DM Cheatsheet - Sets, Relations, & Functions - Shivram Sambhus (Group K, HS25)

1 Introduction: The Building Blocks of Discrete Structures

This document covers the fundamental concepts of sets, relations, and functions as presented in Chapter 3.

1.1 TOC

2 Part I: Sets (Mengenlehre)

A set is an unordered collection of distinct objects. This is the most fundamental structure in mathematics.

2.1 Core Concepts & Notation

- **Element of:** $x \in A$ ("x is an element of set A").
- **Set-Builder Notation:** $\{x \in U \mid P(x)\}$ ("the set of all x in universe U such that property P(x) is true").
- **Empty Set:** $\emptyset$ or $\{\}$ (the unique set with no elements).
- **Cardinality:** $|A|$ (the number of elements in a finite set A).
- **Power Set:** $\mathcal{P}(A)$ (the set of all subsets of A). If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.
- **Russell's Paradox:** The "set of all sets that do not contain themselves", $R = \{A \mid A \notin A\}$, leads to a contradiction ($R \in R \Leftrightarrow R \notin R$). This paradox revealed that not every property can define a set. We must start from an existing set, e.g. $\{x \in B \mid P(x)\}$, not $\{x \mid P(x)\}$.

2.2 Proving Set Equality & Subsets

Core Definitions (Axiom of Extensionality)

- **Subset:** $A \subseteq B \equiv \forall x(x \in A \to x \in B)$.
- **Set Equality:** $A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv (A \subseteq B) \land (B \subseteq A)$.

TA Tip: The Element-Chasing Method This is the standard, rigorous way to prove set relations.

1. **To prove $A \subseteq B$:**
   - Start with "Let x be an arbitrary element of A."
   - Use the definition of A to state properties of x.
   - Logically deduce that x must also satisfy the properties of B (using definitions, logic rules).
   - Conclude with "Therefore, $x \in B$."
2. **To prove $A = B$:**
   - First, prove $A \subseteq B$.
   - Then, prove $B \subseteq A$.
   - Conclude that since both inclusions hold, by definition of equality, $A = B$.

Example: Proving a Distributive Law

Claim: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: We prove this by double inclusion.

Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$.
1. $\Rightarrow x \in A$ and $x \in (B \cup C)$ (Def. of Intersection)
2. $\Rightarrow x \in A$ and ($x \in B$ or $x \in C$) (Def. of Union)
3. $\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) (Distributive Law of Logic)
4. $\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$ (Def. of Intersection)
5. $\Rightarrow x \in (A \cap B) \cup (A \cap C)$ (Def. of Union)

Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$.
1. $\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$ (Def. of Union)
2. $\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) (Def. of Intersection)
3. $\Rightarrow x \in A$ and ($x \in B$ or $x \in C$) (Factoring out $x \in A$ in logic)
4. $\Rightarrow x \in A$ and $x \in (B \cup C)$ (Def. of Union)
5. $\Rightarrow x \in A \cap (B \cup C)$ (Def. of Intersection)

Since both inclusions hold, the sets are equal by definition.

2.3 Set Operations & Properties

Core Operations

- **Union:** $A \cup B = \{x \mid x \in A \lor x \in B\}$
- **Intersection:** $A \cap B = \{x \mid x \in A \land x \in B\}$
- **Difference:** $A \setminus B = \{x \mid x \in A \land x \notin B\}$
- **Cartesian Product:** $A \times B = \{(a, b) \mid a \in A \land b \in B\}$. Creates **ordered pairs**. Note that $A \times B \neq B \times A$ unless $A = B$ or one is empty. The product is not associative: $(A \times B) \times C \neq A \times (B \times C)$.

Laws of Set Algebra

These are direct consequences of the laws of logic.
- **Commutative:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associative:** $(A \cup B) \cup C = A \cup (B \cup C)$
- **Distributive:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **De Morgan's Laws:**
  - $\overline{A \cup B} = \overline{A} \cap \overline{B}$
  - $\overline{A \cap B} = \overline{A} \cup \overline{B}$

3 Part II: Relations (Relationen)

A relation describes a relationship between elements of sets. Formally, a binary relation R from a set A to a set B is any subset of the Cartesian product $A \times B$. We write aRb to mean $(a, b) \in R$.

