Questions. Use Itô's formula the differentials of the processes :

(1) (a) $Y_t = W_t^3$

(b) $Y_t = \sin W_t$

(c) $Y_t = \cos(t^3 + W_t^2)$

(d) $Y_t = e^{W_t^2}$

(e)

$$Y_t = \int_0^t W_\tau^2 \, dW_\tau$$

(f) $Y_t = X_t^2$ where X_t is an Itô process,

$dX_t = a\,dt + b\,dW_t$, and a, b are constants.

Sol-n:

Based on the Differential Form of Itô's Formula when $Y_t = \varphi(W_t)$:

(2) $$d\varphi = \frac{1}{2}\varphi''(W_t)\,dt + \varphi'(W_t)\,dW_t ,$$

then

$$dY_t = \frac{1}{2} Y''(W_t)\,dt + Y'(W_t) \cdot dW_t$$

$$= \frac{1}{2} 6W_t \, dt + 3W_t^2 \, dW_t$$

$$= 3W_t \, dt + 3W_t^2 \, dW_t$$

(3) $$\boxed{dY_t = 3W_t\,dt + 3W_t^2\,dW_t}$$

(b) Similarly like in part (a), formula (2)

$$dY_t = \frac{1}{2} Y_t''(W_t)\,dt + Y_t'(W_t)\,dW_t \qquad \text{where } Y_t = \sin(W_t)$$

$$dY_t = \frac{1}{2}\left[\cos(W_t)\right]'dt + \cos(W_t)\,dW_t$$

$$= \frac{-\sin(W_t)}{2}dt + \cos(W_t)\,dW_t$$

(4) $$dY_t = \frac{-\sin(W_t)}{2}dt + \cos(W_t)\,dW_t$$

(c)

$$Y_t = \cos(t^3 + W_t^2)$$

(5) $$dY_t = \left[\frac{\partial Y_t(W_t, t)}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial x^2}(W_t, t)\right]dt + \frac{\partial Y_t}{\partial x}(W_t, t)\,dW_t$$

$$= \left\{\frac{\partial}{\partial t}\left[\cos(t^3 + W_t^2)\right] + \frac{1}{2}\frac{\partial^2}{\partial W_t^2}\left[\cos(t^3 + W_t^2)\right]\right\}dt$$

$$+ \left\{\frac{\partial}{\partial W_t}\left[\cos(t^3 + W_t^2)\right]\right\}dW_t$$

$$= \left\{-3t^2 \cdot \sin(t^3 + W_t^2) - 2W_t^2 \cos(t^3 + W_t^2)\right\}dt$$

$$- 2W_t \sin(t^3 + W_t^2)\,dW_t$$

(6) $$\boxed{dY_t = \left\{-3t^2 \cdot \sin(t^3 + W_t^2) - 2W_t^2 \cos(t^3 + W_t^2)\right\}dt - 2W_t \cdot \sin(t^3 + W_t^2)\,dW_t}$$

(d) $Y_t = e^{W_t^2}$ $\qquad Y_t = Y_t(W_t)$

Like in part (a), case I,

(7) $$dY_t = \frac{1}{2} Y_t''(W_t)\,dt + Y_t'(W_t)\,dW_t$$

$$= \frac{1}{2}\left(2W_t \cdot e^{W_t^2}\right)'_{W_t} dt + \left(2W_t \cdot e^{W_t^2}\right) dW_t$$

$$= \left[e^{W_t^2} + W_t \cdot 2W_t \cdot e^{W_t^2}\right] dt + \left(2W_t\, e^{W_t^2}\right) dW_t$$

$$= \left[1 + 2W_t^2\right] \underbrace{e^{W_t^2}}_{Y_t} dt + 2W_t \underbrace{e^{W_t^2}}_{Y_t} dW_t$$

so

(8) $$\boxed{dY_t = \left[1 + 2W_t^2\right] Y_t\,dt + 2W_t\, Y_t\,dW_t}$$

(e) $Y_t = \int_0^t W_\tau^2\, dW_\tau$, note this is

Itô process to match

(9) $$Y_t = Y_0 + \int_0^t \mu(X_\tau, \tau)\,d\tau + \int_0^t \sigma(X_\tau, \tau)\,dW_\tau$$

so for our integral in (e), we have

(10) $$Y_t = \underbrace{0}_{Y_0} + \int_0^t \underbrace{0}_{\mu(X_\tau,\tau)} d\tau + \int_0^t \underbrace{W_\tau^2}_{\sigma(X_\tau,\tau)} dW_\tau$$

and differential of Itô process in (9) is as follows:

(11) $dX_t = \underbrace{\mu(X_t, t)}_{0} dt + \underbrace{\sigma(X_t, t)}_{W_t^2} dW_t$, so

(12) $\boxed{dX_t = W_t^2 \, dW_t}$

(f)

$Y_t = X_t^2$ where

(13) $dX_t = \underbrace{a}_{\mu} dt + \underbrace{b}_{\sigma} dW_t$

Let

(14) $\varphi(x) = x^2 \Rightarrow \varphi'(x) = 2x \Rightarrow \varphi''(x) = 2$ & we have

(15) $dY_t = \left[\varphi'(X_t)\mu + \frac{1}{2}\varphi''(x)\sigma^2\right] dt +$

$+ \varphi'(x)\sigma \, dW_t$, so

$dY_t = \left[2X_t \cdot a + \frac{1}{2} \cdot 2 (b)^2\right] dt + 2X_t b \, dW_t$

(16) $\Rightarrow dY_t = \left[2X_t \cdot a + b^2\right] dt + 2X_t b \, dW_t$

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