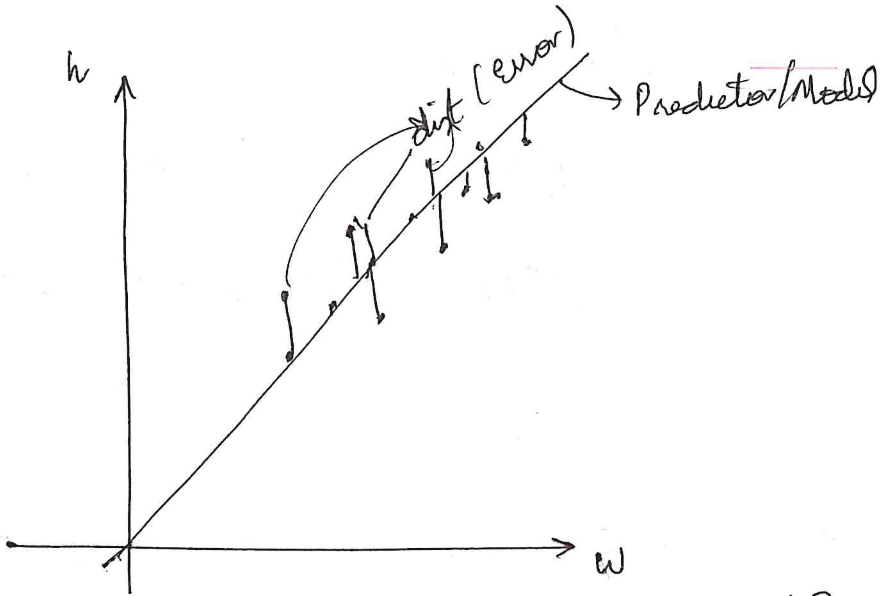
$\therefore$ $f(x)$ has infinitly many maximum's and minimum's.

4.19. APPLICATION OF MAXIMA & MINIMA TO MACHINE LEARNING

→ The concept of maxima & minima is used extensively in ML & AI.

→ Example:- Given the weights and heights of n people

(w_1, h_1)
(w_2, h_2)
(w_3, h_3)
$\vdots$
(w_n, h_n)



→ We would like to have a line or some other function which act as the predictor for heights of a new person if his weight is given.

→ Such a line can be predicted or derived in such a way that the error with the current points $(w_1, h_1), (w_2, h_2), (w_3, h_3)$ —— (w_n, h_n) is minimum.

If we denote the errors as $e_1, e_2, \dots e_n$

then $\sum_{i=1}^{n} e_i$ should be as small as possible.

$$\Rightarrow \sum_{i=1}^{n} (\underset{\substack{\uparrow \\ \text{Actual height}}}{h_i} - \underset{\substack{\uparrow \\ \text{Parameter}}}{m}\underset{\substack{\uparrow \\ \text{Predicted height}}}{w_i})^2 \longleftarrow g(m)$$

★ We would like to find an parameter $\underline{m}$ that minimises $\underline{\underline{g(m)}}$.

→ This problem of predicting/deriving an optimum straight line such that error is minimum, $g(m)$ is minimum is called Linear Regression.

→ The minima is calculated using the same rules of calculus (the best value of m is estimated this way).

4.20. MAXIMA AND MINIMA

SOLVED PROBLEMS

Q1) What is the maximum value of the function $f(x) = 2x^2 - 2x + 6$ in the interval $[0, 2]$?

A: 6
B. 10
C. 12
D. 5.5

Solution.

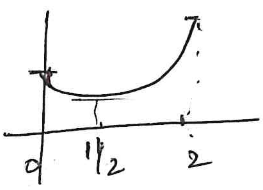
$f'(x) = 4x - 2 = 0 \Rightarrow x = \frac{2}{4} = \frac{1}{2}$.

$f''(x) = 4 > 0$. ∴ At $x = \frac{1}{2}$ the function $f(x)$ has a minima

The given function is a quadratic function and we need to check the boundaries i.e at $x = 0, 2$ one of them will be the maximum point.

$f(0) = 6 \quad f(2) = 4 - 4 + 6 = 6$.

∴ 6 is the Maximum.



Q2) A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct extremum for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is ___ .

