Stat111 W2 P3

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a

Given Z is a uniform normal distribution (N(0,1)), $X = Z^2$

$$F_x(x) = P(Z^2 \leq x) = P(Z < \sqrt{x}) - P(Z < -\sqrt{x}) \tag{1}$$

differentiate both sides, we have

$$f_x(x) = \frac{1}{2\sqrt{x}} f_z(\sqrt{x}) * 2 = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} = \frac{(\frac{1}{2})^{\frac{1}{2}}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{\frac{x}{2}} \tag{2}$$

Thus, we can find

$$Z^2 \sim \chi^2_{(1)} \quad or \quad Gamma(\frac{1}{2}, \frac{1}{2}) \tag{3}$$

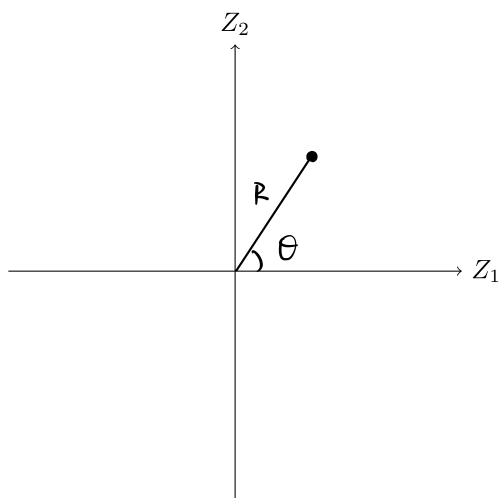
Based on Problem 2 of the same problem set, we can conclude that for all integer k,

$$\sum_{i=1}^{k} Z_i^2 \sim \chi^2_{(k)} \quad or \quad Gamma(\frac{k}{2}, \frac{1}{2}) \tag{4}$$

Thus we can create the transformation between any Gamma distribution and Chi-square distribution that given $X \sim Gamma(\alpha, \lambda)$, let $Y = 2\lambda X$ such that $Y \sim Gamma(\alpha, \frac{1}{2})$ or $\chi^2_{(2\alpha)}$

b

for $Z_1, Z_2 \overset{\text{i.i.d.}}{\sim} N(0,1)$, based on part (a), we can find out $Z_1^2 + Z_2^2 \sim Gamma(1, \frac{1}{2})$ or $Expo(\frac{1}{2})$. We would be able to draw it as



From here, we can find out that given two identical independent unit normal distributions. We can induce two independent exponential distributions and normal distributions where $R = \sqrt{Z_1^2 + Z_2^2}$ and $\theta = arctan\frac{Z_2}{Z_1}$.

c

it is the same as when we were talking about the exponential distribution itself that for $X \sim Poission(1)$, X indicates the number of events that happen in unit time, the joint distribution of R^2 and θ indicates several events happen in unit time.

d

Similar as last question, based on the previous presentation, we can find that a unit uniform distribution can be transformed into exponential distribution such that given i.i.d. uniform distributions U_1 and U_2, since $R^2 = -2ln(U_1)$ follow Exponential($\frac{1}{2}$) distribution, and $\theta = 2\pi U_2$ is a uniform distribution, two independent uniform distribution can be expressed as

$$Z_1 = \sqrt{-2ln(U_1)}cos(2\pi U_2) \tag{5}$$

and

$$Z_2 = \sqrt{-2ln(U_1)}sin(2\pi U_2) \tag{6}$$