Question 8: CIR (Cox–Ingersoll–Ross) Model

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

The CIR model is described by the nonlinear SDE:

$$dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t, \qquad X_0 > 0. \tag{1}$$

where:

- X_t is the state variable (e.g., interest rate, volatility, population).
- $k > 0$ is the speed of mean reversion.
- $\theta > 0$ is the long-term mean level.
- $\sigma > 0$ is the volatility coefficient.
- W_t is a standard Brownian motion.
- X_0 is the initial condition.

This process is always positive provided the Feller condition $2k\theta \geq \sigma^2$ holds.

(iii) Solving the CIR SDE via Transformation

To solve the CIR SDE analytically is challenging, but we can express the solution using a variation of constants: We start by multiplying both sides of (1) by e^{kt}:

$$\begin{aligned}
Z_t &= e^{kt}X_t, && (2)\\
\Rightarrow dZ_t &= ke^{kt}X_t dt + e^{kt}dX_t \\
&= kZ_t dt + e^{kt}\left[k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t\right] \\
&= kZ_t dt + ke^{kt}(\theta - X_t)dt + \sigma e^{kt}\sqrt{X_t}dW_t \\
&= k\theta e^{kt}dt + \sigma e^{kt}\sqrt{X_t}dW_t. && (3)
\end{aligned}$$

$$Z_t = e^{kt} X_t,$$
$$\Rightarrow dZ_t = k\theta e^{kt} dt + \sigma e^{kt} \sqrt{X_t} dW_t. \quad (4)$$

We can "almost"(not in closed explicit form since there is $\sqrt{X_t}$ multiplying dW_t stochastic part.) integrate and solve (4) for X_t:

$$X_t = X_0 e^{-kt} + \theta(1 - e^{-kt}) + \int_0^t \sigma e^{-k(t-s)} \sqrt{X_s} dW_s. \quad (5)$$

This representation reveals the CIR process as a mean-reverting process around θ, with volatility dependent on $\sqrt{X_t}$.

(iv) Graphing the Solution

To visualize the CIR process:

- Set parameters: $k = 1.5$, $\theta = 0.5$, $\sigma = 0.3$, and $X_0 = 0.5$.
- Simulate using the Euler–Maruyama method adapted for square-root diffusion.
- Plot multiple paths to observe mean-reverting behavior and positivity.

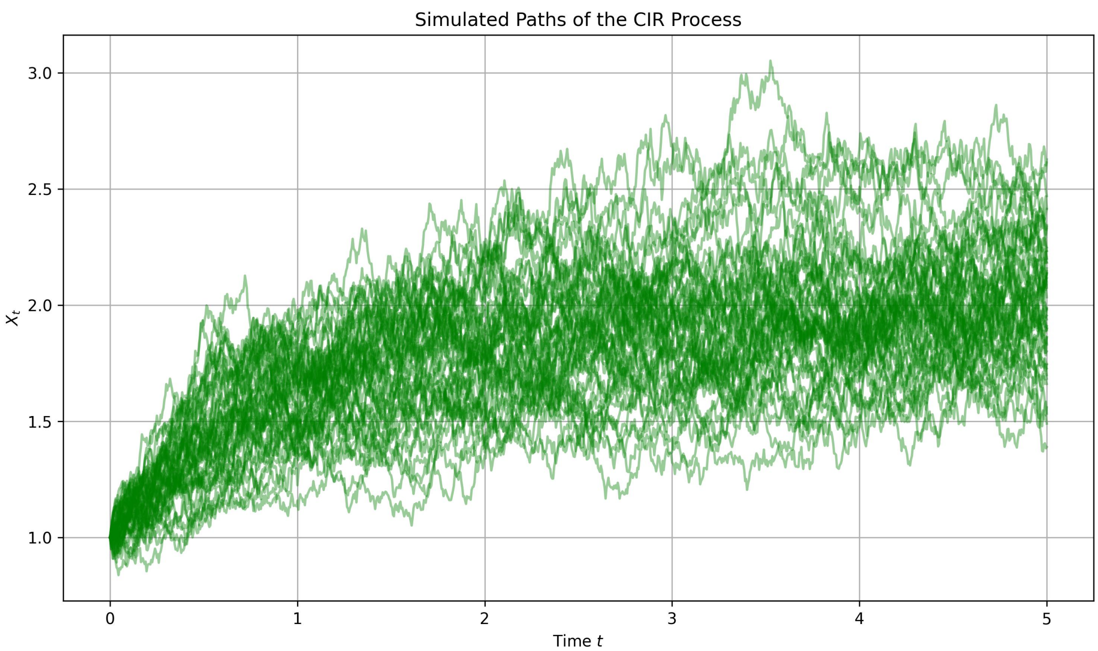

Figure 1: Simulated sample paths of the CIR process with $X_0 = 0.5$, $k = 1.5$, $\theta = 0.5$, $\sigma = 0.3$

