Question3.

Write about Ornstein-Uhlenbeck (OU) process.

Sol-n.

Consider the General form of Linear SDE (5) from Q1:

(1) $$dX_t = [\overbrace{a_1(t)}^{-a=const} X_t + \overbrace{a_2(t)}^{0}]dt + [\overbrace{b_1(t)}^{0} X_t + \overbrace{b_2(t)}^{b=const}]dW_t$$

if

(2) $$a_1(t) = -a = const, \; a_2(t) = 0, \; b_1(t) = 0, \; b_2(t) = b = const,$$

Then we have

Ornstein - Uhlenbeck equation :

(3) $$dX_t = -aX_t \, dt + b\, dW_t \qquad \text{with the IC:}$$
(4) $$X_t(0) = X_o$$

Note (3) is a non-homogeneous SDE, so we need integrating factor as follows:

(5) $$M_t = e^{at}, \quad \text{then change of variable}$$

for (3) is

(6) $$Y_t = X_t M_t, \qquad so$$

$$\varphi(X_t, t) = X_t \cdot M_t = X_t \cdot e^{at}, \qquad \textcircled{1}$$

(7) $$\boxed{\varphi(X_t, t) = X_t \cdot e^{at}} \qquad \text{requires usage of}$$

Itô's formula of case IV:

$$Y_t = \varphi(X_t, t) = X_t \cdot e^{at} \quad \text{with} \quad (3);$$

$$dX_t = \underbrace{-aX_t}_{\mu_t} dt + \underbrace{b}_{\sigma_t} dW_t$$

(8)
$$dY_t = \left[\frac{\partial \varphi(X_t, t)}{\partial t} + \mu_t \frac{\partial \varphi(X_t, t)}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 \varphi(X_t, t)}{\partial x^2} \right] dt$$

$$+ \sigma_t \frac{\partial \varphi(X_t, t)}{\partial x} dW_t, \qquad \text{so} \quad \text{we need}$$

(9) $\begin{cases} \varphi(x,t) = x \cdot e^{at}, \\[2mm] \dfrac{\partial \varphi(x,t)}{\partial t} = \dfrac{\partial}{\partial t}\left(x \cdot e^{at}\right) = axe^{at}; \\[3mm] \dfrac{\partial \varphi(x,t)}{\partial x} = \dfrac{\partial}{\partial x}\left(x e^{at}\right) = e^{at}; \\[3mm] \dfrac{\partial \varphi^2(x,t)}{\partial x^2} = \dfrac{\partial}{\partial x}\left(e^{at}\right) = 0; \end{cases}$

Evaluating expressions @ $x = X_t$ in (9) &
plugging back into (8) gives

(2)

$$dY_t = d(X_t e^{at}) = \left[\overbrace{aX_t e^{at}}^{0} + \overbrace{(-aX_t) \cdot e^{at}}^{\nearrow} + \overbrace{\frac{1}{2} \cdot (b)^2 \cdot 0}^{0} \right] dt +$$

$$+ (b) \cdot e^{at} dW_t \quad \Longrightarrow$$

(10) $\quad dY_t = b e^{at} dW_t \qquad$ which can be directly integrated:

$$\int_0^t dY_\tau = \int_0^t b e^{a\tau} dW_\tau \quad \Longrightarrow$$

(11) $\quad Y_t = Y_0 + \int_0^t b e^{a\tau} dW_\tau, \quad$ but by (3)

$$X_t e^{at} = X_0 \cdot \underbrace{e^{a \cdot 0}}_{=1} + \int_0^t b e^{a\tau} dW_\tau \quad \{ * e^{-at} \text{ of BS:}$$

(12') $\quad X_t = X_0 e^{-at} + e^{-at} \cdot \int_0^t b e^{a\tau} dW_t \qquad$ or

(12) $\quad \boxed{X_t = X_0 e^{-at} + \int_0^t b e^{a(\tau-t)} dW_\tau}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ is the

sol-n for OU eq-n in (3) + (4).

(iii) Plotting several statistic paths

(working with Zach on Wed (Thursday)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ③

(iv) Statistics

$$E[X_t] \overset{\text{by (12)}}{=} E\left[X_0 e^{-at} + \int_0^t b\, e^{a(\tau - t)}\, dW_\tau\right]$$

$$= E\left[X_0 e^{-at}\right] + E\left[\int_0^t b\, e^{a(\tau - t)}\, dW_\tau\right]$$

$$\underbrace{E[\#] = \#}_{} \quad \text{\&} \quad \underbrace{E[\text{Itô Integral}] = 0}_{}$$

$$= X_0 e^{-at}, \quad \text{i.e.}$$

(13)
$$E[X_t] = X_0 e^{-at}$$

$$\text{Var}(X_t) = \text{Var}\left(X_0 e^{-at}\right) + \text{Var}\left[\int_0^t b\, e^{a(\tau - t)}\, dw_\tau\right]$$

$$= \text{Var}\left[\int_0^t b\, e^{a(\tau - t)}\, dw_\tau\right] =$$

$$= E\left[\left(\int_0^t b\, e^{a(\tau - t)}\, dW_\tau\right)^2\right] - \underbrace{\left\{E\left[\int_0^t b\, e^{a(\tau - t)}\, dw_\tau\right]\right\}^2}_{= 0 \text{ by prev. part}}$$

