Question 5: Bivariate Normal Variables

- Introduction

- Difference Between Multivariate & Bivariate - Bivariate $\Rightarrow n=2$

- Joint PDF: $\frac{1}{(2\pi)^{n/2} \cdot |V|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)\right)$

  - X: N-Dimensional Random Vector, represents values of random variables

  - μ: N-Dimensional Vector of EV, represents EVs of each variable

  - $|V|$: Determinant of Covariance Matrix

  - V^{-1}: Inverse of Covariance Matrix

  - $(2\pi)^{n/2} \cdot |V|^{1/2}$: Normalization Constant

- Question 5a

  - $X = (X_1, X_2)^T \Rightarrow$ bivariate normal random vector $\Rightarrow$ Mean vector $\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

  - Covariance Matrix: $V = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ $\quad |V| = \sigma_1^2 \cdot \sigma_2^2 \cdot (1-\rho^2)$

  - $|V|^{1/2} = \sigma_1\sigma_2\sqrt{1-\rho^2} \rightarrow \frac{1}{(2\pi)\cdot\sigma_1\sigma_2\sqrt{1-\rho^2}}$

  - $V^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$

  - $(X-\mu)^T = \begin{bmatrix} X_1-\mu_1 & X_2-\mu_2 \end{bmatrix} \Rightarrow (X-\mu) = \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}$

  - Exponential: $-\frac{1}{2}\begin{bmatrix} X_1-\mu_1 & X_2-\mu_2 \end{bmatrix} \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \cdot \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{bmatrix}$

  - Matrix Multiplication: $-\frac{1}{2} \cdot \frac{1}{1-\rho^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right)\right.$$

Independence: $f(x_1, x_2) = f(x_1) \cdot f(x_2)$

1. Show if $\rho = 0$, x_1 and x_2 are independent

- Substituting $\rho = 0 \rightarrow f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right]$

$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left[-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right)\right] \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left[-\frac{1}{2}\left(\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right] = f(x_1) \cdot f(x_2)$ ✓

2. Show if x_1 and x_2 are independent, ρ must equal 0

- If x_1 & x_2 are independent, $\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}$ must equal $0 \rightarrow \rho = 0$ ✓

Question 5b.

- $Z_1, Z_2 \sim N(0,1)$ $\quad X = Z_1$, $Y = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$ for $\rho \in (-1,1)$

$E[X] = E[Z_1] = 0$ $\quad E[Y] = E[\rho Z_1] + E[\sqrt{1-\rho^2}\, Z_2] = 0 + 0 = 0$ $\quad \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ is

$Var(X) = Var(Z_1) = 1$ $\quad Var(Y) = \rho^2 Var(Z_1) + (1-\rho^2) Var(Z_2) = 1$ $\quad$ multivariate norma

$\rightarrow X \sim N(0,1)$ $\quad \rightarrow Y \sim N(0,1)$

$\begin{bmatrix} X \\ Y \end{bmatrix}$ = Linear Combination of Multivariate Normal Variables = Multivariate (Bivariate) Normal

- $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$ $\quad Var(X) = Var(Z_1) = 1$ $\quad Var(Y) = Var(\rho Z_1 + Z_2\sqrt{1-\rho^2})$

Z_1 and Z_2 are independent, so $Var(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2) = Var(\rho Z_1) + Var(\sqrt{1-\rho^2}\, Z_2)$

$\rightarrow \rho^2 Var(Z_1) + (1-\rho^2) Var(Z_2) \rightarrow Var(Z_1) = Var(Z_2) = 1 \rightarrow Var(Y) = \rho^2 + 1 - \rho^2 = 1$

$\rightarrow \sqrt{Var(X) \cdot Var(Y)} = \sqrt{1 \cdot 1} = 1 \rightarrow Corr(X,Y) = Cov(X,Y)$

$\rightarrow Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY]$

$E[X] = E[Z_1] = 0$ $\quad E[Y] = E[\rho Z_1 + \sqrt{1-\rho^2}\, Z_2] = \rho E[Z_1] + (1-\rho^2) E[Z_2] = 0 + 0 = 0$

$E[XY] = E[Z_1(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)] = E[\rho Z_1^2 + \sqrt{1-\rho^2}\, Z_1 Z_2] = \rho E[Z_1^2] + \sqrt{1-\rho^2}\, E[Z_1 Z_2]$

