Problem 2.

Global coordinate approach

(1) $$\varphi_i(x) = \frac{x - x_{i-\frac{1}{2}}}{h_i}\left[\frac{2}{h_i}(x - x_{i-\frac{1}{2}}) + 1\right], \quad x \in [x_{i-1}, x_i]$$

Solution:

This approach is about finding the quadratic basis functions φ_{i-1} and φ_i on a subinterval (x_{i-1}, x_i) s.t.

(2) $\varphi_{i-1}(x_{i-1}) = 1, \quad \varphi_{i-1}(x_i) = 0, \quad \varphi_i(x_{i-1}) = 0, \quad \varphi_i(x_i) = 1,$

a qudratic function $\varphi_{i-1}(x)$ is in the form

(3) $\varphi_{i-1}(x) = a + bx + cx^2, \quad x \in [x_{i-1}, x_i]$

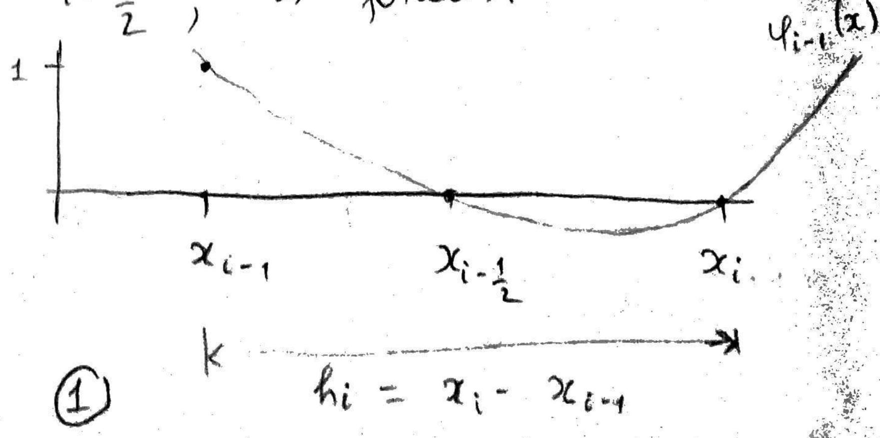
We have now two equations from (2):

(4) $\varphi_{i-1}(x_{i-1}) = 1 \Leftrightarrow a + b x_{i-1} + c(x_{i-1})^2 = 1$

(5) $\varphi_{i-1}(x_i) = 0 \Leftrightarrow a + b x_i + c(x_i)^2 = 0$

We need one more independent eq-n to find unknowns a, b, c. To do this, let's introduce an additional node $x_{i-\frac{1}{2}}$, as follows:

(6) $$\boxed{\left.\begin{array}{l} h_i := x_i - x_{i-1} \\ x_{i-\frac{1}{2}} := \dfrac{x_{i-1} + x_i}{2} \end{array}\right\} \Rightarrow \begin{array}{l} x_i = x_{i-\frac{1}{2}} + \dfrac{h_i}{2} \ \& \\ x_{i-1} = x_{i-\frac{1}{2}} - \dfrac{h_i}{2} \end{array}}$$



So this $x_{i-\frac{1}{2}}$ point satisfies

(7) $\varphi_{i-1}(x_{i-\frac{1}{2}}) = 0 \iff a + b x_{i-\frac{1}{2}} + c x_{i-\frac{1}{2}}^2 = 0$;

Thus we have 3 equations, (4), (5), (7), to find unknowns a, b, c making a 3x3 sytem of linear equations to solve:

$$(8)\ \begin{cases} a + b x_{i-1} + c(x_{i-1})^2 = 1 & \text{------ } (4) \\ a + b x_i + c(x_i)^2 = 0 & \text{---- } (5) \\ a + b x_{i-\frac{1}{2}} + c(x_{i-\frac{1}{2}})^2 = 0 & \text{--- } (7) \end{cases}$$

where $x_i, x_{i-1}, x_{i-\frac{1}{2}}$ are all

(4) − (5) subtraction gives:

$$b(\underbrace{x_{i-1} - x_i}) + c[(x_{i-1})^2 - (x_i)^2] = 1$$

$$-b \cdot h_i + c \cdot (-h_i) \cdot \underbrace{(x_{i-1} + x_i)} = 1 \qquad \} * (-1)$$

$$b h_i + c h_i \cdot (2 x_{i-\frac{1}{2}}) = -1$$

$$(9) \quad \boxed{b h_i + 2 c h_i x_{i-\frac{1}{2}} = -1}$$

Now similarly subtructing eq-n (7) from eq-n (5) yields;

