Q5.

Use counter examples to prove

(a) $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{m.s.} X$

(b) $X_n \xrightarrow{d} X \not\Rightarrow X_n \xrightarrow{P} X$

(c) $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s.} X$

Sol-n:

Assume

(1) $$\mathbb{P}(Y_n) = \begin{cases} \frac{1}{n} & , Y_n = n \\ 1-\frac{1}{n} & , Y_n = 0 \end{cases}$$ as

$n \to \infty$ $\mathbb{P}(Y_n) = 0$ is what it seems like.

Given $\varepsilon > 0$,

(2) $\lim_{n\to\infty} \mathbb{P}(|Y_n - 0| > \varepsilon) = 0 \Rightarrow$

$|Y_n - 0| > \varepsilon \Rightarrow$

$Y_n > \varepsilon$, but

$Y_n = n$ because for $\varepsilon > 0$, the only option Y_n can get is $Y_n = n$. Hence, given $\varepsilon > 0$,

(3) $\mathbb{P}(|Y_n - 0| > \varepsilon) = \mathbb{P}(Y_n = n) = \frac{1}{n} \to 0$ as $n \to \infty$,

thus $Y_n \xrightarrow{P} 0$ by definition. ①.

However, is it true

(4) $$Y_n \xrightarrow{m.s.} 0 \quad ?$$ or by def. of m.s:

(4') $$\longrightarrow \lim_{n\to\infty} E\left((Y_n - 0)^2\right) = 0 \ ?$$

By going through this (4') def-n of m.s. convergence, we have

def. of $E(x) = \sum x f(x)$

(5) $$E\left[(Y_n - 0)^2\right] = E\left[Y_n^2\right] = \sum_n Y_n^2 f(Y_n)$$

$$\overset{\text{by (1)}}{=} 0^2 \cdot \left(1 - \frac{1}{n}\right) + n^2\left(\frac{1}{n}\right)$$

$$= n. \quad \Rightarrow$$

$$\lim_{n\to\infty} E\left[(Y_n - 0)^2\right] = \lim_{n\to\infty}(n) = \infty \neq 0. \quad \Rightarrow$$

by (4') $Y_n \overset{m.s}{\not\longrightarrow} 0$ so

(6) $$Y_n \xrightarrow{P} 0 \not\Rightarrow Y_n \xrightarrow{m.s} 0,$$ establishing

the desired counterexample for

$$X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{m.s.} X$$

∎

(b) $X_n \xrightarrow{d} \;\not\Rightarrow\; X_n \xrightarrow{P} X$

Sol-n:

Let us assume

X is continuous on $[-a, a]$ and symmetric about 0, defined as

(1) $$X_n = \begin{cases} X & \text{if } n \text{ is odd} \\ -X & \text{if } n \text{ is even.} \end{cases}$$

$\forall X \in [-a, a]$ & $n = 1, 2 ..$

When n is odd, we have

(2) $$F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x) = F_X(x)$$

by def-n of CDF of X, and

similarly, when n is even

(3) $$F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}(-X \leq x) =$$
$$\left.\begin{aligned} &= \mathbb{P}(X \geq -x) \\ &= \mathbb{P}(X \leq x) \end{aligned}\right\} \text{by symmetricity of } f(x),$$
$$= F_X(x)$$

Thus

(4) $$F_{X_n}(x) = F_X(x) \quad \forall x \in (-a, a) \;\&\; n = 1, 2, \dots, \infty$$

(5) $$F_{X_n}(x) \xrightarrow{d} F_X(x).$$ Yet,

Now let

(6) $$X_n - X = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -2X & \text{if } n \text{ is even} \end{cases}$$

To show convergence in probability, for using the definition, we need

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0 \quad \text{where } \varepsilon > 0.$$

Consider n is even where $m = 1, 2, 3 \Rightarrow \boxed{n = 2m}$

(7) $|X_{2m} - X| = |-2X|$ and

let $\varepsilon = a/2$ as usual $\varepsilon \to 0$ so $\varepsilon \in [0, a]$.

(8) $$\mathbb{P}(|X_{2m} - X| > \tfrac{a}{2}) = \mathbb{P}(|-2X| > \tfrac{a}{2})$$
$$= \mathbb{P}(|X| > \tfrac{a}{4}) \quad \text{and}$$

$$\lim_{2m \to \infty} \mathbb{P}(|X| > \tfrac{a}{4}) \neq 0.$$

(9) Thus $X_n \overset{P}{\nrightarrow} X$, which establishes, by (5), the desired result

$$X_n \xrightarrow{d} X \not\Rightarrow X_n \xrightarrow{P} X.$$ ∎

(C) WMT: $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s.} X$

Sol n:

Assume

(1) $$\mathbb{P}(Y_n) = \begin{cases} \frac{1}{n}, & Y_n = n \\ 1 - \frac{1}{n}, & Y_n = 0 \end{cases}$$

So it seems like as $n \to \infty$ $\mathbb{P}(Y_n) \to 0$.

Given $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|Y_n - 0| > \varepsilon) = 0$$

$|Y_n - 0| > \varepsilon$

$Y_n > \varepsilon \Rightarrow$

(2) $Y_n = n$ because $\varepsilon > 0$, $n > 0$, the only option for Y_n is n.

So for $\varepsilon > 0$

$$\mathbb{P}(|Y_n - 0| > \varepsilon| = \mathbb{P}(Y_n = n) = \frac{1}{n} \quad \Big\{ \text{Taking } \lim_{n \to \infty}(\cdot)$$

(3) $$\lim_{n \to \infty} \mathbb{P}(|Y_n - 0| > \varepsilon) = \lim_{n \to \infty} \frac{1}{n} = 0 \Rightarrow$$

(4) $\boxed{Y_n \xrightarrow{P} 0}$ by definition.

Yet, proving almost surely convergence

we need to show

(5) $$\mathbb{P}\left(\lim_{n\to\infty}(Y_n = n)\right) = 1.$$,

however

$$\mathbb{P}\left(\lim_{n\to\infty}(Y_n = n)\right) = \mathbb{P}(\infty) \neq 1.$$ so

(6) $$\boxed{Y_n \stackrel{a.s}{\not\longrightarrow} 0}$$

together with (4),

we have $Y_n \xrightarrow{P} 0 \not\Rightarrow Y_n \xrightarrow{a.s} 0$ which

proves

$$X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s} X.$$

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