(v) Statistics of the CIR Process

We now derive the expectation and variance of the CIR process:

$$dX_t = k(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 > 0.$$

Expectation $\mathbb{E}[X_t]$

Take the expectation on both sides of the SDE:

$$\mathbb{E}[dX_t] = \mathbb{E}[k(\theta - X_t)]dt + \underbrace{\mathbb{E}[\sigma\sqrt{X_t}dW_t]}_{=0},$$

since the Itô integral has zero mean. Let $m(t) = \mathbb{E}[X_t]$. Then:

$$\frac{dm}{dt} = k(\theta - m(t)),$$

a first-order linear ODE. Solving it using the integrating factor method:

$$\begin{aligned} \mu(t) &= e^{kt}, \\ \frac{d}{dt}\left(e^{kt}m(t)\right) &= k\theta e^{kt}, \\ e^{kt}m(t) &= \theta e^{kt} + C, \\ m(t) &= \theta + Ce^{-kt}. \end{aligned}$$

Using $m(0) = X_0$, we get $C = X_0 - \theta$. Therefore:

$$\boxed{\mathbb{E}[X_t] = X_0 e^{-kt} + \theta(1 - e^{-kt})}. \tag{6}$$

Variance Var(X_t)

Using the integral form of the CIR process:

$$X_t = X_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma\int_0^t e^{-k(t-s)}\sqrt{X_s}dW_s.$$

Let:

$$M_t := \sigma\int_0^t e^{-k(t-s)}\sqrt{X_s}dW_s.$$

Then:

$$\mathrm{Var}(X_t) = \mathrm{Var}(M_t) = \mathbb{E}[M_t^2],$$

since the first two terms are deterministic.

Apply Itô isometry:

$$\mathrm{Var}(X_t) = \sigma^2\int_0^t e^{-2k(t-s)}\mathbb{E}[X_s]ds.$$

We already know:

$$\mathbb{E}[X_s] = X_0 e^{-ks} + \theta(1 - e^{-ks}).$$

Therefore:

$$\mathrm{Var}(X_t) = \sigma^2 \int_0^t e^{-2k(t-s)} \left(X_0 e^{-ks} + \theta(1 - e^{-ks})\right) ds.$$

Split the integral:

$$\begin{aligned}
\mathrm{Var}(X_t) &= \sigma^2 \left[X_0 \int_0^t e^{-2k(t-s)} e^{-ks} ds + \theta \int_0^t e^{-2k(t-s)} (1 - e^{-ks}) ds\right] \\
&= \frac{\sigma^2 X_0 e^{-kt}}{k}(1 - e^{-kt}) + \frac{\theta\sigma^2}{2k}(1 - e^{-kt})^2.
\end{aligned}$$

Thus:

$$\boxed{\mathrm{Var}(X_t) = \frac{\sigma^2 X_0 e^{-kt}}{k}(1 - e^{-kt}) + \frac{\theta\sigma^2}{2k}(1 - e^{-kt})^2.} \tag{7}$$

Long-Term (Stationary) Behavior

As $t \to \infty$:

$$e^{-kt} \to 0 \quad \Rightarrow \quad \mathbb{E}[X_t] \to \theta, \quad \mathrm{Var}(X_t) \to \frac{\theta\sigma^2}{2k}. \tag{8}$$

Hence, the CIR process has a stationary distribution with mean θ and variance $\frac{\theta\sigma^2}{2k}$.

