Part III: Induction

1. Prove each of the following claims by induction:

   (a) The sum of the first n even number is $n^2 + n$. That is, $\sum_{i=1}^{n} 2i = n^2 + n$.

   Proof by Induction:
   We need to prove p(n): The sum of the first n even number is $n^2 + n$.
   Base Case:
   When $n = 1$, $\sum_{i=1}^{n} 2i = \sum_{i=1}^{1} 2i = 2$ and $n^2 + n = 1 + 1 = 2$, so p(n) is true for $n = 1$.
   Inductive Step:
   Assume p(k) is true for all $k \geq 1$ i.e. $\sum_{i=1}^{k} 2i = k^2 + k$.
   We then show, p(k+1) is true i.e. $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$.
   Now, when $n = k + 1$

$$\begin{aligned}
\text{p(n):} \quad \sum_{i=1}^{n} 2i &= \sum_{i=1}^{k+1} 2i \\
&= \sum_{i=1}^{k} 2i + \sum_{i=k+1}^{k+1} 2i \\
&= k^2 + k + 2(k+1) \quad (\because \text{Using Induction hypothesis}) \\
&= (k^2 + 2k + 1) + k + 1 \\
&= (k+1)^2 + (k+1) \quad \text{So p(k+1) is true.}
\end{aligned}$$

   Hence, p(n): The sum of the first n even number is $n^2 + n$, is true for all $n \geq 1$. □

   (b) $\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$

   Proof by Induction:
   We need to prove p(n): $\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$.
   Base Case:
   When $n = 1$,
   Left Hand Side(LHS) $= \sum_{i=1}^{n} 3^i = \sum_{i=1}^{1} 3^i = 3$, and Right Hand Side(RHS) $= \frac{3}{2}(3^n - 1) = \frac{3}{2} \cdot (3 - 1) = 3$
   Here LHS = RHS so p(n) is true for $n = 1$.
   Inductive Step:
   Assume p(k) is true for all $k \geq 1$ i.e. $\sum_{i=1}^{k} 3^i = \frac{3}{2}(3^k - 1)$.
   We then show, p(k+1) is true i.e. $\sum_{i=1}^{k+1} 3^i = \frac{3}{2}(3^{k+1} - 1)$.

Now, when $n = k + 1$

$$\begin{aligned} \text{p(n): } \sum_{i=1}^{n} 3^i &= \sum_{i=1}^{k+1} 3^i \\ &= \sum_{i=1}^{k} 3^i + \sum_{i=k+1}^{k+1} 3^i \\ &= \tfrac{3}{2}(3^k - 1) + 3^{k+1} \text{ } (\because \text{ Using Induction hypothesis}) \\ &= \frac{3 \cdot 3^k - 3 + 2 \cdot 3^{k+1}}{2} \\ &= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} \\ &= \tfrac{3}{2}(3^{k+1} - 1) \quad \text{So p(k+1) is true.} \end{aligned}$$

Hence, p(n): $\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$, is true for all $n \geq 1$. □

(c) For any integer $n \geq 1$, $5^n - 1$ is divisible by 4.
Proof by Induction:
We need to prove p(n): $5^n - 1$ is divisible by 4 for any $n \geq 1$.
Base Case:
When $n = 1$, $5^1 - 1 = 5 - 1 = 4$ which is divisible by 4. So p(n) is true for $n = 1$.
Inductive Step:
Assume p(k) is true for $k \geq 1$ i.e. $5^k - 1$ is divisible by 4. Then $5^k - 1 = 4z_0$ where $z_0 \in \mathbb{Z}$.
We then show, p(k+1) is true i.e. $5^{k+1} - 1$ is divisible by 4.
Now, when $n = k + 1$

$$\begin{aligned} \text{p(n): } 5^n - 1 &= 5^{k+1} - 1 \\ &= 5^k \cdot 5 - 5 + 4 \\ &= 5(5^k - 1) + 4 \\ &= 5(4z_0) + 4 \text{ } (\because \text{ Using Induction hypothesis}) \\ &= 4(5z_0 + 1) \\ &= 4z_n \quad \text{where, } z_n = 5z_0 + 1 \end{aligned}$$

So $5^{k+1} - 1$ is divisible by 4 i.e. p(k+1) is true.
Hence,
p(n) is true for $n \geq 1$ i.e. $5^n - 1$ is divisible by 4 for any $n \geq 1$. □

2. The function `maxOdd`, given below in pseudocode, takes as input an array A of size n of numbers. It returns the largest *odd* number in the array. If no odd numbers appear in the array, it returns negative infinity ($-\infty$). Using induction, prove that the `maxOdd` function works correctly. Clearly state your recursive invariant at the beginning of your proof.

```
Function maxOdd(A,n)
  If n = 0 Then
    Return -∞
  Else
    Set best To maxOdd(A,n-1)
    If A[n-1] > best And A[n-1] is odd Then
      Set best To A[n-1]
    EndIf
    Return best
  EndIf
EndFunction
```

Proof:
p(n): The function `maxOdd(A,n)` returns the largest odd number in the array `A` and returns $-\infty$ if there are no odd numbers.
<u>Base Case:</u>
p(0): The `maxOdd(A,0)` returns largest odd number from 0 elements of A i.e. `maxOdd` returns $-\infty$.
Here we have $n = 0$,
the first `if-statement` is satisfied so the function will return $-\infty$. Hence, p(0) is True.
<u>Inductive Step:</u>
Assume p(k) is true for $k \geq 0$ i.e. the `maxOdd(A, k)` returns returns the largest odd number in the array `A` with k elements and returns $-\infty$ if there are no odd numbers.
We then show,
p(k+1) is true i.e. the function `maxOdd(A, k+1)` returns returns the largest odd number in the array `A` with (k+1) elements and returns $-\infty$ if there are no odd numbers.
Now,
As we call `maxOdd(A, k+1)`, we will go to `else` case of first `if-condition` and set `best = maxOdd(A, k)` which is correct for k elements from our induction hypothesis. After that, we will move to the next `if-condition`:
If `A[k] = (k+1)th element of A` is an odd number and `A[k] > best` (i.e. it's a larger odd number), it will set the best to be `best = A[k]`. Finally the function will return the value of `best` which will be the (k+1)th element in A if `A[k]` is a larger odd number else it will be the return value from `maxOdd(A, k)`, which will be largest odd number from `(A, k)` or is $-\infty$ and this is true from inductive hypothesis. So, p(k+1) is true.
Hence,
p(n): The function `maxOdd(A,n)` returns the largest odd number in the array `A` and returns $-\infty$ if there are no odd numbers, is true for all $n \geq 0$. □

End of Lab 6 Writings