MATH 588
Introduction to FEM

Homework assignment 1
Date assigned: January 31, 2025
Due date: **February 7, 2025**

Problem

Consider the following second-order ODE:

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = f(x), \quad \text{for } 0 \leq x \leq 1,$$

with boundary conditions:

$$u(0) = 0, \quad u(1) = 0,$$

where $k(x) = 1$ and $f(x) = 2x$.

Tasks

1. **Weak Form Derivation:** Derive the weak form of the given ODE. Write the integral form of the weak equation explicitly.
2. **Finite Element Discretization:**
   (a) Divide the domain $[0, 1]$ into 2 equal elements ($0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$).
   (b) Use linear basis functions for each element. Define the basis functions $\varphi_1(x), \varphi_2(x), \varphi_3(x)$ corresponding to the three nodes at $x = 0, 0.5, 1$.
   (c) Derive the local stiffness matrix for each element. Write out the global stiffness matrix and load vector in matrix form.
3. **Assembly:** Assemble the global stiffness matrix and global load vector from the local contributions. Apply the boundary conditions to modify the system.
4. **Solve:** Write the resulting linear system of equations in the form $A\mathbf{u} = \mathbf{b}$, where $\mathbf{u} = [u_1, u_2, u_3]^T$. Solve the system.
5. **Solve ODE analytically:** Solve given boundary-value problem analytically and compare this solution to the numerical solution.

Deliverables

1. A step-by-step derivation of the weak form.
2. Expressions for the basis functions and the local stiffness matrices.
3. The assembled global stiffness matrix and load vector with applied boundary conditions.
4. A clear and neat presentation of the resulting system of equations.
5. Comparsion of numerical and analytical solutions

Notes

- Assume piecewise linear basis functions. For example:

$$\phi_1(x) = 1 - \frac{x}{0.5}, \quad \phi_2(x) = \frac{x}{0.5} \quad \text{on the first element.}$$

- Show all work for the derivation of integrals and assembly process.
- Use consistent units and notation throughout the assignment.

Submission

Submit your work as a PDF with handwritten or typed solutions.

Math 588. HW # 1.

Sol-n: Using the given info, we have

(1) $\quad -u''(x) = \overset{f(x)}{2x} \qquad x \in (0,1) = \Omega$

(2) $\quad u(0) = 0$,

(3) $\quad u(1) = 0$.

Task 1: (Weak Form Derivation):
Using the given PDE (ODE) in (1), & by letting $v \in V$ s.t.

(4) $\quad L^2(\Omega) = \{ v \in \Omega \mid \int_\Omega v^2 dx = \int_0^1 v^2 dx < \infty \}$ &

(5) $\quad V = H^1(\Omega) = \{ v \in L^2(\Omega) \mid \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \}$ &

(6) $\quad H^1_0(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \}$, so

for $v \in V$ (defined in (5)), let's multiply (BS) both sides of (1) by v & integrate over $\Omega = (0,1)$:

$$-u''(x) \cdot v = (2x) \cdot v \quad \Rightarrow \int_0^1 \text{of BS}$$

(7) $$\underbrace{\int_0^1 -u''(x) \cdot v \, dx}_{\text{IBP}} = \int_0^1 2x \cdot v \, dx$$

$$-u'(x) \cdot v \Big|_0^1 + \int_0^1 u'(x) \cdot v'(x) \, dx = \int_0^1 (2x) \cdot v \, dx \Rightarrow$$

