Question 5: Brownian Bridge Equation

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

We begin with the general linear SDE:

$$dX_t = a_1(t)X_t\,dt + a_2(t)\,dt + b_1(t)X_t\,dW_t + b_2(t)\,dW_t. \tag{1}$$

We now consider the case:

$$a_1(t) = -\frac{1}{1-t} = const, \quad a_2(t) = \frac{b}{1-t}, \quad b_1(t) = 0, \quad b_2(t) = 1,$$

with constant $b > 0$. Then (1) simplifies to the Brownian Bridge SDE:
We consider the following time-inhomogeneous linear SDE:

$$dX_t = \frac{b - X_t}{1-t}\,dt + dW_t, \tag{2}$$

where $t \in [0, 1)$ and W_t is a standard Brownian motion. The initial condition is:

$$X_0 = x_0.$$

Interpretation: The process is attracted to the value b as $t \to 1$, and the drift becomes singular at $t = 1$. The process is a
textbfBrownian bridge from x_0 at $t = 0$ to b at $t = 1$.

(iii) Solving the Brownian Bridge SDE

We solve the linear SDE:

$$dX_t = \frac{b - X_t}{1-t}\,dt + dW_t,$$

using the method of integrating factors. Rearranging:

$$dX_t + \frac{X_t}{1-t}\,dt = \frac{b}{1-t}\,dt + dW_t.$$

Let the integrating factor be $\mu(t) = \frac{1}{1-t}$, so:

$$\mu(t)X_t = Y_t \Rightarrow d(Y_t) = \mu(t)dX_t + X_t d\mu(t).$$

Alternatively, multiply both sides by $\mu(t)$:

$$\frac{d}{dt}\left(\frac{X_t}{1-t}\right) = \frac{b}{(1-t)^2} + \frac{1}{1-t}\,dW_t.$$

Now integrate both sides:

$$\begin{aligned}
X_t &= x_0 \cdot (1-t) + (1-t)\int_0^t \frac{1}{1-s}dW_s + (1-t)\int_0^t \frac{b}{(1-s)^2}ds \\
&= x_0(1-t) + (1-t)\int_0^t \frac{1}{1-s}dW_s + b(1-t)\left(\frac{t}{1-t}\right) \\
&= x_0(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dW_s.
\end{aligned}$$

Thus, the solution is:

$$X_t = x_0(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dW_s. \tag{3}$$

(iv) Graphing the Solution

To simulate and visualize the Brownian bridge:

- Choose constants x_0 and b.
- Simulate paths using the exact solution (3).
- Plot multiple paths to see convergence to b at $t = 1$.

