Question 6: Stochastic Logistic Equation

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

We begin with the deterministic logistic model:

$$\frac{dy}{dt} = a(1-y)y \tag{1}$$

Stochastic Formulation

Assuming the parameter a has both deterministic and stochastic components $a \equiv (a + b\xi_t)$, we obtain the stochastic differential equation:

$$dY_t = a(1-Y_t)Y_t\,dt + b(1-Y_t)Y_t\,dW_t \tag{2}$$

To simplify analysis, we consider the multiplicative noise model:

$$dY_t = \underbrace{a(1-Y_t)Y_t}_{\mu_t}\,dt + \underbrace{bY_t}_{\sigma_t}\,dW_t, \qquad Y_0 = y_0. \tag{3}$$

(iii) Solving the Stochastic Logistic Equation

To solve (3), we use the transformation $Z_t = Y_t^{-1}$. Let $\varphi(Y_t) = Y_t^{-1}$. Then:

$$\varphi'(x) = -x^{-2},$$
$$\varphi''(x) = 2x^{-3}.$$

Applying Itô's lemma:

$$\begin{aligned}
dZ_t &= \left[\varphi'(Y_t)\mu_t + \frac{1}{2}\varphi''(Y_t)\sigma_t^2\right]dt + \varphi'(Y_t)\sigma_t dW_t && (4)\\
&= \left[-Y_t^{-2}\cdot a(1-Y_t)Y_t + \frac{1}{2}\cdot 2Y_t^{-3}(bY_t)^2\right]dt - Y_t^{-2}\cdot bY_t dW_t && (5)\\
&= \left[a(-Y_t^{-1}+1) + b^2Y_t^{-1}\right]dt - bY_t^{-1}dW_t && (6)\\
&= \left[a(1-Z_t) + b^2Z_t\right]dt - bZ_t dW_t. && (7)
\end{aligned}$$

Thus, the linear SDE for Z_t is:

$$dZ_t = (b^2 - a)Z_t dt + a dt - bZ_t dW_t. \tag{8}$$

This is solved using the integrating factor method. Define:

$$M_t = \exp\left(\int_0^t \left[a - \frac{1}{2}b^2\right] d\tau - bW_t\right). \tag{9}$$

Then the solution is:

$$Z_t = \frac{1}{M_t}\left(Z_0 + \int_0^t aM_s ds\right), \qquad Z_0 = y_0^{-1}. \tag{10}$$

Hence, the solution for $Y_t = Z_t^{-1}$ is:

$$Y_t = \frac{M_t}{y_0^{-1} + a\int_0^t M_s ds}. \tag{11}$$

(iv) Graphing the Solution

To visualize the stochastic logistic process:

- Choose constants a, b, and y_0(for the graph below a=2.0, b=0.6, $y_0 = 0.2$ are used, for more see the appendix please).
- Simulate paths using (11).
- Plot multiple sample paths and the expected logistic shape.

