Joint, Marginal, and Conditional Densities

Rice 3.1-3.5, Blitzstein 7.1

2/5/2025

Discrete

Joint CDF:

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

Joint PMF:

$$p_{X,Y}(x,y) = P(X = x, Y = y).$$

Such that

$$\sum_x \sum_y P(X = x, Y = y) = 1.$$

Marginal PMF:

$$P(X = x) = \sum_y P(X = x, Y = y).$$

Conditional PMF:

$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

Continuous

Joint CDF:

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

Joint PDF:

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$$

Such that

$$f_{X,Y}(x,y) \geq 0, \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1.$$

Marginal PDF:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy.$$

Conditional PDF:

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) f_Y(y)}{f_X(x)},$$

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b) As an example, suppose X and Y have joint pdf $f_{xy}(x,y) = \lambda^2 e^{-\lambda y} I_{(0<x<y)}$. Show that the marginal distributions are $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Gamma}(2,\lambda)$.

$f_{xy}(x,y) = \lambda^2 e^{-\lambda y} I_{(0<x<y)}$

↳ so to find marg dist of Y, fix y and integrate over all possible values of x

Marginal density found by integrating wrt opposite variable

$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$f_x(x) = \int_x^{\infty} \lambda^2 e^{-\lambda y}\,dy$ bounds determined by indicator (y must be greater than x)

$f_x(x) = \lambda^2 \int_x^{\infty} e^{-\lambda y}\,dy$

$f_x(x) = \lambda^2 \left. \frac{-e^{-\lambda y}}{\lambda} \right]_x^{\infty} = \lambda e^{-\lambda x} I_{(x>0)}$ marginal pdf shows $X \sim \text{Expo}(\lambda)$

$f_y(y) = \int_0^y \lambda^2 e^{-\lambda y}\,dx = y\lambda^2 e^{-\lambda y} I_{(0<x<y)}$ marg pdf shows $Y \sim \text{Gamma}(2,\lambda)$

c) Show that $Y|X=x$ is a translated Exponential, and $X|Y=y$ is Uniform.

$$f_{y|x}(y|x) = \frac{f_{xy}(x,y)}{f_x(x)} = \frac{\lambda^2 e^{-\lambda y} I_{(0<x<y)}}{\lambda e^{-\lambda x} I_{(0<x)}}$$

$$= \frac{\lambda e^{-\lambda y}}{e^{-\lambda x}}$$

$$= \lambda e^{-\lambda y + \lambda x}$$

$$= \lambda e^{-\lambda(y-x)} I_{(x<y)}$$

↳ exponential pdf shifted along x-axis by x

$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)} = \frac{\lambda^2 e^{-\lambda y}}{y\lambda^2 e^{-\lambda y}}$$

$$= \frac{1}{y} e^{-\lambda y + \lambda y}$$

$$= \frac{1}{y} I_{(0<x<y)}$$

↳ uniform pdf for interval (0,y)

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d) Review the factorization theorem for independence. Show that X and Y are not independent, but that X and $Y - X$ are independent.

factorization theorem for independence:

X, Y independent iff $f_{XY}(x,y) = g_1(x)g_2(y)$

in other words, joint PDF factors into product of marginal PDFs or joint PMF factors into product of marginal PMFs that depend on one variable only

$f_{XY}(x,y) = \lambda^2 e^{-\lambda y} I_{(0<x<y)}$

intuitively indicator shows not independent
range changes based on both variables

$f_X(x) = \lambda e^{-\lambda x} I_{(x>0)}$

$f_Y(y) = y\lambda^2 e^{-\lambda y} I_{(0<x<y)}$

$f_X(x) \propto g_1(x)$

$f_Y(y) \propto g_2(y)$

Show that X and Y - X are independent:

$$X_1 = X \qquad \Rightarrow \qquad X = X_1$$
$$X_2 = Y - X \qquad\qquad Y = X_1 + X_2$$

change of variables $\rightarrow$ $f(x,y) = f(T(u,v))\,|\det DT(u,v)|$

$$\frac{\partial(X,Y)}{\partial(X_1,X_2)} = \begin{bmatrix} \frac{\partial X}{\partial X_1} & \frac{\partial X}{\partial X_2} \\ \frac{\partial Y}{\partial X_1} & \frac{\partial Y}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$|J| = 1$

↳ Jacobian is the determinant of a 2×2 matrix

$f_{X_1X_2}(x_1, x_2) = f_{XY}(x_1, x_1 + x_2) \cdot 1$

$f_{X_1X_2}(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)} I_{(x_1>0,\ x_2>0)}$

$f_{X_1X_2}(x_1, x_2) = \left(\lambda e^{-\lambda x_1} I_{(x_1>0)}\right)\left(\lambda e^{-\lambda x_2} I_{(x_2>0)}\right)$

since $f_{X_1X_2}(x_1, x_2)$ is able to be factored into factors containing only x_1 or x_2, x_1 and x_2 are independent, so X and $Y - X$ are independent

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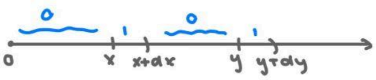
e) A bivariate differential argument approximates $P(X \in [x, x+dx), Y \in [y, y+dy))$ by $f_{xy}(x,y)dxdy$. Use this approach to show that this is the joint pdf for the time of the first event (x) and the time of the second event (y) for a Poisson process with rate λ events per unit time. Show this agrees with what you find setting $X = X_1$ and $Y = X_1 + X_2$, for $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ).

first event at time x

second event at time y



$f_{xy}(x,y)\,dxdy \approx \mathbb{P}(X \in [x, x+dx), Y \in [y, y+dy))$

$= F_{xy}(x+dx, y+dy) - F_{xy}(x,y)$

$\approx \mathbb{P}($0 events in time x, 1 event in time dx, 0 events in time $y-(x+dx)$
1 event in time $(y+dy)-y)$

poisson process tells us each event is independent so total prob = product of individual probs

$$f_{xy}(x,y)\,dxdy \approx \left(\frac{(\lambda x)^0 e^{-\lambda x}}{0!}\right)\left(\frac{(\lambda dx)^1 e^{-\lambda dx}}{1!}\right)\left(\frac{(\lambda(y-(x+dx)))^0}{0!}e^{-\lambda(y-(x+dx))}\right) \times \left(\frac{(\lambda dy)^1}{1!}e^{-\lambda dy}\right)$$

simplify

$f_{xy}(x,y)\,dxdy = \cancel{e^{-\lambda x}}\,\lambda dx\,\cancel{e^{-\lambda dx}}\,e^{-\lambda y + \cancel{\lambda x} + \cancel{\lambda dx}}\;\lambda dy\, e^{-\lambda dy}$

$f_{xy}(x,y)\,dxdy = \lambda^2 dx dy\, e^{-\lambda y}\, e^{-\lambda dy}$

$f_{xy}(x,y) = \lambda^2 e^{-\lambda y}\, I_{(0<x<y)}$

this is the same joint pdf as the one we found when

$X = X_1$

$Y = X_1 + X_2$

for $X_1, X_2 \overset{iid}{\sim} \text{Expo}(\lambda)$