Q4.

Prove the $\downarrow$:

(a) $X_n \xrightarrow{m.s} X \implies X_n \xrightarrow{P} X$

(b) $X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X$

Sol-n. [Preamble]

Recall convergence in probability can be written as

(1) $\lim\limits_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0 \implies$ ^ of B.S.

(2) $\lim\limits_{n \to \infty} \mathbb{P}(|X_n - X|^2 > \varepsilon^2) = 0.$

Then by Markov's inequality

(3) $\mathbb{P}(|X_n - X|^2 > \varepsilon^2) \leq \dfrac{E(|X_n - X|^2)}{\varepsilon^2}$, $\lim\limits_{n\to\infty}(\cdot)$ of B.S

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X|^2 > \varepsilon^2) \leq \underbrace{\lim_{n \to \infty} \frac{E(|X_n - X|^2)}{\varepsilon^2}}_{= 0} \text{ by (2)}, \implies$$

$\lim\limits_{n \to \infty} E(|X_n - X|^2) = 0$ which means the convergence of X_n to X in mean-square sense, hence

$X_n \xrightarrow{P} X \implies X_n \xrightarrow{m.s.} X$. ∎

(b) WMT: $\overbrace{X_n \xrightarrow{P} X}^{(1)} \implies X_n \xrightarrow{d} X$.

Let a be the point where

(2) $$\mathbb{P}(X=a)=0, \text{ and}$$

$$\varepsilon > 0.$$

So we have

(3) $$\mathbb{P}(X_n \le a) = \mathbb{P}(X_n \le a,\ |X_n - X| < \varepsilon) + \mathbb{P}(X_n \le a,\ |X_n - X| \ge \varepsilon)$$

$$\le \mathbb{P}(X \le a+\varepsilon) + \mathbb{P}(|X_n - X| \ge \varepsilon), \text{ also}$$

(4) $$\mathbb{P}(X \le a-\varepsilon) = \mathbb{P}(X \le a-\varepsilon,\ |X_n - X| < \varepsilon) + \mathbb{P}(X \le a-\varepsilon,\ |X_n - X| \ge \varepsilon)$$

$$\le \mathbb{P}(X_n \le a) + \mathbb{P}(|X_n - X| \ge \varepsilon) \implies \text{by this (4)}$$

(5) $$\underline{\mathbb{P}(X \le a-\varepsilon)} - \mathbb{P}(|X_n - X| \ge \varepsilon) \le \underline{\mathbb{P}(X_n < a)} \le \text{by (3)}$$

$$\le \underline{\mathbb{P}(X \le a+\varepsilon)} + \mathbb{P}(|X_n - X| \ge \varepsilon)$$

Then by definition of convergence in probability,

(6) $$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$$

we can write (5) with the underlined terms as

(7) $$\mathbb{P}(X \le a-\varepsilon) \le \mathbb{P}(X_n < a) \le \mathbb{P}(X \le a+\varepsilon)$$

(8) $$\implies \underbrace{\lim_{n\to\infty} \mathbb{P}(X \le a-\varepsilon)}_{\to \mathbb{P}(X \le a)} \le \lim_{n\to\infty} \mathbb{P}(X_n < a) \le \underbrace{\lim_{n\to\infty} \mathbb{P}(X \le a+\varepsilon)}_{= \mathbb{P}(X \le a) \text{ as}}$$

we're assuming $X_n \xrightarrow{P} X$, so $\varepsilon = 0$. $\Rightarrow$

(8) can be written as

(9) $\mathbb{P}(X \leq a) \leq \mathbb{P}(X_n < a) \leq \mathbb{P}(X \leq a)$ $\Rightarrow$

(10) $\boxed{\mathbb{P}(X \leq a) = P(X_n < a)}$

which satisfies the definition of convergence in distribution, hence

(11) $X_n \xrightarrow{d} X.$

i.e.

$$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X.$$

$\blacksquare$