Answer 4(a)

Pmf of Uniform RV, $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ CDF, $F(x) = \int_a^x f(x)\,dx = \int_a^x \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^x dx = \frac{1}{b-a}[x]_a^x = \frac{x-a}{b-a}$ $\therefore F(x) = \begin{cases} 0, & x < \\ \frac{x-a}{b-a}, & \\ 1, & x \end{cases}$

$$\therefore E(X) = \int_a^b x\,f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\cdot\frac{b^2-a^2}{2}$$

$$= \frac{a+b}{2}.$$

$$E(X^2) = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\,\frac{b^3-a^3}{3}$$

$$= \frac{1}{b-a}\cdot\frac{(b-a)(b^2+ab+a^2)}{3}$$

$$= \frac{a^2+ab+b^2}{3}.$$

$$\therefore \mathrm{var}(X) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}.$$

Answer 1(b)

pdf of exponential, $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

$\therefore$ CDF, $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{x} f(t)\,dt = \int_{0}^{x} \lambda e^{-\lambda t}\,dt$

$= \lambda \left[\dfrac{-e^{-\lambda t}}{\lambda}\right]_0^x = \left[-e^{-\lambda x} + e^{0}\right] = 1 - e^{-\lambda x}$

$E(X) = \displaystyle\int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{0}^{\infty} x\,f(x)\,dx = \int_{0}^{\infty} \lambda x e^{-\lambda x}\,dx$

- Using integration by parts:

Let $x = u$ $\therefore$ $dx = du$

Let $dv = \lambda e^{-\lambda x}\,dx$

$\Rightarrow \displaystyle\int dv = \int \lambda e^{-\lambda x}\,dx$

$\Rightarrow v = -e^{-\lambda x}$

Now, $\displaystyle\int u\,dv = uv - \int v\,du$

$\Rightarrow E(X) = \left[-xe^{-\lambda x}\right]_0^{\infty} - \displaystyle\int_0^{\infty} -e^{-\lambda x}\,dx$

$= \left[-x e^{-\lambda x}\right]_0^{\infty} + \left[\dfrac{-e^{-\lambda x}}{\lambda}\right]_0^{\infty} = 0 + \dfrac{e^{0}}{\lambda} = \dfrac{1}{\lambda}$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

let $x^2 = u \Rightarrow 2x = \frac{du}{dx} \Rightarrow 2x\,dx = du$

let $dv = \lambda e^{-\lambda x}\,dx$

$$\Rightarrow v = \int \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]$$

$$\therefore \int u\,dv = uv - \int v\,du$$

$$\therefore E[X] = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, 2x\,dx$$

$$= 0 + 2\int_0^{\infty} x e^{-\lambda x}\,dx$$

$$= 2\int_0^{\infty} x e^{-\lambda x}\,dx$$

Again, let $x = u \Rightarrow dx = du$

let $dv = e^{-\lambda x}\,dx$

$$\Rightarrow v = \int e^{-\lambda x}\,dx = -\frac{e^{-\lambda x}}{\lambda}$$

$$\therefore \int u\,dv = uv - \int v\,du = \left[-x\frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty} + \int_0^{\infty} \frac{e^{-\lambda x}}{\lambda}\,dx$$

$$= 0 + \left[\frac{-e^{-\lambda x}}{\lambda^2}\right]_0^{\infty} = \frac{1}{\lambda^2}$$

$$\therefore E(X^2) = 2\frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

$$\therefore Var(X) = E(X^2) - (E(X))^2$$
$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Answer 4(c)

pdf of a Gaussian, $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$-\infty < x < \infty$

$\therefore$ CDF, $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt$

There is no closed-form solution. So, it is often expressed in terms of standard normal ~~dist~~ RV.

let $Z = \frac{X-\mu}{\sigma}$ which is also a Normal RV with $E(Z)=0$ and $var(Z)=1$

$$\therefore f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

$$\therefore \text{CDF, } \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{(\mu+\sigma z)}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \sigma\, dz$$

from std normal,
$z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$
$\Rightarrow dx = \sigma\, dz$

$$= \int_{-\infty}^{\infty} \frac{\mu+\sigma z}{\sqrt{2\pi}\,\cancel{\sigma}} \exp\left(-\frac{z^2}{2}\right) \cancel{\sigma}\, dz$$

$$= \int_{-\infty}^{\infty} (\mu+\sigma z) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz + \int_{-\infty}^{\infty} \sigma z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz}_{\text{this portion is the total probability distribution of } z \text{ which amounts to } 1.} + \sigma \underbrace{\int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz}_{\text{this portion is an odd function. Its integral b/w symmetric interval equals } 0}$$

$$= \mu \cdot 1 + \sigma \cdot 0$$

$$E(X) = \mu$$

$$\therefore Var(X) = \underbrace{E(X^2)}_{\text{we need only this one.}} - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

from the calculations of $E(X)$, we can use $x = \mu + \sigma z$

$$\therefore E(X^2) = \int_{-\infty}^{\infty} (\mu + \sigma z)^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \mu^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{z^2}{2}\right)} dz}_{\text{total probability, so 1.}} + 2\mu\sigma \underbrace{\int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz}_{\text{integral of odd fn over a symmetric range is zero}} + \sigma^2 \underbrace{\int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz}_{\text{integration by parts shown in the next pg.}}$$

$$\int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Integration by parts:

let $u = z$, $du = dz$

$$dv = z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$\int dv = \int z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

let $w = \frac{z^2}{2} \Rightarrow dw = z\,dz$

$$\therefore v = \int \frac{1}{\sqrt{2\pi}} e^{-w} dw$$

$$= \frac{-1}{\sqrt{2\pi}} e^{-w} = \frac{-1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\therefore \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = \left[z \frac{(-1)}{\sqrt{2\pi}} e^{-z^2/2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

The first part, $e^{-z^2/2}$ decays faster than z grows. So, it equals to 0.

The second part $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ is the CDF of standard normal distribution evaluated over whole interval. So, it equals 1.

$$\therefore E(x^2) = \mu^2(1) + 2\mu\sigma'(0) + \sigma'^2(1)$$

$$= \mu^2 + \sigma'^2$$

$$\therefore Var(x) = E(x^2) - (E(x))^2$$

$$= \mu^2 + \sigma'^2 - \mu^2$$

$$= \sigma'^2$$