**DM Cheatsheet - Logic & Proofs** - Shivram Sambhus (Group K, HS25)

## 1 Introduction: The Structure of Mathematical Reasoning

*This document provides a structured overview of the core tools for formal reasoning in discrete mathematics: propositional logic, predicate logic, and standard proof techniques. The goal is to present a clear, procedural approach to constructing and analyzing mathematical arguments.*

### 1.1 TOC



## 2 Part I: Propositional Logic (Aussagenlogik)

*Propositional logic is the foundation of mathematical reasoning. It deals with propositions (statements that are either true or false) and the logical connectives that combine them.*

### 2.1 Propositions and Connectives

**Core Concepts**

A **proposition** is a declarative sentence with a definite truth value (True/1 or False/0).

| Connective | Symbol | Meaning |
|---|---|---|
| Negation | $\neg P$ | "it is not the case that P |
| Conjunction | $P \wedge Q$ | "P and Q are both true |
| Disjunction | $P \vee Q$ | "at least one of P or Q is true |
| Implication | $P \rightarrow Q$ | "if P is true, then Q is true |
| Biconditional | $P \leftrightarrow Q$ | "P and Q have the same truth value |

### 2.2 Truth Tables and Logical Status

**Procedure: Constructing a Truth Table**

1. Create a column for each atomic proposition ($n$ variables).
2. Create $2^n$ rows to list all possible combinations of truth values.
3. Add columns for complex sub-formulas, building up from simplest to most complex.
4. Fill each new column by applying the definition of its main connective to its constituent columns.

**Example: Truth Table for $(P \vee Q) \rightarrow (P \wedge Q)$**

| $P$ | $Q$ | $P \vee Q$ | $P \wedge Q$ | $(P \vee Q) \rightarrow (P \wedge Q)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

**Definitions of Logical Status**

- **Tautology:** A formula that is always true (final column is all 1s). E.g., $P \vee \neg P$.
- **Contradiction:** A formula that is always false (final column is all 0s). E.g., $P \wedge \neg P$.
- **Contingency:** A formula that is neither a tautology nor a contradiction.
- **Satisfiable:** A formula that is true for at least one assignment of truth values (i.e., not a contradiction).

### 2.3 Logical Equivalences

**Concept**

Two formulas $F$ and $G$ are **logically equivalent** ($F \equiv G$) if they have identical truth tables. This means $F \equiv G$ is a tautology. Equivalences are the rules for algebraic manipulation of logical formulas.

**Fundamental Laws**

- **De Morgan's Laws:** $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$ $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$
- **Distributive Laws:** $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
- **Implication Equivalence:** $P \rightarrow Q \equiv \neg P \vee Q$
- **Contrapositive:** $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
- **Biconditional Equivalence:** $P \equiv Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
- **Double Negation:** $\neg(\neg P) \equiv P$

**TA Tip: The Implication Pitfall**

The expression $P \rightarrow Q$ is only false when a true premise leads to a false conclusion ($T \rightarrow F$). If the premise $P$ is false, the implication is **vacuously true**. This is a common source of confusion but is essential for mathematical reasoning.

## 3 Part II: Predicate Logic (Prädikatenlogik)

*Predicate logic extends propositional logic by introducing variables, predicates, and quantifiers, allowing for statements about properties of objects and relationships between them.*

### 3.1 Predicates and Quantifiers

- **Universe of Discourse ($\mathbb{U}$):** The non-empty set of objects that variables can represent (e.g., integers, people, all cats).
- **Predicate:** A property that becomes a proposition when its variables are assigned values from the UoD. E.g., $P(x) = \text{x} > 3$.
- **Universal Quantifier ($\forall$):** "For all". $\forall x, P(x)$ is true if $P(x)$ is true for every $x$ in the UoD.
- **Existential Quantifier ($\exists$):** "There exists". $\exists x, P(x)$ is true if there is at least one $x$ in the UoD for which $P(x)$ is true.

### 3.2 Nested Quantifiers

**Procedure for Interpretation**

1. Read from left to right. The order is critical.
2. The choice for a variable bound by an inner quantifier can depend on the variables of the outer quantifiers.
3. Think of it as a nested loop or a challenge-response game: $\forall x$ means "for any $x$ an opponent gives you...", $\exists y$ means "...you can find a $y$ such that...".

**Simple Example**

UoD = Integers.

