Question 3

Use Itô's formula to show that the process

(1) $X_t = (W_t - t)\, e^{W_t - t/2}$ is

a martingale.

Sol-n,

Martingale property of Itô Integrals say

Let $T, S \in [0, t_f]$ s.t. $T > S$ &

(2) $I(T) = \int_0^T X_t \, dW_t$, then

(3) $E[I(T) \mid \mathcal{F}_S] = I(S)$.

For $X_t = (W_t - t)\, e^{W_t - \frac{t}{2}}$, we have

$\varphi(x,t) = (x - t)\, e^{x - \frac{t}{2}}$ =>

(4) $\varphi(x,t) = x e^{x - \frac{t}{2}} - t \cdot e^{x - \frac{t}{2}}$ => $\varphi(X_t, t)$

We use Ito formula of case II to apply $\varphi(x,t)$ in (4):

$$\frac{\partial \varphi}{\partial t} = x e^{x - \frac{t}{2}} \cdot \left(-\frac{1}{2}\right) - \left(e^{x - \frac{t}{2}} + t e^{x - \frac{t}{2}} \cdot \left(-\frac{1}{2}\right)\right)$$

(1) $$= \frac{-x e^{x - \frac{t}{2}}}{2} - \frac{2 e^{x - \frac{t}{2}}}{2} + \frac{t \cdot e^{x - \frac{t}{2}}}{2} \Rightarrow$$

(5) $\frac{\partial \varphi}{\partial t} = -e^{x-\frac{t}{2}}\left(\frac{x}{2}+1-\frac{t}{2}\right)$?

$$\frac{\partial \varphi}{\partial x} = e^{x-\frac{t}{2}} + x e^{x-\frac{t}{2}} - t e^{x-\frac{t}{2}}$$

$$= e^{x-\frac{t}{2}}(1+x-t) \quad \Rightarrow$$

(6) $\frac{\partial \varphi}{\partial x} = e^{x-\frac{t}{2}}(1+x-t) \quad \Rightarrow$

$\downarrow$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{x-\frac{t}{2}} + x e^{x-\frac{t}{2}} - t e^{x-\frac{t}{2}}\right]$$

$$= e^{x-\frac{t}{2}} + e^{x-\frac{t}{2}} + x e^{x-\frac{t}{2}} - t e^{x-\frac{t}{2}}$$

$$= e^{x-\frac{t}{2}}(2+x-t) \quad \Rightarrow$$

(7) $\frac{\partial^2 \varphi}{\partial x^2} = e^{x-\frac{t}{2}} \cdot (2+x-t)$

By case II of Itô's formula by (5), (6), (7)

$\downarrow$

$$dX_t = \left[\frac{\partial \varphi}{\partial t}(W_t, t) + \frac{1}{2}\cdot\frac{\partial^2 \varphi}{\partial x^2}(W_t, t)\right]dt + \frac{\partial \varphi}{\partial x}(W_t, t)\, dW_t$$

$$= \left[-e^{x-\frac{t}{2}}\left(\frac{x}{2}+1-\frac{t}{2}\right) + \frac{1}{2}\cdot e^{x-\frac{t}{2}} \cdot (2+x-t)\right]dt + e^{x-\frac{t}{2}}(1+x-t)\, dW_t$$

$\Rightarrow$

$$(8)\quad dX_t = \left\{\left[-\left(\frac{X}{2}+1-\frac{t}{2}\right)+\frac{1}{2}\left(2+X-t\right)\right]e^{X-\frac{t}{2}}\right\}dt + e^{X-\frac{t}{2}}\cdot\left(1+X-t\right)dW_t$$

$$= \left\{\underbrace{\left[-\frac{X}{2}-1+\frac{t}{2}+1+\frac{X}{2}-\frac{t}{2}\right]}_{=0} e^{X-\frac{t}{2}}\right\}dt + e^{X-\frac{t}{2}}\left(1+X-t\right)dW_t$$

$$= \underbrace{0\cdot e^{X-\frac{t}{2}}\,dt}_{\text{determ. part } = 0} + e^{X-\frac{t}{2}}\left(1+X-t\right)dW_t$$

$$= e^{X-\frac{t}{2}}\left(1+X-t\right)dW_t. \Rightarrow$$

$$(9)\quad dX_t = e^{X-\frac{t}{2}}\cdot\left(1+X-t\right)dW_t \quad \Big]_0^T$$

$$\Rightarrow \int_0^T dX_t = \int_0^t e^{X-\frac{t}{2}}\left(1+X-\tau\right)dW_t \Rightarrow$$

$$(10)\quad X_T = X_0 + \underbrace{\int_0^T e^{X-\frac{t}{2}}\cdot\left(1+X-t\right)dW_t}_{\text{Itô integral.}}$$

We've found that X_T is indeed an Itô integral, and we know that Itô Integrals are Martingales, hence X_t in (1) is a Martingale. ∎