$E[Z_1^2] = Var(Z_1) + E[Z_1]^2 = 1 + 0 = 1$ $\quad Z_1, Z_2$ are independent

$E[Z_1 Z_2] = Cov(Z_1, Z_2) + E[Z_1]E[Z_2] = 0 + 0 = 0$

$Cov(X,Y) = E[XY] = \rho(1) + \sqrt{1-\rho^2}(0) = \rho$ Thus, $Corr(X,Y) = Cov(X,Y) = \rho$

Question 5c. $X = Z_1$ $\quad Y = SZ$ $\quad Z \sim N(0,1)$, independent of S

$S \pm 1$ simply changes the sign of Y $\quad S$ & Z are independent, so $Y \sim N(0,1)$ as well

$Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$ $\quad E[X] = E[Y] = 0 \Rightarrow = E[XY]$

$E[XY] = E[Z \cdot SZ] \Rightarrow S$ & Z are independent, so $E[SZ^2] = E[S]\cdot E[Z^2]$

$E[S] = \frac{1}{2}\cdot -1 + \frac{1}{2}\cdot 1 = 0$ $\quad E[Z^2] = Var(Z) + (E[Z])^2 = 1 + 0 = 1$

$Cov(X,Y) = E[XY] = E[S]\cdot E[Z^2] = 0 \cdot 1 = 0 \Rightarrow X$ & Y are uncorrelated

If X and Y are not independent, Y depends on $X \Rightarrow Y$ depends on Z

$Y = SZ \Rightarrow$ depends on the value of Z $(X = Z) \Rightarrow X$ and Y are not independent

By definition, a random vector $\binom{X}{Y}$ is bivariate normal if X and Y

are linear transformations/combinations of iid normal variables

Written in the form $X = a_1 Z_1 + a_2 Z_2$ $\quad Y = b_1 Z_1 + b_2 Z_2$ $\quad a_1, a_2, b_1, b_2$ are constants

X and Y are functions of a single variable (Z_1)

Non-Trivial LC: Z_1 & Z_2 both contribute nonzero weights for either X or Y

If $a_2 = b_2 = 0$ (as in question), X and Y are not independent

Question 5d.

Suppose $E[X] = \mu_X$ and $E[Y] = \mu_Y$, with SDs σ_X and σ_Y

Given X, Y are bivariate normal, they must be normally distributed

Since $X \sim N(\mu_X, \sigma_X^2)$, $Z_X = \frac{X - \mu_X}{\sigma_X}$, so $X = \mu_X + \sigma_X Z_X$

From part b., we know for any correlation value ρ, the vector (X, Y) is bivariate Normal

$\rightarrow X = Z_X,\ Y = \rho Z_X + \sqrt{1-\rho^2}\, Z_Y$, where $Z_X, Z_Y \overset{iid}{\sim} N(0,1)$

$\rightarrow Y = \mu_Y + \sigma_Y Z_Y \rightarrow Y = \mu_Y + \sigma_Y \left(\rho Z_X + \sqrt{1-\rho^2}\, Z_Y\right)$

$\rightarrow Y = \mu_Y + \sigma_Y \left(\rho \frac{X - \mu_X}{\sigma_X} + \sqrt{1-\rho^2}\, Z_Y\right) = \underbrace{\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(X - \mu_X)}_{E[Y|X=x]} + \underbrace{\sigma_Y \sqrt{1-\rho^2}\, Z_Y}_{Var[Y|X=x]}$

Since $Var[Y|X=x] = Var\left(\sigma_Y \sqrt{1-\rho^2}\, Z_Y\right) = \sigma_Y^2 (1-\rho^2)\, Var(Z_Y) = \sigma_Y^2 (1-\rho^2)$

$\rightarrow Y|X=x \sim N\left(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X),\ \sigma_Y^2 (1-\rho^2)\right)$

Linear Regression: $E[Y|X] = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = \underbrace{\mu_Y - \rho \frac{\sigma_Y}{\sigma_X}\mu_X}_{\beta_0} + \underbrace{\rho \frac{\sigma_Y}{\sigma_X}}_{\beta_1} X$

$Y = \beta_0 + \beta_1 X + \varepsilon$