A. 0
B. 1
C. 2
D. 3.

Solution

$f'(x) = 12x^3 - 48x^2 + 48x$

$f'(x) = 0$.

$12x^3 - 48x^2 + 48x = 0$.

$\Rightarrow x^3 - 4x^2 + 4x = 0$

$x(x^2 - 4x + 4) = 0$.

$x(x-2)(x-2) = 0$

$x = 0, 2$ 2 unique points C is the correct option.

$f''(x) = 36x^2 - 96x + 48$

$f''(0) = 48 > 0$

$f''(2) = 36(4) - 96(2) + 48$

$= 144 - 192 + 48$

$= 0$. ∴ only one point is an extremum.

(Q3) If $f(x)$ is defined as follows, what is the minimum value of $f(x)$ for $x \in (0, 2]$?

$$f(x) = \begin{cases} \dfrac{25}{8x} & \text{when } x \le 3/2 \\ x + \dfrac{1}{x} & \text{otherwise.} \end{cases}$$

A. 2.
B. $2\left(\frac{1}{12}\right)$
C. $2\left(\frac{1}{6}\right)$
D. $2\left(\frac{1}{2}\right)$.

Solution

Checking for continuity of this function

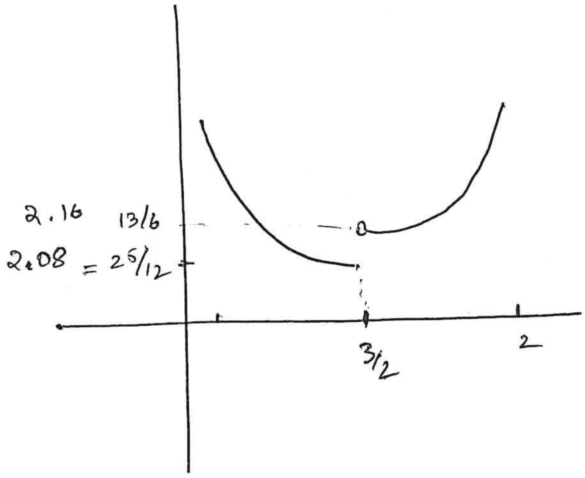
$$\lim_{x \to 3/2^+} f(x) = \lim_{x \to \frac{3}{2}^+} x + \frac{1}{x} = \frac{3}{2} + \frac{2}{3} \Rightarrow \frac{9+4}{6}$$

$$\Rightarrow \frac{13}{6}$$

$$\lim_{x \to 3/2^-} f(x) = \lim_{x \to 3/2^-} \frac{25}{8x} = \frac{25}{\cancel{8}^{4} \times \frac{3}{\cancel{2}}} = \frac{25}{12}.$$

L.H limit $\neq$ R.H limit $\therefore$ the $f(x)$ is not continuous.

Let us draw a rough plot of $f(x)$



Min value is at $x = 3/2$ by looking at the plot $= \frac{25}{12} = 2\frac{1}{12}$ option B.

Q4) Consider the function $f(x) = \sin(x)$ in the interval $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

The number and locations of the local minima are ___.

