Question 7: Geometric Mean-Reverting Process

For the SDE, we are asked to:

(i). Write down the SDE with its initial condition.

(ii). Define and describe all terms, variables, and parameters.

(iii). Solve the SDE using an appropriate method.

(iv). Choose specific parameters and plot several stochastic paths on the same graph.

(v). Study the expectation, variance, autocorrelation, and any other interesting properties.

(vi). Conduct a literature search and discuss one or two applications of the SDE.

(i), (ii) SDE with Initial Condition

The process is determined by:

$$dX_t = k(\theta - \log X_t)X_t dt + \sigma X_t dW_t, \quad X_0 > 0. \tag{1}$$

This is a geometric mean-reverting process. The terms are:

- X_t: state variable (e.g., volatility, population, asset value).
- $k > 0$: rate of mean reversion.
- θ: long-term mean level for $\log X_t$.
- $\sigma > 0$: volatility coefficient.
- W_t: standard Brownian motion.

(iii) Solving the SDE via Transformation

We apply the transformation:

$$Y_t = \log X_t. \tag{2}$$

Using Itô's formula for $\varphi(X_t) = \log X_t$, we know:

$$dY_t = \left(\frac{\mu_t}{X_t} - \frac{1}{2} \cdot \frac{\sigma_t^2}{X_t^2}\right) dt + \frac{\sigma_t}{X_t} dW_t.$$

From (1), we have:

$$\mu_t = k(\theta - \log X_t)X_t, \quad \sigma_t = \sigma X_t.$$

Then:

$$\begin{aligned} dY_t &= \left(\frac{k(\theta - \log X_t)X_t}{X_t} - \frac{1}{2} \cdot \frac{\sigma^2 X_t^2}{X_t^2}\right) dt + \frac{\sigma X_t}{X_t} dW_t & (3) \\ &= \left[k(\theta - Y_t) - \frac{1}{2}\sigma^2\right] dt + \sigma dW_t. & (4) \end{aligned}$$

This is a linear Ornstein–Uhlenbeck (OU)-type SDE for Y_t.

(iv) Graphing the Solution

To simulate X_t:

- Solve the linear SDE for Y_t using the exact OU solution.
- Then exponentiate to recover $X_t = \exp(Y_t)$.
- Use appropriate parameters $k = 1.0$, $\theta = 2.0$, $\sigma = 0.3$, and $X_0 = 1.0$.
- Simulate and plot multiple paths.

Figure 1 shows 100 simulated paths of X_t using the above method.

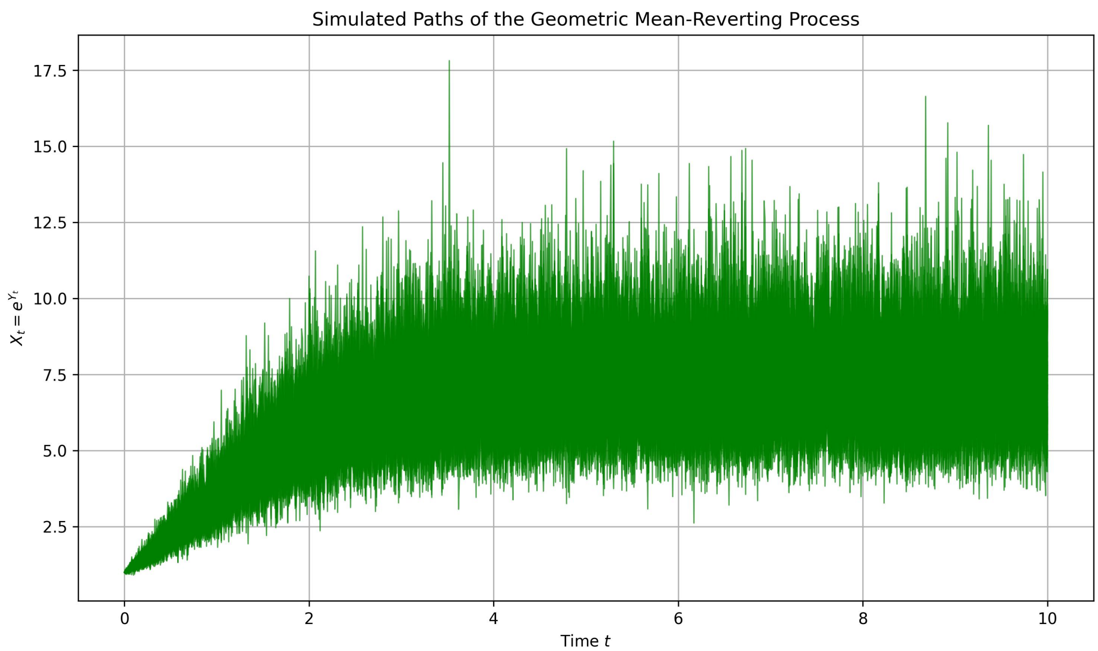

Figure 1: Simulated sample paths of the geometric mean-reverting process

(v) Statistics of the Process

Expectation and Variance of $Y_t = \log X_t$

Since Y_t follows the OU process:

