Stat 544. HW#1

Questions.

Let $A \subseteq \Omega$. Show that $\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$ is a σ-algebra.

Sol-n: Let $A \subseteq \Omega$, & let

(1) $\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$.

WMT: $\mathcal{F}$ is a σ-algebra, i.e. by definition:

(i) $\Omega \in \mathcal{F}$
(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. (Closed under complement-n)
(iii) If $A_i \in \mathcal{F}$, then $\bigcup_{i \in I} A_i \in \mathcal{F}$ (closed under countable unions).

Note that

(i) $\Omega \in \mathcal{F}$ because by (1) it contains Ω.

(ii) Complementations:

$\emptyset \in \mathcal{F}$ & $(\emptyset)^c = \Omega \in \mathcal{F}$ ✓

$\Omega \in \mathcal{F}$ & $(\Omega)^c = \emptyset \in \mathcal{F}$ ✓

$A \in \mathcal{F}$ & $A^c \in \mathcal{F}$. ✓

(iii) Countable unions:

2 groups

$\emptyset, \Omega \in \mathcal{F}$ & $\emptyset \cup \Omega = \Omega \in \mathcal{F}$ ✓

$\emptyset, A \in \mathcal{F}$ & $\emptyset \cup A = A \in \mathcal{F}$ ✓

$\emptyset, A^c \in \mathcal{F}$ & $\emptyset \cup A^c = A^c \in \mathcal{F}$ ✓

$\Omega, A \in \mathcal{F}$ & $\Omega \cup A = \Omega \in \mathcal{F}$ as $(A \subseteq \Omega$ by assumption$)$ ✓

$\Omega, A^c \in \mathcal{F}$ & $\Omega \cup A^c = \Omega \in \mathcal{F}$ ✓

$A, A^c \in \mathcal{F}$ & $A \cup A^c = \Omega \in \mathcal{F}$ ✓

$\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$

3 groups

$\emptyset, \Omega, A \in \mathcal{F}$ & $\emptyset \cup \Omega \cup A = \Omega \in \mathcal{F}$ ✓

$\emptyset, \Omega, A^c \in \mathcal{F}$ & $\emptyset \cup \Omega \cup A^c = \Omega \in \mathcal{F}$ ✓

$\Omega, A, A^c \in \mathcal{F}$ & $\Omega \cup A \cup A^c = \Omega \in \mathcal{F}$ ✓

$\emptyset, A, A^c \in \mathcal{F}$ & $\emptyset \cup A \cup A^c = \Omega \in \mathcal{F}$ ✓

thus $\mathcal{F}$ is closed under its all subsets, which together with (i) & (ii) proves $\mathcal{F}$ to be a σ-algebra. ▨

Question 2.

Let $\Omega = \{a, b, c\}$ be sample space &

(1) $\mathcal{F}_1 = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$ &

(2) $\mathcal{F}_2 = \{\emptyset, \Omega, \{a, b\}, \{c\}\}$.

(a) Show that $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-algebras on Ω.

(b) Is $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ a σ-algebra on Ω?

Sol-n: [Preamble]

Like in Q.1, we go through the def-n of σ-algebras:

$\mathcal{F}_1$ is σ-algebra as

✓ (i) $\Omega \in \mathcal{F}_1$ ✓

✓ (ii) $\emptyset \in \mathcal{F}_1 \Rightarrow \emptyset^c = \Omega \in \mathcal{F}_1$ &

$\Omega \in \mathcal{F}_1 \Rightarrow \Omega^c = \emptyset \in \mathcal{F}_1$ &

$\{a\} \in \mathcal{F}_1 \Rightarrow \{a\}^c = \{b, c\} \in \mathcal{F}_1$;

$\{b, c\} \in \mathcal{F}_1 \Rightarrow \{b, c\}^c = \{a\} \in \mathcal{F}_1$, and finally ✓

