Homework Assignment 3

MATH 588 - Introduction to FEM

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MATH 588
Introduction to FEM

Homework assignment 3
Date assigned: March 9, 2025
Due date: **March 24 , 2025**

Problem 1

Consider a one-dimensional problem with a Neumann boundary condition at $x = 1$:

$$-\frac{d^2p}{dx^2} = f(x), \quad 0 < x < 1$$
$$p(0) = \frac{dp}{dx}(1) = 0$$

Express this problem in a Galerkin variational formulation, formulate the finite element method using piecewise linear functions, and determine the corresponding linear system of algebraic equations for a uniform partition.

Problem 2

Carry out the derivation of the following basis function from the *Global coordinate approach* lecture:

$$\varphi_i(x) = \frac{x - x_{i-1/2}}{h_i}\left(\frac{2}{h_i}\left(x - x_{i-1/2}\right) + 1\right), \quad x \in [x_{i-1}, x_i]\,.$$

Submission

Submit your work as a PDF with handwritten or typed solutions.

Math 588. HW#3.

Problem 1.

One-dim-l- Neumann-boundary pr-m at $x=1$.

(1) $$-\frac{d^2p}{dx^2} = f(x), \quad 0 < x < 1 \;; \; x \in (0,1) =: \Omega$$

(2) $$\underbrace{p(0)}_{D} = \underbrace{\frac{dp}{dx}(1)}_{N} = 0 \quad (3)$$

(i) Galerkin variational for-n?
(ii) F.E.M with piecewise linear f-ns?
(iii) Linear system of equations?

Solution:

(i) Using the given PDE (ODE) in (1), & by letting $v(x) \in V$ s.t.

(4) $$L^2(\Omega) = \{ v \in \Omega : \int_\Omega v^2\,dx = \int_0^1 v^2\,dx < \infty \} \;\&$$

(5) $$V = H^1(\Omega) = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \} \;\&$$

(6) $$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega \}, \text{ so}$$

for $v \in H_0^1(\Omega)$, let's multiply BS of (1) by $v(x)$ & integrate over the domain $\Omega = (0,1)$.

(7) $$-\frac{d^2p}{dx} \cdot v(x) = f(x) \cdot v(x) \quad \Big\} \int_0^1 \cdot \text{ of BS.}$$

(8) $$\int_0^1 -\frac{d^2p}{dx} \cdot v(x)\,dx = \int_0^1 f(x) \cdot v(x)\,dx$$

Applying IBP to the LHS of (8), we get

$$(9)\quad \int_0^1 -p''(x)\,v(x)\,dx = -\int_0^1 v(x)\,d\big(p'(x)\big)$$

$$= -v(x)\cdot p'(x)\Big|_0^1 + \int_0^1 p'(x)\,d\,v(x)$$

$$= -v(1)\cdot \underbrace{p'(1)}_{\text{by (3)} = 0} + \underbrace{v(0)}_{=0 \text{ by (2)}}\cdot p'(0) + \int_0^1 p'(x)\cdot v'(x)\,dx.$$

$$= \underbrace{-v(1)\cdot 0 + 0\cdot p'(0)}_{0} + \int_0^1 p'(x)\cdot v'(x)\,dx$$

$$= 0 + \int_0^1 p'(x)\cdot v'(x)\,dx; \quad \text{so}$$

$$(10)\quad \int_0^1 -p''(x)\,v(x)\,dx = \underbrace{\int_0^1 p'(x)\cdot v'(x)\,dx}$$

plugging this back to (8), we have

$$(11)\quad \underbrace{\int_0^1 p'(x)\cdot v'(x)\,dx}_{:=} = \underbrace{\int_0^1 f(x)\cdot v(x)\,dx}_{:=} \Rightarrow$$

$$(12)\quad \langle p'(x), v'(x)\rangle = \langle f(x), v(x)\rangle$$ by our class notation, where

$$\langle f, g\rangle := \int_0^1 f(x)\cdot g(x)\,dx.$$

This (12) is called <u>Galerkin (weak) formulation</u> of (1)-(3)

