Math 588. HW#3.

Problem 1.

One-dim-l- Neumann-boundary pr-m at $x=1$.

(1) $$-\frac{d^2p}{dx^2} = f(x), \quad 0 < x < 1 \ ; \ x \in (0,1) =: \Omega$$

(2) $$\underbrace{p(0)}_{D} = \underbrace{\frac{dp}{dx}(1)}_{N} = 0 \quad (3)$$

(i) Galerkin variational for-n?

(ii) F.E.M with piecewise linear f-ns?

(iii) Linear system of equations?

Solution:

(i) Using the given PDE (ODE) in (1), & by letting $v(x) \in V$ s.t.

(4) $$L^2(\Omega) = \{ v \in \Omega : \int_\Omega v^2\,dx = \int_0^1 v^2\,dx < \infty \} \ \&$$

(5) $$V = H^1(\Omega) = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \} \ \&$$

(6) $$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \ \text{on} \ \partial\Omega \}, \ \text{so}$$

for $v \in H_0^1(\Omega)$, let's multiply BS of (1) by $v(x)$ & integrate over the domain $\Omega = (0,1)$.

(7) $$-\frac{d^2p}{dx} \cdot v(x) = f(x) \cdot v(x) \quad \Big\} \int_0^1 \cdot \ \text{of BS.}$$

(8) $$\int_0^1 -\frac{d^2p}{dx} \cdot v(x)\,dx = \int_0^1 f(x) \cdot v(x)\,dx$$

Applying IBP to the LHS of (8), we get

(9) $$\int_0^1 -p''(x)\,v(x)\,dx = -\int_0^1 v(x)\,d(p'(x))$$

$$= -v(x)\cdot p'(x)\Big|_0^1 + \int_0^1 p'(x)\,d\,v(x)$$

$$= -v(1)\cdot \underbrace{p'(1)}_{\text{by (3)} \;=0} + \underbrace{v(0)}_{=0 \text{ by (2)}}\cdot p'(0) + \int_0^1 p'(x)\cdot v'(x)\,dx.$$

$$= \underbrace{-v(1)\cdot 0 + 0\cdot p'(0)}_{0} + \int_0^1 p'(x)\cdot v'(x)\,dx$$

$$= 0 + \int_0^1 p'(x)\cdot v'(x)\,dx;$$ so

(10) $$\int_0^1 -p''(x)\,v(x)\,dx = \underbrace{\int_0^1 p'(x)\cdot v'(x)\,dx}_{\text{plugging this back to (8), we have}}$$

(11) $$\underbrace{\int_0^1 p'(x)\cdot v'(x)\,dx}_{:=} = \underbrace{\int_0^1 f(x)\cdot v(x)\,dx}_{:=} \Rightarrow$$

(12) $$\langle p'(x), v'(x)\rangle = \langle f(x), v(x)\rangle$$ by our class notation, where

$$\langle f, g\rangle := \int_0^1 f(x)\cdot g(x)\,dx.$$

This (12) is called <u>Galerkin (weak) formulation of (1)-(3)</u>

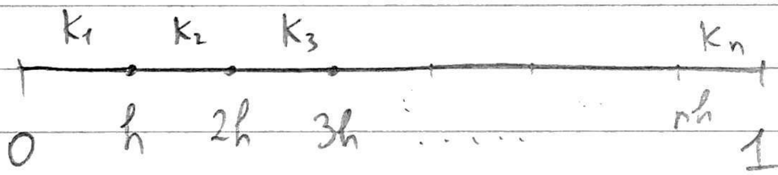
(12) $$\boxed{\langle p'(x), v'(x)\rangle = \langle f(x), v(x)\rangle}$$

So we have (by class note) our Galerkin formulation in (12)

(12) $$\langle p'(x), v'(x) \rangle = \langle f(x), v(x) \rangle$$

Let K_h be the uniform partition of $\Omega = (0,1)$ into triangles. But in our case $\Omega = (0,1)$, a one-dim-l space, the triangles are all collapsed to small subintervals, mesh.

(13) Ω:



and of course by our uniform triangulation,

(13) $$h = \text{diam}(K_i) \qquad \forall\, i = \overline{1,n}.$$

Now we introduce the finite element space, V_h, a discrete analog space of continuos space V.