A. One, at $\frac{\pi}{2}$.

B. One at $\frac{3\pi}{2}$.

C. Two at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

D. Two at $\frac{\pi}{4}$ and $\frac{3\pi}{2}$.

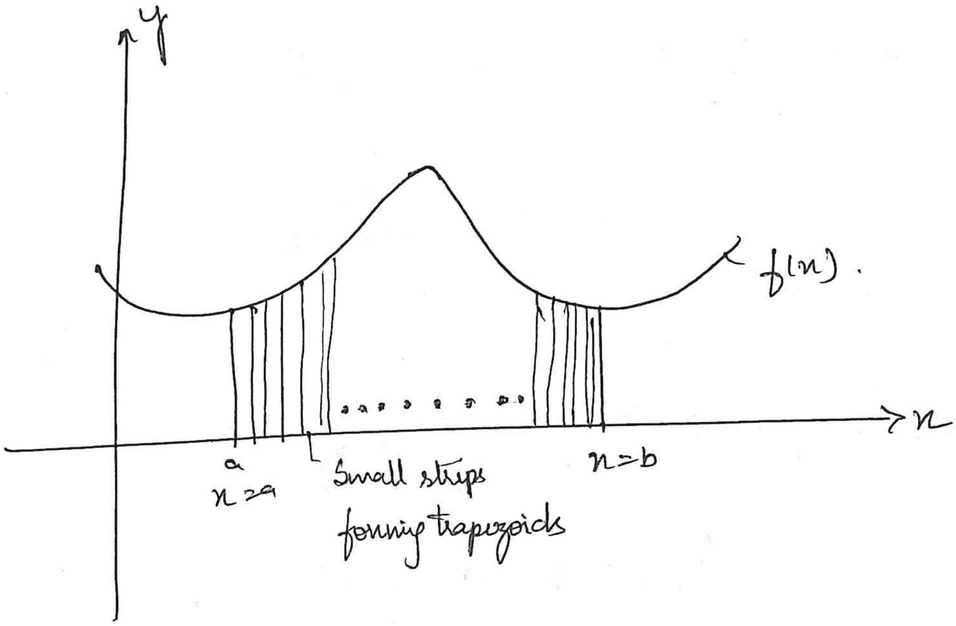
Solution

$$f'(x) = \cos x = 0.$$

$$x = \pi/2, \; 3\pi/2.$$

$f''(x) = -\sin x$ $\quad f''(\pi/2) = -1 < 0.$ local max $\quad f''(3\pi/2) = 1 > 0$ local Min

→ If our function look like the below



Then $\int_a^b f(x) =$ area between $f(x)$ and x-axis, in between $x=a$ and $x=b$.

It can be rewritten as $\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x = \int_a^b f(x)\,dx$ given by Reimann sum.

List of integrals can be found on wikipedia page. http://en.wikipedia.org/wiki/Lists_of_integrals.

4.22 INTEGERAITION BY SUBSTITUTION

(i) $\int x^2 \sqrt{x^3+1}\; dx$

Let $u = x^3+1 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow \frac{du}{3} = x^2 dx$

$$\int x^2\sqrt{x^3+1}\,dx = \int \frac{u^{1/2}}{3}\,du = \frac{1}{3}\cdot\frac{2}{3}\,u^{3/2} + C$$

$$= \frac{2}{9}\left(x^3+1\right)^{3/2} + C$$

If we have the integral in the form $\int f(x)\cdot f'(x)\,dx$

then take $u = f(x)$

$du = f'(x)\,dx$.

$$\int f(x)\,f'(x)\,dx = \int u\,du = \frac{u^2}{2} + C$$

$$= \frac{1}{2}\left(f(x)\right)^2 + C$$

(2) $\int x^2 e^{x^3}\,dx$

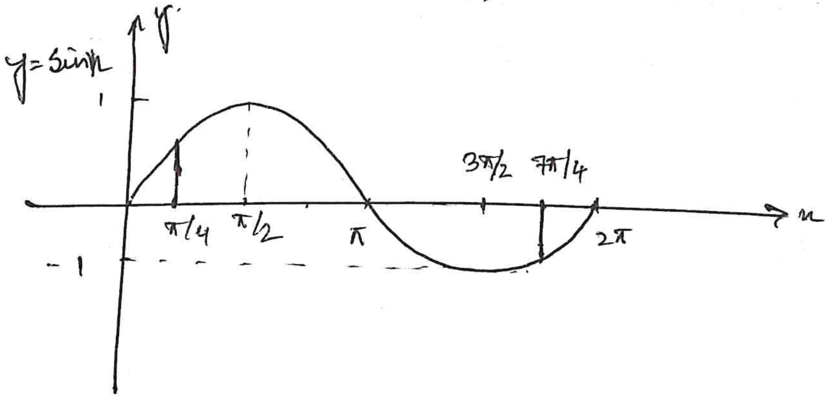
take $u = x^3$

$du = 3x^2\,dx$.

$\frac{du}{3} = x^2\,dx$

$$= \int e^u \frac{du}{3} = \frac{1}{3}e^u + C.$$

Let us check the graph for the range $[\pi/4, \frac{7\pi}{4}]$



At $x = \pi/4$ it is the local minima because it is the least value in its surrounding.

$\therefore$ The correct answer is $x = \pi/4$.

4.21 Inegeration : An Introduction

→ We can think of integration as inverse of derivation or anti derivative

- Also referred as Indefinite Integral.

