HW #5

Question 1. Let

(1) $X = \int_0^1 e^{-\tau} dW_\tau$ and

(2) $Y = \int_0^2 \bar{e}^\tau dW_\tau$.

as X & Y are Wiener process.

Find

(3) $E[X] = E\left[\int_0^1 e^{-\tau} dW_\tau\right] = 0$,   and similarly

(4) $E[Y] = E\left[\int_0^2 e^{-\tau} dW_\tau\right] = 0$

(5) $E[X^2] = E\left[\left(\int_0^1 e^{-\tau} dW_\tau\right)^2\right] \overset{\text{Itô isometry}}{=} E\left[\int_0^1 (e^{-\tau})^2 d\tau\right]$

$= E\left[\int_0^1 e^{-2\tau} d\tau\right]$

$= E\left[-\frac{1}{2} e^{-2\tau} \Big|_0^1\right]$

$= E\left[-\frac{1}{2}\left(e^{-2} - e^0\right)\right]$

$= E\left[\frac{1}{2} - \frac{1}{2e^2}\right]$

$= \frac{1}{2} - \frac{1}{2e^2}$

(6) $\boxed{E[X^2] = \frac{1}{2} - \frac{1}{2e^2}}$

$E[Y^2] = E\left[\left(\int_0^2 e^{-\tau} dW_\tau\right)^2\right] \overset{\text{Itô isometry}}{=}$

$= E\left[\int_0^2 e^{-2\tau} d\tau\right] = E\left[-\frac{1}{2} e^{-2\tau} \Big|_0^2\right] = \frac{1}{2} - \frac{1}{2e^4}$
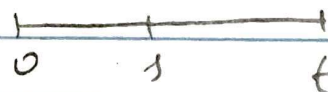
①

$$E[X \cdot Y] = E\left[\int_0^1 e^{-\tau} dw_\tau \cdot \int_0^2 e^{-\tau} dw_\tau\right]$$

$$= E\left[\int_0^1 e^{-\tau} dw_\tau \left\{\int_0^1 e^{-\tau} dw_\tau + \int_1^2 e^{-\tau} dw_\tau\right\}\right]$$

$$= E\left[\left(\int_0^1 e^{-\tau} dw_\tau\right)^2 + \int_0^1 e^{-\tau} dw_\tau \cdot \int_1^2 e^{-\tau} dw_\tau\right]$$

$$= E\left[\left(\int_0^1 e^{-\tau} dw_\tau\right)^2\right] + E\left[\int_0^1 e^{-\tau} dw_\tau \cdot \int_1^2 e^{-\tau} dw_\tau\right]$$

$$= \underbrace{E\left[\int_0^1 e^{-2\tau} d\tau\right]}_{\substack{\text{by part (a)} \\ \frac{1}{2} - \frac{1}{2e^2}}} + \underbrace{E\left[\int_0^1 e^{-\tau} dw_\tau \cdot \int_1^2 e^{-\tau} dw_\tau\right]}_{}$$

or

We know that   for

$0 \le s \le t$

$$\text{Cov}[X, Y] = E[XY] - \underbrace{E[X]}_{=0} \cdot \underbrace{E[Y]}_{=0 \text{ by previous parts}} \, ?$$

(?3) which note

covariance for <u>Wiener processes</u> is (by lecture note ?)

$$\text{Cov}(W_t, W_s) = \min(t, s), \qquad \text{to}$$

(?1)

$$\text{Cov}[X, Y] = \min\left\{\overset{①}{\int_0^1} e^{-\tau} dw_\tau, \overset{②}{\int_0^2} e^{-\tau} dw_\tau\right\} = \min\{1, 2\} = 1$$

rence $1 = E[XY] - 0 \implies \boxed{E[XY] = 1}$ ②