Question 2.

A discrete r.v. X has the ↓ pmf (pdf)

$$(1) \quad p_X(x) = \begin{cases} 0.1 & \text{if } x = 0.2 \\ 0.1 & \text{if } x = 0.4 \\ 0.2 & \text{if } x = 0.5 \\ 0.3 & \text{if } x = 0.7 \\ 0.3 & \text{if } x = 1 \end{cases}$$

Sol-n:

(a) Find the range of the r.v. X?

Def: Range of R.V. X is the set of all possible outcomes (values) that R.V. X can get;

$$(2) \quad \text{Range}(X) = \{0.2, \ 0.4, \ 0.5, \ 0.7, \ 1\}$$

(b) Compute $\mathbb{P}(X \leq 0.5)$ and $Var(X) = ?$

As we know, CDF, $F(x)$ is as follows.

$$(3) \quad F(x) = \mathbb{P}(X \leq x) = \sum_{y: y \leq x} \mathbb{P}(X = y) = \sum_{y: y \leq x} p_X(y);$$

$$\mathbb{P}(X \leq 0.5) = \sum_{y: y \leq 0.5} p_X(y) = p_X(0.2) + p_X(0.4) + p_X(0.5) =$$

$$= 0.1 + 0.1 + 0.2$$

$$= 0.4;$$

$$(4) \quad F(0.5) = \mathbb{P}(X \leq 0.5) = 0.4.$$

Whereas, $Var(X)$ is found by

(5) $Var(X) = E(X^2) - (E(X))^2$ where

(6) $E(X) = \mu_X = \sum_{x \in Range(X)} x \cdot p_X(x)$ & similarly

(7) $E(X^2) = \sum_{x \in Range(X)} x^2 \cdot p_X(x)$ or in general

(8) $E[h(X)] = \sum_{x \in Range(X)} h(x) \cdot p_X(x)$ where $h(X) = X^2$ in our case.

Then

$$E(X) = \sum_{x \in R(X)} x \; p_X(x) = 0.2 \cdot p_X(0.2) + 0.4 \cdot p_X(0.4) +$$

$$+ 0.5 \cdot p_X(0.5) + 0.7 \cdot p_X(0.7) + 1 \cdot p_X(1) =$$

$$= 0.2 \cdot (0.1) + 0.4 \cdot (0.1) + 0.5 \cdot (0.2) + 0.7 \cdot (0.3) + 1 \cdot (0.3) = 0.67$$

(9) $E(X) = \mu_X = 0.67$; and using (7)

$$E(X^2) = (0.2)^2 \cdot p_X(0.2) + (0.4)^2 \cdot p_X(0.4) + (0.5)^2 \cdot p_X(0.5) +$$

$$+ (0.7)^2 \cdot p_X(0.7) + (1)^2 \cdot p_X(1) =$$

$$= (0.2)^2 \cdot (0.1) + (0.4)^2 \cdot (0.1) + (0.5)^2 (0.2) + (0.7)^2 \cdot (0.3) + (1)^2 (0.3)$$

$$= 0.517.$$

(10) $E(X^2) = 0.517$, thus by (5)

(11) $Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2 = 0.52 - (0.67)^2 = 0.0711$

(c) Compute $\mathbb{P}(0.25 \le X \le 0.75)$.

We know that

(12) $\mathbb{P}(a \le X \le b) = F(b) - F(a^-)$

where "a^-" represents the largest possible X value that is strictly less than a.

Or continuing (12),

(13) $$\mathbb{P}(a \le X \le b) = \sum_{y:\, a \le y \le b} \mathbb{P}(X = y) = \sum_{y:\, a \le y \le b} p_X(y), \text{ so}$$

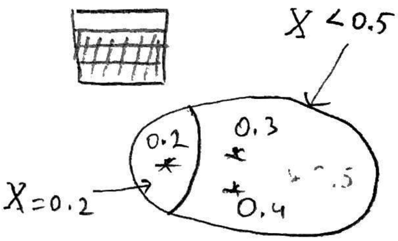
$$\mathbb{P}(0.25 \le X \le 0.75) = \sum_{0.25 \le y \le 0.75} p_X(y) =$$

$$= p_X(0.4) + p_X(0.5) + p_X(0.7)$$

$$= 0.1 + 0.2 + 0.3$$

$$= 0.6.$$

(d) Compute $\mathbb{P}(X = 0.2 \mid X < 0.5)$.



Using conditional probability law

(14) $$\mathbb{P}(X = 0.2 \mid X < 0.5) = \frac{\mathbb{P}[(X = 0.2) \cap (X < 0.5)]}{\mathbb{P}(X < 0.5)} = \frac{\mathbb{P}(X = 0.2)}{\mathbb{P}(X < 0.5)}$$

$$= \frac{p_X(0.2)}{p_X(0.2) + p_X(0.4)} = \frac{0.1}{0.1 + 0.1} = \frac{1}{2}.$$