Covariance Cov(X_s, X_t) for $0 \le s \le t$

Recall the integral representation of X_t:

$$X_t = X_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-u)} \sqrt{X_u}\, dW_u.$$

Let us denote:

$$M_t := \sigma \int_0^t e^{-k(t-u)} \sqrt{X_u}\, dW_u, \quad \text{so} \quad X_t = \mathbb{E}[X_t] + M_t.$$

Then for $0 \le s \le t$, the covariance becomes:

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(M_s, M_t),$$

since $\mathbb{E}[X_s]$ and $\mathbb{E}[X_t]$ are deterministic. We compute this using Itô isometry and the fact that M_s and M_t are stochastic integrals over overlapping intervals.

$$
\begin{aligned}
\operatorname{Cov}(X_s, X_t) &= \mathbb{E}[M_s M_t] \\
&= \mathbb{E}\left[\sigma \int_0^s e^{-k(s-u)} \sqrt{X_u}\, dW_u \cdot \sigma \int_0^t e^{-k(t-v)} \sqrt{X_v}\, dW_v\right] \\
&= \sigma^2 \int_0^s e^{-k(s-u)} e^{-k(t-u)} \mathbb{E}[X_u]\, du \\
&= \sigma^2 \int_0^s e^{-k(s+t-2u)} \mathbb{E}[X_u]\, du.
\end{aligned}
$$

Substitute $\mathbb{E}[X_u] = X_0 e^{-ku} + \theta(1 - e^{-ku})$:

$$
\operatorname{Cov}(X_s, X_t) = \sigma^2 \int_0^s e^{-k(s+t-2u)} \left(X_0 e^{-ku} + \theta(1 - e^{-ku})\right) du.
$$

Split the integral:

$$
\begin{aligned}
\operatorname{Cov}(X_s, X_t) &= \sigma^2 X_0 \int_0^s e^{-k(s+t-u)} du + \sigma^2 \theta \int_0^s e^{-k(s+t-2u)} (1 - e^{-ku}) du \\
&= \frac{\sigma^2 X_0}{k} \left(e^{-kt} - e^{-k(s+t)}\right) + \sigma^2 \theta \int_0^s e^{-k(s+t-2u)} (1 - e^{-ku}) du. \qquad (9)
\end{aligned}
$$

This gives a closed-form expression (the first term) and a semi-explicit integral form (second term), which can be evaluated numerically or approximated if needed.

As $t \to \infty$ and $s \to \infty$, this covariance converges, confirming the process reaches a stationary Gaussian-like regime.

(vi) Applications of the CIR Process

- **Finance – Interest Rate Modeling:** The CIR model is widely used in the term structure of interest rates. It ensures non-negative rates and captures mean-reverting behavior.

- **Credit Risk and Stochastic Volatility:** The CIR process underlies the Heston model for stochastic volatility.

- **Biology and Ecology:** Used in modeling populations under random birth-death processes where the population stays positive.

References

References

[1] Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). *A theory of the term structure of interest rates*. Econometrica: Journal of the Econometric Society, 385–407.

[2] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd Edition.

[3] Bjork, T. (2009). *Arbitrage Theory in Continuous Time*. Oxford University Press.

[4] Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering*. Springer.

Appendix: Python Code for Question 8 (CIR Process)

Listing 1: CIR Process Simulation (Q8)

```
# q8_cir_simulation.py

import numpy as np
import matplotlib.pyplot as plt

# Parameters for the CIR process
k = 1.0            # Mean reversion rate
theta = 2.0        # Long-term mean
sigma = 0.3        # Volatility
x0 = 1.0           # Initial value
T = 5.0            # Total time
N = 1000           # Number of time steps
dt = T / N         # Time step size
t = np.linspace(0, T, N + 1)
M = 50             # Number of sample paths

# Preallocate simulation matrix
X = np.zeros((M, N + 1))
X[:, 0] = x0

# Simulate CIR paths using Euler Maruyama
for i in range(M):
    for j in range(1, N + 1):
        x_prev = X[i, j - 1]
        # Full truncation scheme to preserve positivity
        sqrt_term = np.sqrt(max(x_prev, 0))
        dW = np.random.normal(0, np.sqrt(dt))
        X[i, j] = x_prev + k * (theta - x_prev) * dt + sigma * sqrt_term * dW
        X[i, j] = max(X[i, j], 0)  # Enforce non-negativity

# Plot results
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i], color='green', alpha=0.4)

plt.title("Simulated Paths of the CIR Process")
plt.xlabel("Time $t$")
plt.ylabel("$X_t$")
plt.grid(True)
plt.tight_layout()
```

```
# Save and show the figure
plt.savefig("images/q8_cir_simulation_graph.jpg", dpi=300)
plt.show()
```