Integration by parts

Note on (7) $v(0)=v(1)=0$, so LHS has a zero term:

$$\underbrace{-u'(x)\cdot v\Big|_0^1}_{=0} + \int_0^1 u'(x)\cdot v'(x)\,dx = \int_0^1 (2x)\,v\,dx \Rightarrow$$

$$\int_0^1 u'(x)\cdot v'(x)\,dx = \int_0^1 (2x)\cdot v\,dx \Rightarrow$$

(8) $$\boxed{\langle u'(x), v'(x)\rangle = \langle (2x), v\rangle}$$ where

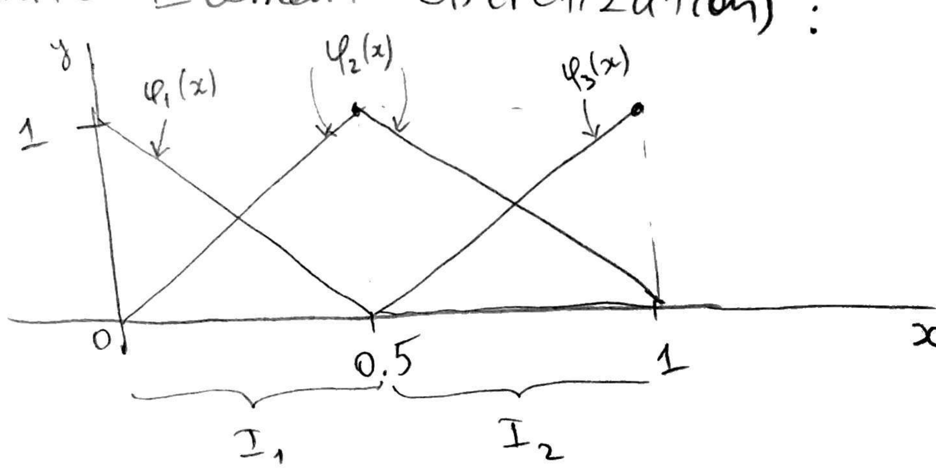
$$\langle f, g\rangle := \int_0^1 f(x)\cdot g(x)\,dx.$$

(8) is called <u>weak formulation</u> of the given PDE in (1)

Task 2. (Finite Element Discretization):

(a)

(9)



(10) $I_1 = [0, 0.5]$, element 1; $I_2 = [0.5, 1]$, element 2.

(b)

(11) $$\varphi_1(x) = \begin{cases} 1 - \dfrac{x}{0.5}, & x \in [0, 0.5] = I_1 \\ 0, & x \in [0.5, 1] = I_2 \end{cases}$$

(12) $$\varphi_2(x) = \begin{cases} \dfrac{x}{0.5}, & x \in [0, 0.5] = I_1 \\ 2 - \dfrac{x}{0.5}, & x \in [0.5, 1] = I_2 \end{cases}$$

(13) $$\varphi_3(x) = \begin{cases} 0, & x \in [0, 0.5] = I_1 \\ \dfrac{x}{0.5} - 1, & x \in [0.5, 1] = I_2 \end{cases}$$

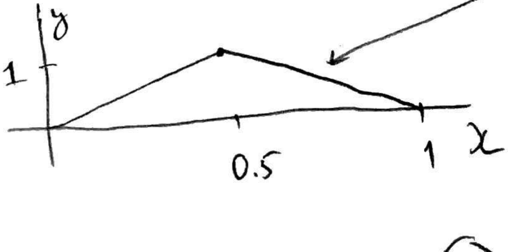
(C) Write out the <u>global stiffness</u> matrix and load vector in matrix form. Derive the local stiffeness matrix for each element (I_1, I_2).

For the considered intervals (elements) I_1, I_2 in (10) & (9), Let $\underline{V_h}$ be the set of functions v s.t.

V_h(i) v is linear on each I_j, & V_h(ii) v is cont-s on $[0,1]$ &

V_h(iii) $v(0) = v(1) = 0$. e.g. of v

(14)



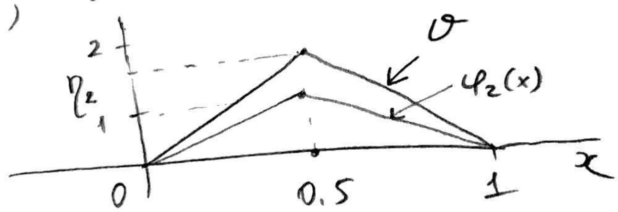
Now by considering (introducing | Bringing) the basis, (11)-(13), functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ on V_h, we have the property

$$(15) \qquad \varphi_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Then a function $v \in V_h$ has the representation

$$(16) \qquad v(x) = \sum_{i=1}^{3} \eta_i \varphi_i(x), \quad x \in [0,1)$$

where $\eta_i = v(x_i)$.



i.e. $v \in V_h$ can be written in a unique way as a L.C. of the basis f-ns φ_i.

