2. **The Differential Argument and the Gamma Distribution** (Blitzstein 5.5, 5.6, 13)
The differential approach to finding a pdf $f_x(x)$ is summarized as follows:

for a continuous variable

This problem continues from the first problem as we move from discrete to continuous variables. For any given discrete variable, the pmf provides the exact probability of X being equal to some number. However, for a continuous variable, the pmf only the provides the probability of X being less than or greater than a certain value. However, as we will see through this problem, if we now add a width to a probabilities, of width dx, we can find the exact probability.

Function

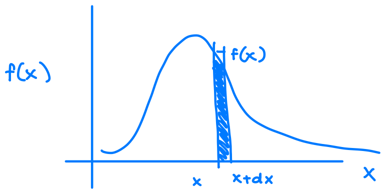
We will see throughout this problem how a discrete variable, like a poisson distribution, which is a process in time with discrete counts, becomes a continuous variable, such as Exponential or Gamma, when we examine the waiting period for the next event.

a) For a continuous random variable X with pdf f_x, explain why, for a small deviation dx,

$$P(X \in [x, x+dx)) \approx f_x(x)dx$$

Show how this becomes an exact equivalence when you divide both sides by dx and let dx approach 0 (or if the density $f_x(x)$ is a constant function). As an example, derive the Exponential density for X by finding the probability the first Poisson event occurs in the interval $[x, x+dx)$. Then divide by dx and take the limit $dx \to 0$. Show this agrees with the result you get by finding the Exponential cdf and differentiating.

Let's start with a general function f(x). If we want to find the pdf at x, we instead want to find the pdf at x to the very small increment of dx. So, we find the area of the rectangle of the pdf evaluated at x to the pdf evaluated at (x + dx).



We can see from the graph that f(x) represents the height and dx represents the width of this rectangle.

Therefore, $f(x)\,dx \approx P(X \in [x, x+dx))$

In order to find this area, we can evaluate the difference in the integrals.

$$f(x)dx \approx F_x(x+dx) - F_x(x)$$

Next, we divide both sides by dx and take the limit as dx approaches 0.

$$f(x) = \lim_{dx \to 0} \frac{F_x(x+dx) - F_x(x)}{dx} = \frac{d}{dx} F(x)$$

We see that this is equivalent to the set up of a derivative and thus the differential approach to find the pmf.

Now, as an example, derive the Exponential density by finding the probability the first poisson event occurs in the interval $[x, x+dx)$.

Count of events in time x $\sim$ Poisson(λx) $\quad$ λ = events per unit rate

x = # time units

first event

$$P(X \in [x, x+dx)) = P(0 \text{ events by time } x) \cdot P(1 \text{ event in time } [x, x+dx))$$

These are poisson and independent so we can multiply the individual pmf's.

$$P(X \in [x, x+dx)) = \frac{(\lambda x)^0 e^{-\lambda x}}{0!} \cdot \frac{(\lambda dx)^1 e^{-\lambda dx}}{1!}$$

$$= \lambda e^{-\lambda(x+dx)} \cdot dx$$

*note: $e^{-dx} \to 1$ as $dx \to 0$ so we can remove it from the equation in the next step

$$f(x)dx \cong \lambda e^{-\lambda x} dx$$

divide both sides by dx

$\to f(x) = \lambda e^{-\lambda x}, \quad x > 0$ $\quad$ (This evaluates to the exponential pmf)

show this agrees with the result you get by finding the exponential cdf and differentiating.

CDF method: $1 - F_x(x) = P(X > x) = P(0 \text{ events by time } x) = e^{-\lambda x}$

$F_x(x) = 1 - e^{-\lambda x}$

$f(x) = \lambda e^{-\lambda x}$ (same result as above).

b) Use a differential argument to derive the Gamma(k, λ) density. Let Y be the time of the kth event for a Poisson process with rate λ. Find the probability of $k-1$ events occurring before time y and a kth event occurring in the interval $[y, y+dy)$. Divide by dy and take the limit as $dy \to 0$. Give intuition for why, if $X_1 \sim$ Gamma(k_1, λ) is independent of $X_2 \sim$ Gamma(k_2, λ), then $X_1 + X_2 \sim$ Gamma$(k_1 + k_2, \lambda)$, and for why the sum is not Gamma if the λ's differ.

We want to find the probability of k-1 events occurring before time y and a k^{th} event occurring in the interval $[y, y+dy)$

These individual and independent events are once again poisson.