If $F(x)$ is differentiable.

then $F'(x) = f(x) = \frac{d\,F(x)}{dx} \Big\} \Rightarrow f(x)dx = dF(x)$.

then

$$\int f(x)\,dx = F(x) + C$$

↳ Constant

eg

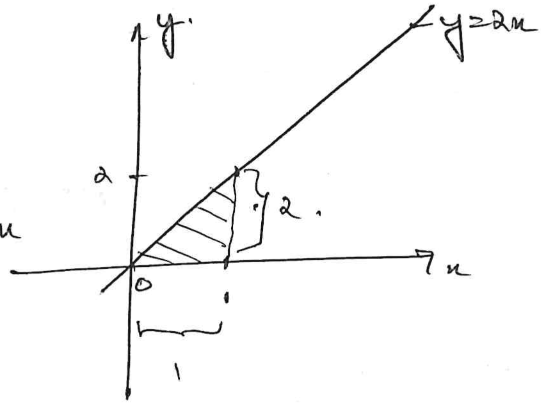
$$F(x) = x^2$$

$$F'(x) = 2x' = 2x = f(x) = \frac{dF}{dx}.$$

$$\int 2x\,dx = x^2 + c$$

Definite Integral

- Represented by $\int_a^b f(x)\,dx$. ex $= \int_0^1 2x\,dx$.
- Geometrically it means area covered by the curve $y = 2x$ and x axis in between the values $x = 0, 1$



By method of definite integral $= \int_0^1 2x\,dx$

$$= \left[x^2 + c\right]_0^1$$

$$= [1 + \cancel{c}] - [0 + \cancel{c}]$$

$$= 1$$

Area $= \frac{1}{2} \times 2 \times 1 \Rightarrow 1$ unit.

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

3. $$\int_0^{\pi/2} \sin^5 x \cos x \, dx$$

$U = \sin x$

$du = \cos x \, dx$.

$x = 0 \quad U = \sin 0 = 0$

$x = \pi/2 \quad U = \sin \pi/2 = 1$

$$= \int_0^1 U^5 \, du = \left[\frac{U^6}{6} \right]_0^1 = \frac{1}{6}$$

4. ln |f| form → widely used.

$\int \frac{f'(x)}{f(x)} dx$ then take $U = f(x)$

$du = f'(x) dx$.

$$= \int \frac{du}{U} = \ln |U| + C = \ln |f(x)| + C$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$U = e^x + e^{-x}$

$du = (e^x - e^{-x}) dx$.

$$= \int \frac{du}{u} \Rightarrow \log |(e^x + e^{-x})| + C$$

(5) $\int \frac{2x+3}{x^2+3x-4}$

$x^2+3x-4 \to u$

$du = 2x+3$.

$$\int \frac{du}{u} = \log|u| + c$$

$$= \underline{\log|x^2+3x-4| + c}.$$

(6) $\int \cot x\, dx. = \int \frac{\cos x\, dx}{\sin x} = \log|\sin x| + c.$

$\cos x\, dx \to du$, $\sin x \to u$

(7) $\int \frac{1}{\ln(x^x)} = \int \frac{1}{x \ln x}\, dx = \ln|\ln x| + c$

$\frac{1}{x}dx \to du$, $\ln x \to u$

4.23 INTEGERATION BY PARTS

→. If we have $u = f(x)$ and $v = g(x)$, then $\int u\,dv = u.v - \int v\,du$.

Proof by product rule of differentiation

$(fg)' = f'g + fg'$ (Product rule)

Integral

$fg = \int f'g\, dx + \int f g'\, dx \to \{ \int u\,dv = uv - \int v\,du.$

① $\int x e^x dx$.

$U = x$ | $dV = e^x dx$

$du = dx$ | $V = e^x$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$= x e^x - e^x + C$$

Here the challange is choosing U?

Rule of thumb: LIATE

L → log

I → Inverse trigonometric

A → Algebraic (x^2, $2x^{10}$)

T → Trigonometric

E → exponential.