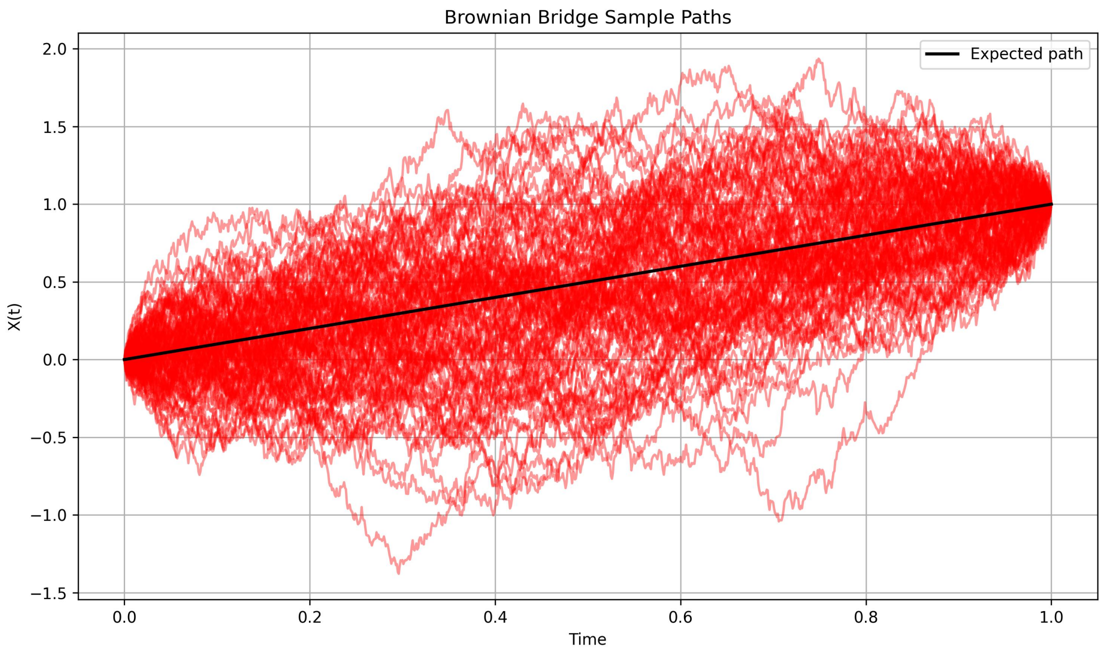

Figure 1: Simulated sample paths of the Brownian Bridge with $X_0 = x_0$ and $X_1 = b$

(v) Statistics of the Brownian Bridge

From (3), we have:

Expectation

Taking expectation:

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\left[x_0(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s} dW_s\right] \\ &= x_0(1-t) + bt + 0 \\ &= (1-t)x_0 + tb.\end{aligned}$$

Variance

Using Itô isometry:

$$\begin{aligned}\mathrm{Var}(X_t) &= \mathrm{Var}\left((1-t)\int_0^t \frac{1}{1-s} dW_s\right) \\ &= (1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds = (1-t)^2 \left[-\frac{1}{1-s}\right]_0^t \\ &= (1-t)^2\left(\frac{1}{1-t} - 1\right) = t(1-t).\end{aligned}$$

Covariance

Let $0 \le s \le t \le 1$.

$$\begin{aligned} \mathrm{Cov}(X_s, X_t) &= (1-s)(1-t)\int_0^s \frac{1}{(1-u)^2}du = (1-s)(1-t)\left(\frac{1}{1-s} - 1\right) \\ &= (1-t)s. \end{aligned}$$

Conclusion

Brownian bridge is a Gaussian process with:

- $\mathbb{E}[X_t] = (1-t)x_0 + tb$
- $\mathrm{Var}(X_t) = t(1-t)$
- $\mathrm{Cov}(X_s, X_t) = (1-t)s$

It interpolates linearly in mean and has a parabolic variance shape.

(vi) Applications of the Brownian Bridge

- **Statistics:** Used in the Kolmogorov-Smirnov and Cramér–von Mises tests. Null distributions of empirical processes often converge to Brownian bridges.
- **Finance:** Employed in simulation algorithms (Brownian bridge construction) to generate conditioned paths ending at a fixed terminal value.
- **Engineering:** Models random processes that start and end at specified values, such as interpolation in stochastic models.

References

References

[1] Billingsley, P. (1999). *Convergence of Probability Measures*. John Wiley & Sons.

[2] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd Edition.

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[4] Shorack, G. R., & Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley.

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Appendix: Python Code for Question 5 (Brownian Bridge)

The following Python script was used to simulate the Brownian Bridge in Part (iv):

Listing 1: Brownian Bridge Simulation (Q5)

```python
# q5_brownian_bridge.py

import numpy as np
import matplotlib.pyplot as plt

# Parameters
x0 = 0          # Starting point X(0)
b = 1           # Endpoint X(1)
T = 1           # Total time (fixed for Brownian Bridge)
N = 1000        # Time steps
M = 100         # Number of sample paths
dt = T / N
t = np.linspace(0, T, N+1)

# Pre-allocate the array for paths
X = np.zeros((M, N+1))

# Simulate M Brownian Bridges
for i in range(M):
    W = np.zeros(N+1)
    dW = np.random.normal(0, np.sqrt(dt), size=N)
    W[1:] = np.cumsum(dW)  # Standard Brownian motion
    # Brownian Bridge formula
    X[i, :] = x0 + (b - x0) * t + W - t * W[-1]

# Expected path (linear interpolation)
X_mean = x0 + (b - x0) * t

# Plot
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i], color='red', alpha=0.4)
plt.plot(t, X_mean, color='black', linewidth=2, label='Expected path')
plt.title('Brownian Bridge Sample Paths')
plt.xlabel('Time')
plt.ylabel('X(t)')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.savefig('images/q5_brownian_bridge_graph.jpg', dpi=300)
```

```
plt.show()
```