Question 3: Ornstein–Uhlenbeck (OU) Process

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

We begin with the general linear SDE:

$$dX_t = a_1(t)X_t\,dt + a_2(t)\,dt + b_1(t)X_t\,dW_t + b_2(t)\,dW_t. \tag{1}$$

We now consider the case:

$$a_1(t) = -a, \quad a_2(t) = 0, \quad b_1(t) = 0, \quad b_2(t) = b,$$

with constants $a > 0$ and $b > 0$. Then (1) simplifies to the Ornstein–Uhlenbeck SDE:

$$dX_t = -aX_t\,dt + b\,dW_t, \tag{2}$$

with initial condition:

$$X_0 = x_0.$$

(iii) Solving the OU SDE via Integrating Factor

To solve (2), we multiply both sides by an integrating factor:

$$M_t = e^{at}, \quad \text{and let} \tag{3}$$
$$Y_t = X_t e^{at} \tag{4}$$

Using Itô's formula for $\varphi(X_t, t) = X_t e^{at}$:

$$\frac{\partial \varphi}{\partial t} = aX_t e^{at}, \tag{5}$$
$$\frac{\partial \varphi}{\partial x} = e^{at}, \tag{6}$$
$$\frac{\partial^2 \varphi}{\partial x^2} = 0. \tag{7}$$

Now apply Itô's lemma:

$$dY_t = \left(\frac{\partial \varphi}{\partial t} + \mu_t \frac{\partial \varphi}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 \varphi}{\partial x^2}\right) dt + \sigma_t \frac{\partial \varphi}{\partial x} dW_t \tag{8}$$

$$= \left(aX_t e^{at} + (-aX_t)e^{at} + 0\right) dt + be^{at} dW_t \tag{9}$$

$$= be^{at} dW_t. \tag{10}$$

$$dY_t = be^{at} dW_t. \tag{11}$$

which we can integrate directly since the $\sigma(X_t, t) = be^{at} = \sigma(t)$, σ only depends on time t. Thus integrating both sides of (11) from 0 to t gives:

$$Y_t = Y_0 + \int_0^t be^{a\tau}\, dW_\tau, \tag{12}$$

$$\tag{13}$$

by (4), and solving for X_t give the final solution.

$$X_t = e^{-at} Y_t = x_0 e^{-at} + \int_0^t be^{-a(t-\tau)}\, dW_\tau. \tag{14}$$

(iv) Graphing the Solution

To visualize the OU process:

- Choose specific values of a, b, and x_0.
- Simulate X_t using either the exact solution (14) or Euler–Maruyama approximation.
- Plot multiple sample paths on the same graph.
- Compute and overlay the analytical mean path $\mathbb{E}[X_t] = x_0 e^{-at}$.

Figure 1 shows 100 simulated paths and the mean trajectory. The black curve is the expected value $\mathbb{E}[X_t]$, while the red paths show different realizations. The code is provided in the Appendix.

