Question 4. Verify the following:

(a) $X_t = e^{W_t}$ solves $dX_t = \frac{1}{2} X_t\, dt + X_t\, dW_t$

(b) $X_t = \frac{W_t}{1+t}$ solves $dX_t = -\frac{1}{1+t} X_t\, dt + \frac{1}{1+t}\, dW_t$

(c) $X_t = \sin W_t$ solves $dX_t = -\frac{1}{2} X_t\, dt + \sqrt{1-X_t^2}\, dW_t$

Sol-n:

(1) (a) $X_t = e^{W_t} \Rightarrow \varphi = e^{W_t} \Rightarrow \varphi = \varphi(W_t)$

(2) $\frac{\partial \varphi}{\partial t} = 0$; $\frac{\partial \varphi}{\partial x} = e^x$; $\frac{\partial^2 \varphi}{\partial x^2} = e^x$

The SDE is as follows:

(3) $dX_t = \frac{1}{2} X_t\, dt + X_t\, dW_t$

by case I of Itô Formula for

$X_t = \varphi(W_t) = e^{W_t}$, we know

by (2)

$$dX_t = \frac{1}{2} \underbrace{\varphi''(W_t)}\, dt + \varphi'(W_t)\, dW_t =$$

$$= \frac{1}{2} \underbrace{e^{W_t}}\, dt + \underbrace{e^{W_t}}\, dW_t$$

in (3).

$$= \frac{1}{2} X_t\, dt + X_t\, dW_t = dX_t$$

(1) Thus $X_t = e^{W_t}$ solves given SDE. $\boxtimes$

(b) (4) $\rightarrow X_t = \frac{W_t}{1+t}$ & we need to show (4)

solves the ↓:

(5) $dX_t = -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dW_t$,

(6) Let $\varphi(W_t, t) = \frac{W_t}{1+t}$ & by

case II of Itô's formula

(7) $dX_t = \left(\frac{\partial\varphi}{\partial t}(W_t, t) + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}(W_t, t)\right] dt +$

$+ \frac{\partial\varphi}{\partial x}(W_t, t) dW_t$, so we need the ↓

terms:

(8) $\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial t}\left[\frac{x}{1+t}\right] = -\frac{x}{(1+t)^2}$ &

(9) $\frac{\partial\varphi}{\partial x}\left[\frac{x}{1+t}\right] = \frac{1}{1+t}$ &

(10) $\frac{\partial^2\varphi}{\partial x^2} = 0$. Putting these (8)-(10) back into

(7) gives

$dX_t = \left[-\frac{W_t}{(1+t)^2}\right] dt + \frac{1}{1+t} dW_t$

$= -\frac{1}{1+t} \cdot \frac{W_t}{1+t} dt + \frac{1}{1+t} dW_t = \overbrace{-\frac{1}{1+t} X_t dt + \frac{1}{1+t} dW_t}^{\equiv dX_t \text{ in } (5)}$

as desired.

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(c)

(11) $X_t = \sin W_t$,

WNT: (11) solves the ↓

(12) $dX_t = -\frac{1}{2} X_t \, dt + \sqrt{1-X_t^2} \, dW_t$

(13) $X_t = \sin(W_t) = \varphi(W_t)$

$X_t = \varphi(W_t)$ is case I.

For $\varphi(x) = \sin(x)$, we have

(14) $dX_t = \frac{1}{2} \varphi''(W_t) \, dt + \varphi'(W_t) \, dW_t$ so

(15) $\begin{cases} \varphi'(x) = \frac{d}{dx}(\sin(x)) = \cos(x) \; \& \\ \varphi'(W_t) = \cos(W_t) \end{cases}$ $\Rightarrow$

(16) $\begin{cases} \varphi''(x) = \frac{d}{dx}\left[\cos(x)\right] = -\sin(x) \; \& \\ \varphi''(W_t) = -\sin(W_t) \end{cases}$

③ Putting (15) & (16) back into (14) & a trig identity gives desired result:

(17) $dX_t = \frac{1}{2} \cdot (-\sin(W_t)) + \cos(W_t) \, dt = \underbrace{-\frac{1}{2} \sin(W_t) + \sqrt{1-\sin^2 W_t} \, dW_t}_{= dX_t \text{ in } (14).}$ ∎