- $\forall x \exists y, x < y$: "For every integer, there is a larger integer." (True, choose $y = x + 1$).
- $\exists y \forall x, x < y$: "There exists an integer that is larger than all integers." (False, no maximum integer exists).

**Harder Example**

UoD = People. $L(x, y) = x$ loves $y$.

- $\forall x \exists y, L(x, y)$: "Everybody loves somebody." (The person loved can be different for each individual).
- $\exists y \forall x, L(x, y)$: "There is somebody who is loved by everybody." (A single, universally loved person exists).

### 3.3 Negating Quantified Statements

**Procedure (De Morgan's for Quantifiers)**

1. Place a $\neg$ in front of the entire quantified statement.
2. "Push" the $\neg$ inward across each quantifier one by one.
3. Each time the $\neg$ passes a quantifier, the quantifier flips ($\forall$ becomes $\exists$, and vice versa).
4. Once inside, apply standard propositional De Morgan's laws to the predicate expression.

**Simple Example**

$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$

$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$

**Harder Example**

**Statement:** "All students who studied passed the exam." $\forall x((S(x) \wedge T(x)) \rightarrow P(x))$

**Negation Procedure:**

1. $\neg(\forall x((S(x) \wedge T(x)) \rightarrow P(x)))$
2. $\equiv \exists x \neg((S(x) \wedge T(x)) \rightarrow P(x))$ (Flip $\forall$, push $\neg$ in)
3. $\equiv \exists x \neg(\neg(S(x) \wedge T(x)) \vee P(x)))$ (Implication law)
4. $\equiv \exists x (S(x) \wedge T(x)) \wedge \neg P(x))$ (De Morgan's & Double Negation)

**Meaning:** "There exists someone who is a student, studied, and did not pass."

## 3.4 Translating Natural Language

### Procedure for Translation

1. Define the Universe of Discourse ($\mathbb{U}$).
2. Define predicates for each property (e.g., $C(x)$ for "$x$ is a cat").
3. Identify the main logical structure ($\forall, \exists, \to, \land$).
4. Translate piece by piece, adhering to the standard patterns.

### Standard Patterns

- "All A's are B's": $\forall x(A(x) \to B(x))$
- "Some A's are B's": $\exists x(A(x) \land B(x))$
- "No A's are B's": $\forall x(A(x) \to \neg B(x))$
- "Not all A's are B's": $\exists x(A(x) \land \neg B(x))$

**The Golden Rule of Translation:**

- Use $\to$ as the main connective with $\forall$.
- Use $\land$ as the main connective with $\exists$.

**Why?**

- $\forall x(A(x) \land B(x))$ means "Everything in the universe is both an A and a B". This is almost always too strong.
- $\exists x(A(x) \to B(x))$ means "There exists something that, if it's an A, is also a B". This is true if there's just one thing in the UoD that is **not** an A, making it too weak and usually not what is intended.

## 4 Part III: Proof Techniques (Beweismuster)

*This section outlines the fundamental strategies for constructing mathematical proofs. A systematic approach involves identifying the claim's structure and selecting the most appropriate technique.*

### 4.1 Direct Proof (Direkter Beweis)

#### Procedure

To prove an implication $P \to Q$:

1. Assume $P$ is true.
2. Use definitions, axioms, and established theorems to build a logical chain of deductions.
3. Conclude that $Q$ must be true.

#### Simple Example

**Claim:** If $n$ is an odd integer, then $n^2$ is odd.

**Proof:** Assume $n$ is odd. By definition, $n = 2k + 1$ for some integer $k$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Let $m = 2k^2 + 2k$. Since $k$ is an integer, $m$ is an integer. Thus, $n^2 = 2m + 1$, which is the definition of an odd number.

### 4.2 Proof by Contraposition (Kontraposition)

#### Procedure

To prove $P \to Q$, instead prove its logically equivalent contrapositive, $\neg Q \to \neg P$. This is often simpler when the conclusion $Q$ is a negative statement.

1. Assume $\neg Q$ is true.
2. Follow logical steps to show that $\neg P$ must be true.

#### Simple Example

**Claim:** For an integer $n$, if $n^2$ is even, then $n$ is even.

**Proof:** The contrapositive is "If $n$ is not even (odd), then $n^2$ is not even (odd)." This is precisely the statement proven in the Direct Proof example above. Since the contrapositive is true, the original statement is true.