(14) $V_h = \{ v :$ (i) v is continuous f-n on $\Omega = (0,1)$
(ii) v is linear on each K_i,
(iii) $v = 0$ on $\partial\Omega$, i.e. $v(0) = 0$ & $v(1) = 0. \}$

No need indeed, but doesn't hurt to have

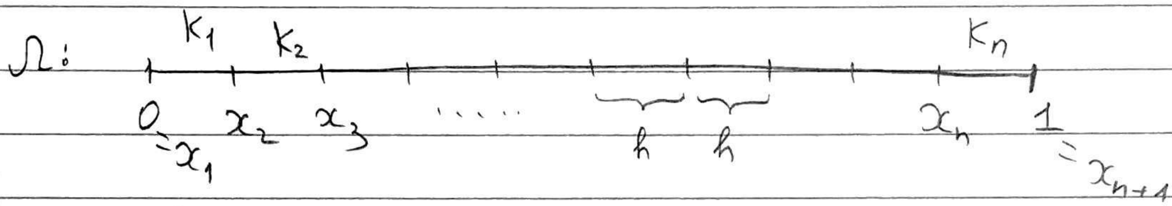
$V_h \subset V$.

The finite element method now can be formulated like in Dirichlet problem in HW#1.

Find $p_h \in V_h$ s.t., using (12)

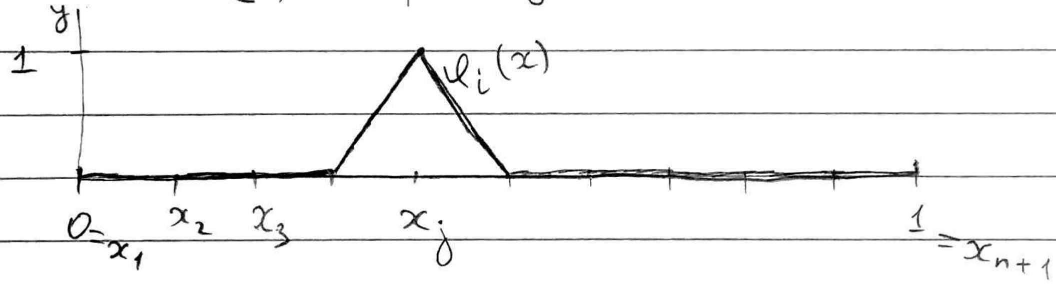
(15) $$< p_h'(x), v'(x) > = < f(x), v(x) > , \quad \forall v \in V_h$$

Denote the vertices (nodes) of triangles (subintervals) in K by $x_1, x_2 \dots x_n$, so (13) is as ↓:

(16) 

The basis functions $\varphi_i \in V_h$, $i = 1, 2, \dots n+1$

(17) $$\varphi_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

(18) 

Let $x_2, x_3 \dots x_n$ be the interior points (nodes) in each K.

Since $p_h(x)$ satisfies (15) $\forall v \in V_h$, then

for choices of v, we can choose easy choice which is basis f-ns φ_i's i.e.

(19) $v = \varphi_1 \in V_h$, $v = \varphi_2 \in V_h$, ... $v = \varphi_{n+1}(x) \in V_h$

(19) satisfies (15), then we can write the ↓:

$< p_h'(x), \varphi_1'(x) > = < f(x), \varphi_1(x) >$ is valid by (15)

$< p_h'(x), \varphi_2'(x) > = < f(x), \varphi_2(x) >$ is valid by (15)

. . . .

$< p_h'(x), \varphi_{n+1}'(x) > = < f(x), \varphi_{n+1}(x) >$ is valid by (15), i.e.

(20) $$< p_h'(x), \varphi_j'(x) > = < f(x), \varphi_j(x) > \qquad j = 1, 2, .. (n+1).$$

Also $p_h \in V_h = \text{span}\{\varphi_1, \varphi_2, .. \varphi_{n+1}\}$, then $p_h(x)$ can be written by L.C. of basis f-ns φ_i's:

(21) $$p_h(x) = \sum_{i=1}^{n+1} z_i \cdot \varphi_i(x) \qquad \text{where}$$

(22) $$\boxed{z_i = p_h(x_i)}$$ ← $p_h(x)$ is discrete, hence captures $p(x)$ at nodle points.