Itô Isometry

$$= E\left[\int_0^t \left(b\, e^{a(\tau - t)}\right)^2 d\tau\right] =$$

Can bring Ⓔ inside integr'n $\int(\cdot)$

by 4.4. p2

④

$$= \int_0^t E\left(6^2 e^{2a(\tau-t)}\right) d\tau$$

$$= 6^2 \frac{1-e^{-2at}}{2a}, \qquad i.e,$$

(14)
$$\boxed{Var(X_t) = 6^2 \cdot \frac{1-e^{-2at}}{2a}}$$

Note that as $t \to \infty$

$$E(X_t) \to 0 \quad \& \quad Var(X_t) \to \frac{6^2}{2a} = const.$$

?1

(15) Auto Correlation $(X_t) =$

?2

↓

How to
calculate
statility
of GM

(4)

Covariance of OU:

By (12)

$$X_t = \underbrace{X_0 e^{-at}}_{g(t)} + \underbrace{be^{-at} \cdot \int_0^t e^{a\tau} dW_\tau}_{Y_t} \; , \; \text{let}$$

so

(15)
$$X_t = g(t) + b e^{-at} \cdot Y_t \; , \qquad \text{then}$$

$$Cov(X_t, X_s) = Cov\left[be^{-at} Y_t, \; (be^{-as}) Y_s \right]$$

$$= b^2 e^{-at} \cdot e^{-as} \; cov(Y_t, Y_s), \qquad \text{and}$$

WLOG, let $s < t$, then

(16)
$$Cov(X_s, X_t) = b^2 e^{-a(s+t)} \cdot Cov(Y_s, Y_t),$$

$$Cov(Y_s, Y_t) = \left[\int_0^s e^{a\tau} dW_\tau, \; \int_0^t e^{a\tau} dW_\tau \right]$$

$$= E[Y_s \cdot Y_t] - E[Y_s] \cdot E[Y_t]$$

$$= E\left[\int_0^s e^{a\tau} dW_\tau \cdot \int_0^t e^{a\tau} dW_\tau \right] - \underset{\nearrow 0 \; \boxed{E[It\hat{o} \, Int] = 0}}{}$$

$$- E\left[\underbrace{\int_0^s e^{a\tau} dW_\tau}_{0} \right] \cdot E\left[\int_0^t e^{a\tau} dW_\tau \right]$$

⑥

as, $s < t$

$$= E\left[\int_0^s e^{a\tau}dW_\tau \cdot \left(\int_0^s e^{a\tau}dW_\tau + \int_s^t e^{a\tau}dW_\tau\right)\right] - 0$$

$$= E\left[\left(\int_0^s e^{a\tau}dW_\tau\right)^2\right] + E\left[\int_0^s e^{a\tau}dW_\tau \cdot \int_s^t e^{a\tau}dW_\tau\right]$$

$\underset{\nearrow}{= d\tau}$ under first integral; $\underbrace{\quad}_{=0}$ ③ why

Itô
Isometry
$$= E\left[\int_0^s e^{2a\tau}d\tau\right] + 0$$

$\boxed{E(\#)=\#}$ arrow

$$= \frac{e^{2a}}{2a}\left[e^{\tau}\Big|_{\tau=0}^{\tau=1}\right] = \frac{e^{2a}}{2a}\cdot\left[e^s - 1\right], \text{ so}$$

(17) $\boxed{Cov(Y_s, Y_t) = \frac{e^{2a}}{2a}\left[e^s - 1\right].}$

Plugging (17) back to (16):

$$Cov(X_s, X_t) = b^2 e^{-a(s+t)} \cdot Cov(Y_s, Y_t)$$

$$= b^2 \cdot e^{-a(s+t)} \cdot \frac{e^{2a}}{2a}\left[e^s - 1\right]$$

$$= \frac{b^2}{2a} e^{a(-s-t+2)} \cdot \left[e^s - 1\right], \Rightarrow$$

(18) $\boxed{Cov(X_s, X_t) = b^2 \cdot \frac{e^{a(-s-t+2)}}{2a}\cdot\left[e^s - 1\right]}$

⑦

assuming $s < t$.

in General, presently with $\min\{s,t\}$:

(19)
$$\text{Cov}(X_s, X_t) = b^2 \cdot \frac{e^{a(-s-t+2)}}{2a} \cdot \left[e^{\min\{s,t\}} - 1\right]$$

Note as $t \to \infty$

(20)
$$\text{Cov}(X_s, X_t) = \frac{b^2}{2a} e^{|t-s|}$$

as desired.