# Ornstein-Uhlenbeck equation

$a_1(t) = -a, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = b$

$dX_t = -aX_t\,dt + b\,dW_t, \quad X_0 = x_0$

Integrating factor. $M_t = e^{at}$

Solution

$$\boxed{X_t = x_0 e^{-at} + \int_0^t b\, e^{a(\tau - t)}\, dW_\tau}$$

**Expectation.** $E(X_t) = x_0 e^{-at}$

**Variance**

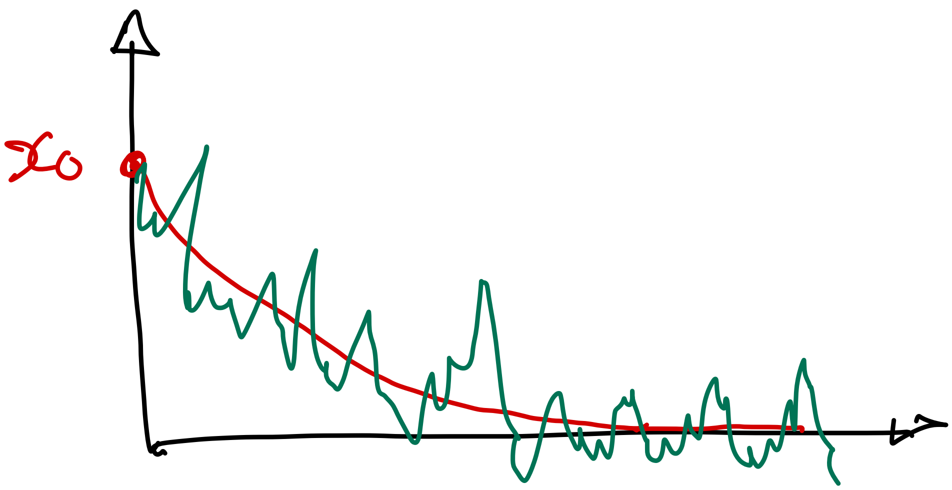
$$Var(X_t) = Var\left(x_0 e^{-at}\right) + Var\left(\int_0^t b\, e^{a(\tau-t)}\, dW_\tau\right)$$

$$= E\left[\left(\int_0^t b\, e^{a(\tau-t)}\, dW_\tau\right)^2\right] = \int_0^t E\left(b^2 e^{2a(\tau-t)}\right) d\tau$$

$$\Rightarrow \quad Var(X_t) = b^2 \, \frac{1 - e^{-2at}}{2a}.$$

Note that as $t \to \infty$

$E(X_t) \to 0$ and $Var(X_t) \to \frac{b^2}{2a}$
constant



# Cov of OU

$$X_t = \underbrace{x_0 e^{-at}}_{g(t)} + \sigma e^{-at} \underbrace{\int_0^t e^{+a\,u}\, dW_u}_{Y_t}$$

$$X_t = g(t) + \sigma e^{-at} Y_t$$

$$\text{cov}(X_t, X_s) = \text{Cov}\left[(\sigma e^{-at}) Y_t, (\sigma e^{-as}) Y_s\right]$$

$$= \sigma^2 e^{-at} e^{-as} \underbrace{\text{cov}(Y_t, Y_s)}$$

$$\text{Cov}(Y_t, Y_s) = \left( \int_0^t e^{+au} \mathbb{1}_{u<t}\, dW_u \, , \int_0^s e^{a\,v} \mathbb{1}_{v<s}\, dW_v \right)$$

$$= \int_0^{\min(s,t)} e^{2av}\, dv = \frac{1}{2a}\left( e^{2a\min(s,t)} - 1 \right)$$

