HW4 - Stat 544

Question 1: Martingale Transform

Show that a martingale transform is a martingale.

Solution:

Given:

- X_i is a martingale
- $Z_n = \sum_{i=1}^{n} A_i(X_i - X_{i-1})$, where A_i is $\mathcal{F}_{i-1}$-measurable

Proof:

1. **Integrability:** Since A_i is $\mathcal{F}_{i-1}$-measurable and X_i is integrable, each term $A_i(X_i - X_{i-1})$ is integrable, so Z_n is integrable.

2. **Adaptedness:** Let $X'_i = Z_i = \sum_{j=1}^{i} A_j(X_j - X_{j-1})$. Since A_j is $\mathcal{F}_{j-1}$-measurable and X_j is adapted, X'_i is adapted.

3. **Martingale Property:**

$$\begin{aligned}
\mathbb{E}[X'_i|\mathcal{F}_{i-1}] &= \mathbb{E}\left[X'_{i-1} + A_i(X_i - X_{i-1})\middle|\mathcal{F}_{i-1}\right] && (1)\\
&= X'_{i-1} + A_i\mathbb{E}[X_i - X_{i-1}|\mathcal{F}_{i-1}] && (2)\\
&= X'_{i-1} + A_i \cdot 0 && (3)\\
&= X'_{i-1} && (4)
\end{aligned}$$

Thus, X'_i is a martingale. ■

Question 2: Scaled Symmetric Random Walk

a) Show the scaled symmetric RW can be written as:

$$X_{[nt]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i \tag{5}$$

b) Show that:

$$\lim_{n\to\infty} X_{[nt]} \overset{d}{\Rightarrow} W_t \tag{6}$$

Solution:

a) For the symmetric random walk:

- $\xi_i = \begin{cases} \Delta x & \text{w.p. } 1/2 \\ -\Delta x & \text{w.p. } 1/2 \end{cases}$
- $X_n(t) = \sum_{i=1}^{n} \xi_i$, with $X_0(t) = 0$
- $\mathbb{E}[X_n(t)] = 0, \quad \mathrm{Var}(X_n(t)) = n(\Delta x)^2 = t\frac{(\Delta x)^2}{\Delta t}$

Setting $\frac{(\Delta x)^2}{\Delta t} = 1$ and $\Delta t = \frac{t}{n}$ gives:

$$X_{[nt]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i \quad \blacksquare \tag{7}$$

b) Convergence follows from:

1. $X_0(t) = 0$ matches $W_0 = 0$
2. Independent increments are preserved in the limit
3. By the Central Limit Theorem: $X_{[nt]} \sim \mathcal{N}(0, t)$ as $n \to \infty$

Thus, $X_{[nt]} \overset{d}{\Rightarrow} W_t$ as $n \to \infty$. $\blacksquare$

Question 3: Asymmetric Random Walk

Let $\xi_i = \begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } q \end{cases}$ and $X_n = \sum_{i=1}^{n} \xi_i$

a) Show $X_n \xrightarrow{a.s.} -\infty$

b) Show $Y_n = \left(\frac{q}{p}\right)^{X_n}$ is a martingale

Solution:

a) By the Strong Law of Large Numbers (SLLN):

$$\lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}[\xi_i] = p - q < 0 \tag{8}$$

$$\Rightarrow X_n \xrightarrow{a.s.} -\infty \quad \blacksquare \tag{9}$$

b) Martingale verification:

1. **Adaptedness:** Y_n is a function of X_n, which is adapted.
2. **Integrability:** Since $X_n \in \mathbb{Z}$ and $0 < \frac{q}{p} < 1$, $Y_n = \left(\frac{q}{p}\right)^{X_n}$ remains bounded (especially since $X_n \to -\infty$), so $\mathbb{E}[|Y_n|] < \infty$.
3. **Martingale Property:**

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = p\left(\frac{q}{p}\right)^{X_n+1} + q\left(\frac{q}{p}\right)^{X_n-1} \tag{10}$$

$$= \left(\frac{q}{p}\right)^{X_n}\left(p\cdot\frac{q}{p} + q\cdot\frac{p}{q}\right) \tag{11}$$

$$= \left(\frac{q}{p}\right)^{X_n}(q+p) = Y_n \quad \blacksquare \tag{12}$$

Question 4: Brownian Motion Properties

Show that for standard BM W_t:

a) $B_t = -W_t$ is BM

b) $B_t = cW_{t/c^2}$ is BM

Solution:

a) Properties for $B_t = -W_t$:

1. $B_0 = -W_0 = 0$
2. B_t has continuous paths (same as W_t)
3. Increments: $B_t - B_s = -(W_t - W_s) \sim \mathcal{N}(0, t-s)$
4. Independent increments preserved under negation

Hence, B_t is Brownian motion. ■

b) Properties for $B_t = cW_{t/c^2}$:

1. $B_0 = cW_0 = 0$
2. B_t has continuous paths (scaling preserves continuity)
3. Increments:

$$B_t - B_s = c(W_{t/c^2} - W_{s/c^2}) \sim \mathcal{N}(0, c^2(t/c^2 - s/c^2)) = \mathcal{N}(0, t-s) \tag{13}$$

4. Independent increments preserved

Hence, B_t is also Brownian motion. ■

Question 5: Quadratic Variation

Let $Z_i := |W_{t_{i+1}} - W_{t_i}|^2 - (t_{i+1} - t_i)$. Then:

A) For $Y_n = \sum_{i=0}^{n-1} Z_i$, show $\mathbb{E}[Y_n] = 0$

B) $2\sum_{i<j} \mathbb{E}[Z_i Z_j] = 0$

C) $\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4] = 3(t_{i+1} - t_i)^2$

D) $\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = (t_{i+1} - t_i)$

Solution:

A) By the result of part (D) (whose proof is shown below), we have

$$\mathbb{E}[Z_i] = \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] - (t_{i+1} - t_i) \tag{14}$$

$$= (t_{i+1} - t_i) - (t_{i+1} - t_i) = 0 \tag{15}$$

$$\Rightarrow \mathbb{E}[Y_n] = \sum_{i=0}^{n-1} \mathbb{E}[Z_i] = 0 \quad \blacksquare \tag{16}$$

B) By independence of Brownian increments:

$$\mathbb{E}[Z_i Z_j] = \mathbb{E}[Z_i] \cdot \mathbb{E}[Z_j] = 0 \quad \text{for } i \neq j \tag{17}$$

So:

$$2\sum_{i<j} \mathbb{E}[Z_i Z_j] = 0 \quad \blacksquare \tag{18}$$

C) Gaussian moment formula:

$$\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^4] = 3\left(\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2]\right)^2 = 3(t_{i+1} - t_i)^2 \quad \blacksquare \tag{19}$$

D)

$$\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = \mathrm{Var}(W_{t_{i+1}} - W_{t_i}) = t_{i+1} - t_i \quad \blacksquare \tag{20}$$

Question 6: Quadratic Variation Convergence

(E) Show $Q_2^{(n)}(W,T) \xrightarrow{m.s.} T$

Solution:

We want to show that:

$$\mathbb{E}\left[\left(Q_2^{(n)}(W,T) - T\right)^2\right] \to 0$$

Let $Z_i := (W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)$ so that:

$$Q_2^{(n)}(W,T) - T = \sum_{i=0}^{n-1} Z_i$$

Then:

$$\mathbb{E}[(Q_2^{(n)} - T)^2] = \mathbb{E}\left[\left(\sum_{i=0}^{n-1} Z_i\right)^2\right] \tag{21}$$

$$= \sum_{i=0}^{n-1} \mathbb{E}[Z_i^2] + 2\sum_{i<j} \mathbb{E}[Z_i Z_j] \tag{22}$$

From Question 5, we know:

$$\mathbb{E}[Z_i^2] = \mathrm{Var}((W_{t_{i+1}} - W_{t_i})^2) = 2(t_{i+1} - t_i)^2$$

$$\mathbb{E}[Z_i Z_j] = 0 \quad \text{for } i \neq j \quad \text{(independent increments)}$$

Thus:

$$\mathbb{E}[(Q_2^{(n)} - T)^2] = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2 \tag{23}$$

$$\leq 2\|\Pi_n\| \sum_{i=0}^{n-1} (t_{i+1} - t_i) \tag{24}$$

$$= 2\|\Pi_n\| T \to 0 \quad \text{as } \|\Pi_n\| \to 0 \quad \blacksquare \tag{25}$$

Question 7: Partition Properties

Prove:

(F) $\lim_{\|\Pi\|\to 0} \sum (t_{j+1} - t_j)^2 = 0$

(G) $\lim_{\|\Pi\|\to 0} \sum (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) = 0$

Solution:

(F) Since $\sum (t_{j+1} - t_j) = T$, we estimate:

$$\sum (t_{j+1} - t_j)^2 \le \|\Pi\| \sum (t_{j+1} - t_j) = \|\Pi\| T \to 0 \quad \text{as } \|\Pi\| \to 0 \quad \blacksquare \tag{26}$$

(G) By the Cauchy-Schwarz inequality:

$$\left| \sum (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) \right| \le \sqrt{\sum (W_{t_{j+1}} - W_{t_j})^2} \cdot \sqrt{\sum (t_{j+1} - t_j)^2} \tag{27}$$

We know:

$$\sum (W_{t_{j+1}} - W_{t_j})^2 = Q_2^{(n)}(W,T) \xrightarrow{m.s.} T, \quad \text{and} \quad \sum (t_{j+1} - t_j)^2 \to 0 \quad \text{as } \|\Pi\| \to 0$$

Therefore, the product converges to zero:

$$\left| \sum (W_{t_{j+1}} - W_{t_j})(t_{j+1} - t_j) \right| \to 0 \quad \text{as } \|\Pi\| \to 0 \quad \blacksquare$$