3.1 Operations on Relations

Key Operations

- **Inverse (R^{-1}):** If $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$. $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
- **Composition ($S \circ R$):** If $R \subseteq A \times B$ and $S \subseteq B \times C$. $S \circ R = \{(a, c) \mid \exists b \in B, (a, b) \in R \land (b, c) \in S\}$. **Intuition:** A path from a to c through some intermediate b. The order is critical: $S \circ R$ means apply R **then** S. It is associative: $(T \circ S) \circ R = T \circ (S \circ R)$.
- **Transitive Closure (R^+):** $R^+ = \cup_{i=1}^{\infty} R^i = R \cup R^2 \cup R^3 \cup \ldots$ **Intuition:** $(a, b) \in R^+$ if there is a path of **any** length (≥ 1) from a to b.

3.2 Properties of Relations on a Set A

Property	Definition ($\forall a, b, c \in A$)	Intuition/Graph
Reflexive	aRa	Every node has a self-loop.
Irreflexive	$a \notin Ra$	No node has a self-loop.
Symmetric	$aRb \to bRa$	If there's an edge from a to b, there's one back (all edges are two-way).
Antisymmetric	$(aRb \land bRa) \to a = b$	No two distinct nodes have edges in both directions between them.
Transitive	$(aRb \land bRc) \to aRc$	If there's a path $a \to b \to c$, there's a direct edge $a \to c$. "Shortcut property".

Note: Antisymmetric is **not** the negation of symmetric. A relation can be both (e.g., equality) or neither.

3.3 Special Types of Relations

Equivalence Relation

A relation that is **Reflexive, Symmetric, and Transitive**.
- **Intuition:** Generalizes "equality". It groups similar elements together.
- **Equivalence Class:** $[a]_R = \{x \in A \mid xRa\}$. The set of all elements equivalent to a.
- **Partition:** The set of all equivalence classes of a set A forms a **partition** of A. This means the classes are non-empty, disjoint ($[a]_R \cap [b]_R = \emptyset$ if not aRb), and their union is A.

Partial Order

A relation R that is **Reflexive, Antisymmetric, and Transitive**.
- **Intuition:** Generalizes $\leq$. It defines a hierarchy where some elements may be **incomparable**.
- **Poset:** A pair (A, R) where R is a partial order on A.

- **Comparable vs Incomparable**: Two elements a, b are comparable if aRb or bRa. Otherwise they are incomparable.
- **Total Order:** A partial order where every pair of elements is comparable.
- **Hasse Diagram:** A simplified graph for a finite poset.
  1. Draw nodes for elements.
  2. If b **covers** a (i.e., $a \prec b$ and no c is between them, $a \prec c \prec b$), draw a line from a to b, with b placed higher.
  3. Omit self-loops (implied by reflexivity) and transitive edges (implied by transitivity). All edges point "up".

3.4 Elements in Posets

Let $(A, \leq)$ be a poset and $S \subseteq A$.

Minimal & Maximal Elements

- $a \in A$ is **minimal** if no element is smaller: $\neg\exists b \in A, b < a$.
- $a \in A$ is **maximal** if no element is larger: $\neg\exists b \in A, a < b$.

Note: There can be many minimal/maximal elements. In a Hasse diagram, these are the "bottom" and "top" elements.

Least & Greatest Elements

- $a \in A$ is **least** if it's smaller than or equal to all other elements: $\forall b \in A, a \leq b$.
- $a \in A$ is **greatest** if it's greater than or equal to all other elements: $\forall b \in A, b \leq a$.

Note: If they exist, they are unique. A least element is the unique minimal element. A greatest element is the unique maximal element.

Bounds for a Subset S

- **Lower Bound:** $a \in A$ is a lower bound of S if $\forall s \in S, a \leq s$.
- **Upper Bound:** $a \in A$ is an upper bound of S if $\forall s \in S, s \leq a$.
- **Greatest Lower Bound (infimum/meet):** glb(S) or $\inf(S)$ is the greatest of all lower bounds.
- **Least Upper Bound (supremum/join):** lub(S) or $\sup(S)$ is the least of all upper bounds.

Lattices

A poset $(A, \leq)$ is a **lattice** if every pair of elements $\{a, b\}$ in A has a unique meet (glb) and a unique join (lub).

4 Part III: Functions (Funktionen)

A function is a special type of relation that maps each element of a domain to exactly one element of a codomain.