$-at + as = -a(t-s)$

$-as + at \quad -a(s-t)$

So,

$$\operatorname{Cov}(X_t, X_s) = \frac{\sigma^2}{2a} e^{-a(t+s)} \left[ e^{2a\min(s,t)} - 1 \right]$$

$$= \frac{\sigma^2}{2a} \left[ e^{-a(t+s) + 2a\min(s,t)} - e^{-a(t+s)} \right]$$

$$= \frac{\sigma^2}{2a} \left[ e^{-a|t-s|} - e^{-a(s+t)} \right]$$

$$E(X_t) = X_0 e^{-at}$$

$$Cov(X_t, X_s) = \frac{\sigma^2}{2a}\left[e^{-a(t-s)} - e^{-(s+t)}\right]$$

$$Var(X_t) = \frac{\sigma^2}{2a}\left[1 - e^{-2t}\right]$$

$t \to \infty$

$$\boxed{\begin{aligned} &E(X_t) = 0 \\ &Var(X_t) = \frac{\sigma^2}{2a} \\ &Cov(X_t, X_s) = \frac{\sigma^2}{2a} e^{|t-s|} \end{aligned}}$$

these three properties make OU unique!

It is the only

$\boxed{\text{Stationar, Markov, Gaussian}}$

# Mean-reverting OU (Vasicek model)

$$a_1 = -a, \quad a_2 = \theta a, \quad b_2 = b, \quad b_1 = 0$$

$$dX_t = a(\theta - X_t)\,dt + b\,dW_t, \quad X_0 = x_0$$

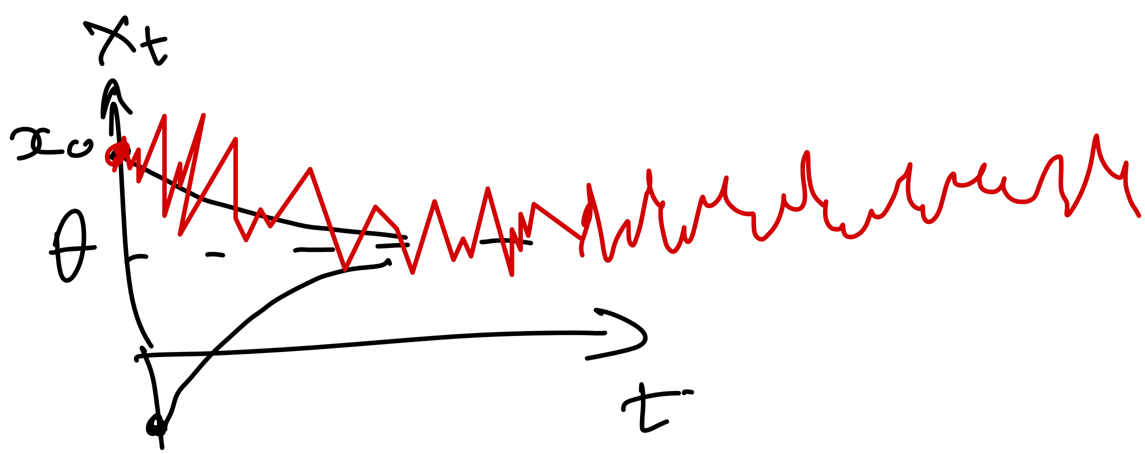
$$\text{IF}: \quad M_t = e^{at}$$

$$X_t = \frac{1}{M_t}\left(x_0 + \int_0^t a\theta M_\tau \, d\tau + \int_0^t b M_\tau \, dW_\tau\right)$$

$$\vdots$$

Solution

$$\boxed{X_t = x_0 e^{-at} + \theta\left[1 - e^{-at}\right] + b\int_0^t e^{a(\tau - t)} \, dW_t.}$$



# Brownian Bridge

$a_1 = -\frac{1}{1-t}$ , $a_2 = \frac{b}{1-t}$ , $b_1 = 0$, $b_2 = 1$

$$\boxed{dX_t = \frac{b - X_t}{1-t}\,dt + dW_t}$$

IF : $\underline{M_t = \frac{1}{1-t}}$ $\qquad \underline{e^{\int a(t)\,dt}}$

$$X_t = (1-t)\left(x_0 + \int_0^t \frac{b\,d\tau}{1-\tau} + \int_0^t \frac{1}{1-\tau}\,dW_\tau\right)$$

$$X_t = (1-t)x_0 + bt + \underbrace{(1-t)\int_0^t \frac{dW_\tau}{1-\tau}}_{\text{converges to } 1}$$

