Q3.

Independently Identically Distributed

Let $X_1, \dots X_n$ be $\overbrace{IID}$ exponential r.v.s with rate $\lambda = 1$. <u>Find n</u> s.t.

(1) $$\mathbb{P}\left(0.9 \le \overline{X}_n \le 1.1\right) \geqslant 0.95$$

where

(2) $$\overline{X}_n = \frac{1}{n}\left(X_1 + X_2 + \dots + X_n\right).$$

Sol-n.

[Preamble].

Let $X_i \sim exp(\lambda = 1)$, $i = 1, 2, \dots, n$. $\Rightarrow$

(3) $$E(\overline{X}_n) = E(X_i) = \frac{1}{\lambda} = \frac{1}{1} = 1. \quad \&$$

(4) $$Var(\overline{X}_n) = \frac{Var(X_i)}{n} = \frac{1/\lambda^2}{n} = \frac{1/1^2}{n} = \frac{1}{n}.$$

By the result of Q2, we know

(5) $$\mathbb{P}\left(\left|\overline{X}_n - \underbrace{E(\overline{X}_n)}\right| \geqslant a\right) \le \frac{1}{(a\lambda)^2} \Rightarrow \text{then by (1),}$$

$$\mathbb{P}\left(0.9 \le \overline{X}_n \le 1.1\right) \geqslant 0.95 \Rightarrow$$

$$\mathbb{P}\left(0.9 - \frac{1}{\lambda} \le \overline{X}_n - \underbrace{\frac{1}{\lambda}} \le 1.1 - \frac{1}{\lambda}\right) \geqslant 0.95 \leftarrow \boxed{\lambda = 1}$$

(6) $$\mathbb{P}\left(-0.1 \le \overline{X}_n - E(X) \le 0.1\right) \geqslant 0.95$$

Now applying Chebyshev's ineq-ty,

(7) $$\mathbb{P}\left(|X - E(X)| \geq k\sigma\right) \leq \frac{1}{k^2},$$

we can write (6) as

(8) $$\underbrace{\mathbb{P}\left(|\bar{X}_n - E(\bar{X})| \leq 0.1\right)}_{} \geq 0.95$$

$$1 - \mathbb{P}\left(|\bar{X}_n - E(\bar{X})| > 0.1\right) \geq 0.95 \quad \Rightarrow$$

$$1 - \mathbb{P}\left(|\bar{X}_n - E(\bar{X})| \geq 0.1\right) \geq 0.95$$

$$\mathbb{P}\left(|\bar{X}_n - E(\bar{X})| \geq 0.1\right) \leq 0.05 \quad \Rightarrow$$

$$\underbrace{\mathbb{P}\left(|\bar{X}_n - E(\bar{X})| \geq 0.1 \cdot \overset{\sigma}{(1)}\right)}_{} \leq 0.05$$

$$\leq \frac{1}{n \cdot (0.1)^2} \leq 0.05 \quad \Rightarrow$$

$$0.0005 \leq \frac{1}{n} \quad \Rightarrow$$

$$\boxed{n \geq 2000}$$

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