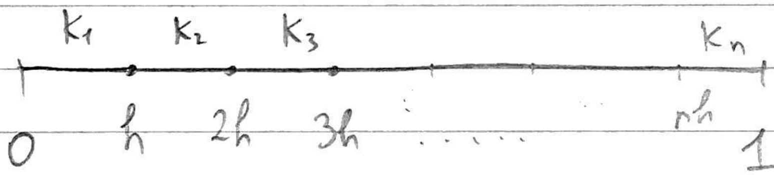
$$(12)\quad \boxed{\langle p'(x), v'(x)\rangle = \langle f(x), v(x)\rangle}$$

So we have (by class note) our Galerkin formulation in (12)

(12) $$\langle p'(x), \vartheta'(x) \rangle = \langle f(x), \vartheta(x) \rangle$$

Let K_h be the uniform partition of $\Omega = (0,1)$ into triangles. But in our case $\Omega = (0,1)$, a one-dim-l space, the triangles are all collapsed to small subintervals, mesh.

(13) Ω:



and of course by our uniform triangulation,

(13) $$h = \mathrm{diam}(K_i) \qquad \forall\, i = \overline{1,n}.$$

Now we introduce the finite element space, V_h, a discrete analog space of continuous space V.

(14) $$V_h = \{ \vartheta : \text{(i) } \vartheta \text{ is continuous f-n on } \Omega = (0,1)$$
$$\text{(ii) } \vartheta \text{ is linear on each } K_i,$$
$$\text{(iii) } \vartheta = 0 \text{ on } \partial\Omega, \text{ i.e. } \vartheta(0) = 0 \ \& \ \vartheta(1) = 0. \}$$

$V_h \subset V$.

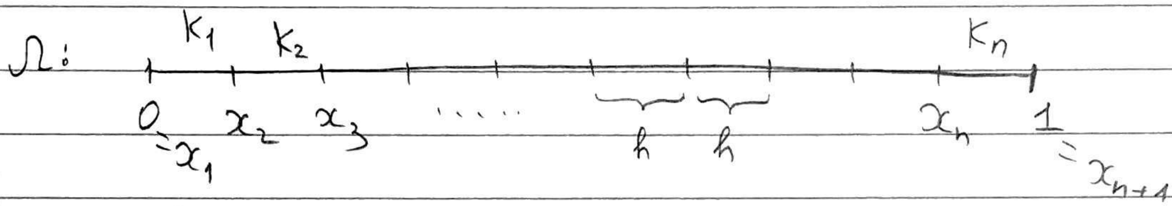
No need indeed, but doesn't hurt to have

The finite element method now can be formulated like in Dirichlet problem in HW#1.

Find $p_h \in V_h$ s.t., using (12)

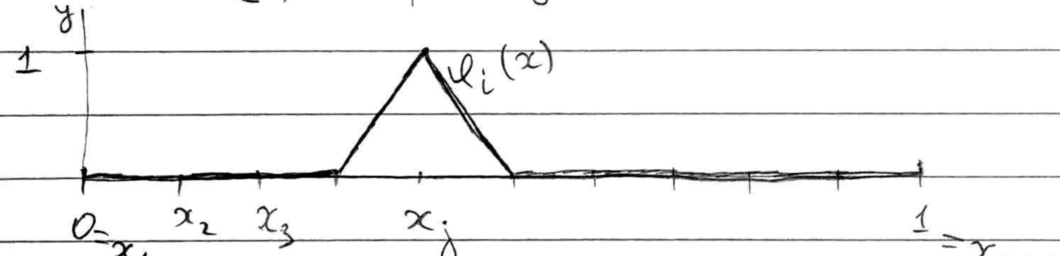
(15) $$< p_h'(x), v'(x) > = < f(x), v(x) > , \quad \underline{\forall v \in V_h}$$

Denote the vertices (nodes) of triangles (subintervals) in K by $x_1, x_2 \dots x_n$, so (13) is as ↓:

(16) 

The basis functions $\varphi_i \in V_h$, $i = 1, 2, \dots n+1$

(17) $$\varphi_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

(18) 

Let $x_2, x_3 \dots x_n$ be the interior points (nodes) in each K.

Since $p_h(x)$ satisfies (15) $\forall v \in V_h$, then

for choices of v, we can choose easy choice which is basis f-ns φ_i's i.e.