$\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$

3 groups

$\emptyset, \Omega, A \in \mathcal{F}$ & $\emptyset \cup \Omega \cup A = \Omega \in \mathcal{F}$ ✓

$\emptyset, \Omega, A^c \in \mathcal{F}$ & $\emptyset \cup \Omega \cup A^c = \Omega \in \mathcal{F}$ ✓

$\Omega, A, A^c \in \mathcal{F}$ & $\Omega \cup A \cup A^c = \Omega \in \mathcal{F}$ ✓

$\emptyset, A, A^c \in \mathcal{F}$ & $\emptyset \cup A \cup A^c = \Omega \in \mathcal{F}$ ✓

thus $\mathcal{F}$ is closed under its all subsets, which together with (i) & (ii) proves $\mathcal{F}$ to be a σ-algebra. ■

Question 2.

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Sol-n: [Preamble]

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$\Omega \in \mathcal{F}_1 \Rightarrow \Omega^c = \emptyset \in \mathcal{F}_1$ &

$\{a\} \in \mathcal{F}_1 \Rightarrow \{a\}^c = \{b, c\} \in \mathcal{F}_1$

$\{b, c\} \in \mathcal{F}_1 \Rightarrow \{b, c\}^c = \{a\} \in \mathcal{F}_1$, and finally ✓

(iii) $\emptyset, \Omega \in \mathcal{F}_1 \Rightarrow \emptyset \cup \Omega = \Omega \in \mathcal{F}_1$

$\emptyset, \{a\} \in \mathcal{F}_1 \Rightarrow \emptyset \cup \{a\} = \{a\} \in \mathcal{F}_1$

$\emptyset, \{b,c\} \in \mathcal{F}_1 \Rightarrow \emptyset \cup \{b,c\} = \{b,c\} \in \mathcal{F}_1$

$$\left.\begin{array}{l} \{a\}, \Omega \\ \{b,c\}, \Omega \end{array}\right\} \Rightarrow \cup A_i = \Omega \in \mathcal{F}_1$$

$$\left.\begin{array}{l} \emptyset, \{a\}, \Omega \\ \emptyset, \{b,c\}, \Omega \\ \{a\}, \{b,c\}, \Omega \\ \emptyset, \{a\}, \{b,c\} \end{array}\right\} \Rightarrow \cup A_i = \Omega \in \mathcal{F}_1$$

&

$\emptyset, \Omega, \{a\}, \{b,c\} \in \mathcal{F}_1 \Rightarrow \emptyset \cup \Omega \cup \{a\} \cup \{b,c\} \in \mathcal{F}_1.$

Hence $\mathcal{F}_1$ is a σ-algebra.

Similarly, $\mathcal{F}_2 = \{\emptyset, \Omega, \{a,b\}, \{c\}\}$ is also proved as

(i) $\Omega \in \mathcal{F}_2$ ✓

(ii) $\emptyset \in \mathcal{F}_2 \Rightarrow \emptyset^c = \Omega \in \mathcal{F}_2$ &

$\Omega \in \mathcal{F}_2 \Rightarrow \Omega^c = \emptyset \in \mathcal{F}_2$ &

$\{a,b\} \in \mathcal{F}_2 \Rightarrow \{a,b\}^c = \{c\} \in \mathcal{F}_2$ &

$\{c\} \in \mathcal{F}_2 \Rightarrow \{c\}^c = \{a,b\} \in \mathcal{F}_2,$ and

(iii)
$$\left.\begin{array}{l} \Omega, \emptyset \in \mathcal{F}_2 \\ \{a,b\}, \emptyset \in \mathcal{F}_2 \\ \{c\}, \emptyset \in \mathcal{F}_2 \end{array}\right\} \Rightarrow \cup A_i \in \mathcal{F}_2 \quad \&$$