Another way of remember. DETAIL (Reverse Order of LIATE)

D → dx

② $\int \frac{\ln x}{x^2} dx$ LIATE

$U = \ln x$ | $dV = \frac{1}{x^2}$

$du = \frac{1}{x} dx$ | $V = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} dx = (\ln x)\left(\frac{-1}{x}\right) - \int -\frac{1}{x}\frac{1}{x} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}$$

$$\Rightarrow -\frac{\ln x}{x} - \frac{1}{x} + C$$

③ $\int \sec^2 x \ln|\sin x| dx$ LIATE

$U = \ln|\sin x|$

$du = \frac{1}{\sin x} \times \cos x \, dx$

$dv = \sec^2 x dx$

$V = \tan x$.

$= \int \sec^2 x \ln|\sin x| dx$

$= \ln|\sin x| \cdot \tan x - \int \tan x \frac{1}{\tan x} dx$

$\Rightarrow \underline{\underline{\tan x \ln|\sin x| - x + C}}$

④ $\int x \cos x \, dx$ LIATE

$U = x$

$dU = dx$.

$dv = \cos x \, dx$

$V = \sin x$

$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$

$\Rightarrow \underline{\underline{x \sin x + \cos x + C}}$

5) $\int e^x \cos x \, dx$.

$u = \cos x$

$du = -\sin x \, dx$.

$dv = e^x dx$

$v = e^x$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

$u = \sin x$ $\quad v = e^x$
$du = \cos x \, dx$ $\quad dv = e^x dx$.

$$= e^x \cos x + e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{\text{original integral lets take it as } I}.$$

$$I = e^x \cos x + e^x \sin x - I$$

$$\Rightarrow 2I = e^x \cos x + e^x \sin x$$

$$I = \frac{e^x \cos x + e^x \sin x}{2}.$$

6) $I = \int \ln x \, dx$

$u = \ln x$.

$du = \frac{1}{x} dx$

$dv = 1 . dx$

$v = x$.

$\Rightarrow \quad I = \int \ln x . 1 \, dx \Rightarrow x \ln x - \int x . \frac{1}{x} \Rightarrow x \ln x - x + c$

(7)

$\int x^3 e^{x^2} dx$

$U = x^3$ ✗ $\quad\quad U = x^2$

$du = 3x^2 dx$ ✗ $\quad du = 2x\,dx$

$dv = x e^{x^2} dx.$

$V = \frac{e^{x^2}}{2}$

$\int x^3 e^{x^2} dx = x^2 \frac{e^{x^2}}{2} - \int \frac{e^{x^2}}{2} \cdot 2x\,dx.$

$\left.\begin{array}{l} w = x^2 \\ dw = 2x\,dx \end{array}\right\}$ Integration by Substitution

$= x^2 \frac{e^{x^2}}{2} - \frac{e^{x^2}}{2} + C.$

$\int e^w \frac{dw}{2}.$

$= \frac{e^w}{2} + C$

$= \frac{e^{x^2}}{2} + C$

Plugging in

4.24 INTEGRATION BY PARTIAL FRACTIONS

→ Partail fractions have many other applications in mathematics apart from Integrations

- Partail Fraction decomposition

- Here the degree of the neumurator is less than the degree of the denominator.

example $f(x) = \dfrac{x}{(x^2-x-2)} = \dfrac{x}{(x+1)(x-2)} = \underbrace{\dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}}$

Decompose into 2 fractions.

$$\Rightarrow \frac{A(x-2)+B(x+1)}{(x+1)(x-2)} = \frac{x}{(x+1)(x-2)}.$$

$$\frac{x(A+B)+(B-2A)}{(x+1)(x-2)} = \frac{x}{(x+1)(x-2)}$$

$$\Rightarrow \begin{array}{l} A+B=1 \\ A+B=0. \end{array} \Rightarrow \begin{array}{l} A = 1/3 \\ B = 2/3. \end{array}$$

$$\frac{x}{x^2-x-2} = \frac{1}{3(x+1)} + \frac{2}{3(x-2)} \;;\; \int \frac{x}{x^2-x-2} = \int \frac{1}{3(x+1)} + \int \frac{2}{3(x-2)}$$

$$= \frac{1}{3} \ln|x+1| + \frac{2}{3} \ln|x-2| + C$$

(1) $\displaystyle\int \frac{1}{x(x+1)}\,dx$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$A(x+1) + Bx = 1$

