Answer 5

$$f_{XY}(x,y) = \begin{cases} c(2x+y) & , \text{ if } 2 \le x \le 6 \text{ and } 0 \le y \le 5 \\ 0 & , \text{ otherwise} \end{cases}$$

first we compute c

CDF, $F_{XY}(x,y) = \int_2^6 \int_0^5 c(2x+y)\,dy\,dx$

$$= \int_2^6 \left[2cxy + \frac{cy^2}{2}\right]_0^5 dx$$

$$= \int_2^6 \left(10cx + \frac{25c}{2}\right) dx$$

$$= \left[10c\frac{x^2}{2} + \frac{25cx}{2}\right]_2^6$$

$$= 10c\frac{6^2}{2} + \frac{25(6)\cdot c}{2} - 10c\frac{2^2}{2} - \frac{25(2)c}{2}$$

$$= 210c$$

This CDF, $F_{XY}(x,y)$ equals 1 as $2 \le x \le 6$ and $0 \le y \le 5$

$\therefore\ 210c = 1$

$\therefore\ c = \frac{1}{210}$

$$f_X(x) = \int_0^5 c(2x+y)\,dy = \left[2cxy + \frac{cy^2}{2}\right]_0^5$$

$$= 2\left(\frac{1}{210}\right)(5)\,x + \frac{1}{210}\,\frac{5^2}{2}$$

$$= \frac{x}{21} + \frac{5}{84}$$

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$$f_Y(\cancel{x}y) = \int_2^6 c(2x+y)\,dx$$

$$= \frac{1}{210}\int_2^6 (2x+y)\,dx = \frac{1}{210}\left[x^2+xy\right]_2^6$$

$$= \frac{1}{210}\left(6^2+6y-2^2-2y\right)$$

$$= \frac{1}{210}(32+4y)$$

$$= \frac{2y}{105}+\frac{16}{105}$$

$$P(X<Y) = \int_2^5\int_2^y c(2x+y)\,dx\,dy$$

$$= \frac{1}{210}\int_2^5\int_2^y (2x+y)\,dx\,dy = \frac{1}{210}\int_2^5\left[x^2+xy\right]_2^y dy$$

$$= \frac{1}{210}\int_2^5 \left(y^2+y^2-4-2y\right)dy = \frac{1}{210}\left[\frac{2y^3}{3} + \cancel{y^3} - 4y - \cancel{2}y^2\right]_2^5$$

$$= \frac{1}{210}\left[\frac{2(5)^3}{3} - 4(5) - 5^2 - \frac{2(2)^3}{3} + 4(2) + 2^2\right]$$

$$= \frac{3}{14}$$

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To compute the covariance matrix we need to find Var(X), Var(Y) and cov(X,Y).

$$\text{Var}(X) = E(X^2) - (E(X))^2$$
$$\text{Var}(Y) = E(Y^2) - (E(Y))^2$$
$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E(X) = \int_2^6 x f_X(x)\,dx = \int_2^6 x\left(\frac{x}{21} + \frac{5}{84}\right) dx = \int_2^6 \left(\frac{x^2}{21} + \frac{5x}{84}\right) dx$$

$$= \left[\frac{x^3}{63} + \frac{5x^2}{168}\right]_2^6 = \frac{6^3}{63} + \frac{5(6)^2}{168} - \frac{2^3}{63} - \frac{5(2)^2}{168} = \frac{268}{63}$$

$$E(X^2) = \int_2^6 x^2 f_X(x)\,dx = \int_2^6 x^2\left(\frac{x}{21} + \frac{5}{84}\right) dx = \int_2^6 \left(\frac{x^3}{21} + \frac{5x^2}{84}\right) dx$$

$$= \left[\frac{x^4}{84} + \frac{5x^3}{252}\right]_2^6 = \frac{6^4}{84} + \frac{5(6)^3}{252} - \frac{2^4}{84} - \frac{5(2)^3}{252} = \frac{1220}{63}$$

$$E(Y) = \int_0^5 y f_Y(y)\,dy = \int_0^5 y\left(\frac{2y+16}{105}\right) dy = \frac{1}{105}\int_0^5 (2y^2 + 16y)\,dy$$

$$= \frac{1}{105}\left[\frac{2y^3}{3} + \frac{16y^2}{2}\right]_0^5 = \frac{1}{105}\left(\frac{2(5)^3}{3} + 8(5)^2\right) = \frac{170}{63}$$

$$E(Y^2) = \int_0^5 y^2 f_Y(y)\,dy = \int_0^5 y^2\left(\frac{2y+16}{105}\right) dy = \frac{1}{105}\int_0^5 (2y^3 + 16y^2)\,dy$$

$$= \frac{1}{105}\left[\frac{2y^4}{4} + \frac{16y^3}{3}\right]_0^5 = \frac{1}{105}\left[\frac{5^4}{2} + \frac{16(5)^3}{3}\right] = \frac{1175}{126}$$

$$E[XY] = \int_2^6 \int_0^5 xy\, f_{XY}(x,y)\, dy\, dx$$

$$= \int_2^6 \int_0^5 xy \frac{1}{210}(2x+y)\, dy\, dx = \frac{1}{210} \int_2^6 \int_0^5 \left(2x^2y + xy^2\right) dy\, dx$$

$$= \frac{1}{210} \int_2^6 \left[x^2y^2 + \frac{xy^3}{3}\right]_0^5 dx = \frac{1}{210} \int_2^6 \left(25x^2 + \frac{125x}{3}\right) dx$$

$$= \frac{1}{210} \left[\frac{25x^3}{3} + \frac{125x^2}{6}\right]_2^6 = \frac{1}{210}\left(\frac{25(6)^3}{3} + \frac{125(6)^2}{6} - \frac{25(2)^3}{3} - \frac{125(2)^2}{6}\right)$$

$$= \frac{80}{7}$$

$$\therefore Var(X) = E(X^2) - (E(X))^2 = \frac{1220}{63} - \left(\frac{268}{63}\right)^2 = 1.27$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{1175}{126} - \left(\frac{170}{63}\right)^2 = 2.04$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{80}{7} - \left(\frac{268}{63} \cdot \frac{170}{63}\right) = -0.05$$

$$\therefore \text{covariance matrix} = \begin{bmatrix} 1.27 & -0.05 \\ -0.05 & 2.04 \end{bmatrix}$$

$$\text{Correlation} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{-0.05}{\sqrt{1.27 \times 2.04}} = -0.031$$

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