$$\left.\begin{array}{l} \{a,b\}, \Omega \in \mathcal{F}_2 \\ \{c\}, \Omega \in \mathcal{F}_2 \\ \{a,b\}, \{c\} \in \mathcal{F}_2 \end{array}\right\} \Rightarrow \cup A_i = \Omega \in \mathcal{F}_2 \quad \&$$

$$\left.\begin{array}{l}\emptyset, \{c\}, \Omega \in \mathcal{F}_2 \\ \emptyset, \{a,b\}, \Omega \in \mathcal{F}_2 \\ \{c\}, \{a,b\}, \Omega \in \mathcal{F}_2 \\ \emptyset, \{c\}, \{a,b\} \in \mathcal{F}_2\end{array}\right\} \Rightarrow \cup A_i = \Omega \in \mathcal{F}_2$$

$$\emptyset, \{a,b\}, \{c\}, \Omega \in \mathcal{F}_2 \Rightarrow \emptyset \cup \{a,b\} \cup \{c\} \cup \Omega = \Omega \in \mathcal{F}_2.$$

Hence, $\mathcal{F}_2$ is a σ-algebra, by definition. $\blacksquare$

(b) Is $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ a σ-algebra?

— NO!

Proof: Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, i.e

(3) $\mathcal{F} = \{\emptyset, \Omega, \{a\}, \{b,c\}, \{a,b\}, \{c\}\}$, & by def-n,

✓ (i) $\Omega \in \mathcal{F}$ ✓

✓ (ii) $\emptyset \in \mathcal{F} \Rightarrow \emptyset^c = \Omega \in \mathcal{F}$ ✓

$\Omega \in \mathcal{F} \Rightarrow \Omega^c = \emptyset \in \mathcal{F}$ ✓

$\{a\} \in \mathcal{F} \Rightarrow \{a\}^c = \{b,c\} \in \mathcal{F}$ ✓

$\{b,c\} \in \mathcal{F} \Rightarrow \{b,c\}^c = \{a\} \in \mathcal{F}$ ✓

$\{a,b\} \in \mathcal{F} \Rightarrow \{a,b\}^c = \{c\} \in \mathcal{F}$ ✓

$\{c\} \in \mathcal{F} \Rightarrow \{c\}^c = \{a,b\} \in \mathcal{F}$ ✓ & finally

(iii)
$$\left.\begin{array}{l}\emptyset, \Omega \in \mathcal{F} \\ \{a\}, \Omega \in \mathcal{F} \\ \{b,c\}, \Omega \in \mathcal{F} \\ \{a,b\}, \Omega \in \mathcal{F} \\ \{c\}, \Omega \in \mathcal{F}\end{array}\right\} \Rightarrow \cup A_i = \Omega \in \mathcal{F}$$ ✓

$$\left.\begin{array}{l} \emptyset, \{a\} \in \mathcal{F} \\ \emptyset, \{b,c\} \in \mathcal{F} \\ \emptyset, \{a,b\} \in \mathcal{F} \\ \emptyset, \{c\} \in \mathcal{F} \end{array}\right\} \Rightarrow \cup A_i \in \mathcal{F} \text{ (since } \emptyset \cup \{a\} = \{a\} \in \mathcal{F})$$

$\{a\}, \{b,c\} \in \mathcal{F} \Rightarrow \{a\} \cup \{b,c\} = \Omega \in \mathcal{F}$

$\{a\}, \{a,b\} \Rightarrow \{a\} \cup \{a,b\} = \{a,b\} \in \mathcal{F}$

(4) $\boxed{\{a\}, \{c\} \in \mathcal{F} \Rightarrow \{a\} \cup \{c\} = \{a,c\} \notin \mathcal{F}}$! $\lightning$

We can see (4) breaks the union-closure property of σ-algebra ; Hence $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is NOT σ-algebra. $\blacksquare$

Problem 3.

Prove that an intersection of multiple σ-algebras is a σ-algebra.