(5) − (7):

$$b \cdot \underbrace{(x_i - x_{i-\frac{1}{2}})} + c \cdot [\underbrace{(x_i - x_{i-\frac{1}{2}})} \cdot (\underbrace{\overset{\text{by (6)}}{x_i} + x_{i-\frac{1}{2}}})] = 0$$

$$b \cdot \frac{h_i}{2} + c \cdot \frac{h_i}{2} \cdot \left(x_{i-\frac{1}{2}} + \frac{h_i}{2} + x_{i-\frac{1}{2}} \right) = 0 \Rightarrow$$

(10) $\Rightarrow$ $$\boxed{b \cdot \frac{h_i}{2} + \frac{c h_i}{2}\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0}$$ by eq. (5)-(7).

So by (9) & (10) we have now 2x2 system to solve for b & c :

$$(11) \quad \begin{cases} b h_i + 2c\, h_i \cdot x_{i-\frac{1}{2}} = -1 \\ \frac{b h_i}{2} + \frac{c h_i}{2}\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0 \end{cases} \quad \} \cdot 2 \text{ of BS:}$$

$$\Rightarrow \begin{cases} b h_i + c h_i (2x_{i-\frac{1}{2}}) = -1 \\ b h_i + c h_i \left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = 0 \end{cases} \quad \} \text{ eq (1) - (2):}$$

$$c h_i \cdot \left(2 \cdot x_{i-\frac{1}{2}} - 2x_{i-\frac{1}{2}} - \frac{h_i}{2}\right) = -1 \Rightarrow$$

$$c = \frac{2}{h_i \cdot h_i} = \frac{2}{h_i^2};$$

(12) $$\boxed{c = \frac{2}{h_i^2}}$$

Then using (10) & substituting c in (12) back into the eq-n (12), we get

$$b = -c\left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right) = -\frac{2}{h_i^2} \cdot \left(2x_{i-\frac{1}{2}} + \frac{h_i}{2}\right)$$

$$= -\frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right); \quad \Rightarrow$$

(13) $\boxed{b = -\frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right)}$ & finally for a,

using eq-n (7), we have

$$a = -b\,x_{i-\frac{1}{2}} - c\cdot\left(x_{i-\frac{1}{2}}\right)^2$$

$$= \frac{1}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right)\cdot x_{i-\frac{1}{2}} - \frac{2}{h_i^2}\cdot\left(x_{i-\frac{1}{2}}\right)^2 =$$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{4x_{i-\frac{1}{2}}}{h_i} + 1 - \frac{2x_{i-\frac{1}{2}}}{h_i}\right]$$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\cdot\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right]; \quad \Rightarrow$$

(14) $\boxed{a = \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right]}$, and putting

these (12), (13), (14) back to (3), we have

(15) $\varphi_{i-1}(x) = a + bx + cx^2$

$$= \frac{x_{i-\frac{1}{2}}}{h_i}\left[\frac{2x_{i-\frac{1}{2}}}{h_i} + 1\right] - \frac{x}{h_i}\left(\frac{4x_{i-\frac{1}{2}}}{h_i} + 1\right) + \frac{2x^2}{h_i^2}$$

$$= \frac{x - x_{i-\frac{1}{2}}}{h_i}\cdot\left[\frac{2}{h_i}\left(x - x_{i-\frac{1}{2}}\right) - 1\right], \quad x \in [x_{i-1}, x_i], \quad \blacksquare$$

as desired.