(19) $$v = \varphi_1 \in V_h, \quad v = \varphi_2 \in V_h, \quad \dots \quad v = \varphi_{n+1}(x) \in V_h$$

(19) satisfies (15), then we can write the ↓:

$< p_h'(x), \varphi_1'(x) > = < f(x), \varphi_1(x) >$ is valid by (15)
$< p_h'(x), \varphi_2'(x) > = < f(x), \varphi_2(x) >$ is valid by (15)
. . . .
$< p_h'(x), \varphi_{n+1}'(x) > = < f(x), \varphi_{n+1}(x) >$ is valid by (15), i.e.

(20) $< p_h'(x), \varphi_j'(x) > = < f(x), \varphi_j(x) > \qquad j = 1, 2, .. (n+1).$

Also $p_h \in V_h = span\{\varphi_1, \varphi_2, .. \varphi_{n+1}\}$, then $p_h(x)$ can be written by L.C. of basis f-ns φ_i's:

(21) $p_h(x) = \sum_{i=1}^{n+1} z_i \cdot \varphi_i(x)$ where

(22) $\boxed{z_i = p_h(x_i)}$ ← $p_h(x)$ is discrete, hence captures $p(x)$ at nodle points.

Then, using (21) the expression of $p_h(x)$, if we consider the inner product of $p_h'(x)$ with all basis f-ns φ_j''s, we come to linear sytem of equations for unknowns $z_1, z_2 .. z_{n+1}$:

(23) $< p_h'(x), \varphi_1'(x) > \overset{(21)}{=} < \left[\sum_{i=1}^{n+1} z_i \cdot \varphi_i(x)\right]', \varphi_1'(x) >$

$= < \sum_{i=1}^{n+1} z_i \varphi_i'(x), \varphi_1'(x) >$

$= z_1 < \varphi_1'(x), \varphi_1'(x) > + z_2 < \varphi_2', \varphi_1' > + .. + z_{n+1} < \varphi_{n+1}', \varphi_1' >$

Similarly

(24) $< p_h'(x), \varphi_2'(x) > = z_1 < \varphi_1'(x), \varphi_2'(x) > + z_2 < \varphi_2'(x), \varphi_2'(x) > +$
$+ .. + z_{n+1} < \varphi_{n+1}'(x), \varphi_2'(x) >$

And continuing this process gives

(25) $<p_h'(x), \varphi_{n+1}'(x)> =$

$= z_1 < \varphi_1'(x), \varphi_{n+1}'(x) > + z_2 < \varphi_2'(x), \varphi_{n+1}'(x) > + .. + z_{n+1} < \varphi_{n+1}'(x), \varphi_{n+1}'(x) >$

Yet on the other hand by (20), we have these left hand sides of (23), (24), (25):

(26) $$< p_h'(x), \varphi_j'(x) > = < f(x), \varphi_j(x) >,$$ hence $j = 1, 2, .., (n+1)$.

we have the ↓ system:

(27) $$\begin{cases} z_1 < \varphi_1', \varphi_1' > + z_2 < \varphi_2', \varphi_1' > + .. + z_{n+1} < \varphi_{n+1}', \varphi_1' > = < f, \varphi_1 > \\ z_1 < \varphi_1', \varphi_2' > + z_2 < \varphi_2', \varphi_2' > + .. + z_{n+1} < \varphi_{n+1}', \varphi_2' > = < f, \varphi_2 > \\ \cdots \\ z_1 < \varphi_1', \varphi_{n+1}' > + z_2 < \varphi_2', \varphi_{n+1}' > + .. + z_{n+1} < \varphi_{n+1}', \varphi_{n+1}' > = < f, \varphi_{n+1} > \end{cases} \Rightarrow$$

(28) $$\begin{bmatrix} < \varphi_1', \varphi_1' > & < \varphi_2', \varphi_1' > & \dots & < \varphi_{n+1}', \varphi_1' > \\ < \varphi_1', \varphi_2' > & < \varphi_2', \varphi_2' > & \dots & < \varphi_{n+1}', \varphi_2' > \\ \cdots & & & \\ < \varphi_1', \varphi_{n+1}' > & < \varphi_2', \varphi_{n+1}' > & \dots & < \varphi_{n+1}', \varphi_{n+1}' > \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} < f, \varphi_1 > \\ < f, \varphi_2 > \\ \cdots \\ < f, \varphi_{n+1} > \end{bmatrix}$$

(29) $$A \cdot \bar{z} = \bar{f}$$ ← is the required matrix eq-n for unknown $\bar{z}$.

where $A = [a_{ij}]_{i,j=1}^{n+1} = < \varphi_j', \varphi_i' >$;

$\bar{z} = [z_1, z_2 .. z_{n+1}]^T$ & $z_i = p_h(x_i) = p(x_i)$; $\boxed{}$

$\bar{f} = [< f, \varphi_1 >, < f, \varphi_2 > .. < f, \varphi_{n+1} >]^T$.

