Week 1

STAT111, Spring 2025, Abe Porschet

1. We want to show that the ratio between any two independent Gamma variables follows an F^* distribution.

   Consider $U \sim \text{Gamma}(\alpha_1, \beta_1)$ and $V \sim \text{Gamma}(\alpha_2, \beta_2)$, independent of each other. Then, using the scaling property of the gamma, we know that $U \sim \text{Gamma}(\alpha_1, \beta_1) = \beta_1 \text{Gamma}(\alpha_1, 1)$ and similarly, $V \sim \text{Gamma}(\alpha_2, \beta_2) = \beta_2 \text{Gamma}(\alpha_2, 1)$. Once we take the ratio between U and V, we can see that it being F^* distributed follows immediately, as U and V are independent Gammas, and can be shown as two Gammas with scaling parameter one, with coefficient $c = \frac{\beta_1}{\beta_2}$

Part (b)

Given $X = \frac{V_1}{V_1+V_2}$, where $V_1 \sim \text{Gamma}(a, 1)$ and $V_2 \sim \text{Gamma}(b, 1)$, we know that $X \sim \text{Beta}(a, b)$.

Step 1: Representation of X

The pdf of X is:

$$f_X(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \quad 0 < x < 1,$$

where $B(a, b)$ is the beta function.

Step 2: Transformation to R

Define $R = \frac{X}{1-X}$. Substituting $X = \frac{V_1}{V_1+V_2}$, we get:

$$R = \frac{\frac{V_1}{V_1+V_2}}{1 - \frac{V_1}{V_1+V_2}} = \frac{V_1}{V_2}.$$

Since $V_1 \sim \text{Gamma}(a, 1)$ and $V_2 \sim \text{Gamma}(b, 1)$, the ratio $\frac{V_1}{V_2}$ follows a scaled F-distribution:

$$R \sim F^*(a, b, 1).$$

Step 3: Deriving the PDF of R

The pdf of R is derived using the transformation method. Let $R = \frac{X}{1-X}$. The inverse transformation is $X = \frac{R}{1+R}$, and the Jacobian of the transformation is:

$$\frac{dx}{dr} = \frac{1}{(1+r)^2}.$$

Thus, the pdf of R is:

$$f_R(r) = f_X\left(\frac{r}{1+r}\right) \cdot \left|\frac{dx}{dr}\right| = \frac{\left(\frac{r}{1+r}\right)^{a-1}\left(1-\frac{r}{1+r}\right)^{b-1}}{B(a,b)} \cdot \frac{1}{(1+r)^2}.$$

Simplifying, we get:

$$f_R(r) = \frac{r^{a-1}}{B(a,b)(1+r)^{a+b}}, \quad r > 0.$$

This is the pdf of the scaled F-distribution $F^*(a, b, 1)$.

Step 4: Deriving the PDF of $Y = cR$

Let $Y = cR$, where $c > 0$. The transformation is $R = \frac{Y}{c}$, and the Jacobian is $\frac{dr}{dy} = \frac{1}{c}$. The pdf of Y is:

$$f_Y(y) = f_R\left(\frac{y}{c}\right) \cdot \left|\frac{dr}{dy}\right| = \frac{\left(\frac{y}{c}\right)^{a-1}}{B(a,b)\left(1+\frac{y}{c}\right)^{a+b}} \cdot \frac{1}{c}.$$

Simplifying, we get:

$$f_Y(y) = \frac{y^{a-1}}{c^a B(a,b)\left(1+\frac{y}{c}\right)^{a+b}}, \quad y > 0.$$

Part (c)

Given $V_1 \sim \chi^2(m_1)$ and $V_2 \sim \chi^2(m_2)$, where V_1 and V_2 are independent, we define:

$$Y = \frac{V_1/m_1}{V_2/m_2}.$$

We will show that Y follows a scaled F^*-distribution.

Step 1: Representation of V_1 and V_2

The χ^2-distribution is a special case of the Gamma distribution:

$$V_1 \sim \text{Gamma}\left(\frac{m_1}{2}, \frac{1}{2}\right), \quad V_2 \sim \text{Gamma}\left(\frac{m_2}{2}, \frac{1}{2}\right).$$

Step 2: Transformation to Y

The random variable Y is defined as:

$$Y = \frac{V_1/m_1}{V_2/m_2}.$$

This is the standard definition of the F-distribution, so:

$$Y \sim F(m_1, m_2).$$

Step 3: Relating F-distribution to F^*-distribution

The scaled F^*-distribution is defined as:

$$F^*(a,b,\lambda) = \frac{U_1/a}{U_2/b},$$

where $U_1 \sim \text{Gamma}(a,\lambda)$ and $U_2 \sim \text{Gamma}(b,\lambda)$.

