Question 5.

Consider the process

(1) $F_t = f(t) \cdot g(W_t)$. Show that

Sol-n:

(a)
$$dF_t = f'(t)\, g(W_t)\, dt + \frac{1}{2} f(t)\, g''(W_t)\, dt + f(t)\, g'(W_t)\, dW_t$$

b $F_t = f(t) \cdot g(W_t) \Rightarrow$ let

(2) $F_t = \underbrace{\varphi(W_t, t)}_{\text{case II}} = f(t) \cdot g(W_t)$ &

(3) $\varphi(x, t) = f(t) \cdot g(x)$, then

we have for case II of Itô's Formula

(4) $$dF_t = \left[\frac{\partial \varphi}{\partial t}(W_t, t) + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2}(W_t, t)\right] dt + \frac{\partial \varphi}{\partial x}(W_t, t)\, dW_t$$

We need the f;

(5) $\frac{\partial \varphi}{\partial t}(x, t) = \frac{\partial}{\partial t}\left[f(t) \cdot g(x)\right] = f'(t) \cdot g(x)$

(6) $\frac{\partial \varphi}{\partial x}(x, t) = \frac{\partial}{\partial x}\left[f(t) \cdot g(x)\right] = f(t) \cdot g'(x)$ &

(7) $\frac{\partial^2 \varphi}{\partial x^2}(x, t) = f(t) \cdot g''(x)$, then putting (6)-(7) to (4):

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(8) $$dF_t = \left[f'(t)\cdot g(W_t) + \frac{1}{2} f(t)\, g'(W_t)\right] dt + f(t)\cdot g'(W_t)\, dW_t$$ as desired

$\square$

(b) Using case II integration form of Itô's formula (or integrating $\int_a^b$ (8) :)

(9) $$\int_a^b dF_t = \int_a^b \left[f'(t)\cdot g(W_t) + \frac{1}{2} f(t)\, g'(W_t)\right] dt + \int_a^b f(t)\cdot g'(W_t)\, dW_t$$

Let's solve (9) for $\int_a^b f(t)\cdot g'(W_t)\, dW_t$ as this is what we need to obtain:

(10) $$\int_a^b f(t)\, g'(W_t)\, dW_t = F_t \Big|_a^b - \int_a^b f'(t)\cdot g(W_t)\, dt - \frac{1}{2}\int_a^b f(t)\cdot g'(W_t)\, dt$$

but we know $F_t = f(t)\cdot g(W_t)$ so subbing it into (10):

yields

$$(11) \quad \int_a^b f(t)\cdot g'(W_t)\,dW_t = \overbrace{f(t)\cdot g(W_t)}^{F_t}\Big|_a^b - \int_a^b f'(t)\cdot g(W_t)\,dt - \frac{1}{2}\int_a^b f(t)\cdot g'(W_t)\,dt$$

as desired. $\boxed{}$

(c) Find

$$(12) \quad \int_0^T \underbrace{e^{\frac{t}{2}}}_{f(t)} \underbrace{\cos W_t}_{g'(W_t)}\,dW_t = ?$$

, in (12), let

$$(13) \quad f(t) = e^{\frac{t}{2}} \quad \&$$

$$(14) \quad g'(W_t) = \cos(W_t),$$

then applying formula (11) in part (b)

gives the desired result:

$$f'(t) = \frac{1}{2} e^{\frac{t}{2}} \; ; \; g(W_t) = \sin(W_t) \; ;$$

$$g''(W_t) = -\sin(W_t)$$

by (11)

$$\int_0^T \underbrace{e^{\frac{t}{2}}}_{f(t)} \underbrace{\cos(W_t)}_{g'(W_t)} dW_t = e^{\frac{t}{2}} \cdot \sin(W_t) \Big|_0^T$$

$$\underbrace{- \int_0^T \frac{1}{2} e^{\frac{t}{2}} \cdot \sin(W_t) dt - \frac{1}{2} \int_0^T e^{\frac{t}{2}} (-\sin(W_t)) dt}_{=0}$$

$$= e^{\frac{t}{2}} \cdot \sin(W_t) \Big|_{t=0}^{t=T} + 0$$

$$= e^{\frac{T}{2}} \sin(W_T) - e^0 \cdot \sin(\overbrace{W_0}^{=0}) =$$

$$= e^{\frac{T}{2}} \sin(W_T) - \sin(0)$$

$$= e^{\frac{T}{2}} \sin(W_T) \, .$$

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