Q7.

(1) Let $\xi_1, \xi_2 ..$ be seq. of IID r.v.s s.t.

$$\xi_i \sim \mathcal{N}(\mu, \sigma^2)$$

(2) Define the r.v.

$$S_n = \sum_i^n \xi_i$$

Check if the ↓ stochastic processes are martingales:

(a) S_n (b) $S_n - \mu n$ (c) S_n^2 (d) $S_n^2 - \sigma^2 n$.

Sol-n: [Preamble]

(a) (i) $S_n = \xi_1 + .. + \xi_n$ is naturally adapted to the filtration $\mathcal{F}_n = \sigma(\xi_1, ... \xi_n)$ consisting of all information upto time n. ✓

(ii) $\xi_i \sim \mathcal{N}(\mu, \sigma^2)$ has a finite expectation, $\Rightarrow$

$E[S_n] < \infty$, i.e. finite expection for S_n. ✓

(iii) $S_n = S_{n-1} + \xi_n \Rightarrow$

linearity of E:

$$E(S_n | \mathcal{F}_{n-1}) = E(S_{n-1} + \xi_n | \mathcal{F}_{n-1})$$

$$= \underbrace{E(S_{n-1} | \mathcal{F}_{n-1})} + \underbrace{E(\xi_n | \mathcal{F}_{n-1})}$$

$$= S_{n-1} + \mu$$

$$\neq S_{n-1} \Rightarrow$$

$E(S_n | \mathcal{F}_{n-1}) \neq S_{n-1}$ so

S_n is not a martingale. $\boxtimes$

(b) $(S_n - \mu n)$ is checked:

(i) $S_n - \mu n$ is adapted to $\mathcal{F}_n$ as it is based on S_n. ✓

(ii) S_n has finite expectation.

$\mu = n \cdot E(\xi_i) < \infty$ which is finite $\Rightarrow$

$S_n - \mu n$ has a finite expectation.

(iii) $E(S_n - \mu n \mid \mathcal{F}_{n-1})$

$$= E\left((S_{n-1} + \xi_n) - \mu n \mid \mathcal{F}_{n-1}\right)$$

$$= \underbrace{E(S_{n-1} \mid \mathcal{F}_{n-1})}_{= S_{n-1}} + \underbrace{E(\xi_n \mid \mathcal{F}_{n-1})}_{+\ \mu} - \underbrace{E(\mu n \mid \mathcal{F}_{n-1})}_{-\ \mu n}$$

$$= S_{n-1} + \mu - \mu n$$

$$= S_{n-1} + \mu(1-n)$$

$$= S_{n-1} - \mu(n-1) \checkmark$$

which shows $S_n - \mu n$ is a martingale. $\boxtimes$

(c) Let's check S_n^2:

Unlike the first two parts (a), (b), let's start this part (& the next) with the (iii) property of martingale:

(iii) $E(S_n^2 \mid \mathcal{F}_{n-1})$

$$= E\left((S_{n-1} + \xi_n)^2 \mid \mathcal{F}_{n-1}\right)$$

$$= E\left(S_{n-1}^2 + 2S_{n-1}\xi_n + \xi_n^2 \mid \mathcal{F}_{n-1}\right) \overset{\text{linearity}}{=\!=}$$

$$= E(S_{n-1}^2 \mid \mathcal{F}_{n-1}) + 2E(S_{n-1}\xi_n \mid \mathcal{F}_{n-1}) + E(\xi_n^2 \mid \mathcal{F}_{n-1})$$

$$= S_{n-1}^2 + 2\mu S_{n-1} + (\sigma^2 + \mu^2) \neq S_{n-1}^2 \quad ?$$

hence S_n^2 is not a martingale. $\blacksquare$

Similarly, for the final part,

(d)

(iii) $E(S_n^2 - \sigma^2 n \mid \mathcal{F}_{n-1})$

$$= \underbrace{E(S_n^2 \mid \mathcal{F}_{n-1})}_{\text{by part (c)}} - \underbrace{E(\sigma^2 n \mid \mathcal{F}_{n-1})}$$

$$= S_{n-1}^2 + 2\mu S_{n-1} + \sigma^2 + \mu^2 - \sigma^2 \cdot \underbrace{E(n)}$$

$$= S_{n-1}^2 + 2\mu S_{n-1} + \sigma^2 + \mu^2 - \sigma^2 \cdot n$$

$$= S_{n-1}^2 - \sigma^2 (n-1) + 2\mu S_{n-1} + \mu^2 \quad ?$$

$$\neq S_{n-1}^2 - \sigma^2 (n-1) . \quad \Rightarrow$$

Thus $S_n^2 - \sigma^2 n$ is not a martingale.

$\blacksquare$