$A + B = 0$

$A = 1$

$B = -1$

$$\Rightarrow \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x(x+1)} = \ln|x| - \ln|x+1| + C$$

(2) $\displaystyle\int \frac{2x+3}{x^2-9}\,dx =$

$$\frac{2x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{(x+3)} \Rightarrow A(x+3) + B(x-3) = 2x+3$$

$A + B = 2$

$3A - 3B = 3 \quad \Rightarrow A = 3/2$

$B = 1/2$

Plug these values back.

$$\int \frac{2x+3}{x^2-9}\,dx = \int \frac{1}{2(x-3)}\,dx + \int \frac{3}{2(x+3)}$$

$$= \frac{1}{2}\ln|x-3| + \frac{3}{2}\ln|x+3| + C$$

(3) $$\int \frac{2x+3}{x^2+3x+9}\,dx$$

Here we can see that numerator is derivative of denominator it is better to go by method of substitution in this case

$U = x^2+3x+9$

$du = (2x+3)dx$

$$\int \frac{du}{U} = \underline{\underline{\ln|x^2+3x+9| + C}}$$

(4) $$f(x) = \frac{x+10}{x^3+x} = \frac{A}{(x)} + \frac{Bx+C}{(x^2+1)}$$

$$\downarrow$$

$$\frac{x+10}{x(x+1)}$$

y- we have 2nd order polynomial which cannot be decomposed using real numbers.

∴ we need to have x term in the numerator.

(8) Decompose into partail fractions

$$f(n) = \frac{1}{(n^3-1)(n^2-1)^2}$$

$$= \frac{1}{(n-1)(n^2+n+1)((n-1)(n+1))^2}$$

$$= \frac{1}{(n-1)^3 (n^2+n+1)(n+1)^2}$$

$$= \frac{A}{(n-1)} + \frac{B}{(n-1)^2} + \frac{C}{(n-1)^3} + \frac{D}{(n+1)} + \frac{E}{(n+1)^2}$$

$$+ \frac{Fn+G}{(n^2+n+1)}$$

→ ** When we have a power of a fraction then we have to consider it in all powers from 1 upto its actual exponent.

for example for $(n-1)^3$ we have to write $(n-1)$, $(n-1)^2$ and $(n-1)^3$.

4.25 PROPERTIES OF DEFINITE INTEGRALS

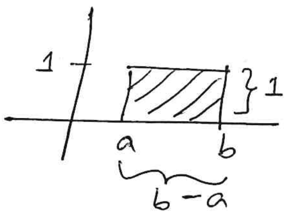
Definite integrals – area under a curve

or

area between a curve and x-axis, when the curve is being represented by the function we are integrating.

Properties of Definite Integrals:

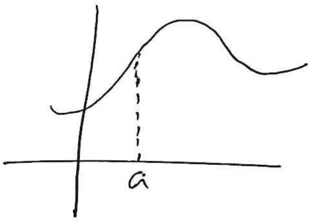
① $\displaystyle\int_a^b 1 \cdot dx = b - a$



② $\displaystyle\int_a^b K \cdot f(x)\, dx = K \int_a^b f(x)\, dx$ → Reimann Sums

③ $\displaystyle\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

④ $\displaystyle\int_a^a f(x)\, dx = 0$



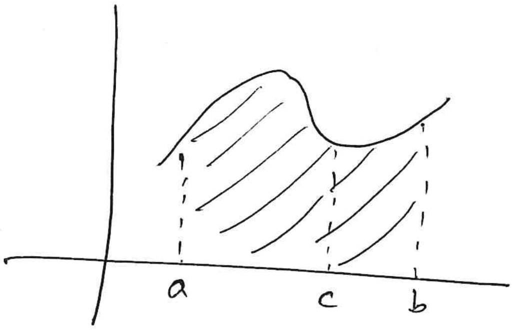
⑤ $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

$\downarrow$

$F(x)\Big|_a^b = F(b) - F(a)$ | $\underbrace{F(x)\Big|_b^a}$ $F(a) - F(b)$

(6) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$c \in [a, b]$



(7) $$\int_a^b f(x)\,dx \geqslant 0 \quad \text{if } f(x) \geqslant 0 \quad [a, b]$$