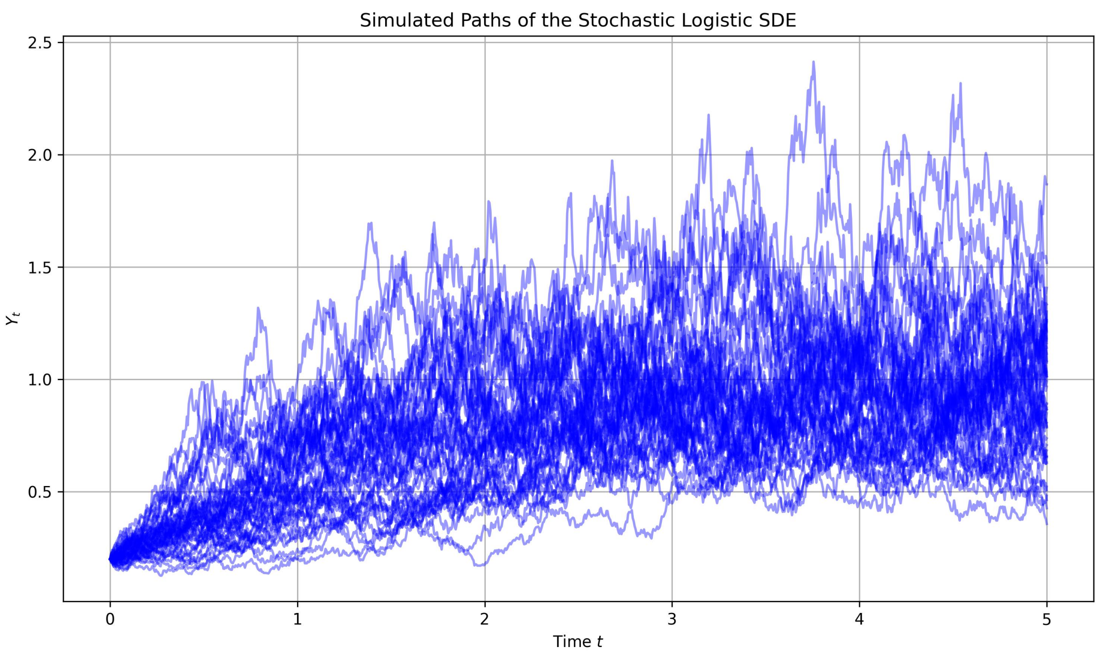

Figure 1: Simulated sample paths of the stochastic logistic process using (11).

(v) Statistics of the Logistic Process

Due to the nonlinearity of the SDE, exact formulas for mean and variance are difficult. But using the representation in (11):

- **Mean:** $\mathbb{E}[Y_t]$ cannot be written explicitly but can be approximated via Monte Carlo averaging of multiple sample paths.
- **Variance:** $\mathrm{Var}(Y_t)$ also must be computed empirically using simulated paths.
- **Long-Term Behavior:** As $t \to \infty$, the solution exhibits stationary-like behavior around a logistic mean $y = 1$.

(vi) Applications of the Stochastic Logistic Model

- **Population Dynamics:** Models biological populations where growth is subject to environmental fluctuations.
- **Epidemiology:** Describes how infection spreads under randomness, especially near saturation.
- **Finance:** Can model asset adoption dynamics under uncertainty (bounded growth).
- **Ecology and Conservation:** Accounts for random events affecting species in limited habitats.

References

References

[1] Allen, L. J. S. (2007). *An Introduction to Stochastic Processes with Applications to Biology*. Chapman and Hall/CRC.

[2] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd Edition.

[3] Mao, X. (2007). *Stochastic Differential Equations and Applications*. Horwood Publishing.

Appendix: Python Code for Question 6 (Stochastic Logistic SDE)

Listing 1: Stochastic Logistic Simulation (Q6)

```
# q6_logistic_simulation.py

import numpy as np
import matplotlib.pyplot as plt

# Parameters
a = 2.0          # deterministic growth rate
b = 0.6          # noise intensity
y0 = 0.2         # initial value
T = 5.0          # total time
N = 1000         # number of time steps
dt = T / N       # time step size
t = np.linspace(0, T, N + 1)
M = 50           # number of sample paths

# Preallocate simulation matrix
Y = np.zeros((M, N + 1))
Y[:, 0] = y0

# Precompute drift coefficient for M_t
alpha = a - 0.5 * b**2

# Simulate paths using exact solution
for i in range(M):
    dW = np.random.normal(0, np.sqrt(dt), size=N)
    W = np.insert(np.cumsum(dW), 0, 0)  # Brownian path with W(0) = 0
    M_t = np.exp(alpha * t - b * W)

    # Compute integral of M_t using left Riemann sum
    integral = np.cumsum(M_t[:-1]) * dt
    denom = y0**-1 + a * integral
    Y[i, 1:] = M_t[1:] / denom

# Plot results
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, Y[i], color='blue', alpha=0.4)

plt.title("Simulated Paths of the Stochastic Logistic SDE")
plt.xlabel("Time $t$")
```

```
plt.ylabel("$Y_t$")
plt.grid(True)
plt.tight_layout()

# Save and show the figure
plt.savefig("images/q6_logistic_simulation_graph.jpg", dpi=300)
plt.show()
```