Then, using (21) the expression of $p_h(x)$, if we consider the inner product of $p_h'(x)$ with all basis f-ns φ_j''s, we come to linear sytem of equations for unknowns $z_1, z_2 .. z_{n+1}$:

(23) $$< p_h'(x), \varphi_1'(x) > \overset{(21)}{=} < \left[\sum_{i=1}^{n+1} z_i \cdot \varphi_i(x)\right]', \varphi_1'(x) >$$

$$= < \sum_{i=1}^{n+1} z_i \varphi_i'(x), \varphi_1'(x) >$$

$$= z_1 < \varphi_1'(x), \varphi_1'(x) > + z_2 < \varphi_2', \varphi_1' > + .. + z_{n+1} < \varphi_{n+1}', \varphi_1' >$$

Similarly

(24) $$< p_h'(x), \varphi_2'(x) > = z_1 < \varphi_1'(x), \varphi_2'(x) > + z_2 < \varphi_2'(x), \varphi_2'(x) > + \ldots + z_{n+1} < \varphi_{n+1}'(x), \varphi_2'(x) >$$

And continuing this process gives

(25) $\langle p_h'(x), \varphi_{n+1}'(x)\rangle =$

$= z_1 \langle \varphi_1'(x), \varphi_{n+1}'(x)\rangle + z_2 \langle \varphi_2'(x), \varphi_{n+1}'(x)\rangle + \dots + z_{n+1} \langle \varphi_{n+1}'(x), \varphi_{n+1}'(x)\rangle$

Yet on the other hand by (20), we have these left hand sides of (23), (24), (25):

(26) $$\langle p_h'(x), \varphi_j'(x)\rangle = \langle f(x), \varphi_j(x)\rangle,$$ hence

we have the ↓ system: $j = 1, 2, \dots, (n+1)$.

(27) $$\begin{cases} z_1 \langle \varphi_1', \varphi_1'\rangle + z_2 \langle \varphi_2', \varphi_1'\rangle + \dots + z_{n+1} \langle \varphi_{n+1}', \varphi_1'\rangle = \langle f, \varphi_1\rangle \\ z_1 \langle \varphi_1', \varphi_2'\rangle + z_2 \langle \varphi_2', \varphi_2'\rangle + \dots + z_{n+1} \langle \varphi_{n+1}', \varphi_2'\rangle = \langle f, \varphi_2\rangle \\ \dots \\ z_1 \langle \varphi_1', \varphi_{n+1}'\rangle + z_2 \langle \varphi_2', \varphi_{n+1}'\rangle + \dots + z_{n+1} \langle \varphi_{n+1}', \varphi_{n+1}'\rangle = \langle f, \varphi_{n+1}\rangle \end{cases} \Rightarrow$$

(28) $$\underbrace{\begin{bmatrix} \langle \varphi_1', \varphi_1'\rangle & \langle \varphi_2', \varphi_1'\rangle & \dots & \langle \varphi_{n+1}', \varphi_1'\rangle \\ \langle \varphi_1', \varphi_2'\rangle & \langle \varphi_2', \varphi_2'\rangle & \dots & \langle \varphi_{n+1}', \varphi_2'\rangle \\ \dots & \dots & \dots & \dots \\ \langle \varphi_1', \varphi_{n+1}'\rangle & \langle \varphi_2', \varphi_{n+1}'\rangle & \dots & \langle \varphi_{n+1}', \varphi_{n+1}'\rangle \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{bmatrix}}_{\bar{z}} = \underbrace{\begin{bmatrix} \langle f, \varphi_1\rangle \\ \langle f, \varphi_2\rangle \\ \dots \\ \langle f, \varphi_{n+1}\rangle \end{bmatrix}}_{\bar{f}}$$

(29) $$A \cdot \bar{z} = \bar{f}$$ ← is the required matrix eq-n for unknown $\bar{z}$.

where $A = [a_{ij}]_{i,j=1}^{n+1} = \langle \varphi_j', \varphi_i'\rangle$;

$\bar{z} = [z_1, z_2 \dots z_{n+1}]^T$ & $z_i = p_h(x_i) = p(x_i)$; $\boxed{}$

$\bar{f} = [\langle f, \varphi_1\rangle, \langle f, \varphi_2\rangle \dots \langle f, \varphi_{n+1}\rangle]^T$.

To determine the linear functions using the property in (17)

(30) $$\varphi_j(x) = \begin{cases} \dfrac{x - x_j}{h}, & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases}$$

Then $a_{ij} = \langle \varphi_i', \varphi_j' \rangle = 0$ if $|i-j| > 0$.

Then the stiffness matrix A in (29) is tridiagonal, symmetric & positive-definite as follows:

(31) $$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

completing the solution. ∎