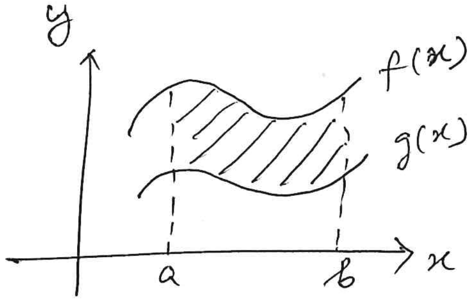
$$\int_a^b f(x)\,dx \leq 0 \quad \text{if } f(x) \leq 0 \quad \text{in } [a, b]$$

(8) Fundamental Theorem of Calculus

anti-derivate $$\left\{ \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \right.$$

$$\text{if } F'(x) = f(x)$$

(9) $$S = \int_a^b (f(x) - g(x))\,dx$$



$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Area between 2-curves

4.26 INTEGRATION : SOLVED PROBLEMS

Q1.)

Let $s = \sum_{i=3}^{100} i \log_2 i$, and

$T = \int_2^{100} x \log_2 x \, dx.$

Which of the following statements is true ?

A. $s > T$

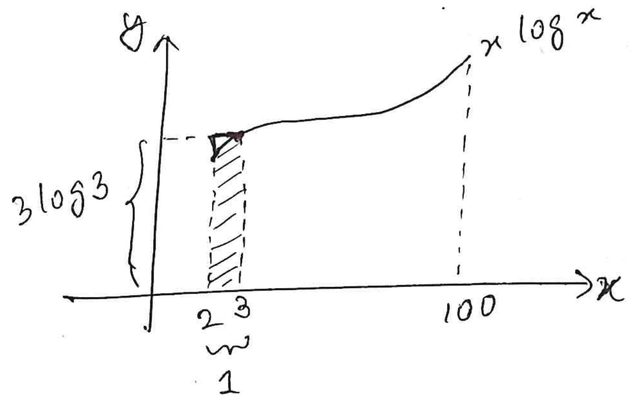
B. $s = T$

C. $s < T$ and $2s > T$

D. $2s < T$

Soln.

As x increases, $x \log x$ also increases. And is always positive in this interval 2 to 100.



$$\sum_{i=3}^{100} i \log_2 i = 3\log 3 + 4\log 4 + 5\log 5 + \cdots + 100 \log 100$$

$$3 \log 3 > \int_2^3 x \log x \, dx$$

$$4\log 4 > \int_3^4 x \log x \, dx$$

$\downarrow$ $\qquad$ $\downarrow$

$\vdots$ $\qquad$ $\vdots$

$$S > T$$

$\therefore$ Option (A).

Q2.)

$$\int_0^{\pi/4} (1-\tan x)/(1+\tan x) \, dx$$

A. 0

B. 1

C. $\ln 2$

D. $1/2 \ln 2$

$$\int_0^{\pi/4} (1-\tan x)/(1+\tan x) \, dx$$

$$= \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} \, dx = \int_1^{\sqrt{2}} \frac{du}{u}$$

$u = \cos x + \sin x$

$du = (-\sin x + \cos x) \, dx$

$x = 0 ; u = 1$

$x = \pi/4 ; \sqrt{2}$

$$= [\ln u]_1^{\sqrt{2}}$$

$$= \ln \sqrt{2} - 0$$

$$= \frac{1}{2} \ln 2$$

Q3)

Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral

$$\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\, dx \; ?$$

A. 0
B. 2
C. $-i$
D. i

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\int_0^{\pi/2} \frac{e^{ix}\, dx}{e^{-ix}} = \int_0^{\pi/2} e^{2ix}\, dx = \left. \frac{e^{2ix}}{2i} \right|_0^{\pi/2} = \frac{1}{2i}(-1-1)$$

$$= \frac{-2}{2i} = \frac{-1}{i}$$

$$= \frac{i^2}{i}$$

$$= i$$

$$e^{2ix} = \cos 2x + i\sin 2x \;\to x = \pi/2 = -1$$

$$\to x = 0 \to 1$$

$\therefore$ option (D)

Q4) The value of the integral given below is

$$\int_0^{\pi} x^2 \cos x \, dx$$

A. -2π

B. π

C. $-\pi$

D. 2π

Soln. We ~~try~~ to use Integration By parts and LIATE principle.

