Stat 544.

HW#6.

Study the following SDE's. For each one:

1. Write down the SDE with initial condition. Describe all therms, variables, & parameters.

2. Use any method to solve it.
3. Choose a set of params & plot several stochastic paths on the same graph.

4. Study the expectation, variance, auto correlation & anything else you find interesting

5. Do some literature search to find one or two application of the SDE. Choose your favourite application and write one or two paragraphs about it.

Questions. Drift - diffusion

(i) The general SDE is the form of

(1) $$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t \qquad (1)$$

if $\int_0^t \mu(X_\tau, \tau)\,d\tau$ & $\int_0^t \sigma(X_\tau, \tau)\,dW_\tau$ exists

(2) $$X_t = X_0 + \int_0^t \mu(X_\tau, \tau)\,d\tau + \int_0^t \sigma(X_\tau, T)\,dW_\tau$$

is called strong sol-n of (1).

For special cases of $\mu(X_t, t)$ & $\sigma(X_t, t)$, we study some types of SDE's, specifically, Let

(3) $$\mu(X_t, t) = a_1(t) X_t + a_2(t) \quad \&$$
(4) $$\sigma(X_t, t) = b_1(t) X_t + b_2(t)$$

Then we have a Linear SDE :

(5) $$dX_t = [a_1(t) X_t + a_2(t)]\,dt + [b_1(t) \cdot X_t + b_2(t)]\,dW_t$$

Case 1: $a_1(t) = b_1(t) = 0$.
Then (5) becomes

1

(6) $dX_t = a_2(t)\,dt + b_2(t)\,dW_t$ & with the IC

(6') $X_t(0) = X_0$

which is drift-diffusion SDE.

(ii) Integrating (6) $\int_0^t (\cdot)\,dt$ gives the solution:

?? How should we come up with change of var-s ?

$$\int_0^t dX_t = \int_0^t a_2(\tau)\,d\tau + \int_0^t b_2(\tau)\,dW_\tau \implies$$

(7) $$X_t = X_0 + \int_0^t a_2(\tau)\,d\tau + \int_0^t b_2(\tau)\,dW_\tau .$$

(iii) Plotting several stochastic paths:

(iv) Statistics: Using (7):

(8) $$E[X_t] = E\left[X_0 + \int_0^t a_2(\tau)\,d\tau + \int_0^t b_2(\tau)\,dW_\tau\right]$$

$$E(X_t) = \underbrace{E(X_0)}_{\#} + \underbrace{E\left[\int_0^t a_2(\tau)\,d\tau\right]}_{\#} + \cancel{E\left[\int_0^t b_2(\tau)\,dW_\tau\right]} \xrightarrow{} 0$$

$E(\#) = \#$ so

$$= X_0 + \int_0^t a_2(\tau)\,d\tau$$

(9) $$E[X_t] = X_0 + \int_0^t a_2(\tau)\,d\tau$$

$$\mathrm{Var}(X_t) \overset{\text{by (7)}}{=\!=}$$

$$= \mathrm{Var}\Big[\underbrace{X_0}_{0} + \underbrace{\int_0^t a_2(\tau)\,d\tau}_{0} + \int_0^t b_2(\tau)\,dW_\tau\Big]$$

$$= \mathrm{Var}\Big[\int_0^t b_2(\tau)\,dW_\tau\Big]$$

$$= E\Big\{\Big[\int_0^t b_2(\tau)\,dW_\tau\Big]^2\Big\} - \Big\{\overbrace{E\Big[\int_0^t b_2(\tau)\,dW_\tau\Big]}^{=0 \text{ by}}\Big\}^2$$

Itô Isometry

?4 where

$$= E\Big[\underbrace{\int_0^t b_2^2(\tau)\,d\tau}_{\#}\Big] - \underbrace{\{0\}^2}_{=0} = \quad E(\#) = \#$$

$$= \int_0^t b_2^2(\tau)\,d\tau \quad \Rightarrow$$

(10) $$\boxed{\mathrm{Var}(X_t) = \int_0^t b_2^2(\tau)\,d\tau}$$

?5 Auto Corrolation $(X_t) =$

How to do it?