Comparing this to Y, we see that:

$$Y = \frac{V_1/m_1}{V_2/m_2} = \frac{V_1/(m_1/2)}{V_2/(m_2/2)}.$$

This matches the form of the scaled F^*-distribution with:

$$a = \frac{m_1}{2}, \quad b = \frac{m_2}{2}, \quad \lambda = \frac{1}{2}.$$

Thus:

$$Y \sim F^*\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{1}{2}\right).$$

Step 4: Deriving the PDF of Y

The pdf of the F-distribution $F(m_1, m_2)$ is:

$$f_Y(y) = \frac{\Gamma\left(\frac{m_1+m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}\right)\Gamma\left(\frac{m_2}{2}\right)}\left(\frac{m_1}{m_2}\right)^{\frac{m_1}{2}} \frac{y^{\frac{m_1}{2}-1}}{\left(1+\frac{m_1}{m_2}y\right)^{\frac{m_1+m_2}{2}}}, \quad y > 0.$$

This matches the pdf of the scaled F^*-distribution, confirming that:

$$Y \sim F^*\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{1}{2}\right).$$

Part (d)

Given $Z \sim N(0,1)$ independent of $V \sim \chi^2(m)$, and $T = \frac{Z}{\sqrt{V/m}} \sim t(m)$, we will show that T^2 follows an F-distribution and derive the pdf of the t-distribution.

Step 1: Show that T^2 is an F-variable

By definition:

$$T = \frac{Z}{\sqrt{V/m}}.$$

Squaring both sides, we get:

$$T^2 = \frac{Z^2}{V/m}.$$

Since $Z \sim N(0,1)$, $Z^2 \sim \chi^2(1)$. Also, $V \sim \chi^2(m)$. Therefore:

$$T^2 = \frac{Z^2/1}{V/m}.$$

This is the ratio of two independent chi-square variables divided by their respective degrees of freedom, which is the definition of the F-distribution. Thus:

$$T^2 \sim F(1,m).$$

Step 2: Derive the pdf of T using symmetry

The t-distribution is symmetric about zero, so we derive its pdf by considering the relationship between T and T^2.

Step 2.1: Relationship between T and T^2

Let $Y = T^2$. Then $Y \sim F(1,m)$, and the pdf of Y is:

$$f_Y(y) = \frac{\Gamma\left(\frac{1+m}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{1}{m}\right)^{\frac{1}{2}} \frac{y^{\frac{1}{2}-1}}{\left(1+\frac{y}{m}\right)^{\frac{1+m}{2}}}, \quad y > 0.$$

Step 2.2: Transformation from Y to T

Since $T = \sqrt{Y}$ or $T = -\sqrt{Y}$, we use the transformation method. The Jacobian of the transformation $y = t^2$ is:

$$\left|\frac{dy}{dt}\right| = 2|t|.$$

Thus, the pdf of T is:

$$f_T(t) = f_Y(t^2) \cdot 2|t|.$$

Substituting the pdf of Y, we get:

$$f_T(t) = \frac{\Gamma\left(\frac{1+m}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{1}{m}\right)^{\frac{1}{2}} \frac{(t^2)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{m}\right)^{\frac{1+m}{2}}} \cdot 2|t|.$$

Simplifying, we obtain:

$$f_T(t) = \frac{\Gamma\left(\frac{1+m}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{1}{m}\right)^{\frac{1}{2}} \frac{1}{\left(1+\frac{t^2}{m}\right)^{\frac{1+m}{2}}} \cdot 2|t| \cdot \frac{1}{|t|}.$$

Further simplification yields:

$$f_T(t) = \frac{\Gamma\left(\frac{1+m}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{1}{m}\right)^{\frac{1}{2}} \frac{1}{\left(1+\frac{t^2}{m}\right)^{\frac{1+m}{2}}} \cdot 2.$$

Finally, using the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get the standard form of the t-distribution pdf:

$$f_T(t) = \frac{\Gamma\left(\frac{1+m}{2}\right)}{\sqrt{m\pi}\,\Gamma\left(\frac{m}{2}\right)} \left(1+\frac{t^2}{m}\right)^{-\frac{1+m}{2}}, \quad -\infty < t < \infty.$$