Question 4.

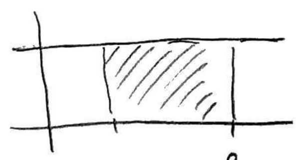
Compute cdf, $E(X)$, and $Var(X)$ for the
(a) uniform,
(b) experiential, and
(c) Gaussian random variables
(See your notes for the definitions of these three distributions)

(b) Exponential $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Sol-n.

(a) pdf of of a uniform distribution X is

$$(1) \quad f(x, a, b) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$



hence cdf of unif. dist. X is

$$(2) \quad F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(y)\,dy = \underbrace{\int_{-\infty}^{a} f(y)\,dy}_{=0} + \int_{a}^{x} f(y)\,dy;$$

$$(3) \quad F(x) = \mathbb{P}(X \leq x) = \int_{a}^{x} f(y)\,dy = \int_{a}^{x} \frac{1}{b-a}\,dy =$$

$$= \frac{1}{b-a}\left[y\right]_{a}^{y=x} = \frac{x-a}{b-a}$$

so

$$(4) \quad F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b. \end{cases}$$

$\square$

The expected value $E(X)$ or mean value μ_X of a unif. distr. X with pdf $f(x)$ is

$$(5) \quad E(X) = \mu_X := \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{a} x \underbrace{f(x)}_{=0} dx + \int_{a}^{b} x \cdot \underbrace{f(x)}_{=\frac{1}{b-a}} dx + \int_{b}^{\infty} x \underbrace{f(x)}_{=0} dx$$

$$= 0 + \int_a^b x \cdot \frac{1}{b-a}\, dx + 0$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_{x=a}^{b} = \frac{a+b}{2};$$

$$(5) \quad E(X) = \mu_X = \frac{a+b}{2};$$

(6) For $Var(X) = E(X^2) - [E(X)]^2$, we need

$$(7) \quad E(X^2) = \int_a^b x^2 \underbrace{f(x)}_{=\frac{1}{b-a}} dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_{x=a}^{b} = \frac{a^2+ab+b^2}{3};$$

$$(8) \quad Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \left[\frac{a+b}{2}\right]^2 = \frac{(a-b)^2}{12}$$

$$Var(X) = \frac{(a-b)^2}{12};$$