$$dY_t = k(\theta - Y_t)dt - \frac{1}{2}\sigma^2 dt + \sigma dW_t, \tag{5}$$

we shift the drift:

$$\tilde{\theta} := \theta - \frac{\sigma^2}{2k},$$

so that:

$$\mathbb{E}[Y_t] = \tilde{\theta} + (Y_0 - \tilde{\theta})e^{-kt}, \tag{6}$$

$$\mathrm{Var}(Y_t) = \frac{\sigma^2}{2k}(1 - e^{-2kt}). \tag{7}$$

Statistics of $X_t = \exp(Y_t)$

Since $X_t = \exp(Y_t)$ and Y_t is Gaussian, X_t is log-normally distributed.

$$\mathbb{E}[X_t] = \exp\left(\mathbb{E}[Y_t] + \frac{1}{2}\mathrm{Var}(Y_t)\right), \tag{8}$$

$$\mathrm{Var}(X_t) = \left(e^{\mathrm{Var}(Y_t)} - 1\right) e^{2\mathbb{E}[Y_t]+\mathrm{Var}(Y_t)}. \tag{9}$$

(vi) Applications of the Geometric Mean-Reverting SDE

- **Finance – Volatility Modeling:** This SDE is used to model volatility processes that revert to a logarithmic mean. It improves over pure GBM models by preventing explosion or vanishing volatility.

- **Econometrics – Inflation/Prices:** Useful for modeling inflation or commodity prices where log-scale stabilization is observed due to central bank targeting or market equilibrium.

- **Environmental Systems:** Logarithmic quantities such as decibel noise levels or chemical concentrations often fluctuate around a mean and are modeled using such SDEs.

- **Biology – Gene Expression:** Log-levels of gene expression are known to fluctuate stochastically and show mean-reversion behavior.

References

References

[1] Karatzas, I., & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer.

[2] Allen, L. J. S. (2007). *An Introduction to Stochastic Processes with Applications to Biology*. Chapman and Hall/CRC.

[3] Gatheral, J. (2014). *The Volatility Surface: A Practitioner's Guide*. Wiley.

[4] Bjork, T. (2009). *Arbitrage Theory in Continuous Time*. Oxford University Press.

Appendix: Python Code for Question 7 (Geometric Mean-Reverting Process)

Listing 1: Geometric Mean-Reverting Simulation (Q7)

```
# q7_geometric_mr_simulation.py

import numpy as np
import matplotlib.pyplot as plt

# Parameters
k = 1.0            # Mean reversion rate
theta = 2.0        # Long-term log mean
sigma = 0.3        # Volatility
x0 = 1.0           # Initial value of X_t
y0 = np.log(x0)    # Initial value of Y_t = log(X_t)

T = 10.0           # Total time
N = 1000           # Number of steps
dt = T / N         # Time step
t = np.linspace(0, T, N + 1)
M = 50             # Number of sample paths

# Precompute parameters
theta_hat = theta - (sigma ** 2) / (2 * k)

# Allocate paths
Y = np.zeros((M, N + 1))
X = np.zeros((M, N + 1))
Y[:, 0] = y0
X[:, 0] = x0

# Simulate paths using exact OU solution for Y_t
for i in range(M):
    dW = np.random.normal(0, np.sqrt(dt), size=N)
    W = np.insert(np.cumsum(dW), 0, 0)  # Brownian path with W(0) = 0

    exp_negkt = np.exp(-k * t)
    mean_Y = theta_hat + (y0 - theta_hat) * exp_negkt
    var_Y = (sigma ** 2 / (2 * k)) * (1 - exp_negkt**2)
    Y[i, :] = mean_Y + np.sqrt(var_Y) * np.random.normal(0, 1, size=N + 1)

    # Recover X_t = e^{Y_t}
    X[i, :] = np.exp(Y[i, :])
```

```python
# Plotting
plt.figure(figsize=(10, 6))
for i in range(M):
    plt.plot(t, X[i, :], lw=0.8, alpha=0.7, color='green')

plt.title("Simulated Paths of the Geometric Mean-Reverting Process")
plt.xlabel("Time $t$")
plt.ylabel("$X_t = e^{Y_t}$")
plt.grid(True)
plt.tight_layout()

# Save and show
plt.savefig("images/q7_geometric_mr_simulation_graph.jpg", dpi=300)
plt.show()
```