### 4.3 Proof by Contradiction (Widerspruchsbeweis)

#### Procedure

To prove a statement $P$:

1. Assume $\neg P$ is true.
2. From this assumption, derive a logical contradiction (a statement of the form $R \land \neg R$).
3. Conclude that the assumption $\neg P$ must be false, hence $P$ is true.

#### Harder Example: Infinitude of Primes

**Claim:** There are infinitely many prime numbers.

**Proof:**

1. Assume for contradiction that there is a finite number of primes. Let them be $p_1, p_2, \ldots, p_n$.
2. Consider the number $N = (p_1 \times p_2 \times \ldots \times p_n) + 1$.
3. $N$ must have a prime factor. Let this prime factor be $p$.
4. This prime $p$ must be one of the primes in our list, so $p = p_i$ for some $i$.
5. This means $p_i$ divides $N$. But $p_i$ also divides the product $p_1 \times \ldots \times p_n$.
6. If $p_i$ divides both numbers, it must divide their difference: $N - (p_1 \times \ldots \times p_n) = 1$.
7. **Contradiction:** No prime number can divide 1.
8. Therefore, the assumption of a finite number of primes is false.

### 4.4 Proof by Cases (Fallunterscheidung)

#### Procedure

1. Partition the problem domain into a set of exhaustive cases $C_1, C_2, \ldots, C_k$.
2. Prove the statement for each case individually.
3. Since the cases cover all possibilities, the statement holds universally.

#### Simple Example

**Claim:** For any integer $n$, $n^2 \geq n$.

**Proof:**

- **Case 1: $n \geq 1$.** Multiplying both sides of $n \geq 1$ by the positive number $n$ gives $n^2 \geq n$.
- **Case 2: $n = 0$.** $0^2 \geq 0$ becomes $0 \geq 0$, which is true.
- **Case 3: $n < 0$.** Here, $n^2$ is non-negative, while $n$ is negative. Any non-negative number is greater than any negative number, so $n^2 > n$.

Since the statement holds in all three exhaustive cases, it is true for all integers.

### 4.5 Proof by Induction (Vollständige Induktion)

#### Procedure (Weak Induction)

To prove $\forall n \geq n_0, P(n)$:

1. **Base Case (Induktionsanfang):** Verify $P(n_0)$ is true.
2. **Inductive Hypothesis (Annahme):** Assume $P(k)$ is true for an arbitrary $k \geq n_0$.
3. **Inductive Step (Schritt):** Using the hypothesis, prove that $P(k+1)$ is also true.

#### Simple Example (Weak)

**Claim:** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n \geq 1$.

**Proof:**

- **Base Case (n=1):** $\sum_{i=1}^{1} i = 1$. And $1\frac{1+1}{2} = 1$. True.
- **Hypothesis:** Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.
- **Step:** Show for $k+1$: $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

(by hypothesis) $= (k+1)(\frac{k}{2} + 1) = (k+1)\frac{k+2}{2}$. This is the required formula for $n = k+1$.

#### Procedure (Strong Induction)

*Strong induction assumes truth for all prior cases up to k, while weak assumes only for k.*

1. **Base Case(s):** Verify $P(n_0)$ (and possibly more initial cases).
2. **Hypothesis:** Assume $P(j)$ is true for **all** integers $j$ where $n_0 \leq j \leq k$.
3. **Step:** Using the hypothesis, prove $P(k+1)$ is true.

#### Harder Example (Strong)

**Claim:** Any postage of $n \geq 12$ cents can be made with 4- and 5-cent stamps.

**Proof:**

- **Base Cases:** $P(12) : 3 \times 4$. True. $P(13) : 2 \times 4 + 1 \times 5$. True. $P(14) : 1 \times 4 + 2 \times 5$. True. $P(15) : 3 \times 5$. True.
- **Hypothesis:** Assume for an arbitrary $k \geq 15$, $P(j)$ is true for all $j$ with $12 \leq j \leq k$.
- **Step:** We want to show $P(k+1)$. Consider the postage for $k-3$. Since $k \geq 15$, $k - 3 \geq 12$. By our strong hypothesis, we know we can make postage for $k-3$. To get postage for $k+1$, we simply add a 4-cent stamp: $(k-3) + 4 = k+1$. Thus, $P(k+1)$ is true.