Stat111 W4 P5

Yihui Wu

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a

To distinguish between conditional expectation given an event and conditional expectation given a random variable, we need to start from their definition. That is, write conditional expectation given an event as $E[X|Y = y_0]$ and write conditional expectation given a random variable as $E[X|Y]$.
$E[X|Y = y_0]$ is a value, while $E[X|Y]$ is a random variable that can be written as a function of y.
We shall use the example with $Poisson(\lambda)$. Let T_i represent the time taken for i^{th} event to happen, thus $T_i \sim Poission(\lambda)$. Let $X \sim T_1$ and $Y \sim T_1 + T_2$. We shall find $(X|Y = y) \sim Unif(y)$, and $E[X|Y = y] = \frac{y}{2}$. Furthermore, $E[X|Y] = \frac{Y}{2}$.

b

Adam's Law (law of total expectation):

$$E[X] = E[E[X|Y]]$$

proof:

$$\begin{aligned} E[E[X|Y]] &= E[\int x f_{X|Y}(x|y)dx] \\ &= \int [\int x f_{X|Y}(x|y)dx] f_Y(y)dy \\ &= \int \int x(f_{X|Y}(x|y)f_Y(y))dxdy \\ &= \int \int x f_{X,Y}(x,y)dxdy \\ &= E[x] \end{aligned}$$

Eve's Law (law of total variance):

$$Var[X] = E[Var[X|Y]] + Var[E[X|Y]]$$

proof:

$$
\begin{aligned}
E[Var(X|Y)] + Var[E(X|Y)] &= E[Var(X|Y)] + E[[E(X|Y)]^2] - [E[E(X|Y)]]^2 \\
&= E[Var(X|Y) + [E(X|Y)]^2] - [E(X)]^2 \\
&= E[E(X^2|Y)] - [E(X)]^2 \\
&= E(X^2) - [E(X)]^2 = E(X)
\end{aligned}
$$

c*

Apply the above equation to the problem we have for part C, we will have

$$
\begin{aligned}
E(X) &= E[E(X|\theta)] \\
&= E(\theta) = \mu
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E[Var(X|\theta)] + Var[E(X|\theta)] \\
&= E(\theta) + Var(\theta) = \mu + \frac{\mu^2}{\alpha}
\end{aligned}
$$

when t approaches infinite, $E(X) = \mu$ and $Var(X) \to \mu$. Then The distribution approaches a Poisson distribution.

d

Similarly, we apply law of total expectation and variance:

$$
\begin{aligned}
E(Y) &= E[E(Y|\theta)] \\
&= E(\theta) = \mu
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= E[Var(Y|\theta)] + Var[E(Y|\theta)] \\
&= E(V) + Var(\theta) = V + A
\end{aligned}
$$

To write θ and Y in terms of two independent standard Normal variables Z_1 and Z_2, write

$$\theta = \sqrt{A}Z_1 + \mu$$

$$Y|\theta = \sqrt{V}Z_2 + \theta$$

$$Y = \sqrt{V}Z_2 + \sqrt{A}Z_1 + \mu$$

e

by Bayesian inference:

$$f_{\theta|Y}(\theta'|Y=y) * f_Y(y) = f_{Y|\theta}(y|\theta=\theta') * f_\theta(\theta')$$

Based on this, we shall find out $\theta|Y \sim N(\frac{\frac{\mu}{A}+\frac{Y}{V}}{\frac{1}{A}+\frac{1}{V}}, \frac{1}{\frac{1}{A}+\frac{1}{V}})$, where $E(\theta|Y=y) = \frac{Ay+V\mu}{A+V}$ and $Var(\theta|Y=y) = \frac{AV}{A+V}$