$P(k-1 \text{ events by time } y) = \frac{e^{-\lambda y}(\lambda y)^{k-1}}{(k-1)!}$ $\sim$ Poisson (λy)

$P(k^{th} \text{ event in } [y, y+dy)) = \lambda dy$ $\sim$ Poisson (λdy) (given that there are k-1 events by time y)

The total probability of having k-1 events by time y and the k^{th} event in $[y, y+dy)$ is

$$P(X \in [y, y+dy)) = \frac{(\lambda y)^{k-1} e^{-\lambda y}}{(k-1)!} \cdot \lambda dy = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} \cdot dy$$ (combine like terms)

divide by dy to get the pdf

$$\frac{P(x \in [y, y+dy))}{dy} = \frac{(\lambda y)^{k-1} e^{-\lambda y}}{(k-1)!} \cdot \lambda$$

Take the limit as $dy \to 0$. This evaluates to the pdf of the Gamma distribution.

$$f_y(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$$

Lastly, give the intuition for why, if $X_1 \sim$ Gamma (k_1, λ) is independent of $X_2 \sim$ Gamma (k_2, λ), then $X_1 + X_2 \sim$ Gamma $(k_1 + k_2, \lambda)$ and for why the sum is not Gamma if the λ's differs.

$X_1 + X_2 \longrightarrow k_1 + k_2$ = total number of events (wait for k_1 events and then wait for k_2)

$X_1 + X_2 \longrightarrow \lambda$ remains the rate of events

if λ_1 and λ_2 were to differ, then they would not be in the same poisson process.

c) For a constant $c > 0$, show that if $X \sim \text{Gamma}(\alpha, \lambda)$, then $Y = cX \sim \text{Gamma}(\alpha, \lambda/c)$. Give intuition based on units of measure. For example, consider X to be a waiting time in hours and $Y = 60X$ to be the time in minutes.

For a constant $c > 0$, show that if $X \sim \text{Gamma}(\alpha, \lambda)$, then $Y = cX \sim \text{Gamma}(\alpha, \lambda/c)$

$X \sim \text{Gamma}(\alpha, \lambda)$ $\quad Y = cX \quad c > 0$

$x = Y/c$

$F_y(y) = P(Y \leq y) = P(cX \leq y)$

$= P(x \leq y/c) = F_x(y/c)$

$f_y(y) = \underbrace{\frac{1}{c} f_x(y/c)}_{\text{chain rule}} = \frac{\lambda^\alpha}{\Gamma(\alpha)} (y/c)^{\alpha-1} e^{-\lambda y/c} \cdot \frac{1}{c}$ * plug into Gamma pdf

$= \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)} \cdot y^{\alpha-1} e^{-(\lambda/c)y}, \quad y > 0$

This confirms $Y = cx \sim \text{Gamma}(\alpha, \lambda/c)$

We can now look at an example of this.

consider X to be a waiting time in hours and $Y = 60X$ to be time in minutes.

Let's look at the set up.

X = time in hours $\quad \lambda$ = events per hour

Y = time in minutes $= 60X \rightarrow$ events per minute $= \frac{\lambda}{60}$

Thus, through the change of variables, we see how this now becomes λ/c or in this case, $\lambda/60$

d) Define the Gamma function and show how it normalizes the Gamma(α, λ) density for any value $\alpha > 0$. Give the recursive property of the Gamma function and the connection to the factorial function.

We have previously defined the Gamma function for an integer K.

$Y \sim \text{Gamma}(K, \lambda) \rightarrow$ Y is the time until K^{th} event

$$f_Y(y) = \frac{\lambda^K}{(K-1)!} y^{K-1} e^{-\lambda y}, \quad y > 0 \quad K = 1, 2, \ldots$$

For an integer K, we believe this pdf is correct because of the story in the previous problems: K-1 events before time x, 1 event in dx, n-K events after x+dx

But what about a non integer K, which I will call α?

non-integer α In general: $\frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, \quad y > 0 \quad \alpha > 0$ (real number)

what we will see is that the Gamma function will normalize the Gamma density.

Example: Let $\lambda = 1$

$$f(x) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}, \quad x > 0$$

for an integer K, we used $\Gamma(K) = (K-1)!$ $K = 1, 2, \ldots$

for non integer α, we can use $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$

This normalizes the probability as $\int_0^\infty f(x)dx = \int_0^\infty \frac{x^{\alpha-1} e^{-x} dx}{\Gamma(\alpha)} = \frac{\int_0^\infty x^{\alpha-1} e^{-x} dx}{\int_0^\infty x^{\alpha-1} e^{-x} dx} = 1$

using a change of variables, we see the normalization regardless of the λ value

$Y = X/\lambda \quad X = \lambda Y \quad Y \sim \text{Gamma}(\alpha, \lambda)$

$$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}, \quad y > 0$$

Give the recursive property of the gamma function.

Recursive property: $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

Broadly: $\Gamma(\alpha+1) = \underbrace{\int_0^\infty x^{\alpha} e^{-x} dx = \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx}_{\text{integration by parts}} = \alpha\Gamma(\alpha)$

connection to the factorial function:

recursive: $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$ $\longrightarrow$ we can think of this like a factorial $\longrightarrow (x+1)! = (x+1) \cdot x!$

$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = 1 = 0!$

rewrite using recursive property

$\Gamma(2) = 1 \cdot \Gamma(1) = 1 = 1!$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 = 2!$

$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 = 6 = 3!$

Thus, we can see that:

$\Gamma(n+1) = n!$

Therefore, the recursive property is equivalent to the factorial function.