To determine the linear functions using the property in (17)

$$(30) \quad \varphi_j(x) = \begin{cases} \dfrac{x - x_j}{h}, & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases}$$

Then $a_{ij} = \langle \varphi_i', \varphi_j' \rangle = 0$ if $|i-j| > 0$.

Then the stiffness matrix A in (29) is tridiagonal, symmetric & positive-definite as follows:

$$(31) \quad A = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

completing the solution. ∎

Problem 2.

Global coordinate approach

(1) $$\varphi_i(x) = \frac{x - x_{i-\frac{1}{2}}}{h_i}\left[\frac{2}{h_i}\left(x - x_{i-\frac{1}{2}}\right) + 1\right], \quad x \in [x_{i-1}, x_i]$$

Solution:

This approach is about finding the quadratic basis functions φ_{i-1} and φ_i on a subinterval (x_{i-1}, x_i) s.t.

(2) $$\varphi_{i-1}(x_{i-1}) = 1, \quad \varphi_{i-1}(x_i) = 0, \quad \varphi_i(x_{i-1}) = 0, \quad \varphi_i(x_i) = 1,$$

a qudratic function $\varphi_{i-1}(x)$ is in the form

(3) $$\varphi_{i-1}(x) = a + bx + cx^2, \quad x \in [x_{i-1}, x_i]$$

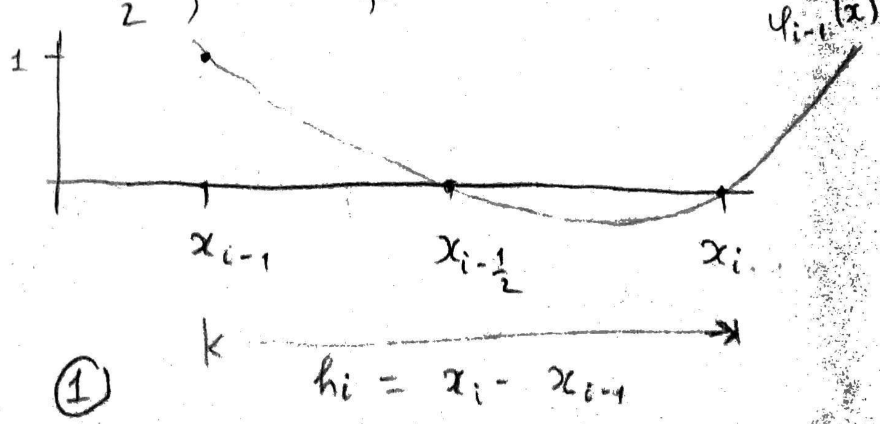
We have now two equations from (2):

(4) $$\varphi_{i-1}(x_{i-1}) = 1 \Leftrightarrow a + b\,x_{i-1} + c(x_{i-1})^2 = 1$$

(5) $$\varphi_{i-1}(x_i) = 0 \Leftrightarrow a + b\,x_i + c(x_i)^2 = 0$$

We need one more independent eq-n to find unknowns a, b, c. To do this, let's introduce an additional node $x_{i-\frac{1}{2}}$, as follows:

(6) $$\boxed{\begin{aligned} &\left.\begin{aligned} h_i &:= x_i - x_{i-1} \\ x_{i-\frac{1}{2}} &:= \frac{x_{i-1} + x_i}{2} \end{aligned}\right\} \Rightarrow \\ &x_i = x_{i-\frac{1}{2}} + \frac{h_i}{2} \quad \& \\ &x_{i-1} = x_{i-\frac{1}{2}} - \frac{h_i}{2} \end{aligned}}$$



So this $x_{i-\frac{1}{2}}$ point satisfies

(7) $\varphi_{i-1}(x_{i-\frac{1}{2}}) = 0 \iff a + b x_{i-\frac{1}{2}} + c x_{i-\frac{1}{2}}^2 = 0$;

