Stat 544 Final Project: Applied Stochastic Processes

Lazizbek Sadullaev

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Question 2: Geometric Brownian Motion

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i) SDE with Initial Condition

We begin with the general linear SDE of the form:

$$dX_t = [a_1(t)X_t + a_2(t)]\,dt + [b_1(t)X_t + b_2(t)]\,dW_t. \tag{1}$$

In this problem, we are interested in the special case where:

$$a_2(t) = 0, \quad b_2(t) = 0,$$

so that (1) reduces to:

$$dX_t = a_1(t)X_t\,dt + b_1(t)X_t\,dW_t, \tag{2}$$

with initial condition:

$$X_0 = x_0. \tag{3}$$

This form defines a **Geometric Brownian Motion (GBM)** — a process where the rate of change is proportional to the current state.

(ii) Solving the SDE via Change of Variable

To solve (2), we apply the logarithmic transformation:

$$Y_t = \ln X_t.$$

Using Itô's formula for a function $\varphi(X_t) = \ln X_t$, we compute:

$$\varphi'(X_t) = \frac{1}{X_t}, \quad \varphi''(X_t) = -\frac{1}{X_t^2}.$$

Apply Itô's Lemma:

$$dY_t = \left[\frac{1}{X_t} \cdot a_1(t)X_t + \frac{1}{2} \cdot \left(-\frac{1}{X_t^2}\right) \cdot b_1^2(t)X_t^2\right] dt + \frac{1}{X_t} \cdot b_1(t)X_t\, dW_t \tag{4}$$

$$= \left[a_1(t) - \frac{1}{2}b_1^2(t)\right] dt + b_1(t)\, dW_t. \tag{5}$$

This is now a drift-diffusion SDE for $Y_t = \ln X_t$, which we can solve by direct integration:

$$Y_t = Y_0 + \int_0^t \left[a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right] d\tau + \int_0^t b_1(\tau)\, dW_\tau \tag{6}$$

$$= \ln X_0 + \int_0^t \left[a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right] d\tau + \int_0^t b_1(\tau)\, dW_\tau. \tag{7}$$

Exponentiating both sides, we get the solution to the GBM:

$$X_t = X_0 \exp\left(\int_0^t \left[a_1(\tau) - \frac{1}{2}b_1^2(\tau)\right] d\tau + \int_0^t b_1(\tau)\, dW_\tau\right). \tag{8}$$

(iii) Constant Coefficient Case: Standard GBM

If $a_1(t) = a$ and $b_1(t) = b$ are constants, then (2) becomes:

$$dX_t = aX_t\, dt + bX_t\, dW_t, \tag{9}$$

with solution:

$$X_t = X_0 \exp\left(\left(a - \frac{1}{2}b^2\right) t + bW_t\right). \tag{10}$$

(iv) Graphing the Solution

To visualize the stochastic dynamics of the GBM:

- We choose constants a, b, and X_0 for drift, diffusion, and initial value respectively.
- We simulate $M = 100$ independent sample paths using either Euler–Maruyama or the exact solution.

- We also compute the analytical mean $\mathbb{E}[X_t] = X_0 e^{at}$.

Figure 1 below shows simulated GBM paths. The black curve shows the theoretical expected value.

