

Stat 544.

HW#6.

Study the following SDE's. For each one:

1. Write down the SDE with initial condition.
Describe all terms, variables, & parameters.
2. Use any method to solve it.
3. Choose a set of params & plot several stochastic paths on the same graph.
4. Study the expectation, variance, auto correlation & anything else you find interesting.
5. Do some literature search to find one or two application of the SDE. Choose your favourite application and write one or two paragraphs about it.

Questions. Drift-diffusion

(i) The general SDE is the form of

$$(1) \quad dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t \quad (1)$$

if $\int_0^t \mu(X_\tau, \tau) d\tau$ & $\int_0^t \sigma(X_\tau, \tau) dW_\tau$ exists

$$(2) \quad X_t = X_0 + \int_0^t \mu(X_\tau, \tau) d\tau + \int_0^t \sigma(X_\tau, \tau) dW_\tau$$

is called strong soln of (1).

For special cases of $\mu(X_t, t)$ & $\sigma(X_t, t)$, we study some types of SDE's, specifically y.

Let

$$(3) \quad \mu(X_t, t) = a_1(t)X_t + a_2(t)$$

$$(4) \quad \sigma(X_t, t) = b_1(t)X_t + b_2(t)$$

Then we have a Linear SDE:

$$(5) \quad dX_t = [a_1(t)X_t + a_2(t)] dt + [b_1(t)X_t + b_2(t)] dW_t$$

Case 1: $a_1(t) = b_1(t) = 0$.

Then (5) becomes

①

(6) $dX_t = a_2(t) dt + b_2(t) dW_t$ & with the IC

(6') $X_t(0) = X_0$

which is drift-diffusion SDE.

(ii) Integrating (6) $\int_0^t (\cdot) dt$ gives the solution:

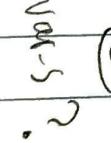

Now
should
we
some
up
etc.
change
of

$$\int_0^t dX_t = \int_0^t a_2(\tau) d\tau + \int_0^t b_2(\tau) dW_\tau \Rightarrow$$

(7) $X_t = X_0 + \int_0^t a_2(\tau) d\tau + \int_0^t b_2(\tau) dW_\tau$.

(iii) Plotting several stochastic paths:

(iv) Statistics: Using (7):


var-s (8) $E[X_t] = E\left[X_0 + \int_0^t a_2(\tau) d\tau + \int_0^t b_2(\tau) dW_\tau\right]$

$$E(X_t) = E(X_0) + E\left[\int_0^t a_2(\tau) d\tau\right] + E\left[\int_0^t b_2(\tau) dW_\tau\right]$$

$$E(\#) = \# \text{ w/ } \# \quad = X_0 + \int_0^t a_2(\tau) d\tau$$

(9) $E[X_t] = X_0 + \int_0^t a_2(\tau) d\tau$

(2)

$$\text{Var}(X_t) \stackrel{\text{by (7)}}{=} \underline{\underline{\text{Var}}}$$

$$= \text{Var} \left[X_0 + \underbrace{\int_0^t a_2(\tau) d\tau}_{0} + \underbrace{\int_0^t b_2(\tau) dW_\tau}_{0} \right]$$

$$= \text{Var} \left[\int_0^t b_2(\tau) dW_\tau \right] \stackrel{=0}{=} \text{by}$$

$$= E \left[\left[\int_0^t b_2(\tau) dW_\tau \right]^2 \right] - \left\{ E \left[\int_0^t b_2(\tau) dW_\tau \right] \right\}^2$$

Ito Isometry

④ where $= E \left[\int_0^t b_2^2(\tau) d\tau \right] - \underbrace{\{0\}^2}_{=0} = E(\#) = \#$

$$= \int_0^t b_2^2(\tau) d\tau \Rightarrow$$

(10)

$$\boxed{\text{Var}(X_t) = \int_0^t b_2^2(\tau) d\tau}$$

⑤

Auto Correlation $(X_t) =$

How to
do it?

③