4.5 NonLinear SDEs

Stochastic logistic model

The deterministic stochastic model reads,

$$\frac{dy}{dt} = a(1-y) \quad (1)$$

Assuming that $a \equiv (a + b\xi_t)$ then we have

$$dY_t = a(1-Y_t)Y_t\,dt + b(1-Y_t)Y_t\,dW_t. \quad (2)$$

To simplify (2), we assume that the stochastic term is

$$bY_t\,dW_t$$

Stochasti logistic model eq.

$$\boxed{dY_t = \underbrace{a(1-Y_t)Y_t}_{\mu_t}\,dt + \underbrace{bY_t}_{\sigma_t}\,dW_t}$$

with $Y_0 = y_0$

a) Apply Itô's formula to: $Z_t = Y_t^{-1}$

(this hint comes from the general solution of the Bernoulli ODE)

This means that

$\varphi(x) = x^{-1}$, $\varphi'(x) = -x^{-2}$, $\varphi''(x) = 2x^{-3}$

and

$$dZ_t = \left(\varphi'(Y_t)\mu_t + \frac{1}{2}\varphi''(Y_t)\sigma_t^2\right)dt$$

$$+ \varphi'(Y_t)\sigma_t\, dW_t$$

$$= \left(-Y_t^{-2}\left(a(1-Y_t)Y_t\right) + \frac{1}{2}2Y_t^{-3}(bY_t)^2\right)dt$$

$$- Y_t^{-2}(bY_t)\,dW_t \to$$

$$dZ_t = \left[a(1-Y_t^{-1}) + b^2 Y_t^{-1}\right]dt - bY_t^{-1}\,dW_t \to$$

$$dZ_t = \left[a(1-Z_t) + b^2 Z_t\right]dt - bZ_t\,dW_t$$

or a Linear SDE:

$$dZ_t = \left[\underbrace{(b^2-a)}_{a_1}Z_t + \underbrace{a}_{a_2}\right]dt + \left(\underbrace{-b}_{b_1}Z_t + \underbrace{0}_{b_2}\right)dW_t$$

With solution *

$$Z_t = \frac{1}{M_t}\left(Z_0 + \int_0^t a M_\tau \, d\tau\right),$$

where $M_t = e^{(a - \frac{1}{2}b^2)t - bW_t}$

thus

$$\boxed{Y_t = \frac{M_t}{y_0^{-1} + a\int_0^t M_\tau \, d\tau}.}$$

* (see previous section)

$$X_t = \frac{1}{M_t}\left(x_0 + \int_0^t \left(a_2(\tau) - b_1(\tau) b_2(\tau)\right) M_\tau \, d\tau + \int_0^t b_2(\tau) M_\tau \, dW_\tau.\right)$$

where

$$M_t = e^{\int_0^t [-a_1(t) + \frac{1}{2} b_1^2(t)] dt - \int_0^t b_1(s) dW_t}$$

Bernoulli equation

$$y' + a(t)y = b(t)y^m.$$

$m=0$, First order ODE

$m=1$, separable

$m \geq 2$. nonlinear.

$\boxed{z = y^{1-m}}$ Change of var.

$$\boxed{z' + (1-m)a(t)z = (1-m)b(t).}$$

First order Linear ODE.

Solve by using the IF method.

Geometric mean-reverting process.

$$\boxed{dX_t = k(\theta - \log X_t)X_t\,dt + \sigma X_t\,dW_t.}$$

Change of Var $Y_t = \log X_t$.

Linear SDE

$$dY_t = \left[k(\theta - Y_t) - \frac{1}{2}\sigma^2\right]dt + \sigma\,dW_t$$

$$\downarrow$$

$$Y_t = \dots \Rightarrow X_t = \dots$$

CIR. (Cox-Ingersoll-Ross)

$$\boxed{dX_t = k(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t.}$$

Use:

$$Z_t = e^{kt}X_t \quad \text{and} \quad Z_t^2 = e^{2kt}X_t^2$$

$$\downarrow$$

$$\boxed{X_t = X_0 e^{-kt} + \theta\left[1 - e^{-kt}\right] + \int_0^t \sigma e^{-k(t-\tau)}\sqrt{X_\tau}\,dW_\tau}$$