Sol-n:

Let $\mathcal{B}$ be an intersection of multiple σ-algebras as follows:

(1) $$\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i \qquad I = \{1, 2, 3, \ldots\}.$$

WTT: $\mathcal{B}$ is a σ-algebra.

By def-n of σ-algebra,

(i) $\Omega \in \mathcal{B}_i \quad \forall i \in I$ as each $\mathcal{B}_i$ is a σ-algebra, hence $\Omega \in \mathcal{B}$.

(ii) Let $A \in \mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i$. $\Rightarrow$
$A \in \mathcal{B}_i \quad \forall i \in I$. $\Rightarrow$ since $\mathcal{B}_i$ is a σ-algebra,
$\Rightarrow A^c \in \mathcal{B}_i \quad \forall i \in I \Rightarrow$
$A^c \in \bigcap_{i \in I} \mathcal{B}_i \quad \forall i \in I \Rightarrow A^c \in \mathcal{B}$, hence
$A^c \in \mathcal{B}$.

(iii) Let $A_j \in \mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i$. $\Rightarrow$
$A_j \in \mathcal{B}_i \quad \forall i \in I$, for any j. Since $\mathcal{B}_i$ is σ-algebra
$\bigcup_{j \in J} A_j \in \mathcal{B}_i \quad \forall i \in I. \Rightarrow$
$\bigcup_{j \in J} A_j \in \bigcap_{i \in I} \mathcal{B}_i = \mathcal{B} \Rightarrow$ Hence by (i)-(iii), $\mathcal{B}$ is a σ-algebra. ▨

Problem 4.

If you toss a fair die what is the probability of having a result that is less than 4 given that it is even.

Sol-n.

$A =$ X is even $= \{X = 2, \quad X = 4, \quad X = 6\}$

$B = X < 4 = \{X = 1, \quad X = 2, \quad X = 3\}$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(X < 4 \cap (X \text{ is even}))}{P(X \text{ is even})} \quad \&$$

$P(A) = P(X \text{ is even}) = \frac{3}{6}$

$$P(B) = P(X < 4) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}.$$

$$P(B \cap A) = P(\overset{B}{(X < 4)} \cap \overset{A}{(X \text{ is even})}) = P(X = 2) = \frac{1}{6}.$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

$$P(B|A) = \frac{1}{3}.$$

Problem 5.

Let B and C be two events of Ω & $\{A_i\}_{i=1}^n$ be a partition of Ω. Assume that (a) B & C are independent given any A_i & (b) C is independent of all A_i. Prove that B & C are independent.

Sol-n. [Preamble].

Let $\{A_i\}$ be a partition of Ω: i.e.

(1) $\bigcup_{i=1}^{n} A_i = \Omega$ $\quad A_m \cap A_n = \emptyset$, $m \neq n$. &

assume

(a) B & C are indep. given any A_i $\Rightarrow$

(2') definition: we say K & L are indep if $P(K \cap L) = P(K) \cdot P(L)$ or equivalently

$$P(K|L) = \frac{P(K\cap L)}{P(L)} = \frac{P(K)\,\cancel{P(L)}}{\cancel{P(L)}} = P(K) \Rightarrow$$

$$P(K|L) = P(K)$$

So

(2) $$P(B \cap C \mid A_i) = P(B|A_i) \cdot P(C \mid A_i)$$ & finally

assume

(b) C is indep of all A_i $\Rightarrow$ by (2') above

(3) $$\begin{cases} P(C \cap A_i) = P(C) \cdot P(A_i) & \forall i \quad \text{or equivalently} \\ P(C|A_i) = P(C). \end{cases}$$

WNT: B & C are indep-t. i.e. by (2')

$$(4)\begin{cases} P(B|C) = P(B) \quad \text{or} \quad P(C|B) = P(C) & \text{or equivalently} \\ P(B\cap C) = P(B)\cdot P(C) \end{cases}$$