The finite element now can now be formulated

Find $u_h \in V_h$ s.t.

$$(M) \qquad F(u_h) \leq F(v), \quad \forall v \in V_h$$

Minimizn Problem (M) is equivalent to finite dim-l variational prob-m:

Find $u_h \in V_h$ s.t.

$$(V) \qquad \langle u_h', v' \rangle = \langle f, v \rangle \quad \forall v \in V_h.$$

Since u_h satisfies (V) $\forall v \in V_h$, then for the choices of v, $v = \varphi_1 \in V_h$, $v = \varphi_2 \in V_h$, $v = \varphi_3 \in V_h$,

(V) still holds:

$$(17) \quad \langle u_h', \varphi_j' \rangle = \langle f, \varphi_j \rangle \qquad j = 1, 2, 3.$$

Since $u_h \in V_h = \text{span}\{\varphi_1, \varphi_2, \varphi_3\}$ by similar argument in (16),

u_h can be written as a L.C. of basis vectors φ_i:

$$(18) \quad u_h(x) = \sum_{i=1}^{3} z_i \varphi_i(x) \quad \text{where}$$

$$(19) \quad \boxed{z_i = u_h(x_i)} \leftarrow \underline{\text{Important result}}$$

Then putting (18) into (17) gives

$$(20) \quad \left\langle \left(\sum z_i \varphi_i(x)\right)', \varphi_j'(x) \right\rangle = \langle f, \varphi_j \rangle \iff$$

$$(21) \quad \sum_{i=1}^{3} z_i \langle \varphi_i'(x), \varphi_j'(x) \rangle = \langle f, \varphi_j \rangle \quad \text{or i.e.}$$

$$(22) \quad \langle \varphi_1', \varphi_j' \rangle \cdot z_1 + \langle \varphi_2', \varphi_j' \rangle z_2 + \langle \varphi_3', \varphi_j' \rangle z_3 = \langle f, \varphi_j \rangle, \; j = 1,2,3.$$

$$(22') \quad \begin{cases} \langle \varphi_1', \varphi_1' \rangle z_1 + \langle \varphi_2', \varphi_1' \rangle z_2 + \langle \varphi_3', \varphi_1' \rangle z_3 = \langle f, \varphi_1 \rangle, & j = 1 \\ \langle \varphi_1', \varphi_2' \rangle z_1 + \langle \varphi_2', \varphi_2' \rangle z_2 + \langle \varphi_3', \varphi_2' \rangle z_3 = \langle f, \varphi_2 \rangle, & j = 2 \\ \langle \varphi_1', \varphi_3' \rangle z_1 + \langle \varphi_2', \varphi_3' \rangle z_2 + \langle \varphi_3', \varphi_3' \rangle z_3 = \langle f, \varphi_3 \rangle, & j = 3. \end{cases}$$

$$(23) \quad \begin{bmatrix} \langle \varphi_1', \varphi_1' \rangle \\ \langle \varphi_1', \varphi_2' \rangle \\ \langle \varphi_1', \varphi_3' \rangle \end{bmatrix} z_1 + \begin{bmatrix} \langle \varphi_2', \varphi_1' \rangle \\ \langle \varphi_2', \varphi_2' \rangle \\ \langle \varphi_2', \varphi_3' \rangle \end{bmatrix} z_2 + \begin{bmatrix} \langle \varphi_3', \varphi_1' \rangle \\ \langle \varphi_3', \varphi_2' \rangle \\ \langle \varphi_3', \varphi_3' \rangle \end{bmatrix} z_3 = \begin{bmatrix} \langle f, \varphi_1 \rangle \\ \langle f, \varphi_2 \rangle \\ \langle f, \varphi_3 \rangle \end{bmatrix}$$

