Question 5.

Two R.V.s X & Y have the ↓ joint pdf :

(1) $$f_{XY}(x,y) = \begin{cases} c\,(2x+y), & \text{if } 2 \le x \le 6 \ \& \ 0 \le y \le 5 \\ 0 & , \text{otherwise} \end{cases}$$

Find $f_X(x)$, $f_Y(y)$, $\mathbb{P}(X < Y)$, and the covariance matrix & correlation.

Sol-n:

By defin-n of a joint probability density f-n, $f_{XY}(x,y)$, we know

(2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dx = 1 .$$

We can use (2) to find c :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \int_{y=0}^{y=5}\int_{x=2}^{x=6} c\,(2x+y)\,dx\,dy = 1 \Rightarrow$$

$$c\int_{y=0}^{y=5}\left[x^2+xy\right]_{x=2}^{x=6} dy = c\int_{y=0}^{y=5} 32+4y\,dy = 1 \Leftrightarrow$$

$$c\left[32y+2y^2\right]_{y=0}^{y=5} = 1 \Rightarrow$$

$$\Rightarrow c = \frac{1}{32\cdot 5 + 2\cdot(5)^2} = \frac{1}{210} \Rightarrow$$

(3) $$\boxed{c = \frac{1}{210}} ;$$

Now to find marginal probability density f-ns $f_X(x)$, $f_Y(y)$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{y=0}^{y=5} c\,(2x+y)\,dy = c\left[2xy+\frac{y^2}{2}\right]_{y=0}^{5}$$

$$= c\left[2x\cdot(5)+\frac{(5)^2}{2}\right] = \frac{1}{210}\left[10x+\frac{25}{2}\right] \Rightarrow$$

(4) $\boxed{f_X(x) = \frac{x}{21}+\frac{5}{84}}$; Similarly

Note the difference here!

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{x=2}^{x=6} c\,(2x+y)\,dx =$$

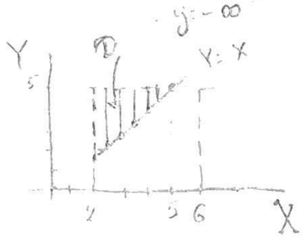
$$= \frac{1}{210}\left[x^2+xy\right]_{x=2}^{x=6} =$$

$$= \frac{1}{210}\,(32+4y) \Rightarrow$$

(5) $\boxed{f_Y(y) = \frac{2y}{105}+\frac{16}{105}}$. Also, by $\mathbb{P}(X\le m) = \int_{x=-\infty}^{x=m} f_X(x)\,dx$ &

For rv's X & Y with 2-dim-rectangle $=\{(x,y): a\le x\le b,\ c\le y\le d\}$

$f_X(x) = \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\,dy$ & $f_Y(x,y) = \int_{x=-\infty}^{x=+\infty} f_{XY}(x,y)\,dx$) hence $\mathbb{P}(X\le m) = \int_{x=-\infty}^{x=m} \underbrace{\int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\,dy}_{=f_X(x)}$



So

$$\underbrace{\mathbb{P}(X<Y)}_{\text{integrate } f_{XY} \text{ over } D,} = \int_{x=2}^{x=5} \int_{y=x}^{y=5} f_{XY}(x,y)\,dy\,dx \Rightarrow$$

$D = \{(x,y):\ 2\le x\le 5,\ x\le y\le 5\}$

(6) $$\mathbb{P}(X<Y) = \int_{x=2}^{x=5} \int_{y=x}^{y=5} \frac{1}{210}(2x+y)\,dy\,dx = \frac{84}{210} = 0.4.$$

For Covariance Matrix, we have

(7) $$\text{Covariance Matrix} = \begin{bmatrix} Cor(X, X) & Cor(X, Y) \\ Cor(Y, X) & Cor(Y, Y) \end{bmatrix} = \begin{bmatrix} Var(X) & Cor(X, Y) \\ Cor(X, Y) & Var(Y) \end{bmatrix}$$

where

(8) $$Cor(X, Y) = E(XY) - E(X) \cdot E(Y)$$ & substituting

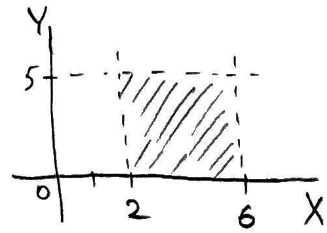
$Y = X$ into (5) gives

$$Cor(X, X) = E(X \cdot X) - E(X) \cdot E(X)$$
$$= E(X^2) - [E(X)]^2 = Var(X); \text{ i.e.}$$