4.1 Definition & Types

Formal Definition

A relation $f \subseteq A \times B$ is a function $f : A \to B$ if it satisfies two conditions:

1. **Totally Defined:** $\forall a \in A, \exists b \in B$ such that $(a, b) \in f$. (Every element in the domain is mapped).
2. **Well-Defined:** If $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. (Each element is mapped to only one output).

- **Image (Range):** $f(S) = \{f(a) \mid a \in S\}$ for a subset $S \subseteq A$. The image of the function is $\mathfrak{I}(f) = f(A)$.
- **Preimage:** $f^{-1}(T) = \{a \in A \mid f(a) \in T\}$ for a subset $T \subseteq B$.

Key Function Types

Type	Formal Definition	Intuition & Cardinality
Injective (one-to-one)	$\forall a_1, a_2 \in A, f(a_1) = f(a_2) \to a_1 = a_2$.	No two inputs map to the same output. For finite sets, $\lvert A\rvert \leq \lvert B\rvert$.
Surjective (onto)	$\forall b \in B, \exists a \in A, f(a) = b$.	Every element in the codomain is "hit". For finite sets, $\lvert A\rvert \geq \lvert B\rvert$.
Bijective	Both injective and surjective.	A perfect one-to-one correspondence. For finite sets, $\lvert A\rvert = \lvert B\rvert$. An inverse function f^{-1} exists iff f is bijective.

5 Part IV: Cardinality & Countability

Cardinality provides a way to compare the sizes of sets, including infinite ones, using functions.

5.1 Comparing Set Sizes

Fundamental Definitions

- **Equinumerous ($A \sim B$):** $|A| = |B|$. There exists a **bijection** $f : A \to B$.
- **Dominates ($A \leq B$):** $|A| \leq |B|$. There exists an **injection** $f : A \to B$.
- **Strictly Dominates ($A < B$):** $|A| < |B|$. There is an injection from A to B, but no bijection.

Schröder-Bernstein Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. **Approach:** If you can find an injection from A to B and another injection from B to A, you can conclude a bijection exists without actually constructing it.

5.2 Countable & Uncountable Sets

Definitions

- **Countable:** A set A is countable if it is finite or countably infinite. Formally, $|A| \leq |\mathbb{N}|$.
- **Countably Infinite:** A set A is countably infinite if $|A| = |\mathbb{N}|$. These are sets whose elements can be listed in an infinite sequence (e.g., $a_0, a_1, a_2, \ldots$).
- **Uncountable:** A set that is not countable. Its elements cannot be put into an infinite list.

Key Results & Proof Techniques

- **Countable Sets:** $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$, the set of all finite-length strings. **Proof Strategy:** To show a set A is countable, find an **injection** from A into a known countable set (like $\mathbb{N}$ or $\mathbb{N} \times \mathbb{N}$). Example: $\mathbb{Q}$ is countable because any rational can be written as $\frac{p}{q}$, mapping to an ordered pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$.
- **Uncountable Sets:** $\mathbb{R}, \mathcal{P}(\mathbb{N})$, the set of infinite binary sequences $\{0, 1\}^\infty$, the interval $[0, 1]$. **Proof Strategy:** Use Cantor's Diagonalization Argument.

Approach: Cantor's Diagonalization Argument Goal: To prove that a set (e.g., infinite binary sequences, $\{0, 1\}^\infty$) is uncountable. **Procedure:**

1. **Assume for contradiction** that the set **is** countable. This implies we can create a complete, infinite list of all its elements. $s_0 = b_{0,0}b_{0,1}b_{0,2}\ldots$ $s_1 = b_{1,0}b_{1,1}b_{1,2}\ldots$ $s_2 = b_{2,0}b_{2,1}b_{2,2}\ldots$ $\vdots$
2. **Construct a "diagonal" enemy:** Create a new sequence, s_{new}, that is guaranteed **not** to be on the list. This is done by making its n-th element different from the n-th element of the n-th sequence in the list. The n-th bit of s_{new} is the **flipped** bit of the n-th bit of s_n. $s_{(\text{new})_n} = 1 - b_{n,n}$
3. **Find the contradiction:** The new sequence s_{new} cannot be in our list. Why?
   - It's not s_0 because it differs in the 0-th bit.
   - It's not s_1 because it differs in the 1st bit.
   - In general, it cannot be s_n for any n because it differs in the n-th bit by construction.
4. **Conclusion:** Our list, which was assumed to be complete, is missing an element. This is a contradiction. Therefore, the initial assumption must be false, and the set is uncountable.