Q6.

Prove all properties of conditional expectation for random variable (r.v.).

Sol-n:

Recall that for

(a) R.V.s X & Y on $(\Omega, \mathcal{F}, \mathbb{P})$

(b) $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ all finite σ-algebras

(c) $a, b \in \mathbb{R}$

the following prop-s hold:

① $E(X|\mathcal{G})(\omega) = \sum_i E(X|G_i) I_{G_i}(\omega)$

$$= \sum_i I_{G_i}(\omega) \sum_{\omega \in \Omega} X(\omega') I_{G_i}(\omega') \cdot \frac{\mathbb{P}(\{\omega'\})}{\mathbb{P}(G_i)}$$

$$= \sum I_{G_i}(\omega) \cdot \sum_{\omega' \in G_i} X(\omega') \cdot \frac{\mathbb{P}(\{\omega'\})}{\mathbb{P}(G_i)}$$

$$= \sum_i I_{G_i}(\omega) X_i \underbrace{\sum_{\omega' \in G_i} \frac{\mathbb{P}(\omega')}{\mathbb{P}(G_i)}}_{=1}$$

$$= \sum_i X_i I_{G_i}(\omega) = X(\omega),$$ thus

$E(X|G) = X$ ✓

①

② $$E(aX+bY\mid G)(\omega) = \sum_i E(aX+bY\mid G_i)\, I_{G_i}(\omega)$$

$$\overset{E(\#)=\#}{=} \sum_i I_{G_i}(\omega) \cdot \sum_{\omega' \in G_i} (aX+bY)(\omega') \frac{\mathbb{P}(\omega')}{\mathbb{P}(G_i)}$$

$$= \sum_i I_{G_i}(\omega) \sum_{\omega' \in G_i} \left(aX(\omega') + bY(\omega')\right) \frac{\mathbb{P}(\omega')}{\mathbb{P}(G_i)}$$

$$= \sum_i I_{G_i}(\omega) \cdot \sum_{\omega' \in G_i} aX(\omega') \cdot \frac{\mathbb{P}(\omega')}{\mathbb{P}(G_i)} + \sum_i I_{G_i}(\omega) \cdot \underbrace{\sum_{\omega' \in G_i} bY(\omega') \frac{\mathbb{P}(\omega')}{\mathbb{P}(G_i)}}_{E(Y\mid G_i)}$$

$$= a\sum_i E(X\mid G_i) \cdot I_{G_i}(\omega) + b\sum_i E(Y\mid G_i) \cdot I_{G_i}(\omega)$$

$$= a\,E(X\mid G)(\omega) + b\,E(Y\mid G)(\omega) . \Rightarrow$$

$$E(aX+bY\mid G) = aE(X\mid G) + bE(Y\mid G).$$

③ $$E(X\mid G)(\omega) = \sum_i E(X\mid G_i)\, I_{G_i}(\omega)$$

$$= \sum_i I_{G_i}(\omega) \underbrace{\sum_{\omega' \in G_i} X(\omega') \cdot \mathbb{P}(\omega')}_{\geq 0 \text{ since } X \geq 0} . \Rightarrow$$

$\Rightarrow E(X\mid G) \geq 0$ if $X \geq 0$.

(4) $$E[E(X|G)] = \sum_{\omega\in\Omega} E(X|G)(\omega)\,\mathbb{P}(\omega)$$

$$= \sum_i E(X|G_i)\cdot\mathbb{P}(G_i)$$

$$= \sum_i \frac{E(X\, I_{G_i})}{\cancel{\mathbb{P}(G_i)}}\cdot\cancel{\mathbb{P}(G_i)}$$

$$= \sum E(X I_{G_i}) = E\Big(X \sum_i I_{G_i}\Big)$$

$$= E(X). \quad \checkmark$$

(5) $$E[X|G](\omega)$$

$$= \sum_i E(X|G_i)\, I_{G_i}(\omega)$$

$$= \sum_i I_{G_i}(\omega) \sum_{i} x(\omega') \frac{\mathbb{P}(\{\omega'\}\cap G_i)}{\mathbb{P}(G_i)}$$

$$= \sum_i I_{G_i}(\omega)\cdot \sum_{\omega\in\Omega} \frac{X(\omega')\,\mathbb{P}(\{\omega'\})\,\mathbb{P}(G_i)}{\mathbb{P}(G_i)}$$

$$\sum_i I_{G_i}(\omega)\cdot E(X) = E(X)\overbrace{\sum_i G_i(\omega)}^{=1}$$

$$= E(X). \quad \Rightarrow$$

$$E(X|G) = E(X). \checkmark$$

⑥ $E[XY|G)(\omega)$

$$= \sum_i E(XY|G_i) \cdot I_{G_i}(\omega)$$

$$= \sum_i I_{G_i}(\omega) \sum_{\omega' \in G_i} X(\omega)\, Y(\omega) \frac{P(\omega')}{P(G_i)}$$

$$= \sum_i I_{G_i}(\omega)\, X_i \sum_{\omega' \in G_i} Y(\omega) \frac{P(\omega')}{P(G_i)}$$

$$= \sum_i I_{G_i}(\omega)\, X_i\, E(Y|G_i)$$

$$= X(\omega)\, E(Y|G)(\omega) . \Rightarrow$$

$$E(XY|G) = X\, E(Y|G) . \checkmark$$

And finally

⑦ $E[E(X|G)|H)(\omega)$

$$= \sum_i E[E(X|G)|H_i] \cdot I_{H_i}(\omega)$$

$$= \sum_i I_{H_i}(\omega) \cdot \sum_{\omega' \in} E(X|G)(\omega') \frac{P(\omega')}{P(H_i)} =$$

$$= \sum_i \frac{I_{H_i}(\omega)}{\mathbb{P}(H_i)} \cdot \sum_m \underbrace{\sum_{\omega \in G_{m_i}} E(X|G)(\omega') \cdot \mathbb{P}(\omega')}$$

$$= \sum_i \frac{I_{H_i}(\omega)}{\mathbb{P}(H_i)} \cdot \sum_{m_i} \underbrace{E(X|G_{m_i})} \cdot \mathbb{P}(G_{m_i})$$

$$= \sum_i \frac{I_{H_i}(\omega)}{\mathbb{P}(H_i)} \cdot \sum_{m_i} \frac{1}{\cancel{\mathbb{P}(G_{m_i})}} \cdot \sum_{\omega' \in G_{m_i}} X(\omega') \cdot \mathbb{P}(\omega') \cdot \cancel{\mathbb{P}(G_{m_i})}$$

$$= \sum_i \frac{I_{H_i}(\omega)}{\mathbb{P}(H_i)} \cdot \underbrace{\sum X(\omega')\, \mathbb{P}(\omega')}$$

$$= \sum_i I_{H_i}(\omega) \cdot \underbrace{\sum_i \frac{X(\omega')\, \mathbb{P}(\omega')}{\mathbb{P}(H_i)}}$$

$$\sum_i I_{H_i}(\omega) \cdot E(X|H_i)$$

$$= E(X|H). \quad \Rightarrow$$

$$E[E(X|G)|H)] = E(X|H). \quad \checkmark$$

which completes all 6 properties' proof. $\blacksquare$