Thus we have 3 equations, (4), (5), (7), to find unknowns a, b, c making a 3×3 sytem of linear equations to solve:

$$(8)\quad \begin{cases} a + b x_{i-1} + c(x_{i-1})^2 = 1 & \text{- - - - - -} \quad (4) \\ a + b x_i + c(x_i)^2 = 0 & \text{- - - -} \quad (5) \\ a + b x_{i-\frac{1}{2}} + c(x_{i-\frac{1}{2}})^2 = 0 & \text{- - -} \quad (7) \end{cases}$$

where x_i, x_{i-1}, $x_{i-\frac{1}{2}}$ are all

(4) − (5) subtraction gives:

$$\underbrace{b(x_{i-1} - x_i)} + c[(x_{i-1})^2 - (x_i)^2] = 1$$

$$-b \cdot h_i + c \cdot (-h_i) \cdot \underbrace{(x_{i-1} + x_i)} = 1 \qquad \} * (-1)$$

$$b h_i + c h_i \cdot (2 x_{i-\frac{1}{2}}) = -1$$

$$(9) \qquad \boxed{b h_i + 2 c h_i x_{i-\frac{1}{2}} = -1}$$

Now similarly subtructing eq-n (7) from eq-n (5) yields;

(5) − (7):

$$\underbrace{b \cdot (x_i - x_{i-\frac{1}{2}})} + c \cdot [\underbrace{(x_i - x_{i-\frac{1}{2}})} \cdot \underbrace{(\overset{\text{by (6)}}{x_i} + x_{i-\frac{1}{2}})}] = 0$$

$$b \cdot \frac{h_i}{2} + c \cdot \frac{h_i}{2} \cdot \left(x_{i-\frac{1}{2}} + \frac{h_i}{2} + x_{i-\frac{1}{2}} \right) = 0 \Rightarrow$$

(10) $\Rightarrow \boxed{b \cdot \frac{h_i}{2} + \frac{c h_i}{2}\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0}$ by eq. (5)-(7).

So by (9) & (10) we have now 2x2 system to solve for b & c :

(11) $$\begin{cases} b h_i + 2c\, h_i \cdot x_{i-\frac{1}{2}} = -1 \\ \frac{b h_i}{2} + \frac{c h_i}{2}\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0 \end{cases}$$ $\times 2$ of BS:

$$\Rightarrow \begin{cases} b h_i + c\, h_i \cdot (2x_{i-\frac{1}{2}}) = -1 \\ b h_i + c h_i \left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0 \end{cases}$$ eq (1) - (2):

$$c h_i \cdot \left(2 \cdot x_{i-\frac{1}{2}} - 2x_{i-\frac{1}{2}} - \frac{h_i}{2}\right) = -1 \Rightarrow$$

$$c = \frac{2}{h_i \cdot h_i} = \frac{2}{h_i^2};$$

(12) $$\boxed{c = \frac{2}{h_i^2}}$$

Then using (10) & substituting c in (12) back into the eq-n (12), we get

$$b = -c\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = -\frac{2}{h_i^2} \cdot \left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right)$$

$$= -\frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right); \Rightarrow$$

(13) $$\boxed{b = -\frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right)}$$ & finally for a,

using eq-n (7), we have

$$a = -b\,x_{i-\frac{1}{2}} - c\cdot\left(x_{i-\frac{1}{2}}\right)^2$$

$$= \frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right)\cdot x_{i-\frac{1}{2}} - \frac{2}{h_i^2}\cdot\left(x_{i-\frac{1}{2}}\right)^2 =$$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{4x_{i-\frac{1}{2}}}{h_i} + 1 - \frac{2x_{i-\frac{1}{2}}}{h_i}\right]$$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\cdot\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right]; \quad \Rightarrow$$

(14) $$\boxed{a = \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right]}$$, and putting

these (12), (13), (14) back to (3), we have

(15) $$\varphi_{i-1}(x) = a + bx + cx^2$$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right] - \frac{x}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right) + \frac{2x^2}{h_i^2}$$

$$= \frac{x - x_{i-\frac{1}{2}}}{h_i}\cdot\left[\frac{2}{h_i}\left(x - x_{i-\frac{1}{2}}\right) - 1\right], \quad x \in [x_{i-1}, x_i],$$ $\blacksquare$

as desired.