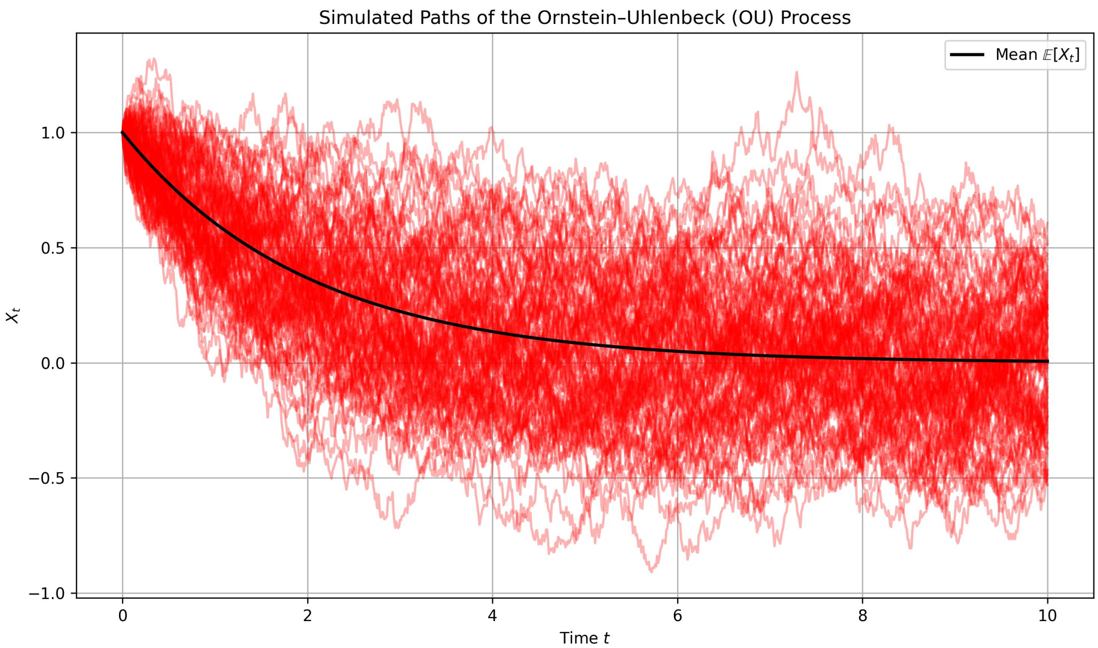

Figure 1: Simulated sample paths of the OU process with mean overlay

(v) Statistics of the OU Process

Expectation

Taking expectation in (14):

$$\mathbb{E}[X_t] = \mathbb{E}\left[x_0 e^{-at} + \int_0^t b e^{-a(t-\tau)} dW_\tau\right] \tag{15}$$

$$= \mathbb{E}\left[x_0 e^{-at}\right] + \mathbb{E}\left[\int_0^t b e^{-a(t-\tau)} dW_\tau\right] \tag{16}$$

$$= x_0 e^{-at} + 0 \tag{17}$$

$$= x_0 e^{-at}, \quad \text{(since Itô integrals have zero mean).} \tag{18}$$

$$\mathrm{Var}(X_t) = \mathbb{E}\left[\left(\int_0^t b_2(\tau)\, dW_\tau\right)^2\right] - \left[\mathbb{E}\left(\int_0^t b_2(\tau)\, dW_\tau\right)\right]^2 = \tag{19}$$

$$= \mathbb{E}\left[\int_0^t (b_2(\tau))^2 \, d\tau\right] - \left[\mathbb{E}\left(Ito\ Integral\right)\right]^2 = \tag{20}$$

$$= \int_0^t (b_2(\tau))^2 \, d\tau - [0]^2 = \tag{21}$$

$$= \int_0^t b_2^2(\tau)\, d\tau. \tag{22}$$

Variance

Since only the stochastic integral contributes to variance:

$$\mathrm{Var}(X_t) = \mathbb{E}\left[\left(\int_0^t be^{-a(t-\tau)}\,dW_\tau\right)^2\right] - \left[\mathbb{E}\left(\int_0^t be^{-a(t-\tau)}\,dW_\tau\right)\right]^2 = \tag{23}$$

$$= \mathbb{E}\left[\int_0^t \left(be^{-a(t-\tau)}\right)^2\,d\tau\right] - \left[\mathbb{E}\left(Ito\ Integral\right)\right]^2 = \tag{24}$$

$$= \mathbb{E}\left[\int_0^t \left(b^2e^{-2a(t-\tau)}\right)\,d\tau\right] - [0]^2 = \tag{25}$$

$$= b^2\int_0^t e^{-2a(t-\tau)}d\tau - 0 \tag{26}$$

$$= \frac{b^2}{2a}\left(1 - e^{-2at}\right). \tag{27}$$

As $t \to \infty$:

$$\mathbb{E}[X_t] \to 0, \quad \mathrm{Var}(X_t) \to \frac{b^2}{2a}.$$

Covariance

Let $s < t$. From the solution:

$$X_t = x_0e^{-at} + be^{-at}\int_0^t e^{a\tau}dW_\tau =: g(t) + be^{-at}Y_t.$$

Then:

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}\left(be^{-as}Y_s, be^{-at}Y_t\right) \tag{28}$$

$$= b^2e^{-a(s+t)}\mathrm{Cov}(Y_s, Y_t). \tag{29}$$

Now compute:

$$Y_t = \int_0^t e^{a\tau}dW_\tau, \quad Y_s = \int_0^s e^{a\tau}dW_\tau.$$

Using Itô isometry:

$$\mathrm{Cov}(Y_s, Y_t) = \mathbb{E}\left[\int_0^s e^{a\tau}dW_\tau \cdot \int_0^t e^{a\lambda}dW_\lambda\right] \tag{30}$$

$$= \int_0^{\min(s,t)} e^{2a\tau}d\tau \tag{31}$$

$$= \frac{1}{2a}\left(e^{2a\min(s,t)} - 1\right). \tag{32}$$

Hence:

$$\mathrm{Cov}(X_s, X_t) = \frac{b^2}{2a}e^{-a(s+t)}\left(e^{2a\min(s,t)} - 1\right). \tag{33}$$

—

This completes the analysis of the Ornstein–Uhlenbeck process. It is a Gaussian, mean-reverting stochastic process with closed-form expressions for the mean, variance, and covariance. It reaches a stationary distribution with variance $\frac{b^2}{2a}$ as $t \to \infty$.

(vi) Literature search and discuss one or two applications of the Statistics of the OU Process SDE

The Ornstein–Uhlenbeck process has many real-world applications:

- **Physics (Langevin dynamics):** Originally developed by Ornstein and Uhlenbeck, the OU process models the velocity of a particle undergoing Brownian motion with friction. The deterministic term represents damping, while the noise models thermal fluctuations.

- **Finance (Interest rate modeling):** The OU process is used in mean-reverting models like the Vasicek model to describe interest rate dynamics. It reflects the tendency of rates to revert to a long-term equilibrium level while fluctuating stochastically.

- **Neuroscience and biology:** OU processes are used to model noisy neural activity and gene regulation processes where variables return to a stable baseline over time.

References

References

[1] Ornstein, L. S., & Uhlenbeck, G. E. (1930). *On the Theory of the Brownian Motion*. Physical Review, 36(5), 823.

[2] Vasicek, O. (1977). *An Equilibrium Characterization of the Term Structure*. Journal of Financial Economics, 5(2), 177–188.

[3] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd Edition.

Appendix: Python Code for Question 3 (OU Process)

The following Python script was used to simulate the paths in Part (iv):

Listing 1: OU Process Simulation (Q3)

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
x0 = 1.0          # Initial value
a = 0.5           # Mean reversion rate
b = 0.3           # Volatility
T = 10.0          # Total time
N = 1000          # Number of time steps
dt = T / N        # Time increment
t = np.linspace(0, T, N + 1)  # Time vector
M = 100           # Number of sample paths

# Preallocate simulation matrix
X = np.zeros((M, N + 1))
X[:, 0] = x0

# Simulate OU paths using exact solution
for i in range(M):
    dW = np.random.normal(0, np.sqrt(dt), size=N)
    W = np.cumsum(dW)
    integrand = np.exp(a * t[:-1]) * dW
    integral = np.cumsum(integrand)
    integral = np.insert(integral, 0, 0)  # Make shape (1001,)
    X[i, :] = x0 * np.exp(-a * t) + b * np.exp(-a * t) * integral

# Compute analytical mean
mean_xt = x0 * np.exp(-a * t)

# Plot the results
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i, :], color='red', alpha=0.3)

plt.plot(t, mean_xt, color='black', linewidth=2, label='Mean $\\mathbb{E}[X_t
plt.title('Simulated Paths of the OrnsteinUhlenbeck (OU) Process')
plt.xlabel('Time $t$')
plt.ylabel('$X_t$')
plt.grid(True)
plt.legend()
```

```
plt.tight_layout()
plt.savefig('images/q3_ou_simulation_graph.jpg', dpi=300)
plt.show()
```