Recall by Law of Total Prob-ties, if $\{A_i\}_{i=1}^n$ is a partition & B & A_i are discrete, then

$$(5)\begin{cases} P(B) = \sum_{i=1}^{n} P(B\cap A_i) & \text{or} \quad \overbrace{P(B\cap A_i) = P(B|A_i)\cdot P(A_i)}^{\text{by conditional prob-ty}} \\ & \text{so equivalently} \\ P(B) = \sum_{i=1}^{n} P(B|A_i)\cdot P(A_i) \end{cases}$$

Consider the following:

$$(6)\quad P(C|B) = \frac{P(C\cap B)}{P(B)} = \frac{P\left(C\cap\left[\bigcup_{i=1}^{n} B\cap A_i\right]\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{n}\left[C\cap B\cap A_i\right]\right)}{P(B)};$$

and by (2), we have

$$(2)\quad P(B\cap C|A_i) = P(B|A_i)\cdot P(C|A_i)$$ & on the other hand

$$(7)\quad P(B\cap C|A_i) = \frac{P(B\cap C\cap A_i)}{P(A_i)},$$ setting the RHS equal to each other gives;

$$P(B|A_i)\cdot P(C|A_i) = \frac{P(B\cap C\cap A_i)}{P(A_i)} \iff$$

$$P(B\cap C\cap A_i) = P(B|A_i)\cdot \underbrace{P(C|A_i)}_{=P(C)}\cdot P(A_i)$$

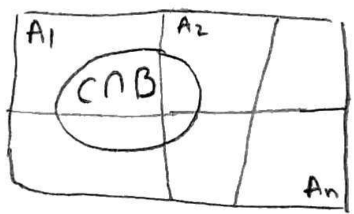
by given info (6);

(8) $\Rightarrow P(B\cap C\cap A_i) = P(B|A_i)\cdot P(C)\cdot P(A_i).$

But now we can apply this result (8) to continue (6) as follows.

Applying Law of Total Probability to sets $(C\cap B)$ & disjoint A_i's

(9) $$P(C|B) = \frac{P\left(\bigcup_{i=1}^{n}\left[(C\cap B)\cap A_i\right]\right)}{P(B)} =$$



$$= \frac{\sum_{i=1}^{n} P(C\cap B\cap A_i)}{P(B)}$$

by (8)

$$= \frac{\sum_{i=1}^{n} P(B|A_i)\cdot P(A_i)\cdot P(C)}{P(B)} =$$

does NOT depend on i

$$= \frac{P(C)\cdot\left[\sum_{i=1}^{n} P(B|A_i)\cdot P(A_i)\right]}{P(B)}$$

$= P(B)$ by (5)

$$= \frac{P(C)\cdot P(B)}{P(B)} = P(C). \Rightarrow$$

(10) $P(C|B) = P(C) \Rightarrow$ thus B & C are independent, as desired in (4). ■

Problem 6.

"Design" an "experiment" & define its probability space.

Solution:

Consider the following experiment:

(0) Sending a packaged item to Arizona by USPS.

(1) The experiment has the following outcomes:

A_1 = The item is delivered on time
A_2 = The item is delivered later than scheduled
A_3 = The item is stolen.

$$\Omega = \{A_1, A_2, A_3\}$$

(2) Define the σ-algebra as

$$\mathcal{F}_1 = \{\emptyset, \Omega, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_1, A_3\}\}$$ or

$$\mathcal{F}_2 = \{\emptyset, \Omega, \{A_3\}, \{A_1, A_2\}\}$$

?1 Are both σ-algebras?

(3)
$\mathbb{P} : \mathcal{F}_2 \to [0, 1]$
$\mathbb{P}(\emptyset) = 0$
$\mathbb{P}(\Omega) = 1$
$\mathbb{P}(A_3) = 0.1$
$\mathbb{P}(\{A_1, A_2\}) = 0.9$

Probability measure of $\mathcal{F}_2$.

Probability space $\{\Omega, \mathcal{F}_2, \mathbb{P}\}$.