**Expectation**: $E(X_t) = (1-t)x_0 + bt$.

**Variance**: $Var(X_t) = Var\left((1-t)\int_0^t \frac{dW_\tau}{1-\tau}\right)$

$$= E\left[(1-t)^2\left(\int_0^t \frac{dW_\tau}{1-\tau}\right)^2\right] = (1-t)^2 \int_0^t E\left[\frac{1}{(1-\tau)^2}\right] d\tau$$

$$\Rightarrow Var(X_t) = t(1-t)$$

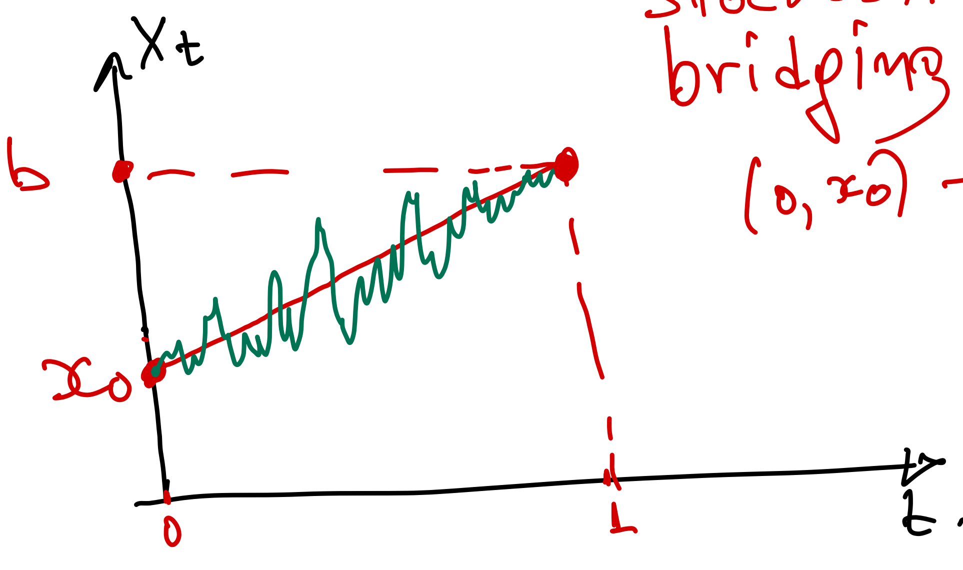
Note that at

$t=0$ $\quad E(X_t) = x_0$

$\qquad Var(X_t) = 0$

$t=1$ $\quad E(X_t) = b$

$\qquad Var(X_t) = 0$




stochastically bridging two points

$(0, x_0) \to (1, b)$

# Solution to the general Linear SDE

(1)

$$dX_t = [a_1(t)X_t + a_2(t)]dt + [b_1(t)X_t + b_2(t)]dW_t$$

$$X_0 = x_0$$

$$\boxed{X_t = \frac{1}{M_t}\left(x_0 + \int_0^t (a_2(\tau) - b_1(\tau)b_2(\tau))M_\tau d\tau + \int_0^t b_2(\tau)M_\tau dW_\tau\right)}$$

## Proof

Let $Y_t$ be the solution to the homogeneous SDE, i.e.,

$$Y_t = e^{\int_0^t (a_1(s) - \frac{1}{2}b_1^2(s))ds + \int_0^t b_1(s)dW_t}$$

with $Y_0 = 1$.

Apply the Itô formula to $M_t = Y_t^{-1}$

i.e., $\varphi(x) = x^{-1}$

Show that you get:

$$dM_t = [-a_1(t) + b_1^2(t)] M_t - b_1(t) M_t \, dW_t \quad (2)$$

with solution:

$$\boxed{M_t = e^{\int_0^t [-a_1(t) + \frac{1}{2} b_1^2(t)] dt - \int_0^t b_1(s) dW_t}} \quad (3)$$

Apply the Integration by parts formula on

$$d(M_t X_t) = (a_2(t) - b_1(t) b_2(t)] M_t \, dt$$
$$+ b_1(t) M_t \, dW_t \longrightarrow$$

$$M_t X_t - \cancel{M_0}^{=1} X_0 = \int_0^t (a_2(s) - b_1(s) b_2(s)] M_t \, ds$$
$$+ \int_0^t b_1(s) M_s \, dW_s$$