Question 4: Mean Reverting (OU) Process

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

We begin with the general linear SDE:

$$dX_t = a_1(t)X_t\,dt + a_2(t)\,dt + b_1(t)X_t\,dW_t + b_2(t)\,dW_t. \tag{1}$$

We now consider the case:

$$a_1(t) = -a = const, \quad a_2(t) = \theta \cdot a, \quad b_1(t) = 0, \quad b_2(t) = b = const,$$

with constants $a > 0$ and $b > 0$. Then (1) simplifies to the Mean Reverting OU SDE:

$$dX_t = a\,(\theta - X_t)\,dt + b\,dW_t, \tag{2}$$

with initial condition:

$$X_0 = x_0.$$

(iii) Solving the Mean Reverting OU SDE via Integrating Factor

To solve (2), we multiply both sides by an integrating factor:

$$\begin{aligned} M_t &= e^{at}, \quad \text{and let} && (3)\\ Y_t &= X_t \cdot M_t = X_t e^{at} && (4) \end{aligned}$$

Using Itô's formula of case IV,

$$\begin{aligned} \varphi(X_t, t) &= X_t e^{at} \quad \text{with (2):} && (5)\\ dX_t &= \mu_t\,dt + \sigma_t\,dW_t, \quad \text{where} && (6)\\ \mu_t &= a\,(\theta - X_t) \quad \text{and} && (7)\\ \sigma_t &= b && (8) \end{aligned}$$

$$\frac{\partial \varphi}{\partial t} = aX_t e^{at}, \tag{9}$$
$$\frac{\partial \varphi}{\partial x} = e^{at}, \tag{10}$$
$$\frac{\partial^2 \varphi}{\partial x^2} = 0. \tag{11}$$

Now apply Itô's formula using (8) with evaluating expressions at $x = X_t$ gives:

$$dY_t = d\left(X_t e^{at}\right) = \tag{12}$$
$$= \left(\frac{\partial \varphi}{\partial t} + \mu_t \frac{\partial \varphi}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 \varphi}{\partial x^2}\right) dt + \sigma_t \frac{\partial \varphi}{\partial x} dW_t \tag{13}$$
$$= \left(aX_t e^{at} + a(\theta - X_t)e^{at} + \frac{1}{2} \cdot (b)^2 \cdot 0\right) dt + (b)e^{at} dW_t \tag{14}$$
$$= a\theta e^{at} dt + be^{at} dW_t. \tag{15}$$

$$dY_t = a\theta e^{at} dt + be^{at} dW_t. \tag{16}$$

which we can integrate directly since the $\mu(X_t, t) = a\theta e^{at}$, and $\sigma(X_t, t) = be^{at} = \sigma(t)$, μ σ only depend on time t. Thus integrating both sides of (16) from 0 to t gives:

$$Y_t = Y_0 + \theta\left[e^{at} - 1\right] + \int_0^t be^{a\tau}\, dW_\tau, \tag{17}$$
$$\tag{18}$$

by (4), and solving for X_t give the final solution.

$$X_t = e^{-at} Y_t \quad \text{hence,}$$
$$X_t = x_0 e^{-at} + \theta\left[1 - e^{-at}\right] + b\int_0^t e^{-a(t-\tau)}\, dW_\tau. \tag{19}$$

(iv) Graphing the Solution

To visualize the OU process:

- Use parameters: $a = 0.7$, $\theta = 2$, $b = 0.5$, $x_0 = 4$.
- Simulate 100 paths using (**??**).
- Overlay the theoretical mean: $\mathbb{E}[X_t] = \theta + (x_0 - \theta)e^{-at}$.

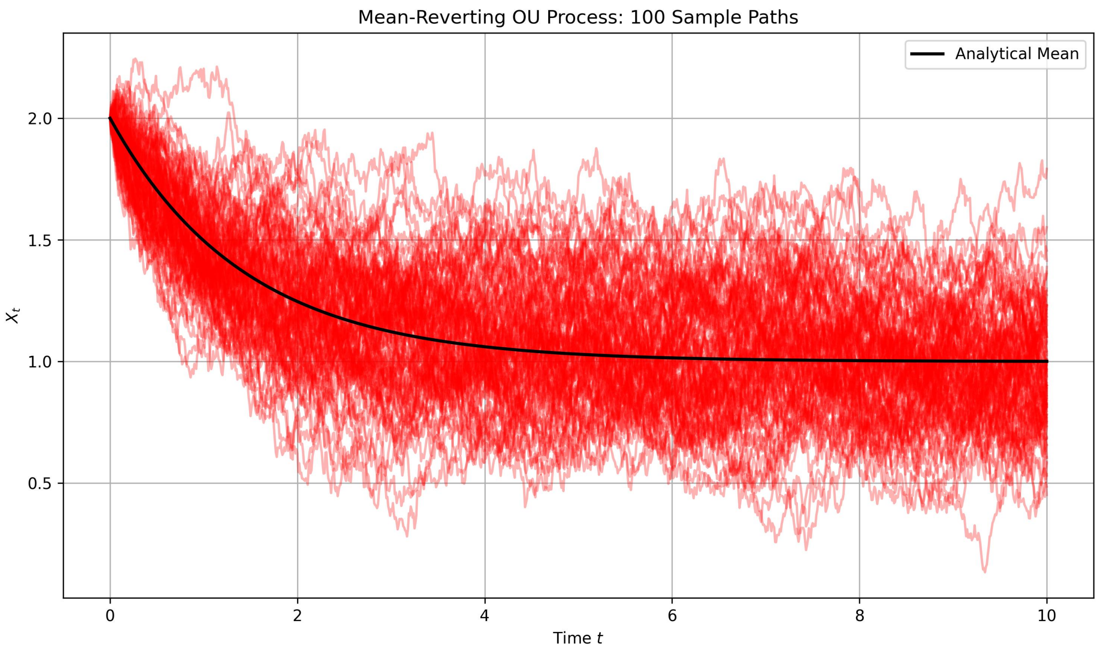

Figure 1: 100 Simulated Paths of Mean-Reverting OU Process with Nonzero Mean

(v) Statistics of the Mean-Reverting OU Process

Expectation

Taking the expectation in (19), and noting that the Itô integral has zero mean:

$$\mathbb{E}[X_t] = \mathbb{E}\left[x_0 e^{-at} + \theta(1 - e^{-at}) + b\int_0^t e^{-a(t-\tau)} dW_\tau\right] \tag{20}$$

$$= x_0 e^{-at} + \theta(1 - e^{-at}) + 0 \tag{21}$$

$$= \theta + (x_0 - \theta)e^{-at}. \tag{22}$$

As $t \to \infty$, $\mathbb{E}[X_t] \to \theta$, confirming that θ is the long-term mean.

Variance

Only the stochastic integral contributes to the variance:

$$\mathrm{Var}(X_t) = \mathbb{E}\left[\left(b\int_0^t e^{-a(t-\tau)} dW_\tau\right)^2\right] \tag{23}$$

$$= b^2 \int_0^t e^{-2a(t-\tau)} d\tau \quad \text{(by Itô isometry)} \tag{24}$$

$$= \frac{b^2}{2a}\left(1 - e^{-2at}\right). \tag{25}$$

As $t \to \infty$, $\mathrm{Var}(X_t) \to \frac{b^2}{2a}$, which is the variance of the stationary distribution.

Covariance and Autocorrelation

Let $0 \leq s \leq t$. The covariance between X_s and X_t is:

$$\mathrm{Cov}(X_s, X_t) = \frac{b^2}{2a} e^{-a(t-s)} \left(1 - e^{-2as}\right), \tag{26}$$

$$\mathrm{Corr}(X_s, X_t) = \frac{\mathrm{Cov}(X_s, X_t)}{\sqrt{\mathrm{Var}(X_s) \cdot \mathrm{Var}(X_t)}} = e^{-a(t-s)}. \tag{27}$$

Thus, the autocorrelation decays exponentially with lag $t - s$, a key feature of mean-reverting processes.

(vi) Applications of the Mean-Reverting OU Process

The mean-reverting Ornstein–Uhlenbeck process has numerous real-world applications, especially in areas where variables fluctuate around an equilibrium value:

- **Finance – Interest Rates and Volatility:** The OU process underlies the Vasicek model for interest rates and the Heston model for stochastic volatility. It captures the realistic behavior of rates or volatility reverting to a mean value rather than drifting indefinitely.

- **Physics – Langevin Dynamics:** In statistical physics, the OU process models the velocity of particles subject to random forces and friction (Brownian motion with drag), particularly in the Langevin equation.

- **Econometrics – Commodity Prices:** Energy commodities like electricity and natural gas often follow mean-reverting dynamics due to supply-demand constraints. OU models are used for pricing energy derivatives.

- **Biology – Homeostatic Control:** Biological systems, such as hormone levels or neuron firing rates, often return to a baseline after perturbation. The OU process models this homeostatic regulation effectively.

References

References

[1] Ornstein, L. S., & Uhlenbeck, G. E. (1930). *On the Theory of the Brownian Motion*. Physical Review, 36(5), 823.

[2] Vasicek, O. (1977). *An Equilibrium Characterization of the Term Structure*. Journal of Financial Economics, 5(2), 177–188.

[3] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer.

[4] Björk, T. (2009). *Arbitrage Theory in Continuous Time*. Oxford University Press.

Appendix: Python Code for Question 4 (Mean-Reverting OU)

The following Python script was used to simulate the sample paths in Part (iv):

Listing 1: Mean-Reverting OU Simulation (Q4)

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
x0 = 2.0              # Initial value
theta = 1.0           # Long-term mean
a = 0.7               # Mean reversion speed
b = 0.3               # Volatility
T = 10                # Time horizon
N = 1000              # Number of time steps
M = 100               # Number of sample paths
dt = T / N            # Time increment
t = np.linspace(0, T, N + 1)  # Time grid

# Preallocate
X = np.zeros((M, N + 1))
X[:, 0] = x0

# Simulate sample paths using the exact solution
for i in range(M):
    dW = np.sqrt(dt) * np.random.randn(N)
    W = np.concatenate(([0], np.cumsum(dW)))
    integral_term = np.array([np.sum(np.exp(a * t[:j]) * dW[:j]) for j in ran
    integral_term = np.insert(integral_term, 0, 0)  # Include initial 0
    X[i, :] = x0 * np.exp(-a * t) + theta * (1 - np.exp(-a * t)) + b * np.exp

# Compute analytical mean
mean_X = theta + (x0 - theta) * np.exp(-a * t)

# Plot
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i, :], color='red', alpha=0.3)
plt.plot(t, mean_X, color='black', linewidth=2.0, label='Analytical Mean')
plt.title('Mean-Reverting OU Process: 100 Sample Paths')
plt.xlabel('Time $t$')
plt.ylabel('$X_t$')
plt.grid(True)
plt.legend()
```

```
plt.tight_layout()
plt.savefig('images/q4_mean_reverting_ou_graph.jpg', dpi=300)
plt.show()
```