Writing this (23) in a matrix eq-n form:

$$(24)\quad \underbrace{\begin{bmatrix} \langle \varphi_1', \varphi_1' \rangle & \langle \varphi_2', \varphi_1' \rangle & \langle \varphi_3', \varphi_1' \rangle \\ \langle \varphi_1', \varphi_2' \rangle & \langle \varphi_2', \varphi_2' \rangle & \langle \varphi_3', \varphi_2' \rangle \\ \langle \varphi_1', \varphi_3' \rangle & \langle \varphi_2', \varphi_3' \rangle & \langle \varphi_3', \varphi_3' \rangle \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} \langle f, \varphi_1 \rangle \\ \langle f, \varphi_2 \rangle \\ \langle f, \varphi_3 \rangle \end{bmatrix}}_{b}$$

$A \cdot z = b$ where

$$(25)\quad A = [a_{ji}], \quad a_{ji} = \langle \varphi_i', \varphi_j' \rangle \equiv \text{stiffness matrix}$$

$$(26)\quad b = [b_j], \quad b_j = \langle f, \varphi_j \rangle \equiv \text{load vector}$$

$$a_{11} = \langle \varphi_1', \varphi_1' \rangle \overset{\text{by (11)-(13)}}{=\!=\!=} \int_0^{0.5} \left(1 - \frac{x}{0.5}\right)' \cdot \left(1 - \frac{x}{0.5}\right)' dx + \int_{0.5}^{1} 0 \, dx$$

$$= \int_0^{0.5} (-2)\cdot(-2)\, dx = [4x]_0^{0.5} = 2 \Rightarrow \boxed{a_{11} = 2}$$

$$a_{12} = \langle \varphi_2', \varphi_1' \rangle = \int_0^{0.5} \overset{\varphi_2}{\left(\frac{x}{0.5}\right)'} \cdot \overset{\varphi_1}{\left(1 - \frac{x}{0.5}\right)'} dx + \int_{0.5}^{1} \overset{\varphi_2}{\left(2 - \frac{x}{0.5}\right)'} \cdot \overset{\varphi_1}{(0)'} dx =$$

$$= \int_0^{0.5} (2)\cdot(-2)\, dx = [4x]_0^{0.5} = -2 \Rightarrow \boxed{a_{12} = -2}$$, also $\exists$ asymmetry

$$a_{21} = \langle \varphi_1', \varphi_2' \rangle \equiv \langle \varphi_2', \varphi_1' \rangle = a_{12} \Rightarrow \boxed{a_{21} = -2}$$

$$a_{13} = \langle \varphi_3', \varphi_1' \rangle = \int_0^{0.5} \overset{\varphi_3'}{(0)'} \cdot \overset{\varphi_1'}{\left(1 - \frac{x}{0.5}\right)'} dx + \int_{0.5}^{1} \overset{\varphi_3'}{\left(\frac{x}{0.5} - 1\right)'} \cdot \overset{\varphi_1'}{(0)'} dx = 0$$

$$\Rightarrow \boxed{a_{13} = 0} \quad \&$$

$$a_{31} = \langle \varphi_1', \varphi_3' \rangle = \langle \varphi_3', \varphi_1' \rangle \equiv a_{13}. \Rightarrow \boxed{a_{31} = 0}$$

by property of inner product

(27)

$$a_{22} = \langle \varphi_2', \varphi_2' \rangle = \int_0^1 \varphi_2' \cdot \varphi_2' \, dx = \int_0^{0.5} \overset{\varphi_2}{\left(\frac{x}{0.5}\right)'} \cdot \overset{\varphi_2}{\left(\frac{x}{0.5}\right)'} dx +$$

$$+ \int_{0.5}^1 \left(2 - \frac{x}{0.5}\right)' \cdot \left(2 - \frac{x}{0.5}\right)' dx = \left[4x\right]_0^{0.5} + \left[4x\right]_{0.5}^1 = 2 + 2 = 4$$

