MATH 588
Introduction to FEM

Homework assignment 1
Date assigned: January 31, 2025
Due date: **February 7, 2025**

Problem

Consider the following second-order ODE:

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = f(x), \quad \text{for } 0 \leq x \leq 1,$$

with boundary conditions:

$$u(0) = 0, \quad u(1) = 0,$$

where $k(x) = 1$ and $f(x) = 2x$.

Tasks

1. **Weak Form Derivation:** Derive the weak form of the given ODE. Write the integral form of the weak equation explicitly.
2. **Finite Element Discretization:**
   (a) Divide the domain $[0, 1]$ into 2 equal elements ($0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$).
   (b) Use linear basis functions for each element. Define the basis functions $\varphi_1(x), \varphi_2(x), \varphi_3(x)$ corresponding to the three nodes at $x = 0, 0.5, 1$.
   (c) Derive the local stiffness matrix for each element. Write out the global stiffness matrix and load vector in matrix form.
3. **Assembly:** Assemble the global stiffness matrix and global load vector from the local contributions. Apply the boundary conditions to modify the system.
4. **Solve:** Write the resulting linear system of equations in the form $A\mathbf{u} = \mathbf{b}$, where $\mathbf{u} = [u_1, u_2, u_3]^T$. Solve the system.
5. **Solve ODE analytically:** Solve given boundary-value problem analytically and compare this solution to the numerical solution.

Deliverables

1. A step-by-step derivation of the weak form.
2. Expressions for the basis functions and the local stiffness matrices.
3. The assembled global stiffness matrix and load vector with applied boundary conditions.
4. A clear and neat presentation of the resulting system of equations.
5. Comparsion of numerical and analytical solutions

Notes

- Assume piecewise linear basis functions. For example:

$$\phi_1(x) = 1 - \frac{x}{0.5}, \quad \phi_2(x) = \frac{x}{0.5} \quad \text{on the first element.}$$

- Show all work for the derivation of integrals and assembly process.
- Use consistent units and notation throughout the assignment.

Submission

Submit your work as a PDF with handwritten or typed solutions.