Solution:

(i) Deriving semidiscrete variational formulation of the BVP:

(1) $$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad 0 < x < 1$$

(2) BC: $u(0,t) = u(1,t) = 0$

(3) IC: $u(x,0) = u_0(x)$, and

$$u(x,0) = u_0 = \begin{cases} 2x, & \text{for } x \in [0, 1/2] \\ 2-2x & \text{for } x \in [1/2, 1] \end{cases}$$

Let the time and the length spaces, $\mathcal{I}$, Ω, be

(4) $$\begin{cases} \mathcal{I} := (0, T) & \text{for some } T > 0, \text{ and} \\ \Omega := (0, 1) & \text{for } x \in \Omega. \end{cases}$$

the admissable space, V, be

(5) $$V := \{ \underbrace{v \in C(\Omega)}_{(i)} : \; v'(x) \text{ is } \underbrace{\text{piecewise continuous}}_{(ii)} \;\&\; \underbrace{\text{bounded}}_{(iii)} \text{ on } \Omega, \text{ and } \underbrace{v = 0}_{(iv)} \text{ on } \partial\Omega \}$$

Now to use the Green's formula to find the variational form for (1) - (3) follows as

find $u : \mathcal{I} \to V$ s.t.

(6) $\displaystyle\int_\Omega \frac{\partial u}{\partial t} v \, dx + \int_\Omega \nabla u \cdot \nabla v(x) \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in V, \ t \in J.$

(3) $u(x, 0) = u_0$ as in (3)

Let V_h be a finite element subspace of V. Replacing V in system (6) by its discrete analog space V_h, we have the finite element method:

find $u_h : J \to V_h$ s.t.

(7) $\displaystyle\int_\Omega \frac{\partial u}{\partial t} v \, dx + \int_\Omega \nabla u_h(x) \cdot \nabla v(x) \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in V_h \ \forall t \in J$

Also using the IC, we have

(3') $\displaystyle\int_\Omega u_h(x,0) \, v(x) \, dx = \int_\Omega u_0 \cdot v(x) \, dx \quad \forall v \in V_h$

This system is discretized in space, but continuous in time yet. So (7) + (3') is called semidiscrete scheme.

Let the basis functions in V_h be denoted by φ_i, $i = 1, 2, \ldots M$, and express $u_h(x) \in V_h$ as

(8) $\displaystyle u_h(x,t) = \sum_{i=1}^{M} u_i(t) \, \varphi_i(x) \qquad (x,t) \in \Omega \times J,$

For $j = 1, 2, \dots M$, we take $v = \varphi_j(x)$ in (7) & utilize (8) to get, for $t \in J$,

$$\int_\Omega \frac{\partial}{\partial t}\left[\sum_{i=1}^{M} u_i(t)\cdot\varphi_i(x)\right]\cdot\varphi_j(x)\,dx + \int_\Omega \nabla\left[\sum_{i=1}^{M} u_i(t)\,\varphi_i(x)\right]\cdot\nabla\varphi_j(x)\,dx = \int_\Omega f\cdot\varphi_j(x)\,dx$$

$$(7'') \Rightarrow \overbrace{\sum_{i=1}^{M}\int_\Omega \varphi_i(x)\cdot\varphi_j(x)\,dx}^{B}\cdot\frac{\partial u_i(t)}{\partial t} + \overbrace{\sum_{i=1}^{M}\int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,dx}^{A}\cdot u_i(t) = \int_\Omega f\cdot\varphi_j(x)\,dx$$

for $i, j = 1, 2, \dots M$ & IC in (3') is

$$(3'') \quad \sum_{i=1}^{M}\int_\Omega \varphi_i(x)\cdot\varphi_j(x)\,dx\cdot u_i(0) = \int_\Omega u_0\cdot\varphi_j(x)\,dx$$

which, in matrix form, is given by

$$(9) \quad B\,\frac{d\,u_i(t)}{dt} + A\cdot u_i(t) = f(t), \quad t \in J$$

$$(9') \quad B\cdot u_i(0) = u_0 \qquad i = 1, 2, \dots M$$

where

the A & B are $M \times M$ matrices, and, $\bar{u}$, $\bar{f}$, and $\bar{u}_0$ are vectors denoted by

$$(10)\quad \begin{cases} A = [a_{ij}], \quad a_{i,j} = \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, dx \\ B = [b_{ij}], \quad b_{ij} = \int_\Omega \varphi_i(x) \cdot \varphi_j(x) \, dx \\ \bar{u} = [u_j] \\ \bar{f} = [f_j], \quad f_j = \int_\Omega f \cdot \varphi_j(x) \, dx \\ \bar{u}_0 = [(u_0)_j], \quad [u_0]_j = \int_\Omega u_0 \, \varphi_j \, dx \end{cases}$$

Both A, B are symmetric & positive definite as is stationary (Laplace eq-n) case. $\boxtimes$

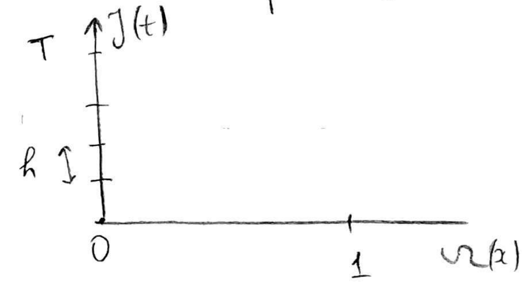
(ii) Using Forward-Euler method to discretize ( ) the system in (9) + (9') in time:

$$(9) \quad B \frac{d\bar{u}(t)}{dt} + A\bar{u}(t) = \bar{f}(t) \quad , \forall t \in J$$

$$(9') \quad B\,\bar{u}(0) = \bar{u}_0$$

Note that for <u>uniform meshing</u> in time space J:

$$(11) \quad \frac{d\bar{u}(t)}{dt} = \frac{\bar{u}(t+h) - \bar{u}(t)}{h} \quad ;$$



for small h (satisfying the stability condition).

Iterating (11) in terms of Euler-Method

$$(12) \quad \bar{u}(t) = \bar{u}(t_i) \quad \& \quad \bar{u}(t+h) = u(t_{i+1}) \quad , \text{so}$$

(9) + (9') is as follows:

$$B \cdot \frac{\bar{u}(t_{i+1}) - \bar{u}(t_i)}{h} + A \cdot \bar{u}(t_i) = \bar{f}(t_i) \quad \Rightarrow$$

$$(13) \quad B \cdot \bar{u}(t_{i+1}) = (B - hA) \cdot \bar{u}(t_i) + h \cdot \bar{f}(t_i)$$

Should we find $\bar{u}(t_{i+1})$ by always solving $Bu = z$ or $u = B^{-1} \cdot z$? 1