$$\Rightarrow \boxed{a_{22} = 4}$$

$$a_{23} = \langle \varphi_3', \varphi_2' \rangle = \int_0^1 \varphi_3' \cdot \varphi_2' \, dx = \int_0^{0.5} \overset{\varphi_3}{(0)'} \cdot \overset{\varphi_2}{\left(\frac{x}{0.5}\right)'} dx + \int_{0.5}^1 \overset{\varphi_3}{\left(\frac{x}{0.5} - 1\right)'} \cdot \left(2 - \frac{x}{0.5}\right)' dx$$

$$= \int_{0.5}^1 (2)(-2)\, dx = \left[-4x\right]_{0.5}^1 = -(4-2) = -2. \Rightarrow \boxed{a_{23} = -2} \Rightarrow$$

$$a_{32} = a_{23} \Rightarrow \boxed{a_{32} = -2},$$ and finally

$$a_{33} = \langle \varphi_3', \varphi_3' \rangle = \int_0^1 \varphi_3' \cdot \varphi_3' \, dx = \int_{0.5}^1 \left(\frac{x}{0.5} - 1\right)' \cdot \left(\frac{x}{0.5} - 1\right)' dx$$

$$= \int_{0.5}^1 (2) \cdot (2)\, dx = \left[4x\right]_{0.5}^1 = 4 - 2 = 2 \Rightarrow \boxed{a_{33} = 2}$$

So combining these calculations into A, we get

(28) $$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \text{global stiffness matrix.}$$

Similarly the load vector in (26), using (1),

$$b_1 = \langle f, \varphi_1 \rangle = \int_0^{0.5} \overset{f}{(2x)} \cdot \left(1 - \frac{x}{0.5}\right) dx + \int_{0.5}^1 (2x) \cdot (0)\, dx = \int_0^{0.5} (2x - 4x^2)\, dx$$

$$= x^2 - \frac{4x^3}{3}\Big|_0^{\frac{1}{2}} = \left(\frac{1}{4} - \frac{4}{3} \cdot \frac{1}{8}\right) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \Rightarrow \boxed{b_1 = \frac{1}{12}}$$

$$b_2 = \langle f, \varphi_2 \rangle = \int_{0.}^{0.5} (2x)\cdot\left(\frac{x}{0.5}\right)dx + \int_{0.5}^{1} (2x)\left(2x - \frac{x}{0.5}\right)dx = \frac{6}{12}$$

$$\boxed{b_2 = \frac{6}{12}}$$

&

$$b_3 = \langle f, \varphi_3 \rangle = \int_0^{0.5} (2x)\cdot(0)\,dx + \int_{0.5}^{1} (2x)\cdot\left(\frac{x}{0.5} - 1\right)dx = \frac{5}{12} \Rightarrow$$

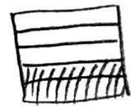
$\boxed{b_3 = \frac{5}{12}}$. Thus the load vector has

$$b = [b_1 \; b_2 \; b_3]^T = \left[\frac{1}{12}, \frac{6}{12}, \frac{5}{12}\right]^T. \tag{29}$$

So (24) can be written as

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1/12 \\ 6/12 \\ 5/12 \end{bmatrix} \tag{30}$$

which is required matrix-vector form for (c)



Task 3. Assembling the global stiffness matrix & load vector & modifying the system using the boundary cond-ns:

Assembled matrix form is in (30). Now let me modify the system using the BC (2), & (3):

$u(0) = 0$ (2) $u(1) = 0$ (3)   & recall $z_i = u_h(x_i)$ (19)

where $u \approx u_h$ for $u \in V$ (continuous space) & $u_h \in V_h$ (discrete analog space of V).

