

Question 5.

Consider the process

(1) $F_t = f(t) \cdot g(w_t)$. Show that

Sol'n:

(a)

$$dF_t = f'(t) g(w_t) dt + \frac{1}{2} f''(t) g''(w_t) dt \\ + f(t) g'(w_t) dw_t$$

b. $F_t = f(t) \cdot g(w_t) \Rightarrow$ let

(2) $F_t = \underbrace{\varphi(w_t, t)}_{\text{case II}} = f(t) \cdot g(w_t) \quad ?$

(3) $\varphi(x, t) = f(t) \cdot g(x)$, then

we have for case II of Itô's Formula

(4) $dF_t = \left[\frac{\partial \varphi}{\partial t}(w_t, t) + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2}(w_t, t) \right] dt + \frac{\partial \varphi}{\partial x}(w_t, t) dw_t$

We need the \dot{t} :

(5) $\frac{\partial \varphi}{\partial t}(x, t) = \frac{\partial}{\partial t} [f(t) \cdot g(x)] = f'(t) \cdot g(x)$

(6) $\frac{\partial \varphi}{\partial x}(x, t) = \frac{\partial}{\partial x} [f(t) \cdot g(x)] = f(t) \cdot g'(x) \quad ?$

① (7) $\frac{\partial^2 \varphi}{\partial x^2}(x, t) = f(t) \cdot g''(x)$, then putting (6)-(7) to (4)

$$(8) \quad dF_t = \left[f'(t) \cdot g(w_t) + \frac{1}{2} f(t) g'(w_t) \right] dt + \\ + f(t) \cdot g'(w_t) dw_t \quad \text{as desired}$$



(b) Using case ii integration form

of If's formula (or integrating \int_a^b (8) :)

$$(9) \quad \int_a^b dF_t = \int_a^b \left[f'(t) \cdot g(w_t) + \frac{1}{2} f(t) g'(w_t) \right] dt + \\ + \int_a^b f(t) \cdot g'(w_t) dw_t$$

Let's solve (9) for $\int_a^b f(t) \cdot g'(w_t) dw_t$ as

this is what we need to obtain :

$$(10) \quad \int_a^b f(t) g'(w_t) dw_t = F_t \Big|_a^b - \int_a^b f'(t) \cdot g(w_t) dt \\ - \frac{1}{2} \int_a^b f(t) \cdot g'(w_t) dt$$

(2) but we know $F_t = f(t) \cdot g(w_t)$ so subbing it into (10) :

yields

(1)

$$\int_a^b f(t) \cdot g'(w_t) dw_t = \underbrace{f(t) \cdot g(w_t)}_{F_t} \Big|_a^b$$

$$- \int_a^b f'(t) \cdot g(w_t) dt$$

$$- \frac{1}{2} \int_a^b f(t) \cdot g'(w_t) dt$$

as desired.



(c)

Find

(2)

$$\int_0^T e^{\frac{t}{2}} \cos w_t dw_t = ?$$

$f(t) \quad g'(w_t)$

, in (2), let

(3)

$$f(t) = e^{\frac{t}{2}}$$

(4)

$$g'(w_t) = \cos(w_t)$$

then applying formula (1) in part (b)

gives the desired result:

③

$$f'(t) = \frac{1}{2} e^{\frac{t}{2}} \quad \text{and} \quad g(w_t) = \sin(w_t) \quad \text{and}$$

$$g''(w_t) = -\sin(w_t)$$

f_g (11)

$$\int_0^T e^{\frac{t}{2}} \underbrace{\cos(w_t)}_{f(t)} dw_t = e^{\frac{t}{2}} \cdot \sin(w_t) \Big|_0^T$$

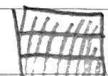
$$-\int_0^T \frac{1}{2} e^{\frac{t}{2}} \cdot \sin(w_t) dt = \frac{1}{2} \int_0^T e^{\frac{t}{2}} (-\sin(w_t)) dt$$

$$= e^{\frac{t}{2}} \cdot \sin(w_t) \Big|_{t=0}^{t=T} + 0$$

$$= e^{\frac{T}{2}} \sin(w_T) - e^0 \cdot \sin(w_0) = 0$$

$$= e^{\frac{T}{2}} \sin(w_T) - \sin(0)$$

$$= e^{\frac{T}{2}} \sin(w_T),$$



(4)