Question 2.

Write about Geometric Brownian motion.

Let's consider a general form Linear SDE (5) from part (a):

(1) $dX_t = [a_1(t)X_t + a_2(t)]\,dt + [b_1(t)X_t + b_2(t)]\,dW_t$

Case 2:

(2) $a_2(t) = b_2(t) = 0$, then (1) becomes

(3) $dX_t = a_1(t) \cdot X_t\,dt + b_1(t)X_t\,dW_t$ with the IC

(4) $X_t(0) = X_0$

. (3)+(4) is called a <u>Geometric Brownian motion</u>.

To come up with the "smart" change of variables we solve the "deterministic" part:

(5) $dX_t = a_1(t)\,X_t\,dt$ (Deterministic part of (3), an ODE)

$$\Downarrow$$

$$\frac{dx}{dt} = a_1(t)x \Rightarrow \underbrace{\frac{dx}{x}}_{} = a_1(t)dt \Rightarrow$$

$$d[\underbrace{\ln(x)}_{}] = a_1(t)\,dt \Rightarrow \overset{(6)}{\boxed{Y_t = \ln(x_t)}}$$

$$d\{Y_t\} = a_1(t)\,dt \Rightarrow \text{"easy" to integrate.}$$

$$\int d(Y_t) = \int a_1(t)\,dt \Rightarrow$$

$$Y_t = \int a_1(t)\,dt \dots$$

(6) $\boxed{Y_t = \ln(X_t)}$

Let's try the change of variable in (6) by Itô Formula to find $dX_t$:

We need to apply Itô's Formula for case iii:

(7) $Y_t = \ln(X_t)$, $dX_t = \mu_t\,dt + \sigma_t\,dW_t$,

(8) $dY_t = \left[\varphi'(X_t)\,\mu_t + \frac{1}{2}\varphi''(X_t)\,\sigma_t^2\right]dt + \varphi'(X_t)\,\sigma_t\,dW_t$

(9) $\varphi(x) = \ln(X_t)$, $\varphi'(x) = \frac{1}{x}$, $\varphi''(x) = -\frac{1}{x^2}$

$$dY_t = \left[\varphi'(X_t)\cdot a_1(t)\,X_t + \frac{1}{2}\varphi''(X_t)\,\overbrace{b_1^2(t)\,X_t^2}^{\sigma_t^2}\right]dt +$$

$$+ \left[\varphi'(X_t)\,\underbrace{b_1(t)\,X_t}_{\sigma_t}\right]dW_t \qquad ; \qquad \Rightarrow \text{using (9)}$$

$$dY_t = \left[\frac{1}{\cancel{X_t}}\,a_1(t)\,\cancel{X_t} + \frac{1}{2}\cdot\left(-\frac{1}{\cancel{X_t^2}}\right)\cdot b_1^2(t)\cdot\cancel{X_t^2}\right]dt + \frac{1}{\cancel{X_t}}\,b_1(t)\,\cancel{X_t}\,dW_t$$

(10) $\Rightarrow dY_t = \left[a_1(t) - \frac{1}{2}b_1^2(t)\right]dt + b_1(t)\,dW_t$

(10) is a drift-diffusion SDE, which can be directly integrated:

(11) $\int_0^t dY_\tau = \int_0^t \left[a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right]d\tau + \int_0^t b_1(\tau)\,dW_\tau \quad \Rightarrow$

$Y_t - Y_0 = \int_0^t \left(a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right)d\tau + \int_0^t b_1(\tau)\,dW_\tau$ ; $Y_t = \ln(X_t)$; $\Rightarrow$

(12) $\ln(X_t) = \ln(X_0) + \int_0^t \left[a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right]d\tau + \int_0^t b_1(\tau)\,dW_\tau \quad \Rightarrow$

e

(13) $$X_t = X_0 \, e^{\int_0^t \left[a_1(\tau) - \frac{b_1^2(\tau)}{2}\right] d\tau + \int_0^t b_1(\tau) dW_\tau}$$

is the solution to the homogenous SDE in (3)+(4) which is called Geometric Brownian motion

In the case that $\begin{cases} a_1(t) = a = \text{const} \\ b_1(t) = b = \text{const} \end{cases}$ & (14)

We have the standard (simple) Geometric Brownian motion (GBM)

(15) $$dX_t = a X_t \, dt + b X_t \, dW_t$$

which we have also solved using the method of matching coefficients.

(GBM appications for last part 4.3. P3)

(iii) Plotting several Stochastic paths:

(iv) Statistics:

Using (13),

$$E[X_t] = E\left[ x_0 \, e^{\int_0^t \left[a_1(\tau) - \frac{b_1^2(\tau)}{2}\right] d\tau + \int_0^t b_1(\tau)\, dW_\tau} \right]$$

?1 How to find its statistics