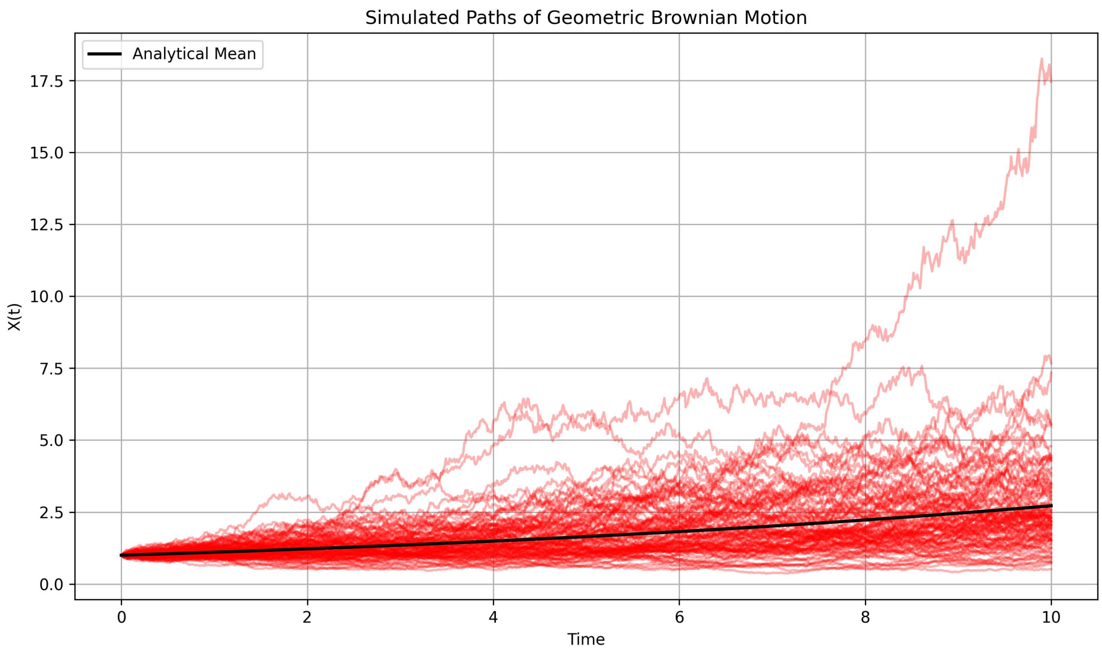

Figure 1: Simulated paths of Geometric Brownian Motion with analytical mean

(v) Statistics of Geometric Brownian Motion

From (9), we know the solution is:

$$X_t = X_0 \exp\left(\left(a - \frac{1}{2}b^2\right)t + bW_t\right).$$

Let's denote $Y_t = \ln X_t$, so that:

$$Y_t = \ln X_0 + \left(a - \frac{1}{2}b^2\right)t + bW_t.$$

Then Y_t is normally distributed:

$$Y_t \sim \mathcal{N}\left(\ln X_0 + \left(a - \frac{1}{2}b^2\right)t,\, b^2 t\right),$$

and hence X_t is lognormally distributed:

$$X_t \sim \text{LogNormal}(\mu = \ln X_0 + (a - \frac{1}{2}b^2)t,\, \sigma^2 = b^2 t).$$

Now, using properties of the lognormal distribution:

- **Expectation:**
$$\mathbb{E}[X_t] = e^{\mu+\frac{1}{2}\sigma^2} = e^{\ln X_0 + (a-\frac{1}{2}b^2)t+\frac{1}{2}b^2 t} = X_0 e^{at}.$$

- **Variance:**
$$\begin{aligned}\mathrm{Var}(X_t) &= \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2} &(11)\\ &= \left(e^{b^2 t} - 1\right) \cdot X_0^2 e^{2at}. &(12)\end{aligned}$$

- **Autocovariance:** Let $s < t$. Then:
$$\mathrm{Cov}(X_s, X_t) = \mathbb{E}[X_s X_t] - \mathbb{E}[X_s]\mathbb{E}[X_t],$$
  and computing:
$$\mathbb{E}[X_s X_t] = X_0^2 \cdot \exp\left(2as + a(t-s) + b^2 s\right) = X_0^2 e^{a(s+t)} e^{b^2 s},$$
  and using:
$$\mathbb{E}[X_s]\mathbb{E}[X_t] = X_0^2 e^{a(s+t)},$$
  so the autocovariance is:
$$\mathrm{Cov}(X_s, X_t) = X_0^2 e^{a(s+t)} \left(e^{b^2 s} - 1\right).$$

(vi) Applications of Geometric Brownian Motion

- **Finance:** GBM is used to model the price of financial assets in the Black–Scholes model. It assumes continuously compounded returns are normally distributed and enables closed-form pricing of European options.

- **Biology and Physics:** GBM describes population dynamics under stochastic growth or systems with multiplicative noise, such as molecular diffusion with growth.

- **Economics:** GBM can model variables such as GDP or inflation, where proportional noise is more realistic than additive noise.

References

References

[1] Black, F., & Scholes, M. (1973). *The Pricing of Options and Corporate Liabilities*. Journal of Political Economy, 81(3), 637–654. `https://doi.org/10.1086/260062`

[2] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd Edition.

Appendix: Python Code for Question 2 (Geometric Brownian Motion)

The following Python script was used to generate the simulation plot in Part (iv):

Listing 1: GBM Simulation (Q2)

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
x0 = 1.0          # Initial value
a = 0.1           # Drift
b = 0.2           # Volatility
T = 10            # Time horizon
N = 1000          # Number of time steps
dt = T / N        # Time step size
t = np.linspace(0, T, N+1)
M = 100           # Number of paths

# Simulate GBM paths
X = np.zeros((M, N+1))
X[:, 0] = x0
for i in range(M):
    W = np.random.randn(N) * np.sqrt(dt)
    W = np.insert(np.cumsum(W), 0, 0)
    X[i, :] = x0 * np.exp((a - 0.5 * b**2) * t + b * W)

# Theoretical expectation
X_mean = x0 * np.exp(a * t)

# Plotting
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i, :], color='red', alpha=0.3)
plt.plot(t, X_mean, color='black', linewidth=2, label='Analytical Mean')
plt.title('Simulated Paths of Geometric Brownian Motion')
plt.xlabel('Time')
plt.ylabel('X(t)')
plt.grid(True)
plt.legend()
plt.tight_layout()
plt.savefig("images/q2_gbm_graph.jpg", dpi=300)
plt.show()
```