(9) $$Var(X) = E(X^2) - [E(X)]^2.$$

Now for

(10) $$f_{XY}(x, y) = \begin{cases} \frac{1}{210}(2x + y), & 2 \le x \le 6 \ \& \ 0 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}$$



joint pdf, the calculations are as follows:

(11) $$E(X) = \int_{x=2}^{x=6} \int_{y=0}^{y=5} x \cdot f_{XY}(x, y)\, dy\, dx = \int_2^6 \int_0^5 x \cdot \frac{1}{210}(2x + y)\, dy\, dx$$

$$= \frac{1}{210} \int_{x=2}^{x=6} \left[2xy + \frac{y^2}{2}\right]_{y=0}^{y=5} dx = \frac{1}{210} \int_{x=2}^{x=6} (10x^2 + 12{,}5x)\, dx = 4.254$$

$\boxed{E(X) = 4.254}$ & similarly

(12) $$E(Y) = \int_{y=0}^{y=5} y \cdot f_Y(y)\, dy \overset{\text{by}}{\underset{(5)}{=}} \int_{y=0}^{y=5} y \cdot \left(\frac{2y + 16}{105}\right) dy = \frac{1}{105} \int_0^5 (2y^2 + 16y)\, dy$$

$$= \frac{1}{105}\left[\frac{2y^3}{3} + 8y^2\right]_{y=0}^{y=5} = \frac{170}{63} \approx 2.698; \quad \boxed{E(Y) = 2.698}$$

Similarly again,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx \overset{\text{by (4)}}{=} \int_2^6 x^2 \cdot f_X(x)\,dx \overset{\text{by (4)}}{=}$$

$$= \int_2^6 x^2 \cdot \left[\frac{x}{21} + \frac{5}{84}\right] dx$$

$$= \frac{x^4}{84} + \frac{5x^3}{252}\Bigg|_{x=2}^{x=6} = \frac{1220}{63} \approx 19.365$$

(13) $\boxed{E(X^2) = 19.365}$;

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy \overset{\text{by (5)}}{=} \int_0^5 y^2 \cdot \left[\frac{2y+16}{105}\right] dy = \frac{1}{105}\int_0^5 (2y^3 + 16y^2)\,dy$$

$$= \frac{1}{105}\left[\frac{2y^4}{4} + \frac{16y^3}{3}\right]_{y=0}^{5} = \frac{1175}{126} \approx 9.325$$

(14) $\boxed{E(Y^2) = 2.698}$, &

$$E[X \cdot Y] = \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} x \cdot y \cdot f_{XY}(x,y)\,dy\,dx =$$

$$= \int_{x=2}^{x=6} \int_{y=0}^{y=5} x\,y \cdot \frac{1}{210}(2x+y)\,dy\,dx = \frac{1}{210}\int_2^6 \int_0^5 (2x^2y + xy^2)\,dy\,dx$$

$$= \frac{1}{210}\int_2^6 \left[x^2y^2 + \frac{xy^3}{3}\right]_{y=0}^{y=5} dx = \frac{1}{210}\int_2^6 \left(25x^2 + \frac{125x}{3}\right) dx$$

$$= \frac{1}{210}\left(\frac{25x^3}{3} + \frac{125x^2}{6}\right)\Bigg|_{x=2}^{x=6} = \frac{80}{7} \approx 11.429$$

(15) $\boxed{E(X \cdot Y) = 11.429}$

Putting these (11) – (15) results into (8) & (9) gives

$Var(X) = E(X^2) - [E(X)]^2 =$ by (13) & (11)

$= 19.365 - (4.254)^2 \approx 1.268$

(15) $\boxed{Var(X) = 1.268}$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$= 9.325 - [2.698]^2 = 2.046$

(16) $\boxed{Var(Y) = 2.046}$

$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$

$= 11.429 - 4.254 \cdot 2.698$

$= -0.05$

(17) $\boxed{Cov(X, Y) = -0.05}$, thus by (7)

(18) Covariance Matrix $= \begin{bmatrix} 1.268 & -0.05 \\ -0.05 & 2.04 \end{bmatrix}$, and

Correlation $= \dfrac{cor(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \dfrac{-0.05}{\sqrt{1.27 \cdot 2.04}} \approx -0.031$

(19) $\boxed{\text{Correlation} = -0.031.}$

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