$u = x^2$

$du = 2x\,dx$

$dv = \cos x\,dx$

$v = \sin x$

$$\therefore \int_0^{\pi} x^2 \cos x\, dx = \int u.dv$$

$$= x^2 \sin x \Big|_0^{\pi} - \int \underbrace{2x}_{u} \underbrace{\sin x\, dx}_{dv}$$

$$= 0 - \int 2x \sin x\, dx$$

$x = u$

$du = dx$

$dv = \sin x\, dx$

$v = -\cos x$

$$= -2 \left\{ -x\cos x \Big|_0^{\pi} - \int -\cos dx \right\}$$

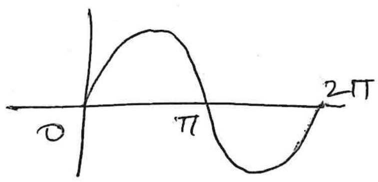
$$= 2x\cos x \Big|_0^{\pi} + \int 2\cos x\, dx$$

$= -2\pi \leftarrow$ Ans

Q5)

If $\int_0^{2\pi} |x \sin x| \, dx = k\pi$, then the value of k is equal to ________

soln.



$$\underbrace{\int_0^{\pi} \underset{u}{x} \underset{dv}{\sin x \, dx}} - \int_{\pi}^{2\pi} +x \sin x \, dx$$

$u = x$

$du = dx$

$dv = \sin x \, dx$

$v = -\cos x$

$$= -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx$$

$$= \pi + \sin x \Big|_0^{\pi} = \pi$$

$$-x \cos x \Big|_{\pi}^{2\pi} + \sin x \Big|_{\pi}^{2\pi}$$

$$= -3\pi$$

$\therefore \pi - (-3\pi)$

$= 4\pi$

Q6) Compute the value of:

$$\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2} \, dx$$

Soln

$u = 1/x$

$du = \frac{-1}{x^2}\, dx$

$$-\int_{\pi}^{\pi/2} \cos u \, du = \int_{\pi/2}^{\pi} \cos u \, du = \sin u \Big|_{\pi/2}^{\pi}$$

$$= 0 - 1$$

$$= -1$$

Q7.) The value of $\int_0^{\pi/4} x \cos(x^2)\, dx$ correct to three decimal places (assuming that $\pi = 3.14$) is ______

$u = x^2$

$du = 2x\,dx$

$$\int_0^{\pi^2/16} \frac{\cos u}{2}\, du = \frac{1}{2} \sin u \Big|_0^{\pi^2/16}$$

$$= \frac{1}{2} \sin(\pi^2/16)$$

$$= \frac{0.5779}{2}$$

$$= 0.289$$

Q 8)

If $f(x) = R \sin\left(\frac{\pi x}{2}\right) + s$. $f'\left(\frac{1}{2}\right) = \sqrt{2}$

and $\int_0^1 f(x)\,dx = \frac{2R}{\pi}$, then the

constants R and s are

A. $\frac{2}{\pi}$ and $\frac{16}{\pi}$

B. $\frac{2}{\pi}$ and 0

C. $\frac{4}{\pi}$ and 0

D. $\frac{4}{\pi}$ and $\frac{16}{\pi}$

Soln.

$$f(x) = R \sin\left(\frac{\pi x}{2}\right) + s$$

$$f'(1/2) = \sqrt{2}$$

$$\int_0^1 f(x)\,dx = \frac{2R}{\pi}$$

$$f'(x) = R \cos\left(\frac{\pi x}{2}\right) * \pi/2$$

$$f'(1/2) = R \cos\left(\pi/2 \cdot 1/2\right) * \pi/2 = \sqrt{2}$$

$$\Rightarrow R = 4/\pi$$

$$f(x) = \int_0^1 \left(R \sin\left(\frac{\pi x}{2}\right) + s\right) dx$$

$$= \int_0^1 \frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right) dx \quad + s x \Big|_0^1$$

$$= \frac{-4}{\pi}\cos\left(\frac{\pi x}{2}\right) \cancel{*} \frac{2}{\pi}\Big|_0^1 + 8x\Big|_0^1 = \frac{8}{\pi^2}$$

$\Rightarrow 8 = 0.$

Ans (C)