Thus, $\underbrace{0}_{} = u(0) \overset{\text{by (19)}}{=} u_h(\overset{x_1}{0}) = u_h(x_1) = \underbrace{z_1}_{} \Rightarrow$

(31) $\boxed{z_1 = 0}$ & similarly

$\underbrace{0}_{} = u(1) \overset{\text{by (19)}}{=} u_h(1) = u_h(x_3) = \underbrace{z_3}_{} \Rightarrow$

(32) $\boxed{z_3 = 0}$

So applying these found z_1 & z_3 into the matrix eq-n in (30) modifies the system to

$z_1 = 0$ by (31)

Note the change to get,

$-2z_2 = -1/4$

(33)
$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 6/12 \\ -1/4 \end{bmatrix}$$

$-2z_2 = -1/4$

$z_2 = \frac{1}{8}$

why (?1)

$z_3 = 0$ by (32)

Task 4 (Solving):

which is a system of one variable, z_2; and we must use the second eq-n to solve for it:

$$-2 \cdot 0 + 4 \cdot z_2 + (-2) \cdot 0 = 6/12 \Rightarrow$$

(34) $4z_2 = \frac{1}{2} \Rightarrow \boxed{z_2 = \frac{1}{8}}$, at last, we have

(35) $z = [z_1, z_2, z_3]^T = [0, \frac{1}{8}, 0]^T$, a numerical sol-n to (33).

Hence, by (18)

$$u_h(x) = \sum_{i=1}^{3} z_i \varphi_i(x) = z_1 \varphi_1(x) + z_2 \cdot \varphi_2(x) + z_3 \cdot \varphi_3(x) =$$

$$= 0 \cdot \varphi_1(x) + \frac{1}{8} \cdot \varphi_2(x) + 0 \cdot \varphi_3(x) =$$

$$= \frac{1}{8} \cdot \varphi_2(x)$$

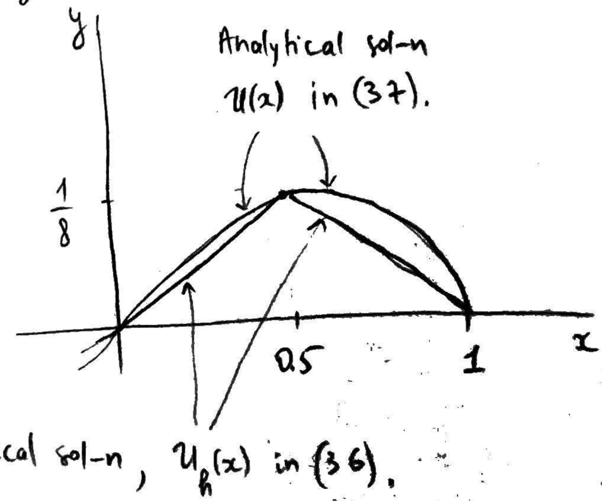
$$= \frac{1}{8} \cdot \begin{cases} \frac{x}{0.5}, & x \in [0, 0.5] = I_1 \\ 2 - \frac{x}{0.5}, & x \in [0.5, 1] = I_2. \end{cases} \Rightarrow$$

$$(36) \quad u_h(x) = \begin{cases} \frac{x}{4}, & x \in [0, 0.5] = I_1 \\ \frac{1}{4} - \frac{x}{4}, & x \in [0.5, 1] = I_2 \end{cases}$$

Task 5 (Solve ODE analytically); By (1) - (3), we have

$$\begin{array}{llll} (1) & u''(x) = -2x & \Rightarrow u'(x) = -x^2 + C_1 & \Rightarrow u(x) = -\frac{x^3}{3} + C_1 x + C_2 \\ (2) & u(0) = 0 & \Rightarrow & u(0) = C_2 = 0 \\ (3) & u(1) = 0 & \Rightarrow & u(1) = C_1 = \frac{1}{3} \end{array} \Rightarrow$$

(37) $\boxed{u(x) = -\frac{1}{3}x^3 + \frac{1}{3}x}$ is the analytical sol-n to the given ODE (1) - (3).



* As expected, the FEM sol-n $u_h(x)$ is <u>well-capturing</u> the $u(x)$, analytical sol-n on the discret pts, showing $z_1 = 0$, $z_2 = \frac{1}{8}$, $z_3 = 0$ for $z_i = u_h(x_i) = u(x_i)$.